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by

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This paper studies auctions designed to maximize the expected revenue of a seller facing risk-averse bidders with unknown preferences. Although we concentrate on auctions where a seller sells a single indivisible item, the principles that emerge apply to a much wider class of "principal-agent" problems, as we argue in Maskin and Riley (1981).

The properties of auctions that are optimal for the seller when buyers are risk neutral and their preferences independently distributed have been much studied (see, for example, Myerson (1981); Maskin and Riley (1980a); Harris and Raviv (1981); and Riley and Samuelson (1981)). One conclusion that emerges from this work is that, for many distributions of preferences (the exceptions are discussed in Remark 8.1), the standard "high bid" and "English" auctions, modified to allow for a seller's reserve price, are equivalent (i.e., they generate the same expected revenue for the seller) and optimal. These classical auctions, however, are not equivalent from the seller's viewpoint when buyers are risk averse (see Theorem 4 below). Moreover, neither is optimal. This is for two essentially conflicting reasons: the desirability of insuring these buyers against risk, and the desirability of exploiting their risk-bearing in order to screen them.

The classical auctions ordinarily confront buyers with risk — i.e., the marginal utility of income if a buyer wins is typically not the same as that if he loses. Clearly, the seller can extract a payment for removing this risk while keeping the buyer at the same utility. Thus, holding utilities fixed, introducing insurance will enhance the seller's revenue. The qualification "holding utilities fixed", however, is crucial since the insurance will usually induce buyers to alter their bidding strategies. Indeed, as we shall see below, the introduction of perfect insurance may so alter buyers' behavior that the seller is better off offering no insurance at all.

Thus, the seller will generally find it optimal not to offer perfect insurance (see, however, the discussion following Theorem 11). The degree of risk-bearing he imposes on buyers is determined by screening considerations. To see the
role of risk in screening, imagine that each buyer either has a high reservation price (eager buyer) or a low reservation price (reluctant buyer) for the item being auctioned. The seller's problem in designing an auction is how to prevent the eager buyers from bidding too low. Suppose that the seller can devise an auction so that buyers who bid low face risk, but that eager buyers who bid low face greater risk than the reluctant ones. Then the seller will derive less revenue from the reluctant buyers than if he offered them complete insurance, but this will be more than made up for by inducing the eager buyers to bid higher than they otherwise would. In the language of the incentives literature, relaxation of the incentive constraint overcompensates for the loss due to risk-bearing. (The incentive constraint is simply the guarantee that an eager buyer should derive at least the expected payoff from bidding high rather than low. For a more comprehensive discussion of the use of risk for screening, see Maskin (1981).)

This two-class example suggests that whereas it is desirable to confront low bidders with risk (to induce the eager buyers to bid high), nothing is gained from a high bidder's bearing much risk. Indeed, these principles are quite general (see Theorems 11 and 12 below). It also suggests that, although buyers bear risk in the high bid and English auctions, the nature of this risk is not optimal, since the eager buyers bear the most risk.

In Theorems 8 and 9 below we show that designing an optimal auction can often be reduced to solving a standard control problem. We then use this fact to derive a number of general properties of optimal auctions, such as the principles above. First, as long as a buyer's marginal utility of income decreases with his eagerness to buy, then the probability of winning the auction (getting the item) and the amount paid if the auction is won increase with a buyer's eagerness (Theorem 10). As for the nature of risk bearing, there are, in principle, two ways in which a seller can confront a buyer with risk: (1) the buyer's marginal utility of income can be made to differ depending on whether he wins or loses, and (2) contingent on winning or losing, his payment can be a random variable. We shall see that under the hypothesis just mentioned
and given that aversion to income risk either decreases or does not increase too fast with eagerness (which, for many utility functions, simply means that absolute risk aversion does not increase too fast with income), method (2) is not desirable (Theorem 9). However, method (1) is. Indeed we show (Theorem 11) that it is desirable for the marginal utility of income in the losing state to exceed that in the winning state for all buyers except the most eager. For the utility functions we consider, this means that those buyers are better off winning than losing (corollary to Theorem 11). This result suggests that sufficiently reluctant buyers might even be made to pay a penalty if they lose (Theorem 13), although when risk aversion is nonincreasing they will pay more if they win than if they lose (Theorem 14). Moreover, sufficiently low bids will be refused by the seller (Theorem 16). On the other hand, very eager buyers will receive a subsidy if they lose (Theorem 13), and the most eager will be perfectly insured (Theorem 12). Notwithstanding this insurance, more eager buyers pay more on average (Theorem 15).

In Section 1 we begin by presenting a general model of auctions when buyers are risk averse. In Section 2 we consider the standard high bid and English auctions. We establish existence and uniqueness of symmetric equilibrium in these auctions (Theorems 2 and 3) and show quite generally (Theorem 4) that when buyers are risk averse, the high bid auction generates greater expected revenue for the seller than the English auction. We argue, moreover, that the seller's preference for the high bid auction is intensified if he is risk averse (Theorem 5). We also consider the "perfect insurance auction", in which buyers' marginal utilities of income are the same whether they win or lose. We show (Theorem 6) that, for an important class of cases, the English and perfect insurance auctions generate the same expected revenue for the seller. In Section 3 we take up optimal auctions and show that the seller's optimization reduces to a straightforward control problem (Theorems 8 and 9). In Section 4, we discuss the properties of optimal auctions mentioned above and also one-buyer auctions (Theorem 17). Finally, Section 5 comprises a few concluding remarks. An Appendix contains the proof of the technically complex Theorem 7.
1. The Model

We consider the problem of a seller who wishes to maximize his expected revenue from the sale of a single item. This formulation assumes that the seller is risk neutral toward revenue. We discuss the reasons for this assumption in Section 5. The formulation also implicitly supposes that the seller himself attaches no value to the item. But the analysis would require only slight modification to accommodate a positive seller's value. The seller chooses a selling procedure, or auction, which is a game among the potential buyers, n in number \((n \geq 1)\). Each buyer \(i\) has a strategy space \(S_i\). On the basis of the \(n\)-tuple of strategies \((s_1, \ldots, s_n)\), the auction assigns buyer \(i\) a probability of winning \(H_i(s_1, \ldots, s_n)\) and requires him to make payment \(\beta_i(s_1, \ldots, s_n)\) if he wins and payment \(\alpha_i(s_1, \ldots, s_n)\) if he loses, where the tildas reflect the possibility that \(\beta_i\) and \(\alpha_i\) are random functions. Feasibility requires that

\[
(1) \quad \sum_{i=1}^{n} H_i(s_1, \ldots, s_n) \leq 1
\]

for all \((s_1, \ldots, s_n)\). To prevent the seller from extracting unlimited payments, we must allow each buyer the option of not participating in the auction. Formally, this option can be expressed by including in each strategy space a null strategy, which ensures the buyer a zero probability of winning and a zero payment independently of what other buyers do.

We shall suppose that a buyer's preferences can be parameterized by the scalar \(\theta \in [0,1]\). We will suppose that the \(\theta\)'s of different buyers are independently and identically distributed\(^3\) according to the c.d.f. \(F\). We assume that \(F'(\theta) > 0\) for all \(\theta \in [0,1]\). Let \(u(-t,\theta)\) be the utility of a buyer of type \(\theta\) who wins the auction and pays the amount \(t\). Let \(w(-t)\) be the utility of a buyer who loses and pays the amount \(t\).\(^4\) Thus, given strategies \((s_1, \ldots, s_n)\), the expected utility of buyer \(i\) with parameter \(\theta\) is
where \( E \) denotes the expectation operator.

We shall suppose that \( u(x, \theta) \) and \( w(x) \) satisfy the following rather innocuous restrictions.

Assumption A.

A1. \( u(x, \theta) \) and \( w(x) \) are thrice continuously differentiable
A2. \( u_1 > 0, w_1 > 0 \)
A3. \( w(0) = 0 \)
A4. \( u_{11} < 0, w_{11} < 0 \)
A5. \( u_2 > 0. \)

Subscripts denote the argument with respect to which a partial derivative is taken.

It is natural to assume that utility is increasing in income; hence A2. Assumption A3 is simply a convenient normalization of preferences. Because we are interested in risk averse buyers, we assume that both \( u \) and \( w \) are concave functions of income (A4). Finally, in A5, we parameterize preferences so that increasing \( \theta \) implies greater utility (greater "eagerness" in the terminology of the introduction).

For some of the results of this paper, we shall require the following more substantive assumptions.

Assumption B

B1. \( u_{12} < 0 \)
B2. \( u_{22} < 0 \)
B3. \( u_{122} > 0 \)
B4. \( u_1(-t_1, \theta) < w_1(-t_2) \) implies \( u(-t_1, \theta) > w(-t_2) \)
B5. \( u_{112} > 0 \)

If we equate \( \theta \) with "wealth", then B1 simply requires that marginal utility of income decline with wealth, whereas B2 stipulates that the gains from increasing wealth should be diminishing. Assumptions B3 and
BA not have such obvious economic interpretations but are nonetheless satisfied by several important models, as we shall see below.

It is commonly thought to be empirically true that people become less risk averse as their well-being increases. One formulation of this "law" asserts that absolute risk aversion declines (or at least does not increase) with income. That is,

$\frac{\partial}{\partial x} \left( \frac{-u_{11}(x, \theta)}{u_1(x, \theta)} \right) = \frac{-u_1 u_{111} + u_2^2}{u_1^2} < 0.$

If we interpret $\theta$ as a measure of well-being, then an alternative formalization is

$\frac{\partial}{\partial \theta} \left( \frac{-u_{11}}{u_1} \right) = \frac{-u_1 u_{112} + u_1 u_{12}}{u_1^2} \leq 0.$

Given Assumptions A and B2, (3) implies that $u_{112} \geq 0$. Thus Assumption B5 simply requires that absolute aversion to risk not increase too fast with $\theta$.

This way of modelling preferences is sufficiently general to incorporate many cases of interest. The case most studied in the existing literature is where the only uncertainty facing a buyer is the outcome of the auction; i.e., where the quality of the item itself is known and has equivalent monetary value. Here is it natural to let $\theta$ represent this monetary value.

Case 1—Certain Quality, Equivalent Monetary Value.

$u(-t, \theta) = U(\theta - t)$

$w(-t) = U(-t)$,

where $U$ is a concave increasing von Neumann-Morgenstern utility function.

Case 1 is just an example of the more general case in which the item is of certain quality and contributes additively to the utility of money but may not have an equivalent monetary value 5:

Case 2

$U(-t, \theta) = U(\theta + \Psi(-t))$

$w(-t) = U(\Psi(-t))$,

where $U$ and $\Psi$ are concave and increasing and $U(0) = \Psi(0) = 0$. 
Another possibility of interest is where the buyer is unsure of the (monetary) value of the object. Assume that the possible values are represented by the random variable $v$ with c.d.f. $K(v|\theta)$, where for all $\theta$ $K(v|\theta)=0$, $K(\bar{v}|\theta)=1$, $K_1(\bar{v}|\theta)=K_1(0|\theta)=0$. To capture the idea that higher values of $\theta$ represent more favorable distributions over $v$, we assume that the distribution for a higher $\theta$ exhibits first order stochastic dominance over that for a lower $\theta$. That is,

$$K_2(v|\theta) \equiv \frac{\partial}{\partial \theta} K(v|\theta) \leq 0, \forall \theta \in [\bar{v}, v]$$

with strict inequality over a subset of non-zero measure. In this case preferences take the form

**Case 3**—Uncertain Quality

$$u(-t,\theta) = \int U(v-t) K_1(v|\theta)dv$$

$$w(-t) = U(-t)$$

As a final illustration, suppose that the item is of certain quality and has equivalent monetary value but also has an intensifying effect - so that higher values of $\theta$ represent a greater ability to derive pleasure, crudely translated into a higher marginal utility of income. A simple example of such an effect is

**Case 4**—Intensification

$$u(-t,\theta) = (\theta + 1) U(\theta - t)$$

$$w(-t) = U(-t).$$

We now derive conditions under which Assumptions A and B are satisfied in each of these four cases. The reader may wish to turn directly to Theorem 1 which summarizes these conditions.

Notice first that the five parts of Assumption A are satisfied by each case as long as $U$ is concave, thrice differentiable, and normalized so that $U(0) = 0$. As for Assumption B1, in cases 1 - 3, $u_{12} < 0$. The preferences of cases 1 and 2 satisfy B2 and, if $U'' > 0$, also B3. (From (2) it can be seen that $U'' > 0$, given nonincreasing absolute risk aversion). The preferences of case 3 satisfy B2 if
Integrating by parts twice and defining

\[ T(v|\theta) = \int \frac{v}{V} K_{122}(x|\theta) \, dx, \]

we have

\[ \int \frac{U'(v-t)K_{122}(v|\theta) \, dv}{V} = \int \frac{U''(v-t)T(v|\theta) \, dv}{V} - U'(v-t)T(v|\theta) . \]

Therefore a sufficient condition for case 3 to satisfy B2 is that \( T(v|\theta) \) be everywhere nonnegative. Note that for the special case \( K(v|\theta) = H(v-\theta) \) we have \( T(v|\theta) = H'(v-\theta) \geq 0 \). This suggests that the restriction \( T(v|\theta) \geq 0 \) is relatively mild.

The preferences of case 3 satisfy B3 if

\[ \int \frac{U'(v-t)K_{122}(v|\theta) \, dv}{V} > 0 \]

Integrating by parts twice we have

\[ \int \frac{U''(v-t)K_{122}(v|\theta) \, dv}{V} = \int \frac{U'''(v-t)T(v|\theta) \, dv}{V} - U''(v-t)T(v|\theta) . \]

Thus, assuming \( U'' \geq 0 \), \( T(v|\theta) \) everywhere non-negative is again a sufficient condition.

Cases 1 and 2 satisfy B4 automatically. To establish that case 3 does also, if absolute risk aversion is nonincreasing in income, we make use of the following Lemma.

**Lemma 1:** Suppose utility \( u = \phi(x,z) \) is an increasing function of income \( x \) and that absolute risk aversion, \( -\phi_1/\phi_1 \), is nonincreasing in \( z \). Then

\[ E(\phi(\bar{x},z)) = \phi(y,z) = E(\phi^-_2(\bar{x},z)) \geq \phi_2(y,z) \]

**Proof:** Since \( \phi_1 > 0 \) we may also define the inverse function \( x = \phi^{-1}(u,z) \) for fixed \( z \). We shall first show that the function

\[ g(u,z) = \phi_2(\phi^{-1}(u,z),z) \]

is a convex function of \( u \). Since \( u = \phi(x,z) \) we have

\[ g(\phi(x,z),z) = \phi_2(x,z) \]
Differentiating with respect to $x$ and rearranging we have

$$g_1(\phi(x,z),z) = \frac{\phi_{21}}{\phi_1} = \frac{3}{2z} \log \phi_1$$

Then

$$g_{11}(\phi(x,z),z) = \frac{3}{2z} \log \phi_1 - \frac{3}{2z} \frac{\phi_{11}}{\phi_1} \geq 0$$

establishing convexity.

Thus

$$E(\phi_2(x,z)) = E(g(u,z))$$

from the definitions of $u$ and $g$

$$\geq g(E(u),z), \text{ by Jensen's Inequality}$$

$$= g(\phi(y,z),z), \text{ by hypothesis}$$

$$= \phi_2(y,z) \text{, from the definition of } g$$

Q.E.D.

We now establish the contrapositive of B4 for Case 3 preferences; that is,

1. $u(-t_1,\theta) \leq w(-t_2) \Rightarrow u_1(-t_1,\theta) \geq w_1(-t_2)$.

For case 3 $u(-t_1,\theta) = \int U(v-t_1) dK(v|\theta)$. Then, since $U$ is an increasing concave function, it is sufficient to show that (4) is true when the left hand side is an equality. That is,

2. $\int U(v-t_1) dK(v|\theta) = U(-t_2) \Rightarrow \int U'(v-t_1) dK(v|\theta) \geq U'(-t_2)$.

Define $\phi(v,-t) \equiv U(v-t)$.

Then $\phi(v,-t)$ is increasing in $v$ and

$$-\frac{\partial}{\partial v} \frac{-\phi_{11}}{\phi_1} = -\frac{\partial}{\partial v} \frac{-U''}{U'} = \frac{\partial}{\partial v} \frac{-U''}{U'} \leq 0$$

if absolute risk aversion is nonincreasing in income.

Thus $\phi(v,-t)$ satisfies the assumptions of Lemma 1. Moreover $\phi_2(v,-t) = U'(v-t)$.

Hence Lemma 1 implies (5) so that B4 is indeed satisfied for case 3.

Turning to Assumption B5 it is readily verified that for cases 1 - 3 a sufficient condition is $U'' \geq 0$.

