EQUIVALENT STATEMENTS TO
EXOTIC P.L. STRUCTURES ON
THE 4-SPHERE

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Let $\Sigma^4$ be a combinatorial piecewise-linear (abbreviated p.l.) 4-manifold with the homotopy type of the 4-sphere. We prove:

Theorem. For each such $\Sigma^4$ there exists a unique p.l. contractible 5-manifold $W_5$ whose boundary is equal to $\Sigma^4$. Furthermore the topological type of $W_5$ is determined by the topological type of $\Sigma^4$.

From this theorem follows the equivalence of statements (1) and (2) of the following three equivalent statements:

1. the 4-sphere has a unique p.l. structure,
2. the 5-ball has a unique p.l. structure, and
3. the 4-ball has a unique p.l. structure.

There is the following relationship with the Schönhflies conjecture.

Theorem. If the 4-ball has a unique p.l. structure then $S^{n-1}$ p.l. embedded in $S^n$ bounds two p.l. n-balls whose interiors are disjoint.

With the assumption that the 4-sphere does not have a unique p.l. structure, non-combinatorial triangulations of $S^5$ and $R^5$ are exhibited.

Finally we embed any p.l. 4-sphere in $R^5$ with the usual linear structure.

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Chapter 1, Introduction

This paper will treat research occasioned by John Stallings' article, "The Piecewise-linear Structure of Euclidean Space." From Stallings' paper we learn that $\mathbb{R}^n$ (n≠4) has a unique piecewise-linear structure. The case n=4 remains unknown and seems beyond the present means of mathematics. However, in an attempt to mimic Stallings' proof for n=4 we consider $\mathbb{R}^4 \subseteq \mathbb{R}^4 \times \mathbb{R} = \mathbb{R}^5$ and proceed with five dimensional engulfing. The engulfing lemma enables us to contain four dimensional compact subsets of $\mathbb{R}^4$ in five dimensional balls. We then take a four dimensional cross section of the 5-ball in the plane of $\mathbb{R}^4$ and hope to obtain a 4-ball containing the original compact set. At this stage certain questions about the piecewise-linear uniqueness of the combinatorial 4-sphere, 4-ball, and 5-ball arise.

Specifically this paper will clarify the uniqueness relationship between the above three combinatorial objects; mention the connection between combinatorial uniqueness and the existence of non-combinatorial triangulations of some simple topological manifolds; and show some ways of embedding these "exotic" objects in standard space. It should be noted that no one has discovered an exotic object.
Chapter 2, Notation and Terminology

We will work in the category of piecewise-linear spaces and a few introductory words are necessary. If $A, B \subseteq \mathbb{R}^n$, euclidean $n$-space, are disjoint subsets we define the associative operation $\text{join } AB = \{ \lambda a + \mu b \mid a \in A, b \in B \}$, where $\lambda$ and $\mu$ are real numbers; $\lambda, \mu$ non-negative; and $\lambda + \mu = 1$. A finite set $\{v_0, v_1, \ldots, v_n\} \subseteq \mathbb{R}^m$ is independent if the vectors $\{v_1 - v_0\}$ are linearly independent in $\mathbb{R}^m$. An $n$-simplex $A \subseteq \mathbb{R}^m$ is the repeated join of $n+1$ points, vertices, which span $A$. A simplex spanned by a subset of the vertices is called a face of $A$.

The next important object is the simplicial complex. A simplicial complex $K$ is a finite collection of simplices in some $\mathbb{R}^n$ satisfying:

1. if $C \in K$ and $B$ is a face of $C$ then $B \in K$, and
2. if $B, C \in K$ then $B \cap C$ is a face of both $B$ and $C$.

A subdivision $L$ of a simplicial complex $K$ has the same underlying subset of $\mathbb{R}^n$ and each simplex of $L$ is contained in a simplex of $K$.

Note that we can represent a simplex merely by letting a vertex set stand for the simplex and by letting all the subsets of the vertex set represent the faces of the simplex. Thinking of simplicial complexes as vertex sets we define a simplicial map $f: K \to L$ to be a map of simplicial complexes $K$ and $L$, considered as vertex sets, such that if $\{v_0, v_1, \ldots, v_n\}$ represents a simplex of $K$ then $\{f(v_0), f(v_1), \ldots, f(v_n)\}$ represents a simplex of $L$. We note that by taking joins such
a map can be extended from a vertex map to a topological map of subsets of two euclidean spaces. Also a map \( f \) of \( K \subset \mathbb{R}^n \) to \( L \subset \mathbb{R}^m \) can yield a map defined on the vertices of \( K \). The map \( f \) is simplicial if \( f \) restricted to the vertices of \( K \) is a simplicial map and \( f \) is linear on each simplex of \( K \). A simplicial isomorphism is a simplicial map that is bijective.

