FINITE PLAYER APPROXIMATIONS TO A CONTINUUM OF PLAYERS

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Number 455 January 1987
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August 1986
Revised: January 1987

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* We are grateful to Andreu Mas-Colell for helpful conversations, and to NSF Grants SES 85-09484 and SES 85-09697 for financial support.

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ABSTRACT

In this paper we are interested in the lower hemi-continuity of the Nash equilibrium correspondence with respect to the number of players. Specifically, given an equilibrium in a game with a continuum of players, and a finite player game that approximates the continuum game, is there an equilibrium of the finite player game that is "close" to the equilibrium of the continuum game?
1. Introduction

In this paper we are interested in the lower hemi-continuity of the Nash equilibrium correspondence with respect to the number of players. Specifically, given an equilibrium in a game with a continuum of players, and a finite player game that approximates the continuum game, is there an equilibrium of the finite player game that is "close" to the equilibrium of the continuum game?

We show that lower hemi-continuity obtains if the players' payoffs are differentiable, strictly concave in their own actions, and moreover have the property that the indirect effect that players have on each other is small in the appropriate sense. This bound on the cross-player effects, when carefully specified, implies that the operator representing the cross partial derivatives of players payoffs with respect to each other's actions is compact. It implies also that the continuum game cannot exhibit a robust indeterminacy, and is connected to similar conditions used to establish determinacy in continuum economics in Kehoe, Levine, Mas-Colell and Zame [1986], and Kehoe, Levine and Romer [1987]. A similar condition is used by Araujo and Scheinkman [1977] in studying the value function in dynamic programming.

The upper hemi-continuity of the Nash correspondence has been studied extensively by Green [1984], and by Fudenberg and Levine [1986]. Our [1986] paper studied lower-hemicontinuity as well, but it considered ε-equilibria, providing conditions for an exact equilibrium of a given game to be an ε-equilibrium of games that are nearby. However, the literature on limits of monopolistically competitive equilibrium, including the papers of Green [1980], Mas-Colell [1982], Novshek and Sonnenschein [1980], and Roberts [1980], have examined lower-hemi-continuity of exact equilibria. These
results depend heavily on there being a fixed finite number of types (although not all of the above papers impose strict concavity in own action). With a finite number of types, the cross partial operator has finite rank, and so is certainly compact. Our formulation is more general, allowing situations where payoffs of individuals depend on their own actions in idiosyncratic ways, and on countably many weighted averages of the actions of others. The model's increased generality may help provide a better understanding of the essence of the lower-hemicontinuity results.

Section 2 of the paper sets up the model, and considers the mathematical issue of when a strategy for finitely many players is "like" a strategy for infinitely many of them. Section 3 develops a version of the implicit function theorem that applies when changes in a parameter change the domain on which the function is defined. This is needed since, depending on the number of players, strategy profiles lie in different spaces. Section 4 extends our topology on finite- and infinite-player games, to consider what it means for a finite dimensional linear operator to be "like" an infinite dimensional one. It concludes with the sufficient conditions for lower-hemi-continuity. Finally, Section 5 shows that these conditions are satisfied when players affect each other only through weighted averages of all players actions.

2. The Model

The space $A$ of players actions is a compact convex subset of a fixed finite dimensional vector space $A^*$ with a fixed norm. We shall assume that $A$ is the closure of its interior. The space of players is a compact Hausdorff topological space $P$, while $P_1,P_2,\ldots$ $P$ are a sequence of distinct closed nested subsets of $P$. We also adopt the convention that
We generally think of $P$ as containing a countable dense set, and $P_n$ as a subset of $n$ players, but this is not essential. A player $p \in P_n$ (or $p \in P$) is called an atom of $P_n(P)$ if $p$ is a connected component of $P_n(P)$; that is $p$ is an isolated point. For concreteness it is helpful to think of $A = [0,1]$, $P = [0,1]$ and $P_n = (0, 1/n, 2/n, \ldots, n-1/n, 1)$. In this example $P$ has no atoms and $P_n$ has only atoms.

A strategy profile of the $n^{th}$ game is a continuous map $x_n : P_n \rightarrow A$ from players to actions. We write $x^P$ in place of $x(p)$. The space of all such maps (in the sup norm) is denoted $X_n$. This is a bounded convex subset of the Banach space $X^*_n$ of continuous mappings from $P_n \rightarrow A^*$ in the sup norm. Moreover, $X_n$ is equal to the closure of its interior. Every continuous map $x : P \rightarrow A^*$ has a continuous restriction $x_n : P_n \rightarrow A^*$. Consequently, there are natural restriction mappings $r_n : X^*_n \rightarrow X^*_n$ (we let $r = r_\infty$ denote the identity). These are obviously continuous linear operators.

If $x \in X^*_n$ then $r_n x \in X^*_n$ "converges" to $x$. More generally, we say for $x_n \in X^*_n$ that $x_n \rightarrow x$ provided $|x_n - r_n x| \rightarrow 0$. This makes precise the idea that a profile in a finite player game converges to a profile in the continuum game.

Let $U \subset X$ be open in $X^*$. Corresponding to $U$ are sets $U_n = r_n U$. Notice that by the Tietze extension theorem, for every $x_n \in X^*_n$ there is $\hat{x}_n \in X^*$ with $r_n \hat{x}_n = x_n$ and $|\hat{x}_n| = |x_n|$. Consequently the continuous linear map $r_n$ is onto all of $X^*_n$, and, by the open mapping principle, $U_n$ is an open set.

