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STATE-DEPENDENT INTELLECTUAL PROPERTY RIGHTS POLICY

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Abstract

What form of intellectual property rights (IPR) policy contributes to economic growth? Should technological followers be able to license the products of technological leaders? Should a company with a large technological lead receive the same IPR protection as a company with a more limited lead? We develop a general equilibrium framework to investigate these questions. The economy consists of many industries and firms engaged in cumulative (step-by-step) innovation. IPR policy regulates whether followers in an industry can copy the technology of the leader and also how much they have to pay to license past innovations. With full patent protection, followers can catch up to the leader in their industry either by making the same innovation(s) themselves or by making some pre-specified payments to the technological leaders.

We prove the existence of a steady-state equilibrium and characterize some of its properties. We then quantitatively investigate the implications of different types of IPR policy on the equilibrium growth rate. The two major results of this exercise are as follows. First, the growth rate in the standard models used in the (growth) literature can be improved significantly by introducing a simple form of licensing. Second, we show that full patent protection is not optimal from the viewpoint of maximizing the growth rate of the economy and that the growth-maximizing policy involves state-dependent IPR protection, providing greater protection to technological leaders that are further ahead than those that are close to their followers. This form of the growth-maximizing policy is a result of the "trickle-down" effect, which implies that providing greater protection to firms that are further ahead of their followers than a certain threshold increases the R&D incentives also for all technological leaders that are less advanced than this threshold.

Keywords: competition, economic growth, endogenous growth, industry structure, innovation, intellectual property rights, licensing, patents, research and development, trickle-down.
JEL classification: O31, O34, O41, L16.

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1 Introduction

How should the intellectual property rights of a company be protected? Should a firm with a large technological lead receive the same intellectual property rights (IPR) protection as a company with a more limited technological lead? These questions are central to many discussions of patent policy. A recent ruling of the European Commission, for example, has required Microsoft to share secret information about its products with other software companies (New York Times, December 22, 2004). There is a similar debate about whether Apple should make iPod's code available to competitors that are producing complementary products. Central to the debate about Microsoft and iPod is the fact that they have a substantial technological lead over their rivals. This implies that the analysis of many of the relevant policy and academic questions requires a framework for the analysis of *state-dependent* patent/IPR protection policy. By state-dependent IPR policy, we mean a policy that makes the extent of patent or intellectual property rights protection conditional on the technology gap between different firms in the industry. Existing work has investigated the optimal length and breadth of patents assuming an IPR policy that does not allow for licensing and is uniform. In this paper, we make a first attempt to develop a framework that incorporates both licensing of existing patents and state-dependent IPR policies.

Our basic framework builds on and extends the step-by-step innovation models of Aghion, Harris and Vickers (1997) and Aghion, Harris, Howitt and Vickers (2001), where a number of (typically two) firms engage in price competition within an industry and undertake R&D in order to improve the quality of their product. The technology gap between the firms determines the extent of the monopoly power of the leader, and hence the price markups and profits. The purpose of R&D by the follower is to catch up and surpass the leader (as in standard Schumpeterian models of innovation, e.g., Reinganum, 1981, 1985, Aghion and Howitt, 1992, Grossman and Helpman, 1991), while the purpose of R&D by the leader is to escape the competition of the follower and increase its markup and profits. As in racing-type models in general (e.g., Harris and Vickers, 1985, 1987, Budd, Harris and Vickers, 1993), a large gap between the leader and the follower discourages R&D by both. Consequently, overall R&D and technological progress are greater when the technology gap between the leader and the follower is relatively small.[1] One may expect that full patent protection may be suboptimal in a world of step-by-step competition; by stochastically or deterministically allowing the follower

[1] Aghion, Bloom, Blundell, Griffith and Howitt (2005) provide empirical evidence that there is greater R&D in British industries where there is a smaller technological gap between firms. See O'Donoghue, Scotchmer and Thisse (1998) for a discussion of how patent life may come to an end because of related innovations.

to use the innovations of the technological leader, the likelihood of relatively small gap between leaders and followers, and thus the amount of R&D may be raised.[2] Based on this intuition, one may further conjecture that state-dependent IPR policy may also be useful and should provide less protection to firms that are technologically more advanced relative to their competitors.

There are two problems with this intuition. First, existing growth models (including those mentioned above) do not allow licensing of the leading-edge technology and the presence of licensing changes the trade-offs underlying the above intuition.[3] Second, the above intuition is derived from models with uniform IPR policy, without a systematic study of the implications of state-dependent IPR policy on R&D incentives.

To investigate these issues systematically, we construct a general equilibrium model with step-by-step innovation, potential licensing of patents and state-dependent IPR policy. In our model economy, each firm can climb the technology ladder via three different methods: (i) by "catch-up R&D," that is, R&D investments applied to a variant of the technology of the leader; (ii) by "frontier R&D," that is, building on the patented innovations of the technological leader for a pre-specified license fee; and (iii) as a result of the expiration of the patent of the technological leader. The second feature is not present in any growth model that we are aware of, but is essential for understanding how IPR policy works in practice.

The combination of these policies allows a variety of different policy regimes. The first is *full patent protection with no licensing*, which corresponds to the environment assumed in existing growth models (e.g., Aghion, Harris, Howitt and Vickers, 2001) and provides full (indefinite) patent protection to technological leaders, but does not allow any licensing agreements (it sets the license fees to infinity). The second is *full patent protection with licensing*, which allows technological followers to build on the leading-edge technology in return for a license fee. We refer to this regime as "full patent protection" because patents never expire and followers have to pay a patent fee equal to the gain in net present value resulting from the use of the leading-edge technology.[4] The third regime is *uniform imperfect patent protection*, which deviates from the previous two benchmarks by allowing either expiration of patents and/or license fees that are less than the full benefit to the follower. The adjective uniform indicates that

[2] This is conjectured, for example, in Aghion, Harris, Howitt and Vickers (2001).

[3] See Scotchmer (2005) for the importance of incorporating these types of licensing agreements into models of innovation.

[4] The alternative would be for the followers to pay a fee equal to the damage to the technological leader, resulting from the loss of the leader's monopoly power after their licensing and further innovation. In practice, licensing fees or patent infringement fees reflect both the benefits to the firm using the knowledge and the damage to the original inventor (see, e.g., Scotchmer, 2005). In our analysis, we allow the license fees to be set at any level, thus incorporating both possibilities.

in this policy environment all industries are treated identically irrespective of the technology gap between the leader and the follower. The final and most interesting policy regime is *state-dependent imperfect patent protection*, which deviates from full patent protection as a function of the technology gap between the leader and the follower in the industry (i.e., it allows technologically more advanced firms to receive more or less protection). Each of these policy regimes captures a different conceptualization of IPR policy and is interesting in its own right. Note however that the last regime is general enough to nest the other three.

We first prove the existence of a stationary (steady-state) equilibrium under any of these policy regimes and characterize a number of features of the equilibrium analytically. For example, we prove that with uniform IPR policy, R&D investments decline when the gap between the leader and the follower increases.

We then turn to a quantitative investigation of growth-maximizing ("optimal") IPR policy, by providing a number of simulations of the equilibrium for plausible parameter values and for different forms of IPR policies.[5] Our quantitative investigation leads to two major results:

1. Allowing for licensing of patents increases the equilibrium growth rate of the economy significantly. Intuitively, without such licensing, a large part of the R&D effort goes to duplication, and followers' R&D does not contribute to the growth rate of the economy. Licensing implies that R&D by all firms—not just the leaders—contributes to growth and also increases the R&D incentives of followers. In our benchmark parameterization, allowing for licensing increases the steady-state equilibrium growth rate of the economy from 1.86% to 2.58% per annum.

2. Growth-maximizing IPR policy is state dependent. In particular, because of the disincentive effect of relaxing IPR protection on R&D, uniform IPR policy (either by manipulating license fees or the duration of patents) has minimal effect on the equilibrium growth rate. In contrast, state-dependent IPR policy can significantly increase innovation and growth in the economy. For example, in our baseline parameterization, optimal state-dependent IPR policy increases the growth rate to 2.96% relative to the growth rate of 2.63% under uniform IPR policy. More important than the quantitative effect of growth-maximizing state-dependent IPR policy is its form. We find in all cases that

[5]Throughout, we simplify terminology by using the terms "optimal" and "growth-maximizing" interchangeably. It is well known that welfare-maximizing and growth-maximizing policies need not coincide, and in fact, in models with quality competition, the equilibrium may involve excessive innovation and growth (e.g., Aghion and Howitt, 1992). Nevertheless, when the discount rate is small, welfare and growth-maximizing policies will be close, which justifies our use of the term "optimal" in this context.

optimal state-dependent IPR policy provides greater protection to technological leaders that are *further ahead* than those that have only a small lead relative to their followers. The reason for this result is the *trickle-down* of R&D incentives. In particular, when a particular state for the leader (say being n^* steps ahead of the follower) is very profitable, this increases the incentives to perform R&D not only for leaders that are $n^* - 1$ steps ahead, but for *all* leaders with a lead of size $n \leq n^* - 1$. Notably, the trickle-down effect can be powerful enough to increase—rather than reduce—the R&D investments by technological leaders relative to the benchmark with full patent protection. Thus, reducing the IPR protection of leaders that are few steps ahead while offering a high degree of protection to firms that are sufficiently advanced relative to their competitors *increases* R&D incentives. Consequently, in contrast to existing models, imperfect state-dependent IPR can increase R&D investments relative to full protection. Moreover, contrary to the conjecture above, the trickle-down effect implies that the optimal state-dependent IPR policy should provide *greater* protection to firms that have a larger technological lead.

Overall, our analysis illustrates the different effects of IPR policy on the equilibrium growth rate of an economy. It shows that allowing for licensing can increase growth significantly and that optimal state-dependent IPR policy can also increase the growth rate of an economy relative to uniform IPR policy. Consequently, the benefits of considering a rich set of IPR policies could be substantial in practice. Our analysis also suggests that the structure of optimal IPR may depend on the equilibrium industry structure (which determines the technology gap between leaders and followers as a function of the underlying technology of the industry). A more detailed analysis of the relationship between industry structure and the optimal form of IPR policy is an area for future research.

Our paper is a contribution both to the IPR protection and the endogenous growth literatures. The industrial organization and growth literatures emphasize that ex-post monopoly rents and thus patents are central to generate the ex-ante investments in R&D and technological progress, even though monopoly power also creates distortions (e.g., Arrow, 1962, Reinganum, 1981, Tirole, 1988, Romer, 1990, Grossman and Helpman, 1991, Aghion and Howitt, 1992, Green and Scotchmer, 1995, Scotchmer, 1999, Gallini and Scotchmer, 2002, O'Donoghue and Zweimuller, 2004).[6]

Much of the literature discusses the trade-off between these two forces to determine the optimal length and breadth of patents. For example, Klemperer (1990) and Gilbert and Shapiro

[6] Boldrin and Levine (2001, 2004) or Quah (2003) argue that patent systems are not necessary for innovation.

(1990) show that optimal patents should have a long duration in order to provide inducement to R&D, but a narrow breadth so as to limit monopoly distortions. A number of other papers, for example, Gallini (1992) and Gallini and Scotchmer (2002), reach opposite conclusions. Another branch of the literature, including the seminal paper by Scotchmer (1999) and the recent interesting papers by Llobet, Hopenhayn and Mitchell (2001) and Hopenhayn and Mitchell (2001), adopts a mechanism design approach to the determination of the optimal patent and intellectual property rights protection system. For example, Scotchmer (1999) derives the patent renewal system as an optimal mechanism in an environment where the cost and value of different projects are unobserved and the main problem is to decide which projects should go ahead. To the best of our knowledge, no other paper in the literature has considered state-dependent patent or IPR policy, which is the focus of the current paper. As a first attempt, we only look at state-dependent patent length and license fees (though similar ideas can be applied to an investigation of the gains from making the breadth of patent awards state-dependent).

Our paper is most closely related to and extends the results of Aghion, Harris and Vickers (1997) and Aghion, Harris, Howitt and Vickers (2001) on endogenous growth with step-by step innovation.[7] Although our model builds on these papers, it also differs from them in a number of significant ways. First, we allow licensing agreements whereby followers can pay a pre-specified license fee for building on the leading-edge technology developed by other firms. We show that such licensing has significant effects on the equilibrium growth rate of the economy. Second, our economy incorporates a general IPR policy that can be state dependent. Third, in our economy there is a general equilibrium interaction between production and R&D, since they both compete for scarce labor.[8] Finally, we provide a number of analytical results for the general model (with or without IPR policy), while Aghion, Harris, Howitt and Vickers (2001) focus on the case where innovations are either "drastic" (so that the leader never undertakes R&D) or very small. They also do not prove the existence of a steady state or characterize the properties of the equilibrium in this class of economies.

Finally, our results are also related to the literature on tournaments and races, for example, Fudenberg, Gilbert, Stiglitz and Tirole (1983), Harris and Vickers (1985, 1987), Choi (1991), Budd, Harris and Vickers (1993), Taylor (1995), Fullerton and McAfee (1999), Baye and Hoppe

[7]Segal and Whinston (2005) analyze the impact of anti-trust policy on economic growth in a related model of step-by-step innovation.

[8]This general equilibrium aspect is introduced to be able to close the model economy without unrealistic assumptions and makes our economy more comparable to other growth models (Aghion, Harris, Howit and Vickers, 2001, assume a perfectly elastic supply of labor). We show that the presence of general equilibrium interactions does not significantly complicate the analysis and it is still possible to characterize the steady-state equilibrium.

(2003), and Moscarini and Squintani (2004). This literature considers the impact of endogenous or exogenous prizes on effort in tournaments, races or R&D contests. In terms of this literature, state-dependent IPR policy can be thought of as state-dependent handicapping of different players (where the state variable is the gap between the two players in a dynamic tournament). To the best of our knowledge, these types of schemes have not been considered in this literature.

The rest of the paper is organized as follows. Section 2 presents the basic environment. Section 3 characterizes the equilibrium under uniform IPR policy. Section 4 extends these results to the case in which IPR policy is state dependent. Section 5 quantitatively evaluates the implications of various different types of IPR policy regimes on economic growth and characterizes the growth-maximizing state-dependent IPR policies. Section 6 concludes, while the Appendix contains the proofs of all the results stated in the text.

2 Model

We now describe the basic environment. The characterization of the equilibrium under the different policy regimes is presented in the next section.

2.1 Preferences and Technology

Consider the following continuous time economy with a unique final good. The economy is populated by a continuum of 1 individuals, each with 1 unit of labor endowment, which they supply inelastically. Preferences at time t are given by

$$\mathbb{E}_t \int_t^{\infty} \exp\left(-\rho\left(s-t\right)\right) \log C\left(s\right) ds, \tag{1}$$

where $\mathbb{E}_t$ denotes expectations at time t, $\rho > 0$ is the discount rate and $C(t)$ is consumption at date t. The logarithmic preferences in (1) facilitate the analysis, since they imply a simple relationship between the interest rate, growth rate and the discount rate (see (2) below).

Let $Y(t)$ be the total production of the final good at time t. We assume that the economy is closed and the final good is used only for consumption (i.e., there is no investment), so that $C(t) = Y(t)$. The standard Euler equation from (1) then implies that

$$g\left(t\right) \equiv \frac{\dot{C}\left(t\right)}{C\left(t\right)} = \frac{\dot{Y}\left(t\right)}{Y\left(t\right)} = r\left(t\right) - \rho, \tag{2}$$

where this equation defines $g(t)$ as the growth rate of consumption and thus output, and $r(t)$ is the interest rate at date t.

The final good Y is produced using a continuum 1 of intermediate goods according to the Cobb-Douglas production function

$$\ln Y(t) = \int_0^1 \ln y(j,t)\, dj, \tag{3}$$

where $y(j,t)$ is the output of jth intermediate at time t. Throughout, we take the price of the final good as the numeraire and denote the price of intermediate j at time t by $p(j,t)$. We also assume that there is free entry into the final good production sector. These assumptions, together with the Cobb-Douglas production function (3), imply that each final good producer will have the following demand for intermediates

$$y(j,t) = \frac{Y(t)}{p(j,t)}, \quad \forall j \in [0,1]. \tag{4}$$

Each intermediate $j \in [0,1]$ comes in two different varieties, each produced by one of two infinitely-lived firms. We assume that these two varieties are perfect substitutes and these firms compete a la Bertrand.[9] Firm $i = 1$ or 2 in industry j has the following technology

$$y(j,t) = q_i(j,t)\, l_i(j,t) \tag{5}$$

where $l_i(j,t)$ is the employment level of the firm and $q_i(j,t)$ is its level of technology at time t. The only difference between two firms is their technology, which will be determined endogenously.

Each consumer in the economy holds a balanced portfolio of the shares of all firms. Consequently, the objective function of each firm is to maximize expected profits.

The production function for intermediate goods, (5), implies that the marginal cost of producing intermediate j for firm i and industry j at time t is

$$MC_i(j,t) = \frac{w(t)}{q_i(j,t)} \tag{6}$$

where $w(t)$ is the wage rate in the economy at time t.

When this causes no confusion, we denote the *technological leader* in each industry by i and the follower by $-i$, so that we have:

$$q_i(j,t) \geq q_{-i}(j,t).$$

[9]Alternatively, we can assume that these two varieties are imperfect substitutes, for example, so that the output of intermediate j is given by

$$y(j,t) = \left[\varphi y_1(j,t)^{\frac{\sigma-1}{\sigma}} + (1-\varphi)\, y_2(j,t)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}},$$

with $\sigma > 1$. This generalization has no effect on any of our qualitative results, but increases notation. We therefore prefer to focus on the case where the two varieties are perfect substitutes.

Bertrand competition between the two firms implies that all intermediates will be supplied by the leader at the "limit" price:[10]

$$p_i(j,t) = \frac{w(t)}{q_{-i}(j,t)}. \tag{7}$$

Equation (4) then implies the following demand for intermediates:

$$y(j,t) = \frac{q_{-i}(j,t)}{w(t)} Y(t). \tag{8}$$

2.2 Technology, R&D and IPR Policy

R&D by the leader or the follower stochastically leads to innovation. We assume that when the leader innovates, its technology improves by a factor $\lambda > 1$.

The follower, on the other hand, can undertake R&D to catch up with the frontier technology or to improve over the frontier technology.[11] The first possibility, catch-up R&D, can be thought of as discovering an alternative way of performing the same task as the current leading-edge technology (applied to the follower's variant of the product). We assume that because this innovation is for the follower's variant of the product and results from its own R&D efforts, it does not constitute infringement of the patent of the leader, and the follower does not have to make any payments to the technological leader in the industry. Therefore, if the follower chooses the first possibility, it will have to retrace the steps of the technological leader (as applied to its own variant of the product), but in return, it will not have to pay a patent fee. For follower firm $-i$ in industry j at time t, we denote this type of R&D by

$$a_{-i}(j,t) = 0.$$

The alternative, frontier R&D, involves followers building on and improving the current leading-edge technology. If this type of R&D succeeds, the follower will have improved the leading-edge technology using the patented knowledge of the technological leader, and thus will have to pay a pre-specified patent fee to the leader.[12] We specify the patent fees below.

