Scheme for Building a ’t Hooft–Polyakov Monopole

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We study a simple quantum mechanical model of a spinning particle moving on a sphere in the presence of a magnetic field. The system has two ground states. As the magnetic field is varied, the ground states mix through a non-Abelian Berry phase. We show that this Berry phase is the path ordered exponential of the smooth SU(2) ’t Hooft–Polyakov monopole. We further show that, by adjusting a potential on the sphere, the monopole becomes a Bogomol’nyi-Prasad-Sommerfield monopole and obeys the Bogomol’nyi equations.

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Introduction.—In 1974, ’t Hooft and Polyakov discovered a new solution of Yang-Mills-Higgs theories [1,2]. From afar, it looks like a Dirac magnetic monopole. However, the configuration is smooth, with the singularity at the origin of the Dirac monopole resolved by the non-Abelian gauge fields.

The spatial profile of the ’t Hooft–Polyakov field configuration depends on the scalar potential for the adjoint-valued Higgs field $\phi$. Among these, one profile is rather special. This occurs when the potential vanishes and, as first shown by Prasad and Sommerfield [3], it is possible to find an exact solution. Later, Bogomol’nyi [4] showed that the non-Abelian field strength, $F_{\mu\nu}$, for this configuration solves the simple, first order, differential equations

$$F_{\mu\nu} = \epsilon_{\mu\nu\rho} D_\rho \phi,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ and $D_\rho \phi = \partial_\rho \phi + [A_\rho, \phi]$. Monopoles of this type are known as BPS, after the three authors named above. The subsequent discovery that these monopole play a special role in supersymmetric theories [5] has resulted in the title “BPS” being ascribed to almost anything associated to supersymmetry.

There is another, more abstract, situation in theoretical physics where the Dirac monopole arises. This is the Berry phase in quantum mechanics. Consider a spin 1/2 particle in a magnetic field $\vec{B}$. The Hamiltonian is given by

$$H = -\vec{B} \cdot \vec{\sigma} - |\vec{B}| l_2,$$

where $\vec{\sigma}$ are the Pauli matrices and $l_2$ is the unit 2 $\times$ 2 matrix, whose presence in the Hamiltonian simply ensures that the ground state energy of this two-state system is normalized to zero. We start in this ground state, $|0\rangle$. We then slowly rotate the magnetic field $\vec{B}$ until, finally, we return to our initial setup. The adiabatic theorem in quantum mechanics tells us that the system remains in the ground state and changes only by a phase. The question is, what is this phase? Since we have normalized the vacuum to zero energy, there is no dynamical contribution. Nonetheless, Berry showed that there is a geometrical phase which depends on the path $\Gamma$ taken in the space of magnetic fields [6,7],

$$|0\rangle \rightarrow \exp\left(-i \oint_{\Gamma} \vec{A} \cdot d\vec{B} \right) |0\rangle. \quad (3)$$

The Abelian Berry connection $\vec{A}$ is defined in terms of the dependence of the ground state on the magnetic field $\vec{B}$,

$$\vec{A} = i \langle 0 | \frac{\partial}{\partial \vec{B}} | 0 \rangle. \quad (4)$$

Berry showed that, for the simple Hamiltonian (2), the connection (4) is that of the Dirac magnetic monopole: $\vec{A} = \vec{A}_\text{Dirac}$. One can form a $U(1)$ field strength from the Berry connection in the usual way $F_{\mu\nu} = \frac{\partial A_\mu}{\partial B_\nu} - \frac{\partial A_\nu}{\partial B_\mu}$. This takes the radial, Dirac monopole form

$$F_{\mu\nu} = \epsilon_{\mu\nu\rho} \frac{B_\rho}{B^3}. \quad (5)$$

Note that there is a potential for confusion here, because $\epsilon_{\mu\nu\rho} F_{\mu\nu}$ is an abstract magnetic monopole over the space of real magnetic fields $\vec{B}$. The field strength $F_{\mu\nu}$ has a singularity at the origin. This is nothing to be afraid of: it simply reflects the fact that the excited state and the ground state become degenerate at $\vec{B} = 0$. Indeed, the very existence of the Berry phase can be traced to this degenerate point in parameter space.

