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This research was supported in part by funds from the General Motors Corporation under the General Motors Grant for Highway Transportation Research. The views expressed in this paper are the author's sole responsibility, and do not reflect those of the General Motors Corporation, the Department of Economics, nor of the Massachusetts Institute of Technology.
ON THE DUALITY BETWEEN ALLOCATION AND VALUATION

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1. INTRODUCTION

There is an intimate relation between linear control and linear programming. Any problem which can be formulated as a discrete-time linear control problem with a finite horizon also can be formulated as a multi-stage linear programming problem and vice-versa. The duality theory of linear programming can be extended to linear control problems [6]. Corresponding to the primal control problem, or maximization problem, is a dual control problem, or minimization problem. The primal system moves forward in time; the dual system moves backward in time. The dual of the dual control problem is the primal control problem. Moreover, the primal and dual control problems can be decomposed into a sequence of primal and dual Hamiltonian programming problems. The purpose of this paper is to provide an economic interpretation of the duality theory for discrete-time linear control problems and to apply the theory to an analysis of dual stability in the dynamic input-output system.

1This research was supported in part by funds from the General Motors Corporation under the General Motors Grant for Highway Transportation Research. I am indebted to E. S. Chase, A. R. Dobell, R. Dorfman, and C. Hamilton for valuable comments.
2. ALLOCATION

The problem of allocating goods over time can be formulated as a primal linear control problem with the aid of linear activity analysis [2,5]. The flow of commodities in period \( k \), \( y(k) \), is assumed to be a linear function of the activity levels in period \( k \), \( u(k) \):

\[
y(k) = C(k)u(k),
\]

where \( C(k) \) is a given matrix describing the technology of the system. The stocks of commodities at the beginning of the period, \( x(k) \), are the primal state variables; and the activity levels are the primal control variables.

The long-run allocation problem is to choose the activity levels over time, \( s_0 \geq 0 \) and \( u(k) \geq 0 \) for \( k = 0, 1, \ldots, N-1 \), so as to maximize net present value,

\[
J = \int_0^{s_0} + \sum_{k=0}^{N-1} [a'(k)x(k) + b'(k)u(k)] + f_Nx(N),
\]

subject to the initial and terminal conditions on the stocks of commodities,

\[
G_0s_0 + x(0) = h_0
\]

and

\[
G_Nx(N) \leq h_N,
\]

the capacity constraints,

\[
D(k)u(k) \leq F(k)x(k) + d(k) \quad k = 0, 1, \ldots, N-1,
\]

where \( x(k) \) is taken as given, and the transformation of stocks,

\[
\Delta x(k) = A(k)x(k) + B(k)u(k) + c_{k+1} \quad k = 0, 1, \ldots, N-1,
\]
where $f_0$, $f_N$, $a(k)$, $b(k)$, $c(k+1)$, $d(k)$, $h_0$, and $h_N$, are given vectors, $A(k)$, $B(k)$, $D(k)$, $F(k)$, $C_0$, and $G_N$ are given matrices, $A(k) = x(k+1) - x(k)$, $0$ is the given primal initial time, and $N$ is the given primal terminal time.

Note that (5) is not a constraint on $x(k)$, since the stock of commodities is taken as given in (5). This is a feature distinguishing linear control from linear programming. The only explicit constraints on stocks are in periods 0 and $N$ as described by (3) and (4). We can allow, however, for constraints on $x(k)$ in all other periods by transforming the constraints on $x(k)$ through (6) into constraints on $u(k-1)$ of the form (5).

Associated with the long-run allocation problem is a sequence of short-run allocation problems, or primal Hamiltonian programming problems. The short-run allocation problem in period $k$ is to choose the activity levels, $u(k) \geq 0$, so as to maximize net income in terms of revenue,

$$j'(k) = a'(k)x(k) + b'(k)u(k) + \Psi'(k+1)[A(k)x(k) + B(k)u(k) + c(k+1)],$$

subject to the capacity constraints, (5), in period $k$, given $x(k)$ and the imputed prices of the stocks of commodities at the beginning of period $k+1$, $\Psi(k+1)$.