[10] If the leader were to charge a higher price, then the market would be captured by the follower earning positive profits. A lower price can always be increased while making sure that all final good producers still prefer the intermediate supplied by the leader i rather than that by the follower $-i$ even if the latter were supplied at marginal cost. Since the monopoly price with the unit elastic demand curve is infinite, the leader always gains by increasing its price, making the price given in (7) the unique equilibrium price.

[11] A third possibility is for the follower to climb the technology ladder step-by-step, meaning that, for example, when the current leader is at some technology rung $n_{ij}(t)$ and the follower itself is at $n_{-ij}(t) < n_{ij}(t) - 1$, it must first discover technology $n_{-ij}(t) + 1$, et cetera. We have investigated this type of environment with "slow catch-up" in a previous version of the paper. Since the general results are similar, we do not discuss this variation to save space.

[12] Clearly, the follower will never license the technology of the leader for production purposes, since this would lead to Bertrand competition and zero ex post profits for both parties.

This strategy is denoted by

$$a_{-i}(j,t) = 1.$$

Throughout, we allow $a_{-i}(j,t) \in [0,1]$ for mathematical convenience, thus a should be interpreted as the probability of frontier R&D by the follower.

R&D by the leader, catch-up R&D by the follower, and frontier R&D by the follower may have different costs and success probabilities. Nevertheless, we simplify the analysis by assuming that all three types of R&D have the same costs and the same probability of success. In particular, in all cases, we assume that innovations follow a controlled Poisson process, with the arrival rate determined by R&D investments. Each firm (in every industry) has access to the following R&D technology:

$$x_i(j,t) = F(h_i(j,t)), \tag{9}$$

where $x_i(j,t)$ is the flow rate of innovation at time t and $h_i(j,t)$ is the number of workers hired by firm i in industry j to work in the R&D process at t. This specification implies that within a time interval of Δt, the probability of innovation for this firm is $x_i(j,t)\Delta t + o(\Delta t)$.

We assume that F is twice continuously differentiable and satisfies $F'(\cdot) > 0$, $F''(\cdot) < 0$, $F'(0) < \infty$ and that there exists $\bar{h} \in (0,\infty)$ such that $F'(h) = 0$ for all $h \geq \bar{h}$. The assumption that $F'(0) < \infty$ implies that there is no Inada condition when $h_i(j,t) = 0$. The last assumption, on the other hand, ensures that there is an upper bound on the flow rate of innovation (which is not essential but simplifies the proofs). Recalling that the wage rate for labor is $w(t)$, the cost for R&D is therefore $w(t)\,G(x_i(j,t))$ where

$$G(x_i(j,t)) \equiv F^{-1}(x_i(j,t)), \tag{10}$$

and the assumptions on F immediately imply that G is twice continuously differentiable and satisfies $G'(\cdot) > 0$, $G''(\cdot) > 0$, $G'(0) > 0$ and $\lim_{x \to \bar{x}} G'(x) = \infty$, where

$$\bar{x} \equiv F(\bar{h}) \tag{11}$$

is the maximal flow rate of innovation (with $\bar{h}$ defined above).

We next describe the evolution of technologies within each industry. Suppose that leader i in industry j at time t has a technology level of

$$q_i(j,t) = \lambda^{n_{ij}(t)}, \tag{12}$$

and that the follower $-i$'s technology at time t is

$$q_{-i}(j,t) = \lambda^{n_{-ij}(t)}, \tag{13}$$

where $n_{ij}(t) \geq n_{-ij}(t)$ and $n_{ij}(t), n_{-ij}(t) \in \mathbb{Z}_+$ denote the technology rungs of the leader and the follower in industry j. We refer to $n_j(t) \equiv n_{ij}(t) - n_{-ij}(t)$ as the *technology gap* in industry j. If the leader undertakes an innovation within a time interval of Δt, then its technology increases to $q_i(j, t+\Delta t) = \lambda^{n_{ijt}+1}$ and technology gap rises to $n_j(t+\Delta t) = n_j(t) + 1$ (the probability of two or more innovations within the interval Δt will be $o(\Delta t)$, where $o(\Delta t)$ represents terms that satisfy $\lim_{\Delta t \to 0} o(\Delta t)/\Delta t$).

When the follower is successful in catch-up R&D (i.e., $a_{-i}(j,t) = 0$) within the interval Δt, then it catches up with the leader and its technology improves to

$$q_{-i}(j, t+\Delta t) = \lambda^{n_{ijt}},$$

and the technology gap variable becomes $n_{jt+\Delta t} = 0$. In contrast, if the follower is successful in frontier R&D and pays the license fee (i.e., $a_{-i}(j,t) = 1$), then it surpasses the leading-edge technology, so we have

$$q_{-i}(j, t+\Delta t) = \lambda^{n_{ijt}+1}$$

and the technology gap variable becomes $n_{jt+\Delta t} = 1$ (and from this point onwards, the labels i and $-i$ are swapped, since the previous follower now becomes the leader).

In addition to catching up with or surpassing the technology frontier with their own R&D, followers can also copy the technology frontier because IPR policy is such that some patents expire. In particular, we assume that patents expire at some policy-determined Poisson rate η, and after expiration followers can costlessly copy the frontier technology, jumping to $q_{-i}(j, t+\Delta t) = \lambda^{n_{ijt}}$.[13]

This description makes it clear that there are two aspects to IPR policy: (i) the length of the patent; (ii) license fees. In our most general policy regime, we allow both of these to be state dependent, so they are represented by the following two functions:

$$\boldsymbol{\eta} : \mathbb{N} \to \mathbb{R}_+$$

and for all $t \geq 0$,

$$\boldsymbol{\zeta}(t) : \mathbb{N} \to \mathbb{R}_+ \cup \{+\infty\}.$$

Here $\eta(n) \equiv \eta_n < \infty$ is the flow rate at which the patent protection is removed from a technological leader that is n steps ahead of the follower. When $\eta_n = 0$, this implies that there is full protection at technology gap n, in the sense that patent protection will never be removed.

[13] Alternative modeling assumptions on IPR policy, such as a fixed patent length of $T > 0$ from the time of innovation, are not tractable, since they introduce a form of delayed-differential equations.

In contrast, $\eta_n \to \infty$ implies that patent protection is removed immediately once technology gap n is reached. Similarly, $\zeta(n,t) \equiv \zeta_n(t)$ denotes the patent fee that a follower has to pay in order to build upon the innovation of the technological leader, when the technology gap in the industry is n steps.[14] Our formulation imposes that $\eta \equiv \{\eta_1, \eta_2, ...\}$ is time-invariant, while $\zeta(t) \equiv \{\zeta_1(t), \zeta_2(t), ...\}$ is a function of time. This is natural, since in a growing economy, license fees should not remain constant. Below, we will require that ζ grows at the same rate as aggregate output in the economy.

When $\zeta_n(t) = 0$, there is no protection because followers can license the leading-edge technology at zero cost.[15] In contrast, when $\zeta_n(t) = \infty$, licensing the leading-edge technology is prohibitively costly. Note however that $\zeta_n < \infty$ does not necessarily imply that patent protection is imperfect. In particular, in what follows we will interpret a situation in which the license fee is equal to the net extra gain from surpassing the leader rather than being neck-and-neck (i.e., being at a technology gap of 0) as "full protection".[16] We also refer to a policy regime as *uniform* IPR protection if both η and $\zeta(t)$ are constant functions, meaning that intellectual property law treats all firms and industries identically irrespective of the technology gap between the leader and the follower (i.e., $\eta^{uni} \equiv \{\eta, \eta, ...\}$ and $\zeta^{uni}(t) \equiv \{\zeta(t), \zeta(t), ...\}$). We also assume that there exists some $\bar{n} < \infty$ such that $\eta_n = \eta_{\bar{n}}$ and $\zeta_n(t) = \zeta_{\bar{n}}(t)$ for all $n \geq \bar{n}$.

Given this specification, we can now write the law of motion of the technology gap in

[14]Throughout, we assume that ζ is a policy choice and firms cannot contract around it. An alternative approach would be to allow firms to bargain over the level of license fees. In this case, it is plausible to presume that the legally-specified infringement penalties or license fees will affect the equilibrium in the bargaining game, so the effect of policies we investigate would still be present. We do not allow bargaining between firms over the license fees in order to simplify the analysis.

[15]Throughout, we interpret $\zeta_n(t) = 0$ as $\zeta_n(t) = \varepsilon$ with $\varepsilon \downarrow 0$, so that followers continue not to license the new technology without innovation (recall the comment in footnote 12).

[16]In other words, we interpret "full protection" to correspond to a situation in which $\zeta_n(t) = V_1(t) - V_0(t)$, where V_1 refers to the net present value of a firm that is one step ahead of its rival and V_0 is the value of a firm that is neck-and-neck with its rival (naturally, $\zeta_n(t) > V_1(t) - V_0(t)$ would also correspond to full protection, since in this case no follower would choose to license). Alternatively, full protection could be interpreted as corresponding to the case in which the follower pays a license fee equal to the loss of profits that it causes for the technology leader (see Scotchmer, 2005). In our model, this would correspond to $\zeta_n(t) = V_0(t) - V_{-1}(t)$, where V_{-1} is the net present value of a firm that is one step behind the technology leader. This expression applies, since without licensing the follower would have made its own innovation, in which case the leader would have fallen to the state of a neck-and-neck firm rather than being surpassed by one step. In all equilibria we compute below, we find that the second amount is significantly less than the first, thus our interpretation of full protection licensing fee is large enough to cover both possibilities. In any case, what value of ζ is designated as "full protection" does not have any bearing on our formal analysis, since we characterize the equilibrium for any ζ and then find the growth-maximizing policy sequence.

industry j as follows:

$$n_j(t+\Delta t) = \begin{cases} n_j(t)+1 & \text{with probability} \quad x_i(j,t)\,\Delta t + a_{-i}(j,t)\,x_{-i}(j,t)\,\Delta t + o(\Delta t) \\ 0 & \text{with probability} \quad \left((1-a_{-i}(j,t))\,x_{-i}(j,t) + \eta_{n_j(t)}\right)\Delta t + o(\Delta t) \\ n_j(t) & \text{with probability} \quad 1-\left(x_i(j,t)+x_{-i}(j,t)+\eta_{n_j(t)}\right)\Delta t - o(\Delta t)) \end{cases} . \tag{14}$$

Here $o(\Delta t)$ again represents second-order terms, in particular, the probabilities of more than one innovations within an interval of length Δt. The terms $x_i(j,t)$ and $x_{-i}(j,t)$ are the flow rates of innovation by the leader and the follower; $a_{-i}(j,t) \in [0,1]$ denotes whether the follower is trying to catch up with a leader or surpass it; and $\eta_{n_j(t)}$ is the flow rate at which the follower is allowed to copy the technology of a leader that is $n_j(t)$ steps ahead. Intuitively, when $a_{-i}(j,t) = 1$, the technology gap in industry j increases from $n_j(t)$ to $n_j(t)+1$ if either the leader is successful (flow rate $x_i(j,t)$) or if the follower is successful (flow rate $x_{-i}(j,t)$).

2.3 Profits

We next write the instantaneous "operating" profits for the leader (i.e., the profits exclusive of R&D expenditures and license fees). Profits of leader i in industry j at time t are

$$\begin{aligned} \Pi_i(j,t) &= [p_i(j,t) - MC_i(j,t)]\,y_i(j,t) \\ &= \left(\frac{w(t)}{q_{-i}(j,t)} - \frac{w(t)}{q_i(j,t)}\right)\frac{Y(t)}{p_i(j,t)} \\ &= \left(1-\lambda^{-n_j(t)}\right)Y(t) \end{aligned} \tag{15}$$

where $n_j(t) \equiv n_{ij}(t) - n_{-ij}(t)$ is the technology gap in industry j at time t. The first line simply uses the definition of operating profits as price minus marginal cost times quantity sold. The second line uses the fact that the equilibrium limit price of firm i is $p_i(j,t) = w(t)/q_{-i}(j,t)$ as given by (7), and the final equality uses the definitions of $q_i(j,t)$ and $q_{-i}(j,t)$ from (12) and (13). The expression in (15) also implies that there will be zero profits in an industry that is neck-and-neck, i.e., in industries with $n_j(t) = 0$. Also clearly, followers always make zero profits, since they have no sales.

The Cobb-Douglas aggregate production function in (3) is responsible for the form of the profits (15), since it implies that profits only depend on the technology gap of the industry and aggregate output. This will simplify the analysis below by making the technology gap in each industry the only industry-specific payoff-relevant state variable.

The objective function of each firm is to maximize the net present discounted value of net profits (operating profits minus R&D expenditures and plus or minus patent fees). In doing this, each firm will take the sequence of interest rates, $[r(t)]_{t\geq 0}$, the sequence of aggregate output levels, $[Y(t)]_{t\geq 0}$, the sequence of wages, $[w(t)]_{t\geq 0}$, the R&D decisions of all other firms and policies as given.

2.4 Equilibrium

Let $\boldsymbol{\mu}(t) \equiv \{\mu_n(t)\}_{n=0}^{\infty}$ denote the distribution of industries over different technology gaps, with $\sum_{n=0}^{\infty} \mu_n(t) = 1$. For example, $\mu_0(t)$ denotes the fraction of industries in which the firms are neck-and-neck at time t. Throughout, we focus on Markov Perfect Equilibria (MPE), where strategies are only functions of the payoff-relevant state variables.[17] This allows us to drop the dependence on industry j, thus we refer to R&D decisions by x_n for the technological leader that is n steps ahead and by a_{-n} and x_{-n} for a follower that is n steps behind. Let us denote the list of decisions by the leader and follower with technology gap n at time t by $\xi_n(t) \equiv \langle x_n(t), p_i(j,t), y_i(j,t)\rangle$ and $\xi_{-n}(t) \equiv \langle a_{-n}(t), x_{-n}(t)\rangle$.[18] Throughout, $\boldsymbol{\xi}$ will indicate the whole sequence of decisions at every state, $\boldsymbol{\xi}(t) \equiv \{\xi_n(t)\}_{n=-\infty}^{\infty}$. We define an allocation as follows:

Definition 1 *(Allocation) Let* $\left\langle \eta,\ [\zeta(t)]_{t\geq 0}\right\rangle$ *be the IPR policy sequences. Then an allocation is a sequence of decisions for a leader that is* $n = 0, 1, ..., \infty$ *step ahead,* $[\xi_n(t)]_{t\geq 0}$, *a sequence of R&D decisions for a follower that is* $n = 1, ..., \infty$ *step behind,* $\left[\xi_{-n}(t)\right]_{t\geq 0}$, *a sequence of wage rates* $[w(t)]_{t\geq 0}$, *and a sequence of industry distributions over technology gaps* $[\boldsymbol{\mu}(t)]_{t\geq 0}$.

For given IPR sequences η and $[\zeta(t)]_{t\geq 0}$, MPE strategies, which are only a function of the payoff-relevant state variables, can be represented as follows

$$
\begin{aligned}
\mathbf{x} &: \mathbb{Z} \times \mathbb{R}_+^2 \rightarrow \mathbb{R}_+, \\
\boldsymbol{a} &: \mathbb{Z}_- \backslash \{0\} \times \mathbb{R}_+^2 \rightarrow [0,1].
\end{aligned}
$$

[17] MPE is a natural equilibrium concept in this context, since it does not allow for implicit collusive agreements between the follower and the leader. While such collusive agreements may be likely when there are only two firms in the industry, in most industries there are many more firms and also many potential entrants, making collusion more difficult. Throughout, we assume that there are only two firms to keep the model tractable.

[18] The price and output decisions, $p_i(j,t)$ and $y_i(j,t)$, depend not only on the technology gap, aggregate output and the wage rate, but also on the exact technology rung of the leader, $n_{ij}(t)$. With a slight abuse of notation, throughout we suppress this dependence, since their product $p_i(j,t)\, y_i(j,t)$ and the resulting profits for the firm, (15), are independent of $n_{ij}(t)$, and only the technology gap, $n_j(t)$, matters for profits, R&D, aggregate output and economic growth.

The first function represents the R&D decision of a firm (both when it is a follower and when it is a leader in an industry) with an n step technology gap ($n \in \mathbb{Z}$), and the aggregate level of output and the wage ($(Y, w) \in \mathbb{R}_+^2$). The second function represents the decision of a follower (as a function of the level of output and the wage in the economy) of whether to direct its R&D to reaching or surpassing the leading-edge technology (or more precisely, it represents the probability with which the follower will choose to undertake R&D to surpass the leading edge technology). Consequently, we have the following definition of equilibrium (below we will define a steady-state equilibrium in terms of a solution to maximization problems):

Definition 2 ***(Equilibrium)*** *Given an IPR policy sequence* $\left\langle \eta, \ [\zeta(t)]_{t\geq 0} \right\rangle$, *a Markov Perfect Equilibrium is given by a sequence* $[\boldsymbol{\xi}^*(t), w^*(t), Y^*(t)]_{t\geq 0}$ *such that (i)* $[p_i^*(j,t)]_{t\geq 0}$ *and* $[y_i^*(j,t)]_{t\geq 0}$ *implied by* $[\boldsymbol{\xi}^*(t)]_{t\geq 0}$ *satisfy (7) and (8); (ii) R&D policies* $[\boldsymbol{a}^*(t), \mathbf{x}^*(t)]_{t\geq 0}$ *are best responses to themselves, i.e.,* $[\boldsymbol{a}^*(t), \mathbf{x}^*(t)]_{t\geq 0}$ *maximizes the expected profits of firms taking aggregate output* $[Y^*(t)]_{t\geq 0}$, *wages* $[w^*(t)]_{t\geq 0}$, *government policy* $\left\langle \eta, \ [\zeta(t)]_{t\geq 0} \right\rangle$ *and the R&D policies of other firms* $[\boldsymbol{a}^*(t), \mathbf{x}^*(t)]_{t\geq 0}$ *as given; (iii) aggregate output* $[Y^*(t)]_{t\geq 0}$ *is given by (3); and (iv) the labor market clears at all times given the wage sequence* $[w^*(t)]_{t\geq 0}$.

2.5 The Labor Market

Since only the technological leader produces, labor demand in industry j with technology gap $n_j(t) = n$ can be expressed as

$$l_n(t) = \frac{\lambda^{-n} Y(t)}{w(t)} \text{ for } n \in \mathbb{Z}_+. \tag{16}$$

In addition, there is demand for labor coming for R&D from both followers and leaders in all industries. Using (9) and the definition of the G function, we can express industry demands for R&D labor as

$$h_n(t) = \begin{cases} G(x_n(t)) + G(x_{-n}(t)) & \text{if } n \in \mathbb{N} \\ 2G(x_0(t)) & \text{if } n = 0 \end{cases}, \tag{17}$$

where $n < 0$ refers to the demand of a follower in an industry with a technology gap of n. Note that in this expression, $x_{-n}(t)$ refers to the R&D effort of a follower that is n steps behind (conditional on its optimal choice of $a_{-n}(t) \in [0,1]$). Moreover, this expression takes into account that in an industry with neck-and-neck competition, i.e., with $n = 0$, there will be twice the demand for R&D coming from the two firms.