In case 4 rather more stringent conditions are required to satisfy Assumption B. For this case $u_{12} = (1+\theta)U'' + U'$ while $u_{22} = (1+\theta)U'' + 2U'$. Then,
since \( \theta \) is nonnegative, B1 and B2 are satisfied if the degree of absolute risk aversion, \(-U''/U'\), exceeds 2. Condition B3 is satisfied if \( u_{122} = (1+\theta)U''' + 2U'' > 0 \).

But
\[
U''' + 2U'' = U' \left[ \frac{U'''}{U'} - 2 \left( -U' \frac{U''}{U'} \right)^2 \right]
\]
\[
> U' \left[ \frac{U'''}{U'} - \left( \frac{U''}{U'} \right)^2 \right] \quad \text{if } -U''/U' > 2
\]
\[
= U' \frac{d}{d(-\theta)} \left( \frac{U''}{U'} \right)
\]

Thus a sufficient condition for B3 is that absolute risk aversion everywhere exceeds 2 and is nonincreasing with income. Condition B4 is satisfied automatically. Finally \( u_{112} = (1+\theta)U''' + U'' > u_{122} \). Therefore if B3 is satisfied so is Assumption B5.

Summarising we have:

**Theorem 1:** Assumptions A and B are satisfied by

- cases 1 and 2 if \( U''' \geq 0 \)
- case 3 if absolute risk aversion is nonincreasing, \( \frac{d}{dx} \left( -U''(x) \right) < 0 \), and \( T(v|\theta) = \int K_{22}(x|\theta) dx > 0 \)
- case 4 if absolute risk aversion is nonincreasing and everywhere exceeds 2.

To predict the outcome of an auction, we must specify a solution concept. We shall assume that the functional forms \( u(\cdot, \cdot) \) and \( w(\cdot) \) and the distribution \( F \) are common knowledge among buyers and seller but that only buyer i knows the value of the parameter \( \theta_i \). In this case, the Bayesian equilibrium of Harsanyi (1968) is appropriate. For this solution concept, the "revelation principle"8 (see Dasgupta, Hammond, and Maskin (1979); Harris and Townsend (1981); and Myerson (1979)) tells us that we can confine our attention, without loss of generality, to auctions where the strategy space coincides with the set of possible parameters, i.e., \([0,1]\), and where there exists an equilibrium in which each buyer plays his true parameter as his strategy. That is,
(6a) \[ \max E_\theta \pi(x|\theta_{-i}) = E_\theta \pi(\theta_i|\theta_{-i}) \]

(6b) \[ \pi(x|\theta_{-i}) = H_i(x,\theta_{-i})u(-\hat{\beta}(x,\theta_{-i}),\theta_i) + (1-H_i(x,\theta_{-i}))w(-\hat{\alpha}(x,\theta_{-i})) \]

and \((x,\theta_{-i}) = (\theta_1, \ldots, x, \ldots, \theta_n)\). Such auctions can be called direct revelation\(^9\) auctions.

Because the buyers are \textit{ex ante} identical, we may confine our attention to symmetric auctions, i.e., those where families of \(H_i, \hat{\beta}_i\) and \(\hat{\alpha}_i\) functions are permutation symmetric.\(^{10}\) Define

\[ G(\theta_i) = \int_{\theta_{-i}} H_i(\theta_i, \theta_{-i}) dF(\theta_j) . \]

Note that, from symmetry, \(G\) requires no subscript. Then (6) becomes

\[ \max x E[G(x)u(-\tilde{b}(x),\theta_i) + (1 - G(x))w(-\tilde{a}(x))] \]

\[ = E[G(\theta_i)u(-\tilde{b}(\theta_i),\theta_i) + (1 - G(\theta_i))w(-\tilde{a}(\theta_i))] , \]

where \(\tilde{b}\) and \(\tilde{a}\) reflect the randomness both from \(\hat{\beta}_i\) and \(\hat{\alpha}_i\) and from \(\theta_{-i}\). They too need no subscripts. Because buyers have the option of not participating,

\[ E[G(\theta_i)u(-\tilde{b}(\theta_i),\theta_i) + (1 - G(\theta_i))w(-\tilde{a}(\theta_i))] > 0 . \]

Since \(u_2 > 0\), we need stipulate the nonparticipation constraint only for \(\theta_i = 0\). Thus,

\[ E[G(0)u(-\tilde{b}(0),0) + (1 - G(0))w(-\tilde{a}(0))] > 0 . \]

Notice that the only characteristics of an auction in which either the seller or the buyers are interested are the functions \(G, \hat{b}, \hat{a}\), as buyer \(i\)'s payoff is

\[ E[G(\theta_i)u(-\tilde{b}(\theta_i),\theta_i) + (1 - G(\theta_i))w(-\tilde{a}(\theta_i))] , \]

and the seller's payoff is

\[ nE \int [G(\theta)\hat{b}(\theta) + (1 - G(\theta))\hat{a}(\theta)]dF(\theta) . \]
For this reason, we shall often represent an auction by the triple \( <G, b, a> \).

2. **Standard Auctions**

Before turning to the optimal choice of \( <G, b, a> \), we first consider the standard auctions, the high bid and English auctions. In the high bid auction, sealed bids are simultaneously submitted by the buyers. The high bidder wins (a tie is broken by a coin flip) and pays his bid. Losers pay (and receive) nothing. In the English auction, bids are submitted successively and openly; each bid must be greater than the preceding one. The winner is the last buyer to bid (again, ties are broken by a randomizing device), and he pays his bid, while losers, again, pay nothing. As Vickrey (1961) argued, the English auction is equivalent, if the \( \theta_i \)'s are independently distributed, to a sealed bid auction in which the higher bidder wins but pays only the second highest bid, i.e., to a "second bid" auction (here we assume the continuous price formulation of footnote 11). Because it is easier to work with, we shall study the second bid formulation. Our ability to do this, however, depends crucially on independence; the two auctions are not equivalent otherwise. Indeed, as Milgrom and Weber (1982) show, buyers who are either risk neutral or exhibit constant absolute risk aversion bid higher, on average, in an English auction than in a second bid auction if, roughly speaking, the \( \theta_i \)'s are positively correlated.

We begin by showing that under some of the conditions discussed in section 1, equilibria in high and second bid auctions exist, are unique and have the property that bids are increasing as functions of \( \theta_i \). Because they are sometimes optimal when buyers are risk neutral, we shall consider high and second bid auctions with *seller reserve prices*, i.e., minimum permissible bids. In the second bid auction with reservice price, \( b^0 \), a winning buyer pays \( b^0 \) if no bid other than his own is greater than \( b^0 \). Because the buyers are *ex ante* identical, it is natural to focus attention on symmetric equilibria. In Remarks 2.3 and 3.2, we discuss the possibility of asymmetric equilibria.

**Theorem 2:** In a high bid auction with \( n \geq 2 \) buyers suppose the seller announces a minimum price, \( b^0 \), such that at least one buyer type is indifferent
between buying and not buying at this price, that is,

(11) \( u(-b^0, \theta^0) = 0 \), for some \( \theta^0 \in [0,1] \)

Then if preferences satisfy assumption A and \( u_{12} < 0 \), those for whom \( \theta < \theta^0 \) will not submit an acceptable bid, and there exists a unique symmetric equilibrium bid function \( b(\theta) \), \( \theta \geq \theta^0 \). Moreover, \( b(\theta) \) is increasing and differentiable.

**Proof:** If an equilibrium with \( b(\theta) \) increasing exists, then, for each \( \theta \), the probability that a buyer with parameter \( \theta \) has of winning is \( G(\theta) \), where

(12) \[
G(\theta) = \begin{cases} 
0 & , \theta < \theta^0 \\
\Phi^{n-1}(\theta) & , \theta \geq \theta^0
\end{cases}
\]

Furthermore, if \( b(\theta) \) is an equilibrium bid function, then, for each \( \theta > \theta^0 \), \( x = \theta \) maximizes

(13) \[
E[u] = G(x)u(-b(x), \theta)
\]

Assuming that \( b(\theta) \) is differentiable and differentiating, we have

\[
\frac{2}{\partial x} E[u] = G'(x)u(-b(x), \theta) - b'(x)G(x)u_1(-b(x), \theta)
\]

\[
= G(x)u_1(-b(x), \theta) \left[ \frac{u(-b(x), \theta)}{u_1(-b(x), \theta)} \frac{G'(x)}{G(x)} - b'(x) \right]
\]

With \( u_{x2} > 0 \) and \( u_{12} < 0 \) the first term in the bracketed expression is strictly increasing in \( \theta \). Thus

\[
\frac{3}{\partial x} E[u] < G(x)u_1(-b(x), \theta) \left[ \frac{u(-b(x), \theta)}{u_1(-b(x), \theta)} \frac{G'(x)}{G(x)} - b'(x) \right] \text{ as } x > \theta
\]

Therefore if \( b(\theta) \) is defined by the differential equation and boundary conditions

(14) \[
\begin{aligned}
b'(\theta) &= \frac{G'(\theta) u(-b(\theta), \theta)}{G(\theta) u_1(-b(\theta), \theta)} \\
b(\theta^0) &= b^0, \text{ where } u(-b^0, \theta^0) = 0
\end{aligned}
\]
we have \( \frac{3}{\theta x} E(u(-b(x), \theta)) \geq 0 \) as \( x \geq \theta \); \( x, \theta \in [0, 1] \).

So \( x = \theta \) yields the global maximum of \( E(u(-b(x), \theta)) \).

This establishes existence.

To prove uniqueness we first show that an equilibrium random bid function \( \tilde{b}(\theta) \) (in principle, an equilibrium could involve mixed strategies) must be increasing for \( \theta \geq \theta_0 \). More precisely, we show that if \( \tilde{b}(\cdot) \) is a deterministic selection from \( \tilde{b}(\cdot) \) (i.e., \( \tilde{b}(\theta) \) is a number in the support of \( \tilde{b}(\theta) \) for all \( \theta \)), then \( \tilde{b} \) must be increasing for \( \theta \geq \theta_0 \). Suppose that \( \tilde{b}(\cdot) \) is decreasing over some interval. Then there exist \( \theta_1 \) and \( \theta_2 \) with \( \theta_1 < \theta_2 \) such that \( \tilde{b}(\theta_1) > \tilde{b}(\theta_2) \). Hence, \( G_1 > G_2 \), where \( G_i \) is the probability that a buyer who bids \( \tilde{b}(\theta_i) \) wins; otherwise a buyer with parameter \( \theta_2 \) would be better off bidding \( \tilde{b}(\theta_2) \).

By definition of equilibrium

\[
(15) \quad G_2 u(-\tilde{b}(\theta_2), \theta_2) \geq G_1 u(-\tilde{b}(\theta_1), \theta_2)
\]

and

\[
(16) \quad G_1 u(-\tilde{b}(\theta_1), \theta_1) \geq G_2 u(-\tilde{b}(\theta_2), \theta_1).
\]

Subtracting (16) from (15), we obtain

\[
(17) \quad G_2 (u(-\tilde{b}(\theta_2), \theta_2) - u(-\tilde{b}(\theta_2), \theta_1)) \geq G_1 (u(-\tilde{b}(\theta_1), \theta_2) - u(-\tilde{b}(\theta_1), \theta_1)).
\]

But, because \( u_1 \leq 0 \) and \( \tilde{b}(\theta_1) > \tilde{b}(\theta_2) \)

\[
(18) \quad u(-\tilde{b}(\theta_2), \theta_2) - u(-\tilde{b}(\theta_2), \theta_1) \leq u(-\tilde{b}(\theta_1), \theta_2) - u(-\tilde{b}(\theta_1), \theta_1).
\]

But since \( G_2 < G_1 \), (18) implies that

\[
G_2 (u(-\tilde{b}(\theta_2), \theta_2) - u(-\tilde{b}(\theta_2), \theta_1)) < G_1 (u(-\tilde{b}(\theta_1), \theta_2) - u(-\tilde{b}(\theta_1), \theta_1)),
\]

contradicting (17). Thus \( \tilde{b}(\cdot) \) is nondecreasing. Now suppose that \( \tilde{b}(\theta) = \hat{b} \) on the interval \([\theta_1, \theta_2]\). Assume that \( \tilde{b}(\theta) < \hat{b} \) for \( \theta < \theta_1 \) and \( \tilde{b}(\theta) > \hat{b} \) for \( \theta > \theta_2 \). Then a parameter \( \theta_2 \) buyer who bids \( \tilde{b}(\theta_2) \) has probability of winning

\[
H = \sum_{k=0}^{n-1} \frac{1}{k+1} \binom{n-1}{k} (F(\theta_2) - F(\theta_1))^k F(\theta_1)^{n-1-k}.
\]
But if the buyer bids $\overline{b}(\theta^2) + \epsilon$, for $\epsilon > 0$, his probability of winning is greater than $\mathbb{P}(\theta^2)^n - 1$, which in turn is greater than $H$. Thus by an infinitesimal increase $\epsilon$ in his bid, the buyer can gain a discrete increase of, at least, $\mathbb{P}(\theta^2)^n - 1 - H$ in his probability of winning, and so $\overline{b}(\theta^2)$ cannot be an equilibrium bid.

Therefore, for $\theta < \theta^0$, $\overline{b}(\theta)$ must be strictly increasing and almost everywhere differentiable. Thus, from our previous argument, $\overline{b}(\theta)$ satisfies the differential equation (14) almost everywhere for $\theta > \theta^0$. In fact, if $\overline{b}(\theta)$ is continuous then it must satisfy (14) everywhere for $\theta > \theta^0$. To see that $\overline{b}(\theta)$ is continuous, suppose to the contrary that there exists $\theta^* > \theta^0$ with

$$\limsup_{\theta < \theta^*} \overline{b}(\theta) < \liminf_{\theta > \theta^*} \overline{b}(\theta).$$

But, for $\epsilon > 0$ sufficiently small, a buyer bidding $\limsup \overline{b}(\theta) - \epsilon$ has a probability of winning that is arbitrarily close to that of one bidding $\liminf \overline{b}(\theta) + \epsilon$. This is impossible, however, since $\limsup \overline{b}(\theta)$ is less than $\liminf \overline{b}(\theta)$, so no one would ever bid $\liminf \overline{b}(\theta) + \epsilon$. So $\overline{b}(\theta)$ is continuous and satisfies (14) everywhere. Since $\overline{b}(\theta)$ was an arbitrary selection, $\overline{b}(\theta)$ must satisfy (14) too. If $\overline{b}(\theta^0) < b^0$, then for $a > 0$ sufficiently small $b(\theta^0 + a) < b^0$. Thus the payoff of a buyer with parameter $\theta^0 + a$ is zero. But if he bids $b^0$, his expected payoff is

$$\mathbb{P}^{n-1}(\theta^0) u(-b^0, \theta^0 + a) > 0,$$

a contradiction. If $\overline{b}(\theta^0) > b^0$, then $u(-\overline{b}(\theta^0), \theta^0) < 0$, also an impossibility. Thus $\overline{b}(\theta^0) = b^0$, and so $\overline{b}(\theta) = b(\theta)$ for $\theta > \theta^0$.

Q.E.D.

Remark 2.1: If the seller gets a minimum price so low that even the least eager buyers ($\theta=0$) would strictly prefer to buy at that price, there is still a unique equilibrium $b(\theta)$ with boundary condition $u(-b(0), 0) = 0$. 

Remark 2.2: Theorem 2 is stated for preferences such that $u_{11} < 0$ and $w_{11} < 0$, but it is clear from the proof that it is true as well if these inequalities hold weakly.

Remark 2.3: This theorem does not discuss the possibility of asymmetric equilibria. Under the hypotheses of theorem, however, one can show (see Maskin and Riley (1982)) that the only equilibrium, symmetric or otherwise, in the class of n-tuples of piecewise continuous functions is the symmetric equilibrium. A crucial hypothesis in this uniqueness result is that the distribution $F$ has bounded support. When the support of $F$ has no upper bound, there can be a continuum of asymmetric equilibria.

Remark 2.4: We can incorporate the possibility that buyers care about others' parameter values by writing the utility of buyer $i$ as $u(-b, \theta_i, \theta_{-i})$. Then if the equilibrium bid function $b(\theta_i)$ is increasing and buyer $i$ is the winner, his expected utility is

$$
\bar{u}(-b, \theta_i) = \mathbb{E}_{\theta} \{u(-b, \theta_i, \theta_{-i}) | \theta_j \neq \theta_i, j \neq i \}
$$

Appealing to the results of Milgrom and Weber (1982) it is readily confirmed that if $u(x, \theta_i, \theta_{-i})$ is a strictly increasing and concave function of $x$, if $3u(x, \theta_i, \theta_{-i})/3x$ is a nonincreasing function, and if parameter values are "affiliated" (roughly speaking, positively correlated), then $\bar{u}$ satisfies Assumption A and $\bar{u}_{12} \leq 0$. We may therefore apply the argument of Theorem 2 to establish existence and uniqueness with affiliated parameter values. This generalizes Milgrom and Weber's existence proof for the case of risk neutral buyers.