The payoff function of the $n^{th}$ game is a function $\pi_n : P_n \times A \times X_n \rightarrow \mathbb{R}$. We write $\pi_n^P(a,x)$ in place of $\pi_n(p,a,x)$. Note that the payoff $\pi_n^P$ depends on $p'$s action both directly, and indirectly through the effect
that it has on the joint distribution of players and actions. If \( p \) is an atom and \( \hat{x}a^p \) is the continuous function derived by replacing \( x^p \) with \( a^p \), then it is sometimes convenient to define \( \pi_n^p(a^p, x) = \pi_n^p(a^p, x\hat{a}^p) \) to be the payoff in which \( a^p \) captures both the direct and indirect effect.

If \( p \) is not atomic, there should be no indirect effect, so we define \( x\hat{a}^p = x, \) and \( \pi_n^p[a^p, x] = \pi_n^p(a^p, x). \)

**Definition:** A Nash equilibrium \( \hat{x} \in X_n \) of the \( n \)th game satisfies for all \( p \in P_n \) and \( x^p \in A \)

\[
\pi_n^p[\hat{x}^p, \hat{x}] \geq \pi_n^p(x^p, \hat{x}).
\]

We shall always assume

**Assumption 1:** For each \( x \pi_n^p[a, x] \) as a function of \( a \) is continuously differentiable and strictly concave; both \( \pi_n^p \) and its derivative with respect to \( a \) are jointly continuous with respect to \( p \) and \( a \).

Let \( \phi_n^p(x) = D_a \pi_n^p[x^p, x] \). Under Assumption 1 \( \phi_n^p(x) \) is continuous in \( p \), so \( \phi_n(x) \in X_n \). Moreover, it is necessary and sufficient for a Nash equilibrium \( x_n \in U_n \) that \( \phi_n(x_n) = 0. \)

We assume that we are given an equilibrium \( \hat{x} \in U \) with \( \phi(\hat{x}) = 0. \) Our goal is to give sufficient conditions relating the \( \phi_n \) to \( \phi \) which guarantee that there are solutions \( \hat{x}_n \in U_n \) with \( \phi_n(\hat{x}_n) = 0 \) and \( \hat{x}_n \to \hat{x}. \)

In other words, we will provide conditions for the Nash equilibrium correspondence to be lower hemi-continuous in the continuum of players limit.

3. **An Implicit Function Theorem**

Suppose that instead of the discrete parameter \( n \) and separate spaces \( X_n \), games are indexed by \( \lambda \) and for all \( \lambda \) the game is played in \( X. \) If
\( \phi(\lambda, x) - 0 \) is the equilibrium condition, \( \phi \) is continuously differentiable in \( U \) and \( D_x \phi(\lambda, \hat{x}) \) is non-singular, the implicit function theorem implies that for \( \lambda \) close enough to \( \hat{\lambda} \) there are solutions \( \hat{x}(\lambda) \) to \( \phi(\lambda, \hat{x}(\lambda)) = 0 \) and \( \hat{x}(\lambda) \to \hat{x} \) as \( \lambda \to \hat{\lambda} \). Our goal is to make sense of how an argument like this might work when games are parameterized by a discrete parameter \( n \), and the spaces \( X_n \) on which the equilibrium condition is defined depend on \( n \).

Now let us consider \( \phi(\lambda, x) \). If this is continuously differentiable then \( \forall \lambda, x \in U, \quad \epsilon > 0 \exists \delta, N \text{ s.t. } \frac{|x' - x|}{|\lambda' - \lambda|} < \delta; \frac{|x' - x|}{|\lambda' - \lambda|} < 1/N \) implies

\[
|D_x \phi(\lambda', x') - D_x \phi(\lambda, x)| < \epsilon.
\]

**Definition:** The discrete family \( \phi_n \) is uniformly differentiable if each \( \phi_n \) is continuously differentiable, and \( \forall x \in U, \quad \epsilon > 0, \exists \delta, N \text{ s.t. } \frac{|x' - x|}{|\lambda' - \lambda|} < \delta, \quad n > N \implies |D\phi_n(r, x') - D\phi_n(r, x)| < \epsilon. \)

**Assumption 2:** The family \( \phi_n \) (including \( n = \infty \)) is uniformly differentiable, and \( \phi_n(r, x) \to \phi(x) \).

Our basic tool is

**The Inverse Function Theorem:** Suppose \( X \) and \( Y \) are Banach spaces and \( A: X \to Y \) is a non-singular continuous linear operator. Let

\[ \hat{U} = \{x \in X \mid |x - \hat{x}| \leq b\} \]

\( b > 0 \), and suppose \( \eta: \hat{U} \to Y \) is Lipshitz with constant \( \kappa \). If

\[ |A\hat{x} + \eta(\hat{x})| \leq b(|A^{-1}|^{-1} - \kappa) > 0 \]

then there is a unique point \( \hat{x} \in \hat{U} \) with

\[ A\hat{x} + \eta(\hat{x}) = 0. \]

Moreover

\[
|\hat{x} - x| \leq \frac{|A\hat{x} - \eta(\hat{x})|}{|A^{-1}|^{-1} - \kappa}.
\]
Proof: This is an implication of the contraction mapping fixed point theorem. Observe that \( A\hat{x} + \eta(\hat{x}) = 0 \) if and only if \( \hat{x} = -A^{-1}\eta(\hat{x}) \).