3. VALUATION

Corresponding to the long-run allocation problem is a valuation problem, or dual linear control problem. The imputed prices of the stocks of commodities at the beginning of the period, $\Psi(k)$, are the dual state variables, and the imputed prices of capacity in the period, $\lambda(k)$, are the dual control variables.

The long-run valuation problem is to choose the capacity prices over time,
\(\gamma_N \geq 0\) and \(\lambda(k) \geq 0\) for \(k = N-1, N-2, \ldots, 0\), so as to minimize the imputed cost of commodities and capacities,

\[
\Phi = h_N \gamma_N + \sum_{k=N-1}^{0} [c'(k+1)\psi(k+1) + d'(k)\lambda(k)] + h_0 \psi(0),
\]  

subject to the initial and terminal constraints on commodity prices,

\[
G_N \gamma_N + \psi(N) = f_N
\]  

and

\[
G_0 \psi(0) \geq f_0,
\]

the no unimputed revenue constraints,

\[
D'(k)\lambda(k) \geq B'(k)\psi(k+1) + b(k) \quad k = N-1, N-2, \ldots, 0,
\]

where \(\psi(k+1)\) is taken as given, and the transformation of prices,

\[
\Delta \psi(k+1) = A'(k)\psi(k+1) + F'(k)\lambda(k) + a(k) \quad k = N-1, N-2, \ldots, 0,
\]

where \(\Delta \psi(k+1) = \psi(k) - \psi(k+1)\), \(N\) is the dual initial time, and \(0\) is the dual terminal time.

Associated with the long-run valuation problem is a sequence of short-run valuation problems, or dual Hamiltonian programming problems. The short-run valuation problem in period \(k\) is to choose the capacity prices, \(\lambda(k) \geq 0\), so as to minimize net income in terms of imputed cost,

\[
\mathcal{X}(k) = c'(k+1)\psi(k+1) + d'(k)\lambda(k)
\]

\[
+ x'(k)[A'(k)\psi(k+1) + F'(k)\lambda(k) + a(k)],
\]

subject to the no unimputed revenue constraints, (11), in period \(k\), given \(\psi(k+1)\) and \(x(k)\).
4. DUALITY

The radical property of duality between the long-run allocation problem and the long-run valuation problem is that the allocation system moves forward in time while the valuation system moves backward in time.\(^2\) The stocks evolve forward as the imputed prices evolve backward.

The sensitivity analysis [6] demonstrates that along an optimal path the imputed price of a stock of commodities measures the effect on net present value of a change in the stock and the stock measures the effect on the imputed cost of commodities and capacities of a change in the price. If the sequences \(\{\hat{x}(k), \hat{u}(k); \hat{s}_0\}\) and \(\{\hat{\Psi}(k+1), \hat{\lambda}(k); \hat{\gamma}_N\}\) are optimal,

\[
\nabla_{x(k)} J[\hat{x}(k)] = (k) \quad k = 0,1,\ldots,N, \tag{14}
\]

and

\[
\nabla_{\Psi(k)} \Phi[\hat{\Psi}(k)] = \hat{x}(k) \quad k = N,N-1,\ldots,0, \tag{15}
\]

where \(\nabla_{x(k)} J[\hat{x}(k)]\) is the gradient of \(J\) with respect to \(x(k)\) evaluated on \(\{\hat{x}(k), \hat{u}(k); \hat{s}_0\}\) if the gradient exists.

