LINEAR DECISION AND CONTROL

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The linear decision problem [3] is to choose at the end of period k-1 a vector of instrument, or control, variables, \( u_{k-1} \), given observations on \( u_{k-2}, u_{k-3}, \ldots \) and on \( y_{k-1}, y_{k-2}, \ldots \), for \( k=1,2,\ldots,N \), where \( u_{k-1} \) is a vector of partly controlled, or endogenous, variables, so as to minimize the expected value of \( J \), where

\[
J = t'u + s'y + \frac{1}{2}[u'Ru + y'Qy + u'Py + y'P'u] 
\]

\[
= \sum_{k=1}^{N} t'_k u_{k-1} + \sum_{k=1}^{N} s'_k y_k + \frac{1}{2} \left( \sum_{k=1}^{N} \sum_{k'=1}^{N} u'_{k-1} R_{k,k'} u_{k'-1} + \sum_{k=1}^{N} \sum_{k'=1}^{N} y' Q_{k,k'} y_{k'} + \sum_{k=1}^{N} \sum_{k'=1}^{N} y' P'_{k,k'} u_{k'-1} \right)
\]

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subject to

\[(2) \quad y = Hu + h,\]

or

\[y_k = \sum_{j=0}^{k-1} H_{k,j+1}u_j + h_{k-l} \quad k=1,2,\ldots,N,\]

where \(s, t, R, Q, P,\) and \(H\) are given, \(R\) and \(Q\) are symmetric, and \(h\) is random with given mean and variance. The purpose of this note is to transform the linear decision problem into a stochastic linear control problem and, then, to derive the optimal decision rules with dynamic programming. The advantages of transformation are in problem formulation, solution, analysis, and generalization.

Consider the reduced form representation of a stochastic dynamic linear system:

\[y_k = \left[ \sum_{j=1}^{n} a_{j,k-l}y_{k-j} + b_{j,k-l}u_{k-j} + c_{j,k-l}v_{k-j} \right] + d_{k-l}e_{k-l} \quad k=1,2,\ldots,\]

where \(v_{k-l}\) is a vector of exogenous variables, \(e_{k-l}\) is a vector of random variables with mean zero and variance \(\Omega_{k-1}, a_{j,k-l}, b_{j,k-l}, c_{j,k-l}, d_{k-l},\) and \(v_{k-j}\) are given for \(j=1,2,\ldots,n\) and \(k=1,2,\ldots,\) and the predetermined variables as of period \(k\) are associated with period \(k-j\) for \(j=1,2,\ldots,n\). A state space representation of the
The system is

\[ x_{k+1} = A_k x_k + B_k u_k + C_k z_k + D_k \varepsilon_k \quad k=0,1, \ldots, \]

where \( x_k \) is the state of the system,

\[
\begin{align*}
    x_k &= [y_k, y_{k-1}, \ldots, y_{k-\lambda+1}, y_{k-\lambda}, y_{k-\lambda-2}, \ldots, u_k, u_{k-1}, u_{k-2}, \ldots, u_{k-\lambda+2}, u_{k-\lambda+1}], \\
    z_k &= [v_k, v_{k-1}, \ldots, v_{k-\lambda+2}, v_{k-\lambda+1}].
\end{align*}
\]

The matrices \( A_k \) and \( B_k \) are defined as follows:

\[
A_k = \begin{bmatrix}
    A_{1,k} & A_{2,k} & \cdots & A_{\lambda-1,k} & A_{\lambda,k} \\
    I & I & \cdots & I & I \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    I & I & \cdots & I & I \\
\end{bmatrix}, \quad B_k = \begin{bmatrix}
    B_{1,k} \\
    I \\
    \vdots \\
    I \\
\end{bmatrix}
\]
\[
\begin{pmatrix}
C_{1,k} & C_{2,k} & \cdots & C_{\lambda-1,k} & C_{\lambda,k} \\
\vdots & \ddots & & \vdots \\
\end{pmatrix}, \quad \begin{pmatrix}
D_{1,k} \\
\vdots \\
\end{pmatrix}
\]

\[
C_k = \begin{pmatrix}
\vdots \\
\end{pmatrix}, \quad D_k = \begin{pmatrix}
\vdots \\
\end{pmatrix}
\]

\[\lambda = \max\{n,N\}, \quad A_{j,k}, B_{j,k}, \text{ and } C_{j,k} \text{ are zero for } j > n, \text{ and } x_0 \text{ is given.}\]

