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IDIOSYNCRATIC PRODUCTION RISK, GROWTH, AND THE BUSINESS CYCLE

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Idiosyncratic Production Risk, Growth, and the Business Cycle

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We examine a neoclassical growth economy with idiosyncratic production risk and incomplete markets. Under a CARA-normal specification for preferences and risk, we characterize the general equilibrium in closed form. Uninsurable production shocks introduce a risk premium on private equity, and typically result in a *lower* steady-state level of capital than under complete markets. In the presence of such risks, the anticipation of future low investment and high interest rates discourages current risk-taking and feeds back into low investment in the present. An *endogenous macroeconomic complementarity* thus arises, which can amplify the magnitude and the persistence of the business cycle. These results – contrasting sharply with those of Aiyagari (1994) and Krusell and Smith (1998) – highlight that idiosyncratic *production* or *entrepreneurial* risks can have significant adverse effects on capital accumulation and aggregate volatility.

Keywords: Entrepreneurial Risk, Investment, Growth, Fluctuations, Precautionary Savings.

JEL Classification: D5, D9, E2, E3, O1.

1

1 Introduction

The standard neoclassical growth model of Ramsey-Cass-Koopmans assumes complete markets, implying that private agents can fully diversify idiosyncratic risks in their production and investment activities. However, large undiversifiable entrepreneurial and production risks are not only paramount in less developed countries, but also quantitatively significant even in the most advanced economies. In a recent study of US private equity, Moskowitz and Vissing-Jørgensen (2001) document that entrepreneurs face a "dramatic lack of diversification" in their investments and an extreme dispersion in their returns.[1] Similarly, moral hazard and agency problems imply that the executives of publicly traded firms are very exposed to the risk in the production and investment decisions they make on behalf of the shareholders. Large idiosyncratic risks are also evident in the accumulation of human or intangible capital.

A natural question is then how the presence of large undiversifiable idiosyncratic risks in entrepreneurial, production, and investment choices, matters for macroeconomic outcomes.[2] In this paper we propose that incomplete risk sharing leads to substantial underaccumulation of capital and generates a powerful amplification and propagation mechanism over the business cycle.

We introduce idiosyncratic production risk and incomplete insurance in an otherwise standard neoclassical growth economy. In our model, each private agent is both a consumer and a producer. The technology is standard neoclassical and fully convex. There are no credit-market imperfections and each agent can invest as much as he likes in his production. However, production is subject to idiosyncratic uncertainty. If markets were complete, individuals would be able to fully diversify their idiosyncratic production risks, and the model would reduce to a standard Ramsey economy. When markets are incomplete, however, individuals must bear at least part of the idiosyncratic risk in their production and investment choices.

Our model belongs to the class of general-equilibrium economies with incomplete markets and

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[1] In the United States, private companies accounted for about half of production and corporate equity in 1998. Moskowitz and Vissing-Jorgensen also observe that "About 77 percent of all private equity is owned by households for whom private equity constitutes at least half of their total net worth. Furthermore, households with private equity ownership invest on average more than 70 percent of their private holding in a single private company, in which the household has an active management interest. [...] Survival rates of private firms are around 34 percent over the first 10 years of the firm's life. Furthermore, even conditional on survival, entrepreneurial investment appears extremely risky, generating a wide distribution of returns."

[2] Castenada *et al.* (1998) and Quadrini (2000) have argued that entrepreneurship is important in explaining the wealth distribution. Heaton and Lucas (2000) have shown that entrepreneurial income is also important empirically to explain portfolios and risk premia.

2

heterogeneous infinitely-lived agents.[3] Models of this class typically suffer from the "curse of dimensionality" because the wealth distribution – an infinite-dimensional object – is a relevant state variable. By considering exponential preferences and Gaussian risks, we render the macro aggregates *independent* from the wealth distribution. The equilibrium dynamics are then characterized in closed form.

The analytical tractability of the model allows us to clearly identify how uninsurable idiosyncratic production risk affects both individual choices and aggregate outcomes. Because exposure to production risk can be controlled through the level of investment, production risk has an *ambiguous* impact on precautionary savings. Most importantly, production risk introduces a risk premium on private equity and therefore *unambiguously* reduces the aggregate demand for investment.[4] As a result, the steady-state levels of capital and income typically decrease with the variance of uninsurable production shocks.

Perhaps more surprisingly, idiosyncratic production risks introduce a kind of *endogenous dynamic macroeconomic complementarity*, which can slow down convergence to the steady state, amplify exogenous aggregate shocks, and increase the persistence of the business cycle. This macroeconomic complementarity has a simple origin. When production is subject to idiosyncratic uncertainty, an individual's willingness to invest depends on his ability to self-insure against undiversifiable shocks to future production. Risk premia and investment demand thus depend on anticipated future credit conditions. In our model, high interest rates and low investment in one period feed back into high risk premia, low risk-taking and low investment in earlier periods. Low investment, low income, and low savings can thus be self-sustaining for long periods of time.

The basis of this mechanism is a particular *pecuniary externality*. If agents save too little in one period, they discourage risk-taking and investment in earlier periods via the interest rate. This externality arises only in the presence of idiosyncratic *production* risks. The macroeconomic complementarity is thus a genuine general-equilibrium implication of a market imperfection.

The macroeconomic complementarity arises in our model even though agents face no borrowing constraints. Our mechanism is thus independent from – and is in fact complementary to – the amplification and propagation mechanisms that are generated by credit-market imperfections.[5]

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[3] Important examples include Aiyagari (1994, 1995), Constantinides and Duffie (1996), Heaton and Lucas (1996), Huggett (1993, 1997), Krusell and Smith (1998), and Ríos-Rull (1996). See Ríos-Rull (1995) and Ljungqvist and Sargent (2001) for a review of this literature.

[4] Our model thus complements earlier research, which has stressed the impact of aggregate production shocks on investment. See for instance Acemoglu and Zilibotti (1997) for the complete market case.

[5] See for instance Bernanke and Gertler (1989, 1990), Kiyotaki and Moore (1997), and Aghion, Banerjee and Piketty (1999).

We share with this literature the idea that market imperfections can have important implications for aggregate outcomes. Our model, however, differs from these earlier approaches in three ways. First, we focus on incomplete insurance, which affects the *willingness* to invest, rather than borrowing constraints, which also affect the *ability* to invest. Second, we consider infinitely-lived agents rather than overlapping generations (OLG) economies; this ensures that our results do not depend on finite-horizon effects and makes our framework directly comparable to the standard RBC framework. Third, the pertinent literature has centered around the dependence of aggregate outcomes on the wealth distribution as the *only* channel through which market incompleteness can generate propagation and amplification; the pecuniary externality identified in this paper has not been considered before and it is thus an innovative finding on its own that incomplete markets can generate amplification and propagation even in the absence of the wealth-distribution channel.

Our analysis concludes that incomplete insurance generates *under*investment in the steady state and increases both the *magnitude* and the *persistence* of aggregate fluctuations. Aiyagari (1994) and Krusell and Smith (1998) also examined the effect of incomplete markets in the neoclassical growth model, but reached a dramatically different conclusion. They argued that incomplete insurance generates *over*investment in the steady state and has no important effect on the business cycle. The contrast between their results and ours has a simple origin. While we view idiosyncratic *production* uncertainty as the main source of risk, Aiyagari (1994) and Krusell and Smith (1998) consider instead the effect of idiosyncratic *endowment* shocks, which do not affect the returns on individual investment. Idiosyncratic endowment shocks influence only precautionary savings. They do not introduce a risk premium on private equity and thus do not generate a pecuniary externality in risk taking. This explains why Aiyagari and Krusell and Smith did not identify the steady-state and business-cycle effects documented in this paper.

The rest of the paper is organized as follows. Section 2 introduces the economy and Section 3 analyzes the individual decision problem. In Section 4 we characterize the general equilibrium in closed form, analyze the steady state, and describe the propagation and amplification mechanism that arises in the presence of idiosyncratic production risk. Section 5 presents some numerical simulations. We conclude in Section 6 with some remarks on the likely robustness of our findings and directions for future research. All proofs are in the Appendix.

2 A Ramsey Economy with Incomplete Risk Sharing

This section introduces a neoclassical growth economy with decentralized production, CARA preferences, Gaussian idiosyncratic uncertainty, and an exogenous incomplete asset span.

2.1 Technology and Idiosyncratic Risks

Time is discrete and infinite, indexed by $t \in \mathbb{N} \equiv \{0, 1, ...\}$. The economy is stochastic and all random variables are defined on a probability space $(\ , \mathcal{F}, \mathbb{P})$. Individuals are indexed by $j \in \mathbb{J} = \{1, ..., J\}$.[6] They are all born at date 0, live forever, and consume a single consumption good in every date.

Each individual is also a producer, or entrepreneur, who operates his own production scheme using his own labor and capital stock. The technology is standard neoclassical. It exhibits constant returns to scale with respect to capital and labor, decreasing returns with respect to the capital/labor ratio, and satisfies the Inada conditions. There are no adjustment costs and no indivisibilities in investment. The individual can invest in a single type of capital. We denote by k_t^j the stock of capital that individual j holds at the beginning of period t, and by l_t^j his effective labor endowment in period t. The output of his production in period t is given by $A_t^j F(k_t^j, l_t^j)$. The production function F is deterministic and common across all individuals. We assume that all individuals have the same labor supply, $l_t^j = \bar{l}$, and thus conveniently consider the function $f(k) \equiv F(k, \bar{l})$.[7] The total factor productivity A_t^j is a random shock specific to individual j. The individual controls k_t^j through his investment at date $t-1$, while A_t^j is observed only at date t. Production and returns are thus subject to idiosyncratic uncertainty. This risk is what we call idiosyncratic *production* risk (or, *technological, entrepreneurial, or investment* risk, depending on our preferable interpretation).

For comparison with *production* risks, it is useful to also introduce *endowment* risks. We let e_t^j denote the exogenous stochastic endowment of consumable good that individual j receives in period t. These shocks model risks that are outside the control of individuals and do not affect production or investment opportunities. The overall non-financial income of individual j in period t is

$$y_t^j = A_t^j f(k_t^j) + e_t^j. \tag{1}$$

The random shock A_t^j is *multiplicative*, while the endowment risk e_t^j is *additive*.

2.2 Asset Structure

Idiosyncratic risks can be partially hedged by trading a limited set of short-lived securities, which are indexed by $m \in \{0, 1, .., M\}$. Purchasing one unit of security m at date t yields a random amount of consumption $d_{m,t+1}$ at date $t+1$. The price of this security at date t is denoted by $\pi_{m,t}$. The vectors $\pi_t = (\pi_{m,t})_{m=0}^M$ and $d_t = (d_{m,t})_{m=0}^M$ are, respectively, the price vector and the payoff

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[6] The model directly extends to economies with a continuum of agents.

[7] f satisfies $f'(k) > 0 > f''(k) \; \forall k \in (0, \infty)$, $\lim_{k \to 0} f'(k) = \infty$, and $\lim_{k \to \infty} f'(k) = 0$.

vector at date t. Security $m = 0$ is a riskless bond, and assets $m \in \{1, ..., M\}$ are risky. The bond delivers $d_{0,t} = 1$ with certainty in every t, and $R_t \equiv 1/\pi_{0,t}$ and $r_t \equiv R_t - 1$ denote respectively the gross and the net interest rate between t and $t + 1$.

The asset span is exogenous and generally incomplete. Default is not allowed. We rule out short-sales constraints or any other kind of credit-market imperfections. Finally, for simplicity, we assume that all assets are in zero net supply.[8] At the outset of every period t, individuals are informed of the contemporaneous realization of asset payoffs, d_t, and idiosyncratic shocks, $\{(A_t^j, e_t^j)\}_{j \in \mathbb{J}}$. Information is thus homogenous[9] across agents and generates a filtration $\{\mathcal{F}_t\}_{t=0}^\infty$. We denote by $\mathbb{E}_t$ the expectation operator conditional on $\mathcal{F}_t$.

2.3 Preferences

To distinguish between intertemporal substitution and risk aversion, we adopt a preference specification that belongs to the Epstein-Zin non-expected utility class. Consider two concave Bernoulli utilities U and Υ. A stochastic consumption stream $\{c_t\}_{t=0}^\infty$ generates a stochastic utility stream $\{u_t\}_{t=0}^\infty$ defined by the recursion

$$U(u_t) = U(c_t) + \beta U[\mathbb{CE}_t(u_{t+1})] \ \ \forall t \geq 0, \tag{2}$$

where $\mathbb{CE}_t(u) \equiv \Upsilon^{-1}[\mathbb{E}_t \Upsilon(u)]$ denotes the certainty equivalence of u. The curvature of Υ thus governs risk aversion, while the curvature of U governs intertemporal substitution. When $\Upsilon = U$, these preferences reduce to standard expected utility, $U(u_0) = \mathbb{E}_0 \sum_{t=0}^\infty \beta^t U(c_t)$. Finally, note that u_t is directly measured in consumption units.

2.4 CARA-Normal Specification

Closed-form certainty-equivalent measures are not possible for general risks and general preferences, but can be obtained in the CARA-normal case.

Assumption 1 (Exponential Preferences) Agents have identical recursive utility (2) with

$$U(c) = -\Psi \exp(-c/\Psi), \qquad \Upsilon(c) = -(1/\Gamma) \exp(-\Gamma c). \tag{3}$$

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[8] Ricardian equivalence holds in our model because there are no credit-market imperfections. As long as public debt is financed by lump-sum taxation, there is thus no loss of generality in assuming that the riskless bond is in zero net supply.

[9] The results of this paper would not be modifed under the alternative assumption that income shocks are privately observed, provided that the structure of the economy remains common knowledge.

Assumption 2 (Gaussian Risks) The idiosyncratic risks $\{(A_t^j, e_t^j)\}_{j \in \mathbb{J}}$ and asset returns d_t are jointly normal.

