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**"Production and Investment  
in an Equilibrium Economy"**

**by**

**Duncan MacRae**

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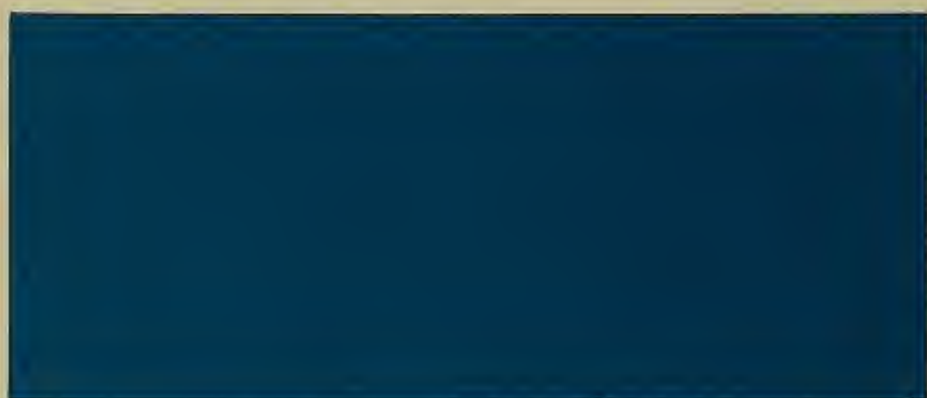
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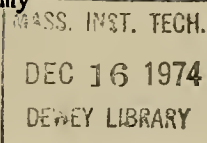
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This paper was supported in part by funds from the General Motors Corporation under the General Motors Grant for Highway Transportation Research. The views expressed in this paper are the author's sole responsibility, and do not reflect those of the General Motors Corporation, the Department of Economics, nor of the Massachusetts Institute of Technology.



# PRODUCTION AND INVESTMENT IN AN EQUILIBRIUM ECONOMY\*

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## INTRODUCTION

In the theory of equilibrium growth investment is treated in a passive manner.<sup>1</sup> Firms in the aggregate are assumed to invest at a rate which will absorb full employment saving. Since planned saving is equal to planned investment in equilibrium, investment is determined by a saving function. Although firms actively produce, they passively invest. They are assumed to produce so as to maximize profits but to invest only so as to maintain full employment equilibrium.<sup>2</sup> But firms do not invest passively, no more so than they produce passively. They both produce and invest so as to maximize the net present value of the firm. The purpose of this paper is to derive some of the implications of this assumption for equilibrium growth.

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\* This research was supported in part by funds from the General Motors Corporation under the General Motors Grant for Highway Transportation Research. I am indebted to E. S. Chase, R. Dorfman, and R. M. Solow for many helpful comments.

<sup>1</sup> See F. H. Hahn and R. C. O. Matthews, "The Theory of Economic Growth: A Survey," Economic Journal, vol. 74 (1964), p. 741.

<sup>2</sup> In F. H. Hahn, "Equilibrium Dynamics with Heterogeneous Capital Goods," Quarterly Journal of Economics, vol. 80 (1966), pp. 633-46, investment is allocated over the varieties of capital goods so as to maximize profits under short-run perfect foresight. The level of investment, however, is determined solely by an "extreme classical savings function."







## THE FIRM

Let us begin by examining the theory of the firm<sup>3</sup> in the context of the two-sector canonical model of capital<sup>4</sup> using continuous-time optimal control theory<sup>5</sup> as a tool of analysis. Imagine a representative firm in an equilibrium economy.<sup>6</sup> The firm produces consumption goods (homogeneous or composite) and homogeneous investment goods with the aid of labor and homogeneous fixed capital. The activities of the firm are production of consumption goods,  $Y_C$ , production of investment goods,  $Y_I$ , allocation of labor to production,  $\bar{L}$ , allocation of capital to production,  $\bar{K}$ , hiring labor,  $L$ , renting capital,  $K_R$ , and gross investment,  $I$ . Renting out is negative renting; and disinvestment is negative investment. The production of consumption goods is related to the allocation of labor and capital to production and production of investment goods by a production-possibility frontier:

$$Y_C = F(\bar{L}, \bar{K}, Y_I), \quad (1)$$

where  $F$  is concave and homogeneous of degree one. Capital allocated to production cannot exceed the available capital, or:

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<sup>3</sup>The model of the firm presented in this paper has evolved from the work of Arrow and Jorgenson. See K. J. Arrow, "Optimal Capital Policy, the Cost of Capital, and Myopic Decision Rules," Annals of the Institute of Statistical Mathematics, Vol. 15 (Tokyo, 1964), and D. W. Jorgenson, "The Theory of Investment Behavior," in R. Ferber, (ed.), Determinants of Investment Behavior. (New York: Columbia University Press for National Bureau of Economic Research, 1967).