If \( K \subset \mathbb{R}^n \) and \( L \subset \mathbb{R}^m \) are simplicial complexes then a p.l. map \( f:K \rightarrow L \) is such that \( f \) is a map of the underlying topological spaces and there exist subdivisions \( K' \) of \( K \) and \( L' \) of \( L \) so that \( f:K' \rightarrow L' \) is simplicial. A p.l. equivalence is a p.l. map that is a simplicial isomorphism upon appropriate subdivision.

We extend the notion of simplicial complex by introducing the idea of triangulation. Let \( X \) be a topological space, \( K \) a simplicial complex, and \( h:K \rightarrow X \) a topological homeomorphism then we say \( K \) is a triangulation of \( X \). We also speak of having put a p.l. structure on \( X \) and confuse the distinction between \( X \) and \( K \). Whenever we speak of uniqueness we mean up to p.l. equivalence unless otherwise mentioned. That is if \( h:K \rightarrow X \) and \( h':K' \rightarrow X \) are two triangulations of \( X \), the p.l. structures on \( X \) are equivalent if and only if \( K \) is p.l. equivalent to \( K' \).

For more of the language of p.l. topology the reader should consult Hudson, Zeeman, or Rourke and Sanderson. When referring to the standard structure on some topological
manifold we mean the p.l. structure induced from the appropriate standard linear object. Thus a standard 4-sphere is triangulated by a member of the class of simplicial complexes simplicially equivalent to the boundary of a 5-simplex. The standard \( \mathbb{R}^n \) is in turn a topological \( \mathbb{R}^n \) triangulated by anything in the class of rectilinear simplicial subdivisions of \( \mathbb{R}^n \) with the usual linear structure. A p.l. structure on a topological manifold that is not p.l. equivalent to the standard structure is called exotic. Unless specifically mentioned all triangulations of manifolds are combinatorial. That is each point has a neighborhood which is a standard \( n \)-ball (p.l. equivalent to an \( n \)-simplex.)

\( S^n \) signifies the topological \( n \)-sphere and \( B^n \) signifies the topological \( n \)-ball, which unless stated to the contrary have the standard p.l. structures.
Chapter 3, Homotopy 4-spheres

We first state several theorems, a corollary, and a definition. The proofs will follow.

**Theorem 3.1** Let $\Sigma^4$ be a compact p.l. manifold with the homotopy type of $S^4$ then $\Sigma^4$ bounds a p.l. 5-manifold $W^5$ such that $W^5 \subset B^{N+5}$ with a product neighborhood.

Two p.l. $n$-manifolds $W_1$ and $W_2$ with the same boundary are cobordant relative to their boundary if there exists a p.l. $n+1$-manifold $W$ such that the boundary of $W$ is

$$W_1 \cup_{\partial W_1} W_1 \times [0,1] \cup_{\partial W_1} W_2$$

where $\partial W_1$ is the boundary of $W_1$.

**Theorem 3.2** The $W^5$ of theorem 3.1 is cobordant relative to its boundary to a contractible p.l. 5-manifold $W^{5'}$.

We omit the proof of this theorem since its is a standard result of surgery (Kervaire and Milnor; Browder).

**Theorem 3.3** The $W^{5'}$ of theorem 3.2 is p.l. unique.

**Theorem 3.4** The topological type of $W^{5'}$ is determined by the topological type of $\Sigma^4$.

**Corollary 3.5** For each p.l. manifold $\Sigma^4$, a homotopy 4-sphere, there exists a unique contractible p.l. 5-manifold $W^5$ with $\partial W^5 = \Sigma^4$. Furthermore the topological type of $W^5$ is determined by the topological type of $\Sigma^4$. 
Proof of theorem 3.3 We suppose that $W_1$ and $W_2$ are contractible p.l. 5-manifolds whose boundaries are the $\Sigma_4$ of theorem 3.1. Let

$$W = W_1 \cup \Sigma_4 \times I \cup \Sigma_4 W_2.$$ 

We observe that $W$ is a p.l. homotopy 5-sphere which by the Poincare conjecture (Rourke and Sanderson, p9) is the standard p.l. 5-sphere. The following lemma verifies this remark. Thus $W$ bounds a standard $B^6$ which gives a cobordism relative to the boundary between $W_1$ and $W_2$. Then by the relative h-cobordism theorem (Rourke and Sanderson, p87) $W_1$ is p.l. equivalent to $W_2$.

