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ON TREES AND LOGS

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On Trees and Logs*

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On Trees and Logs

Abstract

In this paper we critically examine the main workhorse model in asset pricing theory, the Lucas (1978) tree model (LT-Model), extended to include heterogeneous agents and multiple goods, and contrast it to the benchmark model in financial equilibrium theory, the real assets model (RA-Model). Households in the LT-Model trade goods together with claims to Lucas trees (exogenous stochastic dividend streams specified in terms of a particular good) and long-lived, zero-net-supply real bonds, and are endowed with share portfolios. The RA-Model is quite similar to the LT-Model except that the only claims traded there are zero-net-supply assets paying out in terms of commodity bundles (real assets) and households' endowments are in terms of commodity bundles as well. At the outset, one would expect the two models to deliver similar implications since the LT-Model can be transformed into a special case of the RA-Model. We demonstrate that this is simply not correct: results obtained in the context of the LT-Model can be strikingly different from those in the RA-Model. Indeed, specializing households' preferences to be additively separable (over time) as well as log-linear, we show that for a large set of initial portfolios the LT-Model – even with potentially complete financial markets – admits a peculiar financial equilibrium (PFE) in which there is no trade on the bond market after the initial period, while the stock market is completely degenerate, in the sense that all stocks offer exactly the same investment opportunity – and yet, allocation is Pareto optimal. We then thoroughly investigate why the LT-Model is so much at variance with the RA-Model, and also completely characterize the properties of the set of PFE, which turn out to be the only kind of equilibria occurring in this model. We also find that when a PFE exists, either (i) it is unique, or (ii) there is a continuum of equilibria: in fact, every Pareto optimal allocation is supported as a PFE. Finally, we show that most of our results continue to hold true in the presence of various types of restrictions on transactions in financial markets. Portfolio constraints however may give rise other types of equilibria, in addition to PFE. While our analysis is carried out in the framework of the traditional two-period Arrow-Debreu-McKenzie pure exchange model with uncertainty (encompassing, in particular, many types of contingent commodities), we show that most of our results hold for the analogous continuous-time martingale model of asset pricing.

JEL Classifications: D50, G00, G12

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1. Introduction

One of the most commonly employed models in asset pricing theory is the Lucas [15] asset-market tree economy. Investment opportunities in this economy are represented by claims to exogenously specified stochastic dividend streams paid out by firms (Lucas trees) and long-lived real bonds. Households trade in goods and shares of trees or, as we will call them, stocks and bonds so as to maximize their expected lifetime utility defined over intertemporal consumption. Initial endowments of the households are in terms of portfolios of shares of stocks and bonds. By imposing clearing in spot goods and asset markets, one obtains an environment for determining equilibrium asset prices.

In this paper we critically examine the Lucas tree model (LT-Model) extended to include heterogeneous agents and multiple goods. Dividend streams of the trees are specified in terms of a particular good; different trees pay out in different goods. (In fact, our results are easily extended to cover trees which pay out in a variety of goods, as described at the end of Section 2 below.) We weigh equilibrium implications of the LT-Model against those of the benchmark real assets model (RA-Model) in financial equilibrium theory, in which (i) there is no production (and therefore there are no firms); (ii) households diversify risk by trading IOU's whose promised returns are specified in terms of commodity bundles (real assets); and (iii) initial endowments are also commodity bundles. At the outset, one would expect the two models to deliver similar implications since the LT-Model can be transformed into a special case of the RA-Model. Consequently, the wide array of equilibrium results developed in the context of the RA-Model should then readily apply to the LT-Model. It turns out that this is simply not correct: the LT-Model has certain embedded structure that makes it significantly different from the RA-Model, and part of our goal is to highlight this structure and the implications it may lead to.

In particular, specializing households' preferences to be additively separable (over time) as well as *log-linear*, we show that for a large set of initial portfolios the LT-Model − even with potentially complete financial markets − admits a *peculiar financial equilibrium* (PFE) in which all stocks but one are redundant. Put differently, even though returns to the trees − one can think of these as the net output flows of firms involved in production of different commodities − are generally unrelated, goods prices always adjust to make the yields (returns in value terms) from traded claims to the trees perfectly correlated. This result is in sharp contrast to a fundamental implication of the RA-Model (see, in particular, Magill and Shafer [17] for the case of potentially complete financial markets, and Duffie and Shafer [10], as refined by Bottazzi [3], for the case of intrinsically incomplete financial markets): under mild regularity conditions (satisfied in the LT-Model), the matrix of yields on the stocks has full column rank generically in initial endowments (i.e., except for a closed, measure-zero subset of endowments). Furthermore, while in the real asset economy households typically must trade in all assets to achieve equilibrium, in our Lucas tree economy trading in bonds only occurs at the initial date, and the desired objective from trading in stocks can always be achieved by means of a single, fixed portfolio of stocks (for example, consisting of just a single stock).

It then follows that since there are necessarily fewer non-redundant assets in equilibrium than there are states of the world, financial markets are always incomplete. In the RA-Model, when financial markets are incomplete, for given household preferences and asset returns, but for a generic subset of initial endowments, equilibrium allocations are never Pareto optimal (as can be argued, for example, along the lines of Geanakoplos, Magill, Quinzii and Drèze [11]). Strikingly, in the LT-Model, PFE allocations are always Pareto optimal. Also, for a large subset of initial endowments, this peculiar financial equilibrium in our model exists in general, while existence is only generic in the RA-Model (again see Magill-Shafer and Duffie-Shafer, as refined by Bottazzi).

The very peculiar characteristics of equilibria in our economy bring to the fore an important structural difference between the LT- and RA-Models. One of the key features driving our puzzling implications is the specification of endowments. While in the LT-Model, endowments are specified in terms of *shares* of stocks and bonds, in the RA-Model endowments are specified in terms of *commodities*. If in addition to portfolios of shares, households in our model were endowed with bundles of commodities, equilibrium would typically no longer be of the peculiar kind.

It is not unrealistic, however, to have endowments specified in terms of shares of assets. And, in fact, this specification may lead to a number of new results in equilibrium theory. In particular, equilibrium theorists have usually assumed that endowments are *nonnegative*. And while a non-negativity assumption is certainly very defensible in a model with commodity endowments, there is nothing contradictory in dropping this assumption in a model with share endowments, especially when there are no restrictions on asset trade. In our model we allow for short initial positions in some assets. Our log-linear utility specification best highlights one of the implications of this additional degree of freedom. It is a standard result in microeconomics that in a pure-distribution economy with (nonnegative) commodity endowments and log-linear utility, competitive equilibrium is always unique. In contrast, in our model we can find share endowments for which this is no longer true. In fact, we can show that there may even be a *continuum of equilibria*, all of the peculiar type, and supporting all of the Pareto set. The subset of initial portfolios for which this can occur is of a smaller dimension than the space of all initial portfolios, so getting a continuum of equilibria is atypical, but it is nonetheless a distinct theoretical possibility. We fully characterize the errant subset. The proposition about nonuniqueness does not require that there be multiple goods in the economy, it encompasses the one-good case as well.

We then explore the robustness of our results. In particular, we investigate whether the peculiar financial equilibria that we exhibit survive various types of restrictions on transactions in financial markets. We find that for a large class of portfolio constraints, our implications are robust. For example, if households are unconstrained in their bond trades and unconstrained in trading at least one stock (but face arbitrary portfolio constraints on the remaining stocks), the PFE still occur. This is because, in contrast to well-explored single good models with portfolio constraints, there are other markets which are open in addition to asset markets: spot goods markets. So, it is possible to replicate the unconstrained equilibrium allocation by trading in one stock, bonds and goods, thus fully circumventing portfolio constraints.

Finally, we investigate whether there are other (or ordinary) financial equilibria (OFE), apart from the peculiar ones, in our model. At first blush it appears as if this problem should be very similar to the problem of establishing the possibility of sunspot equilibria as in Cass and Shell [6]. Indeed, a natural transformation of the units of the quantities of goods in our model reveals that it is essentially deterministic in the sense that (in the transformed units) the aggregate endowment of each commodity in each date-event is always unity. Then, all PFE in the transformed economy can be identified with the nonsunspot equilibria of Cass-Shell, and the remaining equilibria − OFE − with their sunspot equilibria. It turns out, however, that this suggestive parallel is illusory: the only FE are PFE, whereas, in contrast, in the leading example of the benchmark model of incomplete financial markets (but with asset returns specified in value, i.e., nominal rather than real terms) there is typically a continuum of sunspot equilibria (Cass [4]). On the other hand, even in the presence of portfolio constraints under which (some of) the PFE survive, there may or may not also be OFE: this depends on the nature and scope of the constraints, as we illustrate by example.

Most of the above results have their analogues in continuous time. There, equilibria in the model are peculiar in the sense that, for arbitrary stochastic processes representing dividends paid by the trees, the volatility matrix of securities in the investment opportunity set of the agents is always degenerate. Continuous time offers additional tractability over the original two-period model: we are able to parameterize stochastic processes for the state prices and stochastic weighting for a representative agent in the economy. We feel that this extension may be particularly useful for a further investigation of the effects of portfolio constraints on asset prices and goods allocations in our model.

Closely related to our work is the analysis by Zapatero [19], who uncovers a financial equilibrium of the peculiar variety in the context of a two-country two-good model of asset prices and exchange rates. In fact, it was Zapatero's results which led us to thinking about our trees and logs model. In the same vein is the earlier work of Cole and Obstfeld [7] , who also document occurrence of something like a PFE in an equilibrium international model. Also related is the strand of literature investigating the special structure of preferences belonging to the linear risk tolerance class (see Magill and Quinzii [16], Chapter 3, and the references contained therein). In the context of a one-good model, it has been shown that "effective" market completeness, and hence Pareto optimality obtains in an incomplete financial market when households' preferences display linear risk tolerance with the same coefficient of marginal risk tolerance.

The remainder of the paper is organized as follows. Section 2 describes the economy. Section 3 characterizes the set of equilibria and investigates its properties. Section 4 contains an extension in continuous time. Section 5 outlines a major avenue for future research, while two Appendices contain all proofs.

2. The Economic Environment

Most of our basic framework is very standard in the Finance literature. There are two periods, today and tomorrow, labeled (when useful) $t = 0, 1$ $(= T)$. Uncertainty tomorrow is represented by future states of the world, labeled $\omega = 1, 2, \ldots, \Omega < \infty$, so that it is also natural to represent today as the present state of the world, labeled $\omega = 0$. In our only major departure from the common conventions in asset pricing theory (but the common convention in financial equilibrium theory), we assume here that there are many goods in each state, labeled $g = 1, 2, \ldots, G < \infty$.

Production is described by exogenous stochastic streams of output of each type of good, $\delta^g(\omega)$, all g, all ω, what in Finance have traditionally been viewed as dividend streams from stocks, but more recently as real returns from Lucas trees. The main difference here is that our trees or – as we will usually refer to them – *stocks* correspond one-to-one with the goods, and are accordingly also labeled $g = 1, 2, \ldots, G$. Quantities of stocks are denoted s^{tg}, all g, all t, and are by definition each in initial positive net supply of one unit.

Stocks are the sole source of goods in the economy, as well as one type of investment opportunity. The only other type of investment opportunity is long-lived real *bonds*,[1] each of whose promised returns is also specified in terms of a single good, by definition one unit of that good in each state. The bonds are labeled $\tilde{g} = 1, 2, \ldots, \tilde{G} \leq G$ – where returns from bond $\tilde{g}$ are specified in units of good $g = \tilde{g}$ – and are in zero net supply. Their quantities are denoted $b^{t\tilde{g}}$, all $\tilde{g}$, all

[1] Our particular specification of the alternative available investments to stocks is chosen primarily for expositional convenience. In fact, our results generalize immediately to any real (Economics) or derivative (Finance) assets – as long as they are in zero net supply.

t. Even though the returns on bonds are nonstochastic (and specified equal one unit of particular goods), later on it will be useful to denote them by the abstract notation $\delta^{\tilde{g}}(\omega)$, all $\tilde{g}$, all ω.

Households are the consumer-investors in this economy, and are labeled $h = 1, 2, \ldots, H < \infty$. Each household is endowed with an initial portfolio of assets (b_h^0, s_h^0), and trades on a spot market for goods and assets at spot 0, and then again, after the future state of the world $\omega > 0$ has been realized, on a spot market for goods at spot ω. Short sale of stocks (as well as borrowing) is permitted. Purchase, and therefore also consumption of goods is denoted $c_h^g(\omega)$, all g, all ω, and the terminal portfolio (b_h^1, s_h^1), while spot goods, bond, and stock prices are denoted $p^g(\omega)$, all g, all $\omega, q_b^{\tilde{g}}$, all $\tilde{g}$, and q_s^g, all g, respectively. Both consumption of goods and spot goods prices are always assumed to be strictly positive.

Each household evaluates its actions according to a von Neumann-Morgenstern utility function over present and prospective future consumption

$$u_h(c_h) = \sum_{\omega > 0} \pi(\omega) v_h(c_h(0), c_h(\omega)),$$

where $\pi(\omega) > 0$, $\omega > 0$, with $\sum_{\omega>0} \pi(\omega) = 1$, represent common prior probabilities, and $v_h : \mathbb{R}_{++}^{2G} \to \mathbb{R}$ represents the household's two-period certainty utility function. Expected utility is assumed to satisfy textbook regularity, monotonicity, and convexity assumptions, in particular those (minimally) consistent with additively separable log-linear certainty utility: v_h is C^2, differentiably strictly increasing, and differentiably strictly concave, and satisfies the boundary condition, for every $(c_h^{0\prime}, c_h^{1\prime}) \gg 0$,

$$cl\{(c_h^0, c_h^1) \in \mathbb{R}_{++}^{2G} : v_h(c_h^0, c_h^1) \geq v_h(c_h^{0\prime}, c_h^{1\prime})\} \subset \mathbb{R}_{++}^{2G}.$$

Later on we will specialize to log-linear utility

$$v_h(c_h(0), c_h(\omega)) = \sum_g \alpha_h^{0g} \log c_h^g(0) + \beta_h \sum_g \alpha_h^{1g} \log c_h^g(\omega),$$

so that

$$u_h(c_h) = \sum_g \alpha_h^{0g} \log c_h^g(0) + \sum_{\omega>0} \pi(\omega) \beta_h \sum_g \alpha_h^{1g} \log c_h^g(\omega),$$

with $\alpha_h^{tg} > 0$, all g, and $\sum_g \alpha_h^{tg} = 1$, all t, and $\beta_h > 0$.

