SHORT PERIODIC VARIATIONS IN THE FIRST-ORDER SEMIANALYTICAL SATELLITE THEORY

by

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ABSTRACT

This thesis describes the construction of first-order analytical formulas for the short-periodic variations in the equinoctial orbital elements due to various conservative perturbations. These formulas either can be attached to a large orbit determination system for production computations or executed separately to develop an understanding of the relative importance of the various terms. The considerable algebraic burden is made manageable via the use of a symbolic algebra system, MACSYMA. MACSYMA is employed as a programming language; the code is organized into separate blocks which can produce

-- the short-periodics due to an arbitrary zonal harmonic
-- the m-daily terms due to an arbitrary tesseral harmonic
-- the short-periodics due to an arbitrary tesseral harmonic
-- the short-periodics due to an arbitrary term in the 3rd body potential.

The formulas are in closed-form with the exception of the tesseral short-periodics which are truncated on the orbital eccentricity. Consideration is given to the problem of obtaining physical insight into the relative size of the variations due to the several physical perturbations for three typical Earth orbital flight regimes (circular low altitude, eccentric low altitude and high eccentricity with high altitude) by using the plot package implemented in the MACSYMA system. The compromises necessary in the application of MACSYMA to these various goals are investigated. The limitations of the code on the present MACSYMA machine, the DEC KL-10, are noted.

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Introduction

Estimation and prediction of artificial satellite orbits are two of the more computationally expensive dynamical problems today. Maintaining accurate ephemerides for the ever-increasing number of artificial satellites (which include active scientific, defense, communication, and weather satellites as well as defunct satellites, launch vehicles, and other debris) requires considerable expenditure in computing resources at various space mission centers. The launch mission analysis may also require several hundred trajectories to support lifetime, maneuver, and constraint analysis studies. Finally, construction of physical models for the Earth's gravitational field (which departs from that of a point mass) and the upper atmosphere density relies significantly on solutions to the satellite trajectory problem.

Precise orbital computation requirements are presently met, generally, by integrating numerically Newton's equations of motion

\[ \ddot{\mathbf{r}} + \frac{\mu}{r^3} \mathbf{r} = \mathbf{Q}(t, \mathbf{r}, \mathbf{r}') \]  

(0-1)

or some equivalent set of differential equations (for example, see Reference [1]). In equation (0-1)

\[ \mathbf{r}, \mathbf{r}', \mathbf{r}'' = \text{position, velocity, and acceleration vectors of the satellite} \]
\[ \mu = \text{Earth's gravitational constant} \]
\[ r = \text{magnitude of the position vector} \]
\[ \mathbf{Q} = \text{perturbing acceleration vector} \]

The perturbing acceleration includes the effects of the Earth non-sphericity, lunar and solar point masses, atmospheric drag and solar radiation pressure.

The computational cost of integrating Equation (0-1) numerically is controlled by the stepsize of the process employed. The short periodic variations caused by the perturbing acceleration limit the stepsize to be between 1/100 and 1/200 the orbital period for precise work. In addition, these numerical solutions require starting the computations all over again for each new set of input data.

Analytical solutions to Equation (0-1) are potentially much more efficient than the step by step numerical integration process since the analytical calculations can be carried out in one step, regardless of the prediction interval. However, such analytical solutions (called "General Perturbation Methods" by the astrodynamical community) have been of limited utility in high accuracy operational applications. Current general perturbation methods suffer from the following difficulties:
-- the simplified physical models usually included do not adequately represent the real world
-- the methods are particularly unwieldy for non-conservative perturbations such as atmospheric drag (the empirical structure of the atmospheric density models also complicates the situation)
-- these methods are extremely difficult to upgrade due to the explicit representation of the disturbing function in orbital elements
-- relative inefficiency when the output is required at many closely spaced points.

An alternative to both the conventional numerical and analytical approaches is the Semianalytical Satellite Theory. This approach combines several techniques from astrodynamics and numerical analysis in a very unique way to construct a production-oriented, high accuracy satellite theory. The study of the analytical formulas involved in a Semianalytical Satellite Theory requires a general purpose manipulation language.

The general purpose manipulation system used in this thesis is MACSYMA (Project MAC's SYmbolic MAnipulator). This algebraic language has been chosen because this thesis is the continuation of a previous work using MACSYMA (Reference [13]). Furthermore, MACSYMA is one of the most sophisticated systems among the general purpose algebraic
systems currently available, and it has been developed at the Labora-
tory for Computer Science of MIT and, consequently, an excellent inter-
face with the staff in charge of MACSYMA was possible. One of the most
important characteristics of MACSYMA is that, since May 1972, it has
been nationally available over the ARPA network; consequently, applica-
tion programs can be shared by a large community.

A variation of parameters (VOP) formulation of the orbit predic-
tion problem is considered. As the perturbing accelerations are much
smaller than the central force term, a VOP formulation is most suitable.
In addition, the decoupling characteristics of the VOP equations are
very useful in solving these differential equations.

As stated in the beginning of this introduction, it is desirable
to find a way to eliminate the high frequency components so that the
stepsize of the numerical integration might be increased and the number
of computations considerably decreased. The long period and secular
contributions to the motion can be systematically isolated by applying
the method of averaging to the VOP equations to eliminate the short
period contributions. Once the mean elements are determined corrective
terms must be added which correspond to the short periodic components.

The topic of this thesis is to derive and program in MACSYMA
first order analytical formulas for these corrective terms, expressed
as functions of the mean elements, allowing a first order recovery of
the complete orbital elements as a function of time.
This thesis deals only with the conservative type of perturbation, especially the oblateness of the Earth and the effect of a third body, thus simplifying the form of the VOP equations of motion. However, the application of the VOP equations does require the selection of a set of elements. These are several well known element sets, the best known of which is the set of classical or Keplerian elements. Unfortunately, the VOP equations formulated in these elements present several singularities. In order to avoid the appearance of singularities in the VOP equations of motion for vanishing eccentricity and/or inclination, expansions of the two kinds of disturbing potentials in terms of the equinoctial elements are developed, and this set of non-singular elements is used throughout this thesis.

This thesis is divided into five sections. Section 1 first recalls some general notions about the equinoctial elements and the formulation of the Earth's gravitational potential. Finally, a presentation of the determination to first order of the short periodic functions is given.

Then, the short periodics due to the geopotential are presented in Sections 2, 3 and 4; Section 2 deals with the zonal short periodics, Section 3 with the m-daily effects and Section 4 with the tesseral short periodics. For each of these cases, analytical formulas are derived to first order in a small parameter, the MACSYMA blocks (subroutines) are presented and graphical results for real world problems
are given. A very interesting feature for Sections 2 and 3 is that closed form generating functions of the short periodics are derived thus giving exact general formulas for all elliptical orbits except the retrograde equatorial case.*

Section 5 presents the formulation of the potential of a third body, the analytical formulas for the short periodics due to this third body (using, too, a closed form generating function) and the MACSYMA representation of these short periodics. Finally, plots are obtained for the same real world problems.