Lemma $W$ in the proof of theorem 3.3 is a p.l. homotopy 5-sphere.

Proof It is clear that $W$ is a compact five dimensional p.l. manifold. We must only show it has the homotopy type of $S^5$. Since both $W_1$ and $W_2$ are contractible, $W$ has the homotopy type of the suspension of $\Sigma_4$, which in turn has the homotopy type of the suspension of $S^4$, which is $S^5$.

Proof of theorem 3.4 We suppose that $W_1$ and $W_2$ are contractible p.l. 5-manifolds each of whose boundary is a homotopy 4-sphere. Let $\Sigma_i = \partial W_i$ $(i = 1, 2)$. If $h: \Sigma_1 \rightarrow \Sigma_2$ is a homeomorphism we can use $h$ to induce a new p.l. structure on $\Sigma_2$. By Kirby, thl7, p107 this new triangulation extends to all of $W_2$. Thus by theorem 3.3 $W_1$ and $W_2$ with its new p.l. structure are p.l. equivalent, hence topologically
Proof of theorem 3.1 Since p.l. 4-manifolds admit a smoothing (Cairns) and smooth manifolds triangulate uniquely (Munkres) we can work in the smooth category.

We assume the standard language of smooth topology and smooth bundles. The reader can examine Milnor [1] and [2].

Part 1) \( \sum^4 \) smoothly embeds in \( S^{N+4} \) with a trivial normal bundle. By Whitney's embedding theorem \( \sum^4 \) embeds in \( R^9 \) which gives an embedding into \( S^9 \). So we have

\[ \sum^4 \hookrightarrow S^{N+4}, \quad \text{where } N > 5. \]

At this point we need some notation. For \( N \hookrightarrow M \) smooth manifolds we have:

- \( \tau_M \) is the tangent bundle of \( M \)
- \( \tau_M/N \) is the tangent bundle of \( M \) restricted to \( N \),
- \( N_i \) is the normal bundle of the embedding \( i \),
- \( \mathcal{E}^n \) is the trivial \( n \)-bundle.

We have the following three equations:

1. \( \tau_\sum^4 \oplus \mathcal{E}^1 = \mathcal{E}^5 \)
2. \( N_i \oplus \tau^4 = \tau_\sum^{N+4} |_{\sum^4} \)
3. \( \tau_\sum^{N+4} \oplus \mathcal{E}^1 = \mathcal{E}^{N+5} \)

Equation (1) is a result of Kervaire and Milnor. Equations (2) and (3) are standard smooth results. (Milnor, [2])

We also note that if \( N > \text{dim } K \), where \( K \) is a simplicial
complex, $\mu$ is an $N$-bundle over $K$ then if $\mu$ is stably trivial $\mu$ is trivial. Thus by equation (3) $\tau_{S^N+4} \mid_1(\Sigma^4)$ is trivial. We then add a $\varepsilon^1$ to each side of equation (2) and derive, using equation (1), $\vee_1 \oplus \varepsilon^5 = \varepsilon^{N+5}$. So $\Sigma^4$ has a stably trivial normal bundle which by our above note is trivial. Thus by the tubular neighborhood theorem (Milnor [2]) $i(\Sigma^4)$ has a product neighborhood.

Part ii) There is a smooth map $f:S^{N+4} \rightarrow S^N$ such that there is a regular value $p \in S^N$ and $f^{-1}(p) = \Sigma^4$.

Let $T = \Sigma^4 \times B^N$, the tubular neighborhood of $\Sigma^4$ in $S^{N+4}$.

Let $T: T \rightarrow B^N$ be projection onto $B^N$ and let $c:B^N \rightarrow S^N$ be the map which identifies $B^N$ with a point $q \in S^N$. We now consider $S^{N+4} = T \cup V$, where $V = \text{closure}(S^{N+4} - T)$ and define a map, which can be taken to be smooth, $f:S^{N+4} \rightarrow S^N$ by $f|_V: V \rightarrow q \in S^N$ and $f|_T = c \circ \tau$.