Since one of our primary concerns will be with the relationship between equilibrium allocation and Pareto optimality, for the most part we will concentrate on the situation where there are *potentially complete financial markets*, that is, where $G + \tilde{G} = \Omega$ (so that $\Omega \leq 2G$). However, our main results do not depend on this assumption, and are equally true for the case where $G + \tilde{G} < \Omega$, so that there are *intrinsically incomplete financial markets* $-$ as well as, obviously, the case where $G+\tilde{G} > \Omega$, so that there are necessarily redundant assets. Notice that when assets provide, effectively − as they do in this economy − both initial endowments (of goods) and investment opportunities, having "necessarily redundant assets" (in the conventional sense) is not immaterial; such "redundancy" typically enlarges the set of possible initial endowments.

For certain purposes we will also concentrate on what we will refer to as *the leading example*, where $\Omega = 3$, $G = 2$, $\tilde{G} = 1$, and $H = 2$, the smallest dimensional case with more than one good in which a bond is required in order to provide potentially complete financial markets. This is purely for expositional purposes, where there is no especial insight gained by introducing more generality.

Incidentally, it is well worth stressing that the whole of our main results is still valid when, instead of there being trees paying off in terms of distinct goods, described by $\delta^g(\omega) > 0$, all ω, there are actually firms (perhaps *hybrid trees*), labeled $f = 1, 2, \ldots, F < \infty$, paying off in terms of distinct bundles of goods, described, say, by

$$(\delta^g(\omega) y_f^g, \text{ all } g), \text{ all } \omega,$$

provided only that

$$\sum_f y_f^g = r^g > 0, \text{ all } g,$$

(so that total resources are strictly positive) and, without any loss of generality,

$$\text{rank}[y_f^g, \text{ all } f, \text{ all } g] = F \leq G.$$

Under this interpretation, $Y_f = \{(y_f^g, \text{ all } g)\}$ can be viewed as a typical firm's (single point) production set,[2] and $\{\delta^g(\omega), \text{ all } g\}$, $\omega > 0$, as goods-specific multiplicative aggregate uncertainty − covering the standard case of purely multiplicative aggregate uncertainty, where $\delta^g(\omega) = \delta(\omega)$, all g, $\omega > 0$. Then, without a doubt, the yields

$$\sum_g p^g(\omega)\delta^g(\omega) y_f^g, \text{ all } \omega$$

represent a typical firm's dividend stream (in terms, say, of some appropriate units of account).

We will appeal to this generalization during the course of the following analysis; in fact, the sequencing of this development is partly dictated by a desire to encompass this simple but nonetheless important, interesting extension.[3]

[2] Of course, when some good is an input for some firm, additional restrictions on initial portfolios may be required in order to guarantee, for instance, that households have positive initial endowments of at least some commodities.

[3] In fact, aside from issues concerning the assumptions required to guarantee existence of equilibrium, R_f can be

Finally, we again emphasize that − except for the assumption of many goods − this model, including log-linear utility, is a standard workhorse in Finance, even more so when there are intrinsically incomplete financial markets, or institutionally imposed portfolio restrictions.

3. Characterization of Equilibrium

3.1. Preliminaries

3.1.1. Notation

We adopt the obvious convention for forming vectors (and, similarly, matrices) from indexed scalars or vectors: simply suppress the common index, and write the corresponding set of indexed scalars or vectors in their natural order. Thus, for instance,

$$\begin{aligned} &p(\omega) = (p^g(\omega),\ \text{all } g) \text{ and } p = (p(\omega),\ \text{all } \omega),\ \text{while} \\ &c_h(\omega) = (c_h^g(\omega),\ \text{all } g),\ c_h = (c_h(\omega),\ \text{all } \omega),\ \text{and } c = (c_h, \text{all } h). \end{aligned}$$

Also, modifying the standard convention in mathematics that $x \in \mathbb{R}^n$ is an n-dimensional column vector, we will treat price (e.g., $p(\omega)$) and price-like (e.g., α_h^t) vectors as rows rather than columns of their elements.

3.1.2. Financial Equilibrium

From each household's viewpoint, the returns from an asset are simply a vector of goods − albeit a particular, possibly a very special vector of goods − and their initial portfolio (of assets) represents their initial endowment (of goods). For this reason it is useful to begin by formulating the concept of *financial equilibrium* (FE) in terms of the real asset equivalents of bonds and stocks, initial endowments, and net changes in portfolio holdings. Such a general formulation also highlights the differences between the LT-model and the RA-model, and facilitates comparing properties of their equilibria. Let

$$\Delta_b(\omega) = \overset{\widetilde{G}}{\begin{bmatrix} \begin{bmatrix} \ddots & & 0 \\ & \delta^{\tilde{g}}(\omega) & \\ 0 & & \ddots \end{bmatrix} \\ 0 \end{bmatrix}} \begin{matrix} \widetilde{G} \\ \\ G - \widetilde{G} \end{matrix} \qquad \left(= \begin{bmatrix} I \\ 0 \end{bmatrix} \right)$$

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taken to be an arbitrary closed, convex set, in which case the typical firm should choose

$$r_f(\omega) = \arg\max_{r_f \in R_f} \sum_g p^g(\omega)\delta^g(\omega) r_f^g, \quad \text{all } \omega.$$

A slight further possible extension is to assume that the typical firm's production set depends on the date (though not on the event).

and

$$\Delta_s(\omega) = \overset{G}{\begin{bmatrix} \ddots & & 0 \\ & \delta^g(\omega) & \\ 0 & & \ddots \end{bmatrix}} \; G$$

be the $(G \times \widetilde{G})$- and $(G \times G)$-dimensional matrices representing the goods returns from bonds and stocks, respectively, so that

$$e_h(\omega) = [\Delta_b(\omega)\Delta_s(\omega)](b_h^0, s_h^0)$$

is the initial endowment of household h in state w. Also let

$$z_b = (b_h^1 - b_h^0) \text{ and } z_s = (s_h^1 - s_h^0)$$

be the net change in the portfolio holdings of household h. Then, (p, c, q, z) is a FE if

- households optimize, i.e., given (p, q) (and $\Delta = [\Delta(\omega), \text{ all } \omega] = [[\Delta_b(\omega)\Delta_s(\omega)], \text{ all } \omega]$, according with our convention), for every h, (c_h, z_h) is an optimal solution to the problem

$$\begin{array}{llll} \text{(H)} & \text{maximize}_{c_h, z_h} & u_h(c_h) & \text{with multipliers} \\ & \text{subject to} & p(0)(c_h(0) - e_h(0)) + qz_h = 0 & \lambda_h(0) \\ & \text{and} & p(\omega)(c_h(\omega) - e_h(\omega)) - p(\omega)\Delta(\omega)z_h = 0, \omega > 0, & \lambda_h(\omega) \end{array}$$

  and

- spot goods and asset markets clear, i.e.,

$$\text{(M)} \quad \begin{array}{l} \sum_h (c_h - e_h) = 0, \text{ and} \\ \sum_h z_h = 0. \end{array}$$

For the purpose of presenting and interpreting our main results concerning the structure of FE, it is necessary to introduce two auxiliary concepts: first, the concept of a *certainty equilibrium* (CE) – which is the Walrasian equilibrium in a related two-period, pure-distribution economy that we will refer to as the certainty economy (see Cass-Shell, pp. 207-8) – and second, the device for relating FE to CE, the concept of a puzzling or *peculiar financial equilibrium* (PFE).

3.1.3. Certainty Equilibrium

Consider the two-period, pure-distribution economy without uncertainty for which utility functions, initial endowments, and consumption for each household are v_h, $\overline{e}_h = (\overline{e}_h^0, \overline{e}_h^1)$, and $\overline{c}_h = (\overline{c}_h^0, \overline{c}_h^1)$, respectively, and goods prices (on overall goods markets in period 0) are $\overline{p} = (\overline{p}^0, \overline{p}^1)$. In such a certainty economy, $(\overline{p}, \overline{c})$ is a CE (otherwise known as a Walrasian, competitive, or general equilibrium) if

- households optimize, i.e., given $\overline{p}$, for every h, $\overline{c}_h$ is the optimal solution to the problem

$$(\overline{\text{H}}) \quad \begin{array}{lll} \text{maximize}_{\overline{c}_h} & v_h(\overline{c}_h) & \text{with multiplier} \\ \text{subject to} & \overline{p}(\overline{c}_h - \overline{e}_h) = 0 & \overline{\lambda}_h \end{array}$$

  and

- overall goods markets clear, i.e.,

$$(\overline{\text{M}}) \quad \sum_h (\overline{c}_h - \overline{e}_h) = 0.$$

It will be convenient, when analyzing existence of FE, to have a means of referring to the set of certainty endowments for which CE exists. So, given total resources $\overline{r} = (\overline{r}^0, \overline{r}^1) = \mathbf{1}$, let

$$\overline{E} = \{\overline{e} \in (\mathbb{R}^{2G})^H : \sum_h \overline{e}_h = \overline{r} \text{ and there is a CE}\}.$$

Note that here there is a major departure from the mainstream Walrasian tradition: we consider all conceivable certainty endowments, and, specifically, do not require that they lie in each household's consumption set.

3.1.4. Peculiar Financial Equilibrium

Our first main result concerns the particular kind of FE we refer to as PFE in an economy in which (as in the original economy, the economy described in Section 2)

$$e_h(\omega) = [\Delta_b(\omega)\Delta_s(\omega)](b_h^0, s_h^0), \text{ all } \omega, \text{ all } h, \tag{3.1}$$

but (in sharp contrast to the original economy)

$$\delta^{\widetilde{g}}(\omega) > 0, \text{ all } \widetilde{g}, \text{ and } \delta^g(\omega) = 1, \text{ all } g, \text{ all } \omega, \tag{3.2}$$

that is, $\Delta_b(\omega)$ is essentially unrestricted while $\Delta_s(\omega) = I$. The crucial implication of the second assumption is that, in this economy, total resources, denoted r, are stationary across states

$$r = [r(\omega), \text{ all } \omega] = [\Delta_s(\omega)\mathbf{1}, \text{ all } \omega] = \mathbf{1}.$$

It is then straightforward to apply this result to the original economy with log-linear utility, through a simple transformation of the units of goods.

When $\Delta_s(\omega) = I$, all ω, a FE is a PFE if

(i) *irrelevancy*: $z_{bh} = -b_h^0$, all h, i.e., households completely liquidate their initial portfolio of bonds;

(ii) *degeneracy*: rank $[p(\omega)\Delta_s(\omega),\ \omega > 0] = \text{rank}\ [p(\omega), \omega > 0] = 1$, i.e., households are completely indifferent to which (equally valued) terminal portfolio of stocks they hold; and yet

(iii) *optimality*: rank $[\lambda_h, \text{all } h] = 1$, i.e., the goods allocation is Pareto optimal.

3.2. Existence

The key feature of a PFE which permits a simple characterization is that, effectively, the spot market budget constraints in a FE collapse to the Walrasian budget constraint in a CE with certainty endowments given by the formulas

$$\begin{aligned}\bar{e}_h &= (e_h(0), \sum_{\omega>0} \pi(\omega) e_h(\omega)) \\ &= ([\Delta_b(0)\Delta_s(0)](b_s^0, s_h^0), \sum_{\omega>0} \pi(\omega)[\Delta_b(\omega)\Delta_s(\omega)](b_h^0, s_h^0)), \text{ all } h.\end{aligned} \tag{3.3}$$

This will become obvious when we detail the proof of Proposition 1 in Appendix A. So now let

$$\bar{E}_\Delta = \{\bar{e} \in \bar{E} : \text{ for some } (b_h^0, s_h^0), \text{ all } h, \text{ such that } \sum_h (b_h^0, s_h^0) = (0, \mathbf{1}), \bar{e} \text{ satisfies (3.1)}\}.$$

Note that, generically in Δ, dim $\bar{E}_\Delta = (H-1)(\tilde{G} + G)$, which in the leading example equals 3.

For simplicity, normalize prices so that $p^1(\omega) = 1$, all ω, and $\bar{p}^{11} = 1$ (later on we will find that another price normalization is more useful when analyzing the nature of PFE).

Proposition 1 (Existence of PFE). *Consider an economy which satisfies (3.1) and (3.2), together with the related certainly economy which satisfies (3.3).*

(i) *If* (p, c, q, z) *is a PFE, then* $\bar{e} \in \overline{E}_\Delta$ *and there is a CE* $(\bar{p}, \bar{c})$ *such that*

$$\begin{aligned}&\bar{p} = (p(0), (\lambda_1(1)/\pi(1)\lambda_1(0))p(1)) \\ \textit{and}& \\ &\bar{c}_h = (c_h(0), c_h(1)), \quad \textit{all } h.\end{aligned} \tag{3.4}$$

(ii) *If* $\bar{e} \in \overline{E}_\Delta$ *and* $(\bar{p}, \bar{c})$ *is a CE, then there is a PFE* (p, c, q, z) *such that*

$$\begin{aligned}&p(\omega) = \begin{cases} \bar{p}^0, & \omega = 0 \\ \bar{p}^1/\bar{p}^{11}, & \omega > 0 \end{cases} \\ \textit{and}& \\ &c_h(\omega) = \begin{cases} \bar{c}_h^1, & \omega = 0 \\ \bar{c}_h^2, & \omega > 0, \quad \textit{all } h. \end{cases}\end{aligned} \tag{3.5}$$

Returning now to consideration of the original economy, we observe that if units of goods are converted into per-stock-return units, that is, if, in each state ω, one unit of good g becomes $1/\delta^g(\omega)$ units of good g, then the return matrix for stocks $\Delta_s(\omega)$ becomes simply the identity matrix. Furthermore, with log-linear utility functions, each household's utility in the old and the new units is identical up to an additive constant. This leads immediately to a characterization of FE in such an economy, which we can state succinctly in terms of the "trees and logs" of the paper's title.