In the conclusion, the relative effects of the short periodics on the orbit are compared for the three different typical real world problems studied:

- low altitude circular orbit
- low altitude elliptical orbit
- high altitude and high eccentricity orbit.

Some personal comments are made by the author about his own experience with the MACSYMA system. Finally, a brief description is given of the future work which seems desirable to undertake.

* A retrograde set of equinoctial elements is known but the retrograde case has not been included in the MACSYMA efforts to date.
1. Short Periodics Due to an Arbitrary Term in the Geopotential

In this first section a brief recall of the equinoctial element formulation is presented: definition of the equinoctial elements and of the equinoctial frame, disturbing potential and definition of the special functions involved. Then analytical formulas for the short periodic variations due to an arbitrary term in the potential are derived to first order in the small parameter.

1.1 Equinoctial Orbit Element Formulation

Only direct equinoctial elements and frame are considered in this thesis.

1.1.1 Equinoctial Orbital Elements

In terms of the classical Keplerian elements

\[ a = \text{semi-major axis of the orbit} \]
\[ e = \text{eccentricity of the orbit} \]
\[ i = \text{inclination of the orbit with respect to the equatorial plane} \]
\[ \Omega = \text{longitude of the ascending node measured from the vernal equinox in the equatorial plane} \]
\[ \omega = \text{argument of perigee measured from the line of nodes in the orbital plane} \]
\[ M = \text{mean anomaly}. \]
The equinoctial elements are defined as

\[
\begin{align*}
    a &= a \\
    h &= e \sin(\omega + \Omega) \\
    k &= e \cos(\omega + \Omega) \\
    p &= \tan\left(\frac{i}{2}\right) \sin \Omega \\
    q &= \tan\left(\frac{i}{2}\right) \cos \Omega \\
    \lambda &= M + \omega + \Omega
\end{align*}
\]

(1-1a) (1-1b) (1-1c) (1-1d) (1-1e) (1-1f)

Physically, \(a\) is the orbital semi-major axis, \(h\) and \(k\) are components of the eccentricity vector and orient the perigee vector in the orbital plane, and \(p\) and \(q\) orient the satellite orbital plane with respect to inertial space. The phase angle \(\lambda\) is the orbital mean longitude and locates the satellite in the orbit.

1.1.2 Equinoctial Reference Frame

The orbital reference frame is quite useful in organizing the two-body formulas for position and velocity in terms of the equinoctial elements and in constructing the general representation of the disturbing potential. The equinoctial frame is designated by the orthogonal triad \(\hat{f}, \hat{g}, \hat{w}\) using the satellite plane as reference:

The unit vector \(\hat{f}\) points to a point in the satellite orbit displaced from the ascending node through the angle \(-\Omega\).
The unit vector $\hat{w}$ is identically the unit angular momentum vector.

The unit vector $\hat{g}$ is such that $\hat{g} = \hat{w} \times \hat{f}$. 
In terms of the equinoctial elements the coordinate axes expressed in
the Earth inertial frame are given by

\[ \hat{f} = \frac{1}{1 + p^2 + q^2} \begin{pmatrix} 1 - p^2 + q^2 \\ 2pq \\ -2p \end{pmatrix} \]  \hspace{1cm} (1-2a)

\[ \hat{g} = \frac{1}{1 + p^2 + q^2} \begin{pmatrix} 2pq \\ 1 + p^2 - q^2 \\ 2p \end{pmatrix} \]  \hspace{1cm} (1-2b)

\[ \hat{w} = \frac{1}{1 + p^2 + q^2} \begin{pmatrix} 2p \\ -2q \\ 1 - p^2 - q^2 \end{pmatrix} \]  \hspace{1cm} (1-2c)

1.1.3 Advantages

The use of this set of nonsingular elements avoids the appearance
of singularities in the VOP equations of motion for vanishing eccen-
tricity and/or inclination. Furthermore, other sets of nonsingular ele-
ments previously used were not general enough: the resulting equations
of motion retained a small inclination singularity or general expressions
for an arbitrary harmonic or a complete force model were not obtained,
and when general expressions for the potential were obtained, the
special functions embedded in these expressions were not investigated
in sufficient detail to develop the range of results typically associated
with the special functions of mathematics. With the equinoctial ele-
ments, all these problems do not occur.
1.2 Short-Periodics Due to an Arbitrary Term in the Geopotential

From Reference [4], the disturbing potential due to an arbitrary spherical harmonic pair \((C_{nm}, S_{nm})\) is given in equinoctial variables by

\[
U_{nm} = \sum_{s=-n}^{n} \sum_{t=-\infty}^{\infty} U_{nmst} \tag{1.3}
\]

where

\[
U_{nmst} = \text{Real}\{U_{nmst}^*\} \tag{1.4}
\]

and

\[
U_{nmst}^* = \frac{\mu}{a} \left( \frac{R_e}{a} \right)^n C_{nm}^* \nu_n^m S_{2n}^{(m,s)} (p,q) \gamma^{-n-1,s(h,k)} \exp[j(t\lambda - m\phi)] \tag{1.5}
\]

where

- \(a, h, k, p, q, \lambda\) = equinoctial elements
- \(\mu\) = Earth's gravitational constant
- \(R_e\) = Earth's mean equatorial radius
- \(C_{nm}^* = C_{nm} - jS_{nm}\)
- \(j = \sqrt{-1}\)
- \(\phi\) = Greenwich hour angle
\[ V_{n,s}^m = \text{coefficient which is not a function of the orbital elements} \]

\[ S_{2n}^{(m,s)} = \text{function introduced by the rotation from the equatorial frame to the orbital frame} \]

\[ \gamma_{-n-1,s}^t = \text{Hansen coefficient introduced when the potential is expressed in terms of the mean longitude} \]

The mathematical definitions of these three last quantities are given by

\[ V_{n,s}^m = \frac{(n-s)!}{(n-m)!} P_{n,s}(0) \quad (1-6) \]

where \( P_{n,s} \) is the associated Legendre function of first kind of degree \( n \) and order \( s \).