Note that a high Ψ corresponds to a strong willingness to substitute consumption through time, while a high Γ implies a high risk aversion. The above two restrictions are motivated by the desire for analytical tractability, but do not appear to be critical for any of the main arguments in this paper. They can also be viewed as a kind of approximation to an economy with more general preferences and risk distributions. Section 5 will propose a calibration method that permits our CARA-normal economy to match a constant elasticity of intertemporal substitution and a constant degree of relative risk aversion at the steady state.

We make two additional assumptions.

Assumption 3 (No Persistence) The residuals of the projection of $\{(A_t^j, e_t^j)\}_{j \in \mathbb{J}}$ on the span of d_t are serially uncorrelated.

This will imply that an individual's saving and investment choices are independent of his contemporaneous idiosyncratic income and productivity shocks, a property that will greatly simplify aggregation but is certainly not important for any of our qualitative results. As for our quantitative analysis, we will mimic persistence with a long time interval (see Section 5).

Assumption 4 (No Aggregate Uncertainty) In all dates and events, $\sum_j A_t^j / J = \mathbb{E}_{t-1} A_t^j = \overline{A}$ and $\sum_j e_t^i / J = \mathbb{E}_{t-1} e_t^j = 0$.[10]

This restriction, too, only serves the tractability of the model; it will imply that asset prices and all macro variables are deterministic in equilibrium. We will discuss, however, the implications of our results for an economy with aggregate uncertainty.

3 Decision Theory

This section examines the decision problem of an entrepreneur. We show that the portfolio choice reduces to a mean-variance problem, and derive the optimal saving and investment rules.

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[10]This assumption follows naturally from the Law of Large Numbers when the economy contains an infinity of agents facing independent shocks.

3.1 The Individual's Problem

Consider an individual j in period t. Denote his consumption by c_t^j, his physical-capital investment by i_t^j, his non-financial income by y_t^j, and his portfolio of the bond and the risky assets by $\theta_t^j = (\theta_{m,t}^j)_{m=0}^M$. His budget constraint is

$$c_t^j + i_t^j + \pi_t \cdot \theta_t^j = y_t^j + d_t \cdot \theta_t^j, \tag{4}$$

where y_t^j is given by (1). Capital accumulates according to $k_{t+1}^j = (1-\delta)k_t^j + i_t^j$, where $\delta \in [0,1]$ is the fixed depreciation rate of capital. To simplify notation, we conveniently rewrite the decision problem in terms of stock variables. The quantity

$$w_t^j \equiv A_t^j f(k_t^j) + (1-\delta)k_t^j + e_t^j + d_t \cdot \theta_t^j$$

represents the agent's total *wealth* (or *cash-in-hand*) at date t, inclusive of the capital stock. We then restate the budget constraint (4) as

$$c_t^j + k_{t+1}^j + \pi_t \cdot \theta_t^j = w_t^j. \tag{5}$$

Given a price sequence $\{\pi_t\}_{t=0}^\infty$, agent j chooses an adapted plan $\{c_t^j, k_{t+1}^j, \theta_t^j, w_t^j\}_{t=0}^\infty$ that maximizes utility and satisfies the budget constraint (5).

The indirect utility of wealth $V_t^j(w)$ satisfies the Bellman equation,

$$U[V_t^j(w_t^j)] = \max_{(c_t^j, k_{t+1}^j, \theta_t^j)} U(c_t^j) + \beta\, U\left(\mathbb{CE}_t[V_{t+1}^j(w_{t+1}^j)] \right), \tag{6}$$

subject to (5), and the transversality condition $\lim_{t\to\infty} \beta^t \mathbb{E}_t U[V_t^j(w_t^j)] = 0$. We denote the corresponding optimal consumption rule by $c_t^j(w)$.

3.2 Consumption-Investment Decision

CARA preferences, coupled with no short-sales constraints, imply that the demand for risky assets and productive capital are independent of wealth. We can thus anticipate that the wealth distribution, which is of course stochastic, will not affect the aggregate dynamics. This property, together with Assumption 4, will ensure that all macro variables are deterministic. For this reason, we present in this section the decision theory when the risk premium on financial assets is zero and the interest rate is deterministic.[11]

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[11] See Angeletos and Calvet (2000) for the general analysis. There, we first characterize individual choices for arbitrary asset prices and then prove that there exists an equilibrium such that the interest rate is deterministic and the risk premium on all assets $m \in \{1, ..., M\}$ is zero.

An educated guess is that, along the equilibrium price path, the value function and the optimal consumption rule are linear in wealth:

$$V_t^j(w) = \widehat{a}_t^j \, w + \widehat{b}_t^j, \qquad c_t^j(w) = a_t^j \, w + b_t^j, \tag{7}$$

where $\widehat{a}_t^j, a_t^j > 0$ and $\widehat{b}_t^j, b_t^j \in \mathbb{R}$ are non-random coefficients to be determined. We then infer from (3) and (7) that

$$\mathbb{CE}_t \left[V_{t+1}^j(w_{t+1}^j) \right] = V_{t+1}^j \left[\mathbb{E}_t w_{t+1}^j - \widehat{\Gamma}_t^j \text{Var}_t(w_{t+1}^j)/2 \right], \tag{8}$$

where $\widehat{\Gamma}_t^j \equiv \Gamma \widehat{a}_{t+1}^j$ measures absolute risk aversion in period t with respect to wealth variation in period $t+1$. We henceforth call $\widehat{\Gamma}_t^j$ the *effective* degree of absolute risk aversion at date t.

Without loss of generality, we normalize all risky securities ($m \geq 1$) to have zero expected payoffs. Since there is no risk premium, the assets have zero prices and are thus only used for hedging purposes. For any $(c_t^j, k_{t+1}^j, \theta_{0,t}^j)$, the optimal portfolio $(\theta_{m,t}^j)_{m=1}^M$ is therefore chosen to minimize the conditional variance of wealth. This has a simple geometric interpretation. We project A_{t+1}^j and e_{t+1}^j on the asset span:

$$A_{t+1}^j = \kappa_{t+1}^j \cdot d_{t+1} + \eta_{t+1}^j, \qquad e_{t+1}^j = \xi_{t+1}^j \cdot d_{t+1} + \varepsilon_{t+1}^j.$$

The projections $\kappa_{t+1}^j \cdot d_{t+1}$ and $\xi_{t+1}^j \cdot d_{t+1}$ represent the *diversifiable* components of the idiosyncratic production and endowment shocks. The residuals η_{t+1}^j and ε_{t+1}^j, which are orthogonal to the span of d_{t+1}, correspond to the *undiversifiable* components. Only the latter matter for either individual choices or aggregate outcomes. Their variances,

$$\sigma_A^2 \equiv \text{Var}(\eta_{t+1}^j|\mathcal{F}_t) = \text{Var}(\eta_{t+1}^j), \qquad \sigma_e^2 \equiv \text{Var}(\varepsilon_{t+1}^j|\mathcal{F}_t) = \text{Var}(\varepsilon_{t+1}^j),$$

are thus useful measures of financial incompleteness.[12] The case of complete Arrow-Debreu markets corresponds to $\sigma_A = 0$ and $\sigma_e = 0$. The incomplete-market economies considered by Aiyagari (1994) and Krusell and Smith (1998) correspond to $\sigma_e > 0$ but $\sigma_A = 0$ (additive idiosyncratic income risks but no idiosyncratic production/investment risks). The case we are most interested is when $\sigma_A > 0$ (production/investment risks).

After optimal hedging, individual wealth reduces to $w_{t+1}^j = (\overline{A} + \eta_{t+1}^j) f(k_t^j) + (1-\delta) k_t^j + \varepsilon_{t+1}^j + \theta_{0,t}^j$ and has conditional variance $\text{Var}_t(w_{t+1}^j) = \sigma_e^2 + f(k_{t+1}^j)^2 \sigma_A^2$. It is convenient to define $\Phi(k) \equiv \overline{A} f(k) + (1-\delta)k$ as the *expected production function*; and

$$G(k, \widehat{\Gamma}) \equiv \Phi(k) - \widehat{\Gamma} \left[\sigma_e^2 + f(k)^2 \sigma_A^2 \right] /2$$

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[12]The implicit assumption that η_{t+1}^i and ε_{t+1}^i are stationary, so that σ_A and σ_e are independent of t, can be easily relaxed. Besides, if we wanted to capture the fact that idiosyncratic risk worsens during recessions, we could also let σ_A and σ_e be functions of *aggregate* wealth.

9

as the *risk-adjusted output*. By (8), $(c_t^j, k_{t+1}^j, \theta_{0,t}^j)$ thus maximizes

$$U(c_t^j) + \beta U \left\{ V_{t+1}^j \left[G(k_{t+1}^j, \widehat{\Gamma}_t^j) + \theta_{0,t}^j \right] \right\}$$

subject to the budget constraint, $c_t^j + k_{t+1}^j + \theta_{0,t}^j/R_t = w_t^j$.

Under complete markets, the optimal investment equates the marginal product of capital with the interest rate: $R_t = \Phi'(k_{t+1})$. In the presence of uninsurable production risks, however, the return on investment is discounted for risk. The FOCs with respect to k_t^j and $\theta_{0,t}^j$ imply the key condition for investment demand:

$$R_t = \frac{\partial G}{\partial k}(k_{t+1}^j, \widehat{\Gamma}_t) = \Phi'(k_{t+1}^j) - \widehat{\Gamma}_t^j f(k_{t+1}^j) f'(k_{t+1}^j)\sigma_A^2. \tag{9}$$

The agent equates the *risk-adjusted return* with the risk-free rate. The difference between the expected marginal product of capital and the risk-free rate, $\Phi'(k_{t+1}^j) - R_t$, represents the *risk premium on private equity*. This premium is proportional to the uninsurable production risk σ_A^2 and the effective risk aversion $\widehat{\Gamma}_t$.

The FOC with respect to the riskless rate implies the *Euler equation*,

$$\mathbb{E}_t c_{t+1}^j - c_t^j = \Psi \ln(\beta R_t) + \Gamma \mathrm{Var}_t(c_{t+1}^j)/2. \tag{10}$$

Expected consumption growth thus increases with the variance of consumption. This reflects the standard *precautionary motive* for savings (Leland, 1968; Sandmo. 1970; Caballero, 1990; Kimball, 1990). The sensitivity of savings to consumption risk is governed by risk aversion (Γ), while their sensitivity to the interest rate is governed by intertemporal substitution (Ψ).

The consumption rule and the budget constraint imply that *future* consumption c_{t+1}^j is linear in *current* wealth w_t^j, with slope $(dc_{t+1}^j/dw_{t+1}^j)(dw_{t+1}^j/dw_t^j) = a_{t+1}^j R_t(1 - a_t^j)$. By (10), this slope must be equal to a_t^j. The MPC therefore satisfies the recursion $a_t^j = 1/[1 + (a_{t+1}^j R_t)^{-1}]$. Iterating forward yields

$$a_t^j = \frac{1}{1 + \sum_{s=0}^{+\infty}(R_t R_{t+1}...R_{t+s})^{-1}}. \tag{11}$$

The MPC is thus the inverse of the price of a perpetuity, which delivers one unit of the consumption good in each period $s \geq t$.

Finally, the envelope condition –after simple manipulation– implies that $\widehat{a}_t^j = a_t^j$ and thus $\widehat{\Gamma}_t^j = \Gamma a_{t+1}^j$. We thus infer from (11) that the *effective* absolute risk aversion $\widehat{\Gamma}_t^j$ is an increasing function of *future* interest rates. We summarize our results below

Proposition 1 (Individual Choice) For any path $\{R_t\}_{t=0}^{\infty}$, the value function and the consumption rule are linear in wealth, as in (7), and the coefficients a_t^j and $\widehat{a}_t^j$ are equal and satisfy

(11). The demand for investment is given by

$$R_t = \overline{A} f'(k_{t+1}^j) + (1-\delta) - \widehat{\Gamma}_t^j f(k_{t+1}^j) f'(k_{t+1}^j) \sigma_A^2. \tag{12}$$

Consumption and savings are characterized by the Euler equation,

$$\mathbb{E}_t c_{t+1}^j - c_t^j = \Psi \ln(\beta R_t) + \frac{\Gamma}{2} \mathrm{Var}_t(c_{t+1}^j), \tag{13}$$

where $\mathrm{Var}_t(c_{t+1}^j) = (a_{t+1}^j)^2 \mathrm{Var}_t(w_{t+1}^j) = (a_{t+1}^j)^2 [\sigma_e^2 + f(k_{t+1}^j)^2 \sigma_A^2]$. Finally, the effective risk aversion $\widehat{\Gamma}_t^j \equiv \Gamma \widehat{a}_{t+1}^j$ increases with future interest rates.

Note that higher future interest rates increase the effective risk aversion $\widehat{\Gamma}_t^j$ and thus the risk premium on private investment. We expect than the feedback from future credit conditions to current risk-taking is much more general than our model, as will be discussed in Section 4.3.

3.3   Comparative Statics

Consider the impact of incomplete markets on investment. The optimality condition (12) defines the optimal investment as a function of the contemporaneous interest rate, the effective risk aversion, and the production risk: $k_{t+1}^j = k(R_t, \sigma_A, \widehat{\Gamma}_t^j)$. This function decreases with the interest rate R_t and the production risk σ_A, but is independent from the endowment risk σ_e. When $\sigma_A > 0$, the optimal investment k_{t+1}^j also decreases with the effective risk aversion $\widehat{\Gamma}_t^j$. By (11), effective risk aversion increases with future interest rates. Incomplete markets therefore introduce a negative feedback from *future* interest rates to *current* investment demand.

Proposition 2 (Investment) Investment demand decreases with idiosyncratic production risk: $\partial k_{t+1}^j / \partial \sigma_A < 0$. When $\sigma_A > 0$, higher future interest rates reduce investment demand: $\partial k_{t+1}^j / \partial R_s < 0 \; \forall s > t$. Moreover, production risk and future interest rates are complementary with respect to their impact on investment: $\partial^2 k_{t+1}^j / (\partial \sigma_A \partial R_s) < 0 \; \forall s > t$.

