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Managerial Incentives: On the Near Optimality of Linearity

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Abstract

Managers make efforts and choices. Efficient incentives to induce effort focus on the signal extraction problem of inferring the effort level. Efficient incentives for choices line up the relative payoffs of principal and agent. With choices much more important than the variation in the cost of inducing effort, the optimal payment schedule tends toward proportionality. The argument holds if the control space of the agent has full dimensionality, but not otherwise.

If the agent's choices include a complete set of fair gambles and insurance, the proportional payoff schedule is no more expensive than any other schedule that induces effort.

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Managerial Incentives: On the Near Optimality of Linearity

Peter Diamond[1]

Managers are called on to make efforts and to make choices. In designing incentives to induce effort, efficiency focuses on the signal extraction problem associated with inferring the level of effort made (Hart and Holmstrom, 1987). In contrast, to encourage appropriate choices, efficient incentive design tends to line up the relative payoffs of principal and agent. The presence of choices therefore alters the design of incentives in the standard formulation by introducing an additional factor.[2] In many cases the choices are much more important than the variation in the cost of inducing effort. In this case the optimal payment schedule tends toward proportionality.[3] This point is made in Section I in a simple model with three states of nature where it is proven that as the cost of effort shrinks (relative to gross payoffs), the optimal schedule of payoffs to the agent converges to a linear function of gross payoffs. In this three state model, there is a single variable describing choice and a single relative payment control variable for the principal.

In Section II, a four state model is used to argue that the same logic holds if the control space of the agent has full dimensionality, but not otherwise.[4] If the principal has more degrees of freedom in setting incentives than the agent has degrees of freedom in responding, the extra degrees of freedom are associated with additional first order conditions that block the argument that leads to linearity. The mathematical structure of the argument is that the optimal schedule converges to one of the optima that induce correct choice when there is no cost of effort. When the linear schedule is the unique optimum at zero effort, then the optimal schedule converges to a linear schedule.

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[2] For both choices and efforts, incentives based on outcomes can be supplemented by incentives based on observable aspects of agent behavior, such as gross inefficiencies.

[3] For other approaches to deriving a linear optimal payment schedule, see Holmstrom and Milgrom, 1987, Laffont and Tirole, 1986, and McAfee and McMillan, 1987.

[4] The dimensionality of the control space is an analog in this setup to the "multitask" perspective in Hart and Holmstrom, 1987.

Among the implications of this result is the convergence of the optimal schedule to a schedule that ignores the relative performance of the firm when information is available on the performance of other firms. The needed assumption of full dimensionality implies that the agent can trade expected return for greater correlation with the returns in the industry. When the cost of effort is small, it is better to ignore relative performance in order to induce a higher expected return. In the basic model presented, effort choice is a zero-one variable. In Appendix B, the model is reexamined for the case that effort is a continuous variable.

In Sections I and II, the agent is assumed to be choosing from a set of probabilities of different returns that is well-behaved relative to the choice problem, but is not otherwise specified. In section III, it is assumed that the sets of choices available to the agent (with and without effort) include complete sets of fair gambles and insurance. In this case a different result is possible. If the agent is free to rearrange probabilities across payoffs in any way that preserves the expected gross payoff to the principal, the proportional payoff schedule is no more expensive than any other schedule that induces effort. However, there are many schedules that have this property. In section IV, it is assumed that the sets of choices available to the agent (with and without effort) come from a complete set of fair gambles, but without any insurance. Provided one assumes the monotone likelihood ratio property, the same conclusion holds as in Section III.

I Model

We make the following assumptions. The principal is risk neutral. The agent is risk neutral, but can not receive a negative payment. There is no individual rationality constraint (the expected amount of payment is sufficient to induce supply). There are three states of nature, with payoffs to the principal of $x_3 > 0$, $x_2 > 0$, $x_1 = 0$.[5][6] Assume that the principal chooses three payments to the agent satisfying $s_3 >= 0$, $s_2 >= 0$, $s_1 = 0$.[7] The payment schedule is linear in gross payoffs if $s_2/s_3 = x_2/x_3$.

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[5] Adding a constant to these three payoffs makes no change in the analysis.

[6] In a one period model, one can not consider the issue of the appropriate way to measure the return to the principal. Thus, whether payments to managers should depend on some measure of accounting profits or current or future stock market values, or some combination of these is not addressed. Similarly, the choice of a single period schedule of payments to the agent does not allow consideration of the use of options as opposed to other methods of compensation.

[7] The principal has no reason to use a payment that is the same in all states of nature, since it encourages neither more effort nor correct choices. Thus this restriction amounts to requiring no payment lower than the payment in the state with the lowest gross payoff.