The dual maximum principle [6] shows how the long-run allocation and valuation problems can be decomposed into a sequence of short-run allocation and valuation problems. A feasible solution \(\{\hat{x}(k), \hat{u}(k); \hat{s}_0\}\) (\(\{\hat{\Psi}(k+1), \hat{\lambda}(k); \hat{\gamma}_N\}\)) to the long-run allocation (valuation) problem is an optimal solution if and only if there exists a feasible solution \(\{\hat{\Psi}(k+1), \hat{\lambda}(k); \hat{\gamma}_N\}\) (\(\{\hat{x}(k), \hat{u}(k); \hat{s}_0\}\)) to the long-run valuation (allocation) problem with

\[
\mathcal{A}[\hat{u}(k); \hat{x}(k), \hat{\Psi}(k+1)] = \mathcal{A}[\hat{\lambda}(k); \hat{\Psi}(k+1), \hat{x}(k)] \tag{16}
\]

and satisfying the primal and dual transversality conditions,

\(^2\)The fundamental properties of duality for linear programming [3] are also valid for linear control.
\[ s_0^t \gamma_0 = 0 \]  
and  
\[ \hat{\gamma}_N^t \hat{s}_N = 0, \]

where

\[ \Delta \hat{x}(k) = \nabla_{\Psi(k+1)} \mathcal{H}[\hat{u}(k); \hat{x}(k), \hat{\Psi}(k+1)], \]

\[ \Delta \hat{\Psi}(k+1) = \nabla_{x(k)} \mathcal{H}[\hat{\lambda}(k); \hat{\Psi}(k+1), \hat{x}(k)], \]

\[ \hat{\gamma}_0 = G^t_0 \hat{\psi}(0) - f_0, \text{ and } \hat{s}_N = h_N - G^t_N \hat{\nu}(N). \]

The solutions to the long-run problems, which are duals, also provide the solutions to the short-run problems, which are duals.

This does not mean, however, that the solution to a long-run problem can be generated immediately by solving a sequence of short-run problems. For example, to make the allocation decision today by solving the short-run allocation problem requires a knowledge of future optimal allocation decisions expressed in terms of the imputed price of the stock of commodities tomorrow, \( \hat{\Psi}(k+1) \). Moreover, the solution to the short-run problem may not be unique.

In fact, the primal Hamiltonian may even be identically equal to zero, allowing for an interior solution to the problem. In this way the dual maximum principle is quite similar to the decomposition principle.\(^3\)

The Hamilton-Jacobi inequality [6], then, describes the way the allocation sequence moves forward in time describing the impact of pricing decisions tomorrow on imputed cost today and the valuation sequence moves backward in time describing the impact of allocation decisions today on revenue tomorrow.

If \( \{\hat{x}(k), \hat{u}(k); \hat{s}_0\} \) and \( \{\hat{\psi}(k+1), \hat{\lambda}(k); \hat{\gamma}_N\} \) are optimal solutions to the long-run allocation and valuation problems,

\(^3\)The maximum principle for linear systems may even be considered to be a consequence of the decomposition principle of linear programming [1].
\[ J_k(\hat{x}(k)) - J_{k+1}(\hat{x}(k)) \geq \phi(\hat{\lambda}(k); \hat{\psi}(k+1), \hat{x}(k), \hat{\lambda}(k)) \]
\[ \phi = \phi(\lambda(k); \hat{\psi}(k+1), \hat{x}(k)) \geq \phi_{k+1}(\hat{\psi}(k+1)) - \phi_k(\hat{\psi}(k+1)), \tag{22} \]

where \( J_k(\hat{x}(k)) \) is the optimal value of \( J \) at the beginning of period \( k \) with initial primal state \( \hat{x}(k) \), etc., and \( \phi_{k+1}(\hat{\psi}(k+1)) \) is the optimal value of \( \phi \) at the end of period \( k \) with an initial dual state \( \hat{\psi}(k+1) \), etc.