<sup>4</sup>P. A. Samuelson, "Indeterminacy of Development in a Heterogeneous Capital Model with Constant Saving Propensity," in K. Shell, ed., Essays on the Theory of Optimal Economic Growth. (Cambridge, Mass.: MIT Press, 1967).

<sup>5</sup>L. S. Pontryagin, et al, The Mathematical Theory of Optimal Processes. (New York: Interscience, 1962).

<sup>6</sup>An equilibrium economy is one in which all price and rate of interest expectations are justified and all labor, consumption, production, and investment plans are realized. Moreover, this is an economy with perfect competition, diminishing returns to labor and capital, no economies of mass production, no external economies or diseconomies, and no technological disturbances.



$$\bar{K} \leq K + K_R, \quad (2)$$

where  $K$  is the stock of capital owned by the firm. Similarly, labor allocated to production cannot exceed the labor hired, or:

$$\bar{L} \leq L. \quad (3)$$

Furthermore,  $Y_C, Y_I, \bar{L}, \bar{K}, L \geq 0$ , and  $K_R$  is not sign restricted. Assume that capital depreciates by evaporation at a constant rate  $\mu$ , so that the change in the stock of capital owned by the firm is given by:

$$\dot{K} = I - \mu K, \quad (4)$$

where  $K$  is continuous,  $K \geq 0$ , and  $I$  is not sign restricted. Once the initial stock of capital owned by the firm,  $K_0$ , is given, the behavior of the firm over time is completely determined by the time paths of the activity levels  $Y_C, Y_I, \bar{L}, \bar{K}, L, K_R$ , and  $I$  subject to (1) - (4).

The firm hires labor, rents capital, allocates the labor and available capital, and invests so as to maximize the net present value of the firm:

$$J \equiv \int_0^{\infty} [P_C Y_C + P_I Y_I - WL - RK_R - P_I I] e^{-\rho t} dt, \quad (5)$$

where 
$$\rho \equiv \frac{\int_0^t r(\tau) d\tau}{t},$$

$P_C$  is the price of consumption goods,  $P_I$  is the price of investment goods,  $W$  is the wage rate,  $R$  is the gross rate of rental,  $r$  is the rate of interest, and all prices are assumed to be positive.<sup>7</sup> In an equilibrium the firm takes all prices

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<sup>7</sup>The firm is valued according to the discounted cash flow approach. See M. H. Miller and F. Modigliani, "Dividend Policy, Growth, and the Valuation of Shares," Journal of Business, Vol. 34 (1961), pp. 411-33, for a discussion of the equivalence of this approach to other approaches to the valuation of the firm assuming perfect capital markets, rational behavior, and perfect certainty.



and the rate of interest as given. The firm has expectations about prices and the rate of interest over time for an infinite horizon. In an equilibrium economy these expectations are justified. The problem of the firm,<sup>8</sup> therefore, is to choose the activity levels so as to maximize  $J$  subject to (1) - (4).

The first condition for the maximization of the net present value of the firm is that the firm produce so as to maximize profits i.e. choose the production activity levels,  $Y_C$ ,  $Y_I$ ,  $\bar{L}$ ,  $\bar{K}$ ,  $L$ , and  $K_R$ , so as to maximize

$$H \equiv P_C Y_C + P_I Y_I - WL - RK_R - P_I \mu K \quad (6)$$

subject to (1) - (3) given  $K$ .<sup>9</sup>

The second condition is that the firm invest so that the imputed value of capital owned by the firm is equal to the market cost, and the imputed value of capital not owned by the firm is not greater than the market cost.

The third condition is that the firm invest so that the present value of the stock of capital owned by the firm eventually vanishes.