Turning now to the second bid auction, we define

$$
\sigma(\theta) \quad \text{to be the reservation price of a buyer with parameter } \theta.
$$

That is

$$
(19) \quad u(-\sigma(\theta), \theta) = 0
$$

Theorem 3: In a second bid auction with $n \geq 2$ buyers suppose the seller announces a minimum price, $b^0$, such that at least one buyer is indifferent between buying and not buying at this price, that is

$$
u(-b^0, \theta^0) = 0 \quad \text{for some } \theta^0 \in [0,1]$$
Then if preferences satisfy assumption A there is an equilibrium in which each buyer for whom $\theta > \theta^0$ bids his reservation price while the remainder do not submit acceptable bids. Moreover, this is the unique symmetric equilibrium.

Proof: Given $u(-b^0, \theta^0) = 0$ all those for whom $\theta < \theta^0$ are better off not bidding. Suppose that all buyers use the same bid function $h(\theta)$ in equilibrium and that $h(\theta)$ is increasing. Since $u(-b^0, \theta^0) > 0$ for all $\theta > \theta^0$ we must have $h(\theta) > b^0$ for all $\theta > \theta^0$. Then if buyers bid according to $h(\theta)$, $G(\theta)$ satisfies (12). Thus for all $\theta > \theta^0$, $x = \theta$ maximizes

$$E(u) = u(-b^0, \theta) F^{n-1}(\theta^0) + \int_{\theta^0}^{\theta} u(-h(t), \theta) dF^{n-1}(t)$$

Differentiating (20) by $x$ we obtain

$$\frac{\partial}{\partial x} E(u) = u(-h(\theta), \theta) dF^{n-1}(x) / dx .$$

By assumption $h(\theta)$ is increasing and by Assumption A1 $u_1 > 0$. Thus

$$\frac{\partial}{\partial x} E(u) > u(-h(\theta), \theta) dF^{n-1}(x) / dx , x < \theta .$$

But from the definition of $\sigma(\theta)$, $u(-\sigma(\theta), \theta) = 0$. Moreover, since $u_2 > 0, \sigma(\theta)$ is an increasing function. Then $h(\theta) = \sigma(\theta), \theta > \theta^0$ is an equilibrium of the second bid auction.

To prove uniqueness, suppose $h(\theta)$ is the (possibly random) bid function in a symmetric equilibrium. Let $\overline{h}(\theta)$ be a deterministic selection.

Suppose first that $Pr(\theta > \theta^0, \overline{h}(\theta) > \sigma(\theta)) > 0$. Then there exists $\theta^* > \theta^0$ such that $\overline{h}(\theta^*) > \sigma(\theta^*)$ and for all $\epsilon > 0$

$$Pr[\overline{h}(\theta) \in [\overline{h}(\theta^*) - \epsilon, \overline{h}(\theta^*)]] > 0 .$$

But then a buyer with parameter $\theta^*$ is better off bidding less than $\overline{h}(\theta^*)$, since otherwise there is a positive probability he will pay more than his reservation price.

Next suppose that $Pr(\theta > \theta^0, \overline{h}(\theta) < \sigma(\theta)) > 0$ . Then there exists $\theta^{**} > \theta^0$ such that $\overline{h}(\theta^{**}) < \sigma(\theta^{**})$ and for all $\epsilon > 0$
Pr(\tilde{h}(\theta) \in [\tilde{h}(\theta^{**}), \tilde{h}(\theta^{**}) + \varepsilon]) > 0.

But then a buyer with parameter \theta^{**} is better off bidding more than \tilde{h}(\theta^{**}) since otherwise there is a positive probability that he will lose to a bid less than his reservation price.

Hence, Pr(\tilde{h}(\theta) = \sigma(\theta) \mid \theta > \theta^{\circ}) = 1, and so \tilde{h}(\theta) = \sigma(\theta) for all \theta \in [\theta^{\circ}, 1). Therefore \tilde{h}(\theta) = \sigma(\theta) for all \theta \in [\theta^{\circ}, 1).

Q.E.D.

Remark 3.1: Examining the last line of the proof, we see that equilibrium is not quite unique. Although bids must coincide with \sigma(\theta) for all \theta \in [\theta^{\circ}, 1), they need not for \theta = 1. All that is necessary is that \tilde{h}(1) > \sigma(1).

Remark 3.2: One can also show that there are no asymmetric equilibria with three or more buyers in the second bid auction (see Maskin and Riley (1982)). With two buyers, however, there is a vast family of asymmetric equilibria. For example, consider the continuum of pairs of bid functions in which one buyer announces that if his reservation price \sigma(\theta) exceeds \b^{*}, \theta > \theta^{\circ} he will bid 1, otherwise he will bid his reservation value, and the other buyer bids \min\{\sigma(\theta), \b^{*}\}. It is readily confirmed that all these pairs are equilibria.

Even in the case n=2, however, there is good reason to single out the symmetric equilibrium. Besides its appeal from its very symmetry, the symmetric equilibrium is also the unique dominant strategy equilibrium. Closely related to this point is the fact that it corresponds to the unique (subgame) perfect equilibrium of the English auction (recall that our motivation for examining second bid auctions was their equivalence to English auctions). There is a one-to-one correspondence between the asymmetric equilibria of the second bid and English auctions. However, those in the latter auction fail to be subgame perfect, whereas those in the former are not trembling-hand perfect (see Selten (1975)). For greater elaboration of these points see Maskin and Riley (1982).

Remark 3.3. If the seller sets a minimum price so low that even a buyer with parameter \theta = 0 has a reservation price \sigma(0) > \b^{\circ} the unique symmetric equilibrium is for all buyers to bid their reservation values.
Remark 3.4. Like Theorem 2, Theorem 3 holds for risk neutral buyers, that is, \( u_{11} = 0, \omega_{11} = 0 \).

We are now ready to show that, under weak assumptions, high bid auctions are superior to second bid auctions from the seller's viewpoint. In general, to compare two auctions entails specifying which equilibria in each are to be examined. We shall in fact compare the (unique) symmetric equilibria. However, in view of Remarks 2.3 and 3.2, we need not have made this qualification, since, at least in the case \( n \geq 3 \), equilibrium in both the high bid and second bid auctions is unique.

Theorem 4: Under the assumptions of Theorem 2 the symmetric equilibrium of the high bid auction with reserve price \( b^0 \) generates greater expected revenue for the seller than the symmetric equilibrium of the second bid auction with the same reserve price.

Remark 4.1: For the preferences of case 1 this result has already been established in those papers mentioned in footnote 1.

Proof: Let \( B(\theta) \) be the expected payment by a winner with parameter \( \theta \) in the second bid auction. We shall establish that \( b(\theta) \geq B(\theta) \) for all \( \theta \), with strict inequality for \( \theta \geq \theta^0 \) where \( b(\theta) \) satisfies (14). From Theorem 3 if a buyer with parameter value \( \theta \) is the winner in the second bid auction his payment is a random variable

\[
\tilde{B} = \max\{b^0, \sigma(t)\} \quad \text{where } t \text{ is the highest of the other } \ n-1 \text{ buyer's parameter values}
\]

Therefore the expected payment by the winner, \( B(\theta) \), satisfies

\[
(22) \quad B(\theta) = E(\tilde{B}) = \left[b^0 F_{n-1}(\theta^0) + \int \sigma(t) dF_{n-1}(t)\right]/F_{n-1}(\theta^0).
\]

Differentiating (22) with respect to \( \theta \) we obtain

\[
\frac{dB(\theta)}{d\theta} = \frac{G'(\theta)}{G(\theta)} \left(\sigma(\theta) - B(\theta)\right), \quad \theta \geq \theta^0.
\]

\[
B(\theta^0) = b^0, \quad \text{where } u(-b^0, \theta^0) = 0.
\]
where \( G(\theta) \) satisfies (12); that is, \( G(\theta) = F^{n-1}(\theta) \), for \( \theta \geq \theta^0 \).

Comparing (14) and (23) first note that \( b(\theta^0) = B(\theta^0) \). Thus if we can show that

\[
(24) \quad \frac{u(-b(\theta),\theta)}{u_1(-b(\theta),\theta)} > \sigma(\theta) - b(\theta) \quad \text{whenever } b = B
\]

then (14) and (23) imply that \( b(\theta) > B(\theta) \) for \( \theta > \theta^0 \).

Consider the left and right hand sides of (24) as functions of \( b \). For \( b = \sigma(\theta) \), both sides vanish. The derivative of the left hand side is

\[-1 + uu_{11}/u_1 \]

whereas the derivative of the right hand side is \(-1\). Therefore, because \( b(\theta) < \sigma(\theta) \) we conclude that (24) holds.

\[
\text{Q.E.D.}
\]

The proof of Theorem 4 actually establishes a bit more than the Theorem asserts. The proof indicates that, for each \( \theta > \theta^0 \), \( b(\theta) > B(\theta) \); that is, the high bid auction generates greater expected revenue for each possible value of \( \theta \). From this observation we can draw strong conclusions if the seller is himself risk averse — with strictly concave utility function, \( v(\cdot) \).

In the second bid auction the seller's expected utility is

\[
\frac{1}{E \int v(B(\theta))dF^{n}(\theta)}
\]

\[
\theta^0
\]

\[
< \int v(B(\theta))dF^{n}(\theta) , \quad \text{by Jensen's Inequality, since } B(\theta) = E\tilde{B}(\theta)
\]

\[
\theta^0
\]

\[
\frac{1}{\int v(b(\theta))dF^{n}(\theta)} , \quad \text{since } B(\theta) < b(\theta) , \text{from Theorem 4}.
\]

This last expression is just the seller's expected utility in the 'high bid auction. We have therefore proved

\[
\text{Theorem 5: Under the assumptions of Theorem 2 a risk averse seller strictly prefers the high bid to the second bid auction.}
\]
Remark 5.1: This result was established by Vickrey (1961) for a uniform distribution of risk neutral buyers and by Matthews (1979) for Case 1 preferences.

If buyers are risk averse, one might expect an auction where they are insured against losing to generate more revenue for the seller than the high bid auction. After all, the seller could extract a premium for the insurance in such an auction. As we shall see, however, this is not normally the case because the insurance interferes with the seller's ability to screen. By a perfect insurance auction with reserve price $b^o$ (see Riley and Samuelson (1981) for a discussion of such auctions), we shall mean a triple $(G, b, a)$, where $G$ satisfies (12),

$$b(\theta^o) = b^o,$$

(25) $x = \theta$ maximizes $G(x)u(-b(x), \theta) + (1 - G(x))w(-a(x))$, 

(26) $u_1(-b(\theta), \theta) = w_1(-a(\theta))$, and 

(27) $G(\theta^o)u(-b(\theta^o), \theta^o) + (1 - G(\theta^o))w(-a(\theta^o)) = 0$.

Here we have defined perfect insurance to entail equalization of marginal utilities across states, since that is what a rational risk bearer will attempt to do. For many utility functions, e.g., those of Case I, equalizing marginal utilities is the same as equalizing the utilities themselves.

Theorem 6: For the preferences of Case 1, a perfect insurance auction with reserve price $b^o$ generates the same expected revenue for the seller as the second bid auction with the same reserve price.

Proof: For Case 1 preferences, (26) implies

$$\theta - b(\theta) = -a(\theta).$$

Thus, the first order condition for the maximization, (25), is

$$-GU'(\theta - b)b' + (1 - G)U'(\theta - b)(1 - b') = 0.$$

Hence,

(28) $b' = (1 - G)$
Also, for Case 1 preferences,

\[(29) \quad \phi(\theta) = \theta\]

From (28) and (29), revenue from both the second bid and perfect insurance auctions is independent of the utility function \(U\). In particular, we may assume \(U(x) = x\). But for such risk neutral preferences, we know (see Myerson (1981), Maskin and Riley (1980a)) that two auctions generate the same expected revenue if they share the same \(G\) function and if the most reluctant buyers \((\theta = 0)\) obtain a zero payoff.

Q.E.D.

Theorem 6 implies in particular that, for the preferences of Case 1, the high bid auction generates strictly more revenue than the perfect insurance auction. That this proposition does not hold for general preferences, however, will become clear in Section 4 (see the discussion following Theorem 11).

3. Characterization of the Seller's Optimization Problem

We now consider all possible auction schemes and show that the choice of an optimal auction can be characterized as the solution of a control problem. This is summarized in Theorem 9 at the end of the section. Inferences about the properties of optimal auctions are then drawn in Section 4.

3a. The Buyer's Problem

Let us restrict attention for the time being to deterministic auctions, that is, auctions where the payment by a buyer with parameter \(\theta\) is the deterministic function \(b(\theta)\) if he wins and \(a(\theta)\) if he loses. Let us also assume that \(a(\theta), b(\theta),\) and the probability of winning \(G(\theta)\) are piecewise differentiable.

If "truth-telling" constitutes an equilibrium (as we have already noted, we can assume without loss of generality) then we can express maximized expected utility as

\[(30) \quad W(\theta, \theta) = \max_x W(x, \theta),\]

where

\[W(x, \theta) = G(x)u(-b(x), \theta) + (1-G(x))w(-a(x)).\]
Much of the subsequent analysis makes use of finite approximations to the interval \([0,1]\). For each positive \(m\) and each \(i=1,\ldots,m\), let \(\theta^m_i = \frac{i-1}{m}\) and take \(f^m(\theta^m_i)\) so that \(\sum f^m(\theta^m_i) = 1\) and, if \(\{\theta^m_i\}\) converges to \(\theta\), \(\lim_{m \to \infty} f^m(\theta^m_i) = F'(0)\). Then, for each \(m\) and \(i\), a truth telling equilibrium requires \(V(i,i) = \max_j V(j,i)\), where

\[
V(j,i) = [G_j u(-b_j, \theta_j) + (1-G_j) w(-a_j)].
\]

In this last formula, \(b_j = b_j(\theta^m_i)\), etc. and the superscripts are deleted. In particular, truth-telling implies

\[
(31) \quad V(i,i) - V(i-1,i) \geq 0 \quad i=2,\ldots,m.
\]

In the terminology of the incentive literature, (31) is the adjacent "downward" incentive constraint. We begin by showing that, under certain circumstances, (31) holding with equality is sufficient for a truth-telling equilibrium.

**Lemma 2:** Suppose preferences satisfy assumption A and \(u_{12} \leq 0\). If \(G_\ast\) and \(b_\ast\) are nondecreasing in \(i\) and (31) holds with equality for all \(i\), then \(V(i,i) = \max_j V(j,i)\); i.e., truth telling constitutes an equilibrium.

**Proof:** We first claim that for all \(i\) and all \(j \leq i\)

\[
(32) \quad G_j u(-b_j, \theta_j) + (1-G_j) w(-a_j) \geq G_{j-1} u(-b_{j-1}, \theta_{j-1}) + (1-G_{j-1}) w(-a_{j-1}).
\]

Suppose, to the contrary that there exists \(i\) and \(j \leq i\) such that

\[
(33) \quad G_j u(-b_j, \theta_j) + (1-G_j) w(-a_j) < G_{j-1} u(-b_{j-1}, \theta_{j-1}) + (1-G_{j-1}) w(-a_{j-1}).
\]

From (31), \(j < i\). Subtracting \(V(j,j) = V(j-1,j)\) from (33), we obtain

\[
(34) \quad G_j (u(-b_j, \theta_j) - u(-b_j, \theta_j)) < G_{j-1} (u(-b_{j-1}, \theta_{j-1}) - u(-b_{j-1}, \theta_{j-1})).
\]
Because $G_j \geq G_{j-1}$ and $u_2 > 0$ by hypothesis, (34) implies

$$u(-b_j, \theta_i) - u(-b_j, \theta_j) < u(-b_{j-1}, \theta_i) - u(-b_{j-1}, \theta_j),$$

a contradiction of $u_{12} \leq 0$. Hence (32) holds after all. Similarly, for $j > 1$,

$$G_j u(-b_j, \theta_i) + (1-G_j)w(-a_j) \leq G_{j-1} u(-b_{j-1}, \theta_i) + (1-G_{j-1})w(-a_{j-1}).$$

But (32) and (35) combined with (31) imply $V(i, i) = \max V(j, i)$.