Effort

First we review the standard model, where there is only an effort choice. In the standard formulation of this problem, expending effort changes the probability vector of the gross returns from $[1-f_2-f_3, f_2, f_3]$ to $[1-g_2-g_3, g_2, g_3]$. With the usual formulation, the principal's problem can be stated as

$$(1)\quad \text{Maximize } (x_2-s_2)g_2 + (x_3-s_3)g_3$$

$$\text{s. t. } s_2(g_2-f_2) + s_3(g_3-f_3) >= c,$$

$$s_i >= 0 \text{ for all } i,$$

where c is the cost of effort. Given the linearity of this problem the optimal scheme can pay compensation in just one state of nature. The optimum is found by finding the state or states (other than state 1) for which g_i/f_i is a maximum and paying enough compensation in that state to satisfy the incentive compatibility constraint. If $g_2/f_2 = g_3/f_3$, compensation can be spread across both states; otherwise a linear payment schedule is not optimal.

Effort and Choice

We modify this model by assuming that expending effort generates a set of possible distributions of gross revenues. The agent is then free to (costlessly) choose any element in the set. This structure is meant to capture the idea that effort generates an array of possible strategies for the firm, from which the agent selects one - with all strategies requiring roughly equal managerial effort to execute.

To analyze this problem, we work backwards, beginning with the agent's choice of the probabilities of payoffs. Denote the probability of state two by g. We assume that g is a choice variable (within some range), with $h(g)$ ($h'<0$, $h''<0$) being the (maximal) induced probability of state three. Over the feasible range of values of g, we assume that h' varies sufficiently to result in an interior solution. The probability vector of the three states is $[1-g-h(g), g, h(g)]$. Since there is no payoff in state 1, the payoff to the agent is $s_2g+s_3h(g) = s_3((s_2/s_3)g+h(g))$.[8] Thus, the agent's choice depends only on the ratio of payoffs in states two and three which we denote by $s=s_2/s_3$. Let us write the chosen level of g as $g^*(s)$. Then $g^*(s)$ maximizes $sg+h(g)$ and is defined by the first order condition

$$(2)\quad s + h'(g^*) = 0.$$

Differentiating (2), we see that g^* is monotonically increasing in s.

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[8] With the assumption that h is concave, the maximization problem of the agent is concave and there is a unique solution.

Conditional on effort being expended, we can write expected gross revenue as a function of the relative payoffs, R(s), as

$$(3)\quad R(s) = x_2 g^*(s) + x_3 h(g^*(s)).$$

Differentiating (3) and using (2), we note that R′ is zero if and only if s gives a linear payoff, $s=x_2/x_3$.

$$(4)\quad R'(s) = x_2 g^{*\prime}(s) + x_3 h'(g^*(s)) g^{*\prime}(s)$$
$$= (x_2 - x_3 s) g^{*\prime}(s).$$

Thus the expected gross revenue, $x_2 g + x_3 h(g)$, is maximized at s equal to x_2/x_3. A deviation from linearity results in a loss in expected gross revenue. In this sense, a deviation from linearity is similar to a distorting tax. We note that with proportional increases in x_2 and x_3, we have a proportional increase in R′.

If no effort is made by the agent, the probability vector is $[1-f_2-f_3, f_2, f_3]$. We assume that it is worthwhile to induce effort. Effort can be induced by using payoffs to the agent that give the agent an expected return at least as large as the cost of making effort, c.

$$(5)\quad s_2 g^*(s_2/s_3) + s_3 h(g^*(s_2/s_3)) - s_2 f_2 - s_3 f_3 >= c.$$

Thus we can induce effort with any payoff ratio, s, that satisfies

$$(6)\quad s g^*(s) + h(g^*(s)) - s f_2 - f_3 > 0.$$

That is, for a given ratio, s, that satisfies this inequality, there is a value of s_3 sufficiently large so that the effort inducement constraint (5) is satisfied. For payoff ratios satisfying the condition in (6), by using (5), the expected cost of just inducing effort can be written as:

$$(7)\quad C(s) = s_2 g^*(s) + s_3 h(g^*(s))$$
$$= c[s g^*(s) + h(g^*(s))]/[s g^*(s) + h(g^*(s)) - s f_2 - f_3].$$

We note that C(s) and C′(s) are proportional to c.

We can now state the optimal incentive problem as

$$(8)\quad \text{Maximize}\quad R(s) - C(s).$$

We assume that there is a unique internal solution to this problem. Then, the first order condition is

$$(9)\quad R'(s^*) = C'(s^*).$$

As noted above, preserving the ratio of x_2 and x_3, R′ is proportional to x_3, while C′ is proportional to c. Thus, as the ratio of the cost of effort to gross revenues, c/x_3, decreases to zero, $C'(s)/R'(s)$ tends to zero unless s converges to x_2/x_3. Thus, s^* tends to x_2/x_3; the optimal incentive structure converges to a linear structure. With the cost of inducing effort going to zero (relative to gross payoffs), both s^*_2/x_2 and s^*_3/x_3 tend to zero, but their ratio is well-defined, tending to one.[9]

Note that the result depends on the ratio c/x_3, which can become small either by having c become small or by having x_3 become large. For example, if choices affect the profits of a large firm with profits in the billions, but effort can be induced by expected payoffs in the millions, the cost of effort can be large in absolute terms, while the ratio of cost to gross payoff is small. Note also that the result is that the optimal schedule converges to linear, not that the additional cost of a linear schedule becomes small relative to x_3 (compared with the optimal schedule).