The labor market clearing condition can then be expressed as:

$$1 \geq \sum_{n=0}^{\infty} \mu_n(t) \left[\frac{1}{\omega(t) \lambda^n} + G(x_n(t)) + G(x_{-n}(t)) \right], \tag{18}$$

and $\omega(t) \geq 0$, with complementary slackness, where

$$\omega(t) \equiv \frac{w(t)}{Y(t)} \tag{19}$$

is the labor share at time t. The labor market clearing condition, (18), uses the fact that total supply is equal to 1, and the demand cannot exceed this amount. If demand falls short of 1, then the wage rate, $w(t)$, and thus the labor share, $\omega(t)$, have to be equal to zero (though this will never be the case in equilibrium). The right-hand side of (18) consists of the demand for production (the terms with ω in the denominator), the demand for R&D workers from the neck-and-neck industries ($2G(x_0(t))$ when $n = 0$) and the demand for R&D workers coming from leaders and followers in other industries ($G(x_n(t)) + G(x_{-n}(t))$ when $n > 0$).

Defining the index of aggregate quality in this economy by the aggregate of the qualities in the different industries, i.e.,

$$\ln Q(t) \equiv \int_0^1 \ln q(j,t)\, dj, \tag{20}$$

the equilibrium wage can be written as:[19]

$$w(t) = Q(t)\, \lambda^{-\sum_{n=0}^{\infty} n\mu_n(t)}. \tag{21}$$

2.6 Steady State and the Value Functions

Let us now focus on steady-state (Markov Perfect) equilibria, where the distribution of industries $\mu(t) \equiv \{\mu_n(t)\}_{n=0}^{\infty}$ is stationary, $\omega(t)$ defined in (19) and g, the growth rate of the economy, are constant over time. We will establish the existence of such an equilibrium and characterize a number of its properties. If the economy is in steady state at time $t = 0$, then by definition, we have $Y(t) = Y_0 e^{g^* t}$ and $w(t) = w_0 e^{g^* t}$, where g^* is the steady-state growth rate. The two equations also imply that $\omega(t) = \omega^*$ for all $t \geq 0$. Throughout, we assume that the parameters are such that the steady-state growth rate g^* is positive but not large enough to violate the transversality conditions. This implies that net present values of each firm at all point in time will be finite and enable us to write the maximization problem of a leader that is $n > 0$ steps ahead recursively.

First note that given an optimal policy $\hat{x}$ for a firm, the net present discounted value of a leader that is n steps ahead at time t can be written as:

$$V_n(t) = \mathbb{E}_t \int_t^{\infty} \exp(-r(s-t)) \left[\Pi(s) + Z(s) - w(s)\, G(\hat{x}(s))\right] ds$$

[19] Note that $\ln Y(t) = \int_0^1 \ln q(j,t)\, l(j,t)\, dj = \int_0^1 \left[\ln q_i(j,t) + \ln \frac{Y(t)}{w(t)} \lambda^{-n_j}\right] dj$, where the second equality uses (16). Thus we have $\ln Y(t) = \int_0^1 [\ln q_i(j,t) + \ln Y(t) - \ln w(t) - n_j \ln \lambda]\, dj$. Rearranging and canceling terms, and writing $\exp \int n_j \ln \lambda dj = \lambda^{-\sum_{n=0}^{\infty} n\mu_n(t)}$, we obtain (21).

where $\Pi(s)$ is the operating profit at time $s \geq t$, $Z(s)$ is the patent license fees received (or paid) by a firm which is the leader and $w(s) G(\hat{x}(s))$ denotes the R&D expenditure at time $s \geq t$. All variables are stochastic and depend on the evolution of the technology gap within the industry.

Next taking the equilibrium R&D policy of other firms, $x^*_{-n}(t)$ and $a^*_{-n}(t)$, and equilibrium wage rate, $w^*(t)$, as given, this value can be written as:[20]

$$V_n(t) = \max_{x_n(t)} \left\{ [\Pi_n(t) - w^*(t) G(x_n(t))] \Delta t + o(\Delta t) \right. \tag{22}$$
$$\left. + \exp(-r(t+\Delta t)\Delta t) \begin{bmatrix} (x_n(t)\Delta t + o(\Delta t)) V_{n+1}(t+\Delta t) \\ + \left(\eta_n \Delta t + \left(1 - a^*_{-n}(t)\right) x^*_{-n}(t) \Delta t + o(\Delta t)\right) V_0(t+\Delta t) \\ + \left(a^*_{-n}(t) x^*_{-n}(t) \Delta t + o(\Delta t)\right) (V_{-1}(t+\Delta t) + \zeta_n) \\ + \left(1 - x_n(t)\Delta t - \eta_n \Delta t - x^*_{-n}(t)\Delta t - o(\Delta t)\right) V_n(t+\Delta t) \end{bmatrix} \right\}.$$

The first part of this expression is the flow profits minus R&D expenditures during a time interval of length Δt. The second part is the continuation value after this interval has elapsed. $V_{n+1}(t)$ and $V_0(t)$ are defined as net present discounted values for a leader that is $n+1$ steps ahead and a firm in an industry that is neck-and-neck (i.e., $n = 0$). The second part of the expression uses the fact that in a short time interval Δt, the probability of innovation by the leader is $x_n(t)\Delta t + o(\Delta t)$, where $o(\Delta t)$ again denotes second-order terms. This explains the first line of the continuation value. For the remainder of the continuation value, note that the probability that the follower will catch up with the leader is $\left(1 - a^*_{-n}(t)\right) x^*_{-n}(t) \Delta t + o(\Delta t)$; in particular, if $a^*_{-n}(t) = 1$, this eventually will never happen, since the follower would be undertaking R&D not to catch up but to surpass the leader. This explains the third line, which applies when $a^*_{-n}(t) = 1$. There are two differences between the second and third lines; (i) in the third line, conditional on success by the follower, a leader moves to the position of a follower rather than a neck-and-neck firm (V_{-1} instead of V_0); (ii) it receives the state-dependent patent fee ζ_n. Finally, the last line applies when no R&D effort is successful and patents continue to be enforced, so that the technology gap remains at n steps.

Next, subtract $V_n(t)$ from both sides, divide everything by Δt, and take the limit as $\Delta t \longrightarrow 0$.

[20] Clearly, this value function could be written for any arbitrary sequence of R&D policies of other firms. We set the R&D policies of other firms to their equilibrium values, $x^*_{-n}(t)$ and $a^*_{-n}(t)$, to reduce notation in the main body of the paper.

This yields:

$$
\begin{aligned}
r(t) V_n(t) - \dot{V}_n(t) &= \max_{x_n(t)} \{ [\Pi_n(t) - w^*(t) G(x_n(t))] + x_n(t) [V_{n+1}(t) - V_n(t)] \\
&+ \left(\left(1 - a^*_{-n}(t)\right) x^*_{-n}(t) + \eta_n \right) [V_0(t) - V_n(t)] \\
&+ \left(a^*_{-n}(t) x^*_{-n}(t) + \eta_n \right) [\zeta_n + V_0(t) - V_{-1}(t)] \},
\end{aligned} \tag{23}
$$

where $\dot{V}_n(t)$ denotes the derivative of $V_n(t)$ with respect to time. In steady state, the net present value of a firm that is n steps ahead, $V_n(t)$, will also grow at a constant rate g^* for all $n \in \mathbb{Z}_+$. Let us then define the normalized values as

$$
v_n(t) \equiv \frac{V_n(t)}{Y(t)} \tag{24}
$$

for all $n \in \mathbb{Z}$, which will be independent of time in steady state, i.e., $v_n(t) = v_n$. Similarly, in what follows we assume that license fees are also scaled up by GDP, so that

$$
\zeta_n = \frac{\zeta_n(t)}{Y(t)},
$$

which will ensure the existence of a (stationary) steady-state equilibrium (the use of $\zeta_n(t)$ for the original patent fees and ζ_n for the scaled-up fees should not cause any confusion).

Using (24) and the fact that from (2), $r(t) = g(t) + \rho$, the recursive form of the steady-state value function (23) can be written as:

$$
\begin{aligned}
\rho v_n &= \max_{x_n} \{ \left(1 - \lambda^{-n}\right) - \omega^* G(x_n) + x_n [v_{n+1} - v_n], \\
&+ \left[\left(1 - a^*_{-n}\right) x^*_{-n} + \eta_n \right] [v_0 - v_n] + a^*_{-n} x^*_{-n} [v_{-1} - v_n + \zeta_n] \} \text{ for } n \in \mathbb{N},
\end{aligned} \tag{25}
$$

where x^*_{-n} is the equilibrium value of R&D by a follower that is n steps behind, and ω^* is the steady-state labor share (while x_n is now explicitly chosen to maximize v_n).

Similarly the value for neck-and-neck firms is

$$
\rho v_0 = \max_{x_0} \{ -\omega^* G(x_0) + x_0 [v_1 - v_0] + x^*_0 [v_{-1} - v_0] \}, \tag{26}
$$

while the values for followers are given by

$$
\begin{aligned}
\rho v_{-n} &= \max_{x_{-n}, a_{-n}} \{ -\omega^* G(x_{-n}) + [(1 - a_{-n}) x_{-n} + \eta_n] [v_0 - v_{-n}] \\
&+ a_{-n} x_{-n} [v_1 - v_{-n} - \zeta_n] + x^*_n [v_{-n-1} - v_{-n}] \} \text{ for } n \in \mathbb{N},
\end{aligned} \tag{27}
$$

which takes into account that if the follower decides to build upon the leading-edge technology it will become the new leader, but will have to pay the patent fee ζ_n.

For neck-and-neck firms and followers, there are no instantaneous profits, which is reflected in (26) and (27). In the former case this is because neck-and-neck firms sell at marginal cost, and in the latter case, because followers have no sales. These normalized value functions emphasize that, because of growth, the effective discount rate is $r(t) - g(t) = \rho$ rather than $r(t)$.

The maximization problems in (25)-(26) immediately imply that any steady-state equilibrium R&D policies, $\langle \boldsymbol{a}^*, \mathbf{x}^* \rangle$, must satisfy:

$$a^*_{-n} \begin{cases} = 1 & \text{if } v_1 - \zeta_n > v_0 \\ \in [0,1] & \text{if } v_1 - \zeta_n = v_0 \\ = 0 & \text{if } v_1 - \zeta_n < v_0 \end{cases} \tag{28}$$

and

$$x^*_n = \max\left\{ G'^{-1}\left(\frac{[v_{n+1} - v_n]}{\omega^*}\right), 0\right\} \tag{29}$$

$$x^*_{-n} = \max\left\{ G'^{-1}\left(\frac{(1-a^*_{-n})[v_0 - v_{-n}] + a^*_{-n}[v_1 - v_{-n} - \zeta_n]}{\omega^*}\right), 0\right\} \tag{30}$$

$$x^*_0 = \max\left\{ G'^{-1}\left(\frac{[v_1 - v_0]}{\omega^*}\right), 0\right\}, \tag{31}$$

where $G'^{-1}(\cdot)$ is the inverse of the derivative of the G function, and since G is twice continuously differentiable and strictly concave, G'^{-1} is continuously differentiable and strictly increasing. These equations therefore imply that innovation rates, the x^*_n's, will increase whenever the incremental value of moving to the next step is greater and when the cost of R&D, as measured by the normalized wage rate, ω^*, is less. Note also that since $G'(0) > 0$, these R&D levels can be equal to zero, which is taken care of by the max operator.

The response of innovation rates, x^*_n, to the increments in values, $v_{n+1} - v_n$, is the key economic force in this model. For example, a policy that reduces the patent protection of leaders that are $n+1$ steps ahead (by increasing η_{n+1} or reducing ζ_{n+1}) will make being $n+1$ steps ahead less profitable, thus reduce $v_{n+1} - v_n$ and x^*_n. This corresponds to the standard *disincentive effect* of relaxing IPR policy. In contrast to existing models, however, here relaxing IPR policy can also create a positive *incentive effect*. This positive incentive effect has two components. First, as equation (30) shows, weaker patent protection in the form of lower license fees (lower ζ) may encourage further frontier R&D by the followers, directly contributing to aggregate growth. Second and perhaps somewhat more paradoxically, lower protection for technological leaders that are $n+1$ steps ahead will tend to reduce v_{n+1}, thus increasing $v_{n+2} - v_{n+1}$ and x^*_{n+1}. We will see that this latter effect plays an important role in the form of optimal state-dependent IPR policy. Finally, relaxing IPR protection may also create a beneficial *composition effect*; this is because, typically, $\{v_{n+1} - v_n\}_{n=0}^{\infty}$ is a decreasing sequence, which implies that x_{n-1}^* is higher than x_n^* for $n \geq 1$ (see Propositions 4 and 5 below). Weaker patent protection (in the form of shorter patent lengths) will shift more industries into the neck-and-neck state and potentially increase the equilibrium level of R&D in the economy.[21] The optimal level and structure of IPR policy in this economy will be determined by the interplay of these three forces.

Given the equilibrium R&D decisions $\langle \boldsymbol{a}^*, \mathbf{x}^* \rangle$, the steady-state distribution of industries across states $\boldsymbol{\mu}^*$ has to satisfy the following accounting identities:

$$\left(x_{n+1}^* + x_{-n-1}^* + \eta_{n+1}\right) \mu_{n+1}^* = x_n^* \mu_n^* \text{ for } n \in \mathbb{N}, \tag{32}$$

$$\left(x_1^* + x_{-1}^* + \eta_1\right) \mu_1^* = 2x_0^* \mu_0^* + \sum_{n=1}^{\infty} a_{-n}^* x_{-n}^* \mu_n^*, \tag{33}$$

$$2x_0^* \mu_0^* = \sum_{n=1}^{\infty} \left(\left(1 - a_{-n}^*\right) x_{-n}^* + \eta_n\right) \mu_n^*. \tag{34}$$

The first expression equates exit from state $n+1$ (which takes the form of the leader going one more step ahead or the follower catching up for surpassing the leader) to entry into the state (which takes the form of a leader from the state n making one more innovation). The second equation, (33), performs the same accounting for state 1, taking into account that entry into this state comes from innovation by either of the two firms that are competing neck-and-neck. Finally, equation (34) equates exit from state 0 with entry into this state, which comes from innovation by a follower in any industry with $n \geq 1$.

The labor market clearing condition in steady state can then be written as

$$1 \geq \sum_{n=0}^{\infty} \mu_n^* \left[\frac{1}{\omega^* \lambda^n} + G\left(x_n^*\right) + G\left(x_{-n}^*\right)\right] \text{ and } \omega^* \geq 0, \tag{35}$$

with complementary slackness.

The next proposition characterizes the steady-state growth rate. As with all the other results in the paper, the proof of this proposition is provided in the Appendix.

Proposition 1 *Let the steady-state distribution of industries and R&D decisions be given by* $\langle \boldsymbol{\mu}^*, \boldsymbol{a}^*, \mathbf{x}^* \rangle$, *then the steady-state growth rate is*

$$g^* = \ln \lambda \left[2\mu_0^* x_0^* + \sum_{n=1}^{\infty} \mu_n^* \left(x_n^* + a_{-n}^* x_{-n}^*\right)\right]. \tag{36}$$

[21] Weaker patent protection also creates a beneficial "level effect" by affecting equilibrium prices (as shown in equation (7) above) and by reallocating some of the workers engaged in "duplicative" R&D to production.

This proposition clarifies that the steady-state growth rate of the economy is determined by three factors:

1. R&D decisions of industries at different levels of technology gap, $\mathbf{x}^* \equiv \{x_n^*\}_{n=-\infty}^{\infty}$.

2. The distribution of industries across different technology gaps, $\boldsymbol{\mu}^* \equiv \{\mu_n^*\}_{n=0}^{\infty}$.

3. Whether followers are undertaking R&D to catch up with the frontier or to surpass the frontier, $\boldsymbol{a}^* \equiv \{a_n^*\}_{n=-\infty}^{-1}$.

IPR policy affects these three margins in different directions as illustrated by the discussion above.

We now define a steady-state equilibrium in a more convenient form, which will be used to establish existence and derive some of the properties of the equilibrium.

Definition 3 ***(Steady-State Equilibrium)*** *Given an IPR policy* $\langle \boldsymbol{\eta}, \boldsymbol{\zeta} \rangle$*, a steady-state equilibrium is a tuple* $\langle \boldsymbol{\mu}^*, \mathbf{v}, \boldsymbol{a}^*, \mathbf{x}^*, \omega^*, g^* \rangle$ *such that the distribution of industries* $\boldsymbol{\mu}^*$ *satisfy (32), (33) and (34), the values* $\mathbf{v} \equiv \{v_n\}_{n=-\infty}^{\infty}$ *satisfy (25), (26) and (27), the R&D decisions* $\boldsymbol{a}^*$ *and* $\mathbf{x}^*$ *are given by (28), (29), (30) and (31), the steady-state labor share* ω^* *satisfies (35) and the steady-state growth rate* g^* *is given by (36).*

We next provide a characterization of the steady-state equilibrium, starting first with the case in which there is uniform IPR policy.

3 Uniform IPR Policy

Let us first assume that the only available policy option is uniform IPR policy, whereby $\eta_n = \eta < \infty$ and $\zeta_n = \zeta < \infty$ for all $n \in \mathbb{N}$, which we denote by $\boldsymbol{\eta}^{uni}$ and $\boldsymbol{\zeta}^{uni}$. In this case, (27) implies that the problem is identical for all followers, so that $v_{-n} = v_{-1}$ for $n \in \mathbb{N}$. Consequently, (27) can be replaced with the following simpler equation:

$$\rho v_{-1} = \max_{x_{-1}, a_{-1}} \left\{ -\omega^* G(x_{-1}) + [(1 - a_{-1}) x_{-1} + \eta] [v_0 - v_{-1}] + a_{-1} x_{-1} [v_1 - v_{-1} - \zeta] \right\}, \quad (37)$$

implying optimal R&D decisions for all followers of the form

$$x_{-1}^* = \max \left\{ G'^{-1} \left(\frac{\max \langle [v_0 - v_{-1}], [v_1 - v_{-1} - \zeta] \rangle}{\omega^*} \right), 0 \right\}. \quad (38)$$

Let us denote the sequence of value functions under uniform IPR as $\{v_n\}_{n=-1}^{\infty}$. We start with a number of results that characterize the form of a steady-state equilibrium in this economy.

Proposition 2 *Consider a uniform IPR policy* $\langle \eta^{uni}, \ \zeta^{uni} \rangle$ *and suppose that* $\langle \boldsymbol{\mu}^*, \ \mathbf{v}, \ a^*_{-1}, \ \mathbf{x}^*, \ \omega^*, \ g^* \rangle$ *is a steady-state equilibrium. Then,* $v_{-1} \leq v_0$ *and* $\{v_n\}_{n=0}^{\infty}$ *forms a bounded and strictly increasing sequence converging to some positive value* v_{∞}.

