PARABOLIC CHARACTER SHEAVES, III

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Abstract. The main purpose of this paper is to define a class of simple perverse sheaves (called character sheaves) on certain ind-varieties associated to a loop group. This has applications to a geometric construction of certain affine Hecke algebras with unequal parameters (an affine analogue of another construction by the author), as will be shown elsewhere.


Key words and phrases. Character sheaf, parahoric subgroup, ind-variety, affine Hecke algebra.

1. A Decomposition of $G^d/U_P$

1.1. Let $k$ be an algebraically closed field. In this paper an algebraic variety (or algebraic group) is always assumed to be over $k$. More generally we will consider ind-varieties. An ind-variety is an increasing union $X = \bigcup_{i \in \mathbb{N}} X_i$ of algebraic varieties $X_i$ under closed embeddings $j_i : X_i \to X_{i+1}$. (The sequence $(X_i)$ can be replaced by a sequence $(X'_i)$ such that each $X_i$ is a closed subvariety of some $X'_j$ and each $X'_i$ is a closed subvariety of some $X_j$. This change leads to the same ind-variety.) An algebraic subvariety of $X$ is by definition an algebraic subvariety of $X_i$ for some $i$. We fix a prime number $l$ invertible in $k$. For $X$ as above we can define the category of perverse sheaves on $X$. Its objects are collections $K = (K_i)$, $i \geq i_0$, where $K_i$ is a $\mathbb{Q}_l$-perverse sheaf on $X_i$ for $i \geq i_0$. (The support of $K$ is by the definition the support of $K_i$ for $i \geq i_0$. $K$ is said to be simple if $K_i$ is simple for $i \geq i_0$.)

Let $l = k(\epsilon)$, where $\epsilon$ is an indeterminate. Let $G$ be a connected semisimple almost simple algebraic group. Let $G = G(l)$ and let $G'$ be the derived group of $G$. A subgroup of $G'$ is said to be parahoric if it contains some Iwahori subgroup and is not equal to $G'$. Let $P$ be a parahoric subgroup of $G'$. Note that $P$ can be naturally regarded as a proalgebraic group; it has a canonical normal subgroup $U_P$ which is pronipotent and the quotient $P/U_P$ is naturally a connected reductive algebraic group. Now $G'/U_P$ can be viewed naturally as an ind-variety; indeed $G'/U_P$ is fibred over $G'/P$ (which is known to be an ind-variety) with fibres isomorphic with the algebraic variety $P/U_P$. Note that $P$ acts on $G'/U_P$ by $p : gU_P \mapsto pgp^{-1}U_P$. In
this paper we define a class of simple perverse sheaves (character sheaves) on \( G'/U_P \); these are certain \( P \)-equivariant simple perverse sheaves on \( G'/U_P \) supported by \( P \)-stable algebraic subvarieties (on which \( P \) acts algebraically through a quotient which is an algebraic group). The key to the definition of character sheaves on \( G'/U_P \) is a decomposition of \( G'/U_P \) into \( P \)-stable smooth algebraic subvarieties indexed by a subset of the affine Weyl group. This decomposition is the affine analogue of a decomposition which appeared in \([L1]\); the character sheaves that we define are affine analogues of the parabolic character sheaves of \([L1]\) (see also \([L2]\)).

We will actually consider character sheaves on an ind-variety slightly more general that \( G'/U_P \), namely \( G'/U_P \), where \( \delta \) is an element of the finite abelian group \( D = G/G' \) and \( G' \) is the inverse image of \( \delta \) under the obvious map \( G \to D \).

Now many of the results in \([L1, \text{Section 3}]\) remain valid (with the same proof) if \( G, G' \) are replaced by \( G', G' \), “Borel” is replaced by “Iwahori” and “parabolic” is replaced by “parahoric”. We will sometime refer to a result in \([L1]\) such as \([L1, \text{3.2}]\) by \([L1, \text{"3.2"}]\) with the understanding that the replacements in the previous sentence are made.