In this Letter, we ask whether the smooth non-Abelian ’t Hooft–Polyakov monopole can appear as a Berry connection in simple quantum mechanical systems. The answer, as we shall see, is yes. The concept of the non-Abelian Berry connection was introduced by Wilczek and Zee [8]. This occurs if a system has degenerate eigen-
The Hamiltonian enjoys a $\mathbb{Z}_2$ symmetry,
\[ \tilde{B} \rightarrow -\tilde{B}, \quad \theta \rightarrow \pi - \theta. \]  
\[ (10) \]

The sign flip of the magnetic field acts on the Hilbert space by exchanging spin-up and spin-down states, $|\uparrow\rangle$ and $|\downarrow\rangle$, defined to be the two normalized eigenvectors of $\tilde{B} \cdot \hat{\sigma}$ with eigenvalues $+|\tilde{B}|$ and $-|\tilde{B}|$, respectively.

The $\mathbb{Z}_2$ symmetry guarantees the existence of two ground states for all values of $\tilde{B}$. For $\tilde{B} \neq 0$, the spin-up state is localized near $\theta = 0$, while the spin-down state is localized near $\theta = \pi$. When $\tilde{B} = 0$, both ground states are smeared uniformly over the sphere. However, in contrast to the Hamiltonian (2), there is no extra degeneracy of the ground states when $\tilde{B} = 0$. For arbitrary values of $\tilde{B}$, the two, normalized ground states are a combination of the spin states and a spatial wave function, $\psi(\cos \theta; \tilde{B})$, which depends on the magnitude $B = |\tilde{B}|$.

\[ |1\rangle = \psi(\cos \theta; \tilde{B}) |\uparrow\rangle, \quad |2\rangle = \psi(-\cos \theta; \tilde{B}) |\downarrow\rangle. \]  
\[ (11) \]

Writing $x = \cos \theta$, the spatial wave function $\psi(x; \tilde{B})$ satisfies the Schrödinger equation,
\[ -\frac{\hbar^2}{2m} (1 - x^2) \psi'' + \frac{\hbar^2}{2m} x \psi' - \hbar B x \psi = E_0 \psi, \]  
\[ (12) \]
with $\psi' = d\psi/dx$, and $E_0$ the ground state energy.

We now compute the Berry phase for this quantum mechanical system. The system is prepared in one of the ground states before the magnetic field $\tilde{B}$ is adiabatically varied, traversing a closed loop in parameter space. At the end of this tour, the ground state has undergone a $U(2)$ rotation, defined, as in (6), by the path ordered exponential of the Berry connection,
\[ \tilde{\mathcal{A}}_{ab} = i\langle b| \frac{\partial}{\partial \tilde{B}} |a\rangle, \quad a, b = 1, 2. \]  
\[ (13) \]

To build some intuition, let us start with the diagonal components of the connection. Consider a large magnetic field $B \gg \hbar/m$, which localizes the spatial part of each wave function close to a pole, at $\theta = 0$ or $\theta = \pi$. Here the ground state knows little about the rest of the sphere and sees an effective Hamiltonian of the form (2). This gives rise to a $U(1)$ Berry connection which is equal to that of a Dirac monopole, $\tilde{\mathcal{A}}_{\text{Dirac}}$. In fact, a simple computation reveals that the diagonal components are independent of the spatial wave functions for all values of $B$, and are given by
\[ \tilde{\mathcal{A}}_{11} = \langle 1| \frac{\partial}{\partial \tilde{B}} |1\rangle = \langle 1| \frac{\partial}{\partial \tilde{B}} |1\rangle = \tilde{\mathcal{A}}_{\text{Dirac}}, \]
\[ \tilde{\mathcal{A}}_{22} = \langle 2| \frac{\partial}{\partial \tilde{B}} |2\rangle = \langle 1| \frac{\partial}{\partial \tilde{B}} |1\rangle = -\tilde{\mathcal{A}}_{\text{Dirac}}. \]  
\[ (14) \]

In contrast, the off-diagonal terms describe the tunneling between the two different spin states, and depend on the spatial wave function of the particle $\psi$. They are
\[ A_{12} = \langle 1 | \frac{\partial}{\partial B} | 2 \rangle = f(B) \langle 1 | \frac{\partial}{\partial B} | 1 \rangle, \]

where the function \( f(B) \) is the overlap,

\[
f(B) = 2\pi \int_0^\pi \sin \theta d\theta \psi^* (-\cos \theta; B) \psi (\cos \theta; B). \tag{16}\]

Without specific knowledge of the ground state wave function \( \psi \), we are unable to compute explicitly the profile \( f(B) \) of the non-Abelian Berry monopole. However, on general grounds, we know that \( f(B) \to 0 \) as \( B \to \infty \) since the two spatial wave functions are localized at antipodal points on the sphere. In the opposite limit, \( B = 0 \), the two spatial wave functions coincide and \( f(0) = 1 \).

