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A Graphical Depiction of the Rubinstein-Ståhl Bargaining Solution*

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April 18, 1995

(original version: March, 1993)

Abstract

I illustrate the convergence to Rubinstein's (1982) bargaining outcome of the solutions of the finite-horizon truncations of that game. The depiction is new, and hopefully instructive, and has the flavour of international trade diagrams.

*This title simply owes itself to the fact that by game theory lore and tradition, the finite horizon Rubinstein (1982) model is linked to Stahl's (1972) work. As it turns out, Stahl himself, in his recent (1994) working paper, sets the record straight, disabusing me of this notion. His 1972 model is qualitatively different, restricting focus to finite outcome bargaining with nonstationary and decreasing payoffs, but deriving an order-independent bargaining solution.

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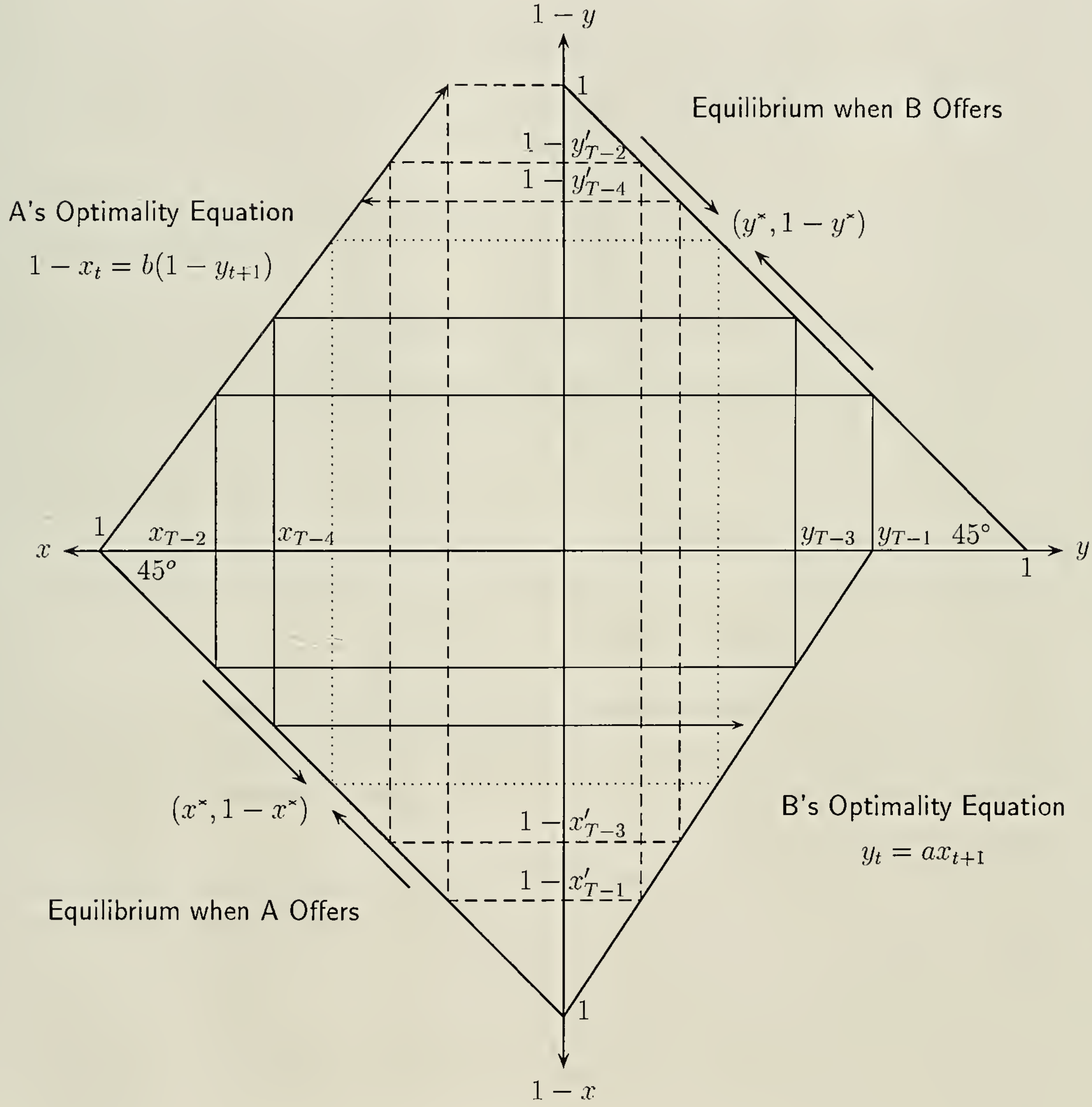
1. INTRODUCTION

