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The Alignment of Objects with Smooth Surfaces: Error Analysis of the Curvature Method

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Abstract

The recognition of objects with smooth bounding surfaces from their contour images is considerably more complicated than that of objects with sharp edges, since in the former case the set of object points that generates the silhouette contours changes from one view to another. The "curvature method", developed by Basri & Ullman [1988], provides a method to approximate the appearance of such objects from different viewpoints. In this paper we analyze the curvature method. We apply the method to ellipsoidal objects and compute analytically the error obtained for different rotations of the objects. The error depends on the exact shape of the ellipsoid (namely, the relative lengths of its axes), and it increases as the ellipsoid becomes "deep" (elongated in the Z-direction). We show that the errors are usually small, and that, in general, a small number of models is required to predict the appearance of an ellipsoid from all possible views. Finally, we show experimentally that the curvature method applies as well to objects with hyperbolic surface patches.

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1 Introduction

Visual object recognition requires the identification of objects observed from different viewpoints. In the alignment approach [Ullman 1989] objects are recognized in a two stage process. First, the pose of the object is determined, and then the appearance of the object from this pose is predicted and compared to the image. Such a method is used in [Fischler & Bolles 1981, Lowe 1985, Faugeras & Hebert 1986, Chien & Aggarwal 1987, Huttenlocher & Ullman 1987, Lamdan *et al* 1987, Thompson & Mundy 1987]. In these works, the method is applied to either planar or polyhedral objects.

Basri & Ullman [1988] extended the alignment approach to handle rigid objects bounded by smooth surfaces. The method is called "the curvature method", and it approximates the appearance of such objects from different viewpoints using the 3-D curvature of points along the contours. The authors showed experimentally that in general a few models are sufficient to predict the appearance of objects from all possible views with high accuracy. This approximation is also the key for showing that the appearance of objects with smooth bounding surfaces can be predicted by linearly combining a small number of views [Ullman & Basri 1991].

In this paper we analyze the curvature method. We apply the method to ellipsoidal objects and derive an expression that describes the errors obtained. We analyze this expression and show that the error depends on the exact shape of the ellipsoid (namely, the relative length of its axes). We show that the errors are usually small, and that, in general, a small number of models is required to predict the appearance of an ellipsoid from all possible views. Finally, we show experimentally that the curvature method is not restricted to contours generated by parabolic surface patches, but it can also handle contours generated by hyperbolic patches.

2 The Curvature Method

In this section we briefly review the curvature method. The difficulty in predicting the appearance of objects with smooth bounding surfaces is described below. The *silhouette* of an object is the set of its bounding contours observed in the image. These points are generated by the *rim* of the object [Koenderink & Van Doorn 1981] (also called the *contour generator* [Marr 1977]), which is the set of all the points on the object's surface whose normal is perpendicular to the visual axis. When the object contains only sharp edges, these edges compose the rim in all views of the object. The silhouette then includes the projection of those edges that are visible. When, however, the object is bounded by smooth surfaces, the rim is not fixed. Instead, it smoothly changes its position on the

object with viewpoint. To illustrate this phenomenon imagine for instance the shape of an egg. When the observer moves slightly, the observed silhouette of the egg is generated by a new set of points along the egg shell, and thus the silhouette appears to change its shape.

To accurately align objects with smooth surfaces one must account for this special property of the objects. Previous approaches to this problem included the construction of detailed 3-D descriptions for the objects using either simple volumetric or surface primitives [Marr & Nishihara 1978, Brown 1981, Dane & Bajcsy 1982, Potmesil 1983, Brady *et al* 1985, Faugeras & Hebert 1986], or using 3-D wires [Baker 1977]. These approaches, when handling objects with relatively complex structures, often require large memory to represent the objects and perform extensive computations to predict their appearances.

Basri & Ullman [1988] suggested a novel approach to predict the appearance of objects with smooth surfaces. In their scheme an object is represented by its silhouette as seen from a particular viewpoint. Every point along the silhouette is associated with its spatial coordinates together with its 3-D curvature. The appearance of the object from other viewpoints is then predicted simply by applying linear transformations to the object. The authors showed experimentally that this method accurately predicts the appearance of objects for relatively large transformations.

The curvature method is based on the following observation. The position change of the rim on the object depends mainly on the 3-D curvature of the object at the rim points. When the curvature is high the rim changes only slightly if at all. (In case the rim lies on a sharp edge, the curvature is infinite and the rim is fixed.) The lower the curvature is, the more significant is the position change of the rim on the object.

Following this observation the curvature can be used to approximate the position change of the rim. The basic idea is illustrated in Figure 1. Consider an object O rotating by a rotation R around the vertical axis, Y. Let p be a point on the object's rim. The figure shows a horizontal section of the object through p. Let r_x be the curvature radius of p in this section, and let $\mathbf{r}_x$ be a vector of length r_x parallel to the X-axis. When the object rotates by R, point p ceases to be a rim point, and it is replaced by a new point, p', approximated by

$$p' \approx R(p - \mathbf{r}_x) + \mathbf{r}_x \tag{1}$$

This approximation holds as long as the circle of curvature provides a good approximation to the section at p. The method handles general 3-D rotation simply by substituting the vector $\mathbf{r}_x$ in Eq. (1) with the radius vector that corresponds to the rotation applied. The radius vectors of a point with respect to all 3-D rotations can be recovered from a single number, the magnitude $\| (r_x, r_y) \|$, where r_x and r_y are the curvature radii with respect to the vertical and the horizontal axes respectively. This number, together with the