In the limit, as $m \to \infty$, (31) becomes

$$-Cu^1 b^- + uG^- - (1-G)w, a^- - wG^- \geq 0,$$

if the (directional) derivative exists, and

$$G(\theta)u(-b(\theta), \theta) + (1-G(\theta))w(-a(\theta)) \geq \lim_{\Delta \to 0} \left[ G(\theta+\Delta)u(-b(\theta+\Delta), \theta) + (1-G(\theta+\Delta))w(-a(\theta+\Delta)) \right],$$

if it does not. Hence we can state

**Lemma 2a:** Suppose preferences satisfy assumption A and $u_{12} \leq 0$. If $G$ and $b$ are nondecreasing and (36) or (37) are satisfied for all $\theta$, then truth-telling constitutes an equilibrium.

Lemmas 2 and 2a, in effect, enable us to ignore all but a single incentive constraint in stating the buyer's optimization problem. Below we shall exploit these results in expressing the optimal auction as the solution to a control theory exercise.

3b. The Seller's Problem

Turning to the seller's problem, we see that the seller's expected revenue is given by

$$n / \int [b(\theta)G(\theta) + a(\theta)(1 - G(\theta))]dF(\theta).$$

Thus, the seller chooses $G$, $b$, $a$ to maximize (38) subject to the buyer's maximization condition (30), the nonparticipation option condition

$$G(0)u(-b(0), 0) + (1 - G(0))w(-a(0)) \geq 0,$$
and the constraint that $G$ be derived via (7) from symmetric probability functions $H_1, \ldots, H_n$, satisfying (1). Obviously because $G$ itself is the probability of winning, it must satisfy $G \geq 0$. For technical reasons, we impose the tighter constraint

(40) $G \geq \varepsilon$.

for $\varepsilon$ very small. But (40) is not enough. The following Theorem characterizes when symmetric $H_i$'s can be found, at least when $G$ is nondecreasing.

**Theorem 7:** Suppose that $G(s)$, the probability of winning with parameter equal to $s$ is piece-wise differentiable and nondecreasing. If $G(s)$ can be generated by a direct revelation auction, then, conditional on having a parameter value of at least $y$, the probability of winning never exceeds the probability that $y$ is the highest parameter value. That is, a necessary condition for there to exist a permutation symmetric family (see footnote 10) of probability functions $H_j(x)$, $j=1,\ldots,n$, satisfying $\sum_j H_j \leq 1$, such that

(7) $G(\theta_i) = \int H_j(\theta_i, \theta_{-i}) \, dF(\theta_j)$ for all $\theta_i$

is

(41) $\int_0^1 G(s) \, dF(s) \leq \int_0^1 F^{-1}(s) \, dF(s)$, $0 \leq y \leq 1$.

Moreover, if $G(s)$ is a step function with finitely many steps, (41) is sufficient.

**Remark 7.1:** That we establish the sufficiency of (41) - the crucial half of the theorem - only for nondecreasing finite step functions is not a real limitation, since we can approximate any nondecreasing, piece-wise differentiable $G$ arbitrarily closely by such a function. In the proof of Theorem 8 it is established that, in the optimal (deterministic) auction $<G,b,a>$, $G$ and $b$ are nondecreasing and

(42) $-G(\theta) u_1(-b(\theta), \theta) b'(\theta) + u(-b(\theta), \theta) G'(\theta) - (1-G(\theta)) w_1(-a(\theta)) a'(\theta)$

$- w(-a(\theta)) G'(\theta) = 0$
for all $0 < 1$. If we replace $G$ by the nondecreasing finite step function $\hat{G}$ and then define $\hat{a}(\theta)$ by the differential equation (42) with the appropriate boundary conditions, then the resulting auction $\langle \hat{G}, b, \hat{a} \rangle$ will generate nearly the same revenue as $\langle G, b, a \rangle$ (if $\hat{G}$ is close to $G$) and will have truth-telling as an equilibrium (by Lemma 2a).

Consider the control problem consisting of choosing $G$, $b$, and $a$ to maximize (38) subject to the buyer first order conditions (36) and (37), the nonparticipation condition (39), and the probability constraints (40) and (41). The Hamiltonian for the control problem is

$$h = (G b + (1-G) a) F + \lambda (-G u b + u G - (1-G) w a - w G )$$

$$+ a (G-c) - \int_0^\theta n(t) dt (G - F^{-1} G')$$

$$+ \gamma (G(0) u(-b(0),0) + (1-G(0)) w(-a(0))),$$

where $\lambda(0)$ is the Lagrange multiplier for (36) and (37), $a(\theta)$ is the multiplier for (40), $n(\theta)$ is the multiplier for (41), and $\gamma$ is the multiplier for (39). If it turns out that, in the solution to this problem, (36) and (37) hold with equality everywhere and $G$ and $b$ are nondecreasing, then by virtue of Lemma 2a and Theorem 7, this solution corresponds to an optimal auction.

Theorem 8 below gives conditions on preferences and $F$ under which these conditions are satisfied. Before stating it, let us consider the finite version of the seller's control problem. The seller's expected revenue (38) becomes

$$n \sum_i [b_i G_i + (1-G_i) a_i] f_i$$

The incentive constraint is (31).

The nonparticipation and nonnegativity constraints (39) and (40) remain the same as before, but the constraint (41) becomes

$$\sum_{j=1}^m (F^{-1}(\check{v}_j^m) - G(\check{v}_j^m) F(\check{v}_j^m) \geq 0).$$
Forming the Lagrangian for this finite optimization problem we have

\[
L = \sum_{i=1}^{m} \left[ (G_i b_i + (1-G_i) a_i) f_i + \lambda_i (G_i u(-b_i, \theta_i) + (1-G_i) w(-a_i)) - G_i u(-b_i, \theta_i) \right. \\
\left. - (1-G_i) w(-a_i) \right] + \alpha_i (G_i - \epsilon) + \eta_i m \sum_{j=1}^{m} (F_i^{-1}(\theta_j) - G_i) f(\theta_j) + \gamma (G_i b_i + (1-G_i) a_i)
\]

Thus the first order conditions are

\[
\frac{\partial L}{\partial G_i} = (b_i - a_i) f_i + \lambda_i (u(-b_i, \theta_i) - w(-a_i)) - \lambda_i + 1 (u(-b_i, \theta_i) - w(-a_i)) \\
\alpha_i - \sum_{j=1}^{m} \eta_i f(\theta_j) = 0,
\]

\[
\frac{\partial L}{\partial b_i} = G_i f_i - \lambda_i G_i u(-b_i, \theta_i) + \lambda_i (1-G_i) w(-a_i) = 0, \quad \lambda_{m+1} = 0
\]

\[
\frac{\partial L}{\partial a_i} = (1-G_i) f_i - \lambda_i (1-G_i) w(-a_i) + \lambda_i + 1 (1-G_i) w(-a_i) = 0, \quad \lambda_{m+1} = 0
\]

together with the constraints (31), (39), (40), and (44).

It can be readily seen that, for all m, there is a solution to the system of conditions (31), (39), (40), (44), (45) - (47). Note in particular that, as \( m \rightarrow \infty \), we have, from (46) and (47),

\[
\lambda(1) G(1) u_1(-b(1), 1) = 0, \quad -\lambda(1) (1-G(1)) w_1(-a(1)) = 0.
\]

Hence

\[
\lambda(1) = 0.
\]

When \( m \rightarrow \infty \), conditions (45) - (47), (31), (40) and (44) become, respectively,

\[
(b-a) F' - \lambda u_2 - \lambda'(u-w) + \theta \int_0^\theta \eta(t) dt F' = 0; \quad G
\]

\[
G(F' + \lambda u_2 + \lambda' u_1) = 0; \quad b
\]

\[
(1-G)(F' + \lambda u_1' + \lambda' u_1) = 0; \quad a
\]

\[
-G u_1 b' + (u-w) G' - (1-G) w_1 a' \geq 0; \quad \lambda \geq 0
\]

\[
G \geq \epsilon; \quad \alpha \geq 0
\]

\[
\int_0^1 (G-F_n^{-1}) dF \leq 0; \quad n \geq 0
\]
where primes denote mere directional derivatives at points of nondifferentiability. That is, for each differential equation in \( m \), there are, in fact, two equations—one for each direction. Of course, because \( a, b, G, \lambda \), and \( \lambda' \) may not be continuous, one of the directional derivatives may, in principle, fail to exist at some points. We are now ready to establish

**Theorem 8:** If \( u \) and \( w \) satisfy assumptions A and B and the solution to the control problem of maximizing (38) subject to (36), (37), and (39) – (41) satisfies

\[
0 \leq j(\theta) = 2 + \frac{F''}{(F')^2} \int_0^1 \frac{w_1(-a(\theta))}{w_1(-a(x))} F'(x) dx
\]

then this solution corresponds to a deterministic auction that is optimal for the seller with \( n(>2) \) buyers.

**Remark 8.1:** Condition (53) requires that the density function, \( F' \), not decline too rapidly with \( \theta \). Indeed, observe that (53) is automatically satisfied if \( F'' > 0 \). Thus in particular, it is satisfied by the uniform distribution. In the limiting case of risk neutrality the condition becomes

\[
0 \leq 2 + \frac{F''}{(F')^2} (1-F) \leq \frac{d}{d\theta} \left( \theta - \frac{1-F}{F'} \right)
\]

From Myerson (1981) and Maskin and Riley (1980a) we know that with risk neutrality (53) guarantees that a high bid or English auction with an appropriately chosen seller's reserve price is optimal. However, it will pay to take \( G' = 0 \) in some intervals when this inequality is violated. With risk aversion, one can show that a violation of (53) may cause the optimal \( G' \) to be negative in places.

Since our methods rely heavily on establishing that \( G' \geq 0 \) (see Lemma 2 and the proof of Theorem 8) condition (53) is indispensable.

From (47) we have

\[
m(\lambda_{i+1} - \lambda_i) = \frac{m_f}{w_1(-a_i)} \quad \text{for} \quad G_i < 1
\]
We obtain a similar expression from (46) if \( G_1 = 1 \). Thus the directional derivatives \( \lambda'_+ \) and \( \lambda'_- \) exist at all points \( \theta \). This implies that \( \lambda(\theta) \) is continuous and that (50) \( \text{--} \) (52) hold in both directions for all \( \theta \). We cannot guarantee that (36) holds in both directions at all points, because \( b, a, \) or \( G \) may, in principle, be discontinuous at some points. However, at such points (37) holds.

We next show that if \( G > 0 \) and the directional derivatives \( G' \) and \( b' \) are defined, then \( G' > 0 \) and \( b' > 0 \). First observe that with \( G < 1 \), (52) can be rewritten as

\[
(54) \quad \frac{\lambda'}{F'} = -\frac{1}{\nu}.
\]

From (41) and because \( G \) is piecewise differentiable, \( G(\theta) \) can equal 1 at only finitely many points. Therefore, because \( \langle G, b, a \rangle \) is piecewise differentiable, if \( G(\hat{\theta}) = 1 \), there exists a sequence \( \{ \theta_n \} \) such that \( \theta_n \to \hat{\theta} \), \( G(\theta_n) \), \( b(\theta_n) \), \( a(\theta_n) \) \( \to \) \( (G(\hat{\theta}), b(\hat{\theta}), a(\hat{\theta})) \), and \( G(\theta_n) < 1 \) for all \( n \). But \( \lambda'(\theta_n), a(\theta_n) \)

satisfies (54) for all \( n \). Hence (54) holds for \( \theta = \hat{\theta} \) and, hence, everywhere.

Substituting (54) into (51) and dividing by \( G \), we have

\[
(55) \quad \frac{\lambda'}{F'} = \frac{u_1 - w_1}{u_1 - w_1}, \quad G > 0.
\]

From (54), \( \lambda' \) is negative. Therefore, since \( \lambda \geq 0 \) everywhere, \( \lambda(\theta) \) must be strictly positive for \( \theta < 1 \). This already shows that (36) and (37) hold with equality for all \( \theta \). Because \( \lambda(\theta) > 0 \) for \( \theta < 1 \), (55) implies that

\[
(56) \quad u_1 - w_1 < 0, \quad 0 < 1, \quad G > 0,
\]

and from assumption B4 it follows that

\[
(57) \quad u - w > 0, \quad \theta < 1, \quad G > 0.
\]

Dividing (54) by (55), we obtain

\[
(58) \quad \frac{\lambda'}{\lambda} = -\frac{u_{12}}{u_1 - w_1}.
\]

Taking the logarithmic derivative of (55) we also have

\[
(59) \quad \frac{\lambda'}{F'} = \frac{u_{12} - u_{122}}{u_1 - w_1} + b' \left[ \frac{u_{11} - u_{112}}{u_1 - w_1} \right] + a' \left[ \frac{w_{11} + w_{11}}{u_1 - w_1} \right].
\]
Substituting from (55) and (58) and rearranging we then have

\[
\begin{align*}
(60) \quad b' \left[ \frac{u_{11}}{u_1 - w_1} - \frac{u_{12}}{u_2} \right] - a' \left[ \frac{w_{11} u_1}{u_1 - w_1} \right] &= \left[ \frac{u_{12}}{u_1 - w_1} \right] (2 + \lambda \frac{w_{11} F''}{(F')^2} ) + \left[ \frac{-u_{12}}{u_2} \right].
\end{align*}
\]

We recall that, with $G > 0$, $\alpha = 0$. Then substituting (54) into (50) dividing by $F'$ and then differentiating by $\theta$, we obtain

\[
\begin{align*}
&b' - a' - \left( \frac{\lambda'}{F'} - \frac{\lambda' F''}{(F')^2} \right) u_2 - \frac{\lambda}{F'} (-u_{12} b' + u_{22}) + \frac{u_2}{w_1} (u-w) \frac{w_{11}}{(w_1)^2} a' + \frac{1}{w_1} (-u_1 b' + w_1 a') - \eta = 0.
\end{align*}
\]

Collecting terms and making use of (55) we can rewrite this as

\[
\begin{align*}
&\frac{(w_1 - w_2) u_{22}}{w_1 u_{12}} + (u-w) \frac{w_{11}}{(w_1)^2} a' + \frac{u_2}{w_1} (1 - \frac{\lambda' w_1}{F'} + \frac{\lambda u F''}{(F')^2}) - \eta = 0.
\end{align*}
\]

Again making use of (54) and then multiplying by $w_1/u_2$ we obtain

\[
\begin{align*}
(61) \quad \frac{[w_1 w_{11} - u_{22}]}{u_2 w_1} a' - \frac{[w_1 (w_1 - u_1) u_{22}]}{w_2 u_{12}} &= \{2 + \frac{\lambda w_1 F''}{(F')^2} \} - \frac{w_1}{u_2} \eta.
\end{align*}
\]

From (56), (57), and Assumptions A and B all the bracketed terms in (60) and (61) are positive. Integrating (54) and recalling (49), we obtain

\[
\lambda(\theta) = \int_0^1 \frac{F'(x)}{w_1(-a(x))} \, dx.
\]

Then the term in braces in (61) can be written as

\[
2 + \lambda w_1 (-a(\theta)) \frac{F''}{(F')^2} = 2 + \frac{F''}{(F')^2} \int_0^1 \frac{w_1 (-a(\theta))}{w_1(-a(x))} F'(x) \, dx \equiv j(\theta).
\]

Given condition (53) this is also nonnegative.

Using (60) and (61) it is straightforward to confirm that $b'$ and $G'$ are positive. First suppose the integral constraint, (41), is not binding. Then $\eta = 0$ and, from (61), $a' > 0$. Then from (60) $b' > 0$. Since $u > w$ it follows also from (36) that $G' > 0$. 
Next suppose that (41) is binding. Then locally \( G = F^{n-1} \), and so \( G' > 0 \).

If \( a' < 0 \) it follows from (36) that \( b' > 0 \). (Remember that \( \lambda \) is strictly positive, so (36) holds with equality). If \( a' > 0 \) then from (60) \( b' > 0 \), as before.

To complete the proof, it suffices to show that \( G, b, \) and \( a \) are continuous.

This combined with our previous demonstration that, for \( G > 0 \), \( G' \) and \( b' \) are positive when defined will imply that \( G \) and \( b \) are nondecreasing.

To see that \( a \) is continuous, suppose, to the contrary, that \( a \) is discontinuous from the right at \( \theta^* \) (the argument is virtually the same if \( a \) is discontinuous from the left).