The theory of bilateral bargaining is arguably the archetypal strategic situation. To quote Schelling (1960), "most conflict situations are essentially bargaining situations." The simple infinite-horizon alternating-offer bargaining model made famous in Rubinstein (1982) is not only of interest to researchers, but also has great pedagogic value for its simplicity. In his classic infinite horizon game, two individuals A and B are bargaining over a 'pie' of size 1. Each period, one player makes an offer, and the other immediately accepts or rejects it. Rubinstein and successors have shown that the following subgame perfect equilibrium is unique: When called upon to offer, each makes an offer that leave the other indifferent between accepting and rejecting, knowing that the other is planning to do likewise next period. Hence, given positive discount factors $a, b < 1$, A always offers the division $(x^*, 1 - x^*)$, while B always offers $(y^*, 1 - y^*)$, where $y^* = ax^*$ and $1 - x^* = b(1 - y^*)$. Rubinstein (1982) was able to depict the above outcome in a simple Edgeworth box diagram with axes x and y.

Binmore (1987a) proved that the agreement reached in the finite-horizon truncation of Rubinstein's bargaining game converges to the unique Rubinstein as the horizon tends to infinity. The purpose of this brief note is purely pedagogical: I provide a simple diagram that illustrates both the Rubinstein outcome and the above convergence. In light of the known purely algebraic arguments for this fact, it is helpful to simply be able to visualize the convergence at work. My diagram is inspired by many two-sector international trade depictions I have seen, which is altogether fitting: After all, Rubinstein's bargaining game *is* about trade.

I should note that Binmore (1987b) provides a wealth of *purely stylized* figures that depict this convergence. Admittedly, one can also use Rubinstein's original (1982) diagram to illustrate this convergence, simply by drawing in the difference equation convergence process. But the final result is rather cluttered, and the intuition for the tatonnment somewhat obscured. My purpose here has been to proceed in a more illuminating fashion. Much in the spirit of international trade analyses that must maintain equilibrium in two separate sectors, my diagram separates the two players' equilibrating processes from their optimizations. This allows one to illustrate other aspects of the Rubinstein outcome.

Figure 1: **Rubinstein Outcome as Dual Limit of Finite Horizon Solutions.** The dashed (resp. undashed) line depicts the subgame perfect offers that would arise in the finite horizon bargaining game in which B (resp. A) offers last, and all outside options are zero. In all four quadrants, A's payoffs are demarcated along the horizontal axes, and B's along the vertical axes, positive radiating away from the origin. It's easy to see how the two cycles converge upon the Rubinstein outcome (the dotted cycle) from opposite sides.



2. A BRIEF DISCUSSION OF THE FIGURE

In Figure 1, the 'optimality equation' for each player describes the maximum he can demand and still leave the other player indifferent about accepting. The two equilibrium quadrants describe what fraction of the pie is left over for the offering player. The backward induction starts at the outside option of each player, assumed for simplicity to be zero. The diagram illustrates the following facts, some well-known:

- Each player is always better off when he offers first;
- The initial proposer is better off in the infinite- rather than the finite-horizon game as the number of periods is even or odd; moreover, this advantage disappears as the horizon tends to infinity;
- Initial offers in the finite-horizon game *monotonically* tend to those in the infinite-horizon one as the horizon game, becoming less generous for even horizons and more generous for odd horizons;
- If the outside option does not bind on the infinite-horizon solution, then the finite-horizon dynamics depicted can still be drawn, and thus the infinite-horizon solution is unaffected.[1]

The diagram can also be used to illustrate the following neat result from §4.6 of Fudenberg and Tirole (1991):

Claim *The infinite-horizon outcome is the only one that survives iterated deletion of conditionally dominated strategies.*

Indeed, the two finite-horizon cycles exactly follow the conditional iterations. One can use the equilibrium quadrants to progressively eliminate those offers that would be rejected by the other player. To avoid clutter, I have abstained from indicating this process in the figure.

Finally, the diagram highlights the fact that the particular source of delay cost ought to produce a stable a fixed point in a difference equation diagram of this sort. Rubinstein's alternative constant fixed cost of delay $c > 0$ does not.[2]

[1] There are actually some subtle timing issues here as to when the outside option can be exercised. See Propositions 3.5 and 3.6 of Osborne and Rubinstein (1990).

[2] See Lemma 3.2 of Osborne and Rubinstein (1990) for a condition guaranteeing uniqueness for general delay costs.

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