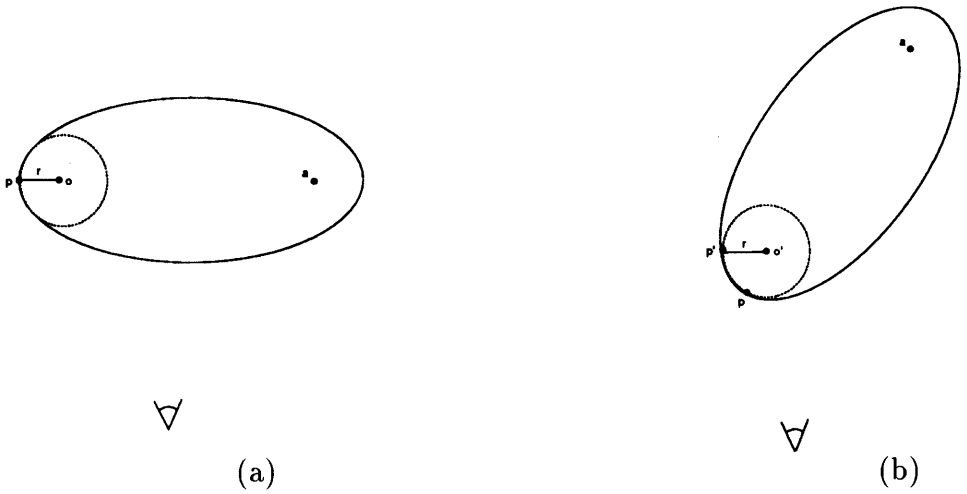

Figure 1: The curvature method. (a) A horizontal section of an ellipsoid. p is a point on the rim, r is the radius of curvature at p, o is the center of the curvature circle, and a is the intersection of the Y-axis with this section. (b) The section rotated. p' is the new rim point, and it is approximated by eq. (1). (Borrowed from [Basri & Ullman 1988]).

spatial coordinates of the point, is the only information required to apply the curvature method.

Consequently, the authors suggested to represent an object by its contour image, as observed from a particular viewing direction. Each point along the silhouette has associated with it, along with its spatial coordinates, the magnitude of its curvature vector. To apply a rotation to the object, the appropriate curvature radius should be computed, and the new position of the point in the image is determined using Eq. (1). Translation and scaling are applied to the object in a straightforward manner. The method was implemented and applied to a number of objects, and it was shown that a few models can predict the appearance of these objects from all possible viewpoints with high accuracy. (See an example in Figure 2.) In addition, Ullman & Basri [1991] showed that the curvature information is implicit in a small number of views of the object. They used this observation to predict the appearance of objects with smooth bounding surfaces by linearly combining a small number of images.

3 Properties of the Curvature Method

The appearance of objects with sharp boundaries (for which the radius of curvature is zero) and of spherical and cylinderal objects is predicted exactly by the curvature

(a) (b) (c)

Figure 2: Application of the curvature method to VW cars. (a) A contour image of a VW car. (b) A prediction of the appearance of the car. (c) Matching the prediction to the actual image (an overlay of (a) and (b)). (Borrowed from [Basri & Ullman 1988]).

method. The appearance of smooth objects with arbitrary structures is, however, only approximated by this method. In order to demonstrate the properties of the curvature method, we applied this method to ellipsoids and analyzed the errors obtained. The analysis is given in this section. We first compute the errors obtained when a canonical ellipsoid rotates around the vertical (Y) axis. We then compute the errors obtained when the same ellipsoid rotates arbitrarily in 3-D space, and show that the errors obtained in the two cases are similar. The error depends on the shape of the ellipsoid, in other words, on the relative length of its axes, and it increases as the ellipsoid becomes "deep" (elongated in the Z-direction). We show that the errors are usually small, and that, in general, a small number of models is required to predict the appearance of an ellipsoid from all possible views.

We start with a brief explanation of the error function used. Consider an ellipsoid rotating about some axis V in the image plane. Let $p_1 = (x_1, y_1)$ be the projected location of some rim point. Following rotation, the rim changes, and the point p_1 is replaced by a new point, $p_2 = (x_2, y_2)$, such that the vector $p_2 - p_1$ is perpendicular to V. Denote the approximated location of p_2 according to the curvature method by $\hat{p}_2 = (\hat{x}_2, \hat{y}_2)$. The observed error is measured by $\| \hat{p}_2 - p_2 \|$. Clearly, if we scale the ellipsoid the observed error would scale as well. We therefore need instead to consider a relative value for the error that is independent of scale.

We define the error as follows. Consider the planar section through p_1 that is perpendicular to the rotation axis V. This section forms an ellipse (or a single point in case

of a tangential plane). Let $p_0 = (x_0, y_0)$ be the center of this ellipse. The relative error is defined by

$$E = \frac{\| \hat{p}_2 - p_2 \|}{\| p_1 - p_0 \|}$$

E reflects the observed error relative to the projected size of the ellipsoid. Notice that E is independent of translation and scale of the ellipsoid.