The next result shows that we can restrict attention to a finite sequence of values:

Proposition 3 *Consider a uniform IPR policy* $\langle \eta^{uni}, \ \zeta^{uni} \rangle$ *and a corresponding steady-state equilibrium* $\langle \boldsymbol{\mu}^*, \ \mathbf{v}, \ a^*_{-1}, \ \mathbf{x}^*, \ \omega^*, \ g^* \rangle$. *Then, there exists* $n^* \in \mathbb{N}$ *such that* $x^*_n = 0$ *for all* $n \geq n^*$.

The next proposition shows that $\mathbf{x}^* \equiv \{x^*_n\}_{n=1}^{\infty}$ is a decreasing sequence, which implies that technological leaders that are further ahead undertake less R&D. Intuitively, the benefits of further R&D are decreasing in the technology gap, since greater values of the technology gap translate into smaller increases in the equilibrium markup (recall (15)).

Proposition 4 *Consider a uniform IPR policy* η^{uni} *and* ζ^{uni}. *Then in any steady-state equilibrium with uniform IPR,* $x^*_{n+1} \leq x^*_n$ *for all* $n \in \mathbb{N}$ *and moreover,* $x^*_{n+1} < x^*_n$ *if* $x^*_n > 0$.

A final comparison is between the R&D levels of neck and-and-neck firms, x^*_0, and the R&D investments of followers and leaders in industries with a technology gap of $n = 1$, x^*_{-1} and x^*_1. The next proposition shows that $x^*_0 > x^*_1$ and $x^*_0 \geq x^*_1$, so that without licensing, neck-and-neck industries are "most R&D intensive".

Proposition 5 *Consider a uniform IPR policy* η^{uni} *and* ζ^{uni}. *Then* $x^*_0 > x^*_1$ *and* $x^*_0 \geq x^*_{-1}$. *Moreover, if* $G'^{-1}\left(\left(1-\lambda^{-1}\right)/\left(\rho+\eta\right)\right) > 0$ *and* $\zeta > 0$, *then* $x^*_0 > x^*_{-1}$.

The condition $G'^{-1}\left(\left(1-\lambda^{-1}\right)/\left(\rho+\eta\right)\right) > 0$ here ensures that there will be positive R&D in the steady-state equilibrium (see Remark 1 below and Lemma 1 in the Appendix).

Next, we prove the existence of a steady-state equilibrium $\langle \boldsymbol{\mu}^*, \ \mathbf{v}, \ a^*_{-1}, \ \mathbf{x}^*, \ \omega^*, \ g^* \rangle$ with $\omega^* < 1$. A steady state with with $\omega^* < 1$ is economically more interesting, since when $\omega^* = 1$, there are no profits, thus it must be the case that $\mu^*_0 = 1$ (so that all firms are neck-and-neck and there are no markups). Equation (36) together with $\mu^*_0 = 1$ and $x^*_n = 0$ for all $n \in \mathbb{Z}_+$ would then imply that there is no economic growth, i.e., $g^* = 0$.

Establishing the existence of a steady-state equilibrium in this economy is made complicated by the fact that the equilibrium allocation cannot be represented as a solution to a maximization problem. Instead, as emphasized by Definition 3, each firm maximizes its value taking the R&D decisions of other firms as given; thus an equilibrium corresponds to a set of

R&D decisions that are best responses to themselves and a labor share (wage rate) ω^* that clears the labor market. We establish the existence of a steady-state equilibrium in a number of steps.

First, note that each x_n belongs to a compact interval $[0, \bar{x}]$, where $\bar{x}$ is the maximal flow rate of innovation defined in (11) above. Now fix a labor share $\tilde{\omega} \in [0,1]$ and a sequence $\langle \tilde{a}_{-1}, \tilde{\mathbf{x}} \rangle$ of (Markovian) steady-state strategies for all other firms in the economy, and consider the dynamic optimization problem of a single firm. Our next result characterizes this problem and shows that given some $\mathbf{z} \equiv \langle \tilde{\omega}, \tilde{a}_{-1}, \tilde{\mathbf{x}} \rangle$, the value function of an individual firm is uniquely determined, while its optimal R&D choices are given by a convex-valued correspondence. In what follows, we denote sets and correspondences by uppercase letters and refer to their elements by lowercase letters, e.g., $a_{-1}(\mathbf{z}) \in A_{-1}[\mathbf{z}]$, $x_n(\mathbf{z}) \in X_n[\mathbf{z}]$.

Proposition 6 *Consider a uniform IPR policy* $\langle \eta^{uni}, \ \zeta^{uni} \rangle$*, and suppose that the labor share and the R&D policies of all other firms are given by* $\mathbf{z} = \langle \tilde{\omega}, \tilde{a}_{-1}, \tilde{\mathbf{x}} \rangle$*. Then the dynamic optimization problem of an individual firm leads to a unique value function* $\mathbf{v}[\mathbf{z}] : \{-1\} \cup \mathbb{Z}_+ \to \mathbb{R}_+$ *and optimal R&D policies* $A_{-1}[\mathbf{z}] \subset [0,1]$ *and* $\mathbf{X}[\mathbf{z}] : \{-1\} \cup \mathbb{Z}_+ \rightrightarrows [0, \bar{x}]$ *are compact and convex-valued for each* $\mathbf{z} \in \mathbf{Z}$ *and upper hemi-continuous in* $\mathbf{z}$ *(where* $\mathbf{v}[\mathbf{z}] \equiv \{v_n[\mathbf{z}]\}_{n=-1}^{\infty}$ *and* $\mathbf{X}[\mathbf{z}] \equiv \{X_n[\mathbf{z}]\}_{n=-1}^{\infty}$ *).*

Now let us start with an arbitrary $\mathbf{z} \equiv \langle \tilde{\omega}, \tilde{a}_{-1}, \tilde{\mathbf{x}} \rangle \in \mathbf{Z} \equiv [0,1]^2 \times [0, \bar{x}]^{\infty}$. From Proposition 6, this $\mathbf{z}$ is mapped into optimal R&D decision sets $A_{-1}[\mathbf{z}]$ and $\mathbf{X}[\mathbf{z}]$, where $a_{-1} \in A_{-1}[\mathbf{z}]$ and $x_n[\mathbf{z}] \in X_n[\mathbf{z}]$. From R&D policies $\langle \tilde{a}_{-1}, \tilde{\mathbf{x}} \rangle$, we calculate $\boldsymbol{\mu}[\tilde{a}_{-1}, \tilde{\mathbf{x}}] \equiv \{\mu_n[\tilde{a}_{-1}, \tilde{\mathbf{x}}]\}_{n=0}^{\infty}$ using equations (32), (33) and (34). Then we can rewrite the labor market clearing condition (35) as

$$\begin{aligned} \omega &= \min\left\{ \sum_{n=0}^{\infty} \mu_n \left[\frac{1}{\lambda^n} + G(\tilde{x}_n)\tilde{\omega} + G(\tilde{x}_{-n}) \right] \tilde{\omega}; 1 \right\}, \\ &\equiv \varphi(\tilde{\omega}, \tilde{a}_{-1}, \tilde{\mathbf{x}}) \end{aligned} \tag{39}$$

where due to uniform IPR, $x_{-n}^* = x_{-1}^*$ for all $n > 0$. Next, define the mapping (correspondence) $\Phi[\mathbf{z}] \equiv (\varphi(\mathbf{z}), \ A_{-1}[\mathbf{z}], \ \mathbf{X}[\mathbf{z}])$, which maps $\mathbf{Z}$ into itself, that is,

$$\Phi \colon \mathbf{Z} \rightrightarrows \mathbf{Z}. \tag{40}$$

That Φ maps $\mathbf{Z}$ into itself follows since $\mathbf{z} \in \mathbf{Z}$ consists of $\tilde{a}_{-1} \in [0,1]$, $\tilde{\mathbf{x}} \in [0, \bar{x}]^{\infty}$ and $\tilde{\omega} \in [0,1]$, and the image of $\mathbf{z}$ under Φ consists of $a_{-1} \in [0,1]$ and $\mathbf{x} \in [0, \bar{x}]^{\infty}$, and moreover, (39) is clearly in $[0,1]$ (since the right-hand side is nonnegative and bounded above by 1). Finally, from Proposition 6, $A_{-1}[\mathbf{z}]$ and $X_n[\mathbf{z}]$ are compact and convex-valued for each $\mathbf{z} \in \mathbf{Z}$, and also upper hemi-continuous in $\mathbf{z}$, and φ is continuous. Using this construction, we can establish:

Proposition 7 *Consider a uniform IPR policy* $\langle \eta^{uni}, \zeta^{uni} \rangle$ *and suppose that* $G'^{-1}\left(\left(1-\lambda^{-1}\right)/\left(\rho+\eta\right)\right) > 0$. *Then a steady-state equilibrium* $\langle \boldsymbol{\mu}^*, \mathbf{v}, a^*_{-1}, \mathbf{x}^*, \omega^*, g^* \rangle$ *exists. Moreover, in any steady-state equilibrium* $\omega^* < 1$. *In addition, if either* $\eta > 0$ *or* $x^*_{-1} > 0$, *then* $g^* > 0$.

Remark 1 If the assumption that $G'^{-1}\left(\left(1-\lambda^{-1}\right)/\left(\rho+\eta\right)\right) > 0$ is relaxed, then there exists a trivial equilibrium in which $x^*_n = 0$ for all $n \in \mathbb{Z}_+$, i.e., an equilibrium in which there is no innovation and thus no growth (this follows from Lemma 1 and from Propositions 4 and 5, which imply that $x^*_0 \geq x^*_n$ for all $n \neq 0$). Moreover, if $\eta > 0$, then this equilibrium would also involve $\mu^*_0 = 1$, so that in every industry two firms with equal costs compete a la Bertrand and charge price equal to marginal cost, leading to zero aggregate profits and a labor share of output equal to 1. The assumption that $G'^{-1}\left(\left(1-\lambda^{-1}\right)/\left(\rho+\eta\right)\right) > 0$, on the other hand, is sufficient to rule out $\mu^*_0 = 1$ and thus $\omega^* = 1$. If in addition the steady-state equilibrium involves some probability of catch-up or innovation by the followers, i.e., either $\eta > 0$ or $x^*_{-1} > 0$, then the growth rate is also strictly positive. A sufficient condition to ensure that $x^*_{-1} > 0$ when $\eta = 0$ is that $G'^{-1}\left(\left(1-\lambda^{-1}\right)/\rho - \zeta\right) > 0$.[22]

An immediate consequence of Proposition 3, combined with (32) is that $\mu_n = 0$ for all $n \geq n^*$ (since there is no innovation in industries with technology gap greater than n^*). Thus the law of motion of an industry can be represented by a finite Markov chain. Moreover, because after an innovation by a follower, all industries jump to the neck-and-neck state (when $a^*_{-1} = 0$) or to the technology gap of one (when $a^*_{-1} = 1$), this Markov chain is irreducible (and aperiodic), thus converges to a unique steady-state distribution of industries. This is stated and proved in the next proposition.

Proposition 8 *Consider a uniform IPR policy* $\langle \eta^{uni}, \zeta^{uni} \rangle$ *and a steady-state equilibrium sequence of R&D decisions* $\langle a^*_{-1}, \mathbf{x}^* \rangle$. *Then, there exists a unique steady-state distribution of industries* $\boldsymbol{\mu}^*$.

[22]To see why this condition is sufficient suppose that $\eta = 0$ and also that $x^*_{-1} = 0$. Then (37) immediately implies $v_{-1} = 0$ and (25) implies $v_1 \geq \left(1-\lambda^{-1}\right)/\rho$. Moreover, from (38) and the fact that $\omega^* \leq 1$, we have $x^*_{-1} \geq G'^{-1}\left(v_1 - v_{-1} - \zeta\right) \geq G'^{-1}\left(\left(1-\lambda^{-1}\right)/\rho - \zeta\right)$. Therefore, $G'^{-1}\left(\left(1-\lambda^{-1}\right)/\rho - \zeta\right) > 0$ contradicts the hypothesis that $x^*_{-1} = 0$, and implies $x^*_{-1} > 0$. The reason why $\eta > 0$ can, under some circumstances, contribute to positive growth is related to the composition effect discussed above.

4 State-Dependent IPR Policy

We now extend the results from the previous section to the environment with state-dependent IPR policy. The main results from the previous section generalize, but the argument is slightly modified.

It is no longer necessarily the case that the sequence of values $\{v_n\}_{n=-\infty}^{\infty}$ is increasing, since IPR policies could be very sharply increasing, making a particular state very unattractive for the leader.[23] For this reason, we do not have the equivalent of Proposition 2. Nevertheless, it can be established that only a finite number of states will have positive weight in the steady-state distribution:

Proposition 9 *Consider the state-dependent IPR policy $\langle \eta, \zeta \rangle$, and suppose that $\langle \mu^*, \mathbf{v}, a^*_{-1}, \mathbf{x}^*, \omega^*, g^* \rangle$ is a steady-state equilibrium. Then there exists a state $n^* \in \mathbb{N}$ such that $\mu^*_n = 0$ for all $n \geq n^*$.*

Because of the reasons highlighted in footnote 23, Proposition 4 may not necessarily hold with state-dependent IPR. Nevertheless, the proofs of these propositions make it clear that as long as $\langle \eta, \zeta \rangle$ is not too far from uniform IPR, the conclusions in these propositions will continue to hold. In fact, our numerical results with optimal state-dependent IPR always verify the conclusions of Proposition 4 (but interestingly not those of Proposition 5; see Figure 5 in Section 5).

Our next result is a generalization of Proposition 6, which shows that each individual firm's maximization problem is well-behaved with state-dependent IPR.

Proposition 10 *Consider the state-dependent IPR policy $\langle \eta, \zeta \rangle$ and suppose that the labor share and the R&D policies of all other firms are given by $\mathbf{z} = \langle \tilde{\omega}, \tilde{\mathbf{a}}, \tilde{\mathbf{x}} \rangle$. Then the dynamic optimization problem of an individual firm leads to a unique value function $\mathbf{v}[\mathbf{z}] : \mathbb{Z} \to \mathbb{R}_+$ and optimal R&D policies $\mathbf{A}[\mathbf{z}] : \mathbb{Z}_- \setminus \{0\} \rightrightarrows [0,1]$ and $\mathbf{X}[\mathbf{z}] : \mathbb{Z} \rightrightarrows [0, \bar{x}]$ are compact and convex-valued for each $\mathbf{z} \in \mathbf{Z}$ and upper hemi-continuous in $\mathbf{z}$ (where $\mathbf{v}[\mathbf{z}] \equiv \{v_n[\mathbf{z}]\}_{n=-1}^{\infty}$, $\mathbf{A}[\mathbf{z}] \equiv \{A_n[\mathbf{z}]\}_{n=-\infty}^{-1}$ and $\mathbf{X}[\mathbf{z}] \equiv \{X_n[\mathbf{z}]\}_{n=-1}^{\infty}$).*

Finally, the next result generalizes Proposition 7 and establishes the existence of a steady-state equilibrium with positive growth.

[23] For example, we could have $\eta_n = 0$ and $\eta_{n+1} \to \infty$, which would imply that $v_{n+1} - v_n$ is negative.

Proposition 11 *Consider the state-dependent IPR policy* $\langle \eta, \zeta \rangle$ *and suppose that* $G'^{-1}\left(\left(1-\lambda^{-1}\right)/\left(\rho+\eta_1\right)\right) > 0$. *Then a steady-state equilibrium* $\langle \boldsymbol{\mu}^*, \mathrm{v}, a^*_{-1}, \mathbf{x}^*, \omega^*, g^* \rangle$ *exists. Moreover, in any steady-state equilibrium* $\omega^* < 1$. *In addition, if either* $\eta_1 > 0$ *or* $x^*_{-1} > 0$, *then* $g^* > 0$.

Although the analysis so far has established the existence of a steady-state equilibrium and characterized some of its properties, it is not possible to determine the optimal state-dependent IPR policy analytically. For this reason, in Section 5, we undertake a quantitative investigation of the form and structure of optimal state-dependent IPR policy using plausible parameter values.

5 Optimal IPR Policy: A Quantitative Investigation

In this section, we investigate the implications of various different types of IPR policies on the equilibrium growth and R&D rates using numerical computations of the steady-state equilibrium. Our purpose is not to provide a detailed calibration of the model economy, but to highlight the broad quantitative characteristics of the model and its implications for optimal IPR policy under plausible parameter values. As we will see, the structure of optimal IPR policy and the innovation gains from such policy are relatively invariant to the range of parameter values we consider.

5.1 Methodology

We take the annual discount rate as 5%, i.e., $\rho_{year} = 0.05$ and without loss of generality, we normalize labor supply to 1. The theoretical analysis considered a general production function for R&D given by (9). The empirical literature typically assumes a Cobb-Douglas production function. For example, Kortum (1993) considers a function of the form

$$\text{Innovation}\,(t) = B_0 \exp\left(\kappa t\right) \left(\text{R\&D inputs}\right)^{\gamma}, \tag{41}$$

where B_0 is a constant and $\exp\left(\kappa t\right)$ is a trend term, which may depend on general technological trends, a drift in technological opportunities, or changes in general equilibrium prices (such as wages of researchers etc.). The advantage of this form is not only its simplicity, but also the fact that most empirical work estimates a single elasticity for the response of innovation rates to R&D inputs. Consequently, they essentially only give information about the parameter γ in terms of equation (41). A low value of γ implies that the R&D production function is more concave. For example, Kortum (1993) reports that estimates of γ vary between 0.1 and 0.6

(see also Pakes and Griliches, 1980, or Hall, Hausman and Griliches, 1988). For these reasons, throughout, we adopt a R&D production function similar to (41):

$$x = Bh^{\gamma} \tag{42}$$

where $B, \gamma > 0$. In terms of our previous notation, equation (42) implies that $G(x) = [x/B]^{\frac{1}{\gamma}} w$, where w is the wage rate in the economy (thus in terms of the above function, it is captured by the $\exp(\kappa t)$ term).[24] Equation (42) does not satisfy the boundary conditions we imposed so far and can be easily modified to do so without affecting any of the results, since in all numerical exercises only a finite number of states are reached.[25] Following the estimates reported in Kortum (1993), we start with a benchmark value of $\gamma = 0.35$, and then report sensitivity checks for $\gamma = 0.1$ and $\gamma = 0.6$. The other parameter in (42), B, is chosen so as to ensure an annual growth rate of approximately 1.9%, i.e., $g^* \simeq 0.019$, in the benchmark economy which features indefinitely-enforced patents and no licensing. This growth rate together with $\rho_{year} = 0.05$ also pins down the annual interest rate as $r_{year} = 0.069$ from equation (2).

We choose the value of λ using a reasoning similar to Stokey (1995). Equation (36) implies that if the expected duration of time between any two consecutive innovations is about 3 years in an industry, then a growth rate of about 1.9% would require $\lambda = 1.05$.[26] We take $\lambda = 1.05$ as the benchmark value, and check the robustness of the results to $\lambda = 1.01$ and $\lambda = 1.2$ (expected duration of 1 year and 12 years, respectively). This completes all of the parameters of the model except the IPR policy.