\[ S_{2n}^{(m,s)}(p,q) = (1 + p^2 + q^2)^{s/2} (p - jq)^{m-s} P_{n+s}^{(m-s,-m-s)}(\gamma) \]

\[ s \leq -m \quad (1-7a) \]
\[ S^{(m,s)}_{2n}(p,q) = \frac{(n+m)! (n-m)!}{(n+s)! (n-s)!} (1 + p^2 + q^2)^{-m} (p-jq)^{m-s} \]

\[ p^{(m-s,s+m)}_{n-m}(\gamma) \]

\[ |s| \leq m \quad (1-7b) \]

\[ S^{(m,s)}_{2n}(p,q) = (-1)^{m-s} (1 + p^2 + q^2)^{-s} (p + jq)^{s-m} \]

\[ p^{(s-m,s+m)}_{n-s}(\gamma) \]

\[ s \geq m \quad (1-7c) \]

where \[ \gamma = \frac{1 - p^2 - q^2}{1 + p^2 + q^2} \]

and \[ p^{(\alpha,\beta)}_n = \text{Jacobi polynomial.} \]

\[ Y^{n,m}_{t}(h,k) = (k + jh)^{m-t} \sum_{\sigma=0}^{\infty} \chi^{n,-m}_{\sigma+m-t,\sigma}(h^2 + k^2)^{\sigma} \]

\[ t \leq m \quad (1-8a) \]

\[ Y^{n,m}_{t}(h,k) = (k - jh)^{t-m} \sum_{\sigma=0}^{\infty} \chi^{n,m}_{\sigma-m+t,\sigma}(h^2 + k^2)^{\sigma} \]

\[ t \geq m \quad (1-8b) \]

where \[ \chi^{n,m}_{\delta,\sigma} = \text{Newcomb operator} \]
The short periodic variations due to $U_{nmst}$ are now developed using the general method of averaging to first order in the small parameter.

Not to be confused with the index $t$, the time will be represented by the variable $T$.

If the perturbing forces are expressed under the form of literal Fourier series of the assumed fast variable $t\lambda - m\Omega$, the VOP equations of motion due to the $U_{nmst}$ term may be expressed as

\begin{align*}
\frac{da_i}{dT} &= \varepsilon F_i(\dot{a}, t\lambda - m\Omega) \quad (i = 1, \ldots, 5) \quad (1-9a) \\
\frac{d\lambda}{dT} &= n + \varepsilon F_6(\dot{a}, t\lambda - m\Omega) \quad (1-9b)
\end{align*}

In equation (1-9) the vector $\dot{a}$ consists of the five slowly varying equinoctial elements $a_i(a, h, k, p, q)$ which describe the instantaneous or osculating ellipse and its orientation with respect to the system of reference. The sixth element $\lambda$ describes the position of the satellite on the osculating ellipse and is referred to as a fast variable. The functions $F_i(\dot{a}, t\lambda - m\Omega) (i = 1, \ldots, 6)$ depend on the perturbing forces which may include atmospheric drag, central body nonspherical gravitational effects, disturbing third body effects, and solar radiation pressure. The quantity $\varepsilon$ is a small parameter of the dynamical
system (e.g., $J_2$ in the central body oblateness perturbation or the parallax factor $\frac{\dot{a}}{a}$ in the third body perturbation). The quantity $n$ is the mean motion of the satellite and depends on the semi-major axis $a$ and the gravitational constant of the central body through Kepler's third law:

$$n = \sqrt[3]{\frac{\mu}{a^3}} \quad (1-10)$$

The averaged equations of motion are required to contain only the long period and secular terms of equation (1-9). These terms are those for which the phase angle $t\lambda - m\Theta$ turns out to be slowly varying. This condition is equivalent to

$$\dot{t}\lambda - m\dot{\Theta} \sim 0 \quad (1-11)$$

or

$$\dot{t}\lambda - m\omega_e \sim 0 \quad (1-12)$$

where $\omega_e$ is the Earth's rotation rate.

This case appears when the ratio of the two fast varying angle rates is close to a rational number or, in other words, when there is resonance between the satellite motion and the Earth's rotation.
Hence, for the averaged equations of motion, the following form in terms of the mean elements \((\overline{a}, \overline{\lambda})\) to first order in \(\varepsilon\) may be assumed:

\[
\frac{d\overline{a}_i}{dT} = \varepsilon \overline{F}_i (\overline{a}, t\overline{\lambda} - m\Theta) \quad \text{for} \quad t\overline{\lambda} - m\omega_e = 0
\]

\((i = 1, \ldots, 5) \quad (1-13a)\)

\[
\frac{d\overline{\lambda}}{dT} = \overline{n} + \varepsilon \overline{F}_6 (\overline{a}, t\overline{\lambda} - m\Theta) \quad \text{for} \quad t\overline{\lambda} - m\omega_e = 0
\]

\((1-13b)\)

where \(\overline{n}\) is the mean-mean motion defined by the mean semimajor axis.

Note that the averaged equations of motion admit terms for which \(s=t=0\). This corresponds to the double averaged term for which all dependence on \(\overline{\lambda}\) and \(\Theta\) is eliminated. In the absence of resonance or for cases when only high order resonance exists, the double-averaged term will constitute the primary contribution to the smoothed dynamic equations.

Each osculating element contains a secular component (mean element) and a fast varying component (short periodics, where the resonant case (1-12) is excluded). The first order representation of the osculating elements in terms of the averaged elements is assumed to be:

\[
a_i = \overline{a}_i + \varepsilon \eta_i (\overline{a}, t\overline{\lambda} - m\Theta) \quad \text{(for} \quad t\overline{\lambda} - m\omega_e \neq 0) \quad (i = 1, \ldots, 6)
\]
where \( a_6 = \lambda \)

and \( \eta_i \) (\( i = 1, \ldots, 6 \)) are the short periodic functions whose period is \( 2\pi \) in \( t\bar{\lambda} - m\Theta \)

By nature of their periodicity, the short periodic functions are assumed to average to zero.

Differentiating the transformation equation (1-14) with respect to time yields

\[
\frac{d a_i}{d\bar{T}} = \frac{d a_i}{d\bar{T}} + \epsilon \left[ \sum_{j=1}^{5} \frac{\partial \eta_i(a, t\bar{\lambda} - m\Theta)}{\partial a_j} \frac{d a_j}{d\bar{T}} + \frac{\partial \eta_i(a, t\bar{\lambda} - m\Theta)}{\partial (t\bar{\lambda} - m\Theta)} \frac{d(t\bar{\lambda} - m\Theta)}{d\bar{T}} \right]
\]

for \( t\bar{\lambda} - m\omega_e \neq 0 \)

\[(i = 1, \ldots, 6) \quad (1-15)\]