Obviously \( \hat{U} \) is a complete metric space; we must show that \(-A^{-1}\eta\) is a contraction of \( \hat{U} \).

First we show that \(-A^{-1}\eta\) maps \( \hat{U} \) into \( \hat{U} \). This follows from the sequence of inequalities

\[
| -A^{-1}\eta(x) - \hat{x} | \leq | -A^{-1}\eta(\hat{x}) - \hat{x} | + | -A^{-1}\eta(x) + A^{-1}\eta(\hat{x}) |
\]

\[
\leq | A^{-1} | ( | \eta(\hat{x}) + A\hat{x} | + | \eta(x) - \eta(\hat{x}) | )
\]

\[
\leq | A^{-1} | ( b(|A^{-1}|^{-1}\kappa) + \kappa b ) - b.
\]

(We obtain the last inequality by substituting the bounds on \( |x - \hat{x}| \) and \( |A\hat{x} - \eta(\hat{x})| \).) Moreover

\[
| -A^{-1}\eta(x) + A^{-1}\eta(x') | \leq | A^{-1} | | \eta(x) - \eta(x') | \leq | A^{-1} | \kappa | x - x' |,
\]

and since \( |A^{-1}|^{-1} - \kappa > 0, \ |A^{-1}| \kappa < 1 \) and so \(-A^{-1}\eta\) is a contraction.

This shows that \( \hat{x} \) exists and is unique.

It remains to estimate \( |\hat{x} - \check{x}| \).

\[
| \check{x} - \hat{x} | = | -A^{-1}\eta(\hat{x}) - \hat{x} |
\]

\[
\leq | -A^{-1}\eta(\hat{x}) - \hat{x} | + | -A^{-1}\eta(\hat{x}) + A^{-1}\eta(\hat{x}) |
\]

\[
\leq | A^{-1} | | \eta(\hat{x}) + A\hat{x} | + | A^{-1} | | \eta(\hat{x}) - \eta(\hat{x}) |
\]

\[
\leq | A^{-1} | | \eta(\hat{x}) + A\hat{x} | + | A^{-1} | \kappa | \hat{x} - \check{x} |.
\]

Solving for \( |\check{x} - \hat{x}| \) we find

\[
|\check{x} - \hat{x}| \leq \frac{|A^{-1}| |\eta(x) - A\hat{x}||}{1 - |A^{-1}| \kappa} = \frac{|\eta(\hat{x}) - A\hat{x}||}{|A^{-1}|^{-1} - \kappa}.
\]

Q.E.D.

We now apply the inverse function theorem to the family \( \phi_n \).
The Implicit Function Theorem: Suppose $\phi_n$ is a uniformly differentiable family, that $\lim \inf |D\phi_n(r_n \hat{x})|^{-1} > 0$, and that $\phi_n(r_n \hat{x}) \to 0$. Then there is $U' \subset U$, and for large enough $n$ a unique $\hat{x}_n$, such that $\phi_n(\hat{x}_n) = 0$ and $\hat{x}_n \in r_n U'$. Moreover, $\hat{x}_n \to \hat{x}$.

Proof: Take $A_n = D\phi_n(r_n \hat{x})$. By assumption $\sigma = \lim \inf |A_n^{-1}|^{-1} > 0$. Now choose $b$ so small and $N$ so large that for $n \geq N$ and $|x - \hat{x}| \leq b$, $|x' - \hat{x}| \leq b$

(3.1) $|D\phi_n(r_n x') - D\phi_n(r_n \hat{x})| \leq \sigma/4$

(3.2) $|\phi_n(r_n \hat{x})| \leq b \sigma/4$

(3.3) $|A_n^{-1}|^{-1} \geq 3\sigma/4$

Define $\eta_n(x_n) = [\phi_n(x_n) - A_n x_n]$. Consequently, $A_n x_n + \eta_n(x_n) = \phi_n(x_n)$. This implies $|A_n r_n \hat{x} + \eta_n(r_n \hat{x})| \leq b \sigma/4$. From the inverse function theorem we need only show $|A_n^{-1} - \kappa_n| \geq \sigma/4$ where $\kappa_n$ is the Lipshitz constant for $\eta_n$ on $|x_n - r_n \hat{x}| \leq b$. Since $|A_n^{-1}| \geq 3\sigma/4$, we need only show $\kappa_n \leq \sigma/2$. We have for $|x - \hat{x}| \leq b; |x' - \hat{x}| \leq b, x_n = r_n x, x_n' = r_n x'$

(3.4) $|\eta_n(x_n') - \eta_n(x_n)| = |\phi_n(x_n') - \phi_n(x_n) - A_n(x_n' - x_n)|$