Under incomplete markets, risk-taking in one period depends on the ability to smooth consumption in future periods. A higher borrowing rate, or a more stringent credit market, in any given period means that self-insurance is more difficult in the period. This in turn raises risk premia and discourages investment in earlier periods. In Section 4.3, we will demonstrate that this feedback generates a kind of *dynamic macroeconomic complementarity*, which can be the source of amplification and persistence over the business cycle.

11

Proposition 2 also states that a high level of production risk and a high level of future interest rates *reinforce* each other's impact on investment. Because recessions tend to predict high investment risks and bad credit conditions, the amplification and persistence effects we document in this paper are likely to be particularly important during recessions.

Consider next the impact of incomplete markets on savings. Although the effect of an increase in endowment risk on precautionary savings is unambiguously positive, the effect of production risk is small or even ambiguous, because exposure to production risk is endogenous. It is indeed possible that, when σ_A increases too much, the individual scales back investment k_{t+1}^j so much that his income risk $f(k_{t+1}^j)^2 \sigma_A^2$ actually *decreases* despite the increase in σ_A. From the Euler equation (13), we conclude:

Proposition 3 (Savings) An increase in endowment risk, σ_e, raises both wealth risk, $\mathrm{Var}_t(w_{t+1}^j)$, and consumption growth, $(\mathbb{E}_t c_{t+1}^j - c_t)$. On the other hand, the impact of the production risk σ_A is generally ambiguous. For example, in the case of a Cobb-Douglas technology with capital share $\alpha = 1/2$, there is a threshold $\overline{\sigma}_A$ such that $\partial Var_t(w_{t+1}^j)/\partial \sigma_A < 0$ if and only if $\sigma_A > \overline{\sigma}_A$.

While the literature has focused on the effect of incomplete markets on precautionary savings, we observe that this channel is ambiguous in the presence of production risks. In contrast, the direct effect on private investment unambiguously reduces investment, and seems likely to dominate in equilibrium.

4 General Equilibrium and Steady State

In this Section we characterize the general equilibrium and the steady state of the economy in closed form.

4.1 Definitions

Recall that there is no *exogenous* aggregate uncertainty (Assumption 4) and that, because of our CARA-normal specification, the wealth distribution is irrelevant. We thus focus on equilibria in which prices are deterministic. If asset prices are deterministic, there can be no risk premium on these assets in equilibrium. Risky assets thus play only one role in the model – the definition of the uninsurable component of idiosyncratic production and endowment risks. Furthermore by

Assumptions 2-4, undiversifiable risks cancel out in the aggregate and are normally distributed:

$$\sum_j \eta_t^j = \sum_j \varepsilon_t^j = 0 \; \forall t, \qquad \begin{pmatrix} \eta_t^j \\ \varepsilon_t^j \end{pmatrix} \sim \mathcal{N}\left[0, \begin{pmatrix} \sigma_A^2 & 0 \\ 0 & \sigma_e^2 \end{pmatrix} \right] \; \forall j, t.$$

σ_A^2 and σ_e^2 are thus sufficient statistics for the structure of risks and the economy can be parameterized by $\mathcal{E} = (\beta, \Gamma, \Psi, F, A, \delta, \sigma_A, \sigma_e)$.

Definition An *incomplete-markets equilibrium* is a *deterministic* price sequence $\{\pi_t\}_{t=0}^\infty$ and a collection of *state-contingent* plans $(\{c_t^j, k_{t+1}^j, \theta_t^j, w_t^j\}_{t=0}^\infty)_{j \in \mathbb{J}}$ such that: The plan $\{c_t^j, k_{t+1}^j, \theta_t^j, w_t^j\}_{t=0}^\infty$ maximizes the utility of each agent j; and asset markets clear in every date and event, $\sum_{j \in \mathbb{J}} \theta_t^j = 0$.

In what follows, we let C_t, W_t and K_{t+1} respectively denote the population averages of consumption, wealth and capital. The initial mean wealth, $W_0 = \sum_{j=1}^J w_0^j / J$, is an exogenous parameter for the economy.

4.2 Equilibrium Characterization

We first observe that (11) implies that all agents share the same MPC and the same effective risk aversion $a_t^j = a_t$ and $\widehat{\Gamma}_t^j = \Gamma a_t$ for all j, t. We next observe two important properties of condition (12). First, because preferences exhibit CARA and there are no borrowing constraints, the optimal investment (k_{t+1}^j) is independent of contemporaneous wealth (w_t^j). This ensures that the wealth distribution does not matter for equilibrium allocations. Second, because technology $(f, \delta, \overline{A})$, investment risk (σ_A), and effective risk aversion (Γa_{t+1}) are identical across agents, all agents choose the same level of investment $k_{t+1}^j = K_{t+1}$ for all j, t. Aggregation is thus straightforward and we conclude:

Theorem 1 (General Equilibrium) There exists an incomplete-markets equilibrium in which the macro path $\{C_t, K_{t+1}, W_t, R_t\}_{t=0}^\infty$ is deterministic and all agents choose identical levels of productive investment. The equilibrium path satisfies in all $t \geq 0$

$$R_t = \Phi'(K_{t+1}) - \Gamma a_{t+1} f(K_{t+1}) f'(K_{t+1}) \sigma_A^2 \tag{14}$$

$$C_{t+1} - C_t = \Psi \ln(\beta R_t) + \Gamma a_{t+1}^2 \left[\sigma_e^2 + f(K_{t+1})^2 \sigma_A^2 \right]/2, \tag{15}$$

$$a_t = \left[1 + \sum_{s=0}^{+\infty} (R_t R_{t+1} ... R_{t+s})^{-1} \right]^{-1}, \tag{16}$$

$$C_t + K_{t+1} = W_t, \tag{17}$$

$$W_{t+1} = \Phi(K_{t+1}). \tag{18}$$

13

Conditions (14) and (16) follow directly from the individual decision problem. Equation (15) is obtained by aggregating the individual Euler equations. If there were no undiversifiable idiosyncratic *production* risks, (14) would reduce to the familiar complete-markets condition $R_t = \Phi'(K_{t+1})$. If *in addition* there were no undiversifiable *endowment* risks, then (15) would reduce to the complete-markets Euler equation, $U'(C_t) = \beta R_t U'(C_{t+1})$. Finally, conditions (17) and (18) are the resource constraint and the production frontier of the economy, which of course remain the same under either complete or incomplete markets.

Under incomplete markets, the *aggregate* consumption growth increases with the variance of *individual* consumption. This reflects the standard precautionary motive and will push down the equilibrium risk-free rate. More interestingly, the presence of uninsurable idiosyncratic *production* risks introduces a *risk premium on private equity*, which pushes down aggregate investment for any given risk-free rate. This risk premium is equal to $\rho_t \equiv \Phi'(K_{t+1}) - R_t = (\Gamma a_{t+1})f(K_{t+1})f'(K_{t+1})\sigma_A^2$ and represents a gap between the *social* and the *private* return to investment.

We finally note from (11) that the effective risk aversion (Γa_{t+1}) and thus the risk premium (ρ_t) in period t increase with R_s for all $s > t$. That is, an anticipated increase in future interest rates raises the risk premium on private equity and thereby decreases the demand for investment. In the next subsection, we show that this feedback generates a kind of *dynamic macroeconomic complementarity*, which induces persistence and amplification in the transitional dynamics.

4.3 Propagation and Amplification: An Endogenous Dynamic Macroeconomic Complementarity

Consider an economy which, starting from the steady state, is hit at a given date t by an unanticipated negative wealth shock. The impact of such a shock in a *complete-markets* Ramsey economy is quite familiar. Consumption and investment fall, interest rates rise, and the economy converges monotonically and asymptotically back to the steady state. The transition is slow under complete markets only because agents seek to smooth consumption through time. But, when markets are *incomplete*, the intertemporal substitution effect is complemented by a *risk-taking effect*. Anticipating high interest rates in the near future, private agents are less willing to invest in risky production. This effect amplifies the fall in initial investment and slows down convergence to the steady state as compared to complete markets.

Below, we illustrate this mechanism in a simplified version of the model.

Example. Suppose that markets are incomplete in period 0 but complete at $t \in \{1, .., \infty\}$. To clarify the presentation, we denote by $I_t = K_{t+1}$ the gross investment chosen by all investors

in period t. Given wealth W_1, we can solve for the Ramsey equilibrium at dates $t \geq 1$, and write period-1 consumption as $C_1 = C^*(W_1)$ and period-1 investment as $I_1 = I^*(W_1)$. By (11), effective risk aversion in period 0, $\widehat{\Gamma}_0 = \Gamma a_1$, is an increasing function of future interest rates, $\{R_t\}_{t=1}^\infty$. In any period $t \geq 1$, the interest rate is equal to the marginal productivity of capital, $R_t = \Phi'(I_t)$. Since investment at $t \geq 2$ increases with I_1, interest rates at all $t \geq 1$ decrease with I_1. We can therefore express the period-0 effective risk aversion as a function of period-1 investment alone, $\widehat{\Gamma}_0 = \widehat{\Gamma}^*(I_1)$.[13] We note that the functions C^* and I^* are increasing in wealth, while $\widehat{\Gamma}^*$ decreases with investment.

We suppose for expositional simplicity that the aggregate endowment $e_t \equiv \sum_j e_t^j / J$ is equal to zero in every period $t \neq 1$. Wealth in period 1 is $W_1 = \Phi(I_0) + e_1$. Using the notation introduced above, we infer that the period-1 investment,

$$I_1 = I^*(\Phi(I_0) + e_1), \tag{19}$$

increases with I_0. More investment in period 0 generates more wealth and therefore more investment in period 1. This is the familiar wealth effect due to intertemporal consumption smoothing.

Since markets are incomplete at $t = 0$, the initial investment I_0 satisfies

$$\Phi'(I_0) - \rho(I_0, I_1) = R_0, \tag{20}$$

where $\rho(I_0, I_1) \equiv \sigma_A^2 f(I_0) f'(I_0) \widehat{\Gamma}^*(I_1)$ is the risk premium on private equity. This relation defines the aggregate demand for investment I_0 as a function of R_0 and I_1.[14] We observe that $\partial I_0 / \partial I_1 = 0$ if $\sigma_A = 0$, but $\partial I_0 / \partial I_1 > 0$ if $\sigma_A > 0$. If production risks are fully insurable ($\sigma_A = 0$), the risk premium on private equity is zero and condition (20) reduces to $\Phi'(I_0) = R_0$. The optimal investment I_0 is then independent of the future investment I_1. When instead $\sigma_A > 0$, the risk premium $\rho(I_0, I_1)$ is a decreasing function of I_1, and the demand for I_0 is therefore increasing in I_1. This is because the anticipation of low (high) investment in the future leads investors to expect high (low) interest rates in later periods, and thus discourages (encourages) risk-taking in the present. The *complementarity* between future and current investment expressed by (20) is the heart of our propagation and amplification mechanism. We stress that it arises only when $\sigma_A > 0$ and that it hinges simply on the property that high risk premia on private equity predict recessions in the near future.

[13] The functions $C^*(W)$ and $I^*(W) \equiv W - C^*(W)$ are simply the optimal consumption and saving rule in the standard Ramsey model. The interest rate is $\Phi'(I)$. The MPC function a^* is determined by the functional equation $a^*(I) \equiv 1/\left(1 + \{\Phi'(I)a^*(\Phi(I))\}^{-1}\right)$, and then $\widehat{\Gamma}^*(I) \equiv \Gamma \cdot a^*(I)$.

[14] The second-order condition of the individual decision problem implies $\partial[\Phi'(I_0) - \rho(I_0, I_1)]/\partial I_0 \equiv \partial^2 G / \partial I_0^2 < 0$. It follows that $\partial I_0 / \partial R_0 = 1/(\partial^2 G / \partial I_0^2) < 0$; and $\partial I_0 / \partial I_1 = (f f')/(\partial^2 G / \partial I_0^2) \cdot \widehat{\Gamma}^{*\prime} \cdot \sigma_A^2 \geq 0$, since $\widehat{\Gamma}^{*\prime} < 0$.

We now consider the impact at date 0 of an *anticipated recession or investment slump* at date 1. To be specific, we assume that the slump originates in an exogenous decrease in date-1 aggregate endowment e_1.[15] For simplicity, we also treat the initial interest rate R_0 as exogenously fixed.[16] When $\sigma_A = 0$, by (20) the optimal investment I_0 is independent of the expected decline in I_1. With R_0 fixed, we conclude that $dI_0/de_1 = 0$, $dW_1/de_1 = 1$, and $dI_1/de_1 = I^{*'}$. Thus, when markets are complete, the anticipation of a recession or an investment slump in period 1 does not affect investment in period 0; and the impact of the exogenous wealth shock on contemporaneous income and investment is not amplified.

On the other hand, when $\sigma_A > 0$, the investment levels I_0 and I_1 are complementary by (20). The anticipation of an exogenous negative wealth shock in period 1 signals high future interest rates and induces private agents to scale down their risky investment in period 0. The reduction in I_0 implies a further reduction in $W_1 = \Phi(I_0) + e_1$, which in turn implies a further reduction in I_1 by the wealth effect (19). The anticipation of the (endogenous) further reduction in W_1 and I_1 feedbacks to even lower I_0, and another round of feedbacks between I_1 and I_0 is initiated. The overall impact of the initial exogenous shock can be quite large. Indeed, $dI_0/de_1 > 0$, $dW_1/de_1 > 1$, and $dI_1/de_1 > I^{*'}$. These impacts are stronger the higher σ_A.

We conclude that, in the presence of undiversifiable idiosyncratic production risks, any given aggregate shock is *propagated* from one period to another via the dynamic complementarity in investment; the contemporaneous impact of the shock is *amplified*; and the dynamic impact of the shock becomes more *persistent*, since it takes more time to revert to the steady state.   ∎

Several remarks are in order about the kind of amplification and propagation in our model. First, incomplete markets generate a particular form of *pecuniary externality*. In the presence of uninsurable production risks, risk taking depends on future interest rates. But, when private agents decide how much to save and invest in any period, they do not internalize the impact that their choices have on the equilibrium interest rate and therefore on aggregate risk taking and investment in earlier periods.