**Notation.** If \( P, Q \) are parahoric subgroups of \( G' \) then so is \( P^Q = (P \cap Q)U_P \) and we have \( U(PQ) = UP(P \cap UQ) \). We say that \( P, Q \) are in good position if \( P^Q = P \) or equivalently \( Q^P = Q \).

### 1.2. Let \( B \) be the set of all Iwahori subgroups of \( G \) (they are actually contained in \( G' \)). Let \( W' \) be the set of \( G' \)-orbits on \( B \times B \) (\( G' \)-acts by conjugation on both factors). For \( B, B' \in B \) we write \( \text{pos}(B, B') = w \) if the \( G' \)-orbit of \((B, B')\) is \( w \). If \( w \in W' \) and \( \text{pos}(B, B') = w \) then \( B/(B \cap B') \) is an algebraic variety of dimension \( l(w) \).

Let \( I = \{w \in W'; l(w) = 1\} \). There is a well defined group structure on \( W' \) in which the elements of \( I \) have order 2 and in which the following property holds: if \( \text{pos}(B, B') = w_1 \), \( \text{pos}(B', B''') = w_2 \) and \( l(w_1w_2) = l(w_1) + l(w_2) \) then \( \text{pos}(B, B'') = w_1w_2 \). This makes \( W' \) into an infinite Coxeter group (an affine Weyl group) in which the length function is \( w \mapsto l(w) \).

Now \( J \subseteq I \) let \( W_J \) be the (finite) subgroup of \( W' \) generated by \( J \). For \( J \subseteq I \), \( K \subseteq I \) let \( W^J, W^J, W^K \) be as in \([L1, \text{2.1}]\) (with \( W \) replaced by \( W' \)).

If \( P \) is a parahoric subgroup of \( G' \), the set of all \( w \in W' \) such that \( w = \text{pos}(B, B') \) for some \( B, B' \in B, B \subseteq P, B' \subseteq P \) is of type \( \delta \). Then \( \delta \) acts transitively (by conjugation) on \( \mathcal{P}_J \) for \( P \in \mathcal{P}_J, Q \in \mathcal{P}_K \), there is a well defined element \( u = \text{pos}(P, Q) \in J W^{JK} \) such that \( \text{pos}(B, B') \geq u \) (standard partial order on \( W' \)) for any \( B, B' \in B, B \subseteq P, B' \subseteq P \) and \( \text{pos}(B, B') = u \) for some \( B_1, B'_1 \in B, B_1 \subseteq P, B'_1 \subseteq Q \); we then have \( B_1 \subseteq P^Q, B'_1 \subseteq Q^P \). We have \( P^Q \in \mathcal{P}_{J \cup \text{Ad}(u)K} \). Also, \( (P, Q) \mapsto u \) defines a bijection between the set of \( G' \)-orbits on \( \mathcal{P}_J \times \mathcal{P}_K \) and \( J W^{JK} \).

### 1.3. Throughout this paper we fix \( \delta \in D \). There is a unique automorphism of \( W' \) (denoted again by \( \delta \)) such that \( \delta(I) = I \) and such that for any \( J \subseteq I \), any \( P \in \mathcal{P}_J \) and any \( g \in G' \) we have \( gP g^{-1} \in \mathcal{P}_{\delta(J)} \). In this subsection we recall some results of Bédard \([B]\) (see also \([L1, \text{Section 2}]\)). For any \( J \subseteq I \) let \( T(J, \delta) \) be the set of all sequences \( t = (J_n, w_n)_{n \geq 0} \), where \( J = J_0 \supset J_1 \supset J_2 \supset \ldots \) and \( w_0, w_1, \ldots \) are
elements of $\mathbf{W}'$ such that $[L1,\,2.2(a)\text{--}(d)]$ hold (with $\mathbf{W}'$, $\delta$ instead of $\mathbf{W}$, $\epsilon$). For $(J_n, w_n)_{n\geq 0}\in \mathcal{T}(J, \delta)$ we have $J_n = J_{n+1} = \cdots = J_\infty$, $w_n = w_{n+1} = \cdots = w_\infty$ for $n$ large and $\text{Ad}(w_\infty)J_\infty = \delta(J_\infty)$. Moreover, $(J_n, w_n)_{n\geq 0} \mapsto w_\infty$ is a bijection

$$T(J, \delta) \sim \delta(J)\mathbf{W}'.$$  

1.4. We fix $J \subsetneq I$. Let

$$Z_{J, \delta} = \{(P, P', gU_P); P \in \mathcal{T}_J, P' \in \mathcal{T}_\delta, gU_P \in \mathbf{G}^3/U_P, gP^{-1} = P'\}.$$  