Corollary to Proposition 1 (PFE with Trees and Logs). *The characterization of PFE in Proposition 1 applies to an economy with trees and logs after conversion to per-tree-return units of goods.*

An economy in which stocks return the same amount of goods in each state is itself not really very interesting. On the other hand, the *trees and logs model* (TL-Model) is intrinsically interesting and − as it turns out − much can be inferred about the finer structure of FE in this model. For this reason we now focus exclusively on the TL-Model, assuming conversion to per-tree-return units (so that hereafter, $\delta^{\widetilde{g}}(\omega) > 0$, all $\widetilde{g}$, all ω, while $\Delta_s(\omega) = I$, all ω). At the same time we will also occasionally concentrate on the leading example. We must emphasize, however, that the Corollary to Proposition 1 is valid for arbitrary dimensionality − including the general case of intrinsically incomplete markets, as well as the special case commonly considered in the Finance literature, where there is a single good.

Before turning to questions of uniqueness and, say, *exclusivity* − that is, whether there are *other* (or "ordinary") *financial equilibria* (OFE) in the TL-Model[4] − it is quite instructive to highlight the peculiarity of the PFE. We accomplish this by, first, contrasting the results reported in Proposition 1 with well-known properties of the RA-Model, and second, relating them to well-known properties of the Cass-Shell sunspot model (SS-Model).

[4]We should mention explicitly, that for the economy of Proposition 1, it is fairly straightforward to establish exclusivity, by slightly modifying Mas-Colell's [18] variant of Cass and Shell's original argument that sunspots can't matter with complete markets. We are indebted to Paolo Siconolfi for bringing this to our attention. However, such an argument relies on a construction which cannot be applied in the case of hybrid trees unless $F = S$, and later on we actually employ this extension of the model in the case where $F < S$. So we have chosen instead to provide an alternative proof which can be so applied (and which relies heavily on the specification of log-linear utility). Our particular argument can also be adapted for other purposes, though the full extent to which this yields interesting results remains to be seen.

3.2.1. The LT-Model v. the RA-Model

The "well-known" properties asserted here can be found − or easily inferred following the lead of related results in the RA-Model literature.[5] We contrast these to the results reported in Proposition 1 applied to the TL-Model. For this purpose, when presenting a result which is (within a well-specified conventional context) true without any qualification we will use the term "general" or "generally". Otherwise, when a result is only true generically (on some given open, full measure set of parameters), we will use the term "typical" or "typically" (in contrast to "exceptional" or "exceptionally"). We also use self-explanatory tables to describe the RA-Model literature. Bear in mind that, looking ahead to subsection 3.5 below (where we establish exclusivity, that OFE can never occur), it is accurate to simply identify PFE with FE in the TL-Model.

1. **Existence**

Existence of FE

FM are / Existence is	typical	only typical
Potentially Complete	Magill-Shafer	Hart [12]
Intrinsically Incomplete	Duffie-Shafer	Cass [5]

In the TL-Model, on the other hand, the operative condition in Proposition 1 − $\overline{e} \in \bar{E}_{\Delta}$ − characterizes the very large set of initial portfolios for which there is generally a PFE (depending, of course, on the other parameters of the model, in particular, $\delta^{\widetilde{g}}(\omega)$, all $\widetilde{g}$, all ω). It is important, and we stress the point, that this condition clearly encompasses much more than just the initial portfolios for which $\overline{e} >> 0$ (see subsection 3.4 below).

2. **Optimality**

By virtue of Arrow's Equivalency Theorem [1], for the RA-Model, when financial markets are potentially complete, Pareto optimality is closely related to the rank of the matrix of asset returns in value terms, or *yields*.[6] So we tabulate both optimality and rank properties for this model.

[5] In fact, many of the counterexamples are so obvious, or so easily constructed based on other results in the financial equilibrium literature that they are hard to find explicit cites for. We will refer to such "well-known" results as "folklore".

[6] For example, in the TL-Model, this matrix is

$$[p(\omega)[\Delta_b(\omega)\Delta_s(\omega)], \omega > 0].$$

Optimality of FE

FM are / Pareto Optimality is	typical	only typical	exceptional
Potentially Complete	Magill-Shafer	folklore	no
Intrinsically Incomplete	no	no	Geanakoplos et al

Matrix of Asset Yields

FM are / Full Rank is	typical	only typical
Potentially Complete	Magill-Shafer	folklore
Intrinsically Incomplete	Duffie-Shafer	folklore

The TL-Model obviously turns all this on its head: *The matrix of asset yields never has full rank, and yet allocation is always Pareto optimal*! FE in this model are very, very peculiar, indeed.

3. **Trade in Assets**

Using the fact that, typically, in the RA-Model the matrix of asset yields has full rank, it is a routine application of the Transversality Theorem to show that, again typically, all assets must be traded by all households. In the TL-Model, contrarily, financial markets are quite inactive. In the first place, households transact on the bond market only to the extent that they completely liquidate their initial positions. In a model with many periods, that is, where $T > 1$, this means that, beyond today, bond markets are completely inactive.[7] In the second place − the point of Proposition 1 − only a single stock market need be active, though, obviously they all can be. So, in this respect as well, PFE are also very, very peculiar!

4. **The Explanation**

Why such striking disparity between the two models? The answer is both very simple and obvious. The TL-Model is an extraordinarily atypical specification of the RA-Model, for two basic reasons: *First, tree returns, and hence total resources are identically one in each state of the world. Second, initial endowments must both (i) lie in the span of the matrix of asset returns, and (ii) add up to the tree returns in each state of the world.* In particular, when there are potentially complete financial markets, it must be the case that if households own (independent) initial endowments, in

[7] By the way (and this should really go without saying!) all the results concerning the discrete date-event version of our model are easily generalized to many periods − provided all assets can be retraded. "Many periods" and "asset retrade" (what is labelled "dynamically ..." in Finance) are of course inherent in the continuous date-event version of the model; see section 4 below.

addition to initial portfolios, then all the anomalies revealed above (typically) simply disappear.[8] What more is there to say, really?

3.2.2. The TL-Model vis-a-vis the SS-Model

For one familiar with the literature on the SS-Model (as one of us, anyway, surely is!), the parallel between PFE and *non*sunspot equilibria (NSE) is inescapable. Both types of equilibrium exhibit stationarity in the precise sense that they are equivalent to CE. Moreover, both are, in their respective economic environments, the only equilibria for which goods allocations are Pareto optimal. This suggests another possible interesting parallel, that between what we have earlier labeled OFE and *sun*spot equilibria (SSE). It turns out, however, that even though there is a strong parallel between the two concepts, it is far from exact. The essential difference is a consequence of the fact that optimality in the TL-Model has nothing to do with financial market completeness, whereas in the SS-Model this is a very significant consideration. Thus, for instance, in the TL-Model, as we will establish in section 3.5 below, there can be no OFE whether financial markets are potentially complete or not, while in the SS-Model there is typically a distinct SSE in the leading example with an incomplete financial market (this can be inferred from the analysis in Cass [4] together with Balasko and Cass [2], pp. 145–9; when asset returns are specified in value terms, there is typically even a continuum of distinct SSE).

We now turn to consideration of another very important implication of the fact that degeneracy and incompleteness of financial markets are part and parcel of the PFE.

3.3. Portfolio Constraints

Financial markets with portfolio constraints have recently become the major area of research in asset pricing theory (see Karatzas and Shreve [14] and references contained therein). The main bulk of this analysis is undertaken in the context of a single-good economy. Rather surprisingly, however, very little is known about the robustness of various implications within a multiple-good setting.

Our objective here is to illustrate the interaction between the spot goods market and portfolio constraints, and to see to what extent the possibility of trade in the real markets can alleviate frictions in the financial markets. Toward that end, we present a straightforward implication of the arguments in the proof of Proposition 1.

[8] In the leading example, for instance, this will be the case if and only if, for some ω', $\omega'' > 0$, $\delta^1(\omega') \neq \delta^1(\omega'')$; see Magill and Shafer, pp. 174-5.

Proposition 2 (Portfolio Constraints). *Consider a class of portfolio constraints under which it is feasible for the households in the economy to liquidate their initial bond holdings in period 0 and invest the proceeds (net of $p(0)c_h(0)$) in some (fixed) portfolio of the stocks. Then in this constrained economy, as long as it is feasible for the households to jointly hold one share of each stock, the relevant PFE − in particular, including those for which initial portfolios also satisfy the constraints, and saving takes place in terms of the (fixed) portfolio of stocks − still obtain.*

In particular, Proposition 2 encompasses the case of restricted participation in the stock market.

Corollary to Proposition 2 (Restricted Participation). *Suppose that* $b_h^{t\widetilde{g}} \in \mathbb{R}$, all $\widetilde{g}$, and $s_h^{tg} \in \mathbb{R}$, *some* g, $t = 0$, 1, all h. *Then, for arbitrary constraints on the remaining stocks, as long as market clearing in those stocks is feasible, the relevant PFE still obtains.*

This result is in striking contrast to the implications of a single-good model with multiple stocks. Portfolio constraints in the TL-economy can be fully circumvented by households trading in the spot goods markets (nonexistent in a single-good model). The policy replicating the unconstrained optimum involves a combination of trades in the assets and the exchange of goods for those paid out by the stocks whose share holdings are constrained.

3.4. Uniqueness

In the TL-Model the question of uniqueness of PFE for given initial portfolios is equivalent to the question of uniqueness of CE for the corresponding initial endowments (3). This question has a very straightforward answer.

It is a routine exercise given in the graduate microeconomic theory sequence to show the following: in the standard 2×2 model of pure distribution with log-linear utility, Walrasian equilibrium is unique. This property stems from the fact that, in this example, the prices which support allocations in the Pareto set define lines which are either parallel − in the borderline case of identical log-linear utility − or intersect outside the Edgeworth-Bowley box. In other words, the only initial endowments for which there are multiple equilibria must lie outside the households' consumption sets − and this violates the spirit of the model.

In the certainty model equivalent of the TL-Model, however, there is absolutely no reason, given the opportunities of both borrowing and short-selling, that initial endowments must lie in the households' consumption sets. This yields an interesting result for the leading example.

Proposition 3 (Uniqueness of PFE). *For the leading example, the CE, and hence the PFE is unique*

- *in the borderline case where* $\alpha_1^t = \alpha_2^t$, $t = 0, 1$ *and* $\beta_1 = \beta_2$, *for all initial endowments* $\bar{e} \in \bar{E}_\Delta$, *but otherwise*

- *in the general case, for all initial endowments except possibly those which lie on a line segment, say,* $\bar{e} \in \bar{L}_\Delta \subset \bar{E}_\Delta$.

Also, for $\bar{e} \in \bar{L}_\Delta$, *every Pareto optimal allocation is supported as a PFE.*

In other words, either the PFE is unique, or there are PFE corresponding to each allocation in the Pareto set (on a relatively small subset of possible initial portfolios, to be sure!).

The intuition behind this result is presented in Figure 1 for the case in which $G = \Omega = 1, \widetilde{G} = 1$, and $H = 2$; the redundant bond is required so the $\bar{e}_1'$ is consistent with portfolio choice (otherwise, for an initial portfolio consisting of just one stock, it must be the case that $\bar{e}_1^0 = \bar{e}_1^1 = s_1^0 > 0$, and the PFE is unique). Note also that in this example, since $G = 1$, $\alpha_h = 1$, $h = 1, 2$.

3.5. Exclusivity

When one first encounters the pervasiveness of PFE − mainly because these financial equilibria are so strange − an immediate, natural reaction is to ask "Just how important is this peculiar phenomenon, anyway?", or more objectively, "Are there other FE which have substantial presence as well?" In this subsection we establish that the answer is a blunt and clear "No!"

Proposition 4 (Exclusivity of PFE). *The only FE are PFE.*

Our particular method of proof for this result (which fully exploits the trees and logs structure) admits an immediate corollary concerning restricted participation.[9]

Corollary to Proposition 4 (Restricted Participation I: Some households are barred from transacting in some stocks). *Suppose that, for* $h < H$, *there is* $G_h \subset \{2, 3, \ldots, G\}$, *such that, for* $g \in G_h$, *household* h *faces the constraint* $s_h^{tg} = 0$, $t = 0, 1$, *while household* H *is unconstrained. Then the only FE is PFE.*

However, matters may become quite complicated with more general portfolio constraints, as the following result indicates.

[9] By generalizing to hybrid trees we also get an immediate corollary concerning the situation where both households can freely transact in the bond, but can only trade stocks in terms of a fixed portfolio with equal shares.