$\leq |\phi_n(x_n') - \phi_n(x_n) - D\phi_n(x_n)(x_n' - x_n)|$

$+ |[D\phi_n(x_n) - D\phi_n(r_n \hat{x})](x_n' - x_n)|$

$\leq (\sigma/4) |x_n' - x_n|$

$+ (\sigma/4) |x_n' - x_n| \leq (\sigma/2) |x_n' - x|$

where $|\phi_n(x_n') - \phi_n(x_n) - D\phi_n(x_n)(x_n' - x_n)| \leq \sigma/4 |x' - x|$ follows from (3.1) and the Fundamental Theorem of Calculus. To see this, write
\[
|\phi_n(x') - \phi_n(x) - D\phi_n(x_n)(x'_n - x_n)| - \\
\int_0^1 [D\phi_n(x_n + t(x'_n - x_n)) - D\phi_n(x_n)](x'_n - x_n) dt \leq [\int_0^1 |D\phi_n(x_n + t(x'_n - x_n)) - D\phi_n(x_n)| dt] |x'_n - x_n|.
\]

From (3.1), and \(|x - \hat{x}|, |x' - \hat{x}| \leq b\), it follows that

\[\int_0^1 |D\phi_n(x_n + t(x'_n - x_n)) - D\phi_n(x_n)| \leq \sigma/4. \quad \text{Q.E.D.}\]

4. Convergence of Operators

We wish to apply the inverse function theorem to conclude that a Nash equilibrium \(\phi(\hat{x}) = 0\) of the continuum game can be approximated by Nash equilibria of games with finitely many players. To do so we must ensure that \(D\phi_n(r_n \hat{x})\) is non-singular for sufficiently large \(n\). If the games were indexed by a continuous parameter \(n\), and played on a common space \(X\), then \(D_x\phi^{-1}(\lambda, \hat{x})\) non-singular would imply that \(D_x\phi^{-1}(\lambda, \hat{x})\) is non-singular for \(\lambda\) sufficiently near to \(\lambda\). In the case at hand, however, \(D\phi(\hat{x})\) non-singular is not sufficient to ensure that \(D\phi_n(r_n \hat{x})\) is non-singular as well.

Suppose \(F_n : X^*_n \rightarrow X^*_n\) and \(F : X^* \rightarrow X^*\) are continuous linear operators. We will say that \(F_n \rightarrow F\) if for every sequence \(x_n \in X^*_n\) with \(|x_n| = 1\) there is a sequence \(\hat{x}_n \in X^*, |\hat{x}_n| = 1\) such that

\[|x_n - r_n \hat{x}_n| \rightarrow 0 \quad \text{such a sequence exists by the Tietze extension theorem).}\]

\[\text{if } |x_m - r_m z| \rightarrow 0 \quad \text{for some subsequence } m \text{ then } \hat{x}_m \rightarrow z.\]

\[|F_n x_n - r_n F\hat{x}_n| \rightarrow 0.\]

A simpler and more obvious definition of convergence would be to require that \(F_n x_n \rightarrow Fx\) for every \(x \in X^*\). If we add the requirement that
$|F_n|$ is uniformly bounded, this can be shown equivalent to our definition of convergence holding for convergent sequences only. We refer to this as **weak** convergence. To show the extra bite from requiring that $F_n$ and $F$ are similar for non-convergent sequences consider the following example. Set $P = [0,1]$, $P_n = (0,1/n, \ldots, 1)$, and $A = [0,1]$. Let $F_n$ be the matrix in which every entry in the first column is one, every entry in the second column minus one, and all remaining entries zero, i.e.,

$$F_n = \begin{bmatrix}
1 & -1 & 0 & \ldots \\
1 & -1 & 0 & \\
\vdots & & &
\end{bmatrix},$$

while $F = 0$. For any fixed $x$, $|r_n x(0) - r_n x(1/n)| \to 0$, since $x$ is continuous. Consequently $|F_n r_n x - r_n F x| = |F_n r_n x| = |r_n x(0) - r_n x(1/n)| \to 0$. In other words $F_n \to 0$ weakly. However, $|F_n| = 2$, so the norm is not continuous. If we allow sequences $x_n$, not necessarily convergent, then we can choose $x_n = (1,0,0,\ldots)$. In this case $F_n x_n = 1$, while $F x = 0$ for all $x$. Thus, $F_n \neq 0$, in our sense. In fact, it can easily be shown that if $F_n \to F$ then $|F| \geq \lim \sup |F_n|$.  

While our definition is a fairly restrictive notion of convergence, it does not follow that if $F_n \to F$ and $F$ is non-singular then $F_n$ is eventually non-singular. A counterexample with $P = [0,1]$ $P = (0,1/n, \ldots, 1)$ and $A = [0,1]$ is the operator

$$(Fx)(p) = \begin{cases} 
x(2p) & 0 \leq p \leq 1/3 \\
x(p/2 + 1/2) & 1/3 \leq p \leq 1.
\end{cases}$$

If $R_n$ maps $X^*$ to its linear extrapolation in $X^*$, the operators $F_n = r_n FR_n$ obviously converge to $F$. Moreover it can be shown that $F$ is non-singular and $F_n$ are always singular. The problem with the operator $F$ is that while it is non-singular, its restriction to the invariant subspace
of functions with \( x(p) = 0 \) for \( 0 \leq p \leq 1/3 \) is one-to-one but not onto. No non-singular finite dimensional operator can be singular on an invariant subspace. In other words, it is unreasonable to think that an operator like \( F \) is a reasonable limit of finite dimensional operators.