To any $(P, P', gU_P) \in Z_{J, \delta}$ we associate a sequence $(J_n, w_n)_{n\geq 0}$ with $J_n \subsetneq I$, $w_n \in \mathbf{W}'$, and a sequence $(P^m, P'^m, gU_{P^m}) \in Z_{J_n, \delta}$. We set

$$P^0 = P, \quad P'^0 = P', \quad J_0 = J, \quad w_0 = \text{pos}(P^0, P'^0).$$  

Assume that $n \geq 1$, that $P^m, P'^m, J_m, w_m$ are already defined for $m < n$ and that $w_m = \text{pos}(P^m_m, P'^m_m)$ for $m < n$, $(P^m, P'^m, gU_{P^m}) \in Z_{J_m, \delta}$ for $m < n$. Let

$$J_n = J_{n-1} \cap \delta^{-1}\text{Ad}(w_{n-1})(J_{n-1}),$$

$$P^n = g^{-1}((P'^{n-1}P^{n-1})g)P_n, \quad P'^n = (P'^{n-1}P^{n-1}) \in \mathcal{T}_\delta,$$

$$w_n = \text{pos}(P^n_n, P'^n_n) \in \delta(J_n)\mathbf{W}'_J.$$  

Note that $gU_{P^n}$ is well defined by $U_{P^n} \cap P^n \subset U_{P^n}$ hence $U_{P^n} \subset U_{P^n}$. This completes the inductive definition. Note that $P' = P'^0 \supset P'^1 \supset \cdots$ hence $(P^n, P'^n)$ is independent of $n$ for large $n$. From $[L1,\,3.2(c)]$ (with $P, P', Z$ replaced by $P^{n-1}, P'^{n-1}, P^n$) we see that $w_n \in \mathbf{W}'_{J_{n-1}}$ for $n \geq 1$. Thus, $(J_n, w_n)_{n\geq 0} \in \mathcal{T}(J, \delta)$. We write $(J_n, w_n)_{n\geq 0} = \beta'(P, P', gU_P)$. For $t \in \mathcal{T}(J, \delta)$ we set

$$^{t}Z_{J, \delta} = \{(P, P', gU_P) \in Z_{J, \delta}; \beta'(P, P', gU_P) = t\}.$$  

The $\mathbf{G}'$-action on $Z_{J, \delta}$ given by

$$h: (P, P', gU_P) \mapsto (hP'h^{-1}, hP'h^{-1}, hgh^{-1}U_{hP'h^{-1}})$$  

preserves the subset $^{t}Z_{J, \delta}$. Clearly, $^{(t)Z_{J, \delta}} \subseteq \mathcal{T}(J, \delta)$ is a partition of $Z_{J, \delta}$.

1.5. Let $J \subsetneq I$. Let $P \in \mathcal{T}_J$. The inclusion $\alpha: \mathbf{G}^3/U_P \to Z_{J, \delta}, \quad a(gU_P) = (P, gP^{-1}gU_P)$ is $P$-equivariant, where $P$ acts on $\mathbf{G}^3/U_P$ by $p, gU_P \mapsto pgP^{-1}U_P$ and on $Z_{J, \delta}$ by restriction of the $\mathbf{G}'$-action. For any $t \in \mathcal{T}(J, \delta)$ we set $^{t}\mathbf{G}'/U_P = a^{-1}(^{t}Z_{J, \delta})$. Clearly, $^{(t)\mathbf{G}'/U_P} \subseteq \mathcal{T}(J, \delta)$ is a partition of $\mathbf{G}'/U_P$ into $P$-stable subsets.