3.1 Rotation Around the Vertical Axis

Let

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

be the surface of a canonical ellipsoid. Let $p_1 = (x_1, y_1)$ be a point on its silhouette. When the ellipsoid rotates about the vertical (Y) axis by an angle θ, p_1 disappears and is replaced by a new contour point $p_2 = (x_2, y_2)$ with an identical y-value, $y_2 = y_1$. Let $\hat{p}_2 = (\hat{x}_2, \hat{y}_2)$ be the approximated position of p_2 according to the curvature method. The horizontal section of the ellipsoid through p_1 is an ellipse centered around $p_0 = (0, y_0)$. Notice that the points p_1, p_2, $\hat{p}_2$, and p_0 all lie on the same horizontal section implying that $y_1 = y_2 = \hat{y}_2 = y_0$. The relative error is therefore reduced to

$$E = \left| \frac{\hat{x}_2 - x_2}{x_1} \right|$$

(For reasons of convenience we ignore the absolute value operation in the discussion below.)

Proposition 1: The error is given by

$$E(\frac{c^2}{a^2}, \theta) = \cos\theta + \frac{c^2}{a^2}(1 - \cos\theta) - \sqrt{\cos^2\theta + \frac{c^2}{a^2}\sin^2\theta} \qquad (2)$$

(A proof is given in Appendix A.)

The expression obtained for the error depends on two parameters the aspect ratio of the ellipsoid, $\frac{c^2}{a^2}$, and the angle of rotation, θ. Consequently, the error is invariant under a uniform scaling of the ellipsoid.

3.2 Properties of the Error

The prediction error obtained by the curvature method for a canonical ellipsoid rotating around the Y-axis vanishes in the following three cases

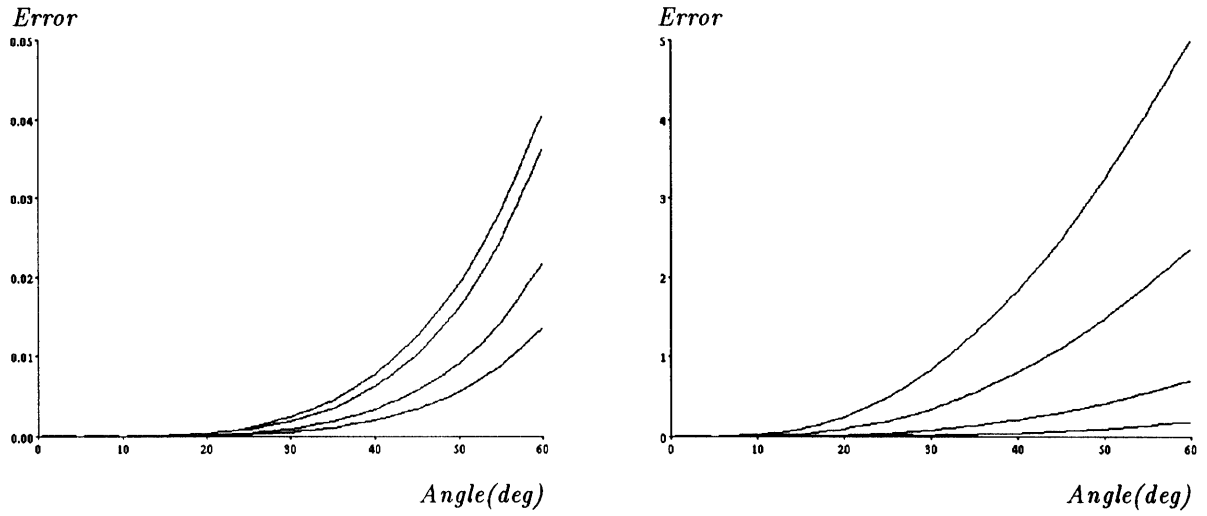

Figure 3: The errors of the curvature method as a function of θ, the angle of rotation. (a) $\frac{c^2}{a^2} = \frac{1}{16}$, $\frac{1}{9}$, $\frac{1}{4}$ and $\frac{1}{2}$. (b) $\frac{c^2}{a^2} = 2$, 4, 9 and 16. (The parameters correspond the curves at increasing heights.)

- $\theta = 0$ (that is, no rotation).
- $\frac{c^2}{a^2} = 1$ (that is, $c = a$, the cross section is a circle).
- $\frac{c^2}{a^2} = 0$ (that is, $c = 0$, a planar ellipsoid).

As a function of θ, the angle of rotation, the error function is symmetric, that is, similar errors are obtained both for positive and negative angles. The absolute value of the error increases monotonically with the absolute value of θ. The partial derivative E_θ also changes monotically with θ, so the error increases faster for larger values of θ. The derivative is given by

$$E_\theta = (1 - \frac{c^2}{a^2}) \sin\theta (\frac{\cos\theta}{\sqrt{\cos^2\theta + \frac{c^2}{a^2}\sin^2\theta}} - 1) \tag{3}$$

and assumes the following values

- $E_\theta(0°) = 0$.
- $E_\theta(90°) = \frac{c^2}{a^2} - 1$.

Figure 3 shows the error as a function of θ for several ellipses.