Cautionary to Proposition 4 (Restricted Participation II: Some households face general constraints on transacting in some stocks). *For the leading example, consider the possibility that Ms. 1 faces a constraint of the form $\phi(s_1^{12}) \geq 0$, where $\phi : \mathbb{R} \to \mathbb{R}$ is differentiable and strictly quasi-concave. Then*

- *if* $\phi(s_1^{12}) = 0 \Rightarrow s_1^{12} \in [0,\, 1]$, *the only FE is PFE, otherwise*
- *for some economy (specified, in part, by ϕ) there is an OFE as well as a unique PFE.*

4. Extension in Continuous Time

We now consider a continuous-time variation on our leading example. Results presented this section are parallel to those of the discrete date-event version. For that reason this section is going to be intendedly dense. The economy has a finite-horizon, $[0, T]$. Uncertainty is represented by a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathcal{P})$, on which is defined a two-dimensional Brownian motion $w(t) = (w_1(t),\, w_2(t))$, $t \in [0, T]$. For simplicity, the Brownian motions w_1 and w_2 are taken to be independent of each other. All stochastic processes are assumed adapted to $\{\mathcal{F}_t; t \in [0, T]\}$, the augmented filtration generated by w. All stated equalities involving random variables hold $\mathcal{P}$-almost surely. Note that this continuous-time specification of the state space is in parallel to that of the discrete-time leading example: there, a random variable was represented by three possible future realizations corresponding to the three branches of the date-event tree; in the continuous-time version, each process would be parameterized by a triple $(\mu,\, \sigma_1,\, \sigma_2)$ − the drift and volatility processes.

The risky stocks pay out the strictly positive dividend stream at rate δ^g, in good g, following an Itô process

$$d\delta^g(t) = \delta^g(t)[\mu_\delta^g(t)\, dt + \sigma_\delta^g(t)\, dw(t)], \qquad g = 1,\, 2\,,$$

where μ_δ^g and $\sigma_\delta^g \equiv (\sigma_{\delta 1},\, \sigma_{\delta 2})^\top$ are arbitrary stochastic processes. The relative price of good 2 (in terms of good 1, the numeraire), p, will be shown in equilibrium to have dynamics

$$dp(t) = p(t)[\mu_p(t)dt + \sigma_p(t)dw(t)],$$

where μ_p and $\sigma_p \equiv (\sigma_{p1},\, \sigma_{p2})^\top$ are (endogenous) drift and volatility processes.

Analogously to those in the discrete-time leading example, investment opportunities are represented by three securities: an instantaneously riskless bond, q_b^1, in zero net supply, and two risky stocks, and the stock prices, q_s^g. The bond price and stock prices are posited to follow

$$dq_b^1(t) = q_b^1(t)r^1(t)dt \; , \qquad q_b^1(0) = 1 \, ,$$
$$dq_s^1(t) + \delta^1(t)dt = q_s^1(t)[\mu_s^1(t)dt + \sigma_s^1(t)dw(t)] \, ,$$
$$dq_s^2(t) + p(t)\,\delta^2(t)dt = q_s^2(t)[\mu_s^2(t)dt + \sigma_s^2(t)dw(t)] \, ,$$

where the interest rate r^1, the drift coefficients $\mu_s = (\mu_s^1, \mu_s^2)$, and the volatility matrix $\sigma = \{\sigma_{ij}, \, i, \, j = 1, \, 2\}$ are to be determined in equilibrium. Under this specification of the investment opportunities, financial markets are (potentially) dynamically complete.

The two households maximize their expected lifetime log-linear utility

$$u_h(c_h) = E\left[\int_0^T e^{-\rho_h t} v_h(c_h(t))dt\right] \qquad h = 1, \, 2, \tag{4.6}$$

where $v_h(c_h(t)) = \alpha_h^1 \log c_h^1(t) + \alpha_h^2 \log c_h^2(t)$ and $\rho_h > 0$, subject to the dynamic budget constraint

$$\begin{aligned} dW_h(t) =& W_h(t)r^1(t)dt - (c_h^1(t) + p(t)c_h^2(t))dt + s_h(t)^\top I_s(t) \begin{pmatrix} \mu_s^1(t) - r^1(t) \\ \mu_s^2(t) - r^1(t) \end{pmatrix} dt \\ &+ s_h(t)^\top I_s(t) \begin{pmatrix} \sigma_s^1(t) \\ \sigma_s^2(t) \end{pmatrix} dw(t), \qquad W_h(0) = b_h(0) + s_h(0)^\top q_s^1(0), \end{aligned} \tag{4.7}$$

where $W_h(t) \equiv b_h(t)q_b^1(t) + s_h(t)^\top q_s(t)$ is the household's wealth and I_s is a 2×2 diagonal matrix with diagonal elements q_s^1 and q_s^2.[10] The volatility matrix in the representation of the investment opportunity set,

$$\Sigma(t) \equiv \begin{pmatrix} \sigma_s^1(t) \\ \sigma_s^2(t) \end{pmatrix} ,$$

is not necessarily invertible. If it is then the two risky stocks are linearly independent and markets are complete, otherwise the two stocks represent the same investment opportunity and markets are intrinsically incomplete. Appealing to the martingale methodology, standard in asset pricing, we deflate household h's wealth by the (possibly personalized – if markets turn out to be incomplete in equilibrium) state prices in order to convert its dynamic budget constraint into a static Arrow-Debreu budget constraint of the form

$$E\left[\int_0^T \xi_h(t)[c_h^1(t) + p(t)c_h^2(t)]dt\right] = E\left[\int_0^T \xi_h(t)[e_h^1(t) + p(t)e_h^2(t)]dt\right] , \tag{4.8}$$

where $\xi_h(t, \omega)$ is the Arrow-Debreu price of a unit of the numeraire good in state $\omega \in \Omega$ at time t per unit of probability $\mathcal{P}$, as perceived by household h, and $e_h^g, \; g = 1, 2$, is, as before, the dividend stream from the initial shareholdings.

[10]For simplicity of exposition, we assume that α_h^1 and α_h^2 are constants. The analysis below can be extended to incorporate time-dependent coefficients in a straightforward fashion.

A *financial equilibrium* is defined as a collection of prices (ξ, p, q) and associated optimal policies (c_h, b_h, s_h, $h = 1, 2$) such that the goods, bond and stock markets clear, i.e., $\forall t \in [0, T]$, for $g = 1, 2$:

$$\sum_h c_h^g(t) = \delta^g(t), \tag{4.9}$$
$$\sum_h b_h^g(t) = 0,$$
$$\sum_h s_h^g(t) = 1.$$

For analytical convenience, we introduce a representative agent with utility

$$U(c; \eta) = E\left[\int_0^T v(c(t), \eta)dt\right],$$

where

$$v(c;\, \eta) = \max_{c_1 + c_2 = c} \eta_1 e^{-\rho_1 t} v_1(c_1) + \eta_2 e^{-\rho_2 t} v_2(c_2),$$

and $\eta_h > 0,\ h = 1, 2$, may be stochastic. If in an equilibrium, η_1 and η_2 are constants, then the allocation is Pareto optimal, otherwise it is not. Since in equilibrium the weights for the representative agent are unique up to a multiplicative constant, we adopt the normalization $\eta_1 = \eta$, $\eta_2 = 1 - \eta$, $\eta \in (0, 1)$.

We are now ready to characterize equilibria in the economy.

Proposition 5 (Characterization of PFE). *If an equilibrium exists in the leading example, it is a PFE. Equilibrium prices are identical across households and are given by*

$$\xi(t) = \frac{\alpha_1^1 \eta e^{-\rho_1 t} + \alpha_2^1 (1 - \eta) e^{-\rho_2 t}}{\delta^1(t)}, \tag{4.10}$$
$$p(t) = \frac{\alpha_1^2 \eta e^{-\rho_1 t} + \alpha_2^2 (1 - \eta) e^{-\rho_2 t}}{\alpha_1^1 \eta e^{-\rho_1 t} + \alpha_2^1 (1 - \eta) e^{-\rho_2 t}} \frac{\delta^1(t)}{\delta^2(t)} \tag{4.11}$$

the equilibrium allocations are

$$c_1^g(t) = \frac{\alpha_1^g \eta e^{-\rho_1 t} \delta^g(t)}{\alpha_1^g \eta e^{-\rho_1 t} + \alpha_2^g (1 - \eta) e^{-\rho_2 t}} \qquad g = 1, 2, \tag{4.12}$$
$$c_2^g(t) = \frac{\alpha_2^g (1 - \eta) e^{-\rho_2 t} \delta^g(t)}{\alpha_1^g \eta e^{-\rho_1 t} + \alpha_2^g (1 - \eta) e^{-\rho_2 t}} \qquad g = 1, 2, \tag{4.13}$$

where the constant weight η is determined from either household's static budget constraint with the optimal consumption allocations (4.12)–(4.13) substituted in, i.e.,

$$E\left[\int_0^T \xi(t)[c_1^1(t) + p(t) c_1^2(t)]dt\right] = E\left[\int_0^T \xi(t)[e_1^1(t) + p(t) e_1^2(t)]dt\right]. \tag{4.14}$$

Furthermore, the prices of stocks expressed in terms of goods they are claims to, and the interest rate are given by

$$q_s^1(t) = \frac{\rho_1\alpha_1^1\eta(e^{-\rho_1 t} - e^{-\rho_1 T}) + \rho_2\alpha_2^1(1-\eta)(e^{-\rho_2 t} - e^{-\rho_2 T})}{\rho_1\rho_2(\alpha_1^1\eta e^{-\rho_1 t} + \alpha_2^1(1-\eta)e^{-\rho_2 t})}\,\delta^1(t), \tag{4.15}$$

$$q_s^2(t)/p(t) = \frac{\rho_1\alpha_1^2\eta(e^{-\rho_1 t} - e^{-\rho_1 T}) + \rho_2\alpha_2^2(1-\eta)(e^{-\rho_2 t} - e^{-\rho_2 T})}{\rho_1\rho_2(\alpha_1^2\eta e^{-\rho_1 t} + \alpha_2^2(1-\eta)e^{-\rho_2 t})}\,\delta^2(t), \tag{4.16}$$

$$r^1(t) = \mu_\delta^1(t) - \frac{\alpha_1^1\,\eta\,\rho_1 e^{-\rho_1 t} + \alpha_2^1(1-\eta)\rho_2 e^{-\rho_2 t}}{\alpha_1^1\,\eta e^{-\rho_1 t} + \alpha_2^1(1-\eta)e^{-\rho_2 t}} - |\sigma_\delta^1(t)|^2$$

Conversely, if there exist ξ, p and η satisfying (4.10)–(4.11) and (4.14), then the associated optimal policies clear all markets.

It is easy to see that the equilibrium is a PFE. Analogously to the discrete date-event version, the relative price of the two goods is proportional the ratio of the dividends. It is easy to see then that in equilibrium the volatility matrix in the representation of the investment opportunity set, $\Sigma(t)$, is degenerate, or, equivalently, the two stocks yield the same investment opportunity. The mapping into the certainty model is also apparent from the characterization in Proposition 5: in per-tree-return units, optimal consumption and prices are deterministic functions of time. Furthermore, since the weight η is constant in equilibrium, the allocation is Pareto optimal.

We now turn to the nonuniqueness of peculiar equilibria.

Proposition 6 (Nonuniqueness). *Consider the set of initial endowments of household 1, e_1, satisfying:*

$$E\left[\int_0^T \left(\alpha_1^1 e^{-\rho_1 t}\,\frac{e_1^1(t)}{\delta^1(t)} + \alpha_1^2 e^{-\rho_1 t}\,\frac{e_1^2(t)}{\delta^2(t)}\right) dt\right] = \frac{1-e^{-\rho_1 T}}{\rho_1}, \tag{4.17}$$

$$E\left[\int_0^T \left(\alpha_1^1 e^{-\rho_2 t}\,\frac{e_1^1(t)}{\delta^1(t)} + \alpha_1^2 e^{-\rho_2 t}\,\frac{e_1^2(t)}{\delta^2(t)}\right) dt\right] = 0. \tag{4.18}$$

On this set of endowments, there is a continuum of PFE with the characterization given by (4.10)–(4.11) and (4.12)–(4.13) for all $\eta \in (0,1)$.

Proposition 6 is an exact analogue of Proposition 3 in the discrete date-event version. Note that for condition (4.18) to be satisfied it is necessary that household 1 be endowed with a short position in one of the securities.

The continuous-time formulation offers additional tractability over the discrete-time version in that one can parameterize the processes for state prices and stochastic weighting in the economy, which proves to be very useful for getting explicit formulas in economies with frictions. Comprehensive investigation of the effects of portfolio constraints in the TL-economy is the focus of a companion project, and is not included in this paper. Here, we just concentrate on a specific constraint: restricted participation in one of the risky securities.

Proposition 7 (Restricted Participation). *Consider the economy where household 1 is restricted from investing in one of the risky stocks, e.g., stock 1, but can take an unrestricted position in the bond and stock 2. Household 2 is unconstrained. Equilibrium in this constrained economy coincides with that of the unconstrained with the characterization given in Proposition 5.*

5. Final Remarks

Our thorough examination of the Lucas tree model when extended to include multiple goods uncovers a variety of puzzling characteristics. In particular, we show that under the maintained assumption of log-linear utility, the only equilibria in the model are peculiar financial equilibria, in which all the stocks represent the same investment opportunity − and yet, nonetheless, allocation is Pareto optimal.

Fairly complete analysis of the effects of portfolio constraints in the general trees and logs economy is a separate issue. In this paper, we merely demonstrate that for a certain large class of portfolio constraints − in contrast to a single-good model − their impact on the economy can be fully alleviated by the possibility of trade in the spot goods markets. This result however must be heavily qualified: even within this class of portfolio constraints, there may be additional financial equilibria in which allocation is not Pareto optimal. Another important class of constraints to consider is the one which leads to allocation which is not Pareto optimal (and therefore financial equilibria which are not peculiar). In this situation constraints on transactions could only, at best, be partially alleviated by trading in the spot goods markets, and it would be of interest to quantify the extent to which trade in goods can circumvent restrictions on trade in assets. Conversely, we should be able to use our framework to investigate the interaction between restrictions on transactions on goods markets (e.g., one cannot transact an unlimited quantity of a particular good, or certain goods have to be purchased concurrently) and transactions on asset markets. This analysis will of course make use of the remarkable tractability of our model: despite the presence of multiple positive-net-supply stocks, it appears possible to explicitly characterize equilibria in an economy encompassing a variety of realistic constraints on transactions.