Second, the pecuniary externality generates a kind of *dynamic macroeconomic complementarity.*

[15] The case of aggregate productivity shocks is very similar but slightly harder to analyze because of the direct effect on the marginal product of capital.

[16] This is equivalent to assuming an infinitely elastic supply for savings. In our model, the supply for savings at $t = 0$ is derived from the contemporaneous Euler equation. The anticipation of a recession in period 1 (lower e_1) increases the period-0 supply of savings under either complete or incomplete markets, but decreases the period-0 demand for investment under only incomplete markets. The endogeneity of the interest rate thus tends to dampen but not to offset the amplification and propagation mechanism we are proposing. The calibrations of Section 5 will show that the mechanism can indeed be quite powerful in general equilibrium.

Because interest rates are endogenous and influence risk taking, the anticipation of low aggregate investment in one period feedbacks into low aggregate investment in earlier periods. Low levels of aggregate investment can thus be self-sustaining for long periods of times. This dynamic macroeconomic complementarity is the basis of amplification and persistence over the business cycle. The reader must be familiar with a standard example of macroeconomic complementarity – an economy with production externalities, as in Bryant (1983) and Benhabib and Farmer (1994).[17] In this world, an individual's marginal productivity of capital is assumed to be increasing in the aggregate stock of capital. There is thus complementarity in investment, like in our model. Unlike our model, however, production externalities are exogenous and *ad hoc*. The complementarity in our model is instead *endogenously* generated by the pecuniary externality we analyzed above; it is a genuine general-equilibrium implication of a market imperfection.

Third, this endogenous macroeconomic complementarity hinges on the fact that idiosyncratic uncertainty affects production/investment. It is thus not present in the economies of Aiyagari (1994) and Krusell and Smith (1998), who consider only endowment risks. This explains why they could not have found the propagation and amplification mechanism we identify in this paper.

We also want to emphasize the fundamental premises driving our result. First, the willingness to invest in risky production depends on the ability to insure or self-insure against future variations in the return of such investment. Second, this ability is lower during recessions. These two premises are obviously much more general than our specific model.[18] They alone imply that the willingness to invest in risky production is lower when a recession is anticipated to persist in the near future, which can slow down recovery and make the recession partially self-fulfilling. We are thus confident that our results are robust to more general frameworks and empirically relevant.[19]

We conclude that the presence of undiversifiable idiosyncratic production risks generates a powerful dynamic macroeconomic complementarity. In the presence of aggregate uncertainty, this mechanism can amplify the impact of an adverse aggregate wealth or productivity shock on contemporaneous output and investment, and can increase the persistence in investment and output over the business cycle.

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[17] Cooper (1999) provides an overview of macroeconomic complementarities.

[18] For example, in models with risk averse agents, incomplete insurance *and* credit markets imperfections, the anticipation of a higher interest rate for borrowing, a higher probability to need credit, or a higher probability to face a binding borrowing constraint, will most likely induce the agent to take less risk and thus to invest less, even if his current funds for investment are not constrained.

[19] The heart of our argument is simply that higher risk premia on private equity predict recessions. This is testable in principle and casual observation suggests it is probably true.

4.4   Steady State

We now demonstrate how the presence of undiversifiable idiosyncratic production risks reduces the capital stock at the steady state. A *steady state* is a fixed point $(C_\infty, W_\infty, K_\infty, R_\infty)$ of the dynamic system (14)-(17). We easily show:

Theorem 2 (Steady State) There exists a steady state.[20]   The consumption level is $C_\infty = \Phi(K_\infty) - K_\infty$, while the interest rate and the aggregate capital stock satisfy

$$R_\infty = \Phi'(K_\infty) - \rho_\infty, \qquad \rho_\infty \equiv \Gamma(1 - R_\infty^{-1}) f(K_\infty) f'(K_\infty) \sigma_A^2, \tag{21}$$

$$\ln(\beta R_\infty) = -\frac{\Gamma}{2\Psi} \sigma_c^2, \qquad \sigma_c^2 \equiv (1 - R_\infty^{-1})^2 \left[\sigma_e^2 + f(K_\infty)^2 \sigma_A^2 \right]. \tag{22}$$

The first equation corresponds to the aggregate demand for productive investment; the second corresponds to the aggregate supply of savings: ρ_∞ is the risk premium on private equity; and σ_c is the standard deviation of individual consumption. We note that $R_\infty = 1/\beta$ when markets are complete ($\sigma_A = \sigma_e = 0$), but $R_\infty < 1/\beta$ in the presence of undiversifiable idiosyncratic risks ($\sigma_A > 0$ and/or $\sigma_e > 0$). The property that the risk-free rate is below the discount rate under incomplete markets has been proposed as a possible solution to the low risk-free rate puzzle (e.g., Weil, 1992; Huggett, 1993; Constantinides and Duffie, 1996; Heaton and Lucas, 1996).

The comparative statics of the steady state can easily be derived from (21)-(22):

Proposition 4 (Comparative Statics) The capital stock (K_∞) unambiguously increases with the endowment risk (σ_e), the discount factor (β), and the mean productivity $(\overline{A})$. On the other hand, the effect of the production risk (σ_A) is generally ambiguous.

We will demonstrate in Section 5 that, for most reasonable parameter values, K_∞ is actually decreasing in σ_A. Proposition 4 and our numerical findings in Section 5 establish that the distinction between endowment and production risks is critical. Consider first the case where $\sigma_e > 0$ but $\sigma_A = 0$, as in Aiyagari (1994) and Krusell and Smith (1998). In this case, there is no risk premium on private investment; the marginal product of capital is always equal to the interest rate; and the steady state reduces to

$$R_\infty = \Phi'(K_\infty) \quad \text{and} \quad \frac{\ln(\beta R_\infty)}{(1 - 1/R_\infty)^2} = -\frac{\Gamma}{2\Psi} \sigma_e^2.$$

A higher endowment risk implies a higher consumption risk, increases the precautionary supply of savings, and thus reduces the interest rate. And, since the interest rate is equal to the marginal

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[20] See the Appendix for a discussion on uniqueness and stability.

product of capital, the capital stock necessarily increases. This is indeed the finding of Aiyagari (1994).

Consider now the case where $\sigma_A > 0$. Production risk introduces a risk premium on private investment and breaks the one-to-one correspondence between the stock of capital and the interest rate. Indeed, σ_A affects both the supply of savings (like σ_e) and the demand for investment (unlike σ_e). If agents scale down their production when σ_A increases, the savings effect can be small, or even ambiguous. The investment effect, on the other hand, is unambiguously negative. An increase in σ_A raises the risk premium on private investment and decreases the capital stock for any given interest rate. It should then be no surprise that the investment effect typically dominates, implying that the general-equilibrium impact of σ_A on K_∞ is typically negative.

There are two cases where the investment effect has to dominate. One is when the precautionary motive itself is weak, implying a small response of the supply of savings to increases in consumption risk. Another is when the interest elasticity of the steady-state supply of savings is not very small, implying that the steady-state interest rate is not very sensitive to shifts in either the supply of savings or the demand for investment. In terms of our model, the latter case corresponds to sufficiently high Ψ. We verify these insights in the numerical simulations of the next Section, which demonstrate that K_∞ decreases with σ_A unless the elasticity of intertemporal substitution is *implausibly* low. Therefore, while Aiyagari (1994) et al. suggested that the economy *over-invests* under incomplete insurance, we instead conclude that the most likely case is that of *under-investment*.

5 Calibration and Numerical Results

In this section, we calibrate our infinite-horizon economy and simulate the impact of uninsurable income risks on the steady state and the speed of convergence to the steady state.

5.1 Calibrated Economies

We first consider technology and risks. We choose a Cobb-Douglas production function, $f(K) = K^\alpha$, $\alpha \in (0, 1)$. We next normalize $\overline{A} = 1$, which permits σ_A to be interpreted as a percentage rate. For instance, when $\sigma_A = 0.25$, the standard deviation of gross output, $\sigma_A f(K_\infty)$, represents 25% of the mean production, $f(K_\infty)$. We accordingly rescale σ_e by the steady-state output; that is, we replace σ_e by $\sigma_e f(K_\infty)$. We can thus interpret both σ_A and σ_e as percentage rates.

We next consider preferences. An important difficulty with exponential preferences is that the degree of *relative* risk aversion and the elasticity of intertemporal substitution are not *invariant*; at

19

consumption level C, the former is $\Gamma \cdot C$ and the latter is Ψ/C. To remedy this problem and obtain a meaningful calibration, we adjust Γ and Ψ as we perturb the rest of the parameters in such a way that the steady state matches some meaningful measures of risk aversion and intertemporal substitution. Under complete markets, for example, the steady-state level of consumption is $C_\infty^* = [(\beta^{-1} + \delta - 1)/\alpha - \delta][\alpha/(\beta^{-1} + \delta - 1)]^{1/(1-\alpha)} \equiv C_\infty^*(\alpha, \beta, \delta)$.[21] If for every (α, β, δ) we calibrate $\Gamma = \gamma/C_\infty^*(\alpha, \beta, \delta)$ and $\Psi = \psi \cdot C_\infty^*(\alpha, \beta, \delta)$, we can then match a degree of relative risk aversion equal to γ and an elasticity of intertemporal substitution equal to ψ at the steady state. The calibration of Γ and Ψ is similar under incomplete markets, but slightly more complicated for the reasons described in Appendix B.

Overall, a *calibrated economy* is parametrized by a vector $\mathcal{E}^{cal} = (\beta, \gamma, \psi, \alpha, \delta, \sigma_A, \sigma_e)$, where β is the discount rate, γ measures the aggregate degree of relative risk aversion, ψ measures the aggregate elasticity of intertemporal substitution, α is the income share of capital, δ is the depreciation rate, and σ_A and σ_c are the standard deviations of uninsurable idiosyncratic production and endowment risks, respectively, as percentages of GDP. (See Appendix B for further details.)

5.2   Calibrated Steady State

Consider a calibrated economy $\mathcal{E}^{cal} = (\beta, \gamma, \psi, \alpha, \delta, \sigma_A, \sigma_e)$. We can easily characterize the comparative statics of the steady state near complete markets

Proposition 5 (Calibrated Steady State) As we move away from complete markets, R_∞ decreases with either σ_e or σ_A; K_∞ increases with σ_e; and K_∞ decreases with σ_A if and only if $\psi > \underline{\psi}$, where $\underline{\psi} \equiv (\beta^{-1} - 1) \left[(\beta^{-1} - 1) + (1 - \alpha)\delta \right] / (2\alpha^2)$.

As we discussed earlier, the endowment risk σ_e affects precautionary savings but not investment demand. As a result, it unambiguously reduces R_∞ and increases K_∞. The effect of the productivity risk σ_A is fundamentally different for two reasons. First, the impact of an increase in σ_A on precautionary savings may be moderated by a decrease in investment. Second, an increase in σ_A raises the risk premium on private equity and therefore reduces the demand for investment. Which force dominates depends on preferences (ψ), technology (α), and markets (σ_A and σ_e). Proposition 5 provides a lower bound for the elasticity of intertemporal substitution, which is consistent with our earlier discussion in Section 4.4. A high ψ implies a high interest elasticity of the supply of savings, which in turn implies that the steady-state interest rate is not very sensitive to variations in consumption risk. Therefore, as long as ψ is not too small, the investment effect dominates.

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[21] This follows from $C_\infty^* = \Phi(K_\infty^*) - K_\infty^*$ and $\Phi'(K_\infty^*) = R_\infty^* = 1/\beta \Rightarrow K_\infty^* = [\alpha/(\beta^{-1} + \delta - 1)]^{1/(1-\alpha)}$.

We note that the lower bound $\underline{\psi}$ is less than 0.20 for all plausible values of (β, α, δ). For example, if we consider annual frequency with a narrow definition for capital, and thus set $(\beta, \alpha, \delta) = (.95, .35, .05)$, then $\underline{\psi} = 0.02$. Similarly, if we consider a longer time interval and a broader definition of capital, and thus set $(\beta, \alpha, \delta) = (.75, .70, .25)$, then $\underline{\psi} = 0.14$. Since most empirical evidence suggest an EIS around 1, and certainly well above 0.20, it seems extremely unlikely in terms of our model that K_∞ may increase with σ_A.

5.3   Numerical Simulations of the Steady State

We have run various simulations and we have identified the following patterns as we vary σ_A. When ψ is *very* small (typically < 0.20), K_∞ is a single-peaked function of σ_A. In that case, completing the markets increases capital accumulation if we start from highly incomplete markets, but may decrease capital accumulation if we are too close to complete markets. On the other hand, for moderate or high ψ (typically > 0.20), the investment effect always dominates and K_∞ is a globally decreasing function of σ_A. In this case, more risk sharing unambiguously increases capital accumulation. Besides, the impact of σ_e is always to reduce R_∞ and increase K_∞.

Figure 1 illustrates the monotonicities of the capital stock K_∞ and the net interest rate $r_\infty \equiv R_\infty - 1$ for a specification typical of RBC models in annual frequency. We set $\beta = 0.95$, $\gamma = 4$, $\psi = 1$, $\delta = 0.05$, and $\alpha = 0.70$ (Panel A) or $\alpha = 0.35$ (Panel B). Panel A thus corresponds to a broad definition of capital (physical and human capital) and Panel B to a narrow definition (physical capital only). In each graph, the solid line corresponds to $\sigma_e = 0$ and the dashed one to $\sigma_e = 50\%$. In all cases, K_∞ monotonically decreases as σ_A varies from 0 to 100% and the drop in K_∞ is higher the higher the income share of capital. We observe, in particular, that the capital stock at $\sigma_A = 100\%$ is about 25% lower than its complete-market value when $\alpha = 0.35$ (Panel B), and 40% lower when $\alpha = 0.70$ (Panel A).