That the three conditions are necessary follows from an application of the maximum principle.<sup>10</sup> For consider the following Hamiltonian system:

$$\mathcal{H} \equiv [P_C Y_C + P_I Y_I - WL - RK_R - P_I I] e^{-\rho t} + \psi_I [I - \mu K] - \gamma_K [I - \mu K], \quad (7)$$

$$\dot{K} = \frac{\partial \mathcal{H}}{\partial \psi_I}, \quad (8)$$

$$\dot{\psi}_I = \frac{-\partial \mathcal{H}}{\partial K} - \rho \psi_I, \quad (9)$$

<sup>8</sup>This is a continuous-time linear optimal control problem with state-control constraints and a variable end point where the functional is given by an improper integral.

<sup>9</sup>Note that depreciation is a fixed cost in the short-run and, hence, can be ignored in the choice of activity levels.

<sup>10</sup>L. S. Pontryagin, op.cit., p. 267.





subject to the transversality condition,

$$\lim_{t \rightarrow \infty} \Psi_I K = 0, \quad (10)$$

and (1) - (3), where

$$\gamma_K = 0 \text{ when } K > 0, \quad (11)$$

$K_0$  is given, and  $Q_R$  is the imputed gross rate of rental. The maximum principle states that the production and investment activity levels which maximize  $J$  subject to (1) - (4) also maximize  $\mathcal{H}$  subject to (1) - (3) given  $K$  and  $\Psi_I$ , where  $K$  and  $\Psi_I$  satisfy (8) - (11), and.

$$\gamma_K \leq 0 \text{ and } \dot{\gamma}_K \geq 0 \text{ when } K = 0. \quad (12)$$

Since the investment activity level chosen by the firm maximizes  $\mathcal{H}$ ,

$$\gamma_K = \Psi_I - P_I e^{-\rho t}. \quad (13)$$

Therefore, from (13) and (7),  $\Psi_I e^{-\rho t} = \mathcal{H}$ ; and the production activity levels which maximize  $J$  also maximize  $H$  given  $K$ . From (9), (13), and the maximization of  $H$ , we obtain

$$r = \frac{R}{P_I} - \mu + \frac{\dot{P}_I}{P_I} + \frac{\dot{\gamma}_K e^{\rho t}}{P_I}. \quad (14)$$

Therefore, either  $P_I = Q_I$ , so that  $r = \frac{R}{P_I} - \mu + \frac{\dot{P}_I}{P_I}$ , when  $K > 0$ ,

or  $P_I > Q_I$ , so that  $r \geq \frac{R}{P_I} - \mu + \frac{\dot{P}_I}{P_I}$ , when  $K = 0$ .

where  $Q_I \equiv \Psi_I e^{\rho t}$  is the imputed value of capital. The firm is indifferent between investment and disinvestment if  $P_I = Q_I$  and holds no capital if  $P_I > Q_I$ . From (12) we see that once  $P_I = Q_I$ , then  $P_I = Q_I$  forever after. Finally, from the transversality condition (10) we obtain the other constraint on investment:





$$\lim_{t \rightarrow \infty} P_I K e^{-\rho t} = 0. \quad (17)$$

If  $K \neq 0$  as  $t \rightarrow \infty$ , then (10) and (17) imply that

$$\lim_{t \rightarrow \infty} Q_I e^{-\rho t} = \lim_{t \rightarrow \infty} P_I e^{-\rho t} = 0. \quad (18)$$

Differentiating  $P_I e^{-\rho t}$  logarithmically we see from (15) that (18) means the own rate of interest must eventually be positive. Multiplying (15) through by  $P_I e^{-(\rho+\mu)t}$ , integrating from  $t$  to  $\infty$ , and using (18) we see that the present imputed value of capital owned by the firm, which is equal to the present value of the market cost, is equal to the integral of the present value of the rate of rental on capital, or:

$$Q_I e^{-\rho t} = P_I e^{-\rho t} = e^{\mu t} \int_t^{\infty} R e^{-(\rho+\mu)\tau} d\tau, \quad (19)$$

which means that if once  $P_I = Q_I = 0$ , then  $P_I = Q_I = 0$  forever after.