The final form representation of the system[3,4] then is given by (2), where

\[
y_k = E_x k', \quad H_{k,j+1} = E_{y_k,j+1} B_j', \\
\phi_{k,j} = \prod_{i=j}^{k-1} A_i, \quad \phi_{k,k} = I,
\]

and

\[
h_{k-1} = \sum_{j=0}^{k-1} E_{y_k,j+1} [C_j z_j + D_j e_j] + E_{y_k,0} x_0.
\]

The linear decision problem transformed into a linear control problem, therefore, is to choose \( u_{k-1} \), given \( x_k \) for \( k=1,2,\ldots,N \) so as to minimize the expected value of \( J \) subject to (3), where
\[
J = \sum_{k=1}^{N} W_k (x_k, u_{k-1}) \,
\]

\[
W_k = \frac{1}{2} x_k^T Q_k x_k + \frac{1}{2} u_{k-1}^T R_k u_{k-1} + x_k^T S_k + u_{k-1}^T L_k
\]

\[
Q_k = \begin{bmatrix}
Q_{k,k} & Q_{k,k-1} & \cdots & Q_{k,k-\lambda+2} & Q_{k,k-\lambda+1} \\
Q_{k-1,k} & \cdots & \cdots & \cdots & \cdots \\
Q_{k-\lambda+2,k} & \cdots & \cdots & \cdots & \cdots \\
Q_{k-\lambda+1,k} & \cdots & \cdots & \cdots & \cdots
\end{bmatrix}
\]

\[
P_k = \begin{bmatrix}
P_{k,k} & P_{k,k-1} & \cdots & P_{k,k-\lambda+2} & P_{k,k-\lambda+1} \\
P_{k-1,k} & \cdots & \cdots & \cdots & \cdots \\
P_{k-\lambda+2,k} & \cdots & \cdots & \cdots & \cdots \\
P_{k-\lambda+1,k} & \cdots & \cdots & \cdots & \cdots
\end{bmatrix}
\]

\[
R_k = \begin{bmatrix}
R_{k,k+1} & R_{k,k-\lambda+2} & R_{k,k-\lambda+1} \\
R_{k+1,k} & \cdots & \cdots & \cdots & \cdots \\
R_{k-\lambda+2,k} & \cdots & \cdots & \cdots & \cdots \\
R_{k-\lambda+1,k} & \cdots & \cdots & \cdots & \cdots
\end{bmatrix}
\]
\[ R_k = R_{k, k}, \]

\[ s_k = \{ s_{k, k} \}, \]

\[ s_k = \{ s_{k, k} \}, \]

\[ t_k = t_{k, k}, \]

\( Q_k \) is symmetric positive semi-definite, \( R_k \) is symmetric positive definite, and \( Q_{k, k'}, P_{k, k'} \) are zero for \( k \) and \( k' < 1 \).

Applying dynamic programming[1] let

\[ \gamma_N \equiv E\{ W_N(x_N, u_{N-1}) | x_{N-1} \} \]

(4) \[ = \frac{1}{2} \left| E\{ x_N' | x_{N-1} \} \right|^2_Q + \frac{1}{2} u_{N-1}' R_{N-1} u_{N-1} + E\{ x_N' | x_{N-1} \} s_{N-1} + u_{N-1}' t_{N-1} \]

\[ + \frac{1}{2} \text{var}\{ x_N' Q_N x_N + x_N' s_{N-1} | x_{N-1} \} \]

(5) \[ = \frac{1}{2} x_{N-1}' A_{N-1} Q_{N-1} x_{N-1} + \frac{1}{2} u_{N-1}' [ B_{N-1}' Q_{N-1} B_{N-1} + R_{N-1}] u_{N-1} \]

\[ + x_{N-1}' [ A_{N-1}' S_N + A_{N-1}' Q_{N-1} C_{N-1} z_{N-1} ] \]

\[ + u_{N-1}' [ B_{N-1}' Q_N A_N x_{N-1} + B_{N-1}' Q_{N-1} z_{N-1} + B_{N-1}' s_{N-1} + t_{N-1} ] \]
\[ + \frac{1}{2}z'_{N-1}C'_{N-1}Q_{N-1}C_{N-1}z_{N-1} + z'_{N-1}C'_{N-1}s_{N} + \frac{1}{2}\text{tr}\{\Omega'_{N-1}D'_{N-1}Q_{N}D_{N-1}\} \]