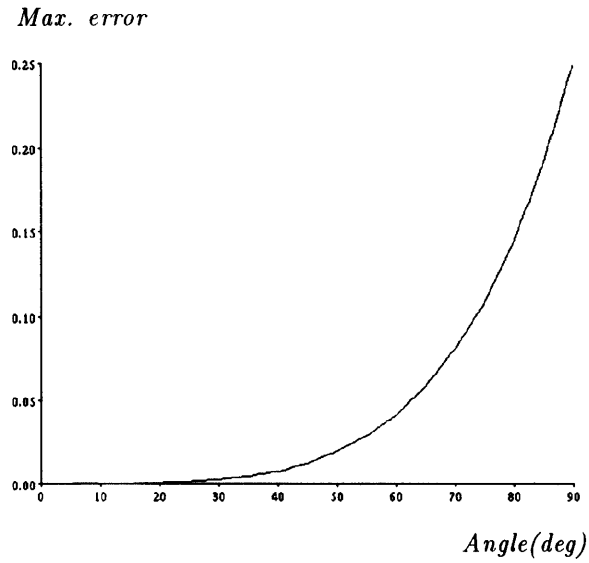

Figure 4: The maximal value of the error for canonical ellipsoids with $c \leq a$ as a function of θ, the angle of rotation.

As a function of $\frac{c^2}{a^2}$, the relative size of the axes of the ellipsoid, the error behaves differently in each of the two ranges: (1) when $c \leq a$, and (2) when $c > a$. In the first case the ellipsoid's width is larger than its depth. The error assumes small values even for fairly large values of θ. The maximal error is obtained when

$$\frac{c^2}{a^2} = \frac{3}{4} - \frac{1}{2(1+\cos\theta)} \tag{4}$$

and it assumes the following values

- 0.24% at 30° ($\frac{c^2}{a^2} = 0.482$).
- 1.26% at 45° ($\frac{c^2}{a^2} = 0.457$).
- 4.14% at 60° ($\frac{c^2}{a^2} = 0.417$).

Figure 4 shows the maximal error as a function of θ.

When $c > a$, the ellipsoid is deeper than it is wide, the error assumes larger values and is unbounded when θ increases to 90°. The partial derivative $E_{\frac{c^2}{a^2}}$ increases monotonically with $\frac{c^2}{a^2}$ and reaches its maximum when $\frac{c^2}{a^2} \to \infty$, where the error increases linearly in

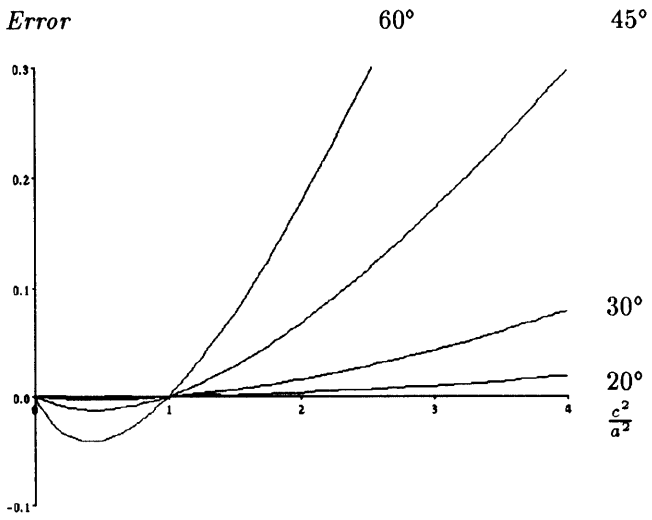

Figure 5: The error as a function of the aspect ratio, $\frac{c^2}{a^2}$, of the ellipsoid for $\theta = 20°$, $30°$, $45°$, and $60°$. (The parameters correspond the curves at increasing magnitude.)

$\frac{c^2}{a^2}$. The derivative is given by

$$E_{\frac{c^2}{a^2}} = (1 - \cos\theta) - \frac{\sin^2\theta}{2\sqrt{\cos^2\theta + \frac{c^2}{a^2}\sin^2\theta}} \tag{5}$$

and assumes the following values

- $E_{\frac{c^2}{a^2}}(0) = -\frac{(1-\cos\theta)^2}{2\cos\theta}$.
- $E_{\frac{c^2}{a^2}}(1) = \frac{(1-\cos\theta)^2}{2}$.
- $\lim_{\frac{c^2}{a^2}\to\infty} E_{\frac{c^2}{a^2}} = 1 - \cos\theta$.

A model for such an ellipsoid would therefore cover only a restricted range of rotations. Larger rotations should be treated by additional models. Figure 5 shows the error as a function of $\frac{c^2}{a^2}$ for several values of θ.

When a complete set of models is prepared for the appearance of an ellipsoid to be predictable from all possible views, it should be considered that following a rotation of $90°$ about the Y-axis, a and c, the axes lengths of the ellipsoid, interchange their roles. Therefore, an ellipsoid with $c < a$ changes after a rotation of $90°$ to an ellipsoid with $c > a$. An ellipsoid with a high aspect ratio, $\frac{c^2}{a^2}$, changes to an ellipsoid with a low aspect

$\frac{c^2}{a^2}$	1%	2%	3%	4%	5%	6%
2	3	3	2	2	2	2
4	3	3	3	3	2	2
9	4	3	3	3	3	2
16	4	3	3	3	3	2
49	4	3	3	3	2	2
100	4	3	3	2	2	2

Table 1: Number of models as a function of allowed error.

ratio. Consequently, the small range of rotations covered by a model for an ellipsoid with a high aspect ratio is compensated by the large range of rotations covered by a model for the same ellipsoid after a 90° rotation. A small number of models is therefore required to represent the ellipsoid from all possible views.

Table 1 shows the number of models required to cover the entire range of rotations about the Y-axis for several ellipsoids. Because of symmetry considerations only rotations up to 90° should be considered. We see from the table that this number is small and does not exceed four even for extreme aspect ratios and allowed error of 1%.