Appendix A: Proofs of the Main Results

We begin by writing down the extended system of equations which provides the whole basis for our formal analysis. This consists of the Lagrange conditions characterizing an optimal solution to each household's optimization problem (H) together with the spot goods and asset market clearing conditions (M). For the time being we will continue to assume that spot goods prices are normalized at each spot in terms of good 1 as the numeraire $p^1(\omega) = 1$, all ω, and all pertinent quantities are expressed in per tree-return units. Also, to avoid unnecessary clutter, where it is obvious from context, "all ω," "$\omega > 0$," or "all h" are understood as given.

A.1 The Extended System of Equations

First-order conditions (FOC's)

$$\sum_{\omega>0} \pi(\omega) D_{c_h(0)} v_h(c_h(0), c_h(\omega)) - \lambda_h(0) p(0) = 0 \tag{A.1}$$

and

$$\pi(\omega) D_{c_h(\omega)} v_h(c_h(0), c_h(\omega)) - \lambda_h(\omega) p(\omega) = 0; \tag{A.2}$$

No-arbitrage conditions (NAC's)

$$\lambda_h(0) q_b - \sum_{\omega>0} \lambda_h(\omega) p(\omega) \Delta_b(\omega) = 0 \tag{A.3}$$

and

$$\lambda_h(0) q_s - \sum_{\omega>0} \lambda_h(\omega) p(\omega) = 0; \tag{A.4}$$

Budget constraints (BC's)

$$p(0)(c_h(0) - e_h(0)) + q z_h = 0 \tag{A.5}$$

and

$$p(\omega)(c_h(\omega) - e_h(\omega)) - p(\omega)\Delta(\omega) z_h = 0; \tag{A.6}$$

Market clearing conditions (MCC's)

$$\sum_h c_h(\omega) - \mathbf{1} = 0 \tag{A.7}$$

and

$$\sum_h z_h = 0. \tag{A.8}$$

Also bear in mind the definition of initial endowments,

$$e_h(\omega) = [\Delta_b(\omega) I](b_h^0, s_h^0). \tag{A.9}$$

Remarks:

1. By virtue of the NAC's (A.3)-(A.4), we can replace the first BC (A.5) by a personalized Walrasian-like BC
$$\sum_{\omega} \lambda_h(\omega)p(\omega)(c_h(\omega) - e_h(\omega)) = 0. \tag{A.5}$$
This fact will prove to be very useful in the course of most of our argument.

2. By virtue of the BC's (A.5)-(A.6) and the MCC's (A.7)-(A.8), $\Omega + 1$ of these equations are redundant (the analogue of Walras' law), for example, Mr. H's BC's. We will explicitly drop these particular redundant equations later on.

3. Taking account of the preceding remark together with the spot goods price normalizations, it follows that there are (at most)
$$\begin{aligned} J &= HG(\Omega + 1) + H(\widetilde{G} + G) + (H - 1)(\Omega + 1) + G(\Omega + 1) + (\widetilde{G} + G) \\ &= HG(\Omega + 1) + H(\widetilde{G} + G) + H(\Omega + 1) + (G - 1)(\Omega + 1) + (\widetilde{G} + G) \end{aligned}$$
independent equations in the J independent variables
$$c_h, z_h, \lambda_h, (p^g(\omega), g > 1, \text{ all } \omega), \text{ and } q.$$
Of course, at a solution corresponding to a PFE, and therefore a Pareto optimal allocation, the NAC's (A.3)-(A.4) are not independent. This means that, with potentially complete financial markets, all of the equations (A.1)-(A.8) can never be independent (since otherwise one would get an immediate contradiction based on Arrow's Equivalency Theorem), and this tends to complicate their analysis.

A.2 Proof of Proposition 1

(i) Suppose that (p, c, q, z) is a PFE. Then, by degeneracy, $p(\omega) = p(1)$, and by irrelevancy,
$$p(\omega)e_h(\omega) + p(\omega)\Delta(\omega)z_h = p(\omega)s_h^1,$$
so that (A.2) and (A.6) become simply
$$\pi(\omega)D_{c_h(\omega)}v_h(c_h(0), c_h(\omega)) - \lambda_h(\omega)p(1) = 0 \tag{A.10}$$
and
$$p(1)(c_h(\omega) - s_h^1) = 0. \tag{A.11}$$

From our textbook assumptions about v_h, it follows that (A.10), (A.11), and (A.7) describe an identical Walrasian equilibrium at each spot $\omega > 0$. Thus, from (A.10) it also follows that
$$c_h(\omega) = c_h(1) \text{ and } \lambda_h(\omega)/\pi(\omega) = \lambda_h(1)/\pi(1), \omega > 0,$$
and from optimality (or, equally well, the NAC (A.4)) that
$$\lambda_h(1)/\pi(1)\lambda_h(0) = \lambda_1(1)/\pi(1)\lambda_1(0).$$

Hence, (A.5) becomes simply

$$p(0)(c_h(0) - [\Delta_b(0)I](b_h^0, s_h^0))+$$

$$(\lambda_1(1)/\pi(1)\lambda_1(0))p(1)(c_h(1) - \sum_{\omega>0} \pi(\omega)[\Delta_b(\omega)I](b_h^0, s_h^0)) = 0, \tag{A.12}$$

while (A1) and (A10) become simply

$$D_{c_h(0)} v_h(c_h(0), c_h(1)) - \lambda_h(0)p(0) = 0 \tag{A.13}$$

and

$$D_{c_h(1)} v_h(c_h(0), c_h(1)) - \lambda_h(0)(\lambda_1(1)/\pi(1)\lambda_1(0))p(1) = 0. \tag{A.14}$$

Finally, making the identifications (3.3)-(3.4) together with $\overline{\lambda}_h = \lambda_h(0)$, we see that (A.12)-(A.14) characterize the optimal solution to $(\overline{H})$, and that these necessarily satisfy $(\overline{M})$, so that this half of the proof is complete.

(ii) Suppose that $\bar{e} \in \bar{E}_\Delta$, and that $(\bar{p}, \bar{c})$ is a CE. Then, given $\bar{p}, (\bar{c}_h, \bar{\lambda}_h)$ solves the analogues of the Lagrange conditions (A.12)-(A.14),

$$\bar{p}(\bar{c}_h - \bar{e}_h) = 0, \tag{A.12$'$}$$

$$D_{\bar{c}_h^0} v_h(\bar{c}_h^0, \bar{c}_h^1) - \bar{\lambda}_h \bar{p}^0 = 0, \tag{A.13$'$}$$

and

$$D_{\bar{c}_h^1} v_h(\bar{c}_h^0, \bar{c}_h^1) - \bar{\lambda}_h \bar{p}^1 = 0, \tag{A.14$'$}$$

with $\bar{e}_h$ satisfying (3.3) for some (b_h^0, s_h^0). Making the identifications (3.5) together with $\lambda_h(0) = \bar{\lambda}_h$,

$$\lambda_1(\omega)/\pi(\omega)\lambda_1(0) = \bar{p}^{11},$$

and, say,

$$s_h^1 = s_h^0 + (\Delta s_h^{01}, 0, \ldots, 0) \text{ such that } \bar{p}^1(c_h^1 - s_h^1) = 0,$$

one can then simply reverse the steps of the preceding argument. Since this procedure is obvious, we omit its details. ■

A.3 Reduction to The True Equations

From here on we will maintain the assumption of log-linear utility. This permits substantial simplification of the extended system of equations (A.1)-(A.8). We also drop Mr. H's BC's as being redundant.

With log-linearity, the FOC's (A.1)-(A.2) become, for all g,

$$\alpha_h^{0g}/c_h^g(0) - \lambda_h(0)p^g(0) = 0 \tag{A.15}$$

and

$$\pi(\omega)\beta_h\alpha_h^{1g}/c_h^g(\omega) - \lambda_h(\omega)p^g(\omega) = 0. \tag{A.16}$$

From (A.15) it follows that

$$\lambda_h(0)p(0)c_h(0) = 1 \tag{A.17}$$

and, together with (A.7) for $\omega = 0$, that

$$p(0) = \sum_h (1/\lambda_h(0))\alpha_h^0. \tag{A.18}$$

Similarly, from (A.16) it follows that, for $\omega > 0$,

$$\lambda_h(\omega)p(\omega)c_h(\omega) = \pi(\omega)\beta_h \tag{A.19}$$

and, together with (A.7), that

$$p(\omega) = \pi(\omega)\sum_h (\beta_h/\lambda_h(\omega))\alpha_h^1. \tag{A.20}$$

What this means − and this is the main advantage of assuming log-linear utility − is that, for all practical purposes, we can ignore the FOC's (A.15)-(A.16) as well as the MCC (A.7): the information these equations contain concerning the household's goods consumption can easily be recovered from the system of equations consisting of (A.3)-(A.6), (A.8), and the spot goods price equations (A.18) and (A.20) (SGP's).

It will be very convenient to record this fact formally, but only after first introducing two additional modifications, (i) substituting, in the appropriate places, for the Lagrange multipliers $\lambda_h(\omega)$ the so-called *stochastic weights*

$$\eta_h(\omega) = \beta_h/\lambda_h(\omega),$$

and (ii) substituting, in the the NAC's (A.3)-(A.4) for $h < H$, for the asset prices q defined by the NAC's (A.3)-(A.4) for $h = H$.

All this manipulation and consequent simplification then leaves us with what we only half-jokingly refer to as *The True Equations* (TTE).

Spot goods prices

$$p(0) = \sum_h \eta_h(0)(\alpha_h^0/\beta_h). \tag{A.21}$$

and

$$p(\omega) = \pi(\omega)\sum_h \eta_h(\omega)\alpha_h^1; \tag{A.22}$$

No arbitrage conditions (for $h < H$)

$$\sum_{\omega>0}(\eta_h(0)/\eta_h(\omega) - \eta_H(0)/\eta_H(\omega))p(\omega)\Delta_b(\omega) = 0 \tag{A.23}$$

and

$$\sum_{\omega>0}(\eta_h(0)/\eta_h(\omega) - \eta_H(0)/\eta_H(\omega))p(\omega) = 0; \tag{A.24}$$

Budget constraints (for $h < H$)

$$(1 + 1/\beta_h) - \sum_{\omega}(1/\eta_h(\omega))p(\omega)[\Delta_b(\omega)I](b_h^0, s_h^0) = 0 \tag{A.25}$$

and

$$\pi(\omega)\eta_h(\omega) - p(\omega)[\Delta_b(\omega)I](b_h^1, s_h^1) = 0; \tag{A.26}$$

Asset market clearing conditions

$$\sum_h (b_h^1, s_h^1) - (0, \mathbf{1}) = 0. \tag{A.27}$$

Finally, we will now find it much more useful to normalize prices according to the formulas

$$\sum_h \eta_h(\omega) = 1. \tag{A.28}$$

Remarks:

1. In deriving (A.25)-(A.26) we also used (A.9), (A.17), and (A.19).

2. The stochastic weights $\eta_h(\omega)$ owe their name to the fact that the FOC's (A.15)-(A.16) can be derived from the social welfare/social planner's problem of maximizing a fictitious representative agent's utility function of the form

$$\sum_h [\eta_h(0) \sum_g (\alpha_h^{0g}/\beta_h) \log c_h^g(0) + \sum_{\omega>0} \pi(\omega)\eta_h(\omega) \sum_g \alpha_h^{1g} \log c_h^g(\omega)]$$

   subject to feasibility of goods allocation (with associated multipliers p). Note that this fact implies that, for goods allocation to be Pareto optimal, it must be the case that

$$\eta_h(\omega) = \eta_h,$$

   which in turn implies that it must be the case that a FE is a PFE.

3. TTE preserve consistency of equations and variables. For this system of equations there are (at most)

$$\begin{aligned} K &= G(\Omega + 1) + (H - 1)(\tilde{G} + G) + (H - 1)(\Omega + 1) + (\tilde{G} + G) + (\Omega + 1) \\ &= H(\tilde{G} + G) + H(\Omega + 1) + G(\Omega + 1) \end{aligned}$$

   independent equations in the K independent variables

$$(b_h^1, s_h^1), \eta_h, \text{ and } p.$$

A.4 Proof of Proposition 2 (and Corollary)

Obvious. ■

A.5 Proof of Proposition 3

For the leading example, in terms of just $\bar{e}_1$ (so that, by definition, $\bar{e}_2 = \mathbf{1} - \bar{e}_1$),

$$\{\bar{e}_1 \in \mathbb{R}^4 : 0 \ll \bar{e}_1 \ll \mathbf{1}\} \subset \bar{E} \subset \mathbb{R}^4,$$

that is, $\bar{E}$ is a full-dimensional subset of $\mathbb{R}^4$. On the other hand,

$$\begin{aligned}\bar{E}_\Delta \quad \subset \quad & \{\bar{e}_1 \in \bar{E} : \text{for some } (b_1^0, s_1^0), \bar{e}_1 = \\ & (\delta^1(0)b_1^0 + s_1^{01}, s_1^{02}, \sum_{\omega>0} \pi(\omega)\delta^1(\omega)b_1^0 + s_1^{01}, s_1^{02})\},\end{aligned}$$

that is, (given $\pi(\omega)$) generically in $\delta^1(\omega)$, all ω, $\bar{E}_\Delta$ is a full-dimensional subset of a 3-dimensional linear subspace in $\mathbb{R}^4$ (noting that, necessarily, $\bar{e}_1^{12} = \bar{e}_1^{02}$).

This said, in order to check for uniqueness of CE in terms of $\bar{e}_1$, we only need to consider solutions to Ms. 1's certainty BC (in terms of *just* her constant stochastic weight $0 < \eta_1 < 1$; see below), but given certainty endowments in the lower-dimensional subset $\bar{E}_\Delta$.