As noted above, we begin with the full patent protection regime without licensing, i.e., $\langle \eta = \{0, 0, ...\}, \zeta = \{\infty, \infty, ...\} \rangle$. We then compare this to an economy with full patent protection and licensing, i.e., $\langle \eta = \{0, 0, ...\}, \zeta = \{\bar{\zeta}, \bar{\zeta}, ...\} \rangle$, where $\bar{\zeta} = v_1 - v_0$.[27] After this,

[24] More specifically, (42) can be alternatively written as

$$\text{Innovation}(t) = Bw(t)^{-\gamma} (\text{R\&D expenditure})^{\gamma},$$

thus would be equivalent to (41) as long as the growth of $w(t)$ can be approximated by constant rate.

[25] For example, we could add a small linear term to the production function for R&D, (42), and also make it flat after some level $\bar{h}$. For example, the following generalization of (42),

$$x = \min\left\{Bh^{\gamma} + \varepsilon h; B\bar{h}^{\gamma} + \varepsilon \bar{h}\right\}$$

for ε small and $\bar{h}$ large, makes no difference to our simulation results.

[26] In particular, in our benchmark parameterization with full protection without licensing, 24% of industries are in the neck-and-neck state. This implies that improvements in the technological capability of the economy is driven by the R&D efforts of the leaders in 76% of the industries and the R&D efforts of both the leaders and the followers in 24% of the industries. Therefore, the growth equation implies that we have $g \simeq \ln \lambda \times 1.24 \times x$, where x denotes the average frequency of innovation in a given industry. A major innovation on average every three years implies a value of $\lambda \simeq 1.05$.

[27] For the interpretation of full patent protection as $\bar{\zeta} = v_1 - v_0$, recall the discussion in footnote 16. Note also

we move to a comparison of the optimal (growth-maximizing) uniform IPR policy η^{uni}, ζ^{uni} to the optimal state-dependent IPR policy. Since it is computationally impossible to calculate the optimal value of each η_n and ζ_n, we limit our investigation to a particular form of state-dependent IPR policy, whereby the same η and ζ applies to all industries that have a technology gap of $n = 5$ or more. In other words, the IPR policy matrix takes the form:

IPR policy $\rightarrow$	*none*	(η_1, ζ_1)	(η_2, ζ_2)	(η_3, ζ_3)	(η_4, ζ_4)	(η_5, ζ_5)
Technology gap: $n \rightarrow$	0	1	2	3	4	5 6 7 8 9 10 11 . . ∞

We checked and verified that allowing for further flexibility (e.g., allowing η_5 and η_6 or ζ_5 and ζ_6 to differ) has little effect on our results.

The numerical methodology we pursue relies on uniformization and value function iteration. The details of the uniformization technique are described in the proof of Proposition 6. On value function iteration, see Judd (1999). In particular, we first take the IPR policies η and ζ as given and make an initial guess for the equilibrium labor share ω^*. Then conditional on ω^*, we generate a sequence of values $\{v_n\}_{n=-\infty}^{\infty}$ (or $\{v_n\}_{n=-1}^{\infty}$ depending on whether there is state-dependent IPR policy or not), and we derive the optimal R&D policies, $\{x_n^*\}_{n=-\infty}^{\infty}$, $\{a_n^*\}_{n=-\infty}^{-1}$ and the steady-state distribution of industries, $\{\mu_n^*\}_{n=0}^{\infty}$. After convergence, we compute the growth rate g^*, and then check for market clearing in the labor market from equation (18). Depending on whether there is excess demand or supply of labor, ω^* is varied and the whole process is repeated until the entire steady-state equilibrium for a given IPR policy is computed. The process is then repeated for different IPR policies.

In the state-dependent IPR case, the optimal (growth-maximizing) IPR policy sequences, η and ζ, are computed one element at a time, until we find the growth-maximizing value for that component, for example, η_1. We then move the next component, for example, η_2. Once the growth-maximizing value of η_2 is determined, we go back to optimize over η_1 again, and this procedure is repeated recursively until convergence.

We now present our simulation results. We start with the parameter values $\lambda = 1.05$ and $\gamma = 0.35$, and then show how the results change when we vary these parameters.

5.2 Full IPR Protection Without Licensing

We start with the benchmark with full protection and no licensing, which is the case that the existing literature has considered so far (e.g., Aghion, Harris, Howitt and Vickers, 2001). In

that at a license fee of $\bar{\zeta}$, followers are indifferent between $a = 0$ and $a = 1$, and in computing the equilibrium in this case we always suppose that they choose $a = 1$. Thus alternatively one might wish to think that $\bar{\zeta} = v_1 - v_0 - \varepsilon$ for $\varepsilon \downarrow 0$.

terms of our model, this corresponds to $\eta_n = 0$ for all n and $\zeta_n = \infty$ for all n. Equation (28) implies that $a^*_{-n} = 0$ for all n. We choose the parameter B in terms of (42), so that the benchmark has an annual growth rate of 1.86%.

The value function for this benchmark case is shown in Figure 1 (with the solid line). The value function has decreasing differences for $n \geq 0$, which is consistent with Proposition 4, and features a constant level for all followers (since there is no state dependence in the IPR policy). Figure 2 shows the level of R&D efforts for leaders and followers in this benchmark (again with the solid lines). This figure shows that the R&D level of a leader declines as the technology gap increases. Moreover, consistent with Propositions 4 and 5, the highest level of R&D is for firms that are neck-and-neck (i.e., at the technology gap of $n = 0$). Since there is no state-dependent IPR policy, all followers undertake the same level of R&D effort, which is also shown in the figure.

Figure 3 shows the distribution of industries according to technology gaps (again the solid line refers to the benchmark case). The mode of the distribution is at the technology gap of $n = 1$, but there is also a concentration of industries at technology gap $n = 0$, because $a^*_{-n} = 0$ implies that innovations by the followers take them to the "neck-and-neck" state.

The first column of Table 1 also reports the results for this benchmark simulation. As noted above, B is chosen such that the annual growth rate is equal to 0.0186, which is recorded at the bottom row. The table also shows the gap between x^*_0 and x^*_{-1} (0.029 versus 0.018), the frequencies of industries with technology gaps of 0, 1 and 2. The steady-state value of ω is 0.952. Since labor is the only factor of production in the economy, ω^* should not be thought of as the labor share in GDP. Instead, $1 - \omega^*$ measures the share of pure monopoly profits in value added. In the benchmark parameterization, this corresponds to 5% of GDP, which is reasonable.[28] Finally, the table also shows that in this benchmark parameterization 3.2% of the workforce is working as researchers, which is also consistent with US data.[29]

5.3 Full IPR Protection With Licensing

We next turn to full IPR protection with licensing. As specified above, we think of full IPR protection with licensing as corresponding to $\eta_n = 0$ for all n (so that patents never expire) and $\zeta_n = \bar{\zeta} \equiv v_1 - v_0$ for all n (so that the license fee for making use of a leading-edge technology is equal to the net present discounted value gap between being a one step ahead

[28] Bureau of Economic Analysis (2004) reports that the ratio of before-tax profits to GDP in the US economy in 2001 was 7% and the after-tax ratio was 5%.

[29] According to National Science Foundation (2006), the ratio of scientists and engineers in the US workforce in 2001 is about 4%.

leader and a neck-and-neck firm). Figures 1-3 show the corresponding value functions, R&D effort levels and distribution of industries for this case (with the dashed lines). Since there is no state-dependent policy the general pattern is similar to that in the economy without licensing. There is no longer a spike in R&D effort at $n = 0$, however, since now firms always prefer to pay the license and jump ahead of the leading-edge technology. This makes the neck-and-neck state no longer special (in fact, as column 2 of Table 1 shows in equilibrium there will be no industries in the neck-and-neck state). More importantly, the level of R&D by followers is considerably higher than in the benchmark case. In particular, x^*_{-1} is now 0.021 rather than 0.018. The resulting growth rate is 2.58% instead of 1.86% in the benchmark. The boost to growth comes not from the increase in the R&D effort overall, but from the fact that the R&D of the followers now also advances the technological frontier of the economy since they can license and build on the leading-edge technology (recall equation (36)). In fact, column 2 of Table 1 shows that this considerably higher growth rate is achieved with a *lower* fraction of the workforce, only 2.6%, working in the research sector.

This result, which is robust across different parameterizations of the model, is the first important implication of our analysis. Relative to existing models of step-by-step innovation, such as Aghion, Harris, Howitt and Vickers (2001), which do not allow for the possibility of licensing, here the R&D effort by followers can directly contribute to economic growth and this increases the equilibrium growth rate of the economy.

5.4 Optimal Uniform IPR Protection

We next turn to optimal (growth-maximizing) IPR policy with licensing. That is, we impose that $\eta_n = \eta$ and $\zeta_n = \zeta$ for all n, and look for values of η and ζ that maximize the growth rate. Column 3 of Table 1 shows that the growth-maximizing values of η and ζ are both equal to 0 in the benchmark parameterization. This corresponds to zero license fees and indefinite life of patents, so that followers can never copy the leading-edge technology without R&D, but they can always advance one step ahead of the leader when they are successful in their R&D efforts (without paying any license fees).

The resulting value function, R&D effort levels and industry distributions according to technology gaps are shown in Figures 4-6 (with the solid lines).

The figures and column 3 of Table 1 show that the growth-maximizing IPR policy discourages leaders (which can be seen from the fact that $v_1 - v_0$ declines significantly), but encourages R&D effort by the followers, since when successful they do not have to pay the license fee. The optimal uniform IPR increases the growth rate by only a little, however. While the growth rate of the economy with full IPR protection with licensing was 2.58%, it is now 2.63%. This increase is associated with a very modest rise in the share of the labor force working in research (from 2.6% to 2.7%).

5.5 Optimal State-Dependent IPR Without Licensing

We next turn to our second major question; whether state-dependent IPR makes a significant difference relative the uniform IPR. We first investigate this question without licensing (so that the comparison is to the benchmark case in column 1). In particular, we set $\zeta_n = \infty$ for all n and look for the combination of $\{\eta_1, ..., \eta_5\}$ that maximizes the growth rate. The results are shown in column 4 of Table 1.

Two features are worth noting. First, the growth rate increases noticeably relative to column 1; it is now 2.03% instead of 1.86%. Nevertheless, this increase is quite small relative to the benefits of licensing. Therefore, state-dependent IPR policy with no licensing is not a substitute for licensing.

Second, we see an interesting pattern (which is in fact quite general in all of our quantitative investigations). The optimal state-dependent policy $\{\eta_1, ..., \eta_5\}$ provides *greater* protection to technological leaders that are further ahead. In particular, we find that the optimal policy involves $\eta_1 = 0.054$, $\eta_2 = 0.005$, and $\eta_3 = \eta_4 = \eta_5 = 0$. This corresponds to very little patent protection for firms that are one step ahead of the followers, which can be seen by comparing $\eta_1 = 0.054$ to $x^*_{-1} = 0.010$. This comparison implies that firms that are one step behind followers are more than *five times* as likely to catch up with the technological leader because of the expiration of the patent of the leader as they are likely to catch up because of their own successful R&D. Then, there is a steep increase in the protection provided to technological leaders that are two steps ahead, and η_2 is 1/10th of η_1. Perhaps even more remarkably, after a technology gap of three or more steps, there is full protection and patents never expire.

This pattern of greater protection for technological leaders that are further ahead may go against a naïve intuition that state-dependent IPR policy should try to boost the growth rate of the economy by bringing more industries with large technology gaps (where leaders engage in little R&D) into neck-and-neck competition. This composition effect is present, but it is dominated by another, more powerful force, the *trickle-down* effect. The intuition for the trickle-down effect is as follows: by providing secure patent protection to firms that are three or more steps ahead of their rivals, optimal state-dependent IPR increases the R&D effort of leaders that are one and two steps ahead as well—despite the fact that these firms now face less secure protection of their own intellectual property. In fact, this is precisely because of low protection for technological leaders that are one or two steps ahead combined with the promise of much greater protection once they reach a technology gap of three steps or more. Mechanically, high levels of η_1 and η_2 reduce v_1 and v_2, while high IPR protection for more advanced firms increases v_n for $n \geq 3$. Consequently, it becomes more beneficial for leaders that are one or two steps ahead to undertake R&D to reach the higher level of IPR protection. This pattern of increased R&D for leaders under state-dependent IPR contrasts with uniform IPR, which always reduces R&D by all technological leaders. The possibility that imperfect state-dependent IPR protection can increase R&D incentives is a novel feature of our approach, since previous models have not considered state-dependent IPR policies.

5.6 Optimal State-Dependent IPR With Licensing

Finally, we turn to the most general policy choice within the context of our approach and investigate which combinations of $\{\eta_1, ..., \eta_5\}$ and $\{\zeta_1, ..., \zeta_5\}$ maximize the growth rate and the contribution of the optimal state-dependent IPR policies to economic growth. The results of this exercise are shown in column 5 of Table 1. Now the comparison should be to the optimal uniform IPR policy with licensing in column 3, where uniform IPR policies η and ζ were chosen to maximize the aggregate growth rate. The value functions, R&D efforts and the industry distribution over different levels of technology gaps in this economy are shown in Figures 4-6 (with the dashed lines).

We see in column 5 that growth-maximizing IPR policy involves $\eta_n = 0$ for all n, so that patents never expire. Nevertheless, IPR protection for technological leaders is not full. In particular, the growth-maximizing policy involves $\zeta_1 = 0$, which implies that followers can build on the leading-edge technology that is one step ahead of their own knowledge without paying any license fees. From there on, ζ increases to $\zeta_2 = 0.82$, which is still less than full patent protection. After three or more steps, however, we have $\zeta_3 = 1.94$ and $\zeta_4 = \zeta_5 = 1.95$, which are very close to full patent protection (since $v_1 - v_0 = 1.96$). The resulting growth rate of the economy is 2.96%, which is significantly higher than the growth rate under uniform IPR policy, 2.63% in column 3. This shows that state-dependent policies can increase the growth rate of the economy significantly.

State-dependent policies again achieve this superior growth performance by exploiting the trickle-down effect, which we already saw in the case without licensing. In particular, ζ_n is an increasing sequence, which implies that technological leaders that are further ahead receive greater protection. As in the previous subsection, this pattern of IPR is used as a way of boosting the R&D effort of technological leaders that are one or two steps ahead of their rivals (see Figure 5). Since these leaders receive little protection and understand that they can increase both their profits and their IPR protection by undertaking further innovations, they have relatively strong innovation incentives and undertake high levels of R&D. This increases the fraction of the labor force working in R&D to 3.9% (which is greater than all the other cases in Table 1) and again illustrates how imperfect state-dependent IPR protection can increase R&D incentives relative to full protection.

The growth rate of the economy also receives a boost from the R&D effort of the followers that are one step behind the leaders, since, thanks to licensing, followers' R&D directly contributes to the advancing the technological frontier of the economy. Figure 5 shows that followers that are one step behind the frontier now have a higher R&D effort than even in the case with growth-maximizing uniform IPR (which involved $\zeta_n = 0$ for all n). The reason for this pattern of R&D efforts is that the growth-maximizing IPR policy and the trickle-down effect have increased the value of being a technological leader. In contrast, the R&D level of followers that are more than once step behind is lower than in the economy with uniform IPR.

Overall, the results show that state-dependent IPR policies can increase the equilibrium growth rate of the economy substantially and that the trickle-down effect is powerful, not only when we consider the economy without licensing, but also in the presence of licensing.

5.7 Robustness

Tables 2-5 show the robustness of the patterns documented in Figures 1-6 and in Table 1. In particular, each of these tables changes one of the two parameters λ and γ (increasing or reducing λ to 1.2 or 1.01, and increasing or reducing γ to 0.6 or 0.1) and show the results corresponding to each one of the five different policy regimes and discussed so far. In each case, we change the parameter B in equation (42) so that the growth rate of the benchmark economy with full IPR protection without licensing is the same as in Table 1, 1.86%.

Notably, the qualitative, and even the quantitative, patterns in Table 1 are relatively robust. In all cases we see a significant increase in the growth rate when we allow licensing. The smallest increase is when $\gamma = 0.6$, presumably because with limited diminishing returns to R&D, incentives were already sufficiently strong without licensing. As a result, in this case, the growth rate increases only from 1.86% to 1.98%. In all other cases, allowing for licensing increases the growth rate to above 2.5%, which is a very large increase relative to the baseline of 1.86%.

Moreover, in all cases, moving to state-dependent IPR policy increases the growth rate further, though the extent of the increase varies depending on parameters. The most modest gain comes when $\gamma = 0.1$, because in this case the economy with full protection and licensing already achieves a very high growth rate of 2.78%, and thus there is only little room left for an increase in growth rate from state-dependent IPR. In all other cases, the increase in the growth rate is quite substantial.

Perhaps, more noteworthy is the fact that in all cases, growth-maximizing state-dependent IPR is shaped by the trickle-down effect. In all of the various parameterizations we have considered, there is little or no protection provided to technological leaders that are one step ahead, but IPR protection grows as the technology gap increases. This is the typical pattern implied by the trickle-down effect.

We therefore conclude that both the substantial benefits of licensing in models of technological competition and the benefits of state-dependent policies, mostly coming from the trickle-down effect, are robust across different specifications.

6 Conclusions

In this paper, we developed a general equilibrium framework to investigate the impact of the extent and form of intellectual property rights (IPR) policy on economic growth. The two major questions we focused on are whether licensing, which allows followers to build on the leading-edge technology in return of a license fee, has a major impact on the equilibrium growth rate and whether the same degree of patent protection should be given to companies that are further ahead of their competitors as those that are technologically close to their competitors.

In our model economy, firms engage in cumulative (step-by-step) innovation. Leaders can innovate in order to widen the technology gap between themselves and the followers, which enables them to charge higher markups. On the other hand, followers innovate to catch up with or surpass the technological leaders in their industry. Followers can advance in three different ways. First, the patent of the technological leader may expire, allowing the follower in the industry to copy the leading-edge technology. Second, each follower can undertake catch-up R&D to improve its own variant of the product to catch up with the leader. Third, each follower can undertake frontier R&D, building on and improving the leading-edge technology; in this case, when successful, it will have to pay a license fee to the technological leader.

In the model economy, IPR policy regulates whether licensing is possible and how costly it will be far followers, and also the length of the patents. We characterized the form of the steady-state equilibrium and proved its existence under general IPR policies. We then used this framework to investigate the form of "optimal" (growth-maximizing) IPR policy quantitatively.

The major findings of this quantitative exercise are as follows:

1. A move from an IPR policy without licensing to one that allows for licensing has a significant effect on the equilibrium growth rate. For the benchmark parameterization of our model, licensing increases the growth rate from 1.86% to 2.58% per annum. This substantial increase in growth rate is robust to a large range of variation in the parameters.