In preparing this table each ellipsoid was initially represented by two models, one taken at its canonical position, the other following a 90° rotation. If the two models did not cover the entire range of rotations, additional models were added at intermediate positions. In this case the value of the error is somewhat different then the canonical case. An expression describing this value is given in Appendix A.

3.3 Rotation in 3-D Space

We now consider the case of a canonical ellipsoid rotating arbitrarily in 3-D space. A rotation in 3-D space can be decomposed into three successive rotations, about the Z-, Y-, and Z-axes. The last rotation can be ignored since it does not deform the image and therefore does not change the errors. Let

$$\frac{(x\cos\alpha + y\sin\alpha)^2}{a^2} + \frac{(-x\sin\alpha + y\cos\alpha)^2}{b^2} + \frac{z^2}{c^2} = 1$$

be the surface of a canonical ellipsoid rotated about the Z-axis by an angle α. We now examine this ellipsoid as it rotates about the Y-axis by an angle θ.

Proposition 2: The error is given by

$$E(\frac{C^2}{A^2}, \theta)$$

in Eq. (2), where

$$\frac{C^2}{A^2} = \frac{c^2}{a^2}\cos^2\alpha + \frac{c^2}{b^2}\sin^2\alpha \tag{6}$$

(A proof is given in Appendix A.) Notice that, depending on α, $\frac{C^2}{A^2}$ assumes any value between $\frac{c^2}{a^2}$ and $\frac{c^2}{b^2}$.

The consequence of Proposition 4 is that after an arbitrary rotation the appearance of an ellipsoid as it is approximated by the curvature method is in general neither better nor worse than its approximated appearance after a rotation about any of the main axes. As a result, if k models are required to cover all rotations around the X-axis, and k (or less) models cover all rotations around the Y-axis, then at most k^2 models are required to cover all possible rotations in 3-D space.

4 Summary

The curvature method was designed to predict the appearance of objects with smooth bounding surfaces from different viewpoints. In this paper we have applied the method to ellipsoidal objects rotating arbitrarily in 3-D space and derived an expression describing the errors obtained. we have analyzed these errors and concluded that they depend on the shape of the ellipsoids, in other words, on the relative length of their axes, The error increases as the ellipsoid becomes "deep" (elongated in the Z-direction). We have shown that the errors are usually small, and that, in general, a small number of models is required to predict the appearance of an ellipsoid from all possible views.

Finally, we would like to add that the curvature method discussed above is not restricted to contours originating from elliptic surface patches. It can as equally handle contours originating from hyperbolic patches - as long as the patches are visible. When, however, a patch is self occluded, a new aspect of the object is observed, and an additional model should be utilized. The treatment of hyperbolic patches is demonstrated in Figure 6. Models of three tori with different radii were prepared analytically. The models were matched to an image that contained the tori in various positions and orientations using the curvature method. It can be seen that although the points of the inner circles of the tori come from hyperbolic patches, their prediction is still accurate.