The three conditions for the maximization of the net present value of the firm are not only necessary but also sufficient in an equilibrium economy. Profit maximization under constant returns to scale implies

$$H = RK - P_I \mu K. \quad (20)$$

From (6) and (20) we can write (5) as

$$J = \int_0^{\infty} [RK - P_I I] e^{-\rho t} dt. \quad (21)$$

Integrating (21) by parts while using (4), (15), and (16) we obtain

$$\begin{aligned} J &= \lim_{T \rightarrow \infty} \int_0^T [RK - P_I I] e^{-\rho t} dt \\ &= P_I(0)K(0) - \lim_{T \rightarrow \infty} P_I(T)K(T)e^{-\rho T}. \end{aligned} \quad (22)$$

From (17) and (22) we see that the net present value is maximized and is equal



to the initial value of the stock of capital owned by the firm, or:

$$\int_0^{\infty} [RK - P_I I] e^{-\rho t} dt = P(0)K(0). \quad (23)$$

Production is a short-run problem of profit maximization which can be divorced from the problem of investment. Since  $H$  is a concave function of the production activity levels, and the resource constraints, (1) - (3), form a convex set, the short-run problem of the firm is a concave programming problem.<sup>11</sup> Either the firm produces no output and rents out all the capital it owns, or it is indifferent between producing and renting more or less. This is the usual result of indeterminacy of output by the firm with constant returns to scale and perfect competition.<sup>12</sup> The size of the firm as measured by  $H$ , however, is well determined by (20) to be equal to net rent on capital owned by the firm, or profit.

Investment, and, hence, the stock of capital owned by the firm are, in general, indeterminate with the exception that the present value of the stock of capital owned by the firm must eventually vanish. This exception prevents the firm from over-accumulating capital.<sup>13</sup> Differentiating  $P_I K e^{-\rho t}$  logarithmically we see from (15) that (17) means the proportionate rate of growth of the stock of capital owned by the firm must eventually be less than the own rate of

<sup>11</sup>The firm hires labor, rents capital, and allocates the labor and available capital between the various activities so that the marginal net revenue of each activity used is equal to its marginal imputed cost and the marginal net revenue of each activity not used is not greater than its marginal imputed cost. See R. Dorfman, P. A. Samuelson, and R. Solow, Linear Programming and Economic Analysis. (New York: McGraw-Hill, 1958).

<sup>12</sup>See P. A. Samuelson, Foundations of Economic Analysis. (Cambridge, Mass.: Harvard University Press, 1947). In general, the firm will be completely specialized in the production of either consumption goods or investment goods; and the choice of technique in production is well determined by the market prices given the technology of the firm.

<sup>13</sup>The optimal capital policy of the firm, therefore, is not perfectly myopic as is the policy described in K. J. Arrow, op.cit.



interest.<sup>14</sup> This means, in particular, that the firm will not invest forever all of the earnings on capital, because it can always do better by less investment, unless it begins with no initial stock of capital. The size of the firm as measured by  $J$  is well determined, however, by (21) to be equal to the value of the initial stock of capital owned by the firm.<sup>15</sup>

If we allow for discontinuous changes in  $K$ , the analysis of production and investment by the firm is essentially unaltered. The behavior of the firm for an infinite horizon continues to be treated as the limiting behavior of the firm for a finite horizon,  $T$ , as the horizon increases without limit, where the firm ceases to exist at  $T$ . Since production is independent of investment, the net present value of the firm is given by

$$J = \lim_{T \rightarrow \infty} \int_0^T [RK - P_I I] e^{-\rho t} dt - \sum_{i \in \mathcal{I}} P_I(t_i) [K^+(t_i) - K^-(t_i)] e^{-\rho t} \quad (24)$$

where

$$K = I - \mu K \quad t \neq t_i, \quad (25)$$

$$K^+(t_i) = \lim_{\substack{t \rightarrow t_i \\ t > t_i}} K(t), \quad K^-(t_i) = \lim_{\substack{t \rightarrow t_i \\ t < t_i}} K(t), \quad (26)$$

$t_i$  are the jump points to be determined,  $0 \leq t_i \leq T$ ,  $\mathcal{I}$  is the set of integers, and  $K \geq 0$ .<sup>16</sup>

<sup>14</sup> See M. H. Miller and F. Modigliani, op. cit., p. 421 for a similar result in the case of constant growth rates.