We note that $\Sigma^4 \rightarrow T$ is sent to a point $p \in S^N$ and that $p$ is a regular value of $f$.

Part iii) Let $S^{N+4} = \partial B^{N+5}$ then $f$ extends to $F:B^{N+5} \rightarrow S^N$ with $F|_B^{N+5} = f$ and with $p$ still a regular value.

Since $\tau_{N+4}(S^N) = 0$ for our range of $N$, (Toda) $f$ extends to $B^{N+5}$. (Conner and Floyd, p20) This extension can be made smoothly while leaving $f$ fixed on $S^{N+4}$. $F$ can be chosen so
that $p$ is still a regular value. (Milnor [2])

Part iv) **Conclusion of proof**

We now have

$$
\begin{align*}
& \mathbb{B}^{N+5} \\
\xrightarrow{F} & \mathbb{S}^{N} \\
\cup & \\
\xrightarrow{F} & \bigcup \\
F^{-1}(p) = W^5 & \xrightarrow{U} W^5
\end{align*}
$$

where $W^5$ is a smooth five manifold and $\partial W^5 = \Sigma^4$. Since our extension preserves transverse regularity at $p$ the normal bundle of $p$ in $\mathbb{S}^{N}$ pulls back onto the normal bundle of $W^5$ in $\mathbb{B}^{N+5}$. The normal bundle of $p$ is clearly trivial so $W^5$ has a trivial normal bundle. That is $W^5$ has a product neighborhood in $\mathbb{B}^{N+5}$. 
Chapter 4, Uniqueness and Schönflies

The combinatorial relationship between $B^5$, the topological 5-ball; $S^4$, the topological 4-sphere; and $B^4$, the topological 4-ball is summarized by

Theorem 4.1 The following statements are equivalent:

a. $B^5$ has a unique p.l. structure
b. $S^4$ has a unique p.l. structure
c. $B^4$ has a unique p.l. structure.

Proof of theorem 4.1 $a \iff b$ follows from corollary 3.5. $b \implies c$. First we note that any combinatorial $B^4$ has the standard $S^3$ as boundary. (Moise) We suppose $B^4$ has an exotic structure and take the cone on its boundary. This yields an $S^4$ with an exotic structure. If not we could remove the cone on the boundary of $B^4$ (a standard p.l. 4-disk) and by Newman's theorem $B^4$ would be standard. (Rourke and Sanderson) $c \implies b$. We suppose $S^4$ has an exotic structure and remove a 4-simplex from a simplicial complex representing the exotic triangulation. What we have left is a four ball which must be exotic. If not we could cone on its boundary and get $S^4$ back as standard.

It is useful to note that we have

Theorem 4.2 $S^4$ bounds a unique p.l. 5-ball. Here $S^4$ has any combinatorial structure. Also the five ball is unique up to isotopy. (Kirby)
We can summarize the preceding two theorems as follows. Each exotic 5-ball comes from a unique exotic 4-sphere. By taking the boundary we get every exotic 4-sphere from a unique exotic 5-ball. We also have a 1-1 correspondence between exotic $B^4$'s and exotic $S^4$'s.

Theorem 4.1 also has connections with a p.l. Schönflies problem. (Rourke and Sanderson, p47) Specifically the problem is:

Suppose $S^{n-1}$ and $S^n$ are standard p.l. spheres and suppose $S^{n-1}$ is p.l. locally flat embedded in $S^n$ then are the closures of the components of $S^n-S^{n-1}$ standard p.l. $n$-balls? (P.l. locally flat means that the closures of these components are p.l. submanifolds of $S^n$)

At the present time the answer is "yes" for $n \neq 4$. If the 4-ball had a unique p.l. structure then if $S^3 \subset S^4$ is p.l. locally flat, it would bound two standard p.l. 4-balls. The reasoning is as follows:

Since p.l. locally flat implies topologically flat, Brown's work implies $S^3$ bounds a topological 4-ball $B^4$ which is triangulated as a combinatorial manifold. Under our assumption $B^4$ has a unique p.l. structure and thus is standard.