Then

\[
\theta^* \equiv \lim_{\Delta \to 0} a*(\theta^*+\Delta) \neq a(\theta^*).
\]

Similarly, define

\[
b^* \equiv \lim_{\Delta \to 0} b(\theta^*+\Delta) \quad \text{and} \quad G^* \equiv \lim_{\Delta \to 0} G(\theta^*+\Delta).
\]

From (55),

\[
\frac{\lambda(\theta^*)}{F'(\theta^*)} = \frac{u_1(-b(\theta^*),\theta^*)-w_1(-a(\theta^*))}{u_{12}(-b(\theta^*),\theta^*)w_1(-a(\theta^*))} = \frac{u_1(-b^*,\theta^*)-w_1(-a^*)}{u_{12}(-b^*,\theta^*)w_1(-a^*)}
\]

For each \( x \in [a(\theta^*), a^*] \), choose \( \hat{b}(x) \) so that \( \hat{b}(a(\theta^*)) = b(\theta^*), \hat{b}(a^*) = b^* \) and

\[
\frac{\lambda(\theta^*)}{F'(\theta^*)} = \frac{u_1(-\hat{b}(x),\theta^*)-w_1(-x)}{u_{12}(-\hat{b}(x),\theta^*)w_1(-x)}.
\]

Such a choice is possible from (62) and Assumption B. Consider the expression

\[
\hat{b}(x) = x - \frac{\lambda(\theta^*)}{F'(\theta^*)} u_2(-\hat{b}(x),\theta^*) + \frac{1}{w_1(-x)} (u(-\hat{b}(x),\theta^*)-w(-x))
\]

Differentiating (64) with respect to \( x \) and using (63), we obtain
\[ b'(x) - 1 + \frac{u(-b(x), \theta^*) - w_1(-x)}{u_{12}'(-b(x), \theta^*)w_1(-x)} u_{12}(-b(x), \theta^*)b'(x) \]
\[ + \frac{1}{w_1(-x)} (u_1(-b(x), \theta^*)b'(x) + w_1(-x)) + \frac{(u(-b(x), \theta) - w(-x))}{(w_1(-x))^2} w_{11}(-x) \]
\[ = \frac{(u(-b(x), \theta^*) - w(-x))}{(w_1(-x))^2} w_{11}(-x). \]

From (56), (57), and (63) the right hand side of (65) is negative. Thus (64) is larger when \( x = a(\theta^*) \) than when \( x = a^* \). From (50), (54), and (55), at \( x = a(\theta^*) \) and \( x = a^* \)

\[ b - a - \frac{\lambda}{\Gamma} u_2 + \frac{u-w}{w_1} = \int_0^\theta n(t)dt - a \]

Since we have just proved that the left hand side is larger at \( x = a(\theta^*) \) than at \( x = a^* \), it therefore follows that \( a(\theta^*) < a^* \). Then

(66) \( G(\theta^*) > G^* = c \)

From (55)

(67) \( \lambda(\theta^*)(u_{12}(-b^*, \theta^*) - u_{12}(-b(\theta^*), \theta^*)) \]
\[ = F'(\theta^*) \left( \frac{u_{12}(-b^*, \theta^*)}{w_1(-a^*)} - \frac{u_{12}(-b(\theta^*), \theta^*)}{w_1(-a(\theta^*))} \right) \]

If \( b^* < b(\theta^*) \), then the left hand side of (67) is nonnegative by Assumption B3. But the right hand side is negative by assumptions A2 and A4. Hence \( b^* > b(\theta^*) \).

From (37) we have
\[
G_u(-b^*, \theta^*) + (1-G^*)w(-a^*) = G(\theta^*)u(-b(\theta^*), \theta^*) + (1-G(\theta^*))w(-a(\theta^*))
\]

Appealing to the mean value theorem we then obtain

\[(68) \ [u(-b(\theta^*), \theta^*) - w(-a(\theta^*))](G^*-G(\theta^*)) = G(\theta^*)u_1(-\hat{b}, \theta^*)(b^*-b(\theta^*)) - (1-G(\theta^*))w_1(-\hat{a})(a^*-a(\theta^*)) = 0 \]

where \(\hat{b} \in [b(\theta^*), b^*]\) and \(\hat{a} \in [a(\theta^*), a^*]\). Because \(b^* - b(\theta^*)\) and \(a^* - a(\theta^*)\) are positive, (57) and (67) imply \(G^* > G(\theta^*)\). But this contradicts (66).

Hence \(a(\theta)\) can not have an upward discontinuity. From virtually the same argument it can be established that \(a(\theta)\) can not have a downward discontinuity.

From (67), if \(a(\theta)\) is continuous at \(\theta^*\) then \(b(\theta)\) must be continuous there.

Finally, from (68) \(G\) is also continuous. This analysis has invoked constraint (40) rather than \(G > 0\). However, by straightforward, although tedious, argument one can draw our same conclusions in the limit where \(\epsilon = 0\).

Q.E.D.

Theorem 8 was not stated for \(n=1\) (one buyer), but it is easy to see that the conclusion holds for this case too, although the proof has to be modified somewhat. (In particular, we can no longer establish that \(\lambda(\theta) > 0\) for all \(\theta < 1\). However the modified proof is actually easier because the "feasibility" constraint (41) simplifies to \(G \leq 1\). The one buyer case is not so restrictive as it may appear. Qualitatively it is identical to the case of a multi-unit, multi-buyer market where no buyer wishes to buy more than one unit and there are at least as many units as buyers.
We mentioned in Remark 8.1 that the density condition (53) is crucial to the conclusion that $G$ is nondecreasing in the optimal auction. It is also essential to ensuring that only the downward adjacent constraint (31) is binding among all the incentive constraints. Violations of (53) can lead to other constraints being binding. Moore (1982) drops condition (53) (but strengthens assumption B). He explicitly introduces all the downward constraints (not just (31)) into the control problem and shows that a solution to the revised control problem automatically satisfies all the upper constraints. He then derives many of the same qualitative properties of optimal auctions that we do (excluding, of course, $G' > 0$).

So far in this section, we have confined our attention to deterministic auctions — ones where $\tilde{b}$ and $\tilde{a}$ are deterministic. That this restriction is justified, assuming the hypotheses of Theorem 8, is confirmed by the following result:

**Theorem 9:** Under the hypotheses of Theorem 8, the optimal auction is a deterministic auction.

**Remark 9.1:** The proof of Theorem 9 is long and complicated but would be considerably simpler if we imposed the stronger hypothesis of nonincreasing absolute risk aversion. (See, e.g., Moore (1982)).

**Proof:** Consider a finite approximation $\{\tilde{b}_1, ..., \tilde{b}_m\}$ for the interval $[0,1]$ and a density function $f$ as in the proof of Theorem 8. Let us examine the control problem

$$\max \sum_{i=1}^{m} n \mathbb{E} \left\{ \tilde{b}_i G_i + (1-G_i) \tilde{a}_i \right\} f_i$$

subject to (40), (41), and

$$E \left\{ G_i u(-\tilde{b}_i, \theta_i) + (1-G_i)w(-\tilde{a}_i) \right\} \geq E \left\{ G_{i-1} u(-\tilde{b}_{i-1}, \theta_{i-1}) + (1-G_{i-1})w(-\tilde{a}_{i-1}) \right\}$$

for all $i$, where, for $i=1$, the righthand side of (69) is defined to be zero. We will show that in the solution to this problem $<\tilde{a}_i, \tilde{b}_i>$ is deterministic for all $i$. Thus, at the optimum, the control problem reduces to the
deterministic one of maximizing $nE\left[b_i G_i + (1-G_i) a_i\right] f_i$ subject to (31), (39), (40), and (44). But, as shown in the proof of Theorem 8, the solution to this latter control problem corresponds, in the limit as $n$ tends to infinity, to an optimal auction. Thus, so must the solution to the former control problem.

We proceed by supposing that $G_i$ has been chosen optimally for all $i$ and that $\tilde{a_i}$ and $\tilde{b_i}$ have been chosen optimally for all $i \neq r$. Then (40) and (44) hold and (69) holds for all $i \neq r, r+1$. The optimal pair $<\tilde{a_r}, \tilde{b_r}>$ is therefore the solution of the following control problem

$$\text{Max } \bar{R} = E \left\{ G \tilde{b_r} + (1-G) \tilde{a_r} \right\}$$

subject to

$$E \left\{ G \tilde{u}(-\tilde{b_r}, \theta_{r}) + (1-G) \tilde{w}(-\tilde{a_r}) \right\} \geq E \left\{ G_{r-1} \tilde{u}(-\tilde{b_{r-1}}, \theta_{r}) + (1-G_{r-1}) \tilde{w}(-\tilde{a_{r-1}}) \right\}$$

and

$$E \left\{ G_{r+1} \tilde{u}(-\tilde{b_{r+1}}, \theta_{r+1}) + (1-G_{r+1}) \tilde{w}(-\tilde{a_{r+1}}) \right\} \geq E \left\{ G \tilde{u}(-\tilde{b_r}, \theta_{r+1}) + (1-G) \tilde{w}(-\tilde{a_r}) \right\}.$$ 

Let $\tilde{x} = \tilde{a_r} - \bar{a_r}$ and $\tilde{y} = \tilde{b_r} - \bar{b_r}$ where $\bar{a_r}$ and $\bar{b_r}$ are the expected values of $\tilde{a_r}$ and $\tilde{b_r}$. Next note that the right hand side of the first constraint and the left hand side of the second constraint are independent of $\tilde{a_r}$ and $\tilde{b_r}$. Then, dropping the subscript $r$ on all terms in $G$, $a$ and $b$, we can rewrite the control problem as

$$\text{Max } \bar{R} = G \bar{b} + (1-G) \bar{a}$$

subject to

$$G \bar{u}(-\bar{b}+\bar{y}, \theta) + (1-G) \bar{w}(-\bar{a}+\bar{x}) \geq k_1 \quad : \quad L$$

and

$$G \bar{u}(-\bar{b}+\bar{y}, \theta_{r+1}) + (1-G) \bar{w}(-\bar{a}+\bar{x}) \leq k_2 \quad : \quad M$$

We now show that the optimum is achieved by setting both $\bar{x}$ and $\bar{y} = 0$.

Since the same argument holds for all $r$ this will complete the proof.

First consider the two constraints, (70) and (71) for some non-degenerate random variables $\bar{x}$ and $\bar{y}$. Since $u$ is concave in its first argument and $w$ is concave, the set of pairs $<\bar{a}, \bar{b}>$ satisfying (70) is convex. Suppose $<\bar{a_o}, \bar{b_o}>$
Fig. 1: The Gains to Deterministic Payments

\[ G \bar{b} + (1-G)a = \bar{R}_o \]
lies in this set. Since \( u \) is decreasing in \( \overline{b} \) and \( w \) is decreasing in \( \overline{a} \) any point \( \langle \overline{a}, \overline{b} \rangle \leq \langle \overline{a}_0, \overline{b}_0 \rangle \) is also in the set. Therefore we can depict the boundary of the set, \( L \), as in Figure 1.

From exactly the same arguments the complement of the set of pairs \( \langle \overline{a}, \overline{b} \rangle \) satisfying (71) is convex. Let \( M \) be the boundary of this set. Suppose the two boundary curves intersect at some point \( D = \langle \overline{a}_D, \overline{b}_D \rangle \). Treating (70) and (71) as equalities and differentiating them at this point, we have

\[
-\frac{da}{db} = \frac{G B_{u_1}(-\overline{b} + y, \theta_r)}{(1-C)Ew_1(-\overline{a} + x)} > \frac{G E_{u_1}(-\overline{b} + y, \theta_{r+1})}{(1-C)Ew_1(-\overline{a} + x)} = \frac{da}{db} \bigg|_{M}
\]

where the inequality follows from our assumption B1 that \( u_{12} < 0 \).

Thus, as depicted, the boundary curve \( L \) is strictly steeper than the boundary curve \( M \) at the intersection point \( D \). It follows also that the intersection point is unique.

Next consider constraint (70) with \( \check{x} = 0 \), that is

\[
(72) \quad GEu(-\overline{b} + y, \check{\theta}_r) + (1-C)w(-\overline{a}) \geq \check{k_1} \quad : \quad L'
\]

Since \( w \) is concave \( w(-\overline{a}) > Ew(-\overline{a} + x) \). Therefore the new boundary curve \( L' \) lies strictly above \( L \). Let \( \overline{a}_D' \) be the point on \( L' \) vertically above \( D \). Then we must have

\[
(73) \quad w(-\overline{a}_D') = Ew(-\overline{a}_D' + \check{x}).
\]

But exactly the same argument holds for constraint (71) with \( \check{y} = 0 \).

That is, the boundary curve \( M' \) lies strictly above \( M \). Moreover the point on \( M' \) vertically above \( D \) satisfies (73). Then this is the intersection point of the two boundary curves \( M' \) and \( L' \). Since expected revenue \( \overline{R} \) is strictly increasing in \( \overline{a} \) and \( \overline{b} \) it follows immediately that setting \( \check{x} = 0 \) raises expected revenue.

We now apply a similar argument to show that setting \( \check{y} = 0 \) further raises expected revenue. Define \( L'' \) to be the boundary of the set of pairs \( \langle \overline{a}, \overline{b} \rangle \) satisfying
Since \( u \) is concave in its first argument

\[
\text{(75)} \quad u(-b, \theta_r) - Eu(-b+y, \theta_r)
\]

is positive. Therefore any point on \( L' \) lies strictly inside the set bounded by \( L'' \). Moreover, from (72) and (74), for any given \( \overline{b} \) and points \( \langle \overline{a}', \overline{b} \rangle \) and \( \langle \overline{a}'', \overline{b} \rangle \) on the boundaries of \( L' \) and \( L'' \) we have

\[
\text{(76)} \quad (1-G) [\omega(-\overline{a}') - \omega(-\overline{a}'')] = G [u(-\overline{b}, \theta_r) - Eu(-\overline{b}+y, \theta_r)]
\]

Thus the larger is (75) the larger is the vertical distance between \( L' \) and \( L'' \).

An identical argument establishes that by setting \( y = 0 \) the boundary curve for constraint (71) shifts upwards from \( M' \) to \( M'' \). Moreover, for any given \( \overline{b} \), the size of the upward shift is given by (76) except that \( \theta_{r+1} \) replaces \( \theta_r \). Then the upward shift from \( M' \) to \( M'' \) is smaller than the shift from \( L' \) to \( L'' \) if and only if

\[
\text{(77)} \quad u(-\overline{b}, \theta_{r+1}) - Eu(-\overline{b}+y, \theta_{r+1}) < u(-\overline{b}, \theta_r) - Eu(-\overline{b}+y, \theta_r)
\]

We shall establish below that (77) holds. In particular it holds at \( \overline{b} = \overline{b}_D \). Then the new intersection point \( \overline{D}'' \) cannot lie to the "northwest" of \( \overline{D}' \) and hence there is some feasible point on the boundary of \( L'' \) which yields strictly greater expected revenue.

The above argument is based on the supposition that the boundary curves \( L \) and \( M \) have an intersection point. However, if not, the upward shift in \( L' \) to \( L'' \) immediately guarantees that maximized expected revenue must rise. It therefore remains only to establish (77).

First note that we may write

\[
u(-\overline{b}, \theta) - u(-\overline{b}+y, \theta) = -\int_{0}^{y} u_1(-\overline{b}+z, \theta) dz
\]

\[
= \int_{0}^{y} [u_1(-\overline{b}, \theta) - u_1(-\overline{b}+z, \theta)] dz = y u_1(-\overline{b}, \theta)
\]
Integrating by parts, we can rewrite the integral as

\[ \int_{0}^{y} [u_{1}(-b,\theta) - u_{1}(-b+z,\theta)]dz = \left[ u_{1}(-b,\theta) - u_{1}(-b+z,\theta) \right]_{z=0}^{y} - \int_{0}^{y} u_{11}(-b+z,\theta)(y-z)dz \]

Combining results and noting again that \( E y = 0 \) we have

\[ u(-b,\theta) - Eu(-b+y,\theta) = E y \left[ - \int_{0}^{y} u_{11}(-b+z,\theta)(y-z)dz \right] \]

Then

\[ \frac{\partial}{\partial \theta} \left[ u(-b,\theta) - Eu(-b+y,\theta) \right] = E y \left[ - \int_{0}^{y} u_{112}(-b+z,\theta)(y-z)dz \right] \]

\[ \leq 0 \text{ by Assumption B5 .} \]

Q.E.D.