<sup>15</sup> See J. Williamson, "Profit, Growth and Sales Maximization," Economica, Vol. 33 (1966), pp. 1-16 for a discussion of the "optimum size" of the firm.

<sup>16</sup> Note that no value is attached to  $K^+(T)$  since  $T$  is the horizon and not the planning period of the firm.





It is shown in an appendix<sup>17</sup> that (17) becomes

$$\lim_{T \rightarrow \infty} P_I(T)K^+(T)e^{-\rho T} = 0. \quad (27)$$

Moreover, if

$$r(t) > \frac{R(t)}{P(t)} - \mu + \frac{\dot{P}(t)}{P(t)} \quad (28)$$

for any  $t$  such that  $0 < t \leq T$ ,  $K^-(t)$  must be zero, and if

$$r(t) = \frac{R(t)}{P(t)} - \mu + \frac{\dot{P}(t)}{P(t)} \quad (29)$$

for any  $t$  such that  $0 \leq t \leq T$ , all  $K^-(t)$  such that  $K^-(t) \geq 0$  are optimal, and once (29) is true it is true forever and the firm is indifferent between continuous and discontinuous changes in  $K$ . Therefore,  $K$  need not change discontinuously to maximize net present value except when (28) is true for  $t = 0$ . Then the firm disinvests the entire initial stock of capital owned by the firm so that (16) applies and

$$J = P_I(0)K_0. \quad (30)$$

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<sup>17</sup> Although the maximum principle cannot be applied directly to a control problem with jumps in the state variable, it can be applied to an ingenious reformulation of the control problem developed by Rishel and Vind. See R. W. Rishel, "An Extended Pontryagin Principle for Control Systems whose Laws Contain Measures," SIAM Journal on Control, Vol. 3 (1965), pp. 191-205, and K. Vind, "Control Systems with Jumps in the State Variables," Econometrica, forthcoming.

<sup>18</sup> Discontinuous changes in production and investment processes have been examined in R. Bellman, Dynamic Programming. (Princeton: Princeton University Press, 1957), p. 207-21, and M. Nerlove and K. J. Arrow, "Optimal Advertising Policy Under Dynamic Conditions," Economica, N. S. Vol. 29 (1962), pp. 129-142 with similar results.



Note that price and rate of interest behavior such that

$$r(t) < \frac{R(t)}{P(t)} - \mu + \frac{\dot{P}(t)}{P(t)} \quad (31)$$

is incompatible with equilibrium. If (31) were true, the firm would immediately attempt to invest an arbitrarily large amount to obtain an arbitrarily large net present value, which is incompatible with both perfect competition and net present value maximization. If there were an upper bound on investment, however, the problem of the existence of an optimum would not become evident until the terminal horizon. If (31) were true for  $t = T$ , the firm would invest the maximum amount until  $T$ , when it would disinvest the entire stock of capital owned by the firm. But this is not a meaningful policy for an infinite horizon.<sup>19</sup>

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<sup>19</sup>See J. Tindbergen, "Optimum Savings and Utility Maximization over Time," Econometrica, Vol. 28 (1960), pp. 481-9, for a similar analysis of investment in an economy where the rate of discount on consumption is less than the rate of return on saving.



## THE ECONOMY

We have examined the production and investment behavior of the representative firm under the assumption of net present value maximization. We have seen that the essential difference between this assumption and the assumption of profit maximization is that the proportionate rate of growth of the stock of capital owned by the firm must eventually be less than the own rate of interest. Now let us derive the implications of this constraint on investment for equilibrium growth in the two-sector canonical model of capital<sup>20</sup> assuming households do all the saving and firms do all the investing.

Since all capital in the economy is owned by some firm, the market cost of capital is equal to the imputed value and (29) is true for all  $t$ . Then investment by firms is indeterminate. Once a saving function (e.g. a proportional saving function) is introduced, however, aggregate investment is completely determined.

Therefore, if the number of firms is finite, the constraint on investment behavior of the representative firm imposed by the assumption of net present value maximization becomes a constraint on the equilibrium saving behavior of the economy. Firms in the aggregate invest at a rate which will absorb full employment saving. Any saving function such that the proportionate rate of growth of the stock of capital is not eventually less than the own rate of interest, however, is incompatible with equilibrium. This means that saving must eventually be less than earnings on capital.