II Generalizations

Four states

In the three state model analyzed, there is a single variable describing choice for the agent and a single relative payment control variable for the principal. This structure prevents the principal from having additional degrees of freedom for encouraging effort beyond the incentive for choice. It is the restriction in the dimensionality of agent choice in some formulations that results in a basically different structure of incentives. To examine this balance between the choice variables of the agent and the control variables of the principal we contrast two different formulations of the four state problem.

Now assume that there are 4 states. Assume that the gross payoffs satisfy $x_1 = 0$, $x_i > 0$, $i=2, 3, 4$. Similarly, assume that $s_1 = 0$, $s_i >= 0$, $i=2, 3, 4$. If we assume that the range of choice of the agent has full dimensionality (given the constraint that probabilities add to one), then we assume that the agent can choose probabilities in two states. That is, the agent can choose both g_2 and g_3. For any choice in the feasible range (and it is assumed that there is a positive range for each variable) there is an implied (maximal) level of probability of state 4, written as $h(g_2, g_3)$. We further assume that the shape of this function is such that the optimum involves a unique interior choice. Then we have the same first order conditions as before and the argument goes through in the same way, leading to the conclusion that the optimal choices of s_2/s_4 and s_3/s_4 converge to x_2/x_4 and x_3/x_4.

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[9] If we did not assume risk neutrality for the agent, we would be examining the utilities associated with each state rather than just the payoff in each state. These utilities might be state-dependent, $u_i(s_i)$, for example if the agent has career concerns and different states impacted differently on future opportunities. As long as the cost of bearing risk associated with the agent's risk aversion is small relative to gross payoffs, we would expect a similar sort of convergence for utilities.

In contrast, if we assume that there is a single control variable for the agent, g_2, then the probabilities in the other states are all functions of this variable, $h_3(g_2)$ and $h_4(g_2)$. That is, in the case examined just above, the four probabilities are $[1-g_2-g_3-h(g_2, g_3), g_2, g_3, h(g_2, g_3)]$. In the case of a single control variable, the probabilities are $[1-g_2-h_3(g_2)-h_4(g_2), g_2, h_3(g_2), h_4(g_2)]$. In the case where the control variables do not span the space of states of nature, the principal has another degree of freedom in setting payments for the agent, and the argument above does not go through. That is, choice depends on the sum $s_3h_3(g_2) + s_4h_4(g_2)$, not on the two terms separately. Thus, there is a continuum of values of s_3 and s_4 that result in the same choice and the selection of the particular combination of s_3 and s_4 is made solely to minimize the cost of inducing effort.

Formal argument[10]

The contrast between the results with and without full dimensionality is striking. This raises the question of the underlying mathematical structure for this result. The intuitive logic follows; a formal proof is given in Appendix A. As the cost of effort goes to zero the set of optimal relative payoffs at each cost converges to the set of optimal relative payoffs when the cost of effort is exactly zero. If the latter set contains a unique optimum, then the optima converge to the linear schedule, which is always an optimum with no cost of effort. However, if the set of optima at zero cost contains other solutions besides the proportional schedule, then the convergence to the set of optima may not involve convergence to the linear schedule. The set of optima with zero cost is the set of supports to the set of available distributions at the point that maximizes expected gross payoffs. With full dimensionality and a lack of kinks, the support for the set of distributions at the point that maximizes expected gross returns will be unique. Then the argument in the text goes through. However, without full dimensionality, there is not uniqueness in the set of supporting prices and convergence will not generally hold.

Relative performance

The model can be extended to a situation where information is available on the performance of other firms in the industry. The critical question remains that of dimensionality - whether the agent has the ability to modify the correlation between the returns to the principal and the returns to other firms. When full dimensionality remains present, the linearity argument goes through, implying that the optimal incentive scheme converges to one that ignores relative performance.

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[10] This argument was made by Jim Mirrlees.

To review this formally, assume that the gross payoff to the principal can be high or low and the gross payoff to the industry can be high or low. Thus there are four states, given by the two-by-two matrix of low and high returns for the principal and the industry. With a zero payment in the event that the principal's payoff is low while the industry payoff is high, the principal has three controls, or two relative controls. If the agent's action space is of full dimensionality, the agent can choose two payoff probabilities. In this case the argument above goes through, with the same result, that in the limit the payoff schedule should ignore the performance of the industry. The problem with trying to use such information to hold down the cost of inducing effort is that it affects the expected gross payoff as the agent responds to the incentive structure by altering the correlation of the principal's payoff with that of the industry.