To formalize this problem, we begin by observing that the analogues of TTE in the certainty economy are identical to (A.21)-(A.28) when $\Omega = 1, G = H = 2$, and $\tilde{G} = 0$ (setting, say, $s_1^{12} = s_1^{02}$) after making appropriate changes in notation (replacing $p(0)$ with $\bar{p}^1, s_1^0$ with $\bar{e}_1^0$, and so on). Hence, after substituting from the SGP equations (A.21)-(A.22) into the BC (A.25) for $h = 1$, and also setting, for convenience, $0 < \eta_1^t = \eta < 1$ and $\eta_2^t = 1 - \eta_1^t = 1 - \eta, t = 0, 1$, finally we find that the question of *non*uniqueness of CE, and a fortiori, PPE boils down to this: when does the linear equation, for $\bar{e}_1 \in \bar{E}_\Delta$,

$$\eta(1 + 1/\beta) - [\eta(\alpha_1^0/\beta_1) + (1 - \eta)(\alpha_2^0/\beta_2)]\bar{e}_1^0 + [\eta\alpha_1^1 + (1 - \eta)\alpha_2^1]\bar{e}_1^1 = 0 \tag{A.29}$$

admit every $0 < \eta < 1$ as a solution? But this will be the case if and only if the pair of equations

$$(\alpha_1^0/\beta_1 - \alpha_2^0/\beta_2)(\bar{e}_1^{01}, \bar{e}_1^{02}) + (\alpha_1^1 - \alpha_2^1)(\bar{e}_1^{11}, \bar{e}_1^{02}) - (1 + 1/\beta) = 0 \tag{A.30}$$

and

$$(\alpha_2^0/\beta_2)(\bar{e}_1^{01}, \bar{e}_1^{02}) + \alpha_2^1(\bar{e}_1^{11}, \bar{e}_1^{02}) = 0 \tag{A.31}$$

(together with the identity $\bar{e}_1^{12} = \bar{e}_1^{02}$) has a solution in $\bar{E}_\Delta$. Since (A.31) but not (A.30) is a homogeneous equation, this is possible only if

$$\text{rank}\begin{bmatrix} \alpha_1^{01}/\beta_1 - \alpha_2^{01}/\beta_2 & \alpha_1^{11} - \alpha_2^{11} & (\alpha_1^{02}/\beta_1 + \alpha_1^{12}) - (\alpha_2^{02}/\beta_2 + \alpha_2^{12}) \\ \alpha_2^{01}/\beta_2 & \alpha_2^{11} & \alpha_2^{02}/\beta_2 + \alpha_2^{12} \end{bmatrix} = 2,$$

that is, only if $\alpha_2^t \neq \alpha_1^t$, some t, or $\beta_2 \neq \beta_1$. The set of such solutions then defines the line segment $\bar{L}_\Delta \subset \bar{E}_\Delta$.[11] Note that, because the coefficients in (A.31) are all positive, any solution must have both positive and negative elements. ■

[11] Of course, there may be no solutions to (A.30)-(A.31) in $\bar{E}_\Delta$, as in the example depicted in subsection 3.4 when there is no redundant bond. However, for the leading example, it is easy to find parameter values for which there are solutions (using the analogues of TTE and the degrees of freedom afforded in choosing $\delta^1(\omega)$, all ω). It is also worth pointing out that (A.29) can also be exploited to give a precise description of $\bar{E}_\Delta$.

A.6 Exclusivity of PFE

A.6.1 Proof of Proposition 4

We want to show that the only solutions to TTE satisfy

$$\eta_h(\omega) = \eta_h(0), \quad \omega > 0, \text{ all } h. \tag{A.32}$$

Consider just the NAC's (A.23)-(A.24) together with the second period BC's (A.26) and the price normalization (A.28). Multiply (A.26) (after replacing h by h') by

$$(\eta_h(0)/\eta_h(\omega) - \eta_H(0)/\eta_H(\omega)),$$

sum over $\omega > 0$, and use (A.23)-(A.24) to simplify, which yields the equations

$$\sum_{\omega>0}(\eta_h(0)/\eta_h(\omega) - \eta_H(0)/\eta_H(\omega))\eta_{h'}(\omega)\pi(\omega) = 0, \quad h < H, \ h' < H. \tag{A.33}$$

But from the NAC (A.24) for $g = 1$ together with the SGP equation (A.22) for $g = 1$ (again after replacing h with h') and (A.33) it also follows that

$$\sum_{\omega>0}(\eta_h(0)/\eta_h(\omega) - \eta_H(0)/\eta_H(\omega))\eta_H(\omega)\pi(\omega) = 0, \quad h < H. \tag{A.34}$$

Focusing on (A.33) for just $h' = h$ (fixed) and (A.34) then yields the equations

$$\sum_{\omega>0}(\eta_h(\omega)/\eta_H(\omega))\pi(\omega) = \eta_h(0)/\eta_H(0) \tag{A.35}$$

and

$$\sum_{\omega>0}(\eta_H(\omega)/\eta_h(\omega))\pi(\omega) = \eta_H(0)/\eta_h(0). \tag{A.36}$$

So now letting $t(\omega) = \eta_h(\omega)/\eta_H(\omega)$, and defining $f(x) = 1/x$, for $x > 0$, a strictly convex function, (A.35)-(A.36) can be rewritten

$$f[\sum_{\omega>0} t(\omega)\pi(\omega)] = f[t(0)]$$

and

$$\sum_{\omega>0} f[t(\omega)]\pi(\omega) = f[t(0];$$

because of Jensen's inequality, this can only be true if

$$t(\omega) = t(0), \omega > 0,$$

or (because h is arbitrary)

$$\eta_h(\omega)/\eta_h(0) = \eta_H(\omega)/\eta_H(0), \quad \omega > 0, \ h < H. \tag{A.37}$$

Finally, to see that (A.37) implies (A.32), suppose that

$$\eta_h(\omega) = \theta(\omega)\eta_h(0) \text{ with } \theta(\omega) > 0, \quad \omega > 0, \text{ all } h.$$

Then (A.28) implies that

$$1 = \sum_h \eta_h(\omega) = \theta(\omega) \sum_h \eta_h(0) = \theta(\omega), \quad \omega > 0,$$

and the proof is complete. ■

A.6.1 Proof of the Corollary to Proposition 4

Since household H can transact freely in every stock, the relevant NAC's imply that (A.33) also obtains in this model. The rest of the argument is then identical to the foregoing. ■

A.6.2 Proof of the Cautionary to Proposition 4

This is detailed separately in Appendix B. ■

A.7 Continuous Time

A.7.1 Proof of Proposition 5

Step 1. We first show that if $(\xi,\, p,\, q,\, c_h,\, b_h,\, s_h,\, h = 1,\, 2)$ is an equilibrium in the model, then the stocks represent the same investment opportunity.

Suppose there exists an equilibrium where none of the risky stocks is redundant. That is, each agent faces an investment opportunity set represented by (4.7) such that the volatility matrix $\Sigma(t) \equiv \begin{pmatrix} \sigma_s^1(t) \\ \sigma_s^2(t) + \sigma_p(t) \end{pmatrix}$ is invertible. Then then households face complete markets, and hence the martingale representation approach of Cox and Huang [8] and Karatzas, Lehoczky and Shreve [13] is applicable.

Household h maximizes (4.6) subject to (4.8) with $\xi_h(t) = \xi(t)$, $\forall h$. Its first-order conditions are given by

$$\alpha_h^1 e^{-\rho_h t}/c_h^1(t) = y_h \xi(t), \tag{A.38}$$

$$\alpha_h^2 e^{-\rho_h t}/c_h^2(t) = y_h \xi(t) p(t), \tag{A.39}$$

where y_h is the multiplier associated with (4.8). This together with goods market clearing (4.9) implies

$$\xi(t) = \frac{\alpha_1^1 e^{-\rho_1 t}/y_1 + \alpha_2^1 e^{-\rho_2 t}/y_2}{\delta^1(t)},$$

$$p(t) = \frac{\alpha_1^2 e^{-\rho_1 t}/y_1 + \alpha_2^2 e^{-\rho_2 t}/y_2}{\alpha_1^1 e^{-\rho_1 t}/y_1 + \alpha_2^1 e^{-\rho_2 t}/y_2} \, \frac{\delta^1(t)}{\delta^2(t)},$$

Making the standard identification with the weights in the Representative Agent in the economy $\eta = 1/y_1$, $1 - \eta = 1/y_2$, we obtain expressions (4.10) and (4.11). Applying Itô's lemma to the above to identify the dynamics of the relative price process $p(t)$, we derive the following for the volatility of $p(t)$:

$$\sigma_p(t) = \sigma_\delta^1(t) - \sigma_\delta^2(t). \tag{A.40}$$

The no-arbitrage condition for the stock prices yields

$$q_s^1(t) = \frac{1}{\xi(t)} E\left[\int_0^T \xi(s)\, \delta^1(s)\, ds \,|\mathcal{F}_t\right] \quad \text{and} \quad q_s^2(t) = \frac{1}{\xi(t)} E\left[\int_0^T \xi(s) p(s)\, \delta^2(s)\, ds \,|\mathcal{F}_t\right]. \tag{A.41}$$

Upon substitution of (4.10) and (4.11) in the above, we explicitly evaluate the conditional expectations to yield (4.15)-(4.16). Expressions for the volatilities of the stock prices are then obtained by applying Itô's lemma to (4.15)-(4.16):

$$\sigma_s^1(t) = \sigma_\delta^1(t), \quad \sigma_s^2(t) = \sigma_\delta^2(t) + \sigma_p(t).$$

This together with (A.40) implies

$$\Sigma(t) = \begin{pmatrix} \sigma_\delta^1(t) \\ \sigma_\delta^1(t) \end{pmatrix}.$$

The volatility matrix $\Sigma(t)$ is not invertible, yielding the desired contradiction.

Step 2. Since there are no equilibria in the model in which $\Sigma(t)$ is invertible, we concentrate on the only remaining possibility for an equilibium: the one in which the two stocks represent the same investment opportunity and hence financial markets are incomplete.

Since one of $(q_s^1,\, q_s^2)$ is redundant, define a composite security, q_S, paying out in good 1. Households' trading strategies for investing in individual securities are indeterminate, however the position in the composite security (consisting of one share of both stocks) would be uniquely identified. The composite security has dynamics

$$dq_S(t) + (\delta^1(t) + p(t)\delta^2(t))dt = q_S[\mu_S(t)dt + \sigma_S(t)dw(t)].$$

In the remainder of the proof, consider an incomplete market $(q_b^1,\, q_S)$.

The first-order conditions to the optimization problem (4.6) subject to (4.8) are

$$\alpha_h^1 e^{-\rho_h t}/c_h^1(t) = y_h \xi_h(t), \tag{A.42}$$

$$\alpha_h^2 e^{-\rho_h t}/c_h^2(t) = y_h \xi_h(t) p(t), \tag{A.43}$$

where the state-price density ξ_h that household h will be facing in equilibrium is now personalized. At the optimum,

$$W_h(t) = \frac{1}{\xi_h(t)} E\left[\int_t^T \left(\xi_h(s) c_h^1(s) + \xi_h(s) p(s) c_h^2(s)\right) ds \,|\, \mathcal{F}_t\right].$$

Substituting in the first-order conditions (A.42)–(A.43), we have

$$\begin{aligned} W_h(t) &= \frac{1}{\xi_h(t)} E\left[\int_t^T \left(\frac{\alpha_h^1 e^{-\rho_h s}}{y_h} + \frac{\alpha_h^2 e^{-\rho_h s}}{y_h}\right) ds \,|\, \mathcal{F}_t\right] \\ &= \frac{e^{-\rho_h t} - e^{-\rho_h T}}{y_h\, \rho_h\, \xi_h(t)}. \end{aligned} \tag{A.44}$$

Hence

$$c_h^1(t) = \frac{\rho_h \alpha_h^1 e^{-\rho_h t}}{e^{-\rho_h t} - e^{-\rho_h T}} \, W_h(t) \,, \tag{A.45}$$

$$c_h^2(t) = \frac{\rho_h \alpha_h^2 e^{-\rho_h t}}{(e^{-\rho_h t} - e^{-\rho_h T}) \, p(t)} \, W_h(t) \,. \tag{A.46}$$

The dynamic budget constraint that household h is facing is similar to (4.7), except that now there is a single composite risky security available for investment

$$dW_h(t) = W_h(t) r^1(t) dt - (c_h^1(t) + p(t) c_h^2(t)) dt + s_h(t)(\mu_S(t) - r^1(t)) dt + s_h(t) \sigma_S(t) dw(t) \,.$$

This combined with (A.45)–(A.46) gives

$$dW_h(t) = W_h(t)[r^1(t) - \frac{\rho_h e^{-\rho_h t}}{e^{-\rho_h t} - e^{-\rho_h T}} + \phi_h(t)(\mu_S(t) - r^1(t))] dt + W_h(t) \phi_h(t) \sigma_S(t) dw(t) \,, \tag{A.47}$$

where ϕ_h denotes the proportion of the household's wealth invested in the composite security. Solving the above stochastic differential equation for $\ln W_h(t)$, we obtain

$$\begin{aligned} \ln W_h(t) = \ln W_h(0) &+ \int_0^t [r^1(s) - \frac{\rho_h e^{-\rho_h s}}{e^{-\rho_h s} - e^{-\rho_h T}} + \phi_h(s)(\mu_S(s) - r^1(s)) - \frac{1}{2} |\phi_h(s) \sigma_S(s)|^2] \, ds \\ &+ \int_0^t \phi_h(s) \sigma_S(s) dw(s) \,. \end{aligned} \tag{A.48}$$

Household h is solving

$$\begin{aligned} &\max_{c,\,\phi} E \int_0^T e^{-\rho_h t} [\alpha_h^1 \ln c_h^1(t) + \alpha_h^2 \ln c_h^2(t)] dt \\ &= E \int_0^T e^{-\rho_h t} [\ln \frac{\rho_h e^{-\rho_h t}}{e^{-\rho_h t} - e^{-\rho_h T}} + \alpha_h^1 \ln \alpha_h^1 + \alpha_h^2 \ln \alpha_h^2 - \alpha_h^2 \, p(t) + (\alpha_h^1 + \alpha_h^2) \ln W_h(0) - \alpha_h^2 p(t) \\ &\quad + r^1(s) - \frac{\rho_h e^{-\rho_h s}}{e^{-\rho_h s} - e^{-\rho_h T}} + \phi_h(s)(\mu_S(s) - r^1(s)) - \frac{1}{2} |\phi_h(s) \sigma_S(s)|^2] dt \,, \end{aligned}$$

where we made use of (A.45)–(A.46) in the first equality and (A.48) in the second. Since $W_h(0)$, p and r^1 are taken as given by a household, the optimization problem of solving for the trading strategies becomes a pointwise problem

$$\max_{\phi_h(t)} \phi_h(t)(\mu_S(t) - r^1(t)) - \frac{1}{2} |\phi_h(t) \sigma_S(t)|^2$$

yielding at the optimum

$$\phi_h(t) = (\sigma_S(t) \sigma_S^\top(t))^{-1} (\mu_S(t) - r^1(t)). \tag{A.49}$$

Note that the proportion of wealth invested in the composite security is identical for both households.