Thus, the dynamically inefficient paths of consumption described by Phelps,<sup>21</sup> along which the proportionate rate of growth is eventually greater than the own

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<sup>20</sup>P. A. Samuelson, op.cit., pp. 220-1.

<sup>21</sup>E. S. Phelps, Golden Rules of Economic Growth. (New York: W. W. Norton and Co., 1966).





rate of interest, cannot occur. Moreover, paths of consumption where the rate of growth is eventually equal to the rate of interest, corresponding to the Golden Rule path, are inconsistent with equilibrium. Although these paths are dynamically efficient paths of consumption, they are not optimal from the point of view of the firm.

The incompatibility of Golden Rule equilibrium and net present value maximization would seem to conflict with Debreu's result that a Pareto optimum is a valuation equilibrium.<sup>22</sup> Radner has shown, however, that the linear function approach to present value maximization with an infinite horizon, adopted by Debreu, is not necessarily equivalent to the "scalar product" approach, which is used in this paper.<sup>23</sup> The cases in which the two approaches give different results are "boundary cases," corresponding to Golden Rule paths.

Another way to see the nature of this incompatibility is to consider the problem of optimal saving behavior in a centrally planned economy described by Koopmans.<sup>24</sup> He shows that if the rate of discount on consumption per capita is zero, a necessary condition for eligibility of a path is that it eventually approaches the Golden Rule path. The problem of optimal saving behavior with a zero discount rate, therefore, is transformed into a utility maximization problem with a fixed end point equal to the Golden Rule stock of capital. If a central planner with a zero discount rate imposes this fixed end point on the firms, instead of allowing them a variable end point, these paths are compatible with equilibrium.

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<sup>22</sup>G. Debreu, "Valuation Equilibrium and Pareto Optimum," Proc. Nat. Acad. Sci. of the U.S.A., Vol. 40 (1954), pp. 588-92.

<sup>23</sup>R. Radner, "Efficiency Prices for Infinite Horizon Production Programmes," Review of Economic Studies, Vol. 34 (1967), pp. 51-66.

<sup>24</sup>See T. C. Koopmans, "On the Concept of Optimal Economic Growth," in Econometric Approach to Development Planning. (Amsterdam: North-Holland, 1965), pp. 225-87.





If the number of firms is infinite, the constraint on investment by the representative firm is not necessarily a constraint on the investment behavior of the economy. If growth in the stock of capital is accompanied by growth in the number of firms or never ending buck-passing,<sup>25</sup> both the dynamically inefficient paths of consumption and the Golden Rule path may be compatible with equilibrium.

### CONCLUSIONS

One implication of net present value maximization for equilibrium growth in the two-sector canonical model of capital is that the dynamically inefficient paths of consumption are incompatible with equilibrium if a finite number of firms do all investing. A second implication is that the dynamically efficient paths approaching the Golden Rule path are also incompatible with equilibrium. This result depends upon the approach taken to present value maximization with an infinite horizon. A problem of decentralizing allocation over time may arise, therefore, not only with an infinite number of firms but also with an infinite horizon.

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<sup>25</sup> See T. C. Koopmans, Three Essays on the State of Economic Science. (New York: McGraw-Hill, 1957), p. 112. Another paradox of infinity referred to by Samuelson as the tulip mania phenomenon in "Intertemporal Price Equilibrium," Weltwirtschaftliches Archiv. Vol. 79 (1957), pp. 181-219, could occur if a market value is attached to  $K^+(T)$  and  $T$  is allowed to increase without limit. But this would imply  $T$  is the planning period of the firm, and not the terminal horizon.



APPENDIX

Following Vind<sup>26</sup> the problem of the firm with jumps in  $K$  can be reformulated as an optimal control problem without discontinuous changes in the state variable by introducing an additional concept of time. Let  $t$  be real time and  $\tau$  be analytic time. Then the problem is to choose  $u_0$ ,  $I$ , and  $v$  over analytic time so as to maximize

$$J = \lim_{T \rightarrow \infty} \int_{\tau_0}^{\tau_1} u_0 [R(t)K - P_I(t)I] e^{-\rho(t) \cdot t} + [1 - u_0] P_I(t) v e^{-\rho t} d\tau \quad (A-1)$$