For example, if $J = \emptyset$, $P \in \mathcal{B}$ and we identify $\mathcal{T}(\emptyset, \delta) = \mathbf{W}'$ as in 1.3(a), then for $w \in \mathbf{W}'$ we have $^{w}\mathbf{G}'/U_B = \{gU_B \in \mathbf{G}'/U_B; \text{pos}(gBg^{-1}, B) = w\}$.  

1.6. Let $t \in \mathcal{T}(J, \delta)$. For any $r \geq 0$ let $t_r = (J_n, w_n)_{n \geq r} \in \mathcal{T}(J_r, \delta)$. Consider the sequence of $\mathbf{G}'$-equivariant maps

$$t_0 Z_{J_0, \delta} \overset{\partial_0}{\longrightarrow} t_1 Z_{J_1, \delta} \overset{\partial_1}{\longrightarrow} t_2 Z_{J_2, \delta} \overset{\partial_2}{\longrightarrow} \cdots, \quad \text{(a)}$$

where for $r \geq 0$ and any $(P, P', gU_P) \in t_r Z_{J_r, \delta}$ we set

$$\partial_r(Q, Q', gU_Q) \mapsto (g^{-1}(Q'Q)g, Q', gU_{g^{-1}(Q'Q)g}) \in t_{r+1}Z_{J_{r+1}, \delta}. $$
Note that for sufficiently large $r$, $\vartheta_r$ is the identity map. (Recall that $J_r = J_{r+1} = \cdots = J_\infty$, $w_r = w_{r+1} = \cdots = w_\infty$ for $r$ large and $\text{Ad}(w_\infty)(J_\infty) = \delta(J_\infty)$.) Note that

(b) each fibre of $\vartheta_r$ is isomorphic to the affine space $(U_Q \cap U_{Q'})\setminus(U_Q \cap Q')$ (a closed subset of the algebraic variety $U_{Q'}\setminus Q'$), where

$$(Q, Q') \in \mathcal{P}_{J_r} \times \mathcal{P}_{\delta(J_r)}, \quad \text{pos}(Q', Q) = w_r.$$ (See [L1, "3.12(b)"]). Also,

(c) the map induced by $\vartheta_r$ from the set of $G'$-orbits on $t_r Z_{J_r, \delta}$ to the set of $G'$-orbits on $t_{r+1} Z_{J_{r+1}, \delta}$ is a bijection.

(See [L1, "3.12(c)"]).

Let $P \in \mathcal{P}_{J_r}$. For any $r \geq 0$ we set $\Delta_r = \{(Q, Q', gU_Q) \in t_r Z_{J_r, \delta}; Q \subset P\}$. Now $P$ acts on $\Delta_r$ by restriction of the $G'$-action on $t_r Z_{J_r, \delta}$. We have a cartesian diagram

$$
\begin{array}{ccc}
\Delta_r & \xrightarrow{c_r} & t_r Z_{J_r, \delta} \\
\downarrow{b_r} & & \downarrow{\vartheta_r} \\
\Delta_{r+1} & \xrightarrow{c_{r+1}} & t_{r+1} Z_{J_{r+1}, \delta},
\end{array}
$$

where $c_r, c_{r+1}$ are the obvious inclusions and $b_r$ is the restriction of $\vartheta_r$. Using this and (b) we see that

(d) each fibre of the $P$-equivariant map $b_r : \Delta_r \to \Delta_{r+1}$ is isomorphic to the affine space $(U_Q \cap U_{Q'})\setminus(U_Q \cap Q')$, where

$$(Q, Q') \in \mathcal{P}_{J_r} \times \mathcal{P}_{\delta(J_r)}, \quad \text{pos}(Q', Q) = w_r.$$ We show:

(e) the map induced by $b_r$ from the set of $P$-orbits on $\Delta_r$ to the set of $P$-orbits on $\Delta_{r+1}$ is a bijection.