We are now ready to show that the state-price densities driving the investment opportunity sets of the two households are identical. It is given by (A.44) that

$$\xi_h(t) = \frac{e^{-\rho_h t} - e^{-\rho_h T}}{y_h \rho_h W_h(t)}.$$

Parameterizing the household-specific state-price density in the standard fashion by the interest rate r_h^1 and market price of risk θ_h^1 processes, applying Itô's lemma to both sides of this equality and simplifying we have

$$\begin{aligned}
\xi_h(t)[-r_h^1(t)dt - \theta_h^1(t)dw(t)] &= -\frac{e^{-\rho_h t}}{y_h W_h(t)} \\
&- \frac{e^{-\rho_h t} - e^{-\rho_h T}}{y_h \rho_h W_h(t)}[r^1(t) - \frac{\rho_h e^{-\rho_h t}}{e^{-\rho_h t} - e^{-\rho_h T}} + \phi_h(t)(\mu_S(t) - r^1(t))]dt \\
&- \frac{e^{-\rho_h t} - e^{-\rho_h T}}{y_h \rho_h W_h(t)}\phi_h(t)\sigma_S(t)dw(t) + \frac{e^{-\rho_h t} - e^{-\rho_h T}}{y_h \rho_h W_h(t)}|\phi_h(t)\sigma_S(t)|^2 dt \\
= - \xi_h(t)[r^1(t)dt - \phi(t)\sigma_S(t)dw(t)]\,,
\end{aligned}$$

where we used (A.47) and (A.49). So, $r_h^1(t) = r^1(t)$ and $\theta_h^1(t) = \phi(t)\sigma_S(t)$, $\forall\, h = 1, 2$. The two households face identical state-prices densities, hence markets are effectively complete, the weight η in the representative agent is constant, and a Pareto optimal allocation obtains.

We can then proceed with the same derivations as in Step 1 of this proof to derive (4.10)–(4.11) and then the no-arbitrage prices of redundant securities (4.15)–(4.16) from (A.41). (4.12)–(4.13) follow from (A.38)–(A.39) combined with (4.10)–(4.11). The constant η reflects an initial allocation of wealth and is determined from either household's static budget constraint with equilibrium quantities substituted in. Finally, to determine the interest rate, we apply Itô's lemma to (4.10) yielding

$$d\xi(t) = \xi(t)[(-\mu_\delta^1(t) - \frac{\alpha_1^1\,\eta\,\rho_1 e^{-\rho_1 t} + \alpha_2^1(1-\eta)\rho_2 e^{-\rho_2 t}}{\alpha_1^1\,\eta e^{-\rho_1 t} + \alpha_2^1(1-\eta)e^{-\rho_2 t}} + |\sigma_\delta^1(t)|^2)dt - \sigma_\delta^1(t)dw(t)]$$

and identify the negative of the drift term with the interest rate in the economy. ■

A.7.2 Proof of Proposition 6

The weight η is determined from either household's budget constraint with optimal quantities substituted in, e.g., household 1's:

$$E\left[\int_0^T \xi(t)[c_1^1(t) + p(t)c_1^2(t)]dt\right] = E\left[\int_0^T \xi(t)[e_1^1(t) + p(t)e_1^2(t)]dt\right].$$

Substituting (A.38)–(A.39) and (4.10)–(4.11), we have

$$\begin{aligned}
E\left[\int_0^T e^{-\rho_1 t}\left(\frac{\alpha_1^1}{y_1} + \frac{\alpha_1^2}{y_1}\right)dt\right] &= E\left[\int_0^T \frac{\alpha_1^1\,\eta\, e^{-\rho_1 t} + \alpha_2^1(1-\eta)e^{-\rho_2 t}}{\delta_1(t)} e_1^1(t)dt\right] \\
&+ E\left[\int_0^T \frac{\alpha_1^2\,\eta\, e^{-\rho_1 t} + \alpha_2^2(1-\eta)e^{-\rho_2 t}}{\delta_2(t)} e_1^2(t)dt\right].
\end{aligned}$$

Rearranging and using $\eta = 1/y_1$, we arrive at

$$\frac{1-e^{-\rho_1 T}}{\rho_1} = E\left[\int_0^T e^{-\rho_1 t}\alpha_1^1 \frac{e_1^1(t)}{\delta^1(t)}dt\right] + \frac{1-\eta}{\eta}E\left[\int_0^T e^{-\rho_2 t}\alpha_2^1 \frac{e_1^1(t)}{\delta^1(t)}dt\right]$$
$$+E\left[\int_0^T e^{-\rho_1 t}\alpha_1^2 \frac{e_1^2(t)}{\delta^2(t)}dt\right] + \frac{1-\eta}{\eta}E\left[\int_0^T e^{-\rho_2 t}\alpha_2^1 \frac{e_1^2(t)}{\delta^2(t)}dt\right]. \qquad \text{(A.50)}$$

Due to (4.17) the sum of the first and third terms on the right-hand side of the last expression is $\frac{1-e^{-\rho_1 T}}{\rho_1}$, while the sum of the second and fourth is zero due to (4.18). Hence (A.50) is satisfied $\forall\, \eta \in (0,1)$. ■

A.7.3 Proof of Proposition 7

Obvious. ■

Appendix B: An Example of an OFE with a Portfolio Constraint

B.1 Further Simplification of TTE

B.1.1 For H>1

$(b_H^1,\, s_H^1)$ only appear in (A.27), so the latter can be used to define the former.

B.1.2 For H=2

We can use (A.28) to substitute

$$\eta_1(\omega) = \eta(\omega) \text{ and } \eta_2(\omega) = 1 - \eta_1(\omega) = 1 - \eta(\omega), \quad \text{all } \omega$$

into (A.21)-(A.26). This permits rewriting

$$\begin{aligned}
&\eta_1(0)/\eta_1(\omega) - \eta_2(0)/\eta_2(\omega) = \\
&\eta(0)/\eta(\omega) - (1-\eta(0))/(1-\eta(\omega)) = \\
&\frac{\eta(0)(1-\eta(\omega)) - (1-\eta(0))\eta(\omega)}{\eta(\omega)(1-\eta(\omega))} = \\
&\frac{\eta(0)-\eta(\omega)}{1-\eta(\omega)}\frac{1}{\eta(\omega)}
\end{aligned}$$

and thereby simplifying (A.23)-(A.24), which in turn permits "renormalizing" $p(\omega)/\eta(\omega) \to p(\omega)$

in (A.21)-(A.26), resulting finally in the system of equations

$$\alpha_1^0/\beta_1 + \frac{1-\eta(0)}{\eta(0)}\alpha_2^0/\beta_2 - p(0) = 0, \tag{B.1}$$

$$\pi(\omega)\alpha_1^1 + \frac{1-\eta(\omega)}{\eta(\omega)}\pi(\omega)\alpha_2^1 - p(\omega) = 0, \quad \omega > 0, \tag{B.2}$$

$$\sum_{\omega>0}\left(\frac{\eta(0)-\eta(\omega)}{1-\eta(\omega)}\right)p^1(\omega)/\delta^1(\omega) = 0 \tag{B.3}$$

$$\sum_{\omega>0}\left(\frac{\eta(0)-\eta(\omega)}{1-\eta(\omega)}\right)p^g(\omega) = 0, \quad g = 1,2, \tag{B.4$'$}$$

$$(1+1/\beta_1) - \sum_{\omega>0}(p^1(\omega)/\delta^1(\omega),\, p(\omega))(b_1^0,\, s_1^0) = 0\,, \quad \text{and} \tag{B.5$'$}$$

$$1 - ((p^1(\omega)/\delta^1(\omega),\, p(\omega))/\pi(\omega))(b_1^1,\, s_1^1) = 0\,, \quad \omega > 0. \tag{B.6}$$

Note: The yield matrix has become

$$Y = [\eta(\omega)(p^1(\omega)/\delta^1(\omega),\, p(\omega)),\, \omega > 0],$$

which, since $\pi(\omega) > 0$, $\eta(\omega) > 0$, $\omega > 0$, necessarily has the same rank as the matrix which now defines Ms. 1's second period budget constraints

$$[(p^1(\omega)\delta^1(\omega),\, p(\omega))/\pi(\omega),\, \omega > 0].$$

B.2 The Leading Example When Ms. 1 Faces an Arbitrary Constraint on Transacting in Stock 2

Let $\phi : \mathbb{R} \to \mathbb{R}$ be differentiable and strictly quasi-concave, and add a constraint of the form $\phi(s_1^{12}) \geq 0$ to Ms. 1's optimization problem.

Note: There is an issue as to whether this requires − for logical consistency in describing the economy − adding the requirement for initial portfolios $\phi(s_1^{02}) \geq 0$. For simplicity, we sidestep this issue here, and permit $s_1^{02} \in \mathbb{R}$.

Note: It will be established that such a constraint can only be (effectively) binding at s_1^{12} if $s_1^{12} \notin [0,1]$.

Let $\theta \geq 0$ be the multiplier associated with this constraint. If it is binding in a FE, then two changes are required of TTE.

Note: Bear in mind that, in terms of the original extended system of equations,

$$\eta_h(\omega) = \beta_h/\lambda_h(\omega),$$

and that, for present purposes, we have substituted $p(\omega)$ for $p(\omega)/\eta(\omega)$.

(i) The NAC's (B.4′) become

$$\sum_{\omega>0}\left(\frac{\eta(0)-\eta(\omega)}{1-\eta(\omega)}\right)p^g(\omega) = \begin{cases} 0, & \text{if } g = 1, \\ -\mu\eta(0), & \text{if } g = 2, \quad \text{and} \end{cases} \tag{B.4}$$

(ii) Ms. 1's first period BC (B.5′) becomes

$$(1+1/\beta_1) - \sum_{\omega>0}(p^1(\omega)/\delta^1(\omega),\, p(\omega))(b_1^0,\, s_1^0) - \mu(s_1^{12} - s_1^{02}) = 0\,, \tag{B.5}$$

where μ is defined by

$$\beta_1\mu = \theta D\phi(s_1^{12}).$$

This suggests the following approach to constructing an OFE. First, observing that neither μ nor $p(0)$ appears in the system consisting of (B.2), (B.3), (B.4) for $g = 1$, and (B.6) (say, *System **I***, use System **I**, first, to construct values of the variables $p(\omega)$ and $\eta(\omega) \neq \eta(0),\ \omega > 0$ such that

$$\text{rank}[(p^1(\omega)\delta^1(\omega),\, p(\omega))/\pi(\omega)] = 3,$$

and then construct $(b_1^1,\, s_1^1)$. Second, given values for these variables, use (B.1), (B.4) for $g = 2$, and (B.5) (say, *System* **II**) to construct values of the remaining variables $p(0)$ and μ (which, by Proposition 4, is necessarily nonzero). Finally, simply specify any ϕ such that $\phi(s_1^{12}) = 0$ and $\text{sign}\, D\phi(s_1^{12}) = \text{sign}\,\mu$.