One important class of operators for which the finite and infinite dimensional versions are similar is the diagonal operators. Let \( L(A) \) be the linear operators from \( F \) to itself. A diagonal operator \( F_n : X^* \rightarrow X^* \) is representable by a continuous map \( f_n : P_n \rightarrow L(A) \), such that

\[
(F_n x)(p) = f_n(p)x(p),
\]

that is, \( F_n \) is the operator induced by pointwise multiplication with \( f_n \). Similarly we can define a diagonal operator \( F : X^* \rightarrow X^* \). These operator arise naturally as the derivative of \( \phi \) with respect to the players' own actions: the diagonal is simply the second derivative of players payoffs with respect to their own actions.

It is easy to show that for a diagonal operator

\[
|F_n| = \sup_{p \in P_n} |f_n(p)|, \quad \text{and} \quad |F_n^{-1}| = (\sup_{p \in P_n} |f_n^{-1}(p)|)^{-1}.
\]

Moreover, for diagonal operators \( F_n \rightarrow F \) and \( F \) non-singular implies

\[
\lim \inf |F_n^{-1}|^{-1} > 0. \quad \text{This follows from Lemma 4.1:} \quad \text{If} \ F_n \quad \text{and} \ F \quad \text{are diagonal} \ F_n \rightarrow F \quad \text{if and only if}
\]

\[
\sup_{p \in P_n} |f_n(p) - f(p)| \rightarrow 0.
\]

**Proof:** That \( \sup_{p \in P_n} |f_n(p) - f(p)| \rightarrow 0 \) implies \( F_n \rightarrow F \) is straightforward. We show the converse. Suppose there is a subsequence with
\[ |\tilde{x}(p_n) - x(p_n)| > \epsilon. \]

Choose \( x_n \) so that \( x_n(p_n) = 1 \). We may assume \( \tilde{x}_n \) has \( |F_n x_n - x_n F \tilde{x}_n| \to 0 \) and \( |F| |\tilde{x}_n(p_n) - 1| \leq \epsilon/2 \). Then

\[
|F_n x_n - x_n F \tilde{x}_n| \\
|\tilde{x}_n(p_n) - 1| \\
|\tilde{x}(p_n)| \\
|\tilde{x}_n(p_n) - x(p_n)| \\
|f(p_n) - f(p_n)\tilde{x}(p_n)| \\
|f(p_n) - f(p_n)| - |f(p_n)(1 - \tilde{x}_n(p_n))| \geq \epsilon/2.
\]

This contradiction establishes the Lemma. Q.E.D.

Notice that as far as diagonal operators are concerned, we could have used weak convergence, rather than convergence.

We now characterize a broad class of cases for which \( F \) non-singular and \( F_n \to F \) imply that \( \lim \inf |F_n^{-1}|^{-1} > 0 \). An operator \( C \) on a Banach space is compact if the closure of the image of an arbitrary bounded set is compact. A broad class of compact operators can be found by taking strong limits of operators with a finite dimensional range; in some Banach spaces, all compact operators have this form. As a result, it is natural to think of the derivative of a player's incentives \( \phi \) with respect to what everyone else does as compact provided that there are only finitely many important channels of interaction. A formal model of what this means is found in the next section. From a mathematical point of view, compact operators are important, because, like diagonal operators, they can be approximated by finite dimensional operators.

**Proposition 4.2:** Suppose \( F_n \to F; \ F_n = B_n + C_n; F = B + C; B_n \to B \) are diagonal; \( F, B \) non-singular and \( C \) compact. Then \( \lim \inf |F_n^{-1}|^{-1} > 0 \).

First we prove a preliminary lemma.
Lemma 4.3: Under the hypotheses of Proposition 4.2 \( B_n^{-1} C_n \rightarrow B^{-1} C \).

Proof: Let \( |x_n| = 1 \). Since \( F_n \rightarrow F \), there is a sequence \( |\tilde{x}_n| = 1 \) in \( X \) satisfying (4.1) to (4.3). In other words

\[
(4.4) \quad |B_n x_n + C x_n - r_n B \tilde{x}_n - r_n C \tilde{x}_n| \rightarrow 0.
\]

We will show

\[
(4.5) \quad |B_n^{-1} C x_n - r_n B^{-1} C \tilde{x}_n| \rightarrow 0.
\]

By Lemma 4.1 and \( B \) non-singular \( |B_n^{-1}| \) is eventually bounded above. Consequently from (4.4)

\[
\frac{|B_n^{-1}|}{|B_n x_n + C x_n - r_n B \tilde{x}_n - r_n C \tilde{x}_n|} \rightarrow 0
\]

implying

\[
|x_n + B_n^{-1} C x_n - B^{-1} r_n B \tilde{x}_n - B^{-1} r_n C \tilde{x}_n| \rightarrow 0.
\]

This expression is at least as great as

\[
|B_n^{-1} C x_n - r_n B^{-1} C \tilde{x}_n| - |x_n - r_n \tilde{x}_n| - |r_n \tilde{x}_n - B_n^{-1} r_n B \tilde{x}_n + r_n B^{-1} C \tilde{x}_n - B^{-1} r_n C \tilde{x}_n|.
\]