Perks

The model above does not have explicit representation of decisions by the agent that directly affect the agent's utility at the cost of expected gross returns. Excessive numbers of limousines and jets are representative examples of such actions. More expensive are new corporate headquarters, or relocation of corporate headquarters for the pleasure of the agent. Even more expensive might be empire building. Since there is no limit on the ability to waste and since it is inevitable that top management will have only a small share of the variation of returns in a large firm, the presence of such possible actions calls for attempts to monitor them directly. Thus, just as the principal sets the cash reimbursement schedule of the agent and needs to check that the agent does not receive more cash compensation than the agent is entitled to, so the principal needs to monitor the noncash reimbursement of the agent. With sufficiently effective monitoring, the argument above should remain, so that complicated reimbursement schedules, as called for by the model with only effort, remain unnecessary, and possibly ineffective.

Another decision that involves large returns is that of replacement of the agent by another agent. The model above has not considered such a possibility, one that complicates the analysis by having nonfinancial returns associated with replacement.

III Fair gambles and insurance

In the usual formulation of the principal-agent problem, (1), the optimal schedule selects the state (or states) for which g_i/f_i is a maximum and pays enough compensation in that state to induce effort. If state i is the lowest cost state, then, paralleling the argument leading to (7), the cost of inducing effort is $cg_i/(g_i-f_i) = c(g_i/f_i)/((g_i/f_i)-1)$. Note that the function $cz/(z-1)$ is decreasing in z. Thus, the principal rewards the agent in the state where g_i/f_i is largest. Only if $g_2/f_2 = g_3/f_3$, can the optimum include compensation in both states.

V Concluding remarks

In many settings, the cost of inducing managers to work hard is far less important than encouraging them to make the right choices from the set made available by their hard work. For example, the managers of large firms have effort costs which are very small compared to the range of possible profits of the firms, which can be in the billions. In modelling this situation, it was assumed that agents were risk neutral, but could not be paid a negative amount. This structure was designed to capture several aspects of large firms. One is that the wealth of management is indeed small relative to the value of the firm. Thus there is no way that the payoff to management can vary as a significant fraction of gross payoffs to the firm without having very large expected payoffs to the management. Thus the focus is on the structure of payoffs to the agent relative to gross payoffs, not the level of the ratio. Convergence of the payoffs to the agent to zero, relative to gross payoffs, is an implication of large firms without a large enough set of suitably wealthy individuals.

One assumption of the basic model is that agents make optimal choices. If the variation in returns were sufficiently small, agents might not bother making optimal choices, since choosing might not be costless, as was assumed. The convergence of the optimal schedule to one that is linear with positive slope, rather than to a flat salary, makes this concern not seem important.

Assuming the agents are well-enough paid in the worst state to be risk neutral reflects two presumed aspects of a more basic model. One is that it is probably efficient for a large firm to absorb the risk of agents as a way of holding down the cost of inducing a supply of suitable agents. Second is that the risk aversion of agents will lead to poor choices for a risk neutral principal. Thus significant minimal payoff takes care of the risk aversion problem (except that of losing the job) at relatively small cost to the firm. Given the assumptions made, a proportional payoff schedule (in utilities) will be nearly optimal. If, as also assumed, managers are paid sufficiently well to be risk neutral, then a proportional payoff schedule will be nearly optimal.

The second argument made in the paper is that the ability of managers to alter the probability structure of expected payoffs gives management an opportunity to take advantage of some nonlinearities in the payment schedule, implying that a nonlinear structure would result in higher expected costs to the principal. Thus, when the ability to manipulate probabilities is large enough, the proportional schedule is optimal even if the cost of inducing effort is significant.

Appendix A: Proof of convergence[12]

The argument in the text for the convergence of the optimal schedule to a linear schedule was made in calculus terms in a model with three states of nature. We now sketch a noncalculus proof of the same argument, without the restriction to three states of nature. We will see that the argument goes through when there is a unique revenue maximizing schedule, but not necessarily otherwise. We assume that it is worthwhile to induce effort.

Assume n states of nature, and write as G the n-1 dimensional choice set (omitting state 1) of the agent, assuming effort. We assume that G is a convex set. Let us define C(s) as the cost to the principal of the optimal choice of g by the agent in the presence of a reward schedule s:

A1 $C(s) = \text{Maximum } g.s \text{ for } g \text{ in } G.$

Note that C(s) is homogeneous of degree 1 in s. We define G(s) as the set of probabilities that achieve this maximum:

A2 $G(s) = \{ g \mid g.s = C(s) \}.$

We note that the set G(s) is homogeneous of degree zero in s.

Similarly, we define the gross revenue generated by a reward schedule, s, assuming that indifference on the part of the agent results in a choice that maximizes gross revenue for the principal.