2. State-dependent IPR also leads to a significant improvement in the equilibrium growth rate. In our benchmark parameterization, the growth rate of the economy increases from 2.57% under the growth-maximizing uniform IPR policy to 2.93% under state-dependent IPR policy. Perhaps more interesting than this substantial impact on the equilibrium growth rate is the form of the optimal state-dependent IPR policy. Contrary to a naïve intuition, we find that the growth-maximizing IPR policy provides greater protection to firms that are further ahead of their rivals than those that are technologically close to their competitors. Underlying this form of the optimal IPR policy is *the trickle-down effect*. The trickle-down effect implies that providing greater protection to sufficiently advanced technological leaders not only increases their R&D efforts, but also raises the R&D efforts of all technological leaders that are less advanced than this level, because the reward to innovation now includes the greater protection that they will receive once they reach this higher level of technology. Our results suggest that the trickle-down effect is powerful both with and without licensing, and its form and magnitude are relatively insensitive to the exact parameter values used in the quantitative investigation.

The analysis in this paper suggests that a move to a richer menu of IPR policies, in particular, a move towards optimal state-dependent policies with licensing may significantly increase innovation and economic growth. The results also show that the form of optimal IPR policy may depend on the industry structure (and the technology of catch-up within the industry). Nevertheless, these conclusions are based on a quantitative evaluation of a rather simple model. It would be interesting to investigate the robustness of these results to different models of industry dynamics and study whether the relationship between the form of optimal IPR policy and industry structure suggested by our analysis also applies when variation in industry structure has other sources (for example, differences in the extent of fixed costs). The most important area for future work is a detailed empirical investigation of the form of optimal IPR policy, using both better estimates of the effects of IPR policy on innovation rates and also structural models where the effect of different policies on equilibrium can be evaluated.

Appendix: Proofs

Proof of Proposition 1. Equations (19) and (21) imply

$$Y(t) = \frac{w(t)}{\omega(t)} = \frac{Q(t)\,\lambda^{-\sum_{n=0}^{\infty} n\mu_n^*(t)}}{\omega(t)}.$$

Since $\omega(t) = \omega^*$ and $\{\mu_n^*\}_{n=0}^{\infty}$ are constant in steady state, $Y(t)$ grows at the same rate as $Q(t)$. Therefore,

$$g^* = \lim_{\Delta t \to 0} \frac{\ln Q(t+\Delta t) - \ln Q(t)}{\Delta t}.$$

Now note the following: during an interval of length Δt (i) in the fraction μ_n^* of the industries with technology gap $n \geq 1$ the leaders innovate at a rate $x_n^* \Delta t + o(\Delta t)$; (ii) in the same industries, the followers innovate at the rate $a_{-n}^* x_{-n}^* \Delta t + o(\Delta t)$; (iii) in the fraction μ_0^* of the industries with technology gap of $n = 0$, both firms innovate, so that the total innovation rate is $2x_0^* \Delta t + o(\Delta t)$); and (iv) each innovation increase productivity by a factor λ. Combining these observations, we have

$$\ln Q(t+\Delta t) = \ln Q(t) + \ln \lambda \left[2\mu_0^* x_0^* \Delta t + \sum_{n=1}^{\infty} \mu_n^* x_n^* \Delta t + o(\Delta t) + \sum_{n=1}^{\infty} a_{-n}^* x_{-n}^* \Delta t + o(\Delta t) \right].$$

Subtracting $\ln Q(t)$, dividing by Δt and taking the limit $\Delta t \to 0$ gives (36). ■

Proof of Proposition 2. Let $\left\langle a_{-1}, \{x_n\}_{n=-1}^{\infty} \right\rangle$ be the R&D decisions of the firm and $\{v_n\}_{n=-1}^{\infty}$ be the sequence of values, taking the decisions of other firms and the industry distributions, $\{x_n^*\}_{n=-1}^{\infty}$, $\{\mu_n^*\}_{n=-1}^{\infty}$, ω^* and g, as given. By choosing $x_n = 0$ for all $n \geq -1$, the firm guarantees $v_n \geq 0$ for all $n \geq -1$. Moreover, since flow profit satisfy $\pi_n \leq 1$ for all $n \geq -1$, $v_n \leq 1/\rho$ for all $n \geq -1$, establishing that $\{v_n\}_{n=-1}^{\infty}$ is a bounded sequence, with $v_n \in [0, 1/\rho]$ for all $n \geq -1$.

Proof of $v_1 > v_0$: Suppose, first, $v_1 \leq v_0$, then (31) implies $x_0^* = 0$, and by the symmetry of the problem in equilibrium (26) implies $v_0 = v_1 = 0$. As a result, from (30) we obtain $x_{-1}^* = 0$. Equation (25) implies that when $x_{-1}^* = 0$, $v_1 \geq \left(1 - \lambda^{-1}\right)/(\rho + \eta) > 0$, yielding a contradiction and proving that $v_1 > v_0$.

Proof of $v_{-1} \leq v_0$: Suppose, to obtain a contradiction, that $v_{-1} > v_0$.
If $v_1 - \zeta \leq v_0$, (30) yields $x_{-1}^* = 0$. This implies $v_{-1} = \eta v_0/(\rho + \eta)$, which contradicts $v_{-1} \geq v_0$ since $\eta/(\rho+\eta) < 1$. Thus we must have $v_1 - \zeta > v_0$, which implies that $a_{-1}^* = 1$. Imposing $a_{-1}^* = 1$, the value function of a neck-and-neck firm can be written as:

$$\begin{aligned} \rho v_0 &= \max_{x_0} \left\{ -\omega^* G(x_0) + x_0 [v_1 - v_0] + x_0^* [v_{-1} - v_0] \right\}, \qquad (43) \\ &\geq \max_{x_0} \left\{ -\omega^* G(x_0) + x_0 [v_1 - v_0] \right\}, \\ &\geq \max_{x_0} \left\{ -\omega^* G(x_0) + x_0 [v_1 - v_{-1} - \zeta] \right\}, \\ &\geq -\omega^* G\left(x_{-1}^*\right) + x_{-1}^* [v_1 - v_{-1} - \zeta], \\ &\geq -\omega^* G\left(x_{-1}^*\right) + x_{-1}^* [v_1 - v_{-1} - \zeta] + \eta [v_0 - v_{-1}], \\ &= \rho v_{-1}, \end{aligned}$$

which contradicts the hypothesis that $v_{-1} > v_0$ and establishes the claim.

Proof of $v_n < v_{n+1}$: Suppose, to obtain a contradiction, that $v_n \geq v_{n+1}$. Now (29) implies $x_n^* = 0$, and (25) becomes

$$\rho v_n = \left(1 - \lambda^{-n}\right) + x_{-1}^* \left[a_{-1}^* (v_{-1} + \zeta) + \left(1 - a_{-1}^*\right) v_0 - v_n \right] + \eta [v_0 - v_n] \qquad (44)$$

Also from (25), the value for state $n+1$ satisfies

$$\rho v_{n+1} \geq \left(1-\lambda^{-n-1}\right) + x_{-1}^{*}\left[a_{-1}^{*}\left(v_{-1}+\zeta\right)+\left(1-a_{-1}^{*}\right) v_0 - v_{n+1}\right] + \eta\left[v_0 - v_{n+1}\right]. \tag{45}$$

Combining the two previous expressions, we obtain

$$\begin{aligned} & \left(1-\lambda^{-n}\right) + x_{-1}^{*}\left[a_{-1}^{*}\left(v_{-1}+\zeta\right)+\left(1-a_{-1}^{*}\right) v_0 - v_n\right] + \eta\left[v_0 - v_n\right] \\ \geq\ & 1-\lambda^{-n-1} + x_{-1}^{*}\left[a_{-1}^{*}\left(v_{-1}+\zeta\right)+\left(1-a_{-1}^{*}\right) v_0 - v_{n+1}\right] + \eta\left[v_0 - v_{n+1}\right]. \end{aligned}$$

Since $\lambda^{-n-1} < \lambda^{-n}$, this implies $v_n < v_{n+1}$, contradicting the hypothesis that $v_n \geq v_{n+1}$, and establishing the desired result, $v_n < v_{n+1}$. Consequently, $\{v_n\}_{n=-1}^{\infty}$ is nondecreasing and $\{v_n\}_{n=0}^{\infty}$ is (strictly) increasing. Since a nondecreasing sequence in a compact set must converge, $\{v_n\}_{n=-1}^{\infty}$ converges to its limit point, v_{∞}, which must be strictly positive, since $\{v_n\}_{n=0}^{\infty}$ is strictly increasing and has a nonnegative initial value. This completes the proof. ■

Proof of Proposition 3. The first-order condition of the maximization of the value function (25) implies:

$$G'\left(x_n\right) \geq \frac{v_{n+1}-v_n}{\omega^{*}} \text{ and } x_n \geq 0,$$

with complementary slackness. $G'(0)$ is strictly positive by assumption. If $\left(v_{n+1}-v_n\right)/\omega^{*} < G'(0)$, then $x_n = 0$. Proposition 2 implies that $\{v_n\}_{n=-1}^{\infty}$ is a convergent and thus a Cauchy sequence, which implies that there exists $\exists n^{*} \in \mathbb{N}$ such that $v_{n+1}-v_n < \omega^{*} G'(0)$ for all $n \geq n^{*}$. ■

Proof of Proposition 4. From equation (29),

$$\delta_{n+1} \equiv v_{n+1} - v_n < v_n - v_{n-1} \equiv \delta_n \tag{46}$$

is sufficient to establish that $x_{n+1}^{*} \leq x_n^{*}$.

Let us write:

$$\bar{\rho} v_n = \max_{x_n}\left\{\left(1-\lambda^{-n}\right) - \omega^{*} G\left(x_n\right) + x_n^{*}\left[v_{n+1}-v_n\right] + x_{-1}^{*}\left[a_{-1}^{*}\left(v_{-1}+\zeta\right)+\left(1-a_{-1}^{*}\right) v_0\right] + \eta v_0\right\}, \tag{47}$$

where $\bar{\rho} \equiv \rho + x_{-1}^{*} + \eta$. Since x_{n+1}^{*}, x_n^{*} and x_{n-1}^{*} are maximizers of the value functions v_{n+1}, v_n and v_{n-1}, (47) implies:

$$\begin{aligned} \bar{\rho} v_{n+1} &= 1-\lambda^{-n-1} - \omega^{*} G\left(x_{n+1}^{*}\right) + x_{n+1}^{*}\left[v_{n+2}-v_{n+1}\right] + x_{-1}^{*}\left[a_{-1}^{*}\left(v_{-1}+\zeta\right)+\left(1-a_{-1}^{*}\right) v_0\right] + \eta v_0, \quad (48) \\ \bar{\rho} v_n &\geq 1-\lambda^{-n} - \omega^{*} G\left(x_{n+1}^{*}\right) + x_{n+1}^{*}\left[v_{n+1}-v_n\right] + x_{-1}^{*}\left[a_{-1}^{*}\left(v_{-1}+\zeta\right)+\left(1-a_{-1}^{*}\right) v_0\right] + \eta v_0, \\ \bar{\rho} v_n &\geq 1-\lambda^{-n} - \omega^{*} G\left(x_{n-1}^{*}\right) + x_{n-1}^{*}\left[v_{n+1}-v_n\right] + x_{-1}^{*}\left[a_{-1}^{*}\left(v_{-1}+\zeta\right)+\left(1-a_{-1}^{*}\right) v_0\right] + \eta v_0, \\ \bar{\rho} v_{n-1} &= 1-\lambda^{-n+1} - \omega^{*} G\left(x_{n-1}^{*}\right) + x_{n-1}^{*}\left[v_n-v_{n-1}\right] + x_{-1}^{*}\left[a_{-1}^{*}\left(v_{-1}+\zeta\right)+\left(1-a_{-1}^{*}\right) v_0\right] + \eta v_0. \end{aligned}$$

Now taking differences with $\bar{\rho} v_n$ and using the definitions of δ_n's, we obtain

$$\begin{aligned} \bar{\rho} \delta_{n+1} &\leq \lambda^{-n}\left(1-\lambda^{-1}\right) + x_{n+1}^{*}\left(\delta_{n+2}-\delta_{n+1}\right) \\ \bar{\rho} \delta_n &\geq \lambda^{-n+1}\left(1-\lambda^{-1}\right) + x_{n-1}^{*}\left(\delta_{n+1}-\delta_n\right). \end{aligned}$$

Therefore,

$$\left(\bar{\rho} + x_{n-1}^{*}\right)\left(\delta_{n+1}-\delta_n\right) \leq -k_n + x_{n+1}^{*}\left(\delta_{n+2}-\delta_{n+1}\right), \tag{49}$$

where

$$k_n \equiv (\lambda-1)^2 \lambda^{-n-1} > 0.$$

Now to obtain a contradiction, suppose that $\delta_{n+1} - \delta_n \geq 0$. From (49), this implies $\delta_{n+2} - \delta_{n+1} > 0$ since k_n is strictly positive. Repeating this argument successively, we have that if $\delta_{n'+1} - \delta_{n'} \geq 0$, then $\delta_{n+1} - \delta_n > 0$ for all $n \geq n'$. However, we know from Proposition 2 that $\{v_n\}_{n=0}^{\infty}$ is strictly increasing and converges to a constant v_∞. This implies that $\delta_n \downarrow 0$, which contradicts the hypothesis that $\delta_{n+1} - \delta_n \geq 0$ for all $n \geq n' \geq 0$, and establishes that $x^*_{n+1} \leq x^*_n$. To see that the inequality is strict when $x^*_n > 0$, it suffices to note that we have already established (46), i.e., $\delta_{n+1} - \delta_n < 0$, thus if equation (29) has a positive solution, then we necessarily have $x^*_{n+1} < x^*_n$. ■

Proof of Proposition 5. *Proof of* $x_0^* \geq x_{-1}^*$: Suppose first that $\zeta > v_1 - v_0$. Then (28) implies $a_{-1}^* = 0$, and (26) can be written as

$$\rho v_0 = -\omega^* G(x_0^*) + x_0^* [v_{-1} + v_1 - 2v_0]. \tag{50}$$

We have $v_0 \geq 0$ from Proposition 2. Suppose $v_0 > 0$. Then (50) implies $x_0^* > 0$ and

$$\begin{aligned} v_{-1} + v_1 - 2v_0 &> 0 \\ v_1 - v_0 &> v_0 - v_{-1}. \end{aligned} \tag{51}$$

This inequality combined with $a_{-1}^* = 0$, (31) and (38) yields $x_0^* > x_{-1}^*$. Suppose next that $v_0 = 0$. Inequality (51) now holds as a weak inequality and implies that $x_0^* \geq x_{-1}^*$. Moreover, since $G(\cdot)$ is strictly convex and x_0^* is given by (31), (50) then implies $x_0^* = 0$ and thus $x_{-1}^* = 0$.

We next show that when $\zeta \leq v_1 - v_0$, $x_0^* \geq x_{-1}^*$. In this case, $a_{-1}^* = 1$ is an optimal policy, so that

$$\begin{aligned} \rho v_0 &= -\omega^* G(x_0^*) + x_0^* [v_1 - v_0] + x_0^* [v_{-1} - v_0] \\ \rho v_{-1} &\geq -\omega^* G(x_0^*) + x_0^* [v_1 - v_{-1} - \zeta] + \eta [v_0 - v_{-1}]. \end{aligned}$$

Subtracting the second expression from the first, we obtain

$$\rho [v_0 - v_{-1}] \leq x_0^* [v_{-1} + \zeta - v_0] + (x_0^* + \eta) [v_{-1} - v_0],$$

and therefore

$$[v_0 - v_{-1}] \leq [v_{-1} + \zeta - v_0].$$

Proposition 2 implies that $v_{-1} \leq v_0$, and therefore $v_{-1} + \zeta \geq v_0$. Next observe that with $a_{-1}^* = 1$, (31) and (38) imply that $x_0^* \geq x_{-1}^*$ if and only if $v_1 - v_0 \geq v_1 - v_{-1} - \zeta$, or equivalently if and only if $v_{-1} + \zeta \geq v_0$. Thus we have established that $x_0^* \geq x_{-1}^*$ both when $\zeta > v_1 - v_0$ and when $\zeta \leq v_1 - v_0$.

Lemma 1 and proof: We next first prove a lemma, which will be used in the rest of the Appendix.

Lemma 1 *Suppose that* $G'^{-1}\left(\left(1-\lambda^{-1}\right)/(\rho+\eta)\right) > 0$, *then* $x_0^* > 0$ *and* $v_0 > 0$.

Proof of Lemma: Suppose, to obtain a contradiction, that $x_0^* = 0$. The first part of the proof then implies that $x_{-1}^* = 0$. Then (25) implies

$$\rho v_1 \geq 1 - \lambda + \eta [v_0 - v_1].$$

Equation (26) together with $x_0^* = 0$ gives $v_0 = 0$, and hence

$$v_1 - v_0 \geq \frac{1 - \lambda^{-1}}{\rho + \eta}.$$

Combined with this inequality, (31) implies

$$\begin{aligned} x_0^* &\geq \max\left\{G'^{-1}\left(\frac{1-\lambda^{-1}}{\omega^*(\rho+\eta)}\right), 0\right\}, \\ &\geq \max\left\{G'^{-1}\left(\frac{1-\lambda^{-1}}{\rho+\eta}\right), 0\right\}, \end{aligned}$$

where the second inequality follows from the fact that $\omega^* \leq 1$. The assumption that $G'^{-1}\left(\left(1-\lambda^{-1}\right)/(\rho+\eta)\right) > 0$ then implies $x_0^* > 0$, thus leading to a contradiction and establishing that $x_0^* > 0$. Strict convexity of $G(\cdot)$ together with $x_0^* > 0$ then implies $v_0 > 0$.■

Proof of $x_0^* > x_{-1}^*$ when $G'^{-1}\left(\left(1-\lambda^{-1}\right)/(\rho+\eta)\right) > 0$ and $\zeta > 0$: Given Lemma 1, $G'^{-1}\left(\left(1-\lambda^{-1}\right)/(\rho+\eta)\right) > 0$ implies that $x_0^* > 0$. Then the first part of the proof implies that when $\zeta > v_1 - v_0$, $x_0^* > x_{-1}^*$. Next suppose that $0 < \zeta < v_1 - v_0$. Then the same argument as above implies that $x_0^* > x_{-1}^*$ if and only if $v_1 - v_0 > v_1 - v_{-1} - \zeta$, or equivalently if and only if $v_{-1} + \zeta > v_0$. Suppose this is not the case. Then from the first part of the proof, we have that $x_0^* = x_{-1}^* = 0$, and thus $v_{-1} = v_0 = 0$, which implies $v_{-1} + \zeta > v_0$ and thus $x_0^* > x_{-1}^*$. This yields a contradiction and completes the proof that $x_0^* > x_{-1}^*$ when $\zeta > 0$ and $G'^{-1}\left(\left(1-\lambda^{-1}\right)/(\rho+\eta)\right) > 0$.