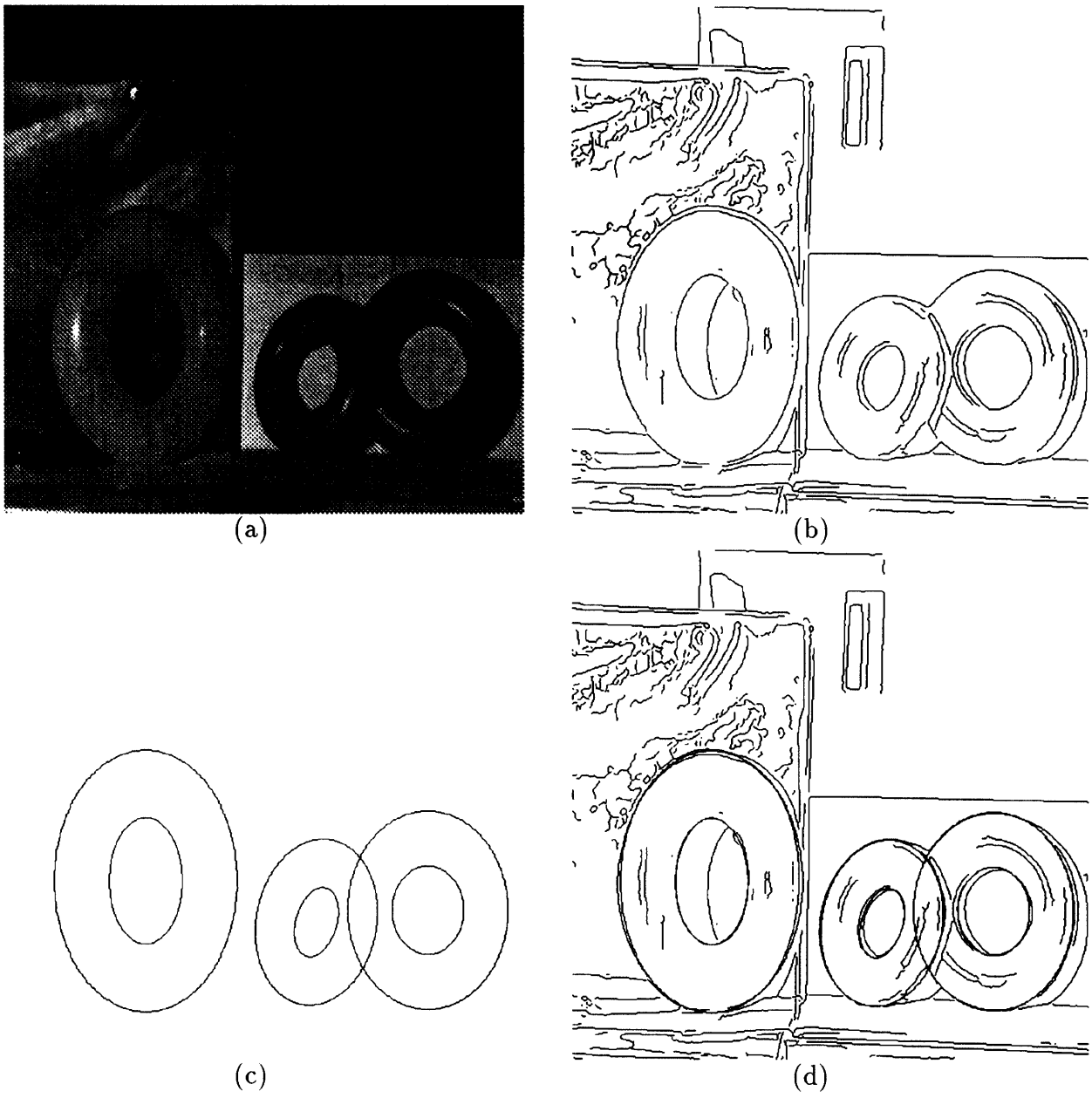

Figure 6: (a) A picture of three tori. (b) A contour image of the tori. (c) A prediction of the appearance of the three tori. (b) Matching the prediction to the actual image.

Appendix A

In this appendix we derive an expression of the error obtained when the curvature method is applied to a canonical ellipsoid rotating about the vertical axis. We then show that a similar error is obtained when the ellipsoid is rotating about any axis in space. Finally, we compute the error resulting from applying the curvature method to a non-canonical ellipsoid.

A.1 Rotation about the Vertical Axis

Let

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

be the surface of a canonical ellipsoid. Let $p_1 = (x_1, y_1)$ be a point on its silhouette. Assume the ellipsoid is rotating about the vertical (Y) axis by an angle θ. Let $p_2 = (x_2, y_2)$ be the appeared position of p_1 following the rotation, and let $\hat{p}_2 = (\hat{x}_2, \hat{y}_2)$ be the approximated position of p_2 according to the curvature method. The relative error for the case of an ellipsoid that is rotating about the Y-axis is given by

$$E = \frac{\hat{x}_2 - x_2}{x_1}$$

Proposition 3: The error is given by

$$E(\frac{c^2}{a^2}, \theta) = \cos\theta + \frac{c^2}{a^2}(1 - \cos\theta) - \sqrt{\cos^2\theta + \frac{c^2}{a^2}\sin^2\theta}$$

Proof: The rim of a canonical ellipsoid contains the surface points for which $z = 0$. Therefore, the silhouette is defined by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

After the ellipsoid is rotated by an angle θ around the Y-axis, it is described by

$$\frac{(x\cos\theta - z\sin\theta)^2}{a^2} + \frac{y^2}{b^2} + \frac{(x\sin\theta + z\cos\theta)^2}{c^2} = 1$$

And its silhouette is given by

$$\frac{x^2}{a^2\cos^2\theta + c^2\sin^2\theta} + \frac{y^2}{b^2} = 1$$

The position of $p_2 = (x_2, y_2)$ is therefore

$$\begin{aligned} x_2 &= \frac{x_1}{a}\sqrt{a^2\cos^2\theta + c^2\sin^2\theta} \\ y_2 &= y_1 \end{aligned}$$

Next we calculate $\hat{p}_2$. Denote the surface of the canonical ellipsoid by the form $F(x, y, z) = 1$. According to Basri & Ullman [1988] the curvature radius with respect to the Y-axis is given by

$$r_x = -\frac{F_x}{F_{zz}} = -\frac{c^2 x}{a^2}$$

When the ellipsoid rotates around the Y-axis by an angle θ, the position of p_2 is estimated by the curvature method to be

$$\begin{aligned} \hat{x}_2 &= x_1\cos\theta - \frac{c^2 x_1}{a^2}(1 - \cos\theta) \\ \hat{y}_2 &= y_1 \end{aligned}$$

Consequently, the relative error is given by

$$E(\frac{c^2}{a^2}, \theta) = \cos\theta + \frac{c^2}{a^2}(1 - \cos\theta) - \sqrt{\cos^2\theta + \frac{c^2}{a^2}\sin^2\theta}$$

The error is therefore a function of θ and $\frac{c^2}{a^2}$.

A.2 Rotation in 3-D Space

In this section we consider the case of a canonical ellipsoid rotating arbitrarily in 3-D space. A rotation in 3-D space can be decomposed into three successive rotations, about the Z-, Y- and Z-axes. The last rotation can be ignored since it does not deform the image and therefore does not change the errors. (The first rotation cannot be ignored since it determines the actual axis of the second rotation.) Let

$$\frac{(x\cos\alpha + y\sin\alpha)^2}{a^2} + \frac{(-x\sin\alpha + y\cos\alpha)^2}{b^2} + \frac{z^2}{c^2} = 1$$

be the surface of a canonical ellipsoid rotated about the Z-axis by an angle α. We now examine this ellipsoid as it rotates about the Y-axis by an angle θ.