A3 $R(s) = \text{Maximum } g.x \text{ for } g \text{ in } G(s).$

Note that R(s) is maximized over s when s is proportional to x since G(s) is a subset of G and maximal revenue over G comes from a choice in G(x).

Define as S the set of rewards that result in selection of the revenue maximizing probabilities over the entire set G.

A4 $S = \{ s \mid R(s) = R(x) \}$

We note that the set S is homogeneous of degree zero in s. If G has a unique supporting hyperplane at the revenue maximizing probabilities, then S contains only the point x and its scalar multiples. If G is not of full dimension, then there will be additional points in S.

We can now write the principal's problem as choosing g and s to:

A5 $\text{Maximize } g.x - C(s)$

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[12] This approach and proof were provided by Jim Mirrlees.

subject to $C(s) - f.s = c$, g in $G(s)$, $s>=0$.

Equivalently, we can consider the problem of choosing relative rewards, s, constrained to add to one, and a scalar multiple, z. Thus we restate the principal's problem as choosing g, s and z to:

A6 Maximize $g.x - zC(s)$
subject to $z[C(s) - f.s] = c$, g in $G(s)$, $s>=0$, $1.s = 1$.

Using the moral hazard constraint to eliminate z, we can restate the principal's problem when $c>0$ as choosing g and s to:

A7 Maximize $g.x - cC(s)/[C(s) - f.s]$
subject to $C(s) > f.s$, g in $G(s)$, $s>=0$, $1.s = 1$.

In order to make the limit argument, we now consider a sequence of solutions to this problem as c varies. Consider a sequence of values of c, c_i, which converges to zero. For each c_i, select a pair of values of g and s, g_i and s_i, which maximize the principal's net revenue, that is solve A7. Next consider a subsequence of values of c (retaining the same notation) for which the subsequence s_i converges to some value, named t. Next, consider a subsequence of this subsequence (retaining the same notation) for which the subsequence g_i converges to some value, named h. We will have completed the proof if we show that t is in S. That is, we will have shown that as c decreases to zero, every convergent sequence of optimal rewards converges to a point in S. If there is a unique supporting hyperplane to G at the revenue maximizing point, then the limit point t is proportional to x.

Lemma 1. Lim sup $R(s_i) <= R(t)$.

From the optimality of s_i and g_i, we have $g_i.s_i >= g.s_i$ for all g in G. Passing to the limit we have $h.t >= g.t$ for all g in G. Thus we have h in $G(t)$, implying that $h.x <= R(t)$. With s_i converging to t, we have $g_i.x$ converging to $h.x$, completing the proof.

Lemma 2. Lim sup $R(s_i) >= R(x)$.

Let x' be proportional to x and satisfying $1.x'=1$. From the optimality of s_i and g_i, we have $R(s_i) - cC(s_i)/[C(s_i) - f.s_i] >= R(x') - cC(x')/[C(x') - f.x']$. Taking the limit as c goes to zero completes the proof.

Thus we have shown that $R(t) = R(x)$, implying that s_i converges to one of the points in S. If G has a unique supporting hyperplane, then s_i converges to proportionality with x, that is, t is a linear payoff schedule. If G does not have a unique support, because of a kink or less than full dimensionality, then the limit of s_i need not be proportional to x.

Appendix B: Continuous effort

The discussion above was made simpler by the discrete nature of the effort choice. In this section, I briefly turn to the case of a continuous adjustment of effort. Let us denote effort by e and its cost to the agent by ce. First we review the problem if there is no choice variable, only an effort variable. In this case, we write the probabilities as functions of effort, $h_i(e)$. In this setting, the agent maximizes $s_2h_2(e) + s_3h_3(e) - ce$. This gives the first order condition for the agent's choice

B1 $s_2h'_2 + s_3h'_3 = c.$

From the first order condition, we can write the chosen level of effort as a function of the payments, $e^*(s_2, s_3)$. We note that the derivatives of e^* with respect to payments are proportional to the values of h':

B2 $e^*_i = -h'_i/(s_2h''_2 + s_3h''_3).$

We can now state the principal's problem as

B3 $\mathrm{Max}\ (x_2 - s_2)h_2(e^*(s_2, s_3)) + (x_3 - s_3)h_3(e^*(s_2, s_3)).$

The first order conditions for the optimal incentive schedule are

B4 $-h_2 + (x_2-s_2)h'_2e^*_2 + (x_3-s_3)h'_3e^*_2 = 0,$

$-h_3 + (x_2-s_2)h'_2e^*_3 + (x_3-s_3)h'_3e^*_3 = 0.$

Using (B1), we can rewrite the first order conditions, (B4), as:

B5 $-h_2 + (x_2h'_2 + x_3h'_3 - c)e^*_2 = 0,$

$-h_3 + (x_2h'_2 + x_3h'_3 - c)e^*_3 = 0.$

or, using (B2):

B6 $-h_2(s_2h''_2 + s_3h''_3) + (x_2h'_2 + x_3h'_3 - c)h'_2 = 0,$

$-h_3(s_2h''_2 + s_3h''_3) + (x_2h'_2 + x_3h'_3 - c)h'_3 = 0.$

Thus, unless h'_i/h_i is the same for both states, the optimum offers a payment in only one state of nature. This is similar to the situation with a discrete choice of effort level.