Since by (4.1) \( |x_n - r_n \tilde{x}_n| \rightarrow 0 \), it suffices to show that the final negative term goes to zero. Since \( |\tilde{x}_n| = 1 \), it is bounded above in absolute value by

\[
(|B_n^{-1}| + |C|) |r_n B^{-1} - B_n^{-1} r_n|.
\]

Since \( |B_n^{-1}| \) is bounded, we need only show \( |r_n B^{-1} - B_n^{-1} r_n| \rightarrow 0 \). Since \( B \) and \( B_n \) (and their inverse) are diagonal

\[
|r_n B^{-1} - B_n^{-1} r_n| = \sup_{p \in \mathbb{P}_n} |B^{-1}(p) - B_n^{-1}(p)|.
\]

Since \( B_n \rightarrow B \), the final expression goes to zero by Lemma 4.1. Q.E.D.
Proof of Proposition 4.2: If the proposition is not true we can find $x_n \in X$ with $|x_n| = 1$ and $|F_n x_n| \to 0$. Since $|B_n^{-1}|$ is bounded, this implies $|B_n^{-1} F_n x_n| \to 0$. Consequently

\begin{equation}
|x_n - B_n^{-1} C_n x_n| \to 0.
\end{equation}

(4.6)

By Lemma 4.3, choose $\tilde{x}_n \in X$ with $|\tilde{x}_n| = 1$, $|x_n - r_n \tilde{x}_n| \to 0$ and

\begin{equation}
|B_n^{-1} C_n x_n - r_n B_n^{-1} C_n \tilde{x}_n| \to 0.
\end{equation}

(4.7)

Notice that this step is where we need strong rather than weak convergence: there is no reason $x_n$ should have a convergent subsequence. From (4.6)

\begin{equation}
|x_n + r_n B_n^{-1} C_n \tilde{x}_n| \to 0.
\end{equation}

(4.8)

Since $C$ is compact, and $\tilde{x}_n$ is bounded, $B_n^{-1} C \tilde{x}_m \to z$ for a properly chosen subsequence. This implies $r_m B_n^{-1} C \tilde{x}_m \to r_m z$. Since $|E_n| \to 0$, (4.8) implies $|x_m + r_m z| \to 0$. Then from (4.2),

$\tilde{x}_n \to z$.

Since $B_n^{-1} C \tilde{x}_m \to z$, we conclude $|\tilde{x}_m + B_n^{-1} C \tilde{x}_m| \to 0$. Since $|\tilde{x}_m| = 1$ this contradicts $I + B_n^{-1} C = B_n^{-1}(B+C) = B_n^{-1} F$ non-singular. Q.E.D.

Now we provide conditions on $D_\phi_n(x)$ that permit us to appeal to Proposition 4.2. Recall that in the payoff function $\pi_n(a,x)$, player p's own action affects his payoff both directly and through its "indirect effect" on the vector $x$. We have already assumed that $\pi_n[a,x]$, which incorporates the indirect effects into the first argument, is concave in $x$. We will now also assume that the direct effects themselves are concave:

**Definition:** $B_n(x)$ is the diagonal operator whose entries are $\frac{D^2}{a_x} \pi_n(a,x)$. 
Assumption 3: For all $x$, $B_n(x)$ is strictly negative definite, and $B_n(rx) 	o B(x)$.

From Proposition 4.2 we see that we should also assume

Assumption 4: $D\phi_n(rx) 	o D\phi(x)$; $D\phi(\hat{x}) - B(\hat{x})$ is compact.

The condition $D\phi(\hat{x}) - B(\hat{x})$ compact is a restriction on the indirect effect that players can have on each other. In the next section we show that it is satisfied in the case where $\pi_n^D(a,x)$ depends on $x$ only through the average value of $x$.

Under these assumptions we can use Proposition 4.2 to show that the Nash correspondence is lower hemi-continuous at the continuum of players limit.

Theorem (Lower Hemicontinuity): If $\phi(\hat{x}) = 0$, $D\phi(\hat{x})$ is non-singular, and Assumptions 1 through 4 are satisfied, then there exists an $N$ and a sequence $\hat{x}_n$, $n > N$, such that $\phi_n(\hat{x}_n) = 0$ and $\hat{x}_n \to \hat{x}$.

Finally, let us note a useful fact: to show $D\phi_n(rx) 	o D\phi(x)$ it suffices to show that $D\phi_n(rx) - B_n(rx)$ converges to $D\phi(x) - B(x)$. This follows from

Lemma 4.4: Suppose $F = B + C$; $F = B + C$, $B_n$ and $B$ are diagonal $B_n \to B$ and $C_n \to C$. Then $F_n \to F$.

Proof: Given $x \in X^*$, $|x| = 1$, find $\tilde{x} \in X^*$, $|\tilde{x}| = 1$, as guaranteed by the fact that $C_n \to C$. Then since $B_n$ and $B$ are diagonal, it follows that $|B_n x_n - r B_\tilde{x}_n| \to 0$. Consequently $|F_n x_n - r F_\tilde{x}_n| \to 0$ as required.

Q.E.D.
5. **Compactness With Weighted Averages**

This section shows that the sufficient conditions of Section 4 are satisfied by games in which each player cares only about a weighted average of his opponents' actions, and not about the play of each individual opponent. We will begin by considering a single weighted average. This is, of course, the case in Cournot competition with a single homogeneous good, which is the game in which lower hemi-continuity has been most extensively studied. Below, we extend these results to the case in which payoffs depend on a finite number of weighted averages or even infinitely many.