A general $n$-dimensional p.l. Schönflies problem exists in which the assumption of locally flat is dropped. The problem is:

If $S^{n-1}$ is p.l. embedded in $S^n$ are the closures of the
components of $S^n \cdot S^{n-1}$ p.l. standard n-balls?
These problems are inductively related. Specifically, if the general Schönflies problem is true in dimension $n-1$ then any $S^{n-1}$ p.l. embedded in $S^n$ is p.l. locally flat. This can be seen by examining links of points in $S^{n-1}$ problem.

The general Schönflies problem is true in dimensions $n = 1, 2, 3$, being just a rephrased Jordan curve problem. As noted above the $n=3$ case implies $S^3$ p.l. embedded in $S^4$ is p.l. locally flat.

So if we assume $B^4$ has a unique p.l. structure we have affirmatively answered the general Schönflies problem in dimension 4. However, a "yes" answer for dimension 4 implies "yes" for all higher dimensions. We see this as follows:

For $n \geq 5$ the Poincaré conjecture (Rourke and Sanderson, p9) implies $S^n$ has a unique p.l. structure. Let

$$\Sigma^m = B^m \cup \bigcup_{S^{m-1}} \text{cone}S^{n-1} \text{ for } m \geq 6.$$ 

$\Sigma^m$ is a p.l. homotopy m-sphere and thus is p.l. equivalent to the standard p.l. $S^m$. By Newman's theorem (Rourke and Sanderson) since $\text{cone}S^{m-1}$ is a standard ball, $B^m$ is a standard m-ball. We have also shown that $B^5$ is standard if and only if the boundary of $B^5$ is a standard $S^4$. Now the general Schönflies problem in dimension $n = 4$ implies a standard $S^4$ p.l. embedded in $S^5$ is p.l. locally flat. Brown's work implies $S^4$ bounds two topological 5-balls,
which are p.l. manifolds with a standard $S^4$ as boundary. By remarks above these 5-balls are standard. The induction clearly works for all higher dimensions. Thus we have the following theorem.

**Theorem 4.3** If $B^4$ has a unique p.l. structure then any $S^{n-1}$ p.l. embedded in $S^n$ bounds two standard p.l. n-balls whose interiors are disjoint.
Chapter 5, Non-combinatorial triangulations

Up to here we have referred only to combinatorial p.l. manifolds, that is each point has a neighborhood which is p.l. equivalent to the standard ball. We should note that no one has produced a triangulation of a manifold that is not combinatorial. Here again the p.l. uniqueness of $S^4$ is germane.

**Theorem 5.1** We assume that an exotic $S^4$ exists. Then $S^5$ has a non-combinatorial triangulation and $R^5$ has a non-combinatorial triangulation.

**Proof of theorem 5.1** The suspension of an exotic $S^4$ yields a non-combinatorial triangulation of $S^5$. If we remove one of the suspension points we obtain a non-combinatorial triangulation of $R^5$. 

Chapter 6, Embedding

We conclude with a simple embedding result. Here $S^4$ and $B^5$ are combinatorial but possibly exotic.

Theorem 6.1 $S^4$ and $B^5$ p.l. embed in the standard $R^5$.

Proof of theorem 6.1 Since every $S^4$ bounds a corresponding $B^5$ (theorem 4.2) it suffices to p.l. embed $B^5$ into $R^5$.

We do this by taking

$$\Sigma = B^5 \cup_{S^4} B^5$$

$\Sigma$ is a combinatorial 5-sphere which has a unique p.l. structure. We remove a point from one copy of $B^5$. So

$$R^5 \simeq S^5 - \{\text{pt}\} \simeq B^5 \cup_{S^4} B^5 - \{\text{pt}\}$$

where $R^5$ is standard and $\simeq$ signifies p.l. equivalence.
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Biographical Note

I, Ralph Alexander Gerra, Junior, was born on July 29, 1948, in Brooklyn, New York. I attended school in Bethlehem, Pennsylvania and graduated as the valedictorian of my high school class. I received my B.A. summa cum laude with highest honors in mathematics from Williams College in June, 1970. I studied a wide selection of subjects and truly tried to broaden my mind.

I entered the Ph.D. at MIT as a Woodrow Wilson Fellow and as a National Science Foundation Graduate Fellow. At MIT I became dissatisfied with the prospects of doing mathematics as a profession rather than as an enjoyment. As a result I have decided to study law and maintain my amateur status as a mathematician.