Now let us assume that the probability of state three is a function of both the effort undertaken and the choice of probability of state two, h(g,e).[13] In this setting, the agent maximizes $s_2 g + s_3 h(g,e) - ce$. This gives the pair of first order conditions for the agent's effort and choice:

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[13] We are ignoring the impact of effort on the available range of values of g.

B7 $$s_3 h_e = c,$$

$$s_2 + s_3 h_g = 0.$$

From the first order conditions, we can write the choice variables as functions of the payments, $g^*(s_2, s_3)$ and $e^*(s_2, s_3)$.

We can now state the principals problem as

B8 $$\text{Max } (x_2 - s_2)g^*(s_2, s_3) + (x_3 - s_3)h(g^*(s_2, s_3), e^*(s_2, s_3)).$$

The first order conditions for the optimal incentive schedule are

B9 $$-g^* + (x_2 - s_2)g^*_2 + (x_3 - s_3)(h_g g^*_2 + h_e e^*_2) = 0,$$

$$-h(g^*, e^*) + (x_2 - s_2)g^*_3 + (x_3 - s_3)(h_g g^*_3 + h_e e^*_3) = 0.$$

Following the same tack as previously, let us group the first order conditions between terms relating to choice and those relating to effort:

B10 $$x_2 g^*_2 + x_3 h_g g^*_2 = g^* + s_2 g^*_2 + s_3(h_g g^*_2 + h_e e^*_2) - x_3 h_e e^*_2,$$

$$x_2 g^*_3 + x_3 h_g g^*_3 = h(g^*, e^*) + s_2 g^*_3 + s_3(h_g g^*_3 + h_e e^*_3) - x_3 h_e e^*_3.$$

Now using the first order conditions for the agent's problem, we can write this as:

B11 $$(x_2 - x_3(s_2/s_3))g^*_2 = g^* + s_3 h_e e^*_2 - x_3 h_e e^*_2,$$

$$(x_2 - x_3(s_2/s_3))g^*_3 = h(g^*, e^*) + s_3 h_e e^*_3 - x_3 h_e e^*_3.$$

or

B12 $$(x_2 - x_3(s_2/s_3))g^*_2 = g^* + (s_3 - x_3)h_e e^*_2,$$

$$(x_2 - x_3(s_2/s_3))g^*_3 = h(g^*, e^*) + (s_3 - x_3)h_e e^*_3.$$

If the right hand sides of these two equations go to zero, then s_2/s_3 converges to x_2/x_3. But the right hand sides of equations (B12) are precisely the first order conditions for the agent's efforts in the model without choice, equation (B4). So, we have a similar conclusion to that above - when the variation in the cost of inducing optimal effort becomes small relative to gross payoffs, the incentive schedule converges to linear.

The exploration of sufficient conditions for the effort choice not to be important at the margin is more complicated when effort is a continuous variable than when it is a zero-one variable, and I will not examine it in detail. In the latter case, it was sufficient to have the cost of effort become small relative to gross payoffs, c/x_3 go to zero. In this case, one needs to have assumptions that limit the increase in effort, since ce/x_3 need not go to zero when c/x_3 does. Thus, we would want to make the plausible assumption that h_e goes to zero at a finite level of e. We might also want to rule out the possibility that changes in effort have important effects on the slope of the tradeoff between probabilities in both states. That is, we might also want to assume that h_{ge} goes to zero as well.

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In this formulation, expending effort changes the probability vector of the gross returns from $[1-f_2-f_3, f_2, f_3]$ to $[1-g_2-g_3, g_2, g_3]$. In the analysis above, we added a choice variable that was only available if effort was expended. We now consider the case where choice is available whether or not the agent expends effort. However, we now assume that the choice set of the agent is defined by the ability to take all possible fair gambles and fair insurance policies (or hedging possibilities). That is, we assume the agent can rearrange the probabilities of the different states in any way that preserves the expected gross payoff to the principal.

Proposition. Assume that there are n states, with $0=x_1<x_2<x_3<...<x_n$. Assume that whether effort is exerted or not, the agent has access to all fair gambles and insurance. Then the linear schedule costs no more than any other schedule.

We proceed by first examining schedules that pay the agent in just one state, then considering all alternatives.