5. DYNAMIC INPUT-OUTPUT

As an example, let us analyze the dynamic Leontief system \([7,8]\) with the theory we have developed. The technology of the system is described by
\[
\begin{bmatrix}
I - A^* + B^* \\
-B^*
\end{bmatrix} v(k) \geq
\begin{bmatrix}
x(k+1) \\
-x(k)
\end{bmatrix},
\tag{23}
\]

where \( A^* \) is the input-output matrix, \( B^* \) is the stock-flow matrix, \( v(k) \) is the vector of activity levels, and \( x(k) \) is the vector of commodity stocks at the beginning of period \( k \). The problem is to choose \( v(k) \geq 0 \) for \( k = 0, 1, \ldots, N-1 \) so as to maximize the terminal value of the stock of commodities, \( p'x(N) \), subject to (23),
\[
x(0) = x_0, \tag{24}
\]
\[
x(k) \geq 0 \quad k = 1, 2, \ldots, N-1, \tag{25}
\]
and
\[
x(N) \geq 0, \tag{26}
\]

where \( p \) and \( x_0 \) are given.

If we introduce a stock disposal activity, \( w(k) \geq 0 \), then we can rewrite (23) as
\[ x(k+1) = [I - A^* + B^*]v(k) - w(k) \quad (27) \]

and
\[ B^*v(k) \leq x(k). \quad (28) \]

The stock constraint (25), then, becomes
\[ - [I - A^* + B^*]v(k) + w(k) \leq 0 \quad k = 0, 1, \ldots, N-1. \quad (29) \]

The long-run allocation problem, therefore, is to choose the activity levels over time, \( v(k) \geq 0 \) and \( w(k) \geq 0 \) for \( k = 0, 1, \ldots, N-1 \), so as to maximize terminal value,
\[ J = p'x(N) \quad (30) \]
subject to the initial and terminal conditions on the stocks of commodities,
\[ x(0) = x_0 \quad (31) \]
and
\[ - x(N) \leq 0, \quad (32) \]
the capacity constraints,
\[
\begin{bmatrix}
B^* & 0 \\
-C^* & I
\end{bmatrix}
\begin{bmatrix}
v(k) \\
w(k)
\end{bmatrix}
\leq
\begin{bmatrix}
I \\
0
\end{bmatrix}
x(k) \quad k = 0, 1, \ldots, N-1, \quad (33)
\]
where \( C^* = I - A^* + B^* \) and \( x(k) \) is given, and the transformation of stocks,
\[
\Delta x(k) = -x(k) + [C^*, -I] \begin{bmatrix}
v(k) \\
w(k)
\end{bmatrix} \quad k = 0, 1, \ldots, N-1. \quad (34)
\]

The long-run valuation problem is to choose the capacity prices, \( \gamma_N \geq 0, \kappa(k) \geq 0, \) and \( \mu(k) \geq 0 \) for \( k = N-1, N-2, \ldots, 0 \), so as to minimize the imputed cost of the initial stocks of commodities,
\[ \phi = x_0 \psi(0), \quad (35) \]
subject to the initial constraints on commodity prices,

$$- \gamma_N + \psi(N) = p,$$  \hspace{1cm} (36)

the no unimputed value constraints,

$$\begin{bmatrix} B^* & -C^* \\ 0 & I \end{bmatrix} \begin{bmatrix} \kappa(k) \\ \mu(k) \end{bmatrix} \geq \begin{bmatrix} C^* \\ -I \end{bmatrix} \psi(k+1) \hspace{1cm} k = N-1, N-2, \ldots, 0, \hspace{1cm} (37)$$

where \(\psi(k+1)\) is taken as given, and the transformation of prices,

$$\Delta \psi(k+1) = -\psi(k+1) + [I, 0] \begin{bmatrix} \kappa(k) \\ \mu(k) \end{bmatrix} \hspace{1cm} k = N-1, N-2, \ldots, 0. \hspace{1cm} (38)$$

The short-run allocation problem is to choose \(v(k) \geq 0\) and \(w(k) \geq 0\) so as to maximize net income in terms of value,

$$\mathcal{H}(k) = \psi'(k+1) \left( -x(k) + [C^*, -I] \begin{bmatrix} v(k) \\ w(k) \end{bmatrix} \right), \hspace{1cm} (39)$$

subject to the capacity constraints, (33), given \(x(k)\) and \(\psi(k+1)\).