A restriction like \( u_{112} > 0 \) on the rate at which absolute risk aversion can increase with \( \theta \) is essential for the conclusion that the optimal auction is deterministic. To see that randomization may pay if \( u_{112} \) is negative suppose that \( \theta \) can take on two values \( \theta_1 \) and \( \theta_2 \) \( (\theta_2 > \theta_1) \) where

\[ u(-t,\theta_1) = \theta_1 - t \]

\[ u(-t,\theta_2) = \log(1+\theta_2 - t) . \]

and

\[ w(-t) = -t . \]

Since \( u \) is risk neutral for \( \theta=\theta_1 \) and risk averse for \( \theta=\theta_2 \), risk aversion is increasing with \( \theta \). Suppose that there is just one buyer. Consider a scheme in which the seller offers to sell the item (with probability one) if the buyer accepts either of the following two payments schedules:
\[
\begin{align*}
\bar{b}_1 &= \begin{cases} 
0, & \text{with probability } 1 - \frac{\theta_1}{1+\theta_2} \\
\frac{\theta_1}{1+\theta_2}, & \text{with probability } \frac{\theta_1}{1+\theta_2}
\end{cases} \\
b_2 &= \theta_2
\end{align*}
\]

It is readily confirmed that if \( \theta = \theta_1 \), the buyer opts for \( \bar{b}_1 \) and that if \( \theta = \theta_2 \) the buyer prefers \( b_2 \). Moreover, given these choices, the scheme extracts all buyer surplus. Thus the scheme is certainly optimal. Furthermore, it is evident that no scheme where \( \bar{b}_1 \) is deterministic can extract all surplus. Hence randomization is essential.

Theorem 9 establishes that the first order conditions \( M \) are necessary for a maximum. They need not be sufficient, however, because, although the objective function is concave and the constraints (39) - (41) are convex, the incentive constraint (36) (or (31) in the finite problem) is nonconvex. Indeed, without that nonconvexity, establishing that the optimal \( \bar{a} \) and \( \bar{b} \) are deterministic would be trivial and would not require any assumptions about how risk aversion changes in \( \theta \) (only that the buyer actually be risk averse); we could simply replace \( \bar{a} \) and \( \bar{b} \) by their certainty equivalents.

4. Properties of Optimal Auctions

The proof of Theorem 8, in addition to demonstrating that designing an optimal auction reduces to a conceptually simple and standard control problem, establishes and suggests certain interesting properties of optimal auctions. We now present some of these properties explicitly.

Theorem 10: Under the hypotheses of Theorem 8, the probability of winning and the amount a buyer pays if he wins in an optimal auction are increasing functions of his eagerness to buy, if the probability of his winning is positive. That is, \( b' > 0 \) and \( G' > 0 \) if the constraint \( G > 0 \) is not binding.

Proof: Established in the proof of Theorem 8.

Theorem 11: Under the hypotheses of Theorem 8, the marginal utility of income in an optimal auction is lower when a buyer wins than when he loses.
That is, \( u_1(-b(\theta), \theta) < w_1(-a(\theta)) \), if \( \theta^0 < \theta < 1 \),
where \( \theta^0 = \inf \{ \theta \mid G(\theta) > 0 \} \).

**Proof:** For \( \theta \) such that \( 0 < G(\theta) < 1 \), the result follows from (56).
Equation (55) holds for \( \theta \in (\theta^0, 1] \). By continuity, it holds at \( \theta = \theta^0 \) as well.
Because \( G(\theta^0) < 1 \), \( \lambda(\theta^0) > 0 \). Thus, since \( u_{12} < 0 \) and \( \lambda'(\theta^0) < 0 \), \( u_1 < w_1 \) at \( \theta = \theta^0 \).

Q.E.D.

Theorem 11 establishes that, under the hypotheses of Theorem 8, it is desirable for the seller to make all buyers, except the most conceivably eager (\( \theta = 1 \)) and those who have no chance of winning (\( G = 0 \)), bear risk in order to exploit this risk for screening. The result that an optimal incentive scheme introduces "inefficiency" for all values of the unknown parameter \( \theta \) but one, is a very general principle in the incentives literature. In the optimal income tax literature (see Mirrlees (1971)), for example, it implies that all but the very ablest agent should face a positive marginal tax rate.

The main interest of Theorem 11, therefore, is its description of the nature of the inefficiency, namely, that \( u_1 < w_1 \). The direction of the inequality \( u_1 < w_1 \) is due to the hypothesis B1, i.e., \( u_{12} < 0 \). If B1 holds and \( u_1(-b(\theta), \theta) < w_1(-a(\theta)) \) for given \( \theta \), then the difference between \( u_1(-b(\theta), \theta) \) and \( w_1(-a(\theta)) \) is greater for \( \hat{\theta} > \theta \) than for \( \theta = \hat{\theta} \). In other words, by having a \( \theta \)-buyer bear risk, the seller can relax the incentive constraint (36) by making \( (G(\theta), b(\theta), a(\theta)) \) appear still riskier for buyers with parameters greater than \( \theta \).

From this reasoning, it is evident that when \( u_{12} = 0 \), i.e., when preferences take the form
\[
\begin{align*}
u(-t, \theta) &= \theta - v(t) \\
w(-t) &= -v(t),
\end{align*}
\]
there is no value to buyers bearing risk. It is easy to show that for such preferences, the optimal auction entails full insurance. This is to be contrasted with Theorem 6 which demonstrates that, for Case 1 preferences, a perfect insurance auction is not only suboptimal but inferior to the high bid auction.
Corollary: Under the hypotheses of Theorem 8, a buyer is strictly better off in an optimal auction when he wins than when he loses. That is, \( u(-b(e), e) > w(-a(e)) \) for \( e^0 < 0 < 1 \), where \( e^0 = \inf(\theta \mid G(\theta) > 0) \).

Proof: Follows directly from Theorem 11 and B4. Q.E.D.

In contrast with Theorem 11, the next result shows that the most eager buyer possible (\( \theta = 1 \)) should be perfectly insured.

Theorem 12: Under the hypotheses of Theorem 8, the most conceivably eager buyer is perfectly insured against losing in an optimal auction. That is,

\[ u_1(-b(1), 1) = w_1(-a(1)). \]

Remark 12.1 For Case 1 preferences with constant absolute risk aversion, this result is established by Matthews (1982).

Proof: Follows directly from (49) and (55). Q.E.D.

Theorem 12 is in general false when \( n = 1 \), as Matthews (1982) illustrates with Case 1 preferences and constant absolute risk aversion. Intuitively, a high bidder in a multi-buyer auction must be insured against losing because there may always be a higher bidder. But in a one-buyer auction, a sufficiently high bidder will have a probability one chance of winning (see Theorem 17).

Next we consider the behavior of \( a \), the fee a buyer pays if he loses. We observe that for low values of \( \theta \) where \( G \) is positive, \( a \) is positive and increasing, whereas \( a \) is negative for high \( \theta \)'s. Since, from Theorem 10, \( b \), the buyer's "bid", is increasing in \( \theta \), we conclude that if a buyer bids low, he is penalized for losing in an optimal auction but is compensated for losing if he bids high.
Theorem 13: Let \((G,b,a)\) be an optimal auction. Let \(\theta^0\) be the infimum of all \(\theta\)'s such that the constraint (39), \(G > 0\), is not binding. Then under the hypotheses of Theorem 8 \(a(\theta) > 0\) and \(a'(\theta) > 0\) for \(\theta(>\theta^0)\) sufficiently close to \(\theta^0\). If, in addition, \(u_1 = w_1\) implies \(u = w\), then for \(\theta\) sufficiently close to 1, \(a(\theta) < 0\).

Remark 13.1: The first but not the second assertion of Theorem 13 holds for one-buyer auctions. The hypothesis "\(u_1 = w_1\) implies \(u = w\)" clearly holds for Case 1 preferences and, under constant absolute risk aversion, for those of Case 3. (See (5) and the subsequent argument. The inequality in Lemma 1 holds with equality under constant absolute risk aversion).

Remark 13.2: One simple way of instituting a positive \(a\) - so that losers as well as winners pay - is to introduce a nonrefundable entry fee. For more on the desirability of entry fees see Maskin and Riley (1980).

Proof: We first observe that the nonparticipation constraint (39) must be binding, otherwise, we could increase \(a\) without altering \(G\) and \(b\) and augment the seller's expected revenue.

First suppose that \(G(\theta^0) = 0\). Then (39) implies that

\[(78) \quad a(\theta^0) = 0\]

since \(w(0) = 0\). If the feasibility constraint (41) is not binding at \(\theta^0\), then, from the argument in the proof of Theorem 8, \(a'(\theta) > 0\) for all \(\theta(>\theta^0)\) sufficiently close to \(\theta^0\). In view of (78), this implies that \(a(\theta) > 0\) for all \(\theta(>\theta^0)\) sufficiently close to \(\theta^0\). If (41) is binding at \(\theta^0\), then locally \(G = F^{n-1}\), and so \(G'(\theta^0) > 0\). Since (36) holds with equality,

\[(79) \quad a' = \frac{(u - w)G'}{w_1} \quad \text{at } \theta = \theta^0.\]

Now from the corollary to Theorem 11, \(u > w\). Therefore, (78) and (79) imply that \(a(\theta)\) and \(a'(\theta)\) are positive for \(\theta(>\theta^0)\) close to \(\theta^0\).
Next suppose that $G(\theta^0) > 0$. Then $\theta^0 = \theta_0$, from the definition of $\theta^0$. Because, as already was observed, $u > w$ at $\theta = \theta^0$, the equality of the nonparticipation constraint implies that $w(-a(\theta^0)) < 0$, and so $a(\theta^0) > 0$. Because (41) cannot be binding at $\theta^0$, $a'(\theta^0) > 0$.

From Theorem 12, $u_1(-b(1),1) = w_1(-a(1))$. If $u_1 = w_1$ implies $u = w$, then $w(-a(1)) > 0$. Therefore $a(1) < 0$, and so $a(\theta) < 0$ for $\theta$ near 1.

Q.E.D.

Theorem 14: Suppose that condition (53) is satisfied. If the preferences of Case 1 exhibit nonincreasing absolute risk aversion, a buyer pays at least as much if he wins as if he loses in an optimal auction. That is, $b \geq a$, and the inequality is strict if risk aversion is strictly decreasing.

Proof: If we substitute for $\lambda'$ and $\lambda$ using (54) and (55) and divide by $F'$, (50) becomes

\[ (80) \quad b - a = \left( \frac{u_1 - w_1}{u_1 w_1} \right) u_2 - \left( \frac{u - w}{w_1} \right) + \int_0^\theta n(t) dt \]

if $G > 0$. Because preferences take the Case 1 form, the first two terms on the right-hand side of (80) can be written as

\[ (81) \quad \frac{U'(\theta - b(\theta)) - U'(a(\theta))}{U'(\theta - b(\theta))U'(a(\theta))} U'(\theta - b) - \left[ \frac{U(\theta - b(\theta)) - U(-a(\theta))}{U'(a(\theta))} \right] \]

Rearranged, (81) becomes

\[ (82) \quad \frac{U'(\theta - b(\theta))(U(\theta - b(\theta)) - U(-a(\theta)))}{U'(\theta - b(\theta))U'(a(\theta))} \left[ \frac{U'(\theta - b) - U'(a(\theta))}{U'(\theta - b(\theta))U'(a(\theta))} \right] \]

We will show that the bracketed factor in (82) is negative for decreasing absolute risk aversion, implying that (81) is positive. From the corollary to Theorem 11, $U(\theta - b) > U(-a)$, implying that $\theta - b(\theta) > -a(\theta)$. Thus we can show that the bracketed factor is negative by establishing that

\[ (83) \quad \frac{U'(x_2) - U'(x_1)}{U(x_2) - U(x_1)} < 0 \quad \frac{U''(x_2)}{U'(x_1)} < 0 \]
when \( x_2 > x_1 \). Since \( v = U(x) \) is increasing, define \( x = U^{-1}(v) \), \( x_1 = U^{-1}(v_1) \), and \( x_2 = U^{-1}(v_2) \). Also take
\[
g(v) = U'(U^{-1}(v)).
\]

Arguing exactly as in the proof of Lemma 1, we know that
\[
g'(v) = \frac{U''(x)}{U'(x)}
\]
and that \( g'' \) is positive if absolute risk aversion is decreasing. But if \( g'' > 0 \), then
\[
\frac{g(v_2) - g(v_1)}{v_2 - v_1} < g'(v_2),
\]
and so (83) holds. Thus (81) is positive. Because
\[
\int_0^\theta \eta(t)dt > 0,
\]
(80) implies that \( b > a \).

**Q.E.D.

**Corollary:** Under the hypotheses of Theorem 14, \( b > 0 \) if \( G > 0 \).

**Proof:** From Theorem 13, \( a(\theta^0) \geq 0 \), where \( \theta^0 \) is the infimum of all \( \theta \) for which \( G(\theta) > 0 \) is not binding. From Theorem 14, \( b(\theta^0) > 0 \). Thus, from Theorem 10, \( b(\theta) > 0 \) for all \( \theta > \theta^0 \).

**Q.E.D.

We next study the expected revenue generated from a given buyer.

**Theorem 15:** Under the hypotheses of Theorem 8, the expected revenue from a given buyer is a nondecreasing function of his willingness to buy. That is,
\[
R(\theta) = G(\theta)b(\theta) + (1-G(\theta))a(\theta)
\]
is a nondecreasing function of \( \theta \). Furthermore, under the additional hypotheses of Theorem 14, expected revenue is a strictly increasing function if the probability of winning is positive but less than 1. That is, \( R'(\theta) \) for \( 0 < G < 1 \).

**Proof:** In a finite approximation, let \( R_1 = G_1b_1 + (1-G_1)a_1 \). For a sufficiently fine approximation, we know, from the proof of Theorem 8, that if \( <G_1,b_1,a_1> \)
is the solution to the seller's problem, then \( G_i \) and \( b_i \) are nondecreasing in \( i \) and (31) holds with equality for all \( i \). If, for some \( j \), \( R_j > R_{j+1} \), define the alternative auction \( <\hat{G}_i, \hat{b}_i, \hat{a}_i> \) so that

\[
(\hat{G}_i, \hat{b}_i, \hat{a}_i) = \begin{cases} 
(G_j, b_j, a_j), & \text{if } i = j + 1 \\
(G_i, b_i, a_i), & \text{otherwise}
\end{cases}
\]

Clearly \( <\hat{G}_i, \hat{b}_i, \hat{a}_i> \) satisfies (39) and (40). It satisfies (44) since \( \hat{G}_i < G_i \) for all \( i \). Because by hypothesis \( <G_i, b_i, a_i> \) satisfies (31) with equality, Lemma 2 implies that \( <\hat{G}_i, \hat{b}_i, \hat{a}_i> \) satisfies (31). Thus \( <\hat{G}_i, \hat{b}_i, \hat{a}_i> \) satisfies all the constraints of the buyer's maximization. Furthermore, it generates greater expected revenue than \( <G_i, b_i, a_i> \), since \( \hat{R}_j > R_{j+1} \) and \( \hat{R}_i = R_i \) for \( i \neq j+1 \), a contradiction of the optimality of \( <G_i, b_i, a_i> \). Thus \( R_i \) must be nondecreasing everywhere, and, in the limit, \( R(\theta) \) is nondecreasing.

Next, assume the hypotheses of Theorem 14. We have

\[
R'(\theta) = Gb' + (1 - G)a' + (b - a)G'
\]

Since (31) holds with equality,

\[
GU'(\theta - b)b' + (1 - G)U'(-a)a' = (U(\theta - b) - U(-a))G'
\]

Dividing through by \( U'(-a) \), we have

\[
(84) \quad G \frac{U'(\theta - b)b' + (1 - G)a'}{U'(-a)} = \frac{(U(\theta - b) - U(-a))G'}{U'(-a)}
\]

From Theorem 11, \( U'(\theta - b) < U'(-a) \) for \( \theta < 1 \). From Theorem 10, \( b' > 0 \). Thus, from (84)

\[
G b' + (1 - G)a' > \frac{(U(\theta - b) - U(-a))G'}{U'(-a)}
\]

From Theorem 10, \( G' > 0 \). From the Corollary to Theorem 11, \( U(\theta - b) - U(-a) > 0 \). Thus,

\[
G b' + (1 - G)a' > 0
\]
From Theorems 10 and 14,

\[(b - a)G' > 0 .\]

Hence \(R' > 0 .\) 

Q.E.D.

Theorem 15 has analogues in many other "monopoly" problems. In the optimal tax literature, for example, its counterpart is the property that taxes should be increasing in individuals' skill (see, e.g., Mirrlees (1971)). Theorem 15 is less obvious than many of these counterparts, however, because of the feasibility constraint (37) and because there is a two-dimensional vector of payments \((b(\theta), a(\theta))\) rather a single function relating \(\theta\) to a payment.