The cartesian map above induces a commutative diagram

$$
\begin{array}{ccc}
(P\text{-orbits on } \Delta_r) & \to & (G'\text{-orbits on } t_r Z_{J_r, \delta}) \\
\downarrow & & \downarrow \\
(P\text{-orbits on } \Delta_{r+1}) & \to & (G'\text{-orbits on } t_{r+1} Z_{J_{r+1}, \delta}).
\end{array}
$$

The horizontal maps are clearly bijections. The right vertical map is a bijection by (c). It follows that the left vertical map is a bijection. This proves (e).

If $k$ is replaced by a finite field $F_q$ with $q$ elements and if the Frobenius map acts trivially on $W'$, $\delta$, then $^4G^\delta/U_P$ becomes a finite set with cardinality equal to $\#(P/U_P)q^{\ell(w)}$, where $w \in \delta^{(J)} W'$ corresponds to $t$ under 1.3(a). (This is seen by an argument similar to that in [L2, 8.20].)

1.7. We now make a digression. Let $L$ be a group and let $E$ be a set with a free transitive left $L$-action $(l, e) \mapsto le$, and a free transitive right $L$-action $(e, l) \mapsto el$ such that $(le)l' = l(el')$ for $l, l' \in L, e \in E$. For any $e \in E$ we define a map $\tau_e : L \to L$ by $\tau_e(l)e = el$. Note that $\tau_e$ is a group automorphism of $L$. Assume further that $E_0 := \{e \in E; \tau_e^N = 1 \text{ for some } N \geq 1\}$ is nonempty. Let $e_0 \in E_0$ and
let \( d \geq 1 \) be such that \( \tau_d^0 = 1 \). We consider the group \( L_{\gamma_d} \) (semidirect product), where \( \gamma_d \) is the cyclic group of order \( d \) with generator \( \omega \) and \( \omega l = \tau_{\gamma_d}(l) \omega \) for any \( l \in L \). Define a bijection \( f: E \xrightarrow{\sim} L_\omega \) (\( L_\omega \) is an \( L \)-coset in \( L_{\gamma_d} \)) by \( le_0 \mapsto l \omega \). For any \( l \in L, e \in E \) we have \( f(le_0 l^{-1}) = lf(e)l^{-1} \) (in the right side we use the group structure of \( L_{\gamma_d} \)). Thus, the set \( E \) with the \( L \)-action \( l: e \mapsto le_0 l^{-1} \) is isomorphic to the coset \( L_\omega \) in \( L_{\gamma_d} \) with the \( L \)-action given by \( L \)-conjugacy in the group structure of \( L_{\gamma_d} \).

1.8. We return to the setup in 1.6. For \( r \geq 0 \) sufficiently large, \( t_r \) is independent of \( r \); we denote it by \( t_\infty \). We have \( t_\infty = (J_\infty', w_\infty')_{n \geq 0} \), where \( J_\infty' = J_\infty, w_\infty' = w_\infty \) for all \( n \geq 0 \) and \( \operatorname{Ad}(w_\infty)(J_\infty) = \delta(J_\infty) \). It follows that

\[
n_{L_{\gamma_d}} \ast \{ (Q, Q'), gU_Q \} = \{ (Q, Q'), gU_Q \} \in \mathcal{O}, gU_Q \in G^d/U_Q, gQg^{-1} = Q' \}
\]

where

\[
\mathcal{O} = \{ (Q, Q') \in \mathcal{P}_{J_\infty} \times \mathcal{P}_{J_\infty'}; \operatorname{pos}(Q', Q) = w_\infty \}.
\]

Note that any \( Q, Q' \) with \( (Q, Q') \in \mathcal{O} \) are in good position. Hence

\[
\Delta := \{ (Q, Q'), gU_Q \} \in \mathcal{O}, gU_Q \in G^d/U_Q, gQg^{-1} = Q', Q \subset P \}.
\]