subject to 
$$\frac{dK}{d\tau} = u_0 [I - \mu K] + [1 - u_0] v, \quad (A-2)$$

$$\frac{dt}{d\tau} = u_0, \quad (A-3)$$

$K \geq 0$ ,  $I$  and  $v$  not sign restricted,  $0 \leq u_0 \leq 1$ ,  $t(\tau_0) = 0$ ,  $t(\tau_1) = T$ , and  $K(\tau_0) = K_0$ , where  $\tau_0$  is given and  $\tau_1$  is free,

$$\tau_1^- = t_1 + \sum_{j < i} v_0(t_j) \quad \text{and} \quad \tau_1^+ = t_1 + \sum_{j \leq i} v_0(t_j). \quad (A-4)$$

The optimal Hamiltonian system is given by:

$$\max_{u_0, I, v} \Omega = u_0 \phi + (1 - u_0) [\Psi_I - P_I(t) e^{-\rho(t) \cdot t}] v = 0. \quad (A-5)$$

$$\frac{d\Psi_I}{d\tau} = - u_0 \frac{\partial \mathcal{H}}{\partial K}, \quad (A-6)$$

and 
$$\frac{d\Psi_t}{d\tau} = - u_0 \frac{\partial \mathcal{H}}{\partial \Psi_t} - (1 - u_0) \left[ r(t) - \frac{\dot{P}_I(t)}{P_I(t)} \right] P_I e^{-\rho(t) \cdot t} v, \quad (A-7)$$

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<sup>26</sup>K. Vind, ibid.



where 
$$\mathcal{H} = [R(t)K - P_I(t)I]e^{-\rho(t) \cdot t} + \Psi_I [I - \mu K] - \gamma_K [I - \mu K] \quad (\text{A-8})$$

with 
$$\gamma_K = 0 \text{ if } K > 0, \text{ and } \phi = \Psi_t + \mathcal{H}.$$

If  $u_0 = 1$ , the results of pp. 4-8 follow immediately. If  $u_0 = 0$ , real time stops at  $t_1$ , and the present imputed value of capital owned by the firm remains constant, i.e.

$$\frac{dt}{d\tau} = 0 \quad \text{and} \quad \frac{d\Psi_I}{d\tau} = 0. \quad (\text{A-9})$$

Then  $\phi \leq 0$  and

$$\max_{u_0, I, v} \Omega = [\Psi_I - P_I(t_k)e^{-\rho(t_1) \cdot t_1}]v = 0. \quad (\text{A-10})$$

If there is to be a jump in  $K$ , the imputed value of capital must be equal to the market value of capital, i.e. if  $v \neq 0$ ,

$$\Psi_I = P_I e^{-\rho(t_1) \cdot t_1}. \quad (\text{A-11})$$

Then 
$$\frac{d\phi}{d\tau} = - \left\{ r(t_1) - \left[ \frac{R(t_1)}{P_I(t_1)} - \mu + \frac{\dot{P}_I(t_1)}{P_I(t_1)} \right] \right\} P_I(t_1) e^{-\rho(t_1) \cdot t_1} \frac{dK}{d\tau}. \quad (\text{A-12})$$

If (29) is true,  $\frac{d\phi}{d\tau} = 0$ . Then we must have  $\phi(\tau) = 0$ , otherwise  $u_0 = 0$  forever and real time never starts again. But if (29) is true, and  $\phi = 0$ , the firm is indifferent between continuous and discontinuous changes in  $K$ .

If (28) is true,  $\text{sign } \frac{d\phi}{d\tau} = - \text{sign } \frac{dK}{d\tau}$ . Either  $\phi = 0$ ,  $0 \leq u_0 \leq 1$ ,  $\frac{dK}{d\tau} = 0$ ,  $\frac{d\phi}{d\tau} = 0$ , and  $K = 0$  or  $\phi < 0$ ,  $u_0 = 0$ ,  $\frac{dK}{d\tau} < 0$ ,  $\frac{d\phi}{d\tau} > 0$ , until  $K = 0$ . Hence, the only jump that is required is when  $K_0 > 0$  and (28) is true for  $t = 0$ . Then  $\tau_0^- = 0$  and real time starts again at  $\tau_0^+ = 0 + v_0(0)$ , where  $v_0(0)$  is finite. The transversality condition is now given by (27).