Minimizing \( \gamma_{N} \) with respect to \( u_{N-1} \) given \( x_{N-1} \) we obtain

\[ u_{N-1}^* = -[B'_{N-1}Q_{N}B_{N-1} + R_{N}]^{-1}[B'_{N-1}Q_{N}A_{N-1}x_{N-1} + B'_{N-1}Q_{N}C_{N-1}z_{N-1} + B'_{N-1}s_{N} + t_{N}] \]

where \( u_{N-1}^* \) is the optimal \( u_{N-1} \). Then substituting (6) into (5) and rearranging terms we get

\[ \gamma_{N}^* = \frac{1}{2}x'_{N-1} [K_{N-1} - Q_{N-1}]x_{N-1} + x'_{N-1}[s_{N-1} - s_{N}] + \frac{1}{2}z'_{N-1}C'_{N-1}[Q_{N} - Q_{N}B_{N-1}B'_{N-1}Q_{N}B_{N-1} + R_{N}]^{-1}[B'_{N-1}Q_{N}C_{N-1}z_{N-1} + z'_{N-1}C'_{N-1}[s_{N} - Q_{N}B_{N-1}B'_{N-1}Q_{N}B_{N-1} + R_{N}]^{-1}[B'_{N-1}s_{N} + t_{N}]] + \frac{1}{2}[B'_{N-1}s_{N} + t_{N}]^\prime [B'_{N-1}Q_{N}B_{N-1} + R_{N}]^{-1}[B'_{N-1}s_{N} + t_{N}] + \frac{1}{2}\text{tr}\{\Omega'_{N-1}D'_{N-1}Q_{N}D_{N-1}\} , \]

where

\[ K_{k-1} = Q_{k-1} + A'_{k-1}K_{k}A_{k-1} + A'_{k-1}K_{k}B_{k-1}[B'_{k-1}Q_{k}B_{k-1} + R_{k}]^{-1}B'_{k-1}K_{k}A_{k-1} \]
and

\[ g_{k-1} = s_{k-1} + A'_{k-1}g_k + A'_{k-1}K_{k-1}C_{k-1}z_{k-1} \]

\[- A'_{k-1}K_{k-1}B_{k-1} \left[ B'_{k-1}Q_{k-1}B_{k-1} + R_k \right]^{-1} \left[ B_{k-1}Q_{k-1}C_{k-1}z_{k-1} + B'_{k-1}s_k + t_k \right] \]

for \( k=N \) with \( K_N = 0 \) and \( g_N = s_N \), and \( \gamma^*_N \) is the optimal value of \( \gamma_N \). Then

\[ \gamma_{k-1} = E \left\{ W_{k-1} \left[ x_{k-1}, u_{k-2} \right] + \gamma^*_k \left| x_{k-2} \right. \right\} \]

\[ = \frac{1}{2} \left| \left| E \left\{ x_{k-1} \left| x_{k-2} \right. \right\} \right| \right|_{K_{k-1}}^2 + \frac{1}{2} u'_{k-2} R_{k-1} u_{k-2} \]

\[ + E \left\{ x'_{k-1} \left| x_{k-2} \right. \right\} g_{k-1} + u'_{k-2} t_{k-1} \]

\[ + \text{var} \left\{ x'_{k-1} K_{k-1} x_{k-1} + x'_{k-1} g_{k-1} \left| x_{k-2} \right. \right\} + \kappa_k \]

for \( k=N \), where

\[ \kappa_k = \sum_{j=k}^{N} \left\{ \frac{1}{2} [C_{j-1} z_{j-1}]' [K_{j} - K_{j} B_{j-1} \left[ B'_{j-1} K_{j} B_{j-1} + R_j \right]^{-1} B'_{j-1} K_{j} \left[ C_{j-1} z_{j-1} \right] \]