Proposition 4: The error is given by

$$E(\frac{C^2}{A^2}, \theta)$$

where

$$\frac{C^2}{A^2} = \frac{c^2}{a^2}\cos^2\alpha + \frac{c^2}{b^2}\sin^2\alpha$$

Proof: In order to prove this proposition we have to show that every horizontal section of the ellipsoid defined above is an ellipse with an aspect ratio $\frac{C^2}{A^2}$ as given in the proposition.

Any nonempty intersection of an ellipsoid and a plane is either a point or an ellipse. The section is nonempty when $y^2 \leq a^2\sin^2\alpha + b^2\cos^2\alpha$, and is a point when a strict equality holds. Given the canonical ellipsoid following its rotation about the Z-axis by an angle α, we show that the boundaries of its horizontal section can be represented as

$$\frac{(x - x_0)^2}{A^2} + \frac{z^2}{C^2} = 1$$

which describes a canonical ellipse displaced along the X-axis. To establish the above relation, we show that for a constant value of y the surface equation of the rotated ellipsoid reduces to equation of the displaced ellipse. The two equations are identical if there exists a constant $k \neq 0$ such that the following equation system holds

$$\begin{aligned}
kC^2 &= \frac{\cos^2\alpha}{a^2} + \frac{\sin^2\alpha}{b^2} \\
kC^2x_0 &= y\sin\alpha\cos\alpha(\frac{1}{b^2} - \frac{1}{a^2}) \\
kA^2 &= \frac{1}{c^2} \\
kC^2(A^2 - x_0^2) &= 1 - y^2(\frac{\sin^2\alpha}{a^2} + \frac{\cos^2\alpha}{b^2})
\end{aligned}$$

We obtain a system of four equations in four unknowns, A^2, C^2, x_0 and k. We now show that when $y^2 < a^2\sin^2\alpha + b^2\cos^2\alpha$ this system has a unique solution with positive values for A^2 and C^2.

Denote the right side of the four equations by

$$\begin{aligned}
p &= \frac{\cos^2\alpha}{a^2} + \frac{\sin^2\alpha}{b^2} \\
q &= y\sin\alpha\cos\alpha(\frac{1}{b^2} - \frac{1}{a^2}) \\
r &= \frac{1}{c^2} \\
s &= 1 - y^2(\frac{\sin^2\alpha}{a^2} + \frac{\cos^2\alpha}{b^2})
\end{aligned}$$

The solution to the system above is given by

$$\begin{aligned} x_0 &= \frac{kC^2x_0}{kC^2} = \frac{q}{s} \\ A^2 &= \frac{kC^2(A^2 - x_0^2) + kC^2x_0^2}{kC^2} = \frac{ps + q^2}{p^2} \\ C^2 &= \frac{kC^2(A^2 - x_0^2) + kC^2x_0^2}{kA^2} = \frac{ps + q^2}{pr} \\ k &= \frac{kA^2}{A^2} = \frac{p^2r}{ps + q^2} \end{aligned}$$

Notice that $p, r > 0$. This system therefore has a unique solution with positive values for A^2 and C^2 when $ps + q^2 > 0$. This inequality is satisfied when $y^2 < a^2 \sin^2\alpha + b^2 \cos^2\alpha$.

Now, we can compute the value of the ratio $\frac{C^2}{A^2}$ from this equation system by dividing the first equation by the third one

$$\frac{C^2}{A^2} = \frac{c^2}{a^2}\cos^2\alpha + \frac{c^2}{b^2}\sin^2\alpha$$

Therefore, any horizontal section of this ellipsoid is an ellipse with an aspect ratio of $\frac{C^2}{A^2}$, and since translation does not affect the results of the curvature method, the error is given by

$$E(\frac{C^2}{A^2}, \theta)$$

Where A^2 and C^2 are the parameters of the ellipse, and θ is the rotation angle around the Y-axis.

A.3 Intermediate Models

In this section we derive an expression of the error obtained when the curvature method is applied to an ellipsoid that is rotated about the Y-axis (rather than a canonical ellipsoid). This computation is required for constructing Table 1 in Section 3.2.

Let

$$\frac{(x\cos\alpha - z\sin\alpha)^2}{a^2} + \frac{y^2}{b^2} + \frac{(x\sin\alpha + z\cos\alpha)^2}{c^2} = 1$$

be the surface of a canonical ellipsoid rotated about the Y-axis by an angle α. Assume this ellipsoid is modeled by the curvature method. We consider now the error produced by using this model as the ellipsoid rotates about the Y-axis by an angle θ.

Proposition 5: The relative error is given by

$$E_\alpha(\frac{c^2}{a^2}, \theta) = \cos\theta + z' \sin\theta + r'(1 - \cos\theta) - x''$$

where

$$\begin{aligned} z' &= -\frac{\sin\alpha\cos\alpha(1 - \frac{c^2}{a^2})}{\cos^2\alpha + \frac{c^2}{a^2}\sin^2\alpha} \\ r' &= \frac{\frac{c^2}{a^2}}{(\cos^2\alpha + \frac{c^2}{a^2}\sin^2\alpha)^2} \\ x'' &= \sqrt{\frac{\cos^2(\alpha+\theta) + \frac{c^2}{a^2}\sin^2(\alpha+\theta)}{\cos^2\alpha + \frac{c^2}{a^2}\sin^2\alpha}} \end{aligned}$$

Proof: Let $p_1 = (x_1, y_1)$ be a point on the silhouette of the ellipsoid. Let z_1 be its depth value, and let r_1 be its curvature value with respect to the Y-axis. Then

$$\begin{aligned} x_1 &= \frac{x}{a}\sqrt{a^2\cos^2\alpha + c^2\sin^2\alpha} \\ y_1 &= y \\ z_1 &= \frac{-x\sin\alpha\cos\alpha(a^2 - c^2)}{a\sqrt{a^2\cos^2\alpha + c^2\sin^2\alpha}} \\ r_1 &= \frac{xac^2}{(a^2\cos^2\alpha + c^2\sin^2\alpha)^{\frac{3}{2}}} \end{aligned}$$

where $p = (x, y)$ is the corresponding point on the silhouette of the ellipsoid in its canonical position.