We now take \( A = [0,1] \), \( P = [0,1] \) and \( P_n = (0,1/n,2/n,\ldots,n-1/n,1) \). We suppose that

\[
\pi_n^P(p,a,x_n) = g(p,a, \sum_{q \in P_n} h(q)x_n^{q}/n)
\]

and \( \pi(p,a,x) = g(p,a, \int h(q)x^q dq) \), where the weighting function \( h: P \to \mathbb{R} \) is \( C^1 \) and \( g: P \times A \times A \to \mathbb{R} \) is \( C^2 \) and concave in its second argument.

We will show that in this case Assumptions 1 through 4 are satisfied.

For \( n < \infty \) Assumption 1, concavity and continuity, follows from the fact that

\[
g(p,a, \sum_{q \in P_n \setminus p} h(q)x_n^{q}/n + h(p)a/n)
\]

is \( C^2 \) jointly in \( p \) and \( q \), and, for large enough \( n \), strictly concave in \( a \). For \( n = \infty \) Assumption 1 follows immediately from the properties of \( g \).

For notational convenience define the averages

\[
\tilde{x}_n = \sum_{q \in P_n} h(q)x_n^{q}/n
\]

and \( \tilde{x} = \int h(q)x^q dq \). Notice that if \( x_n \to x \) then ordinary integration theory implies \( \tilde{x}_n \to \tilde{x} \). We now verify Assumptions 2 through 4. We begin by
computing $\phi_n^P(x)$ and $\phi^P(x)$:

\begin{align}
\phi_n^P(x) & = D_2g(p,x_n^P,\ddot{x}_n) + h(p)D_3g(p,x_n^P,\ddot{x}_n)/n \\
\phi^P(x) & = D_2g(p,x^P,\ddot{x}).
\end{align}

Now we will differentiate (5.1), and show that $D_n\phi_n(r_n,x)$ and $D\phi(x)$ can be decomposed as $B_n + E_n + C_n$ and $B + C$, respectively, in a way that satisfies Assumptions 3 and 4. Let $B_n$ be the diagonal matrix whose $p^{th}$ entry, $b_n(p)$, is

$$b_n(p) = D_2g(p,x_n^P,\ddot{x}_n),$$

and let $E_n$ be the diagonal matrix whose $p^{th}$ entry is

$$e_n(p) = (h(p)/n) D_3g(p,x_n^P,\ddot{x}_n).$$

To define $C_n$, set

$$c_n(p) = D_2g(p,x_n^P,\ddot{x}_n) + (h(p)/n)D_3g(p,x_n^P,\ddot{x}_n).$$

Let $c_n$ be the corresponding column vector, and set

$$h_n = (h(0)/n,h(1/n)/n,\ldots,h(1)/n).$$

Then $C_n = c_nh_n$.

In the continuum limit, $B$ is as before, the diagonal matrix containing the $D_2g$ terms, and $C = ch$, where

$$c = D_2g(p,x_n^P,\ddot{x})$$

and $h$ is the linear functional defined by $hx = \int h(q)x(q)dq$.

Then $D_n\phi_n = B_n + E_n + C_n$, $D\phi = B + C$. Clearly $B_n \to B$, and since $B_n$ is strictly negative definite, Assumption 3 is satisfied.

Next observe that $D\phi(\ddot{x}) = B - ch$ has rank one, and is thus compact.

To satisfy Assumption 4, we must also show that $D_n\phi_n(r_n,x) \to D\phi(x)$, which is equivalent to $E_n + C_n \to C$. Since $E_n$ clearly converges to $0$ and is
diagonal, it suffices to show \( C_n \rightarrow C \).

To do this, we assume that we are given a sequence \( x_n \) with \( |x_n| = 1 \). We must construct a continuous approximation \( \tilde{x}_n \) with \( |	ilde{x}_n| = 1 \), such that

1. \( |x_n - r_n \tilde{x}_n| \rightarrow 0 \).
2. \( \tilde{x}_n \) converges if \( x_n \) does, and
3. \( |C_n x_n - r_n C \tilde{x}_n| \rightarrow 0 \).

Recall that \( C_n = C / n \) and \( C = \text{ch} \). Since \( c_n \rightarrow c \), it will be sufficient to choose \( \tilde{x}_n \) so that the scalar difference \( |h_n x_n - h \tilde{x}_n| \rightarrow 0 \). That is, the approximating \( \tilde{x}_n \) must yield essentially the same average with the limiting weight-functional \( h \) as \( x_n \) does with \( x_n \). Moreover, \( \tilde{x}_n \) must be continuous. If \( x_n \) has a (continuous) limit \( x \), then we can simply take \( \tilde{x}_n = x \). However, we must also worry about non-convergent \( x_n \). To do this, we set \( \tilde{x}_n \) to be a continuous approximation to the step-function associated with \( x_n \), that is, \( \tilde{x}_n \) will have \( n \) constant segments centered at \((p_1, p_2, \ldots, p_n)\). Along the \( k \)th segment, we will set \( \tilde{x}_n(p) = x_n(p_k) \). We then connect these segments by linear interpolation. Thus as \( n \) grows, \( \tilde{x}_n \) is an increasingly steeply sloped continuous function. To ensure that

\( |h_n x_n - h \tilde{x}_n| \rightarrow 0 \), the area in which \( \tilde{x}_n - x_n \) must go to zero faster than \( 1/n \); \( 1/n^2 \) will do. [The computations are available from the authors upon request.]