Proof of $x_0^* > x_1^*$: To prove that $x_0^* > x_1^*$, let us write the value functions v_2, v_1 and v_0 as in (48) in the proof of Proposition 4:

$$\begin{aligned}
\bar{\rho} v_2 &= 1-\lambda^{-2} - \omega^* G(x_2^*) + x_2^*[v_3 - v_2] + x_{-1}^*\left[a_{-1}^*(v_{-1}+\zeta) + \left(1-a_{-1}^*\right)v_0\right] + \eta v_0, \\
\bar{\rho} v_1 &\geq 1-\lambda^{-1} - \omega^* G(x_2^*) + x_2^*[v_2 - v_1] + x_{-1}^*\left[a_{-1}^*(v_{-1}+\zeta) + \left(1-a_{-1}^*\right)v_0\right] + \eta v_0, \\
\bar{\rho} v_1 &\geq 1-\lambda^{-1} - \omega^* G(x_0^*) + x_0^*[v_2 - v_1] + x_{-1}^*\left[a_{-1}^*(v_{-1}+\zeta) + \left(1-a_{-1}^*\right)v_0\right] + \eta v_0, \\
\bar{\rho} v_0 &= -\omega^* G(x_0) + x_0^*[v_1 - v_0] + \eta v_0 + x_{-1}^* v_0 + x_0^*[v_{-1} - v_0].
\end{aligned}$$

Now taking differences with $\bar{\rho} v_n$ and using the definitions of δ_n's as in (46) in the proof of Proposition 4, we obtain

$$\begin{aligned}
\bar{\rho}\delta_2 &\leq \lambda^{-1}\left(1-\lambda^{-1}\right) + x_2^*(\delta_3 - \delta_2), \qquad (52) \\
\bar{\rho}\delta_1 &\geq \left(1-\lambda^{-1}\right) + x_0^*(\delta_2 - \delta_1) + x_{-1}^*\left[a_{-1}^*(v_{-1}+\zeta) + \left(1-a_{-1}^*\right)v_0 - v_0\right] - x_0^*[v_{-1} - v_0], \\
\bar{\rho}\delta_1 &\geq \left(1-\lambda^{-1}\right) + x_0^*(\delta_2 - \delta_1) + x_{-1}^* a_{-1}^*\zeta + \left(x_{-1}^* a_{-1}^* - x_0^*\right)[v_{-1} - v_0], \\
\bar{\rho}\delta_1 &\geq \left(1-\lambda^{-1}\right) + x_0^*(\delta_2 - \delta_1) + \left(x_{-1}^* a_{-1}^* - x_0^*\right)[v_{-1} - v_0].
\end{aligned}$$

Next note that Proposition 2 implies that $v_{-1} - v_0 \leq 0$. Moreover, the first part of the first part of the proof has established that $x_{-1}^* - x_0^* \leq 0$. Combining this with $a_{-1}^* \leq 1$ establishes that $\left[x_{-1}^* - x_0\right][v_{-1} - v_0] \geq 0$, and the last inequality then implies

$$\bar{\rho}\delta_1 \geq \left(1-\lambda^{-1}\right) + x_0^*(\delta_2 - \delta_1).$$

Now combining this inequality with the first inequality of (52), we obtain

$$(\bar{\rho} + x_0^*)(\delta_2 - \delta_1) \leq -\left(1-\lambda^{-1}\right)^2 + x_2^*(\delta_3 - \delta_2). \qquad (53)$$

Proposition 2 has established $\delta_2 > \delta_3$, so that the right-hand side is strictly negative, therefore, we must have $\delta_2 - \delta_1 < 0$, which implies that $x_0^* > x_1^*$ and completes the proof. ■

Proof of Proposition 6. Fix $\mathbf{z} = \left\langle \tilde{\omega}, \{\tilde{x}_n\}_{n=-1}^{\infty}, \{\tilde{a}_n\}_{n=-\infty}^{-1} \right\rangle$, and consider the optimization problem of a representative firm, written recursively as:

$$\begin{aligned}
\rho v_n = & \max_{x_n \in [0,\bar{x}]} \left\{ \left(1-\lambda^{-n}\right) - \tilde{\omega} G(x_n) + x_n[v_{n+1} - v_n] \right. \\
+ & \left. \tilde{x}_{-1}\left(\tilde{a}_{-1}[v_{-1} - v_n + \zeta] + (1-\tilde{a}_{-1})[v_0 - v_n]\right) + \eta[v_0 - v_n] \right\} \text{ for } n \in \mathbb{N}
\end{aligned}$$

$$\begin{aligned}
\rho v_0 &= \max_{x_0 \in [0,\bar{x}]} \left\{ -\tilde{\omega} G(x_0) + x_0[v_1 - v_0] + \tilde{x}_0[v_{-1} - v_0] \right\} \\
\rho v_{-1} &= \max_{x_{-1} \in [0,\bar{x}],\ a_{-1} \in [0,1]} \left\{ -\tilde{\omega} G(x_0) + x_{-1}\left(a_{-1}[v_1 - v_{-1} - \zeta] + (1-a_{-1})[v_0 - v_{-1}]\right) \right. \\
& \left. + \eta[v_0 - v_{-1}] \right\}.
\end{aligned}$$

We now transform this dynamic optimization problem into a form that can be represented as a contraction mapping using the method of "uniformization" (see, for example, Ross, 1996, Chapter 5). Let $\tilde{\xi} = \left\langle \{\tilde{x}_n\}_{n=-1}^{\infty}, \{\tilde{a}_n\}_{n=-\infty}^{-1} \right\rangle$ and $p_{n,n'}\left(\xi \mid \tilde{\xi}\right)$ be the probability that the next state will be n' starting with state n when the firm in question chooses policies $\xi \equiv \left\langle \{x_n\}_{n=-1}^{\infty}, \{a_n\}_{n=-\infty}^{-1} \right\rangle$ and the R&D policy of other firms is given by $\tilde{\xi}$. Using the fact that, because of uniform IPR policy, $\langle x_{-n}, a_{-n}\rangle = \langle x_{-1}, a_{-1}\rangle$ for all $n \in \mathbb{N}$, these transition probabilities can be written as:

$p_{-1,0}\left(\xi \mid \tilde{\xi}\right) = \frac{(1-a_{-1})x_{-1}+\eta}{x_{-1}+\eta}$	$p_{-1,1}\left(\xi \mid \tilde{\xi}\right) = \frac{a_{-1}x_{-1}}{x_{-1}+\eta}$	
$p_{0,-1}\left(\xi \mid \tilde{\xi}\right) = \frac{\tilde{x}_0}{x_0+\tilde{x}_0}$	$p_{0,1}\left(\xi \mid \tilde{\xi}\right) = \frac{x_0}{x_0+\tilde{x}_0}$	
$p_{n,-1}\left(\xi \mid \tilde{\xi}\right) = \frac{a_{-1}\tilde{x}_{-1}}{x_n+\tilde{x}_{-1}+\eta}$	$p_{n,0}\left(\xi \mid \tilde{\xi}\right) = \frac{(1-a_{-1})\tilde{x}_{-1}+\eta}{x_n+\tilde{x}_{-1}+\eta}$	$p_{n,n+1}\left(\xi \mid \tilde{\xi}\right) = \frac{x_n}{x_n+\tilde{x}_{-1}+\eta}$

Uniformization involves adding fictitious transitions from a state into itself, which do not change the value of the program, but allow us to represent the optimization problem as a contraction. For this purpose, define the transition rates ψ_n as

$$\psi_n\left(\xi \mid \tilde{\xi}\right) = \begin{cases} x_n + x_{-1} + \eta & \text{for } n \in \{1,2,...\} \\ x_{-1} + \eta & \text{for } n = -1 \\ 2x_n & \text{for } n = 0 \end{cases}.$$

These transition rates are finite since $\psi_n\left(\xi \mid \tilde{\xi}\right) \leq \psi \equiv 2\bar{x} + \eta < \infty$ for all n, where $\bar{x}$ is the maximal flow rate of innovation defined in (11) in the text (both $\bar{x}$ and η are finite by assumption).

Now following equation (5.8.3) in Ross (1996), we can use these transition rates and define the new transition probabilities (including the fictitious transitions from a state to itself) as:

$$\tilde{p}_{n,n'}\left(\xi \mid \tilde{\xi}\right) = \begin{cases} \frac{\psi_n(\xi|\tilde{\xi})}{\psi} p_{n,n'}\left(\xi \mid \tilde{\xi}\right) & \text{if } n \neq n' \\ 1 - \frac{\psi_n(\xi|\tilde{\xi})}{\psi} & \text{if } n = n' \end{cases}.$$

This yields equivalent transition probabilities

$\tilde{p}_{-1,-1}\left(\xi \mid \tilde{\xi}\right) = 1 - \frac{x_{-1}+\eta}{2\bar{x}+\eta}$	$\tilde{p}_{-1,0}\left(\xi \mid \tilde{\xi}\right) = \frac{(1-a_{-1})x_{-1}+\eta}{2\bar{x}+\eta}$	$\tilde{p}_{-1,1}\left(\xi \mid \tilde{\xi}\right) = \frac{a_{-1}x_{-1}}{2\bar{x}+\eta}$	
$\tilde{p}_{0,-1}\left(\xi \mid \tilde{\xi}\right) = \frac{\tilde{x}_0}{2\bar{x}+\eta}$	$\tilde{p}_{0,0}\left(\xi \mid \tilde{\xi}\right) = 1 - \frac{x_0+\tilde{x}_0}{2\bar{x}+\eta}$	$\tilde{p}_{0,1}\left(\xi \mid \tilde{\xi}\right) = \frac{x_0}{2\bar{x}+\eta}$	
$\tilde{p}_{n,0}\left(\xi \mid \tilde{\xi}\right) = \frac{(1-\tilde{a}_{-1})\tilde{x}_{-1}+\eta}{2\bar{x}+\eta}$	$\tilde{p}_{n,n}\left(\xi \mid \tilde{\xi}\right) = 1 - \frac{x_n+\tilde{x}_{-1}+\eta}{2\bar{x}+\eta}$	$\tilde{p}_{n,n+1}\left(\xi \mid \tilde{\xi}\right) = \frac{x_n}{2\bar{x}+\eta}$	$\tilde{p}_{n,-1}\left(\xi \mid \tilde{\xi}\right) = \frac{\tilde{a}_{-1}\tilde{x}_{-1}}{2\bar{x}+\eta}$

and also defines an effective discount factor β given by

$$\beta \equiv \frac{\psi}{\rho + \psi} = \frac{2\bar{x} + \eta}{\rho + 2\bar{x} + \eta}.$$

Also let the per period return function (profit net of R&D expenditures) be

$$\hat{\Pi}_n(x_n) = \begin{cases} \frac{1-\lambda^{-n} - \tilde{\omega}G(x_n)}{\rho+2\bar{x}+\eta} & \text{if } n \geq 1 \\ \frac{-\tilde{\omega}G(x_n)}{\rho+2\bar{x}+\eta} & \text{otherwise} \end{cases}. \tag{54}$$

Using these transformations, the dynamic optimization problem can be written as:

$$\begin{aligned} v_n &= \max_{x_n,\, a_n} \left\{ \hat{\Pi}_n(x_n) + \beta \sum_{n'} \tilde{p}_{n,n'}\left(\xi_n \mid \tilde{\xi}\right) \tilde{v}_{n'} \right\}, \text{ for all } n \in \mathbb{Z}, \\ &\equiv T\tilde{v}_n, \text{ for all } n \in \mathbb{Z}. \end{aligned} \tag{55}$$

where $\mathbf{v} \equiv \{v_n\}_{n=-1}^{\infty}$ and the second line defines the operator T, mapping from the space of functions $\mathbf{V} \equiv \{\mathbf{v} : \{-1\} \cup \mathbb{Z}_+ \to \mathbb{R}_+\}$ into itself. T is clearly a contraction, thus, for given $\mathbf{z} = \langle \tilde{\omega}, \tilde{a}_{-1}, \{\tilde{x}_n\}_{n=-1}^{\infty} \rangle$, possesses a unique fixed point $\mathbf{v}^* \equiv \{v_n^*\}_{n=-1}^{\infty}$ (e.g., Stokey, Lucas and Prescott, 1989).

Moreover, $x_n \in [0, \bar{x}]$, $a_{-1} \in [0,1]$, and v_n for each $n = -1, 0, 1, ...$ given by the right-hand side of (55) is continuous in a_n and x_n (a_n applying only for $n = -1$), so Berge's Maximum Theorem (Aliprantis and Border, 1999, Theorem 16.31, p. 539) implies that the set of maximizers $\langle A_{-1}^*, \{X_n^*\}_{n=-1}^{\infty} \rangle$ exists, is nonempty and compact-valued for each $\mathbf{z}$ and is upper hemi-continuous in $\mathbf{z} = \langle \tilde{\omega}, \tilde{a}_{-1}, \{\tilde{x}_n\}_{n=-1}^{\infty} \rangle$. Moreover, concavity of v_n in a_n and x_n for each $n = -1, 0, 1, ...$ implies that $\langle A_{-1}^*, \{X_n^*\}_{n=-1}^{\infty} \rangle$ is also convex-valued for each $\mathbf{z}$, completing the proof. ■

Proof of Proposition 7. We first show that the mapping Φ: $\mathbf{Z} \rightrightarrows \mathbf{Z}$ constructed in (40) has a fixed point, and then establish that when $G'^{-1}\left(\left(1 - \lambda^{-1}\right) / (\rho + \eta)\right) > 0$ this fixed point corresponds to a steady state with $\omega^* < 1$.

First, it has already been established that Φ maps $\mathbf{Z}$ into itself. We next show that $\mathbf{Z}$ is compact in the product topology and is a subset of a locally convex Hausdorff space. The first part follows from the fact that $\mathbf{Z}$ can be written as the Cartesian product of compact subsets, $\mathbf{Z} = [0,1] \times [0,1] \times \prod_{n=-1}^{\infty} [0, \bar{x}]$. Then by Tychonoff's Theorem (e.g., Aliprantis and Border, 1999, Theorem 2.57, p. 52; Kelley, 1955, p. 143), $\mathbf{Z}$ is compact in the product topology. Moreover, $\mathbf{Z}$ is clearly nonempty and also convex, since for any $\mathbf{z}, \mathbf{z}' \in \mathbf{Z}$ and $\lambda \in [0,1]$, we have $\lambda \mathbf{z} + (1 - \lambda) \mathbf{z}' \in \mathbf{Z}$. Finally, since $\mathbf{Z}$ is a product of intervals on the real line, it is a subset of a locally convex Hausdorff space (see Aliprantis and Border, 1999, Lemma 5.54, p. 192).

Next, φ is a continuous function from $\mathbf{Z}$ into $[0,1]$ and from Proposition 6, $A_{-1}(\mathbf{z})$ and $X_n(\mathbf{z})$ for $n \in \{-1\} \cup \mathbb{Z}_+$ are upper hemi-continuous in $\mathbf{z}$. Consequently, $\Phi \equiv \langle \varphi[\mathbf{z}], A_{-1}[\mathbf{z}], \mathbf{X}[\mathbf{z}] \rangle$ has closed graph in $\mathbf{z}$ in the product topology. Moreover, each one of $\varphi(\mathbf{z})$, $A_{-1}(\mathbf{z})$ and $X_n(\mathbf{z})$ for $n = -1, 0, ...$ is nonempty, compact and convex-valued. Therefore, the image of the mapping Φ is nonempty, compact and convex-valued for each $\mathbf{z} \in \mathbf{Z}$. The Kakutani-Fan-Glicksberg Fixed Point Theorem implies that if the function Φ maps a convex, compact and nonempty subset of a locally convex Hausdorff space into itself and has closed graph and is nonempty, compact and convex-valued $\mathbf{z}$, then it possesses a fixed point $\mathbf{z}^* \in \Phi(\mathbf{z}^*)$ (see Aliprantis and Border, 1999, Theorem 16.50 and Corollary 16.51, pp. 549-550). This establishes the existence of a fixed point $\mathbf{z}^*$ of Φ.

To complete the proof, we need to show that the fixed point, $\mathbf{z}^*$, corresponds to a steady state equilibrium. First, since $a_n\left(\omega^*, a_{-1}^*, \{x_n^*\}_{n=-1}^{\infty}\right) = a_{-1}^*$ and $x_n\left(\omega^*, a_{-1}^*, \{x_n^*\}_{n=-1}^{\infty}\right) = x_n^*$ for $n \in \{-1\} \cup \mathbb{Z}_+$, we have that given a labor share of ω^*, $\langle a_{-1}^*, \{x_n^*\}_{n=-1}^{\infty} \rangle$ constitutes an R&D policy vector that is best response to itself, as required by steady-state equilibrium (Definition 3). Next, we need to prove that the implied labor share ω^* leads to labor market clearing. This follows from the fact that the fixed point involves $\omega^* < 1$, since in this case (39) will have an interior solution, ensuring labor market clearing. Suppose, to obtain a contradiction, that $\omega^* = 1$. Then, as noted in the text, we must have $\mu_0^* = 1$. From (32), (33) and (34), this implies $x_n^* = 0$ for $n \in \{-1\} \cup \mathbb{Z}_+$. However, Lemma 1 implies that this is not possible when $G'^{-1}\left(\left(1 - \lambda^{-1}\right) / (\rho + \eta)\right) > 0$. Consequently, (39) cannot be satisfied at $\omega^* = 1$, implying that $\omega^* < 1$. When $\omega^* < 1$, the labor market clearing condition (35) is satisfied at ω^* as an equality, so ω^* is an equilibrium given $\{x_n^*\}_{n=-1}^{\infty}$, and thus $\mathbf{z}^* = \left(\omega^*, a_{-1}^*, \{x_n^*\}_{n=-1}^{\infty}\right)$ is a steady-state equilibrium as desired.

Finally, if $\eta > 0$, then (34) implies that $\mu_0^* > 0$. Since $x_0^* > 0$ from Lemma 1, equation (36) implies $g^* > 0$. Alternatively, if $x_{-1}^* > 0$, then $g^* > 0$ follows from (36), completing the proof. ■

Proof of Proposition 8. We will establish that there exists n^* such that $x_{n^*}^* = 0$ and $x_n^* > 0$ for all $n > n^*$, which will imply that the states $n < n^*$ are transient and can be ignored, and that $\{\mu_n^*\}_{n=0}^{\infty}$ forms a finite and irreducible Markov chain over the states $n = 0, 1, ..., n^*$.