We have maps \( \Delta \xrightarrow{\sim} \Delta' \xrightarrow{\sim} \Delta'' \), where \( \Delta' = \{ (Q, Q') \in \mathcal{O}, Q \subset P \}, \Delta'' = \{ Q \in \mathcal{P}_{J_\infty}; Q \subset P \} \), \( e(Q, Q'), gU_Q = (Q, Q') \), \( e'(Q, Q') = Q \). We can regard \( \Delta, \Delta', \Delta'' \) as smooth algebraic varieties so that \( \Delta'' \) is isomorphic to a partial flag manifold \( P/Q \) (where \( Q \in \Delta'' \) of \( P/U_P \), \( e' \) is an affine space bundle and \( e \) is a fibration with fibres isomorphic to \( Q/U_Q \) (where \( Q \in \Delta'' \)). We can regard \( \Delta_0, \Delta_1, \Delta_2, \ldots \) as smooth algebraic varieties such that \( b_r: \Delta_r \rightarrow \Delta_{r+1} \) is an affine space bundle for any \( r \geq 0 \) (note that \( \Delta_r = \Delta \) for \( r \) large is a smooth algebraic variety). We can identify \( \ast G^d/U_P \) with \( \Delta_0 \) by \( gU_P \mapsto (P, gQg^{-1}, gU_P) \). It follows that \( \ast G^d/U_P = \Delta_0 \) is naturally a smooth algebraic variety, an iterated affine space bundle over \( \Delta \) via the map

\[
b := \ldots b_2b_1b_0: \ast G^d/U_P \rightarrow \Delta.
\]

In fact, it is an algebraic subvariety of the ind-variety \( G^d/U_P \).

Now the \( P \)-action on \( \Delta \) induces a transitive \( P \)-action on \( \Delta' \). Hence if we fix \( (Q, Q') \in \Delta' \) and we set \( E = \{ gU_Q \in G^d/U_Q, gQg^{-1} = Q' \} \), we have a \( P \)-equivariant isomorphism

\[
\chi: P \times_{Q \cap Q'} E \xrightarrow{\sim} \Delta, (p, gU_Q) \mapsto (pQp^{-1}, pQ'p^{-1}, pqp^{-1}U_Qp^{-1}Q').
\]

Let \( L_t = (Q \cap Q')/U_Q \subset Q' \). Since \( Q, Q' \) are in good position, the obvious homomorphisms \( Q/U_Q \xrightarrow{\sim} L_t \xrightarrow{\sim} Q'/U_{Q'} \) are isomorphisms.

Note that the \( (Q \cap Q') \)-action on \( E \) given by \( h: gU_Q \mapsto hgh^{-1}U_Q \) factors through an \( L_t \)-action (called “conjugation”). On \( E \) we have a free transitive right \( L_t \)-action given by \( l: gU_Q \mapsto gU_Q \cdot l = gQU_Q \) (where \( gU_Q = i(l) \)) and a free transitive left \( L_t \)-action given by \( l: gU_Q \mapsto l \cdot gQU_Q = gQlU_Q \) (where \( gQU_Q = i'(l) \)), so that the “conjugation” \( L_t \)-action above is \( l: gU_Q \mapsto l \cdot gQU_Q \cdot l^{-1} \). The left and right \( L_t \)-actions on \( E \) satisfy the hypotheses of 1.7 with \( L = L_t \). Hence, by 1.7, the conjugation \( L_t \)-action on \( E \) is isomorphic to the \( L_t \)-action on the connected component \( C_t = L_t \).
of a semidirect product \( L_t \gamma_d \) (an algebraic group with identity component \( L_t \) and with cyclic group of components \( \gamma_d \) of order \( d \) with generator \( \omega \)).

Thus \( \chi \) can be viewed as an isomorphism \( \chi^1: P \times_{Q' \cap Q'} C_t \xrightarrow{\sim} \Delta \), where \( Q \cap Q' \) acts on \( C_t \) via its quotient \( L_t \), by conjugation.

This isomorphism induces a bijection between the set of \( P \)-orbits on \( \Delta \) and the set of \( P \)-orbits on \( P \times_{Q' \cap Q'} C_t \) which can be naturally identified with the set of \( L_t \)-orbits (for conjugation) on \( C_t \). Composing this with the bijections in 1.6(e) we obtain a bijection between the set of \( P \)-orbits on \( ^tG^d/U_P = \Delta_0 \) and the set \( L_t \backslash C_t \) of \( L_t \)-orbits (for conjugation) on \( C_t \). Putting together these bijections (for various \( t \)) we obtain a bijection between the set of \( P \)-orbits on \( G^d/U_P \) and the set \( \bigsqcup_{t \in T(\delta, \delta)} L_t \backslash C_t \).