For the sake of the presentation, the following definitions are made:

\[
\tilde{F}_i = \tilde{F}_i(a, t\bar{\lambda} - m\Theta) \quad \text{for} \quad t\bar{\lambda} - m\omega_e = 0 \quad (1-16)
\]

\[
\tilde{\eta}_i = \tilde{\eta}_i(a, t\bar{\lambda} - m\Theta) \quad \text{for} \quad t\bar{\lambda} - m\omega_e \neq 0 \quad (1-17)
\]
Substitution of the assumed form of the averaged equations of motion (1-13) into equation (1-15) gives, after a truncation to first order in $\varepsilon$

$$
\frac{d\lambda_i}{dT} = \varepsilon \left[ F_i \left( \frac{\partial \eta_i}{\partial (t\lambda - m\Theta)} \right) \right] (i = 1, \ldots, 5) \tag{1-18a}
$$

and

$$
\frac{d\lambda}{dT} = \tilde{n} + \varepsilon \left[ F_6 \left( \frac{\partial \eta_6}{\partial (t\lambda - m\Theta)} \right) \right] \tag{1-18b}
$$

Substitution of equation (1-14) into equation (1-9), followed by an expansion about the mean elements to first order in $\varepsilon$ leads to another expression for $\frac{d\lambda_i}{dT}$:

$$
\frac{d\lambda_i}{dT} = \varepsilon F_i (\tilde{a}, \tilde{t}\lambda - m\Theta), \quad (i = 1, \ldots, 5) \tag{1-19a}
$$

and

$$
\frac{d\lambda}{dT} = n + \varepsilon F_6 (\tilde{a}, \tilde{t}\lambda - m\Theta) \tag{1-19b}
$$
Equating Equations (1-19b) and (1-18b) leads to first order to

\[ \varepsilon F_6(\bar{\alpha}, t\bar{\lambda} - m\phi) = \bar{n} - n + \varepsilon \left[ F_6 + (t\bar{n} - m\omega_e) \frac{\partial \eta_6}{\partial (t\bar{\lambda} - m\phi)} \right] \]

(1-20)

But the difference between the mean and the osculating mean motion can easily be derived to first order from Equation (1-10)

\[ \bar{n} - n = \mu^{1/2} (\bar{a}_1^{-3/2} - a^{-3/2}) \]

(1-21)

For the semimajor axis the mean to osculating transformation (1-14) to first order is given by

\[ a_1 = \bar{a}_1 + \varepsilon \eta_1 \]

(1-22)

Therefore, to first order

\[ a_1^{-3/2} = \bar{a}_1^{-3/2} \left( 1 - \frac{3}{2} \frac{\varepsilon \eta_1}{\bar{a}_1} \right) \]

(1-23)

Substitution of Equation (1-23) into Equation (1-21) gives

\[ \bar{n} - n = \mu^{1/2} \bar{a}_1^{-3/2} \left( 1 - \frac{3}{2} \frac{\varepsilon \eta_1}{\bar{a}_1} \right) \]

(1-24)
or

\[
\bar{n} - n = \left( \frac{3n}{2a_1} \right) \varepsilon \eta_1 \quad (1-25)
\]

Hence, substitution of Equation (1-25) into Equation (1-20) to first order leads to

\[
\varepsilon \left[ \bar{F}_6(\bar{a}, t\bar{\lambda}-\bar{m}\bar{\Theta}) - \frac{3n}{2a_1} \eta_1 \right] = \varepsilon \left[ \bar{F}_6 + (\bar{n}-m\omega) \frac{\partial \eta_6}{\partial (t\bar{\lambda}-m\Theta)} \right]
\]

\[ (1-26a) \]

Similarly, equating Equations (1-18a) and (1-19a) to first order yields

\[
\varepsilon [\bar{F}_i(\bar{a}, t\bar{\lambda}-m\Theta)] = \varepsilon \left[ \bar{F}_i + (\bar{n}-m\omega) \frac{\partial \eta_i}{\partial (t\bar{\lambda}-m\Theta)} \right]
\]

\[ (i = 1, ..., 5) \quad (1-26b) \]

It may be seen by inspection of Equation (1-26) that

\[
\bar{F}_i = \bar{F}_i(\bar{a}, t\bar{\lambda}-m\Theta) = F_i(\bar{a}, t\bar{\lambda}-m\Theta) \quad (i = 1, ..., 6)
\]

for \( t\lambda-m\omega = 0 \) for \( t\lambda-m\omega = 0 \)

\[ (1-27) \]
Therefore

\[ F_i(\vec{a}, t\lambda - m\omega) - \bar{F}_i = F_i(\vec{a}, t\lambda - m\omega) \]

for \( t\lambda - m\omega_e \neq 0 \) \((i = 1, \ldots, 6)\) (1-28)

Substitution of Equation (1-28) into Equation (1-26) gives the short periodic functions

\[ \epsilon \eta_i = \frac{1}{t_{n-m\omega_e}} \int t_{\lambda-m\omega} \epsilon F_i(\vec{a}, t\lambda - m\omega) \, d(t\lambda - m\omega), \quad (i = 1, \ldots, 5) \]

\[ \epsilon \eta_6 = \frac{1}{t_{n-m\omega_e}} \int t_{\lambda-m\omega} \left[ \epsilon \left( F_6(\vec{a}, t\lambda - m\omega) - \frac{3n\epsilon \eta_1(\vec{a}, t\lambda - m\omega)}{2a_1} \right) \right] d(t\lambda - m\omega) \]

Equation (1-29) thus gives a complete structure for obtaining the first order short periodic variations from the original equation of motion (1-9).

For gravitational perturbations, the equations of motion can be expressed in a more compact way using the Poisson brackets and the disturbing potential. Assuming that the disturbing potential is given by
\( U_{nmst} \) [Equations (1-4) and (1-5)], then the osculating element rates can be expressed by

\[
\varepsilon F_i = - \sum_{j=1}^{6} (a_i, a_j) \frac{\partial U_{nmst}}{\partial a_j} \quad (i = 1, \ldots, 6)
\]

(1-30)

This representation has the characteristic that only the scalar function \( U_{nmst} \) depends on the rapidly varying phase angle \( t\lambda - m\Theta \). Taking advantage that the Poisson brackets do not depend on the fast variable \( t\lambda - m\Theta \), the substitution of Equation (1-30) into Equation (1-29a) gives

\[
\varepsilon \eta_i = \frac{-1}{t\lambda - m\omega_e} \sum_{j=1}^{6} (\bar{a}_i, \bar{a}_j) \int_{0}^{t\lambda - m\Theta} \frac{\partial U_{nmst}}{\partial a_j} \, d(t\lambda - m\Theta)
\]

\[
\varepsilon \eta_i = \frac{-1}{t\lambda - m\omega_e} \sum_{j=1}^{6} (\bar{a}_i, \bar{a}_j) \int_{0}^{t\lambda - m\Theta} \frac{\partial U_{nmst}}{\partial a_j} \, d(t\lambda - m\Theta)
\]

(1-31)