First, recall that Proposition 2 has established that $\{v_n\}_{n \in \mathbb{Z}_+}$ is strictly increasing. Then it follows from the proof of Proposition 3 that there exists a state $n^{**} \in \mathbb{N}$ such that $x_n^{**} = 0$ for all $n \geq$

n^{**}. Now let $n^* = \min_{n\in\{0,...,n^{**}\}} \{n \in \mathbb{N}: v_{n+1} - v_n \leq \omega^* G'(0)\}$. Such an n^* exists, since the set $\{0, ..., n^{**}\}$ is finite and nonempty because of the assumption that $G'^{-1}\left(\left(1-\lambda^{-1}\right)/(\rho+\eta)\right) > 0$. Then by construction $x_n^* > 0$ for all $n < n^*$ and $x_{n^*}^* = 0$ as desired. Now denoting the probability of being in state $\tilde{n}$ starting in state n after τ periods by $P^\tau(n, \tilde{n})$, we have that $\lim_{\tau\to\infty} P^\tau(n, \tilde{n}) = 0$ for all $\tilde{n} > n^*$ and for all n. Thus we can focus on the finite Markov chain over the states $n = 0, 1, ..., n^*$, and $\{\mu_n^*\}_{n=0}^{n^*}$ is the limiting (invariant) distribution of this Markov chain. Given a_{-1}^* and $\{x_n^*\}_{n=-1}^{n^*}$, $\{\mu_n^*\}_{n=0}^{n^*}$ is uniquely defined. Moreover, the underlying Markov chain is irreducible (since $x_n^* > 0$ for $n = 0, 1, ..., n^* - 1$, so that all states communicate with $n = 0$ or $n = 1$). Therefore, by Theorem 11.2 in Stokey, Lucas and Prescott (1989, p. 62) there exists a unique stationary distribution $\{\mu_n^*\}_{n=0}^{\infty}$. ■

Proof of Proposition 9. There are two cases to consider. First, suppose that $\{v_n\}_{n\in\mathbb{Z}_+}$ is strictly increasing. Then it follows from the proof of Proposition 3 that there exists a state $n^* \in \mathbb{N}$ such that $x_n^* = 0$ for all $n \geq n^*$, and as in the proof of Proposition 8, states $n \geq n^*$ are transient (i.e., $\lim_{\tau\to\infty} P^\tau(n, \tilde{n}) = 0$ for all $\tilde{n} > n^*$ and for all n), so $\mu_n^* = 0$ for all $n \geq n^*$.

Second, in contrast to the first case, suppose that there exists some $n^{**} \in \mathbb{Z}_+$ such that $v_{n^{**}} \geq v_{n^{**}+1}$. Then, let $n^* = \min_{n\in\{0,...,n^{**}\}} \{n \in \mathbb{N}: v_{n+1} - v_n \leq \omega^* G'(0)\}$, which is again well defined. Then, optimal R&D decision (29) immediately implies that $x_n^* > 0$ for all states with $n < n^*$, and since $x_{n^*}^* = 0$, all states $n > n^*$ are transient and $\lim_{\tau\to\infty} P^\tau(n, \tilde{n}) = 0$ for all $\tilde{n} > n^*$ and for all n, completing the proof. ■

Proof of Proposition 10. The proof follows closely that of Proposition 6. In particular, again using uniformization, the maximization problem of an individual firm can be written as a contraction mapping similar to (55) there. The finiteness of the transition probabilities follows, since $\psi_n\left(\boldsymbol{\xi} \mid \tilde{\boldsymbol{\xi}}\right) \leq \psi \equiv 2\bar{x} + \max_n\{\eta_n\} < \infty$ (this is a consequence of the fact that $\bar{x}$ defined in (11) is finite and $\max_n\{\eta_n\}$ is finite, since each $\eta_n \in \mathbb{R}_+$ and by assumption, there exists $\bar{n} < \infty$ such that $\eta_n = \eta_{\bar{n}}$). This contraction mapping uniquely determines the value function $\mathrm{v}[\mathrm{z}]: \mathbb{Z} \to \mathbb{R}_+$.

Berge's Maximum Theorem (Aliprantis and Border, 1999, Theorem 16.31, p. 539) again implies that each of $A_n(\mathrm{z})$ for $n \in \mathbb{Z}_-\setminus\{0\}$ and $X_n(\mathrm{z})$ for $n \in \mathbb{Z}$ is upper hemi-continuous in $\mathrm{z} = \langle\tilde{\omega}, \tilde{\mathbf{a}}, \tilde{\mathbf{x}}\rangle$, and moreover, since v_n for $n \in \mathbb{Z}$ is concave in a_n and x_n, the maximizers of $\mathrm{v}[\mathrm{z}]$, $\mathbf{A} \equiv \{A_{-n}\}_{n=1}^{\infty}$ and $\mathbf{X} \equiv \{X_n\}_{n=-\infty}^{\infty}$, are nonempty, compact and convex-valued. ■

Proof of Proposition 11. The proof follows that of Proposition 7 closely. Fix $\mathrm{z} = \left\langle\tilde{\omega}, \{\tilde{a}_n\}_{n=-\infty}^{-1}, \{\tilde{x}_n\}_{n=-\infty}^{\infty}\right\rangle$, and define $\mathbf{Z} \equiv [0,1] \times \prod_{n=-1}^{\infty}[0,1] \times \prod_{n=-\infty}^{\infty}[0,\bar{x}]$. Again by Tychonoff's Theorem, $\mathbf{Z}$ is compact in the product topology. Then consider the mapping $\Phi: \mathbf{Z} \rightrightarrows \mathbf{Z}$ constructed as $\Phi \equiv (\varphi, \mathbf{A}, \mathbf{X})$, where φ is given by (39) and $\mathbf{A}$ and $\mathbf{X}$ are defined in Proposition 10. Clearly Φ maps $\mathbf{Z}$ into itself. Moreover, as in the proof of Proposition 7, $\mathbf{Z}$ is nonempty, convex, and a subset of a locally convex Hausdorff space. The proof of Proposition 10 then implies that Φ has closed graph in the product topology and is nonempty, compact and convex-valued in z. Consequently, the Kakutani-Fan-Glicksberg Fixed Point Theorem again applies and implies that Φ has a fixed point $\mathrm{z}^* \in \Phi(\mathrm{z}^*)$. The argument that the fixed point z^* corresponds to a steady-state equilibrium is identical to that in Proposition 7, and follows from the fact that within argument identical to that of Lemma 1, $G'^{-1}\left(\left(1-\lambda^{-1}\right)/(\rho+\eta_1)\right) > 0$ implies that $x_0^* > 0$. The result that $\omega^* < 1$ then follows immediately. Finally, as in the proof of Proposition 7, either $\eta_1 > 0$ or $x_{-1}^* > 0$ is sufficient for $g^* > 0$. ■

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Table 1. Benchmark Results

	Full IPR Protection without licensing	Full IPR Protection with licensing	Optimal Uniform IPR with licensing	Optimal State Dependent without licensing	Optimal State Dependent with licensing
λ	1.05	1.05	1.05	1.05	1.05
γ	0.35	0.35	0.35	0.35	0.35
η_1	0	0	0	0.054	0
η_2	0	0	0	0.005	0
η_3	0	0	0	0	0
η_4	0	0	0	0	0
η_5	0	0	0	0	0
ζ_1	∞	3.52	0	∞	0
ζ_2	∞	3.52	0	∞	0.82
ζ_3	∞	3.52	0	∞	1.94
ζ_4	∞	3.52	0	∞	1.95
ζ_5	∞	3.52	0	∞	1.95
$v_1 - v_0$	2.70	3.52	1.71	1.59	1.96
x^*_{-1}	0.018	0.021	0.023	0.010	0.026
x^*_0	0.029	0.033	0.023	0.022	0.024
μ^*_0	0.238	0	0	0.440	0
μ^*_1	0.326	0.481	0.506	0.195	0.446
μ^*_2	0.189	0.253	0.253	0.135	0.232
ω^*	0.952	0.931	0.936	0.961	0.937
Researcher ratio	0.032	0.026	0.027	0.028	0.039
g^*	0.0186	0.0258	0.0263	0.0203	0.0296

Note: This table gives the results of the benchmark numerical computations with $\rho = 0.05$, $\lambda = 1.05$, $\gamma = 0.35$ under five different IPR policy regimes. It reports the steady-state equilibrium values of the difference in the values $v_1 - v_0$; the R&D rate of a follower that is one step behind, x^*_{-1}; the R&D rate of neck-and-neck competitors, x^*_0; fraction of industries in neck-and-neck competition, μ^*_0; fraction of industries at a technology gap of $n = 1$; the value of "labor share," ω^*; the ratio of the labor force working in research; and the growth rate, g^*. It also reports the growth-maximizing uniform and state-dependent IPR policies with or without licensing. See text for details.

Table 2. $\lambda = 1.01$

	Full IPR Protection without licensing	Full IPR Protection with licensing	Optimal Uniform IPR with licensing	Optimal State Dependent without licensing	Optimal State Dependent with licensing
λ	1.01	1.01	1.01	1.01	1.01
γ	0.35	0.35	0.35	0.35	0.35
η_1	0	0	0	0.272	0
η_2	0	0	0	0.017	0
η_3	0	0	0	0	0
η_4	0	0	0	0	0
η_5	0	0	0	0	0
ζ_1	∞	0.19	0	∞	0
ζ_2	∞	0.19	0	∞	0.04
ζ_3	∞	0.19	0	∞	0.10
ζ_4	∞	0.19	0	∞	0.10
ζ_5	∞	0.19	0	∞	0.10
$v_1 - v_0$	0.14	0.19	0.09	0.08	0.10
x^*_{-1}	0.090	0.106	0.107	0.053	0.126
x^*_0	0.139	0.162	0.107	0.103	0.115
μ^*_0	0.245	0	0	0.452	0
μ^*_1	0.330	0.496	0.501	0.191	0.455
μ^*_2	0.186	0.251	0.251	0.137	0.232
ω^*	0.991	0.987	0.987	0.993	0.988
Researcher ratio	0.008	0.007	0.007	0.007	0.010
g^*	0.0186	0.0257	0.0257	0.0203	0.0293

Note: This table gives the results of the benchmark numerical computations with $\rho = 0.05$, $\lambda = 1.01$, $\gamma = 0.35$ under five different IPR policy regimes. It reports the steady-state equilibrium values of the difference in the values $v_1 - v_0$; the R&D rate of a follower that is one step behind, x^*_{-1}; the R&D rate of neck-and-neck competitors, x^*_0; fraction of industries in neck-and-neck competition, μ^*_0; fraction of industries at a technology gap of $n = 1$; the value of "labor share," ω^*; the ratio of the labor force working in research; and the growth rate, g^*. It also reports the growth-maximizing uniform and state-dependent IPR policies with or without licensing. See text for details.

Table 3. $\lambda = 1.2$

	Full IPR Protection without licensing	Full IPR Protection with licensing	Optimal Uniform IPR with licensing	Optimal State Dependent without licensing	Optimal State Dependent with licensing
λ	1.20	1.20	1.20	1.20	1.20
γ	0.35	0.35	0.35	0.35	0.35
η_1	0	0	0	0.014	0
η_2	0	0	0	0.002	0
η_3	0	0	0	0.001	0
η_4	0	0	0	0	0
η_5	0	0	0	0	0
ζ_1	∞	24.85	0	∞	0
ζ_2	∞	24.85	0	∞	5.69
ζ_3	∞	24.85	0	∞	12.25
ζ_4	∞	24.85	0	∞	14.75
ζ_5	∞	24.85	0	∞	14.98
$v_1 - v_0$	20.11	24.85	13.69	12.40	14.99
x^*_{-1}	0.005	0.006	0.007	0.003	0.008
x^*_0	0.008	0.010	0.007	0.006	0.007
μ^*_0	0.221	0	0	0.418	0
μ^*_1	0.318	0.452	0.523	0.206	0.445
μ^*_2	0.199	0.260	0.261	0.142	0.236
ω^*	0.806	0.743	0.783	0.847	0.760
Researcher ratio	0.068	0.056	0.070	0.058	0.090
g^*	0.0186	0.0261	0.0284	0.0200	0.0306

Note: This table gives the results of the benchmark numerical computations with $\rho = 0.05$, $\lambda = 1.2$, $\gamma = 0.35$ under five different IPR policy regimes. It reports the steady-state equilibrium values of the difference in the values $v_1 - v_0$; the R&D rate of a follower that is one step behind, x^*_{-1}; the R&D rate of neck-and-neck competitors, x^*_0; fraction of industries in neck-and-neck competition, μ^*_0; fraction of industries at a technology gap of $n = 1$; the value of "labor share," ω^*; the ratio of the labor force working in research; and the growth rate, g^*. It also reports the growth-maximizing uniform and state-dependent IPR policies with or without licensing. See text for details.

Table 4. $\gamma = 0.1$

	Full IPR Protection without licensing	Full IPR Protection with licensing	Optimal Uniform IPR with licensing	Optimal State Dependent without licensing	Optimal State Dependent with licensing
λ	1.05	1.05	1.05	1.05	1.05
γ	0.1	0.1	0.1	0.1	0.1
η_1	0	0	0	0.291	0
η_2	0	0	0	0	0
η_3	0	0	0	0	0
η_4	0	0	0	0	0
η_5	0	0	0	0	0
ζ_1	∞	3.11	0	∞	0
ζ_2	∞	3.11	0	∞	0.77
ζ_3	∞	3.11	0	∞	1.90
ζ_4	∞	3.11	0	∞	1.90
ζ_5	∞	3.11	0	∞	1.90
$v_1 - v_0$	2.20	3.11	1.64	0.48	1.90
x^*_{-1}	0.022	0.023	0.024	0.016	0.025
x^*_0	0.024	0.026	0.024	0.021	0.024
μ^*_0	0.314	0	0	0.777	0
μ^*_1	0.333	0.497	0.501	0.096	0.492
μ^*_2	0.172	0.251	0.251	0.058	0.248
ω^*	0.944	0.916	0.917	0.981	0.917
Researcher ratio	0.008	0.007	0.008	0.003	0.010
g^*	0.0186	0.0278	0.0280	0.0220	0.0286

Note: This table gives the results of the benchmark numerical computations with $\rho = 0.05$, $\lambda = 1.05$, $\gamma = 0.1$ under five different IPR policy regimes. It reports the steady-state equilibrium values of the difference in the values $v_1 - v_0$; the R&D rate of a follower that is one step behind, x^*_{-1}; the R&D rate of neck-and-neck competitors, x^*_0; fraction of industries in neck-and-neck competition, μ^*_0; fraction of industries at a technology gap of $n = 1$; the value of "labor share," ω^*; the ratio of the labor force working in research; and the growth rate, g^*. It also reports the growth-maximizing uniform and state-dependent IPR policies with or without licensing. See text for details..

Table 5. $\gamma = 0.6$

	Full IPR Protection without licensing	Full IPR Protection with licensing	Optimal Uniform IPR with licensing	Optimal State Dependent without licensing	Optimal State Dependent with licensing
λ	1.05	1.05	1.05	1.05	1.05
γ	0.6	0.6	0.6	0.6	0.6
η_1	0	0	0	0.070	0.007
η_2	0	0	0	0.019	0
η_3	0	0	0	0.007	0
η_4	0	0	0	0.003	0
η_5	0	0	0	0	0
ζ_1	∞	5.24	5.24	∞	0
ζ_2	∞	5.24	5.24	∞	0.59
ζ_3	∞	5.24	5.24	∞	1.69
ζ_4	∞	5.24	5.24	∞	2.13
ζ_5	∞	5.24	5.24	∞	2.35
$v_1 - v_0$	4.38	5.24	5.24	2.25	2.35
x_{-1}^*	0.011	0.015	0.015	0.002	0.027
x_0^*	0.053	0.070	0.070	0.020	0.021
μ_0^*	0.092	0	0	0.327	0.037
μ_1^*	0.257	0.413	0.413	0.094	0.242
μ_2^*	0.193	0.252	0.252	0.074	0.128
ω^*	0.942	0.937	0.937	0.9320	0.9307
Researcher ratio	0.073	0.044	0.044	0.087	0.010
g^*	0.0186	0.0198	0.0198	0.0230	0.0303

Note: This table gives the results of the benchmark numerical computations with $\rho = 0.05$, $\lambda = 1.05$, $\gamma = 0.6$ under five different IPR policy regimes. It reports the steady-state equilibrium values of the difference in the values $v_1 - v_0$; the R&D rate of a follower that is one step behind, x_{-1}^*; the R&D rate of neck-and-neck competitors, x_0^*; fraction of industries in neck-and-neck competition, μ_0^*; fraction of industries at a technology gap of $n = 1$; the value of "labor share," ω^*; the ratio of the labor force working in research; and the growth rate, g^*. It also reports the growth-maximizing uniform and state-dependent IPR policies with or without licensing. See text for details.

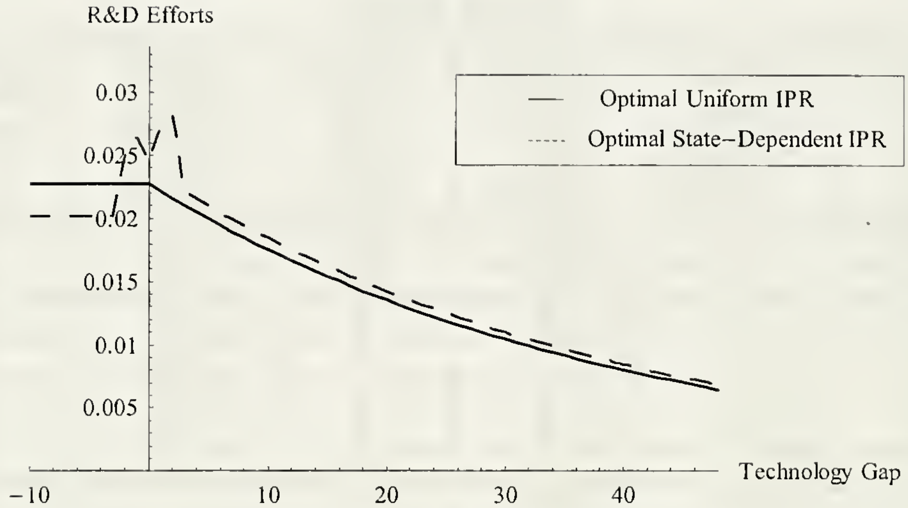

FIGURE 5. R&D EFFORTS.

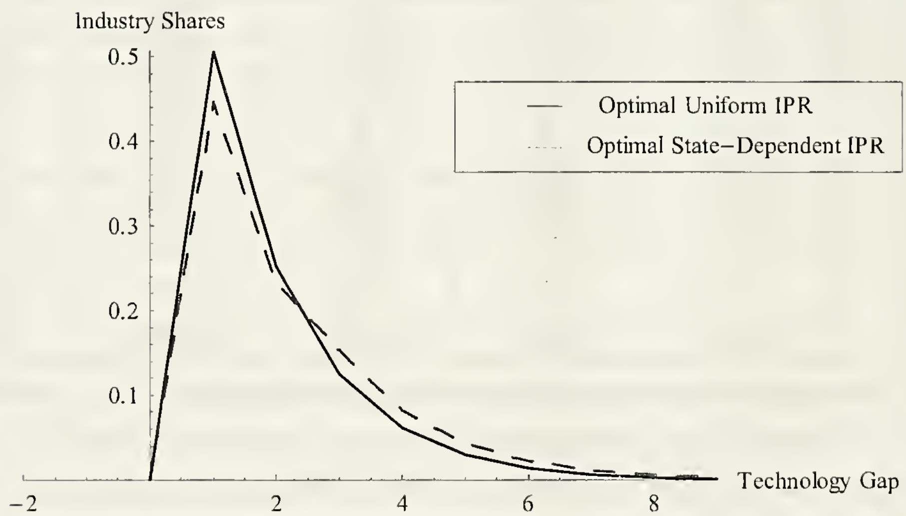

FIGURE 6. INDUSTRY SHARES.