We next demonstrate that, at least for Case 1 preferences exhibiting nonincreasing absolute risk aversion, the seller will find it advantageous to set a positive reserve price — that is, he will refuse to sell to a buyer with a \(\theta\) less than some positive level \(\theta^0 .\)

**Theorem 16:** Under the hypotheses of Theorem 14, there exists \(\theta^0 > 0\) such that \(G(\theta) = 0\) for \(\theta < \theta^0 .\)

**Proof:** Suppose the theorem is false, and \(G(\theta) > 0\) for all \(\theta > 0 .\) Then \(a(0) = 0\) in (50). From the Corollary to Theorem 11,

\[\theta - b > - a \quad \text{for } 0 < G < 1 .\]

Therefore, letting \(\theta\) tend to zero, we have

\[0 \geq b(0) - a(0) .\]

From Theorem 14,
\[ b(0) - a(0) \geq 0 , \]

and so

\[ b(0) - a(0) = 0 . \]

Equation (50) at \( \theta = 0 \) thus reads

\[ (85) \quad -\lambda u'(-b(0)) = 0 . \]

But \( \lambda(0) > 0 \). Thus (85) is impossible.

Q.E.D.

Theorem 16 applies as well to the case of a single buyer. In this case, we can establish a corresponding result for high values of \( \theta \); namely, that for sufficiently high \( \theta \), the probability of winning is one.

**Theorem 17:** Under the hypotheses of Theorem 14, if \( n=1 \), then there exists \( \theta^* < 1 \) such that, for all \( \theta > \theta^* \), \( G(\theta) = 1 \) in an optimal auction.

**Proof:** From the argument of Theorem 8,

\[ (50) \quad (b-a)F' - \lambda u_2 - \lambda'(u-w) + a - \int_0^\theta \eta(t)dt F' = 0 \]

Suppose \( G < 1 \), for all \( \theta < 1 \). Then

\[ (86) \quad \int_0^1 \eta(t)dt = 0 . \]

Recall that

\[ (49) \quad \lambda(1) = 0 . \]

Therefore, since \( \lambda = \frac{u_1-w_1}{u_1} F' \) for \( 0 < G < 1 \) (see (55)), \( u_1 + w_1 \) as \( \theta \to 1 \), and so

\[ (87) \quad u + w , \text{ as } \theta \to 1 . \]

From (87), we conclude that

\[ 1 - b(1) = -a(1) . \]

and thus

\[ (88) \quad b(1) - a(1) > 0 . \]
But from (49) and (86) - (88), the left-hand side of (50) is positive for \( \theta = 1 \), an impossibility.

Q.E.D.

Theorems 16 and 17 and the continuity of \( G \) permit us to conclude that, at least for Case 1 preferences with nonincreasing absolute risk aversion, an optimal auction divides the unit interval into three nondegenerate subintervals: the lowest interval has \( G = 0 \); the middle interval has \( 0 < G < 1 \); the upper interval has \( G = 1 \). The middle interval is perhaps the most interesting. We have taken \( G(\theta) \) to be the probability of winning. In the one buyer case we could alternatively interpret \( G(\theta) \) as the probability that the item does not "fall apart", i.e., the "quality" of the item. The non-degeneracy of the middle interval then implies that there are values of \( \theta \) for which the seller will offer less than top quality, even though quality is costless to provide. This result hinges crucially on risk aversion. As Riley and Zeckhauser (1983) show, the optimal \( G \) equals either 0 or 1 for all values of \( \theta \) if the buyer is risk neutral.

5. Concluding Remarks

We have been most concerned in this paper with elucidating the interplay between insurance and screening considerations in models of incomplete information with risk averse agents. We have studied auctions in particular, but as they are formally very similar to a variety of other monopoly problems, the principles that emerge all apply elsewhere.

We have discussed the roles of most of our assumptions, but it is worth returning to two of them. First, by assuming that the seller maximizes expected revenue, we implicitly suppose that he is risk neutral. For the case of a single buyer this assumption makes no qualitative difference. Indeed, for this case, we could have presented Theorems 8 - 11 and 13 - 17 for a risk averse seller with only slightly modified proofs. The assumption of seller risk neutrality is, however, crucial to our methods for two or more buyers. Risk neutrality means that the seller's payoff depends on the
underlying probabilities, $H_i$, only through the marginal distribution $G$. Thus we can work directly with $G$ rather than with the analytically more difficult $H_i$'s.

For much the same reason, the hypothesis that the $\theta_i$'s are distributed independently is highly simplifying. Indeed, without independence, a buyer's marginal probability of winning depends not just on his bidding behavior but on his parameter. Thus, again, we are forced to work with the $H_i$ functions. It is easy to see that the seller can exploit any correlation among the $\theta_i$'s. To take an extreme example, suppose that the value of $\theta_i$ were the same for all buyers. Even if the seller did not know this value, he could extract all surplus from buyers by operating a second bid auction.

As Myerson (1981) suggests, it is possible, even with imperfect (but nonzero) correlation, to construct auctions one of whose equilibria extracts all surplus (at least, if $\theta$ can take on only discrete values) when buyers are risk neutral. Maskin and Riley (1980b) show that at least in the case where $\theta$ assumes only two values, such auctions can be constructed with a unique equilibrium. How this result fares for more general distributions, and what optimal auctions look like with correlation when buyers are risk averse remain conjectural.
FOOTNOTES

1. Matthews (1982) studies much the same problem when buyers' preferences belong to our Case 1 and satisfy constant absolute risk aversion. Moore (1982) establishes many of our results without invoking the density condition (53) but under somewhat stronger conditions on preferences.

2. This last result has been established in special cases by various others, including Butters (1975), Holt (1980), Matthews (1979), and Riley and Samuelson (1981).

3. The assumption of identical distributions is inessential; however, the independence assumption is crucial to the methods and results of this paper (see Section 5).

4. Our formulation in terms of u and w assumes that buyers' preferences over money are identical in the event they lose. This assumption is inessential for the results, but it somewhat simplifies the analysis. The formulation also implies that buyers' appraisals of the item are not influenced by what other buyers think. This simplification is also not crucial. For treatment that allows for interdependence of tastes, see the predecessor of this paper, Maskin and Riley (1980a).

5. Let \( z = 0 \) represent no purchase and \( z = 1 \) represent purchase of the object. Let \( x \) represent income. Then the CES utility function

\[
(\theta z^\alpha + x^\alpha)^{1/\beta}, \quad 0 < \alpha < \beta
\]

is a simple example of Case 2 preferences.

6. In Case 3

\[
u_{12}(-t, \theta) = \int U'(v - t)K_{12}(v|\theta) dv = U'K_2\left|_\tilde{v} \right. - \int U''K_2 dv.
\]

But \( K(v|\theta) = 0 \) for all \( \theta \) and \( K(\tilde{v}|\theta) = 1 \) for all \( \theta \). Therefore the first term is zero. Also \( K_2 \leq 0 \) by assumption. Thus \( u_{12} < 0 \).

7. In Case 3

\[
u_{112}(-t, \theta) = -\int U''K_2 dv > 0
\]

from the same kind of integration by parts as in Footnote 6.

8. This term is due to Myerson (1979). For an informal discussion of the application of this principle to auction design see Riley and Samuelson (1981).

9. This term is borrowed from Dasgupta, Hammond and Maskin (1979).

10. A family of functions \( \{f_1, \ldots, f_n\} \), where each function \( f_i \) has \( n \) arguments, is permutation symmetric if, for all \( i \) and \( j \) and all vectors \( x \) and \( y \) in the domains of \( f_i \) and \( f_j \),

\[
f_i(x) = f_j(y)
\]

if \( x_i = y_j \), \( x_j = y_i \), and \( \forall k \neq i, j \quad x_k = y_k \).

To see that it suffices to consider permutation symmetric families of
H_1's, \alpha_i's and \theta_i's, consider the case of two buyers. Suppose an asymmetric auction A_1 were optimal. By symmetry, the auction A_2 obtained from A_1 by reversing the roles of the buyers is also optimal. But then the symmetric auction A_12 obtained by flipping a coin to decide which of the two auctions, A_1 or A_2, to play is also optimal. The argument generalizes to more than two buyers.

11. This description of the English auction is ill-specified because rational buyers may wish to raise their bids by infinitesimals. This problem can be avoided either by postulating a minimum quantity by which bids must be raised, or by adopting the following continuous price formulation. Suppose that the seller quotes a price that rises continuously over time. At any instant, a buyer can choose either to stay in or to drop out (forever). The winner is the last buyer to remain (again, ties are broken by a randomizing device), and he pays the price prevailing at the time the penultimate buyer drops out. Losers, as usual, pay nothing.

12. Actually, Vickrey did not treat equivalence for the case where Bayesian equilibrium is the solution concept (he worked instead with dominant strategies). For such a treatment see Remark 3.2 following Theorem 3.

13. For joint density functions f(\theta_1, \ldots, \theta_n) which are twice differentiable and non-zero for all \theta_i \in [0,1], i = 1, \ldots, n, the n variables are affiliated if

\[ \frac{\partial^2}{\partial \theta_i \partial \theta_j} \log f(\theta_1, \ldots, \theta_n) \geq 0 \text{ for all } i, j = 1, \ldots, n. \]

14. Even in the case n = 2, it is not necessary to single out any particular equilibrium in the second bid auction since, as one can easily confirm, all asymmetric equilibria are dominated, from the seller's perspective, by the symmetric equilibrium.

15. That these functions when chosen optimally are piecewise differentiable, given the differentiability of \( u \) and \( F \), is a standard result from control theory.

16. If a buyer with parameter value \( \theta \) chooses \( x \) his expected utility from the auction \( \langle G, b, a \rangle \) is

\[ E(x, \theta) = G(x) \left[ \theta - v(b(x)) \right] - (1 - G(x))v(a(x)) \]

Since \( v \) is strictly convex there exists \( \delta(x) > 0 \) (with strict inequality whenever \( a(x) \neq b(x) \) and \( 0 < G(x) < 1 \)) such that

\[ E(x, \theta) = G(x) \left[ \theta - v(c(x) + \delta(x)) \right] - (1 - G(x))v(c(x) + \delta(x)) \]

where \( c(x) = G(x)b(x) + (1 - G(x))a(x) \).

Thus expected revenue can be increased whenever \( b \neq a \).
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Theorem 7: Suppose that \( G(s) \), the probability of winning with parameter equal to \( s \), is piecewise differentiable and nondecreasing. A necessary condition for there to exist a permutation symmetric family of probability functions \( H_j(x) \), \( j = 1, \ldots, n \), satisfying \( \sum_{j} H_j < 1 \) such that

\[
G(s) = \int_{x_{-1}} H_i(s, x_{-1}) \, dx_{-1} \quad \text{for all } s
\]

is

\[
\frac{1}{n} \int G(s) \, dF(s) \leq \frac{1}{n} \int F^{n-1}(s) \, dF(s) \quad \text{for all } 0 < y < 1.
\]

Moreover, if \( G(s) \) is a step function with finitely many steps, (37) is sufficient.

We prove Theorem 7 in four steps.

We first show that we can eliminate \( F \) from the statement of the problem.

Lemma A1:

Suppose we can establish that, for all nonnegative and nondecreasing functions \( G(y) \) on \([0,1]\), if there exist probability functions \( H_1, \ldots, H_n \) satisfying the symmetry condition,

(i) \( H_i(x) = H_j(x') \), if \( x'_i = x_j \), \( x'_j = x_i \), and \( x'_k = x_k \), \( k \neq i, j \)

and the feasibility condition

(ii) \( \sum_{i=1}^{n} H_i < 1 \),

such that

(iii) \( G(s) = \int_{x_{-1}} H_i(s, x_{-1}) \, dx_{-1} \),

then

(iv) \( \int (G(s) - s^{n-1}) \, ds < 0 \), for all \( y \in [0,1] \).

Suppose, furthermore, that the converse holds if \( G \) is a finite step function.

Then Theorem 7 must hold.
Proof:

Since \( F \) is a continuous strictly increasing function we can define
\[ \theta_i = F^{-1}(x_i), \]
a strictly increasing function from \([0,1]\) to \([0,1]\).

For any \( H_i(\theta) \) we can define
\[ H_i(x) = \tilde{H}_i(F^{-1}(x_1), \ldots F^{-1}(x_n)). \]

Then the \( H_i \)'s satisfy conditions (i) and (ii) if and only if the \( \tilde{H}_i \)'s do.
Similarly, for any \( G(t) \), we can define \( G(x_i) = \tilde{G}(F^{-1}(x_i)) \). Then \( G \) is nondecreasing if and only if \( \tilde{G} \) is, and \( G \) satisfies (iii) and (iv) if and only if \( \tilde{G} \) satisfies
\[
\frac{1}{\theta} \int (G(t) - F^{n-1}(t)) \, dF(t) \leq 0, \quad 0 \leq \theta \leq 1
\]
and
\[
\tilde{G}(\theta_i) = \int \tilde{H}_i(\theta_i, \theta_i, \ldots) \, dF(\theta_i).
\]

Q.E.D.

We next establish the theorem in one direction.

Lemma A2:

If the probability functions \( H_i \), \( i = 1, \ldots, n \), satisfy (i) and (ii) and if \( G \) is defined by (iii) then (iv) is satisfied.

Proof:

Let \( X_r = \{ x \mid x_1 \in [y,1], i \leq r; x_1 \in [0,y), \text{otherwise} \} \).

Over \( X_r \) the symmetry of \( H_i \) implies that
\[
\int_{H_1} dx = \ldots = \int_{H_r} dx = \frac{1}{r} \int_{X_r} \sum_{i=1}^{r} H_i \, dx
\]
\[ x \in X_r, \quad x \in X_r, \quad x \in X_r, \quad i = 1, \ldots, r \]

Then, since \( \sum_{i=1}^{r} H_i \leq 1 \)
\[
\int_{H_i} dx \leq \frac{1}{r} (1-y)^{r-1} y^{n-r}, \quad \text{for } i = 1, \ldots, r.
\]
Moreover there are \( \binom{n-1}{r-1} \) ways to choose exactly \( r-1 \) components of \( (x_2, \ldots, x_n) \) to lie in \( [y, 1] \). Thus by the symmetry of \( H_1, \ldots, H_n \),

\[
\frac{1}{n} \sum_{r=1}^{n} (\frac{n}{r}) (1-y)^r n^{-r}
\]

Thus by the symmetry of \( H_1, \ldots, H_n \),

\[
\frac{1}{n} \sum_{r=1}^{n} (\frac{n}{r}) (1-y)^r n^{-r}
\]

Next, we establish the theorem in the other direction for the case where

(iv) holds with equality at \( y=0 \).

Lemma A3

For any nondecreasing finite step function \( G(s) \) satisfying the integral constraint (iv) for all \( y \) and with equality at \( y=0 \) there exist probability functions \( H_i(x), i=1, \ldots, n \), satisfying the symmetry and feasibility conditions (i) and (ii) such that

(iii) \( G(s) = \int H_i(s,x) \, ds \).

Proof:

First note that if the \( H_i(x) \)'s are symmetric in the sense of (i) and satisfy (ii) then

\[
\frac{1}{n} \sum_{j=1}^{n} H_j(x) = \frac{1}{n} \int H_j(x) \, dx
\]

Also since \( G(s) \) satisfies the integral constraint (iv) with equality at \( y=0 \),

\[
\frac{1}{n} \int G(s) \, ds = \frac{1}{s^{n-1}} \left|_{s=0}^{s=1} \right. = \frac{1}{n}
\]
It follows that the probability functions $H_j(x)$ must satisfy the adding up condition,

$$
\sum_{j=1}^{n} H_j(x) = 1.
$$

If $G$ is a finite step function satisfying the hypotheses of the lemma, we can write

$$(vi) \quad G(s) = G_i, \quad y_i \leq s < y_{i+1}, \ i=1\ldots, m$$

where $y_1 = 0$ and $y_{m+1} = 1$.

From (vi),

$$
G_i y_2 = \int_0^{y_2} G(s) ds = \int_0^{y_2} s^{n-1} ds = G(s) ds,
$$

where the last equality is just the requirement that (iv) should hold with equality at $y = 0$.