The Dirac monopole connection \( \mathcal{A}^{\text{Dirac}} \) necessarily contains a singularity along a half-line, known as the Dirac string. In the present context, this arises because it is not possible to globally define a basis of spin states \( |\uparrow\rangle \) and \( |\downarrow\rangle \) for all values of \( B \). Therefore any explicit computation of the components of \( \mathcal{A}_{ab} \), using the basis shown in (14) and (15), necessarily suffers from the Dirac string. However, there does exist a gauge in which the non-Abelian connection \( \mathcal{A} \) is free from the Dirac string. To demonstrate this, one must first choose a \( \hat{B} \) dependent basis for \( |1\rangle \) and \( |\downarrow\rangle \), then rotate \( \mathcal{A} \) using a suitable singular gauge transformation \( [10,11] \). The result is the non-Abelian Berry connection which takes the rotationally covariant form,

\[
\mathcal{A}_\mu = \epsilon_{\mu \nu \rho} \frac{B_\nu \sigma^\rho}{2B^2} [1 - f(B)]. \tag{17}\]

This is the connection of a 't Hooft–Polyakov monopole. Note, first, that it is an \( su(2) \) connection, rather than \( u(2) \). Moreover, and most importantly, the asymptotic behavior of \( f(B) \) described above guarantees that, as \( B \to \infty \), it reduces to the Dirac monopole for a \( U(1) \subset SU(2) \). Yet the field strength is smooth at \( B = 0 \).

The BPS monopole.—Any deformation of the Hamiltonian (8) that preserves the vacuum degeneracy will again lead to a 't Hooft–Polyakov monopole with a different profile function \( f(B) \). For example, we may add a spin-blind potential to the Hamiltonian. Something special happens for the potential given by

\[
V(\theta) = \frac{1}{2} m B^2 \sin^2 \theta. \tag{18}\]

For this choice, the Schrödinger equation simplifies. The ground state energy is \( E_0 = 0 \), and it is a simple matter to find the exact wave functions. They are given by (11), with

\[
\psi (\cos \theta; B) = \left( \frac{Bm}{2\pi \sinh (2Bm/h)} \right)^{1/2} e^{(Bm/h)\cos \theta}. \tag{19}\]

Equations (16) and (17) then tell us the exact Berry connection for this system:

\[
\mathcal{A}_\mu = \epsilon_{\mu \nu \rho} \frac{B_\nu \sigma^\rho}{2B^2} (1 - \frac{2Bm}{h} \coth (\frac{2Bm}{h}) - 1). \tag{22}\]

which is precisely the form of the Higgs field for the BPS monopole solution of \( SU(2) \) Yang-Mills Higgs theory (1). The magnetic field \( \vec{B} \) plays the role of the spatial position, while the analog of the Higgs expectation value is \( 2m/h \).

Discussion.—The fact that one can write an equation, such as (1), to describe the Berry connection is intriguing. Typically, the only way to compute the Berry connection is through the direct definition (7), but to do this one first needs to compute the exact ground states as a function of the parameters. If the Berry connection can be shown to obey an equation—for example, of the form (1)—then one can circumvent this step. In fact, this shortcut is known to happen in supersymmetric quantum mechanics. This was first shown some years ago [12–14] for quantum mechanical models where the Berry connection was shown to obey a formula known as the \( tr^2 \) equation. More recently, other supersymmetric models have been exhibited for which it can be shown that the Berry connection satisfies the Bogomol’nyi equation (1), or generalizations thereof [15,16]. Interestingly, the Berry phase of supersymmetric quantum mechanics of this type has recently arisen in the context of black hole microstates [17].

Although the model described in this Letter is not supersymmetric, it may not surprise the reader to learn that, for the specific potential (18), it may be shown to be the truncation of a supersymmetric quantum mechanics [18]. The emergence of the Bogomol’nyi monopole equation thus, once again, nods towards the existence of an underlying supersymmetry.

In this Letter we have introduced a simple-quantum mechanical model of a particle moving on the sphere which gives rise to a smooth, non-Abelian Berry connection taking the form of a 't Hooft–Polyakov monopole. However, the periodic form of the Hamiltonian (8) makes it clear that it can also be given the interpretation of a particle moving in a one-dimensional potential lattice with an applied periodic magnetic field. Indeed, the exact ground states (19) are of the required Bloch form with vanishing crystal momentum. The monopole Berry phase derived in
this Letter then describes the response of the Bloch states to adiabatic changes in the magnetic field. It would be interesting to study the effects of this smooth non-Abelian Berry phase on the band structure of such 1d crystals in more detail. Related systems, where the Berry phase depends on the reciprocal lattice, were previously studied in [19].

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