The above simulations assume that an idiosyncratic production shock lasts only one year. With short transitory shocks and infinite lives, agents can easily self-insure. Hence, the effect of such risks can not be dramatic. The effect should be much stronger if idiosyncratic shocks are highly persistent, the ability to self-insure is limited, and investment is subject to long irreversibilities or adjustment costs. Persistence in idiosyncratic production shocks and long irreversibilities are certainly empirically valid assumptions, especially if we think of human capital formation, large R&D projects, or other investments that involve specialization, indivisibilities, and long horizons. Unfortunately, we can not *explicitly* introduce in the model persistent idiosyncratic production risks and irreversibilities or adjustment costs, for then the model loses its tractability.[22] Nonetheless, we

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[22] With persistent productivity shocks, an individual's optimal investment becomes a function of his contempo-

21

can *implicitly* capture these effects in our simulations, by simply increasing the length of a time period. We then interpret the interval between t and $t + 1$ as the horizon of an investment project and the average life of an idiosyncratic productivity shock.

Figure 2 considers the example of a 5-year investment horizon. As previously, we let the discount rate and the depreciation rate to be approximately 5% per year, implying $\beta = 0.75$ and $\delta = 0.25$ over the 5-year period; we also set $\gamma = 4$, $\psi = 1$, and $\alpha \in \{0.35, 0.70\}$. The effect of σ_A on K_∞ is now really dramatic. At $\sigma_A = 100\%$, the capital stock is 30% of what it would have been under full insurance when $\alpha = 0.35$ (Panel B); it is 15% of the complete-markets level when $\alpha = 0.7$ (Panel A).

Figures 1 and 2 demonstrate another related point. In contrast to Aiyagari (1994), incomplete markets imply *both* a low risk free rate *and* a low capital stock. And, the risk free rate in an incomplete-markets economy can be a largely downward biased proxy of the marginal product of capital. In Panel B of Figure 2, for example, the marginal product of capital is 18% per year when $\sigma_A = 100\%$, as compared to a risk-free rate of just 4% per year.

The simulations also provide a useful insight on the interaction between endowment and production risks. The dashed lines in Figures 1 and 2 correspond to $\sigma_e = 50\%$ and the solid ones to $\sigma_e = 0$. We observe that the steady state becomes less sensitive to σ_e as σ_A increases. This is because when σ_A is large, individuals are already holding a buffer stock that can be used to self-insure against both investment and endowment risks. The precautionary effect of σ_A similarly diminishes with σ_e, implying that the investment effect dominates more easily when there is already a lot of precautionary saving in the economy.

We also observe that the impact of production risk on the capital stock tends to dominate that of endowment risk when the two types of risk take comparable value. In Figures 1 and 2, the capital stock K_∞ at $\sigma_A = \sigma_e = 50\%$ is well below the complete-markets one. That is, when production and endowment risks make equal contributions to idiosyncratic income variation, the adverse investment effect of production risk tends to dominate the precautionary savings effect of *both* types of income risk. This finding is suggestive of the fact that the dynamic macroeconomic complementarity we have discussed works as a 'multiplier' for the steady-state impact of σ_A.

We conclude that the quantitative impact of σ_A on K_∞ is clearly quite large within our model.[23]

raneous idiosyncratic productivity. This is obviously not important for our *qualitative* findings, but it complicates aggregation and precludes a closed-form solution of the general equilibrium.

[23] The numerical results we present in Figures 1 and 2 are not very sensitive to either ψ or γ. A higher ψ weakens the effect of σ_A on R_∞ and strengthens its impact on K_∞, because it raises the interest elasticity of savings. On the other hand, γ tends to have a small ambiguous effect, since a higher γ increases both the precautionary motive and the risk premium on investment.

Unfortunately, we do not know of any obvious empirical analogues of σ_A or σ_e. We know, however, that idiosyncratic production, entrepreneurial, and investment risks appear to be quite substantial in reality. For example, survival rates of private firms in the United States are only 34% over the first 10 years of a firm's life. And, even conditional on survival, the distribution of returns to entrepreneurial activity is extremely wide. On the other hand, private savings are overall very low in the United States. These facts suggest that substantial underinvestment is indeed the most likely scenario.

5.4 Numerical Results on Persistence

We argued earlier that the presence of uninsurable production risks generate a dynamic macroeconomic complementarity, which can be the source of amplification and propagation over the business cycle. We now illustrate this effect by examining the effect of σ_A on the speed of convergence to the steady state.

In Appendix C, we linearize the dynamic system (14)-(18) around the steady state of a calibrated economy $\mathcal{E}^{cal} = (\beta, \psi, \gamma, \alpha, \delta, \sigma_A, \sigma_e)$. We calculate the stable eigenvalue λ of the linearized system and approximate the local dynamics by $\log(K_{t+1}/K_\infty) = \lambda \log(K_t/K_\infty)$. The *convergence rate* is $1-\lambda$. Incomplete insurance slows down convergence to the steady state if $1-\lambda$ decreases with σ_A. Numerical simulations show that such is the case for a wide range of plausible parameter values.

Consider our earlier example of a 5-year investment project (Figure 2). We calibrate the model with $\beta = 0.75$ (discount rate $\approx 5\%$ per year), $\delta = 0.25$ (depreciation rate $\approx 5\%$ per year), $\gamma = 4$, $\psi = 1$, and $\alpha \in \{0.35, 0.70\}$. In Figure 3 we then plot the convergence rate and the half-life of an aggregate wealth shock,[24] as we vary σ_A from 0 to 100%. Clearly, the convergence rate decreases rapidly with σ_A. With a narrow definition of capital ($\alpha = 0.35$, Panel B), the half-life of a shock almost doubles as σ_A increases from 0 to 100%. The effect is even stronger when incompleteness affects both physical and human capital ($\alpha = 0.7$, Panel A).

These are substantial effects on the convergence rate, which suggest that undiversifiable productivity risks can play a useful role in generating endogenous persistence over the business cycle in standard RBC models with aggregate uncertainty. We finally stress once again that these results depend on idiosyncratic risks affecting production and investment choices, thereby introducing the kind of dynamic macroeconomic complementarity we discussed before.[25] That is why our persis-

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[24] The half-life T of a deviation from the steady state is defined by $\lambda^T = 1/2$; that is, $T = \square \log_2 \lambda$.

[25] Figure 3 demonstrates the asymmetry between production and endowment risk. On the one hand, endowment risk does not introduce the kind of dynamic macroeconomic complementarity that production risk does. On the other hand, the precautionary motive tends to boost up savings below the steady state and thus to speed up convergence.

tence findings could not have been identified by Krusell and Smith (1998).

6   Concluding Remarks

This paper examined a standard neoclassical growth economy with heterogeneous agents, decentralized production, and uninsurable production and endowment risks. Under a CARA-normal specification for preferences and risks, we obtained closed-form solutions for individual choices and the aggregate dynamics. We found that uninsurable production shocks introduce a risk premium on private equity and reduce the aggregate demand for investment. As a result, the steady-state capital stock tends to be lower under incomplete markets, despite the low risk-free rate induced by the precautionary motive. Undiversifiable idiosyncratic production risks also generate a powerful dynamic macroeconomic complementarity between future and current investment. Originating in a pecuniary externality, this mechanism amplifies the impact of an exogenous aggregate shock on output and investment, slows down convergence to the steady state, and increases the persistence of the business cycle. The complementarity relies critically on the fact that uninsurable risks affect private returns to production. This explains the contrast between this paper and earlier research on economies with endowment shocks (Aiyagari, 1994; Krusell and Smith, 1998).

The model rules out any wealth effects on risk-taking because of the chosen CARA specification.[26] We expect that wealth effects should strengthen our results in more general frameworks. For instance, investment demand would be weak during a recession not only because high borrowing rates and stringent credit conditions are expected in the near future, but also because agents are poor and less willing to engage in risky projects. More generally, the CARA-normal specification appears to play no essential role in our argument that undiversifiable production risks introduce a negative feedback from future borrowing rates and credit conditions to current investment.

Credit-market imperfections and non-convexities in production have been viewed by many authors as a source of persistence in the business cycle. Although these departures from the neoclassical growth model are not considered here, we find that incomplete risk sharing *alone* is sufficient to generate underinvestment and introduce a powerful propagation mechanism. The presence of uninsurable production risks reduces the individual's *willingness* to invest. Introducing borrow-

Therefore, while the convergence rate tends to *decrease* with σ_A, it tends to *increase* with σ_e. Furthermore, the dynamic macroeconomic complementarity appears to dominate the convergence effect of precautionary savings. For instance, when σ_A and σ_e are equal, the convergence rate is lower than under complete markets.

[26] Krusell and Smith (1998) have argued that wealth effects are quantitatively small. This may suggest that we do not miss much by abstracting from them. We expect, however, that wealth effects will be quantitatively significant when credit constraints are accompanied by idiosyncratic technological shocks.

ing constraints would in addition restrict the *ability* to undertake risky projects, which should make investment more sensitive to future credit conditions. We thus expect that credit constraints would only reinforce both the steady-state and the business-cycle effects documented in the paper. Moreover, our findings stress that it is not only contemporaneous credit conditions that matter for investment, but also anticipated future credit conditions.

A detailed quantitative assessment of our mechanism would require the calibration of a standard RBC economy with decentralized production, borrowing constraints, isoelastic preferences, and both aggregate and idiosyncratic productivity shocks. This can be done by combining the framework of this paper with the numerical analysis of Krusell and Smith (1998). Wealth effects and financial constraints are then likely to reinforce our quantitative findings.

Finally, like Aiyagari (1994) and Krusell and Smith (1998), we also treated incomplete markets as exogenous to the model. There is a long tradition in models where incomplete markets originate in private information,[27] while some more recent contributions study economies where incomplete risk sharing originates in the lack of commitment.[28] Although beyond the scope of this paper, it is important to extend this line of work to production economies with idiosyncratic production risk and endogenous incomplete markets.

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[27] See for example Townsend (1982), Green (1987), Banerjee and Newman (1991), Atkeson and Lucas (1992), and Cole and Kocherlakota (2001).

[28] See Kehoe and Levine (1993), Kocherlakota (1996), and Alvarez and Jermann (2000).

Appendix A: Proofs

Proof of Proposition 1 (Individual Choice)

We have already presented the main steps the main text. A more detailed derivation follows below, maintaining the assumption that there is no risk premium on financial assets. A general treatment of the individual's decision problem, under arbitrary prices, is presented in Angeletos and Calvet (2000).

Following (8), the portfolio problem reduces to a simple mean-variance problem. Since the asset structure includes a riskless bond, it is w.l.o.g to assume that $\mathbb{E}_t d_{m,t+1} = 0, \forall m \geq 1$. In the absence of a risk premium, it follows that $\pi_{m,t} = 0, \forall m \geq 1$. We thus rewrite the Bellman equation (6) as

$$\max_{(c_t^j, k_t^j, l_t^j, \theta_{0t}^j)} U(c_t^j) + \beta U \left\{ V_{t+1}^j \left[\mathbb{E}_t w_{t+1}^j - \frac{\Gamma \widehat{a}_{t+1}^j}{2} \min_{(\theta_{1t}^j, \ldots, \theta_{Jt}^j)} \text{Var}_t(w_{t+1}^j) \right] \right\}. \tag{23}$$

The above suggests a two-step solution We successively solve for the optimal portfolio of risky assets and the optimal consumption-investment choices.

Given any $(c_t^j, k_t^j, \theta_{0,t}^j)$, the optimal portfolio $(\theta_{j,t}^j)_{j=1}^J$ minimizes the conditional variance of wealth, $\text{Var}_t(w_{t+1}^j) = \text{Var}_t \left[A_{t+1}^j f(k_{t+1}^j) + e_{t+1}^j + \sum_{j=1}^J d_{j,t+1}^j \theta_{j,t}^j \right]$. The FOCs imply

$$\theta_{m,t}^j = -Cov_t \left[d_{m,t+1} ; A_{t+1}^j f(k_{t+1}^j) + e_{t+1}^j \right], \ \forall m \geq 1.$$

This result has a natural geometric interpretation. For all t, we can project (regress) A_{t+1}^j and e_{t+1}^j on the asset span. This yields $A_{t+1}^j = \kappa_{t+1}^j \cdot d_{t+1} + \eta_{t+1}^j$ and $e_{t+1}^j = \xi_{t+1}^j \cdot d_{t+1} + \varepsilon_{t+1}^j$, where $\kappa_{t+1}^j, \xi_{t+1}^j$ are deterministic constants and $\eta_{n,t+1}^j, \varepsilon_{t+1}^j$ are random variables, $\mathcal{F}_{t+1}$-measurable and orthogonal to the span of d_{t+1}. The optimal portfolio hedges fully the *diversifiable* component of idiosyncratic risks

$$(\theta_{m,t}^j)_{m=1}^M = -\left[f(k_{t+1}^j)\kappa_{t+1}^j + \xi_{t+1}^j \right] \cdot d_{t+1}. \tag{24}$$

The residuals represent the *undiversifiable* risks. We let $\mathbb{E}_t A_{t+1}^j = \overline{A}$, $\mathbb{E}_t e_{t+1}^j = 0$. $\sigma_A^2 \equiv \text{Var}_t(\eta_{t+1}^j) = \text{Var}(\eta_{t+1}^j)$, and $\sigma_e^2 \equiv \text{Var}_t(\varepsilon_{t+1}^j) = \text{Var}(\varepsilon_{t+1}^j)$. By (24), wealth after hedging reduces to $w_{t+1}^j = (\overline{A} + \eta_{t+1}^j)f(k_{t+1}^j) + (1-\delta)k_{t+1}^j + \varepsilon_{t+1}^j + \theta_{0,t}^j$. Thus, $\mathbb{E}_t w_{t+1}^j = \overline{A} f(k_{t+1}^j) + (1-\delta)k_{t+1}^j + \theta_{0,t}^j$ and $\text{Var}_t(w_{t+1}^j) = \sigma_e^2 + f(k_{t+1}^j)^2 \sigma_A^2$.