Interchanging the differentiation and integration operations (see Reference [23]):

\[
\varepsilon \eta_i = \frac{-1}{t\lambda - m\omega_e} \sum_{j=1}^{6} (\bar{a}_i, \bar{a}_j) \int_{0}^{t\lambda - m\Theta} \frac{\partial U_{nmst}}{\partial a_j} \, d(t\lambda - m\Theta)
\]

(1-32)
Defining
\[ S_{nmst} = \int t^{t\lambda - m\Theta} \frac{U_{nmst}}{t^n - m\omega_e} \left\{ \begin{array}{l} \mathbf{t} \lambda - m\omega_e \neq 0 \\
\end{array} \right. \] (1-33)

and
\[ \Delta a_i = a_i - \bar{a}_i = \varepsilon \eta_i \quad (i = 1, \ldots, 6) \] (1-34)

Equation (1-32) can be rewritten as
\[ \Delta a_i = - \sum_{j=1}^{6} (\bar{a}_i, \bar{a}_j) \frac{\partial S_{nmst}}{\partial a_j} \quad (i = 1, \ldots, 5) \] (1-35)

It must be emphasized that the partial derivative of \( S_{nmst} \) with respect to the semimajor axis does not operate on the quantity \( t^n - m\omega_e \). Finally, the structure of \( U_{nmst}^* \) allows the expressions
\[ S_{nmst}^* = \frac{U_{nmst}^*}{j(t^n - m\omega_e)} \] (1-36)

and
\[ S_{nmst} = \text{Real}\{S_{nmst}^*\} \] (1-37)
For the short periodic variations in the mean longitude, substituting Equation (1-30) for \( i = 6 \) in the Equation (1-29b), using the definition of the generating function \( S_{nmst} \) (1-33) and the invariance of the Poisson brackets, and interchanging differentiation and integration once more yields

\[
\Delta \lambda = - \sum_{j=1}^{6} (\bar{\lambda}, \bar{a}_j) \left[ \frac{\partial S_{nmst}}{\partial \bar{a}_j} - \frac{3n}{2(tn - m\omega_e - a_1)} \int_{\bar{t}^\lambda - m\omega_e \neq 0} t\bar{\lambda} - m\omega_e \right] \eta_1 d(t\bar{\lambda} - m\omega_e) \tag{1-38}
\]

Equation (1-35) for \( i = 1 \) gives

\[
\Delta a_1 = \varepsilon \eta_1 = - \sum_{j=1}^{6} (\bar{a}_1, \bar{a}_j) \left[ \frac{\partial S_{nmst}}{\partial \bar{a}_j} \right] \int_{\bar{t}\lambda - m\omega_e \neq 0} t\bar{\lambda} - m\omega_e \neq 0 \tag{1-39}
\]

From Reference [6]:

\[
(\bar{a}_1, \bar{a}_j) = 0 \quad (j = 1, \ldots, 5) \tag{1-40a}
\]

\[
(a_1, \bar{\lambda}) = \frac{-2}{n \bar{a}_1} \tag{1-40b}
\]
Therefore $\Delta a_1$ becomes

$$\Delta a_1 = \varepsilon \eta_1 = \frac{2}{n} \frac{\partial S_{nmst}}{\partial \lambda} \left( \frac{t}{\lambda - m \omega e} \right)$$

(1-41)

Using the definition of $S_{nmst}$ (1-33)

$$\varepsilon \eta_1 = \frac{2}{n} \frac{t}{\lambda - m \omega e} \frac{U_{nmst}}{tn-m \omega e}$$

(1-42)

Substitution of Equation (1-42) into Equation (1-38), using again the definition of $S_{nmst}$ (1-33) leads to

$$\Delta \lambda = \sum_{j=1}^{6} (\lambda, a_j) \frac{\partial S_{nmst}}{\partial a_j} - \frac{3t}{(t - m \omega e) a_1} \frac{S_{nmst}}{a_1^2}$$

(1-43)

The short periodic variations due to $U_{nmst}$ can be summarized as -- for the five slowly variables $(a, h, k, p, q)$:

$$\Delta a_i = - \sum_{j=1}^{6} (a_i, a_j) \frac{\partial S_{nmst}}{\partial a_j}$$

(i = 1, ..., 5)
Substitution of Equation (1-5) into (1-33) gives an explicit form of the short periodic generating function

\[ S_{nmst} = \left( \frac{R_{e}}{a} \right)^{n} C_{nm}^{*} v_{nmst}^{m} s_{n,s}^{(m,s)} s_{2n(p,q)}^{y-n-1,s} (h,k) \]

\[ \times \exp[j(t\lambda - m\omega)] \]

\[ \frac{1}{j(t\tau - m\omega)} \]

for \( t\lambda - m\omega \neq 0 \) \hspace{1cm} (1-44)

The total short periodic generating function due to the \( n,n \) harmonic pair is given by

\[ S_{nm}^{*} = \sum_{s=-n}^{n} \sum_{t=-\infty}^{+\infty} S_{nmst} \]

where the terms identified in Equation (1-12) must be excluded.
At this point, three separate cases are identified:

-- zonal short periodics
-- Tesseral m-daily effects
-- Tesseral short periodics.

The next three sections deal with these cases. For each case, analytical forms are derived, MACSYMA blocks are developed and graphical results are presented for three real world problems.
2. **Zonal Harmonic Short Periodics**

The zonal harmonic coefficients correspond to terms with \( m \) equal to zero and \( t \) not equal to zero, in this case

\[
C_{nm} = C_{no} = -J_n
\]

\[
S_{nm} = S_{no} = 0
\]

Thus for zonals, the generating function (1-45) reduces to

\[
S_{no}^{*} = \frac{-\mu}{na} \left( \frac{R}{a} \right)^n J_n \sum_{s=-n}^{n} \gamma_{n,s}^{0} \gamma_{n,s}^{0} (p,q) \]

\[
x \sum_{t=\infty}^{+\infty} \frac{\gamma_{n-1,s}^{0} (h,k) \exp(i\lambda t)}{jt} \quad (2-1)
\]

where the term for which \( t = 0 \) corresponding to the double averaged term is excluded.

This generating function is an infinite series in the eccentricity due to the range of the index \( t \). For small eccentricity cases, the fact that the \( \gamma_{n-1,s}^{0} (h,k) \) are proportional to \( e^{s-t} \) (see Equations (33) and (34) in Reference [4]) allows truncation of the summation over \( t \). But for high eccentricity cases, convergence is a serious issue due to the Hansen coefficients. To improve the analytical formulation, two closed forms of the generating function (2-1) are introduced. These employ finite expansions and are discussed in the two following subsections.

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2.1 Closed Form Representation for the Zonal Short Periodic Generating Functions in the \(p,q\) Elements

It is possible to modify Equation (2-1) to obtain a closed form in the slowly varying elements, the averaged mean longitude and the averaged true longitude.