Finally, we verify Assumption 2. Set \( F'_n = D_\phi(x, x') \). Evidently

\[
|F_n - F'_n| \leq \sup_{p \in \mathbf{P}_n} |b_n(p) - b'_n(p)| + |e_n(p) - e'_n(p)| + |c_n h_n - c'_n h_n|.
\]

Now \( |c_n h_n - c'_n h_n| \) is less than \( |c_n - c'| |h_n|_1 \), where \( |c_n - c'| \) is measured in the sup or \( L_\infty \) norm, and \( |h_n|_1 \) is measured in the dual or \( L_1 \) norm. It is clear that for \( \epsilon_n \rightarrow 0 \).
\[ \sup_{p \in P_n} |b_n(p) - b'(p)| \leq \sup_{p \in P} |b(p) - b'(p)| + \epsilon_n, \]
\[ \sup_{p \in P_n} |e_n(p) - e'(p)| \leq \epsilon_n \]
and similarly \[ |c_n - c'_n| \leq |c - c'| + \epsilon_n \] respectively. Consequently, we have the uniform bound
\[ |F_n - F'| \leq \sup_{p \in P} |b(p) - b'(p)| + |c - c'| |h|_{\infty} + 3 \epsilon_n \]
and the first half of Assumption 2 follows from the fact that \( b \) and \( c \) are continuous in \( x \). The last half, that \( \phi_n(r, x) \to \phi(x) \) is immediate from (5.1).

Next we consider several weighted averages. In this case we write \( g(p, a, x^1, \ldots, x^k) \) where \( \hat{x}^i = \int h^i(q) x^i dq \) is the \( i \)th weighted average, and similarly in the finite case. In this case
\[ \phi^p_n = D_2 g(p, x^p_n, \hat{x}^1_n, \ldots, \hat{x}^k_n) + \sum_{i=1}^k h^i(p) D_{1+i} g(p, x^p_n, \hat{x}^1_n, \ldots, \hat{x}^k_n) \]
while only the arguments of \( \phi^p \) in (5.1) change. The definition of \( b_n \) and \( b \) are unchanged, we have
\[ e_n^i(p) = \frac{(h^i(p)/n)}{D_{1+i}^2} g(p, x^p_n, \hat{x}^1_n, \ldots, \hat{x}^k_n) \]
and
\[ c_n^i(p) = D_{2(2+i)} g(D, x^p_n, \hat{x}^1_n, \ldots, \hat{x}^k_n) + \sum_{j=1}^k \frac{(h^j(p)/n)}{D_{(j+1)(i+1)}} g(p, x^p_n, \hat{x}^1_n, \ldots, \hat{x}^k_n) \]
while
\[ c^i(p) = D_{2(2+i)} g(p, x^p, \hat{x}^1, \ldots, \hat{x}^k). \]
We write \( D\phi_n = B_n + \sum_{i=1}^k E^i_n + \sum_{i=1}^k C^i_n \) and \( D\phi = B + \sum_{i=1}^k C^i \). The proof
is as in the case of a single average, with one exception: we must show that \( c^i_n - c^i \) implies that \( \sum_{i=1}^{k} c^i_n \to \sum_{i=1}^{k} c^i \).

Define \( d_{\infty}(c_n, c) = |c_n - r_n c|_\infty \) to be the "sup" distance between \( c_n \) and \( c \), let \( \tilde{x}_n \) be the extrapolation of \( x_n \) described above, and set

\[
d_1(h_n, h) = \sup |h_n x_n - h \tilde{x}_n| \]

to be the "linear functional" distance. Since \( C_i^i_n = c_i^i h_n^i_n \) and \( C_i = c_i^i h^i \), we have

\[
\left| \sum_{i=1}^{k} C_i^i n x_n - \sum_{i=1}^{k} r_n C_i n \tilde{x}_n \right| \leq \sum_{i=1}^{k} |h_n^i|_1 d_{\infty}(c_n^i, c^i) + |c^i| d_1(h_n^i, h_i^i).
\]

We showed above \( d_{\infty}(c_n^i, c^i) \to 0 \), and it can be shown that \( d_1(h_n^i, h_i^i) \to 0 \) (that is the gap between \( h_n x_n \) and \( h \tilde{x}_n \) is uniform in \( x_n \)), and \( |h_n^i|_1 \to |h^i| \).

This reasoning may also be extended to a function of infinitely many averages \( g(p, x^1, x^2, \ldots) \) provided that when \( k = \infty \) above, the sums converge absolutely. In other words, the later averages in the sequences have relatively little impact on utility. An important case is where \( h^1 \) is uniform; \( h^2 \) is uniform on \([1/2, 1]\), \( h^3 \) on \([1/4, 1/2]\), \( h^4 \) on \([3/4, 1]\) and so forth. In this case \( g \) depends on all of \( x \), but variations in \( x \) over very small intervals do not make very much difference.
References


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