\[ + [C_{j-1} z_{j-1}]' [g_{j} - K_{j} B_{j-1} \left[ B'_{j-1} K_{j} B_{j-1} + R_j \right]^{-1} B_{j-1} g_{j} + t_j] \]

\[ + \frac{1}{2} [B_{j-1} g_{j} + t_j] ' \left[ B'_{j-1} Q_{j} B_{j-1} + R_j \right]^{-1} [B_{j-1} g_{j} + t_j] \]

\[ + \frac{1}{2} \text{tr} \{ \Omega_{j-1} D'_{j-1} Q_{j} D_{j-1} \} \} \]

and with the exception of \( \kappa_k \) is of the same form as (4).
Hence,

\begin{equation}
(11) \quad u_{k-1}^* = - [B_k' K_{k-1} B_{k-1} + R_k]^{-1} [B_k' K_{k-1} A_{k-1} x_{k-1} + B_k' C_{k-1} z_{k-1} + B_k' g_k + t_k] \\
\text{for } k = N, N-1, \ldots, 1,
\end{equation}

\begin{equation}
(12) \quad \gamma_{k-1}^* = \frac{1}{2} x_{k-2}' [K_{k-2} - Q_{k-2}] x_{k-2} + x_{k-2}' [g_{k-2} - s_{k-2}] + \kappa_{k-1} \\
\text{for } k = N+1, N, \ldots, 2,
\end{equation}

where $K_{k-1}, g_{k-1}, \gamma_{k-1},$ and $\kappa$ are determined by (7), (8), (9), and (10), respectively, the Riccati matrix, $K_{k-1},$ is symmetric positive semi-definite, $\gamma_{k-1}^*$ is the optimal value of $\gamma_{k-1},$ and the optimal evolution of the state of the system is described by

\[ x_{k+1}^* = [I - B_k' K_{k+1} B_k + R_{k+1}]^{-1} [B_k' K_{k+1} A_k x_k^* + C_k z_k] \\
- B_k' [B_k' K_{k+1} B_k + R_{k+1}]^{-1} [B_k' g_{k+1} + t_{k+1}] \\
+ D_k \epsilon_k \\
\text{for } k = 0, 1, \ldots, N-1.\]

The structure of the optimal strategy may be seen immediately from (11). Certainty equivalence holds; hence, $D_{k-1}$ and $\Omega_{k-1}$ play no role in the strategy, although the expected gain from minimizing expected utility[3], or the expected value of perfect information on $\epsilon_{k-1}[2], is$

\[ \frac{1}{2} \text{tr} \{ \Omega_{k-1} D'_{k-1} Q_{k-1} D_{k-1} \}. \]

There is feedback on $x_{k-1}$ and $z_{k-1},$ which are directly observable, rather than on past errors[3,5]. There is feedforward on
future $O_k$, $R_k$, $A_k$, and $B_k$ through $K_k$, and on future $s_k$, $t_k$, $C_k$, and $z_k$ through $g_k$. Computation of the strategy is done recursively through (7) and (8) and involves inversion of matrices only of order equal to the number of control variables. From (12) we see that the optimal expected value of $J$ is $y^*_1$ and the imputed price of $x^*_k$ for $k=N,N-1,...,1$ is $K_k x^*_k + p_k$.

The problem may be generalized to include objective functions of the form

\begin{equation}
J = \hat{c}'[u-\hat{u}] + \hat{s}'[y-\hat{y}] + \frac{1}{2}[[u-\hat{u}]'R[u-\hat{u}] + [y-\hat{y}]'Q[y-\hat{y}]
+ [u-\hat{u}]'P[y-\hat{y}] + [y-\hat{y}]'P'[u-\hat{u}]],
\end{equation}

where $\hat{u}$ and $\hat{y}$ are the desired levels of $u$ and $y$, respectively, since (13) is of the same form as (1) with

\begin{align*}
\hat{t} &= \hat{c} - R\hat{u} - P\hat{y}, \\
\hat{s} &= \hat{s} - Q\hat{y} - P'\hat{u}
\end{align*}

and the addition of a constant term. Moreover, since the state-space representation, in contrast to the final form representation, corresponds directly to the reduced form and the state of the system is observable the problem may also be generalized to include unknown parameters or errors in variables[1].
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