Payoff in one state

Let us denote the mean gross payoffs without and with effort by m_f and m_g. With a proportional payoff schedule, gambles that do not change the expected return to the principal, do not change the expected payoff to the agent. Thus we can ignore fair gambles in evaluating the proportional schedule and note, from the argument leading to (7), that the cost of inducing effort with the proportional schedule is

$$(10) \quad C = cm_g/(m_g-m_f) = c(m_g/m_f)/((m_g/m_f)-1).$$

Next consider a payoff schedule that gives compensation only in state i. The agent wants to concentrate as much probability as possible on this state. If x_i exceeds the mean return, the agent maximizes the expected payoff by taking gambles so that all probability is concentrated on states i and 1. If x_i is less than the mean return, the agent maximizes the expected payoff by taking gambles so that all probability is concentrated on states i and n. Let us denote the probabilities of payoffs after such gambles by f′ and g′ in the cases that effort is not and is expended. Then we have two cases (ignoring the state with a payoff precisely equal to the mean, in which case the probability of such a state can be set to 1).

$$(11) \quad \text{If } x_i > m_f, \text{ then } f'_i = m_f/x_i.$$

$$\text{If } x_i < m_f, \text{ then } f'_i = (x_n-m_f)/(x_n-x_i).$$

The same rule holds for g.

To induce effort using a schedule that pays A in state i, we note that A must satisfy

$$(12) \quad A(g'_i-f'_i) = c.$$

Thus, the expected cost satisfies

(13) $C = Ag'_i = cg'_i/(g'_i-f'_i) = c(g'_i/f'_i)/((g'_i/f'_i)-1)$.

We noted that the cost, C, is decreasing in g'/f'.

We have three cases as x_i exceeds m_g, lies between m_f and m_g, and is less than m_f. In these three cases, we have:

(14) If $x_i > m_g > m_f$, $g'_i/f'_i = m_g/m_f$.

If $m_f < x_i < m_g$, $g'_i/f'_i = (x_n-m_g)x_i/(x_n-x_i)m_f < m_g/m_f$.

If $x_i < m_f < m_g$, $g'_i/f'_i = (x_n-m_g)/(x_n-m_f) < m_g/m_f$.

Comparing (13) and (10) and using (14), we conclude that costs are at least as high with payoff to the agent in a single state as with proportional payoffs. In the first case, where the payoff is in a state with high return, we have the same cost for the schedule paying in the single state as for the schedule with proportional payoffs. In the other two cases, costs are higher with the payoff in a single state than with proportional payoffs. Thus, with fair gambles, the agent is relatively better able to take advantage of gambles when there is the lower expected return without effort. For example, in the case $x_i < m_f < m_g$, the ratio of probabilities of collecting, g'_i/f'_i, converges to one as x_n rises without limit. The convergence of the probabilities makes it expensive to induce effort with this schedule.

General case

Schedules that pay the agent in just one state are at least as expensive as the proportional schedule. We extend the argument to more complicated schedules using the following argument. First we argue that the linearity of this problem implies that an optimum can be found where the agent puts probability weight on no more than two states. In turn, this implies that costs are not increased and effort not discouraged by setting payoffs equal to zero in states in which the agent puts no probability if effort is taken. An argument directly comparing costs, parallel to that above, completes the proof.

We begin with the agent's problem after exerting effort:

(15) Maximize $g'_2s_2+g'_3s_3+\dots+g'_ns_n$

$$\text{subject to } g'_2x_2+g'_3x_3+\dots+g'_nx_n = m_g,$$

$$g'_2+g'_3+\dots+g'_n <= 1,$$

$$g'_i >= 0 \text{ for all } i.$$

This is a linear programming problem. We can conclude that an optimum can be found by the agent in which probability is put on no more than two states. In order to preserve the mean payoff to the principal, these two states, i and j, say, must lie on either side of m_g:

(16) $x_i < m_g < x_j$.

Denote the probability placed on state i by g′, with the probability on state j being 1-g′. Then, from the constraint on expected gross returns, we have

(17) $g' = (x_j - m_g)/(x_j - x_i)$.

The mean return without effort, m_f, might lie in the interval between x_i and x_j, or might be less than x_i. In the former case, denoting the no-effort probabilities by f′ and 1-f′, we have:

(18) $f' = (x_j - m_f)/(x_j - x_i)$.

In this case, the cost of just inducing effort is

(19) $s_j - (s_j - s_i)g'$,

where

(20) $(s_j - s_i)(f' - g') = c$.

Thus the cost of just inducing effort using payoffs only on states i and j is

(21) $s_j - (s_j - s_i)g' = s_j - cg'/(f' - g')$.

This is minimized at $s_j = 0$, implying we are back in the case with payment in a single state, which does not cost less than the proportional schedule.