We now turn to the optimal consumption, savings, and investment decision. We define $\Phi(k) \equiv \overline{A}f(k) + (1-\delta)k$ and $G(k, \Gamma \widehat{a}) \equiv \Phi(k) - \frac{\Gamma \widehat{a}}{2} \left[\sigma_e^2 + f(k)^2 \sigma_A^2 \right]$. It follows that $\mathbb{E}_t w_{t+1}^j = \Phi(k_{t+1}^j) + \theta_{0,t}^j$ and $\mathbb{E}_t w_{t+1}^j - \Gamma \widehat{a}_{t+1}^j \text{Var}_t(w_{t+1}^j)/2 = G(k_{t+1}^j, \Gamma \widehat{a}_{t+1}^j) + \theta_{0,t}^j$. Combining with (23), we conclude that the optimal $(c_t^j, k_{t+1}^j, \theta_{0,t}^j)$ maximizes

$$U(c_t^j) + \beta U \left\{ V_{t+1}^j \left[G(k_{t+1}^j, \Gamma \widehat{a}_{t+1}^j) + \theta_{0,t}^j \right] \right\}, \tag{25}$$

26

subject to $c_t^j + k_{t+1}^j + \theta_{0,t}^j/R_t = w_t^j$. The FOCs with respect to k_t^j and $\theta_{0,t}^j$ give

$$U'(c_t^j) = \beta U' \left\{ V_{t+1}^j \left[G(k_{t+1}^j, \Gamma \widehat{a}_{t+1}^j) + \theta_{0,t}^j \right] \right\} \widehat{a}_{t+1}^j \frac{\partial G}{\partial k}(k_{t+1}^j, \Gamma \widehat{a}_{t+1}^j),$$
$$U'(c_t^j) = \beta U' \left\{ V_{t+1}^j \left[G(k_{t+1}^j, \Gamma \widehat{a}_{t+1}^j) + \theta_{0,t}^j \right] \right\} \widehat{a}_{t+1}^j R_t.$$

(Without loss of generality, we assume interior solutions throughout our analysis.) Combining the above two yields $R_t = \partial G/\partial k$, that is, condition (12).

Next, the *envelope condition* is $U'[V_t^j(w_t^j)] \cdot \partial V_t^j(w_t^j)/\partial w = U'(c_t^j)$. Using (3) and (7), this reduces to $c_t^j = \widehat{a}_t^j w_t^j + \widehat{b}_t^j - \Psi \ln \widehat{a}_t^j$. We infer that $a_t^j = \widehat{a}_t^j$ and $b_t^j = \widehat{b}_t^j - \Psi \ln \widehat{a}_t^j$. Using (3) and (7), we rewrite the FOC with respect to $\theta_{0,t}^j$ as

$$\begin{aligned} U'(c_t^j) &= \beta R_t U' \left\{ V_{t+1}^j \left[\mathbb{E}_t w_{t+1}^j - \Gamma \widehat{a}_{t+1}^j \mathrm{Var}_t(w_{t+1}^j)/2 \right] \right\} \widehat{a}_{t+1}^j = \\ &= \beta R_t U' \left\{ \widehat{a}_{t+1}^j \mathbb{E}_t w_{t+1}^j - \Gamma (\widehat{a}_{t+1}^j)^2 \mathrm{Var}_t(w_{t+1}^j)/2 + \widehat{b}_{t+1}^j - \Psi \ln \widehat{a}_t^j \right\}. \end{aligned}$$

Using $\widehat{a}_{t+1}^j = a_{t+1}^j$, $\widehat{b}_{t+1}^j - \Psi \ln \widehat{a}_{t+1}^j = b_t^j$, and the consumption rule, the above reduces to

$$U'(c_t^j) = \beta R_t U' \left[\mathbb{E}_t c_{t+1}^j - \Gamma \mathrm{Var}_t(c_{t+1}^j)/2 \right].$$

This gives the Euler condition (13).

Finally, the consumption rule and the budget constraint imply that $dc_{t+1}^j/dw_t^j = a_{t+1}^j R_t(1 - a_t^j)$. Plugging this relation in the Euler condition, we infer that $a_t^j = 1/[1 + (a_{t+1}^j R_t)^{-1}]$. Iterating forward yields (11). **QED**

Proof of Proposition 2 (Savings)

Since $\mathrm{Var}(w) = \sigma_e^2 + \sigma_A^2 f(k)^2$, $\partial \mathrm{Var}(w)/\partial \sigma_e^2 > 0$ but $\partial \mathrm{Var}(w)/\partial \sigma_A^2 = f(k)^2 + [2\sigma_A^2 f(k) f'(k)][\partial k/\partial \sigma_A^2] \lesseqgtr 0$. Let $f(k) = \sqrt{k}$ and, w.l.o.g., $\delta = 1$. Then, $G(k, \widehat{\Gamma}) = (2\sqrt{k})^{-1}[\overline{A} - \widehat{\Gamma} \sigma_A^2 \sqrt{k}]$ and $R = \partial G(k, \widehat{\Gamma})/\partial k$ implies $k = \overline{A}^2/(2R + \widehat{\Gamma} \Gamma \sigma_A^2)^2$. Hence, $\mathrm{Var}(w) = \sigma_e^2 + \sigma_A^2 \overline{A}^2/(2R + \widehat{\Gamma} \sigma_A^2)^2$ and $\partial \mathrm{Var}(w)/\partial \sigma_A^2 < 0 \Leftrightarrow \sigma_A^2 > 2R/\widehat{\Gamma}$. **QED**

Proof of Proposition 3 (Investment)

By the implicit function theorem, the first-order condition, $R = \partial G/\partial k$, implies $\partial k/\partial \sigma_A^2 = -\mu \widehat{\Gamma}$ and $\partial k/\partial \widehat{\Gamma} = -\mu \sigma_A^2$, where $\mu \equiv f(k) f'(k)/(-\partial^2 G/\partial k^2)$. The second-order condition implies $\partial^2 G/\partial k^2 < 0$ and thus $\mu > 0$. With some but not serious loss of generality, we can ignore the dependence of μ on $\widehat{\Gamma}$ and σ_A^2. We then conclude that $\partial k/\partial \sigma_A^2$ is proportional to $-\widehat{\Gamma}$ and $\partial k/\partial \widehat{\Gamma}$ is proportional to $-\sigma_A^2$. (The first is exact when $\widehat{\Gamma} \approx 0$ and the second when $\sigma_A \approx 0$.) Finally, from (11), $\partial \widehat{\Gamma}_t/\partial R_s > 0$ for all $s > t$. **QED**

27

Proof of Theorem 1 (General Equilibrium)

Existence is proved in Angeletos and Calvet (2000), by taking the limit of finite-horizon economies. Here, we only characterize the general equilibrium. First, we note that (11) implies $a_t^j = a_t = \widehat{a}_t^j = \widehat{a}_t$, $\forall t, j$. Given this, (12) implies $k_t^j = K_t$, $\forall t, j$. (12) then reduces to (14) and (13) reduces to

$$\mathbb{E}_t c_{t+1}^j - c_t^j = \Psi \ln(\beta R_t) + \frac{\Gamma a_{t+1}^2}{2}[\sigma_e^2 + f(K_{t+1})\sigma_A^2].$$

Aggregating the above across all $j \in \mathbb{J}$, and using the fact that

$$\frac{1}{J}\sum_{j \in \mathbb{J}} \mathbb{E}_t c_{t+1}^j = \mathbb{E}_t \left\{ \frac{1}{J}\sum_{j \in \mathbb{J}} w_{t+1}^j - \frac{1}{J}\sum_{j \in \mathbb{J}} k_{t+1}^j - \pi_t \cdot \frac{1}{J}\sum_{j \in \mathbb{J}} \theta_t^j \right\} = W_{t+1} - K_{t+1} = C_{t+1},$$

we get (15). Finally, (18) and (17) follow from aggregating the budget constraints and using Assumption 4 (no aggregate uncertainty). **QED**

Proof of Theorem 2 (Steady State)

The steady state is defined by the system (21)-(22). The second condition implies $R_\infty \leq 1/\beta$. Since $\alpha_\infty > 0$, we also have $R_\infty > 1$. Since $R_\infty > 1$, the first equation implies $\overline{A}F'(K_\infty) + 1 - \delta > 1$, or equivalently $K_\infty < \widehat{K} \equiv (F')^{-1}(\delta/\overline{A})$. Therefore, R_∞ is bounded between 1 and $1/\beta$ and K_∞ is bounded between 0 and $\widehat{K}$. We find it useful to consider the functions $m_1(1, \beta^{-1}] \to [0, +\infty)$ and $m_2[0, \widehat{K}) \to [\sigma_e^2, \sigma_e^2 + f(\widehat{K})^2\sigma_A^2)$, defined by $m_1(R) \equiv (2\Psi/\Gamma)(1 - R^{-1})^{-2}\ln[1/(R\beta)]$ and $m_2(K) \equiv \sigma_e^2 + f(K)^2\sigma_A^2$. We observe that m_1 is decreasing in R and m_2 is increasing in K. For any $K \in [0, \widehat{K})$, the equation $m_1(R) = m_2(K)$ has unique solution, $R_1(K) \equiv m_1^{-1}[m_2(K)]$, which maps $[0, +\infty)$ onto $(1, m_1^{-1}(\sigma_e^2/2)] \subseteq (1, \beta^{-1}]$. Similarly, the first equation (21) implicitly defines a function $R_2(K)$ that maps $(0, \widehat{K}]$ onto $[1, +\infty)$. This function is decreasing, due to the second-order condition of the individual's problem.

Consider the function $\Delta(K) = R_1(K) - R_2(K)$. When $K \to 0$, we observe that $R_1(K)$ is bounded and $R_2(K) \to +\infty$, implying $\Delta(K) \to -\infty$. We also note that $\Delta(\widehat{K}) = R_1(\widehat{K}) - 1 > 0$. The graphs of the functions R_1 and R_2 therefore intersect at least once and there exists at least one steady state.

The above proves existence. The system (21)-(22) and the local dynamics allow us to analyze also multiplicity and stability. We earlier discussed how incomplete markets introduce an dynamic macroeconomic complementarity. If this complementarity is strong enough, multiple steady states, endogenous cycles, or indeterminacy may arise. In Angeletos and Calvet (2000), we show that such complex dynamics arise only for *very* large σ_A and σ_e. When instead σ_A and σ_e are close to zero, uniqueness and stability of the *incomplete-markets* steady state follow, by continuity, from uniqueness and stability of the *complete-markets* steady state. In this paper, we consider only the

plausible case where σ_A and σ_e are not *too* large. This also ensures that our calibration of the steady state in Section 5 is meaningful. **QED**

Proof of Proposition 5 (Comparative Statics)

We used the functions R_1 and R_2 define in the proof of Theorem 2 to analyze the monotonicity of the steady state with respect to the economy's parameters. We consider the case $|R_2'(K_\infty)| > |R_1'(K_\infty)|$, which is necessarily satisfied when the steady-state is unique. An increase in σ_e or β pushes down the function $R_1(K)$ and leaves the function $R_2(K)$ unchanged. The steady state is therefore characterized by a lower interest rate and a higher capital stock. Similarly, an increase in $1-\delta$ and $\overline{A}$ pushes up $R_2(K)$, also leading to a lower interest rate and a higher capital stock. An increase in Γ or σ_A pushes down both $R_1(K)$ and $R_2(K)$, reflecting the fact that $\Gamma\sigma_A$ enters in both the demand for investment and the supply of savings. Γ and σ_A can therefore have ambiguous effects, as verified in simulations. **QED**

Proof of Proposition 6 (Calibrated Steady State)

Consider a calibrated economy $\mathcal{E}^{cal} = (\beta, \gamma, \psi, \alpha, \delta, \sigma_A, \sigma_e)$. Let $q_t \equiv f(K_t)/K_t = K_t^{\alpha-1}$ denote the output-capital ratio at date t, implying $f'(K_\infty) = \alpha q_\infty$ and $C_\infty/K_\infty = (q_\infty-\delta)$ at the steady state. The calibration of Γ and Ψ (see Appendix C) implies $\Gamma C_\infty = \gamma$ and $\Psi/K_\infty = \psi(q^*-\delta)$, $q^* \equiv (\beta^{-1}-1 + \delta)/\alpha$. The steady-state system (21)-(22) thus reduces to

$$R_\infty = (1-\delta) + \alpha q_\infty(1-\gamma\Delta\sigma_A^2), \quad \ln(\beta R_\infty) = -\frac{\gamma\Delta^2}{\psi\nu}(\sigma_A^2 + \sigma_e^2). \tag{26}$$

where $\Delta \equiv [(R_\infty-1)/R_\infty][q_\infty/(q_\infty-\delta)]$ and $\nu \equiv 2(q^*-\delta)/(q_\infty-\delta)$. In Angeletos and Calvet (2001) we prove that a steady state exists for any calibrated economy $\mathcal{E}^{calib}$. When $\sigma_A = \sigma_e = 0$ (complete markets), $\Delta = (1-\beta)q^*/(q^*-\delta)$, $\nu = 2$, $R_\infty = R^* \equiv 1/\beta$, and $q_\infty = q^* \equiv (\beta^{-1}-1 + \delta)/\alpha$, like in the standard Ramsey model. When σ_A^2 and σ_e^2 are positive but close to 0, the first-order variations in R_∞ and q_∞ are obtained by keeping Δ and ν constant in (26). We thus get $d(\ln R_\infty) = -(\gamma\Delta^2)/(\psi\nu) \cdot d(\sigma_A^2)$ and $dq_\infty = \gamma\Delta q_\infty(1-\underline{\psi}/\psi)d(\sigma_A^2)$, where $\underline{\psi} = (q^*-\delta)(\beta^{-1}-1)/(2\alpha)$. It follows that $dq_\infty/d(\sigma_A^2) > 0$ and thus $dK_\infty/d\sigma_A < 0$ if and only if $\psi > \underline{\psi}$. **QED**

Appendix B: Calibrating Γ and Ψ

In this Appendix we describe the calibration of Γ and Ψ that permit us to map the preference characteristics of our exponential-utilities economy to the preference characteristics of an isoelastic-utilities economy.