The generating function for the nonsingular Hansen coefficients (References [4] and [11]) is given by

\[
\left( \frac{r}{a} \right)^n \exp(jsL) = \sum_{t=-\infty}^{+\infty} Y^n_{t,s} \exp(jt\lambda) \tag{2-2}
\]

where \(L\) is the true longitude. Replacing \(n\) with \(-n+1\) in Equation (2-2) and subtracting the quantity \(Y_{0}^{-n-1,s}\) from both sides of the result gives

\[
\left( \frac{a}{r} \right)^{n+1} \exp(jsL) - Y_{0}^{-n-1,s} = \sum_{t=-\infty}^{+\infty} Y_{t}^{-n-1,s} \exp(jt\lambda) \tag{2-3}
\]

An integration of Equation (2-3) with respect to the mean longitude leads to

\[
\int_{\lambda}^{\lambda} \left( \frac{a}{r} \right)^{n+1} \exp(jsL) \, d\lambda - \lambda Y_{0}^{-n-1,s} = \sum_{t=-\infty}^{+\infty} Y_{t}^{-n-1,s} \frac{\exp(jt\lambda)}{jt} \tag{2-4}
\]
The right hand side of Equation (2-4) is the infinite summation in Equation (2-1). Thus finding a closed form for Equation (2-1) is equivalent to finding a closed form for the integral in Equation (2-4).

Introducing the change of variable (see Appendix A)

\[ d\lambda = \frac{1}{\sqrt{1-h^2-k^2}} \left( \frac{r}{a} \right)^2 dL \]  

(2-5)

Therefore

\[ \int_{\lambda}^{\lambda} \left( \frac{a}{r} \right)^{n+1} \exp(jsL) \, d\lambda = \frac{1}{\sqrt{1-h^2-k^2}} \int_{L}^{L} \left( \frac{a}{r} \right)^{n+1} \exp(jsL) \, dL \]  

(2-6)

From Reference [11] we have the expansion

\[ \left( \frac{a}{r} \right)^n = \sqrt{1-h^2-k^2} \sum_{i=-n}^{n} \gamma_0^{n-2,i} \exp(ijL) \]  

(2-7)

Subsitution of Equation (2-7) in Equation (2-6) gives

\[ \int_{\lambda}^{\lambda} \left( \frac{a}{r} \right)^{n+1} \exp(jsL) \, d\lambda = \sum_{i=-(n-1)}^{n-1} \gamma_0^{n-1,-i} \int_{L}^{L} \exp((i+s)jL) \, dL \]  

(2-8)
Therefore, Equation (2-4) becomes

\[ \sum_{t=-\infty}^{+\infty} y_{t}^{-n-1,s} \exp(jt\lambda) = \sum_{i=-(n-1)}^{n-1} y_{0}^{-n-1,-i} \exp[(i+s)jL] \]

\[ + (L - \lambda) y_{0}^{-n-1,s} \]

All the Hansen coefficients used here have the subscript zero and for this case there exists a simple closed form representation of these coefficients (References [8] and [10]).

Substitution of Equation (2-9) into Equation (2-1) leads to a closed form of the zonal short periodic generating function

\[ S_{n0} = \text{Real} \left\{ \frac{-\mu}{na} \left( \frac{R e}{a} \right)^{n} J_{n} \sum_{s=-n}^{n} v_{0}^{s} s_{2n}^{(0,s)} (p,q) \right\} \]

\[ + \left\{ \sum_{i=-(n-1)}^{n-1} y_{0}^{-n-1,-i} (h,k) \exp[(i+s)jL] \right\} \]

\[ + (L - \lambda) y_{0}^{-n-1,s} (h,k) \right\} \]

(2-10)

2.2 Closed Form Representation for the Zonal Short Periodic Generating Function in the Direction Cosines of the Inertial Z Axis

The starting point is to modify the expression of the potential in order to introduce the direction cosines of the inertial z axis
(expressed in the equinoctial frame) rather than the p and q elements. This generating function uses new special functions and coefficients, but contains less terms than in the formulation in the p, q elements, leading thus to a more compact formula.

The general expression for the disturbing potential due to a general harmonic pair, in terms of the radial distance \( r \), the latitude \( \phi \) and the longitude \( \lambda \) relative to an earth-fixed coordinate system is given by

\[
U_{nm} = \frac{\mu}{r} \sum_{m=0}^{n} \left( \frac{Re}{r} \right) (C_{nm} \cos m\lambda + S_{nm} \sin m\lambda)P_{nm}(\cos \phi)
\]

(2-11)

where \( P_{nm} \) = associated Legendre function of first kind of degree \( n \) and order \( m \).

for the zonals \( m = 0 \), \( C_{n0} = -J_n \) and \( S_{nm} = 0 \). Therefore

\[
U_{n0} = -\frac{\mu}{r} J_n \left( \frac{Re}{r} \right)^n P_n (\cos \phi)
\]

(2-12)

where \( P_n \) = Legendre polynomial and \( \phi \) = latitude
If \( \hat{r} \) = unit vector from the earth to the satellite

and \( \hat{z} \) = unit vector for the inertial z axis

Then:

\[
\sin \phi = \cos (\frac{\pi}{2} - \phi) = \hat{r} \cdot \hat{z}
\]

(2-13)

In the direct equinoctial frame \((\hat{f}, \hat{g}, \hat{w})\) these vectors are given by

\[
\hat{r} = \begin{pmatrix}
\cos L \\
\sin L \\
0
\end{pmatrix}
\]

(2-14)
\hat{z} \text{ in the inertial reference frame is expressed as}
\[
\hat{z} = \begin{pmatrix}
\alpha \\
\beta \\
\gamma
\end{pmatrix}
\]  
(2-15a)

Therefore \( \alpha, \beta, \gamma \) can be obtained in terms of equinoctial elements from Equations (2-15b) and (1-2)

\[
\alpha = \hat{z} \cdot f = \frac{-2p}{1 + p^2 + q^2}
\]  
(2-16a)

\[
\beta = \hat{z} \cdot g = \frac{2q}{1 + p^2 + q^2}
\]  
(2-16b)

\[
\gamma = \hat{z} \cdot w = \frac{1 - p^2 - q^2}{1 + p^2 + q^2}
\]  
(2-16c)

Substituting Equations (2-14) and (2-15a) into Equation (2-13) yields

\[
\sin \phi = \alpha \cos L + \beta \sin L
\]  
(2-17)

Substitution of Equation (2-17) in Equation (2-12) leads to

\[
U_{n0} = -\frac{\mu}{r} J_n \left( \frac{R_e}{r} \right)^n P_n(\alpha \cos L + \beta \sin L)
\]  
(2-18)
The next step is to develop a formula which will transform the expression \( P_n(\alpha \cos L + \beta \cos L) \) into a Fourier series expansion.