Let $p_2 = (x_2, y_2)$ be the appeared position of p_1 after a rotation around the Y-axis by an angle θ, p_2 is given by

$$\begin{aligned} x_2 &= \frac{x}{a}\sqrt{a^2\cos^2(\alpha+\theta) + c^2\sin^2(\alpha+\theta)} \\ y_2 &= y \end{aligned}$$

Let $\hat{p}_2 = (\hat{x}_2, \hat{y}_2)$ be the position of p_2 approximated by the curvature method

$$\begin{aligned} \hat{x}_2 &= x_1\cos\theta + z_1\sin\theta + r_1(1 - \cos\theta) \\ \hat{y}_2 &= y \end{aligned}$$

Since $y_1 = y_2 = \hat{y}_2$, the error is defined by

$$E_\alpha = \frac{\hat{x}_2 - x_2}{x_1}$$

Let

$$\begin{aligned} z' &= \frac{z_1}{x_1} \\ r' &= \frac{r_1}{x_1} \\ x'' &= \frac{x_2}{x_1} \end{aligned}$$

and we obtain the expressions given in the proposition. Notice that

$$E_0(\frac{c^2}{a^2}, \theta) = E(\frac{c^2}{a^2}, \theta)$$

in Eq. (2).

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References

Baker, H., 1977. Three-dimensional modeling. *Proc. 5th Int. Joint Conf. on Artificial Intelligence (Cambridge, Mass., Aug 22-25)*, IJCAI, 649-655.

Basri R. & Ullman S., 1988. The alignment of objects with smooth surfaces. *Proc. of 2nd Int. Conf. of Computer Vision, Florida, Dec.*, 482-488.

Brady, M., Ponce., J., Yuille, A. & Asada, H., 1985. Describing surfaces. *Computer Vision, Graphics, and Image Processing, 32*, 1-28.

Brown, C.M., 1981. Some mathematical and representational aspects of solid modeling. *IEEE Transactions on Pattern Analysis & Machine Intelligence*, Vol. PAMI-3, No. 4, July.

Chien, C.H. & Aggarwal, J.K., 1987. Shape recognition from single silhouette. *Proc. of ICCV Conf. (London)* 481-490.

Dane, C. & Bajcsy, R., 1982. An object-centered three-dimensional model builder. *Proc. 6th Int. Conf. Pat. Recog., Munich, West Germany, Oct 19-22*, 348-350.

Faugeras, O.D. & Hebert, M., 1986. The representation, recognition and location of 3-D objects. *Int. J. Robotics Research* 5(3), 27-52.

Fischler, M.A. & Bolles, R.C., 1981. Random sample consensus: a paradigm for model fitting with application to image analysis and automated cartography. *Communications of the ACM* 24(6), 381-395.

Huttenlocher, D.P. & Ullman, S., 1987. Object recognition using alignment. *Proc. of ICCV Conf. (London)* 102-111.

Koenderink, J.J., & Van Doorn, A.J., 1979. The internal representation of solid shape with respect to vision. *Biol. Cybernetics 32*, 211-216.

Koenderink, J.J., & Van Doorn, A.J., 1981. The shape of smooth objects and the way contours end. *Perception 11*, 129-137.

Lamdan, Y., Schwartz, J.T., & Wolfson, H., 1987. On recognition of 3-D objects from 2-D images. *Courant Institute of Mathematical Sciences, Robotics Technical Report 122.*

Lowe, D.G., 1985. Three-dimensional object recognition from single two-dimensional images. *Robotics research Technical Report 202, Courant Institute of Math. Sciences, N.Y. University.*

Marr, D., 1977. Analysis of occluding contour. *Phil. Trans. R. Soc. Lond. B 275*, 483-524.

Marr, D. & Nishihara, H.K., 1978. Representation and recognition of the spatial organization of three dimensional shapes. *Proc. Roy. Soc. B 200*, 269-291.

Potmesil, M., 1983. Generating models of solid objects by matching 3-D surface segments. *Proc. 8th Int. Joint Conf. Art. Intell. (Karlsruhe, West Germany, Aug 8-12)*, 1089-1093.

Thompson, D.W. & Mundy J.L., 1987. Three dimensional model matching from an unconstrained viewpoint. *Proc. IEEE Int. Conf. on robotics and Automation,* Raleigh, N.C., 208-220.

Ullman S., 1989. Aligning pictorial descriptions: an approach to object recognition. *Cognition, Vol. 32, No. 3*, 193-254. Also: 1986, *A.I. Memo 931, The Artificial Intelligence Lab., M.I.T.*.

Ullman, S. & Basri, R., 1991. Recognition by linear combinations of models. *IEEE Trans. on Pattern Analysis and Machine Intelligence, Vol. 13, No. 10*, 992-1006.