We first consider Γ. We observe that the degree of relative risk aversion at the steady-state level of aggregate consumption is ΓC_∞. We pick some constant γ and restrict the incomplete-markets economy $\mathcal{E}$ so that $\Gamma C_\infty = \gamma$. The parameter γ thus corresponds to a constant measure of relative risk aversion at the aggregate level, which permits a meaningful calibration of the model.

We next consider Ψ. We observe that the elasticity of intertemporal substitution (EIS) at the steady-state level of aggregate consumption is Ψ/C_∞. In analogy to what we did for risk aversion, we could pick some constant ψ and restrict $\mathcal{E}$ such that $\Psi/C_\infty = \psi$. This calibration, however, is problematic for the following reason. The convergence rate under *complete* markets (see Appendix C) is given by

$$g = 1 - 2\left\{1 + \beta + \frac{\Psi}{K_\infty}\beta^2(1-\alpha)(\beta^{-1}+\delta-1) + \sqrt{\left[1 + \beta + \frac{\Psi}{K_\infty}\beta^2(1-\alpha)(\beta^{-1}+\delta-1)\right]^2 - 4\beta}\right\}^{-1}.$$

The above is a function of Ψ/K_∞ and (α, β, δ). When markets are *incomplete*, the convergence rate g depends as well on σ_A and σ_e. The *direct* effect of σ_A on the convergence rate g captures the macroeconomic complementarity we have discussed in Section 4. But, since K_∞ is a function of σ_A, there is also an *indirect* effect through the dependence of g on Ψ/K_∞. If we were to calibrate $\mathcal{E}$ so that Ψ/C_∞ remains invariant at some prespecified level, then we would be permitting Ψ/K_∞ to vary with σ_A. We would thus be compounding the direct and the indirect effect of σ_A on g. Instead, we want to control for the indirect effect (which is an unfortunate implication of our CARA-normal specification) and partial out the direct effect (which captures the dynamic macroeconomic complementarity introduced by uninsurable idiosyncratic production risk). To control for the indirect effect, we must keep Ψ/K_∞ constant as we vary σ_A. This in turn requires a meaningful parametrization of Ψ/K_∞. We observe that under complete markets $C_\infty/K_\infty = (q^*-\delta)$, where $q^* \equiv (\beta^{-1}-1+\delta)/\alpha$. Therefore, restricting the elasticity of intertemporal substitution, Ψ/C_∞, to equal ψ at the complete-markets steady state implies $\Psi/K_\infty = \psi C_\infty/K_\infty = \psi(q^*-\delta)$. The latter suggests how to calibrate Ψ. We pick some constant ψ and restrict $\mathcal{E}$ so that Ψ/K_∞ remains invariant at $\psi(q^*-\delta)$. The parameter ψ thus measures the elasticity of intertemporal substitution around the complete-markets steady state and our calibration permits a meaningful assessment of the impact of market incompleteness on the convergence rate.[29]

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[29] We think that the calibration of Ψ we propose above is the most reasonable one. The alternative of calibrating the incomplete-markets steady state so that $\Psi/C_\infty = \psi$ yields similar results in the following sense: The convergence rate is a decreasing function of σ_A and an increasing function of Ψ/K_∞. Because K_∞ typically decreases with σ_A, the overall effect of σ_A on g turns out to be non-monotonic in some simulations. However, if we look at the *difference* between the incomplete-markets convergence rate and the *shadow complete-markets* convergence rate, where the latter is defined as the convergence rate of a complete-markets economy that has the same Ψ/K_∞ as the incomplete-

Finally, as discussed in Section 5, we chose a Cobb-Douglas specification for the technology and rescale σ_A and σ_e as percentages of output. We thus define:

Definition (Calibrated Economies) A calibrated economy $\mathcal{E}^{cal} = (\beta, \gamma, \psi, \alpha, \delta, \sigma_A, \sigma_e)$ corresponds to an incomplete-markets economy $\mathcal{E} = (\beta, \Gamma, \Psi, F, \delta, \overline{A}, \sigma'_A, \sigma'_e)$ such that $\Gamma C_\infty = \gamma$, $\Psi/K_\infty = \psi(q^*-\delta)$, $q^* \equiv (\beta^{-1}-1+\delta)/\alpha$, $F(K,L) = K^\alpha L^{1-\alpha}$, $\overline{A} = 1$, $\sigma'_A = \sigma_A$, and $\sigma'_e = \sigma_e F(K_\infty)$.

Appendix C: Local Dynamics Around the Steady State

We first observe that equilibrium paths can be calculated by a backward recursion of the *state vector* $z_t = (a_t, C_t, W_t)$.

Lemma (Equilibrium Recursion) For any state vector $z_{t+1} = (a_{t+1}, C_{t+1}, W_{t+1}) \in (0,1] \times R \times [0, +\infty)$, there exists a generically unique $(a_t, C_t, W_t, K_{t+1}, R_t) \in (0,1) \times R^2 \times R_+^2$ satisfying the equilibrium recursion (14)-(17).

Proof. Given $z_{t+1} = (a_{t+1}, C_{t+1}, W_{t+1})$ for some t, and with $L_t = 1 \forall t$, we proceed as follows. First, we solve equation (14), namely $R_t = \partial G(K_{t+1}, \Gamma a_{t+1})/\partial K$, for K_{t+1} as a function of (R_t, a_{t+1}). This equation might assume multiple solutions. However, generically, only one of them corresponds to a global maximum of the individuals problem. Second, we substitute K_{t+1} from the above into (18), namely $W_{t+1} = \Phi(K_{t+1}, 1)$, to obtain an equation in (R_t, a_{t+1}, W_{t+1}). We solve this equation for R_t as a function of (a_{t+1}, W_{t+1}). We then plug this back into $R_t = \partial G(K_{t+1}, \Gamma a_{t+1})/\partial K$, to get K_{t+1} as a function of (a_{t+1}, W_{t+1}). Finally, equations (16)-(17) then assign unique values to a_t, C_t, and W_t. *QED*

We then let **H** denote the equilibrium recursion mapping; $z_t = \mathbf{H}(z_{t+1})$ is implicitly defined by

$$
\begin{aligned}
a_t &= 1/[1 + (a_{t+1}R_t)^{-1}], \\
C_t &= C_{t+1} - \Psi\ln(\beta R_t) - \Gamma a_{t+1}^2 \left[f(K_{t+1})^2 \sigma_A^2 + \sigma_e^2 \right]/2, \\
W_t &= C_t + K_{t+1},
\end{aligned}
$$

---

markets economy, then this difference is always negative and increasing in σ_A. That is, the 'true' impact of σ_A is indeed to slow down convergence.

where $R_t = \partial G(K_{t+1}, \Gamma a_{t+1})/\partial K$ and $K_{t+1} = \Phi^{-1}(W_{t+1})$. The Jacobian of $\mathbf{H}$ is

$$\mathbf{D}_z\mathbf{H} = \begin{bmatrix} \frac{\partial a_t}{\partial a_{t+1}} & 0 & \frac{\partial a_t}{\partial W_{t+1}} \\ \frac{\partial C_t}{\partial a_{t+1}} & 1 & \frac{\partial C_t}{\partial W_{t+1}} \\ \frac{\partial C_t}{\partial a_{t+1}} & 1 & \frac{\partial (C_t+K_{t+1})}{\partial W_{t+1}} \end{bmatrix}.$$

We observe that $\partial K_{t+1}/\partial W_{t+1} = 1/\Phi'(K_{t+1}) > 0$, and

$$\frac{\partial R_t}{\partial a_{t+1}} = -\Gamma f(K_{t+1}) f'(K_{t+1})\sigma_A^2 \le 0,$$

$$\frac{\partial R_t}{\partial W_{t+1}} = \frac{1}{\Phi'(K_{t+1})}\left\{ f''(K_{t+1})\left[\overline{A} - \Gamma a_{t+1} f(K_{t+1})\sigma_A^2\right] - a_{t+1}[f'(K_{t+1})]^2\Gamma\sigma_A^2\right\} < 0.$$

Let $\chi(v) \equiv 1/(1 + v^{-1}) = v/(1 + v)$ and note that $\chi'(v) = 1/(1 + v)^2 = [\chi(v)/v]^2$. Since $a_t = \chi(a_{t+1}R_t)$, we infer that

$$\frac{\partial a_t}{\partial a_{t+1}} = \left(\frac{a_t}{a_{t+1}R_t}\right)^2\left(R_t - a_{t+1}\left|\frac{\partial R_t}{\partial a_{t+1}}\right|\right),$$

$$\frac{\partial a_t}{\partial W_{t+1}} = -\left(\frac{a_t}{a_{t+1}R_t}\right)^2 a_{t+1}\left|\frac{\partial R_t}{\partial W_{t+1}}\right| < 0.$$

Future propensity a_{t+1} has an ambiguous effect on current propensity a_t. There is both a positive *direct* effect (due to the complementarity of future and current consumption) and a negative *indirect* effect (the precautionary motive causes a decline in the current interest rate R_t). Since $C_t = C_{t+1} - \Psi\ln(\beta R_t) - \Gamma a_{t+1}^2\left[f(K_{t+1})^2\sigma_A^2 + \sigma_e^2\right]/2$, we infer that

$$\frac{\partial C_t}{\partial a_{t+1}} = \frac{\Psi}{R_t}\left|\frac{\partial R_t}{\partial a_{t+1}}\right| - \Gamma a_{t+1}[f(K_{t+1})^2\sigma_A^2 + \sigma_e^2]$$

$$\frac{\partial C_t}{\partial W_{t+1}} = \frac{\Psi}{R_t}\left|\frac{\partial R_t}{\partial W_{t+1}}\right| - \Gamma a_{t+1}^2\sigma_A^2\frac{f(K_{t+1})f'(K_{t+1})}{\Phi'(K_{t+1})}$$

An increase in a_{t+1} and W_{t+1} leads to a decline in the current interest rate, which has a positive effect on current consumption. On the other hand, the increase in a_{t+1} and W_{t+1} implies that the agent bears more risk between time t and time $t + 1$.

Let $\mathbf{I}$ denote the identity matrix. The characteristic polynomial, $P(x) = \det(\mathbf{D}_z\mathbf{H} - x\mathbf{I})$, can be rewritten as

$$P(x) = \left(\frac{\partial a_t}{\partial a_{t+1}} - x\right)\left\{x^2 - \left[1 + \frac{\partial(C_t + K_{t+1})}{\partial W_{t+1}}\right]x + \frac{\partial K_{t+1}}{\partial W_{t+1}}\right\} + x\frac{\partial C_t}{\partial a_{t+1}}\frac{\partial a_t}{\partial W_{t+1}}.$$

The roots of P are the eigenvalues of the *backward* dynamical system. (The eigenvalue λ considered in Section 5 thus satisfies $P(1/\lambda) = 0$ and $1/\lambda > 1$.) Since $P(-\infty) = +\infty$ and $P(+\infty) = -\infty$,

32

There always exists a real eigenvalue. Simple calculation shows that $P(1) > 0$ if and only if $|K_2'(R_\infty)| < |K_1'(R_\infty)|$. Thus when there is a unique steady state, the Jacobian matrix $\mathbf{D}_z\mathbf{H}$ has an eigenvalue in $(1, +\infty)$, and the dimension of the stable manifold is at least 1.

When markets are complete, $\sigma \equiv (\sigma_A, \sigma_e) = 0$, we know that

$$\left.\frac{\partial a_t}{\partial a_{t+1}}\right|_{\sigma=0} = \beta, \qquad \left.\frac{\partial C_t}{\partial a_{t+1}}\right|_{\sigma=0} = 0, \quad \text{and} \quad \left.\frac{\partial C_t}{\partial W_{t+1}}\right|_{\sigma=0} > 0.$$

The characteristic polynomial thus reduces to $P(x)|_{\sigma=0} = (\beta - x)\, Q(x)$, where

$$Q(x) = x^2 - x\left\{1 + \left.\frac{\partial(C_t + K_{t+1})}{\partial W_{t+1}}\right|_{\sigma=0}\right\} + \left.\frac{\partial K_{t+1}}{\partial W_{t+1}}\right|_{\sigma=0}.$$

Obviously, $x = \beta$ is the one eigenvalue. This is contained in the interval $(0, 1)$ and thus corresponds to an unstable manifold. We next observe that $Q(0) > 0$ and $Q(1) < 0$. We infer the quadratic Q has one root in the interval $(0, 1)$ and one root in $(1, +\infty)$. Overall, the Jacobian matrix $\mathbf{D}_z\mathbf{H}$ has two eigenvalues in the interval $(0, 1)$ and one eigenvalue larger than 1. The stable manifold under complete markets has thus dimension 1. We can explicitly calculate the root that corresponds to the stable manifold. We note that, when markets are complete, $R_\infty = \Phi'(K_\infty) = 1/\beta$ and

$$\left.\frac{\partial K_{t+1}}{\partial W_{t+1}}\right|_{\sigma=0} = \frac{1}{\Phi'(K_\infty)} = \beta,$$

$$\left.\frac{\partial R_t}{\partial W_{t+1}}\right|_{\sigma=0} = \frac{\overline{A}f''(K_\infty)}{\Phi'(K_\infty)} = \overline{A}\beta f''(K_\infty) < 0,$$

$$\left.\frac{\partial C_t}{\partial W_{t+1}}\right|_{\sigma=0} = -\frac{\Psi}{R_\infty}\left.\frac{\partial R_t}{\partial W_{t+1}}\right|_{\sigma=0} = -\Psi\overline{A}\beta^2 f''(K_\infty) > 0.$$

The polynomial $Q(x)$ therefore reduces to

$$Q(x) = x^2 - \left[1 + \beta - \Psi\beta^2\overline{A}f''(K_\infty)\right]x + \beta.$$