The short-run valuation problem is to choose \(\kappa(k) \geq 0\) and \(\mu(k) \geq 0\) so as to minimize net income in terms of cost,

$$\mathcal{J}(k) = x'(k) \left( -\psi(k+1) + [I, 0] \begin{bmatrix} \kappa(k) \\ \mu(k) \end{bmatrix} \right), \hspace{1cm} (40)$$

subject to the no unimputed revenue constraints, (37), given \(\psi(k+1)\) and \(x(k)\).

Along an optimum path the imputed value of the stock of commodities, \(\psi(k)'x(k)\), is constant, since \(J = \Phi\) and \(\mathcal{H} = \mathcal{J}\), and equal to the terminal value, \(p'x(N)\), from the transversality condition. Moreover, \(\psi(k)\) is not negative for all \(k\).
6. DUAL STABILITY

The radical property of duality between allocation and valuation over time has significance for the relative stability of the dynamic input-output system and its dual. To see this we must examine the meaning of stability in an optimizing model.

First, an optimum path per se is neither stable nor unstable. It is characterized by a number of conditions, some given and some derived, that make it determinate. To talk about stability we must relax one of these conditions. For example, we can assume the initial stock of commodities is no longer given and examine the effect of perturbing the initial stock. Or we can drop one of the conditions of optimality, such as the transversality condition, and see what is the effect of perturbing price or quantity. The meaning of stability will depend upon which course we decide to take.

Second, the problem we have analyzed has a finite horizon and terminal valuation. Under appropriate assumptions on the technology a strong turnpike theorem can be proved, however, for both the dynamic input-output system and its dual [8].

Third, the radical property of duality leads to opposite time arrows in the long-run allocation and valuation systems. If we consider both systems together, there is a question of which time arrow to use.

This leads us to a discussion of the behavior of the dynamic input-output system and its dual about the turnpike. There are a number of ways we can describe the behavior of the system away from the turnpike.

One approach is to employ the dual maximum principle to derive \( u(k) = [v(k), w(k)] \) and \( \lambda(k) = [\kappa(k), \mu(k)] \) as a function of \( x(k) \) and \( \psi(k+1) \) and then substitute these relations into the transformation of stocks and prices.
equations, ignoring the transversality condition. The solution to the short-run problems, however, may not be unique. Moreover, the question of which time arrow to use is still with us. Nevertheless, we could examine the effect of perturbing $x(0)$ or $\Psi(N)$. We would observe the "saddlepoint" behavior associated with the turnpike.

Another approach is to use the closed-loop feedback rules which are optimal along the turnpike to describe behavior off of the turnpike. If $B^*$ is non-singular and $A^*$ is indecomposable, the rules are:

\[ v(k) = [B^*]^{-1} x(k) \]  \hspace{1cm} (41)

and \[ \kappa(k) = [B^*]^{-1} C^* \Psi(k+1), \]  \hspace{1cm} (42)

with \[ w(k) = u(k) = 0 \]  \hspace{1cm} (43)

Substituting (41) - (43) into (34) and (38) we obtain

\[ \Delta x(k) = D^* x(k) \]  \hspace{1cm} (44)

and \[ \Delta \Psi'(k+1) = \Psi'(k+1) D^*, \]  \hspace{1cm} (45)

where \[ D^* = [B^*]^{-1} C^*. \]  \hspace{1cm} (46)

We have then a radical dual stability theorem: for systems of order greater than one, the allocation system is globally relatively stable if and only if the valuation system is also globally relatively stable. The proof follows immediately from Jorgenson's dual stability theorem [4] and the radical property of duality.
7. CONCLUSION

The radical property of duality between allocation and valuation over time is that the allocation system moves forward in time while the valuation system moves backward in time. This property has a natural economic interpretation and leads to a decomposition of the long-run allocation and valuation problems into a sequence of short-run allocation and valuation problems. Moreover, in the case of the dynamic input-output system it yields a radical dual stability theorem, which inverts the established theorem.
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