Using the standard addition formula for the Legendre polynomials we have

\[
P_n[\sin y \sin y' + \cos y \cos y' \cos(x - x')] =
\]

\[
= P_n(\sin y) P_n(\sin y') + 2 \sum_{m=1}^{\infty} \frac{(n-m)!}{(n+m)!} P_{nm}(\sin y) x P_{nm}(\sin y') \cos m(x - x')
\]

(2-19)

The form \((\alpha \cos L + \beta \sin L)\) is exactly equivalent to the form \([\sin y \sin y' + \cos y \cos y' \cos(x - x')]\) if the following definitions are made:

\[
\begin{align*}
\sin y &= 0 \quad \text{(2-20a)} \\
\cos y &= 1 \quad \text{(2-20b)} \\
\cos y' &= \sqrt{\alpha^2 + \beta^2} \quad \text{(2-20c)} \\
\sin y' &= \sqrt{1 - \alpha^2 - \beta^2} = \gamma \quad \text{(2-20d)} \\
x &= L \quad \text{(2-20e)} \\
\cos x' &= \frac{\alpha}{\sqrt{\alpha^2 + \beta^2}} \quad \text{(2-20f)} \\
\sin x' &= \frac{\beta}{\sqrt{\alpha^2 + \beta^2}} \quad \text{(2-20g)}
\end{align*}
\]

\(\sqrt{1 - \alpha^2 - \beta^2}\) has been replaced by \(\gamma\) in Equation (2-20d) since \(\hat{z}\) is a unit vector.
Using Moivre’s formula and Equations (2-20f) and (2-20g)

\[
\left( \cos x' + j \sin x' \right)^m = \left( \alpha^2 + \beta^2 \right)^{m/2} (\alpha + j\beta)^m = \\
= \cos m x' + j \sin m x' \tag{2-21}
\]

At this point two polynomial functions are defined

\[
C_{m}^{\alpha \beta} = \text{Re}(\alpha + j\beta)^m \tag{2-22a}
\]
\[
S_{m}^{\alpha \beta} = \text{Im}(\alpha + j\beta)^m \tag{2-22b}
\]

Therefore

\[
\cos m(n - n') = \cos m(L - x') - \cos mL \cos mx' \\
+ \sin mL \sin mx' \tag{2-23}
\]

Using Equations (2-21) and (2-22), Equation (2-23) becomes

\[
\cos m(x - x') = \left( \alpha^2 + \beta^2 \right)^{-m/2} \left( C_{m}^{\alpha \beta} \cos mL + S_{m}^{\alpha \beta} \sin mL \right) \tag{2-24}
\]
Substitution of Equations (2-20) and (2-24) into (2-19) results in

\[ P_n(\alpha \cos L + \beta \sin L) = P_n(0) P_n(\gamma) + 2 \sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} \]

\[ \times P_{nm}(0) (\alpha^2 + \beta^2)^{-m/2} P_{nm}(\gamma) \left( C_m^\alpha \cos mL + S_m^\alpha \sin mL \right) \]

The associated Legendre function is defined as

\[ P_{nm}(x) = (1 - x^2)^{m/2} \frac{d^m}{dx^m} P_n(x) \]  \hspace{1cm} (2-26)

An additional definition is introduced (see Appendix C)

\[ Q_{nm}(x) = \frac{d^m}{dx^m} P_n(x) \]  \hspace{1cm} (2-27)

Comparison of Equation (2-26) with (2-27) shows that

\[ (1 - x^2)^{-m/2} P_{nm}(x) = Q_{nm}(x) \]  \hspace{1cm} (2-28)

Using Equations (2-20d) and (2-28), Equation (2-25) reduces to

\[ P_n(\alpha \cos L + \beta \sin L) = \sum_{m=0}^{n} K_m \frac{(n-m)!}{(n+m)!} Q_{nm}(0) Q_{nm}(\gamma) \]

\[ \times (C_m^\alpha \cos mL + S_m^\alpha \sin mL) \]  \hspace{1cm} (2-29)
where $K_m$ takes the value 1 for $m$ equals zero and the value 2 for $m$ greater than zero.

Substitution of Equation (2-29) into $U_{n0}$ (2-18) gives

$$U_{n0} = -\frac{\mu}{r} J_n \left( \frac{Re}{r} \right)^n \sum_{m=0}^{n} K_m \frac{(n-m)!}{(n+m)!} Q_{nm}(0) Q_{nm}(\gamma)$$

$$\times \left( C_m^\alpha \cos mL + S_m^\alpha \sin mL \right) \quad (2-30)$$

Defining the new coefficient (see Appendix C)

$$V_{nm} = \frac{(n-m)!}{(n+m)!} Q_{nm}(0) \quad (2-31)$$

And rearranging Equation (2-30)

$$U_{n0} = \text{Re} \left\{ - \frac{\mu}{a} J_n \left( \frac{Re}{a} \right)^n \sum_{m=0}^{n} K_m V_{nm} Q_{nm}(\gamma) \left( \frac{a}{r} \right)^{n+1} \right\}$$

$$\times \left( C_m^\alpha + jS_m^\alpha \right) \exp(-jmL) \quad (2-32)$$

Using the generating function for the nonsingular Hansen coefficient (2-2) and excluding the term with $t$ equals zero (1-12)

$$U^*_{n0} = -\frac{\mu}{a} \left( \frac{Re}{a} \right)^n J_n \sum_{m=0}^{n} K_m V_{nm} Q_{nm}(\gamma) \left( C_m^\alpha + jS_m^\alpha \right)$$

$$\times \sum_{t=-\infty}^{\infty} \gamma^{-n-1-m} \eta(h,k) \exp(jt\lambda) \quad (2-33)$$
From Equation (1-33) the short periodic generating function is

\[
S_{no}^* = \frac{-\mu}{a} \left( \frac{R}{a} \right)^n J_n \sum_{m=0}^{n} K_m V_{nm} Q_{nm}(\gamma)(C_m^\alpha + jS_m^\alpha) \\
\times \sum_{t=-\infty}^{+\infty} \gamma^{n-1,-m}_{t}(h,k) \frac{\exp(jt\lambda)}{jt}
\]

(2-34)

Applying Equations (2-4) through (2-8) gives the closed form representation for the zonal short periodic generating function

\[
S_{no} = \text{Real} \left\{ -\frac{\mu}{na} \left( \frac{R}{a} \right)^n J_n \sum_{m=0}^{n} K_m V_{nm} Q_{nm}(\gamma) \\
\times \left\{ (C_m^\alpha + jS_m^\alpha) \sum_{i=-(n-1)}^{n-1} \gamma^{n-1,-i}_{0}(h,k) \frac{\exp[(i-m)jL]}{(i-m)^j} \\
+ (\bar{C} - \bar{\lambda}) \gamma^{n-1,0}_{0}(h,k) \right\} \right\}
\]