If the production function is Cobb-Douglas, $f(K) = K^\alpha$, then $\overline{A}f''(K_\infty) = (\alpha-1)\overline{A}f'(K_\infty)/K_\infty = -(1-\alpha)(\beta^{-1}+\delta-1)/K_\infty$. Therefore, with complete markets and a Cobb-Douglas technology, the stable eigenvalue is $\lambda = 1/x$ and

$$x = \frac{1}{2}\left\{1 + \beta + \frac{\Psi}{K_\infty}\beta^2(1-\alpha)(\beta^{-1}+\delta-1) + \sqrt{\left[1 + \beta + \frac{\Psi}{K_\infty}\beta^2(1-\alpha)(\beta^{-1}+\delta-1)\right]^2 - 4\beta}\right\}.$$

Finally, when markets are incomplete, the stable manifold is one-dimensional as long as σ_A and σ_e are not very large. (Locally, this follows from our finding that the cubic $P(x)$ has only one root outside $(0, 1)$ when $\sigma_A = \sigma_e = 0$ and by continuity of the $P(x)$ in σ_A and σ_A. More generally, we

check numerically that this is the case for all our simulations in Section 5.) The incomplete-markets convergence rate is a function of Ψ/K_∞, like the complete-markets one, but it is also of function of σ_A and σ_e per se. Since $P(x)$ is a cubic, there is a closed-form solution for the incomplete-markets convergence rate. We omit the formula, however, because it is *too* long and not interesting. For our numerical simulations in Section 5, we use Mathematica to solve analytically for the convergence rate in general and then evaluate it for the particular numerical values.
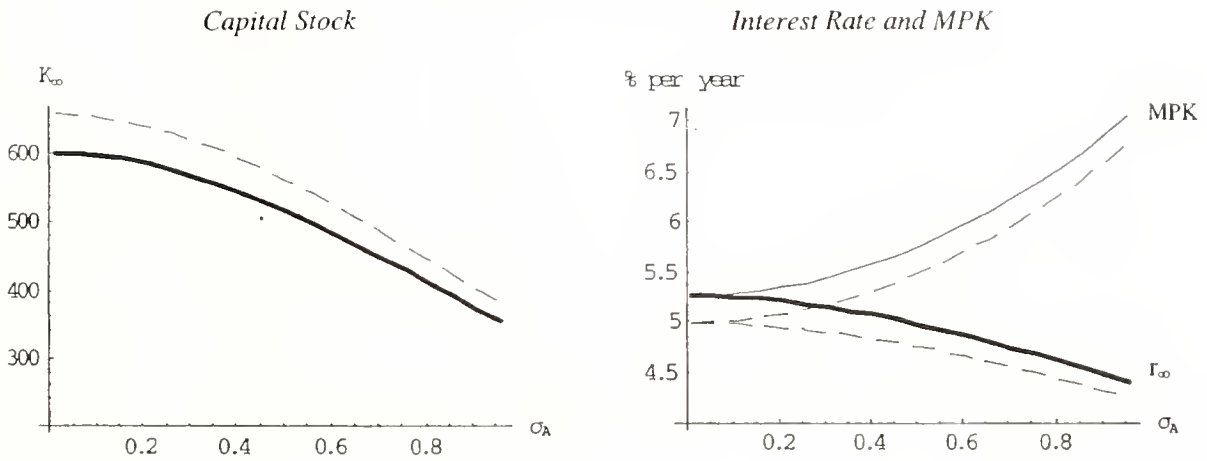
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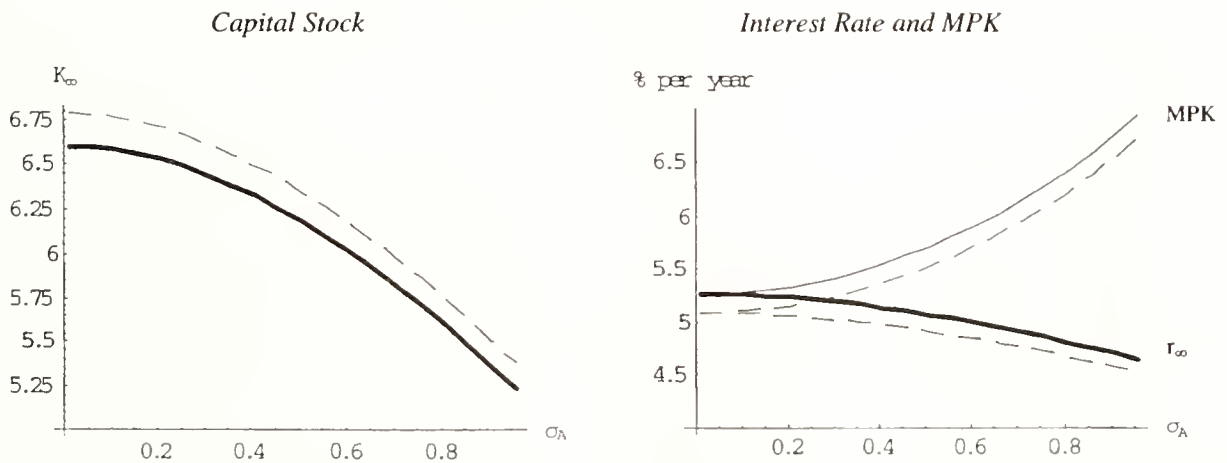
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Figure 1.A ($\alpha = 0.70$)

Capital Stock

Interest Rate and MPK

Figure 1.B ($\alpha = 0.35$)

Capital Stock

Interest Rate and MPK

FIGURE 1. We perform an RBC calibration of the model with a time period of one year. The discount rate is 5% per year, the depreciation rate is 5% per year, the degree of relative risk aversion is 4, and the elasticity of intertemporal substitution is 1. The income share of capital is 70% in Panel A and 35% in Panel B. The solid lines correspond to $\sigma_e = 0$ (no idiosyncratic endowment risk) and the dashed ones to $\sigma_e = 50\%$ (of steady-state GDP). The plots show the steady-state level of the capital stock, the interest rate, and the marginal product of capital (MPK), as idiosyncratic production risk σ_A varies between zero and 100% of steady-state GDP.
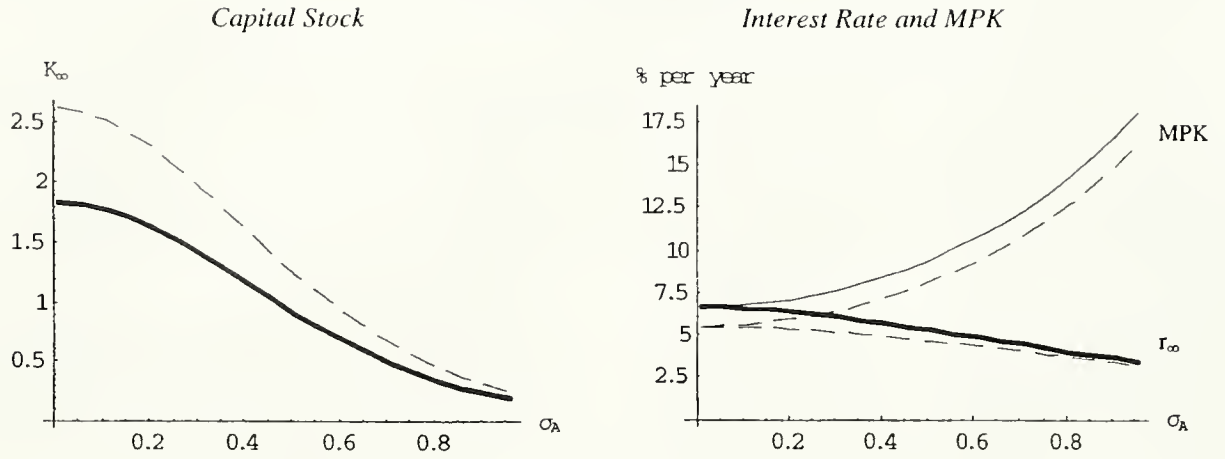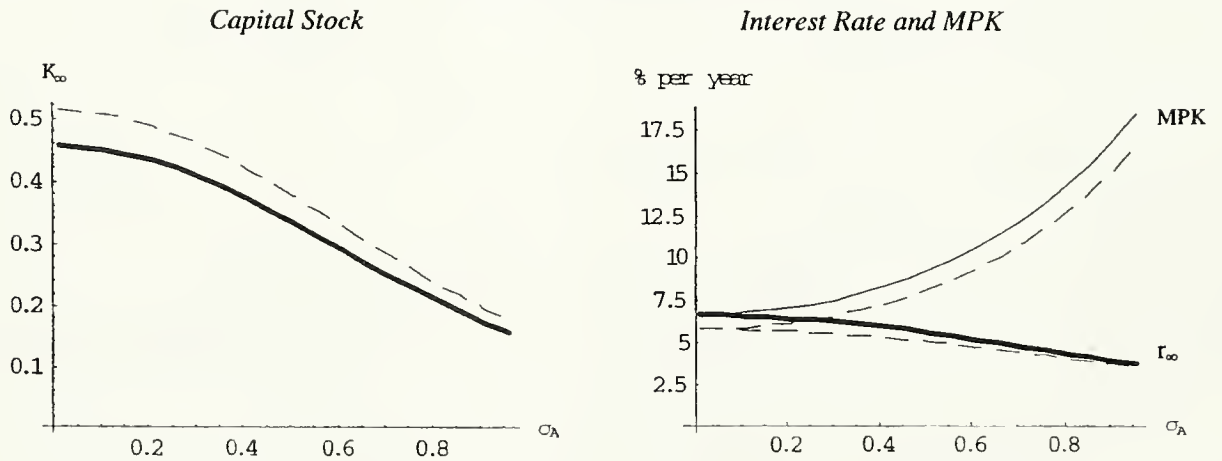
Figure 2.A ($\alpha = 0.70$)

Capital Stock

Interest Rate and MPK

Figure 2.B ($\alpha = 0.35$)

Capital Stock

Interest Rate and MPK

FIGURE 2. We assume the same parameter values as in Figure 1, but now use a five year time period (for both the length of an investment project and the duration of an idiosyncratic production shock). The solid lines correspond to $\sigma_e = 0$ and the dashed ones to $\sigma_e = 50\%$. The plots show the steady-state level of the capital stock, the interest rate, and the marginal product of capital (MPK), as idiosyncratic production risk σ_A varies between zero and 100%.
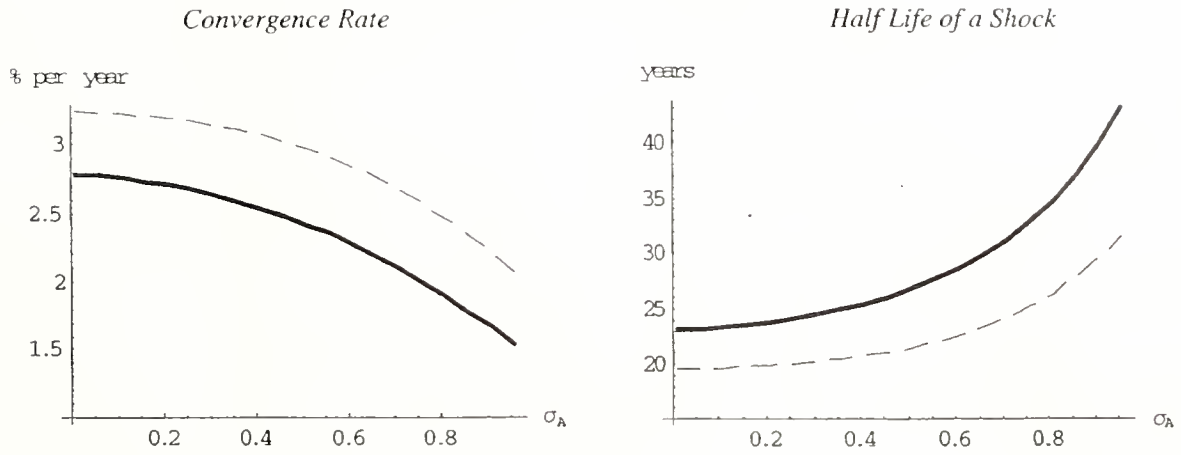
Figure 3.A ($\alpha = 0.70$)

Convergence Rate

Half Life of a Shock

Figure 3.B ($\alpha = 0.35$)

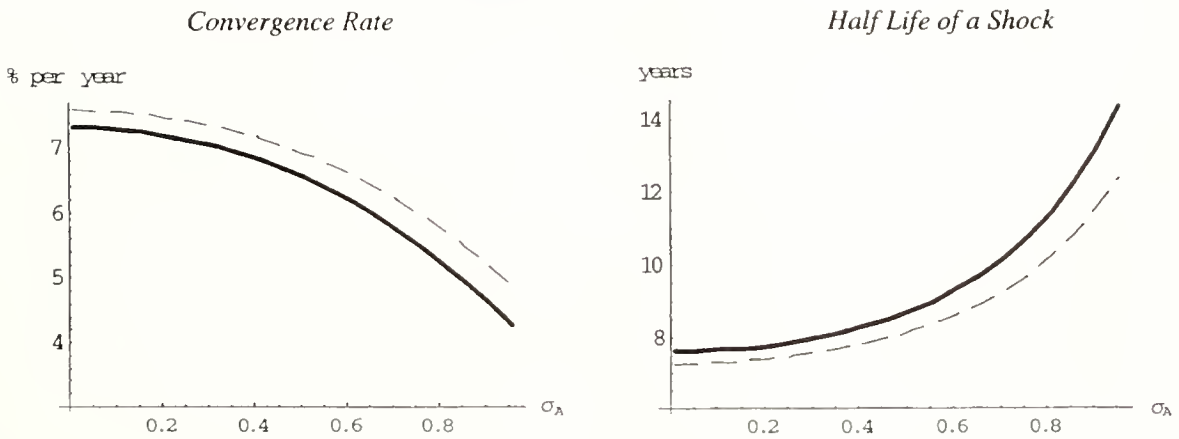*Convergence Rate*

Half Life of a Shock

FIGURE 3. Assuming the same parameters as in Figure 2, we plot the convergence rate and the half-life of the deviation from the steady state as idiosyncratic production risk σ_A varies between zero and 100%. The solid lines correspond to $\sigma_e = 0$ and the dashed ones to $\sigma_e = 50\%$.