Notice that, assuming that s_1^{12} has been fixed by $\phi(s_1^{12}) = 0$, the system (B.1)-(B.6) consists of 15 equations in the 15 variables

$$p(\omega),\ \text{all}\ \omega,\ \eta(\omega),\ \text{all}\ \omega,\ \mu,\ b_1^1\ \text{and}\ s_1^{11},$$

given the 12 independent parameters

$$0 < \alpha_h^{t0},\ \alpha_h^{t1} = 1 - \alpha_h^{t0} < 1,\ t = 0,\, 1,\ \beta_h > 0,\ h = 1, 2,\ \delta^1(\omega) > 0,\ \text{all}\ \omega,$$
$$\text{and}\ 0 < \pi(1),\ \pi(2),\ \pi(3) = 1 - \pi(1) - \pi(2) < 1.$$

B.3 Results

Claim 1. *If (B.2), (B.3), (B.4) and (B.6) has a solution in which $\eta(\omega) \neq \eta(0)$, (some) $\omega > 0$, then $s_1^{12} \notin [0, 1]$.*

Claim 2. *There is an economy for which (B.1)-(B.6) has a solution satisfying $\eta(\omega) \neq \eta(0)$, $\omega > 0$ (i.e., and OFE for an appropriately specified portfolio constraint function ϕ), as well as a unique solution satisfying $\eta(\omega) = \eta(0)$, $\omega > 0$ (i.e., a unique PFE).*

Proof of Claim 1. Let

$$\psi_h = \begin{cases} \sum_{\omega>0} \frac{\eta(0)-\eta(\omega)}{1-\eta(\omega)}\,\pi(\omega), & \text{if}\ h = 1, \\ \sum_{\omega>0} \frac{\eta(0)-\eta(\omega)}{\eta(\omega)}\,\pi(\omega), & \text{if}\ h = 2. \end{cases}$$

Then substituting from (B.2) into (B.4) yields the system

$$\alpha_1^{11}\psi_1 + \alpha_2^{11}\psi_2 = 0,$$
$$(1-\alpha_1^{11})\psi_1 + (1-\alpha_2^{11})\psi_2 = -\mu\,\eta(0),$$

which can only have a solution with $\mu \neq 0$ if $\alpha_2^{11} \neq \alpha_1^{11}$, and has the solution

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \frac{1}{\alpha_1^{11} - \alpha_2^{11}} \begin{pmatrix} -\alpha_2^{11}\mu\eta(0) \\ \alpha_1^{11}\mu\eta(0) \end{pmatrix}$$

if $\alpha_2^1 \neq \alpha_1^1$. Hence,

$$\mu \neq 0 \;\Rightarrow\; \operatorname{sign}\psi_2 = -\operatorname{sign}\psi_1 \neq 0. \tag{B.7}$$

Note: Here it is useful to recall that the analogue of (B.6) also holds for $h = 2$:

$$\frac{1-\eta(\omega)}{\eta(\omega)} - ((p^1(\omega)\delta^1(\omega),\, p(\omega))/\pi(\omega))(b_2^1,\, s_2^1) = 0,\; \omega > 0. \tag{B.6$'$}$$

(This can be inferred from (B.2) and (B.6) together with the identity $(b_2^1,\, s_2^1) = (-b_1^1,\, 1 - s_1^1)$, bearing in mind our having "renormalized" $p(\omega)/\eta(\omega) \to p(\omega)$ earlier.)
Now multiplying each of the equations in (B.6) and (B.6$'$) by

$$\frac{\eta(0) - \eta(\omega)}{1 - \eta(\omega)}\pi(\omega),$$

summing over $\omega > 0$, and using (B.3)-(B.4) yields the property

$$\psi_h = \begin{cases} \mu\eta(0)s_1^{12}, & \text{if } h = 1, \\ \mu\eta(0)s_2^{12}, & \text{if } h = 2. \end{cases} \tag{B.8}$$

Since we have the identity $s_2^{12} = 1 - s_1^{12}$, both (B.7) and (B.8) can obtain with $\eta_1(\omega) \neq \eta(0)$, (some) $\omega > 0$, ($\Rightarrow\; \mu \neq 0$) only if $s_1^{12} \notin [0, 1]$. ■

Proof of Claim 2. This merely requires displaying an example. See the following calculation. ■

B.4 Calculation

Step 1. Existence of a solution to System I in which $\eta(\omega) \neq \eta(0),\ \omega > 0$.

- **Parameter Values**

  $\alpha_1^{11} = 1/3,\ \alpha_2^{11} = 2/3,$
  $\delta^1(1) = 1,\ \delta^1(2) = 1,\ \delta^1(3) = 5/3,$
  $\pi(1) = 5/12,\ \pi(2) = 4/12,\ \pi(3) = 3/12.$

- **Proposed Solution**

  For simplicity we set $\eta(2) = \eta(3) \;\Rightarrow\; p(2)/\pi(2) = p(3)/\pi(3)$, i.e., we impose identical spot market equilibrium for $\omega = 2, 3$.

$$p(1)/\pi(1) = (7/3,\ 5/3),\ p(2)/\pi(2) = p(3)/\pi(3) = (5/9,\ 5/9).$$

$$\eta(0) = 1/2,\ \eta(1) = 1/4,\ \eta(2) = \eta(3) = 3/4.$$

- **Checking the solution for (B.2), (B.3), and (B.4),** $g = 1$

For (B.2):

$$p(1)/\pi(1) = (\alpha_1^{11},\, 1-\alpha_1^{11}) + \frac{1-\eta(1)}{\eta(1)}(\alpha_2^{11},\, 1-\alpha_2^{11}) = (1/3,\, 2/3) + \frac{3/4}{1/4}(2/3,\, 1/3) = (7/3,\, 5/3)$$

$$p(2)/\pi(2) = p(3)/\pi(3) = (\alpha_1^{11},\, 1-\alpha_1^{11}) + \frac{1-\eta(2)}{\eta(2)}(\alpha_2^{11},\, 1-\alpha_2^{11}) = (5/9,\, 7/9)$$

For (B.3):

$$\begin{aligned}\sum_{\omega>0}\frac{\eta(0)-\eta(\omega)}{1-\eta(\omega)}p^1(\omega)/\delta^1(\omega) &= \frac{1/4}{3/4}(5/12)(7/3) - \frac{1/4}{1/4}(4/12)(5/9)(1/2) - \frac{1/4}{1/4}(3/12)(5/9)(5/3)\\ &= 0\end{aligned}$$

For (B.4), $g = 1$:

$$\begin{aligned}\sum_{\omega>0}\frac{\eta(0)-\eta(\omega)}{1-\eta(\omega)}p^1(\omega) &= \frac{1/4}{3/4}(5/12)(7/3) - \frac{1/4}{1/4}(4/12)(5/9) - \frac{1/4}{1/4}(3/12)(5/9)\\ &= 0\end{aligned}$$

- **Verifying full rank of the yield matrix, i.e., the solution for (B.6)**

$$\begin{aligned}&[(p^1(\omega)/\delta^1(\omega),\, p^1(\omega),\, p^2(\omega))/\pi(\omega),\, \omega > 0]\\ &= \begin{bmatrix} 7/3 & 7/3 & 5/3 \\ 5/18 & 5/9 & 7/9 \\ 25/3 & 5/9 & 7/9 \end{bmatrix}\\ &= \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/9 & 0 \\ 0 & 0 & 1/9 \end{bmatrix}\begin{bmatrix} 7 & 7 & 5 \\ 5/2 & 5 & 7 \\ 25/3 & 5 & 7 \end{bmatrix}\end{aligned}$$

Consider the system

$$7v_1 + 7v_2 + 5v_3 = 0, \tag{B.9}$$

$$(5/2)v_1 + 5v_2 + 7v_3 = 0, \text{ and} \tag{B.10}$$

$$(25/3)v_1 + 5v_2 + 7v_3 = 0. \tag{B.11}$$

We know that Y will have full rank if and only if the only solution to (B.9)-(B.11) is $v_1 = v_2 = v_3 = 0$. However

$$\text{(B.10) and (B.11)} \;\Rightarrow\; v_1 = 0,\; v_3 = -\,(5/7)v_2$$

while

$$v_1 = 0 \text{ and (B.9)} \;\Rightarrow\; v_3 = -\,(7/5)v_2,$$

which are consistent only if $v_2 = v_3 = 0$ as well.

- **Checking that the solution to**

$$[(p(\omega)/\delta^1(\omega),\, p(\omega))/\pi(\omega),\ \omega > 0](b_1^1,\, s_1^1) = 1 \tag{B.12}$$

satisfies $s_1^{12} \notin [0,1]$

(B.12) becomes (using $(b_1^1,\, s_1^{11},\, s_1^{12}) \to (b,\, s^1,\, s^2)$)

$$\begin{bmatrix} 7 & 7 & 5 \\ 5/2 & 5 & 7 \\ 25/3 & 5 & 7 \end{bmatrix} \begin{pmatrix} b \\ s^1 \\ s^2 \end{pmatrix} = \begin{pmatrix} 3 \\ 9 \\ 9 \end{pmatrix}$$

or

$$7b + 7s^1 + 5s^2 = 3 \tag{B.13}$$

$$(5/2)b + 5s^1 + 7s^2 = 9,\ \text{and} \tag{B.14}$$

$$(25/3)b + 5s^1 + 7s^2 = 9. \tag{B.15}$$

From (B.13)-(B.15) it follows that

$$\text{(B.14) and (B.15)}\ \Rightarrow b = 0\ \text{(sic)},\ s^2 = 9/7 - (5/7)s^1.$$

The latter and (B.13) yield $7s^1 + 5(9/7 - (5/7)s^1) = 3$, so that

$$s^1 = -1,\ \text{and}\ s^2 = 2 > 1.$$

Step 2. Given the solution to System I in which $\eta(\omega) \neq \eta(0),\ \omega > 0$, existence of a solution to System II for which $(b_1^0,\, s_1^0)$ is positive but small (which implies a unique PFE).

This is a "laydown." Use (B.4) for $g = 2$ to calculate μ. It is then easily verified that, for $b_1^0 = 0$ and $s_1^{02} = 0$, and for any choice of $0 < \alpha_1^{01} < 1$, $0 < \alpha_2^{01} < 1$, and $\delta^1(0) > 0$, when β_2 is sufficiently small, the solution to (B.5) satisfies $0 < s_1^{01} < 1$ (so that $0 < \bar{e}_1 = ((s_1^{01},\, 0),\, (s_1^{01},\, 0)) < \mathbf{1}$, and there is a unique CE, hence a unique PFE).

Note: Given the sign of μ (in the example, $\mu\eta(0) = 5/27$) and $s_1^{12} \notin [0,1]$, choose any ϕ such that

$$\phi(s_1^{12}) = 0 \text{ and sign } D\phi(s_1^{12}) = \text{sign}\,\mu.$$

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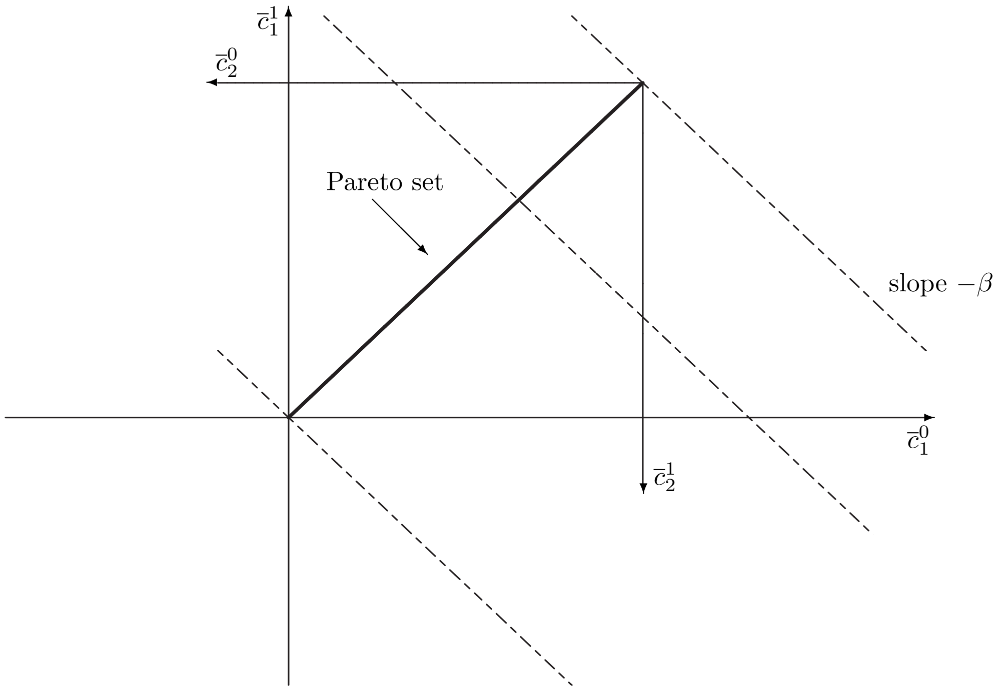
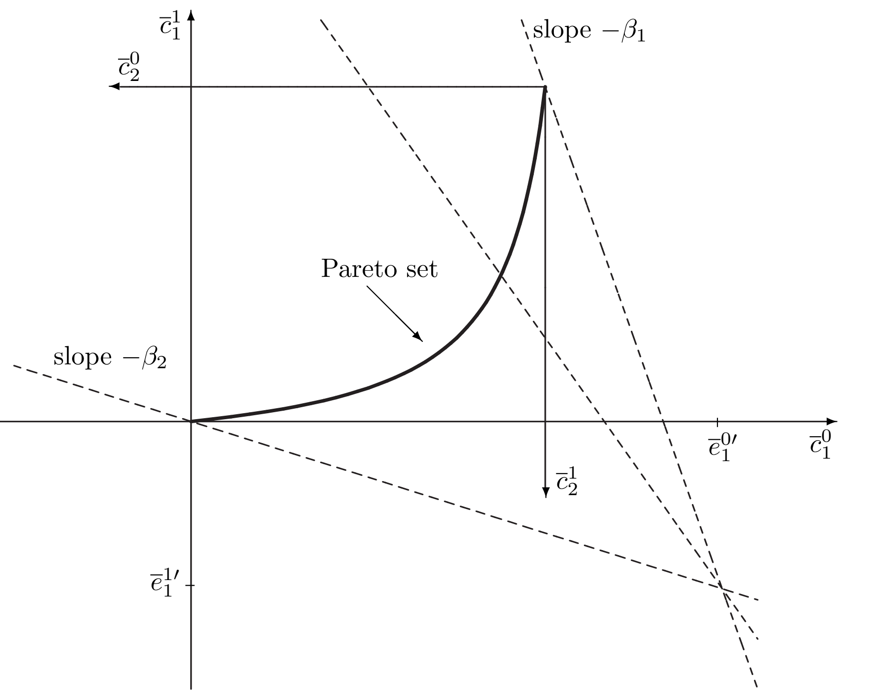


(a) Unique Equilibrium. $\beta_1 = \beta_2 = \beta$.



(b) Continuum of Equilibria. $\beta_1 > \beta_2$.

Figure 1. Equilibria in the TL-Model. The Edgeworth-Bowley box is presented for the (certainty) case of $G = \Omega = 1$, $\tilde{G} = 1$, $H = 2$. The thick solid line depicts the Pareto set, the dotted lines correspond to the prices which support allocations in the Pareto set. $(\bar{e}_1^{0\prime}, \bar{e}_1^{1\prime})$ is the endowment point for which a continuum of equilibria obtains.