1.9. Let \( V \) be a 2-dimensional vector space over \( l = k((\epsilon)) \) with a fixed volume element \( \omega \). Let \( A = k[[\epsilon]] \). Let \( X' \) be the set of all \( A \)-lattices in \( V \). Let \( X \) be the set of all \( L \in X' \) such that for some \( A \)-basis \( e_1, e_2 \) of \( L \) we have \( e_1 \wedge e_2 = \omega \). Then \( G = SL(V) \) acts transitively on \( X \). Let \( L \in X \) and let \( P = \{ g \in G: gL = L \} \). Note that \( G = G' \) is as in 1.1 and \( P \) is a parahoric subgroup of \( G \). In our case the set \( G'/U_P \) may be identified with the set \( Y \) of all pairs \((L', g)\), where \( L' \in X \) and \( g \) is an isomorphism of \( k \)-vector spaces \( L/\epsilon L \xrightarrow{\sim} L'/\epsilon L' \) preserving the volume elements induced by \( \omega \). We have a partition \( Y = \bigsqcup_{n \in \mathbb{N}} Y_n \), where \( Y_n = \{ (L', g) \in Y: \text{dim}_k L/(L \cap L') = n \} \). If \( n > 0 \) then \( Y_n \) has a further partition \( Y_n = Y'_n \sqcup Y''_n \) defined as follows. Let \((L', g) \in Y_n \). Let \( L_1 \) be the unique lattice in \( X' \) such that \( L \cap L' \subset L_1 \subset L \), \( \text{dim} \frac{L}{L_1} = 1 \) and let \( L_2 \) be the unique lattice in \( X' \) such that \( L \cap L' \subset L_2 \subset L' \), \( \text{dim} \frac{L'}{L_2} = 1 \). Let \( L_1 = \frac{L_1}{\epsilon L_0} \) (a line in \( L/\epsilon L \)) and let \( L_2 = L_2/\epsilon L' \) (a line in \( L'/\epsilon L' \)). We say \( (L', g) \in Y'_n \) if \( g(L_1) = L_2 \); we say that \((L', g) \in Y''_n \) if \( g(L_1) \neq L_2 \). This defines the partition of \( Y_n \).

The subvarieties \( Y_0, Y'_n(n > 0), Y''_n(n > 0) \) are precisely the pieces \( ^tG'/U_P \) of \( Y = G^d/U_P \). If \( k \) is replaced by a finite field \( F_0 \) with \( q \) elements then \( Y_0, Y'_n, Y''_n \) become finite sets with cardinal \( q(q^2 - 1), q^{2n}(q^2 - 1), q^{2n+1}(q^2 - 1) \) respectively.

2. Character Sheaves on \( G^d/U_P \)

2.1. We preserve the setup of 1.8. Let \( K \) be a character sheaf on \( C_t \) (the definition of [L3] applies since \( C_t \) is a connected component of an algebraic group whose identity component is \( L_t \), a reductive algebraic group). Then \( K \) is \((Q \cap Q')\)-equivariant, where \( Q \cap Q' \) acts on \( C_t \) through its quotient \( L_t \) and there is a well defined \( P \)-equivariant simple perverse sheaf \( K' \) on \( P \times_{Q' \cap Q'} C_t \) such that \( h_1^*K' \xrightarrow{\sim} h_2^*K \) up to shift (here \( P \times_{Q' \cap Q'} C_t \xrightarrow{h_1} P/(U_Q \cap U_{Q'}) \times C_t \xrightarrow{h_2} C_t \) are the obvious maps).