Inequality (iv) also implies that

$$
\int_0^{y_2} G(s) ds \leq \int_0^{y_2} s^{n-1} ds.
$$

Therefore $G_i$ must satisfy the constraint

$$
(vii) \quad G_i \leq \frac{1}{y_2} \int_0^{y_2} s^{n-1} ds = \frac{1}{y_2} y_2^{n-1}.
$$

Moreover, using (iv) once more

$$
G_m (1-y_m) = \int_{y_m}^{1} G(s) ds \leq \int_{y_m}^{1} s^{n-1} ds = \frac{1}{y_m} (1-y_m^n).
$$

Hence $G_m$ must satisfy the constraint
(viii) \[ G_m = \frac{1}{n} \frac{(1-y_m^n)}{1-y_m} \]

Next define \( H_1(x) \) to be the probability of winning in a second bid auction modified so that, for all \( i \), all bids in the interval \([y_i, y_{i+1})\) are treated as equal. Then,

\[
H_1(x) = \begin{cases} 
\frac{1}{1+c}, & \text{if for some } i, y_i < x_j < \max_{j=1, \ldots, n} \{x_j\} < y_{i+1} \\
& \text{where } c \text{ is the number of components of } x_{i-1} \\
& \text{which are elements of } [y_i, y_{i+1}) \\
0, & \text{otherwise}
\end{cases}
\]

We shall find it useful to modify this probability function in the following manner. Define

\[
X_r = \{x \mid y_{r-1} < x < \max_{j=1, \ldots, n} \{x_j\} < y_{r+1} \text{ and at least one component of } x_{i-1} \text{ is an element of } [y_{r-1}, y_{r+1}) \}.
\]

For \( p = (p_1, \ldots, p_{n-1}) \in P = \{p \mid 0 \leq p_i \leq \frac{1}{1+c} \text{ for all } i\} \), take

\[
H_1(x) = \begin{cases} 
\bar{H}_1(x) ; x \notin X_r \\
\frac{1}{1+c} ; x_1 \in [y_{r-1}, y_r) , \text{ c components of } x_{i-1} \text{ are elements of } [y_{r-1}, y_r) \text{ and no component of } x_{i-1} \text{ is an element of } [y_r, y_{r+1}) , c \geq 1.
\end{cases}
\]

\[
\bar{H}_1(x) = \begin{cases} 
\frac{1}{1+c} - p_c ; x_1 \in [y_{r-1}, y_r) , \text{ c components of } x_{i-1} \text{ are elements of } [y_{r-1}, y_r) \\
& \text{and d components of } x_{i-1} \text{ are elements of } [y_r, y_{r+1}) , d \geq 1.
\end{cases}
\]

\[
\frac{1}{1+c} + \left(\frac{c-d}{1+d}\right)p_c ; x_1 \in [y_r, y_{r+1}) , \text{ c components of } x_{i-1} \text{ are elements of } [y_r, y_{r+1}) \\
& \text{and d components of } x_{i-1} \text{ are elements of } [y_r, y_{r+1}).
\]
It is a straightforward although tedious exercise to confirm that if \( H^r_j(x), j = 2, \ldots, n \) are defined symmetrically, then for any choice of \( p = (p_1, \ldots, p_{n-1}) \in P \), the \( H^r_j(x) \) are non-negative functions satisfying both the symmetry condition, (i), and the adding up condition, (v). With these preliminaries completed we now suppose the lemma to be true for any \((m-1)\)-step function and show that it must then also hold for any \( m \)-step function. Since it holds trivially for any 1-step function (set \( H^r_j(x) = 1/n \)) this will establish the lemma.

Consider any nondecreasing \( m \)-step function \( G(s) \) defined by (vi).

We can, in effect, delete the \( r \)th step by defining

\[
G^r(s) = \begin{cases} 
G_1, & y_1 \leq s < y_{i+1}, \; i = 1, \ldots, r-2 \\
\frac{y_r-y_{r-1}}{y_{r+1}-y_{r-1}} G_{r-1} + \frac{y_{r+1}-y_r}{y_{r+1}-y_{r-1}} G_r, & y_{r-1} \leq s < y_{r+1} \\
G_1, & y_1 \leq s < y_{i+1}, \; i = r+1, \ldots, m.
\end{cases}
\]

Since \( G(s) \) is nondecreasing so is \( G^r(s) \). By inductive hypothesis there exist probability functions \( H^r_j(x) \) such that (i), (iii) and (v) are satisfied.

Define

\[
\tilde{H}^r_j(x) = \begin{cases} 
H^r_j(x), & x \notin X_r \\
H^r_j(x), & x \in X_r
\end{cases}
\]

for some \( p = (p_1, \ldots, p_{n-1}) \in P \).

We will show that there exist choices of \( p \) and \( r \) such that

\[
G^r(s) = \int \tilde{H}^r_j(y_i, x_{i-1})dx_{i-1} \quad i = 1, \ldots, m+1
\]

Notice that for any choice of \( p \) and \( r \), (xiv) holds for all \( i \neq r-1, r \) by definition of \( \tilde{H}^r_j(x) \) and \( H^r_j(x) \). Also, since both the \( H^r_j(x) \) and \( \tilde{H}^r_j(x) \) functions satisfy the adding up condition so must the \( \tilde{H}^r_j(x) \). Hence, from symmetry,

\[
\int_0^1 G(s)ds = \frac{1}{n} = \int \tilde{H}^r_j(x)dx.
\]
Since (xiv) holds for all \( i \neq r-1, r \) we have

\[(xv) \quad (y_r - y_{r-1}) \left[ g_{r-1} \int \hat{H}_1^{r} (y_{r-1}, x_{r-1}) \, dx_{r-1} \right] + (y_r - y_{r+1}) \left[ g_r \int \hat{H}_1^{r} (y_r, x_{r-1}) \, dx_{r-1} \right] = 0.\]

Thus, if (xiv) holds for \( i=r \) it must also hold for \( i=r-1 \).

First set \( p=(0,0,\ldots,0) \) in (xi) and (xiii). Then, for all \( x_{r-1} \), it follows from the definitions of \( \hat{H}_1^{r} \) and \( H_1^{r} \) that

\[(xvi) \quad \hat{H}_1^{r} (y_{r-1}, x_{r-1}) = H_1^{r} (y_r, x_{r-1}).\]

Hence, since \( G(s) \) is nondecreasing, (xv) and (xvi) imply that

\[(xvii) \quad G_r > \int \hat{H}_1^{r} (y_r, x_{r-1}) \, dx_{r-1} \text{ when } p=(0,0,\ldots,0).\]

Suppose that for all \( p \in P \), and all \( r \),

\[ G_r > \int \hat{H}_1^{r} (y_r, x_{r-1}) \, dx_{r-1} \]

From (xv) we must therefore have

\[ G_{r-1} > \int \hat{H}_1^{r} (y_{r-1}, x_{r-1}) \, dx_{r-1} \]

Hence

\[(xviii) \quad G_r - G_{r-1} > \int [H_1^{r} (y_r, x_{r-1}) - \hat{H}_1^{r} (y_{r-1}, x_{r-1})] \, dx_{r-1}.\]

From the definition of \( \hat{H}_1^{r} \)

\[(xix) \quad \hat{H}_1^{r} (y_r, x_{r-1}) - \hat{H}_1^{r} (y_{r-1}, x_{r-1}) = H_1^{r} (y_r, x_{r-1}) - H_1^{r} (y_{r-1}, x_{r-1}), \text{ if } (y_r, x_{r-1}) \in X_r \]

Again from definition,

\[(xx) \quad \hat{H}_1^{r} (y_r, x_{r-1}) - \hat{H}_1^{r} (y_{r-1}, x_{r-1}) = H_1^{r} (y_r, x_{r-1}) - H_1^{r} (y_{r-1}, x_{r-1}) = 0; \text{ if } (y_r, x_{r-1}) \notin X_r \]

If \( p = (\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}) \), then for all \( x \)

\[ H_1^{r} (x) = H_1 (x). \]
Hence,

\[ (x_{11}) \quad \hat{H}_1^{(r)}(y_r, x_{-1}) - \hat{H}_1^{(r)}(y_{r-1}, x_{-1}) = \hat{H}_1(y_r, x_{-1}) - \hat{H}_1(y_{r-1}, x_1), \quad \text{for } p = \left( \frac{1}{2}, \ldots, \frac{1}{n} \right) \]

Therefore

\[ (x_{12}) \quad \hat{H}_1(y_r, x_{-1}) - \hat{H}_1(y_{r-1}, x_{-1}) = 0 \quad \text{if } (y_r, x_{-1}) \notin X_r. \]

Combining (x_{10}) - (x_{12}) we obtain

\[ \hat{H}_1^{(r)}(y_r, x_{-1}) - \hat{H}_1^{(r)}(y_{r-1}, x_{-1}) = \hat{H}_1(y_r, x_{-1}) - \hat{H}_1(y_{r-1}, x_{-1}), \quad \text{for } p = \left( \frac{1}{2}, \ldots, \frac{1}{n} \right). \]

Hence, from (x_{18})

\[ G_r - G_{r-1} > \int [\hat{H}_1(y_r, x_{-1}) - \hat{H}_1(y_{r-1}, x_{-1})] \, dx_{-1} \quad \text{for } p = \left( \frac{1}{2}, \ldots, \frac{1}{n} \right). \]

Summing over \( r \), we obtain

\[ (x_{33}) \quad G_m - G_1 > \int [\hat{H}_1(y_m, x_{-1}) - \hat{H}_1(y_1, x_{-1})] \, dx_{-1} \quad \text{for } p = \left( \frac{1}{2}, \ldots, \frac{1}{n} \right). \]

But from the definition of \( \hat{H}_1(x) \),

\[ \int \hat{H}_1(y_1, x_{-1}) \, dx_{-1} = \frac{1}{n} \frac{y_{n-1}}{2} \]

and

\[ \int \hat{H}_1(y_m, x_{-1}) \, dx_{-1} = \sum_{c=0}^{n-1} \frac{n-1}{1+c} \frac{1}{1-c} (1-y_m)^c \frac{n-1-c}{y_m} \]

Thus we may rewrite (x_{33}) as
(xxiv) \( G_m - G_1 > \frac{1}{n} \left( 1 - \frac{y^n}{1 - y_m} \right) - \frac{1}{n} y^{n-1} \).

But (xxiv) contradicts (vii) and (viii). Thus we conclude that there exist \( p \in P \) and \( r \) such that \( G_r = \int_{y_r}^{y_{r-1}} (y_r^* x_{r-1}) dx_{r-1} \), to complete the induction.

Q.E.D.

The final step is to extend the previous lemma to \( G' \)'s for which the constraint (iv) does not hold with equality.

Lemma A4: Lemma A3 is also true if the integral constraint (iv) holds with strict inequality at \( y = 0 \).

Proof: Suppose that \( G(s) \) is a nondecreasing finite step function satisfying the integral constraint (iv) and with strict inequality at \( y = 0 \).

We first claim that there exists a finite step function \( G \geq G \) such that

( xxv ) \( G \) is nondecreasing and

( xxvi ) \( \int_y^{y_1} (G(z) - z^{n-1}) dz \leq 0 \) for all \( y \),

where (xxvi) holds with equality for \( y = 0 \).

Let us write \( G(s) \) as in (vi). Define the step function \( G^*(s) \) so that for \( s \in [y_i, y_{i+1}) \)

\[
G^*(s) = \frac{1}{y_{i+1} - y_i} \int_{y_i}^{y_{i+1}} z^{n-1} dz
\]

Clearly \( G^*(s) \) is nondecreasing and satisfies (iv) everywhere and with equality at \( y_i, i = 1, \ldots, m \).

Let \( Y_1 \) be the set of "crossing points" of \( G(s) \) and \( G^*(s) \). That is,

\[
Y_1 = \{ y_i | \text{for all } \epsilon > 0 \text{ sufficiently small } (G(y_i + \epsilon) - G^*(y_i + \epsilon))(G(y_i - \epsilon) - G^*(y_i - \epsilon)) < 0 \}
\]

If \( Y_1 \) is empty, then from (iv), \( G(s) < G^*(s) \) for all \( s \), and so we can take \( G(s) = G^*(s) \) to establish the claim. Therefore, assume that \( Y_1 \) is nonempty.
Let

\[
Y_2 = \{ y_1 \in Y_1 \mid \int_0^{y_1} \max \{ G^*(s), G(s) \} ds + \int_0^{1} G(z) dz \geq \int_0^{1} z^{-1} dz \}.
\]

Take

\[
y^{**} = \begin{cases} 
1, & \text{if } Y_2 \text{ is empty} \\
\min Y_2, & \text{if } Y_2 \text{ is nonempty}
\end{cases}
\]

and

\[
y^* = \begin{cases} 
\max \{ y \in Y_1 \mid y < y^{**} \}, & \text{if } \{ y \in Y_1 \mid y < y^{**} \} \text{ is nonempty} \\
0, & \text{otherwise}.
\end{cases}
\]

Define

\[
\tilde{G}(s) = \begin{cases} 
\max \{ G(s), G^*(s) \}, & s < y^* \\
\lambda G(s) + (1-\lambda) \max \{ G(s), G^*(s) \}, & y^* \leq s \leq y^{**} \\
G(s), & s > y^{**}
\end{cases}
\]

for choice of \( \lambda \) between 0 and 1. If \( \lambda = 1 \), then by choice of \( y^* \),

\[
\int_0^{1} \tilde{G}(z) dz < \int_0^{1} z^{-1} dz.
\]

Suppose \( \lambda = 0 \). If \( Y_2 \) is empty, then \( y^{**} = 1 \), and so

\[
(\text{xxix}) \quad \int_0^{1} \tilde{G}(z) dz \geq \int_0^{1} z^{-1} dz.
\]

Thus, if \( Y_2 \) is empty, \( \tilde{G}(z) dz \geq \int_0^{1} z^{-1} dz \). If \( Y_2 \) is nonempty then \( y^{**} \in Y_2 \), and (xxvii) implies that,

\[
\int_0^{1} \tilde{G}(z) dz \geq \int_0^{1} z^{-1} dz.
\]

Again, \( \int_0^{1} \tilde{G}(z) dz \geq \int_0^{1} z^{-1} dz \). Thus we may choose \( \lambda < 1 \) so that

\[
(\text{xxx}) \quad \int_0^{1} \tilde{G}(z) dz = \int_0^{1} z^{-1} dz.
\]

Because \( y^* \in Y_1 \cup \{0\} \) and \( y^{**} \in Y_1 \cup \{1\} \), \( \tilde{G}(s) \) is nondecreasing. It is obvious from (xxix) that \( \tilde{G}(s) \geq G(s) \). Thus it remains only to show that (xxvi) holds.

Suppose that

\[
(\text{xxxi}) \quad \int_0^{1} (G(z) - z^{-1}) dz > 0 \text{ for some } \bar{y}.
\]

If \( \tilde{G}(\bar{y}) > \bar{y}^{-1} \), then \( \int_0^{1} (G(z) - z^{-1}) dz > 0 \).
where \( \bar{y} \in [y_i, y_{i+1}) \). If \( \tilde{G}(\bar{y}) \leq \bar{y}^{n-1} \), then \( \int_{y_i}^{\bar{y}} (\tilde{G}(z) - z^{n-1}) dz > 0 \),

where \( \bar{y} \in [y_{i-1}, y_i) \). Therefore, we may as well assume that \( \bar{y} = y_i \) for some i. If \( \bar{y} < y^* \), then because \( \tilde{G}(s) \geq G^*(s) \) for all \( s < y^* \) and

\[
\int_{0}^{\bar{y}} G^*(z) dz = \int_{0}^{y} z^{n-1} dz \quad \text{(since (iv) holds with equality for } G^* \text{ at } y = 0) \]

\( y = \bar{y} \) and \( y = 0 \),

\[
(\text{xxxii}) \quad \int_{0}^{\bar{y}} \tilde{G}(z) dz > \int_{0}^{y} z^{n-1} dz.
\]

But (xxx) and (xxxii) together imply that

\[
(\text{xxxiii}) \quad \int_{0}^{1} \tilde{G}(z) dz > \int_{0}^{1} z^{n-1} dz,
\]

which contradicts (xxx).

Suppose \( y \in [y^*, y^{**}] \). By definition of \( y^* \), either \( G(s) \leq G^*(s) \) for all \( s \in (y^*, y^{**}) \) or \( G(s) > G^*(s) \) for all \( s \in (y^*, y^{**}) \). If the former, then \( \tilde{G}(s) \leq G^*(s) \) for \( s \in (y^*, y^{**}) \), and so

\[
(\text{xxxiv}) \quad \int_{y^{**}}^{y} (\tilde{G}(z) - z^{n-1}) dz \leq 0.
\]

From (xxx) and (xxxiv)

\[
\int_{y^{**}}^{1} (\tilde{G}(z) - z^{n-1}) dz > 0
\]

which contradicts the fact that \( \tilde{G}(s) = G(s) \) for \( s > y^{**} \).

If the latter, then \( \tilde{G}(s) = G(s) \) for \( s > y^* \), which contradicts (xxx). Finally, if \( y > y^{**} \), (xxx) is impossible, since \( \tilde{G}(s) = G(s) \) for all \( s > y^{**} \). We conclude that (xxx) is impossible and that the claim is established.

From Lemma A3 there exist probability functions \( \tilde{H}_j \) satisfying the counterparts of (i) - (iii) for \( \tilde{G} \). Define \( Q(z) = G(z) / \tilde{G}(z) \) and take
\[ h_j(x) = q(x) \] 

It is immediate that the \( h_j \)'s satisfy (i) - (iii).

Q.E.D.
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