In the remaining case, $m_f < x_i$. If $s_i/x_i < s_j/x_j$, after exerting effort, the agent will do at least as well using states 1 and j rather than states i and j. But this involves a payoff in just one state, and does not cost less than the proportional schedule. Thus we are left with the case that there are positive payoffs in two states and the state with the lower gross payoff has at least as high a relative payoff as the state with the higher gross payoff. Note that with effort, the agent puts probability on states i and j, while without effort, the agent puts probability on states 1 and i. In this case, we have

(22) $f' = m_f/x_i$,

while g′ satisfies (18). The cost of effort is equal to the difference between expected returns with and without effort, while the expected cost to the principal of inducing effort is the expected return with effort, which equals the cost of effort plus the expected return without effort.

(23) $C = c + s_i m_f / x_i$.

Thus lowering s_i while raising s_j to preserve the incentive for effort lowers expected cost to the principal. This argument holds as long as the relative payoff is higher on state i. Thus we have not raised the cost to the principal by going to a schedule with the same relative payoffs. But this case is covered by the argument just above.

Thus we can conclude that when the set of alternative choices available to the agent is the set of all fair gambles and hedges, the proportional payoff schedule does at least as well as any other schedule. Note that in this case, there are many equally good schedules. The linear schedule is one of these, but so, too, are the schedules that pay in just one state of nature, provided that the state with the payoff has a gross return at least as large as the mean return when making effort.

IV Fair Gambles and the Monotone Likelihood Property

In the argument above, we have assumed no restriction on the ability of the agent to move probability across the states of nature other than preserving the mean. This assumption might be viewed as a rich set of alternative investments that have a frontier with this property. Alternatively, the ability to move probability can be viewed as coming from the availability of both fair gambles and fair insurance. There is an asymmetry between gambles and insurance in that the former can be done after the agent knows the realized before-gamble return, while the latter requires ex ante arrangements. Thus it is natural to consider the case where the agent can take any fair gamble, but has no insurance available at all. That is, we assume that decreasing the probability in some state requires increasing the probabilities in states with both higher and lower returns. This implies that probability can be moved out of any state except the ones with the highest and lowest returns.

With just this assumption, it is not the case that the linear schedule can do as well as any other schedule. Consider a three state model where without the availability of gambles, the unique optimum is to pay the agent only in state 2. Then, with this payoff schedule, the availability of gambles does not alter what the agent will do, with or without effort. Thus, this remains the unique optimal schedule. To analyze this situation further, we also assume the monotone likelihood ratio property.[11]

Proposition. Assume that there are n states with $0=x_1<x_2<x_3<...<x_n$. Assume that all states have positive probability without and with effort: $f_i>0$, $g_i>0$. Assume that

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[11] For analysis of this problem without gambles, see Innes, 1990.

$g_1/f_1<g_2/f_2<g_3/f_3<\ldots<g_n/f_n$. Assume that whether effort is exerted or not, the agent has access to all fair gambles, but no insurance. Then the linear schedule costs no more than any other schedule.

The proof proceeds in three steps. First we observe that the linear schedule costs exactly the same as the schedule that pays only in state n. Second, we show that the schedule paying only in state n is strictly better than the schedule paying only in one state other than state n. Third, we argue that any schedule with a relative payoff in some state higher than that in state n can be improved upon.

With a payoff only in state n, the absence of insurance is of no consequence and the argument is the same as that used above. Similarly, we note that any schedule that has no state with a higher relative payoff than in state n costs the same as the schedule with payoff only in state n. This follows from the fact that with such a schedule, the agent would shift all probability to states 1 and n.

Now consider the schedule with a payoff only in state $i>1$. The agent will take gambles so there is no probability on any state between 1 and i. Thus, the probabilities of receiving payoffs, f'_i and g'_i satisfy

$$(24)\quad \begin{aligned} f'_i &= (f_1x_1+f_2x_2+f_3x_3+\ldots f_ix_i)/x_i, \\ g'_i &= (g_1x_1+g_2x_2+g_3x_3+\ldots+g_ix_i)/x_i. \end{aligned}$$

Thus we see that g'_i/f'_i is increasing in i. Since the cost is decreasing in g'_i/f'_i, as we saw above, this completes the second step in the proof.

Now consider an arbitrary payoff schedule which has a higher relative payoff in some state than in state n. Let state i be the state with the highest index (highest gross payoff) that has a higher relative payoff than does state n, $s_i/x_i > s_n/x_n$. Since relative payoffs on states between i and n are lower than those at i and n, gambles are taken until there is no probability on states between i and n. For any state with index below i, none of the probability is moved to state n. These statements hold both with and without effort. Thus, there is more probability placed on state n with effort than without effort. Consider slightly raising s_n and lowering all other payoffs, with the decreases in other payoffs proportional to gross payoffs, x_i, and done so that the payoff with effort is unchanged. The payoff to gambles not transferring probability to state n is unchanged. Thus there are no changes in gambles, and the inducement to effort now exceeds the cost of effort, allowing a reduction in all payoffs, and so in the cost of the schedule.

From the argument, we see that all schedules for which the relative payoff in every state is no larger than the relative payoff in the highest state have the same cost. This includes the linear schedule. This observation completes the proof.

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