Now \( K' \) can be viewed as a \( P \)-equivariant simple perverse sheaf on \( \Delta \) (via \( \chi' \)). Since \( \chi: G^d/U_P \to \Delta \) is a \( P \)-equivariant iterated affine space bundle we deduce that \( b^*K[d] \) is a \( P \)-equivariant simple perverse sheaf on \( ^tG^d/U_P \) (for a well defined \( d \in \mathbb{Z} \)). Let \( \mathcal{G}^d/U_P \) be the closure of \( ^tG^d/U_P \) in \( G^d/U_P \) (a \( P \)-stable algebraic subvariety of \( G^d/U_P \)). Let \( \hat{K} \) be the simple perverse sheaf on \( \mathcal{G}^d/U_P \) whose restriction to \( ^tG^d/U_P \) is \( b^*K'[d] \). We can view \( \hat{K} \) as a simple perverse sheaf on \( G^d/U_P \) with support contained in \( \mathcal{G}^d/U_P \).
Let $\mathcal{C}_{P,\delta}$ be the class of simple perverse sheaves on $G^\delta/U_P$ consisting of all $\hat{K}$ as above (for various $\mathfrak{t}$). The set of isomorphism classes in $\mathcal{C}_{P,\delta}$ is in bijection with the set of pairs $(\mathfrak{t}, K)$, where $\mathfrak{t} \in \mathcal{T}(J, \delta)$ and $K$ is a character sheaf on $\mathcal{C}_\mathfrak{t}$ (up to isomorphism). The objects of $\mathcal{C}_{P,\delta}$ are called character sheaves on $G^\delta/U_P$.

2.2. Let $J \subset J' \subset I$ and let $P \in \mathcal{P}_J, Q \in \mathcal{P}_{J'}$ be such that $P \subset Q$. Consider the diagram

$$
\begin{array}{cccc}
G^\delta/U_P & \xrightarrow{f_1} & E_1 & \xrightarrow{f_2} & E_2 & \xrightarrow{f_3} & E_3 & \xrightarrow{f_4} & G^\delta/U_Q,
\end{array}
$$

where $E_1 = Q/U_Q \times G^\delta/U_P, \quad E_2 = \{(P', gU_P) \in \mathcal{P}_J \times G^\delta/U_P; \quad P' \subset Q\}$,

$E_3 = \{(P', gU_Q) \in \mathcal{P}_J \times G^\delta/U_Q; \quad P' \subset Q\}$

are ind-varieties and

$f_1(xU_Q, gU_P) = gU_P, \quad f_2(xU_Q, gU_P) = (xP^{-1}, xgx^{-1}UxP^{-1}),$

$f_3(P', gU_Q) = (P', gU_P), \quad f_4(P', gU_Q) = gU_Q.$

Note that $f_1$ is a principal fibration with group $Q/U_Q; \quad f_2$ is a principal fibration with group $P/U_Q; \quad f_3$ is an affine space bundle with fibres isomorphic to $U_P/U_Q; \quad f_4$ is a proper map.

Below we shall extend in an obvious way the standard operations for derived categories of $\mathcal{T}$-sheaves on algebraic varieties to the case of ind-varieties. Let $K \in \mathcal{C}_{P,\delta}$. Then $f_1^* K[d_1] = f_2^* K'[d_2]$ for a well defined simple perverse sheaf $K'$ on $E_2$. Here $d_1 = \dim Q/U_Q, \quad d_2 = \dim P/U_Q$. We set $\text{ind}(K') = f_4^* f_3^* K'$. By the decomposition theorem, $\text{ind}(K)$ is a direct sum of simple perverse sheaves (with shifts) on $G^\delta/U_Q$. More precisely, it is a direct sum of character sheaves (with shifts) on $G^\delta/U_Q$. (This is shown by arguments similar to those in [L1, Sections 4 and 6].)

Conversely, for any character sheaf $\hat{K}$ on $G^\delta/U_Q$ there exists a character sheaf $K$ on $G^\delta/U_P$ (where $P$ is an Iwahori subgroup contained in $Q$) such that some shift of $\hat{K}$ is a direct summand of $\text{ind}(K)$. (This is shown by arguments similar to those in [L1, Section 4].)

References


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