(2-35)

2.3 Short Periodic Variations in the Equinoctial Elements

Although two equivalent closed forms for the generating function have been achieved, it has been at the expense of computing the true longitude each time the analytical formulas are used to compute the short periodics. Therefore the solution of Kepler's equation in terms of equinoctial elements must be included and is developed in the next subsection.
For the zonals where \( m \) equals zero, Equations (1-35) and (1-43) become

\[
\Delta a_i = - \sum_{j=1}^{6} (\bar{a}_i, \bar{a}_j) \frac{\partial S_{\text{host}}}{\partial a_j} \quad (i = 1, \ldots, 5) \quad (2-36a)
\]

\[
\Delta \lambda = - \sum_{j=1}^{6} (\bar{\lambda}, \bar{a}_j) \frac{\partial S_{\text{host}}}{\partial a_j} - \frac{3S_{\text{host}}}{n \ a^2} \quad (2-36b)
\]

Because the coefficients of \( S_{\text{host}} \) no longer depend on \( t \), \( S_{\text{host}} \) in Equation (2-36) can be replaced with \( S_{\text{no}} \) defined in Equations (2-10) or (2-35) or eventually in Equation (2-1) if the user wants to use a truncation mechanism. The formulas for \( \Delta a_i \) take the following form

\[
\Delta a_i = - \sum_{j=1}^{6} (\bar{a}_i, \bar{a}_j) \frac{\partial S_{\text{no}}}{\partial a_j} \quad (i = 1, \ldots, 5) \quad (2-37a)
\]

\[
\Delta \lambda = - \sum_{j=1}^{6} (\bar{\lambda}, \bar{a}_j) \frac{\partial S_{\text{no}}}{\partial a_j} - \frac{3S_{\text{no}}}{n \ a^2} \quad (2-37b)
\]

As \( S_{\text{no}} \) includes the true longitude and as the true longitude is a function of the three equinoctial elements \( h \), \( k \) and \( \lambda \), it must be emphasized that for \( j = 2, 3, \) and 6
\[ \frac{\partial S_{no}}{\partial h} \text{ must be replaced by } \frac{\partial S_{no}}{\partial h} + \frac{\partial S_{no}}{\partial \bar{I}} \frac{\partial \bar{I}}{\partial h} \]

\[ \frac{\partial S_{no}}{\partial k} \text{ must be replaced by } \frac{\partial S_{no}}{\partial k} + \frac{\partial S_{no}}{\partial \bar{I}} \frac{\partial \bar{I}}{\partial k} \]

\[ \frac{\partial S_{no}}{\partial \lambda} \text{ must be replaced by } \frac{\partial S_{no}}{\partial \lambda} + \frac{\partial S_{no}}{\partial \bar{I}} \frac{\partial \bar{I}}{\partial \lambda} \]

See Appendix A for the derivation of the partial derivatives of the true longitude with respect to \( h, k, \) and \( \lambda \).

2.4 MACSYMA Representation

The generating functions (2-1) and (2-35) as well as the short periodic variations (2-37) contain several coefficients and special functions. Some of these have already been programmed in MACSYMA in a previous work (Reference [13]). The others (appearing in the case using the direction cosines) are new and are presented in detail. As these appear not only in the case of the zonal short periodics but also in all the cases treated in this thesis, they will be called fundamental blocks and functions.
2.4.1 Fundamental Blocks and Functions

For the zonal short periodics the fundamental blocks used and already programmed by Zeis are:

-- VFUNCT\([N,R,M]\) which computes the rational number \(V_{n,r}^m\).
-- SFUNCT\([N,M,S] (P,Q)\) which computes the function \(S_{2n}^m (p,q)\).
-- HANSEN2\([N,M,-1] (H,K)\) which computes the modified Hansen coefficient \(V_{0,0}^n (h,k)\) exactly.
-- POISSON\((I,J,-1)\) which computes the Poisson bracket \((i,j)\) of the equinoctial elements \(i\) and \(j\) exactly.

In the generating function (2-35) we must consider a new coefficient \(V_{nm}\) and three new functions, \(Q_{nm}(Y)\), \(C_m^\alpha\beta\), and \(S_m^\alpha\beta\) whose MACSYMA representations are now presented.

2.4.1.1 \(C(N,X,Y)\) and \(SI(N,X,Y)\)

A listing of the functions and some examples are presented. The formulas used are the definition of \(C_n^{XY}\) and \(S_n^{XY}\) [Equation (2-22)].
Listing of \( C(N,X,Y) \) and of \( SI(N,X,Y) \)

\[ C(N,X,Y):= \]

\[
\text{/*}
\text{THIS FUNCTION COMPUTES THE POLYNOMIAL } C_N = \text{Re}((X+jY)^N) \text{ */}
\]

\[
\text{/*PROGRAMMER: J-P.KANIECKI,MIT-FEBRUARY 1979 */}
\]

\[
\text{EXPAND(RALPART}((X+\%I*Y)^N))\$
\]

\[ SI(N,X,Y):= \]

\[
\text{/*}
\text{THIS FUNCTION COMPUTES THE POLYNOMIAL } S_N = \text{Im}((X+jY)^N) \text{ */}
\]

\[
\text{/*PROGRAMMER: J-P.KANIECKI,MIT-FEBRUARY 1979 */}
\]

\[
\text{EXPAND(IMAGPART}((X+\%I*Y)^N))\$
\]

Usage of These Functions

(C1) \( C(2,X,Y); \)

(D1) \[
2 \quad 2 \\
X - Y
\]

(C2) \( SI(2,X,Y); \)

(D2) \[
2 \quad X \quad Y
\]

(C3) \( C(5,X,Y); \)

(D3) \[
4 \quad 3 \quad 2 \quad 5 \\
5 \quad X \quad Y - 10 \quad X \quad Y + X
\]

(C4) \( SI(5,X,Y); \)

(D4) \[
5 \quad 2 \quad 3 \quad 4 \\
Y - 10 \quad X \quad Y + 5 \quad X \quad Y
\]
2.4.1.2 $Q(N,M,X)$

A listing of this block and several examples are presented. The recursive formula used to generate the $Q_{nm}(x)$ function is the one of Appendix C, for fixed order and varying degree [Equation (C-4)] with the starting values given by Equations (C-10) and (C-14), taking advantage of the condition (C-16).

Listing of $Q(N,M,X)$

```
Q(N,M,X) := BLOCK([], /*THIS BLOCK COMPUTES THE FUNCTION $Q_{N,M}(X)$ */ /*PROGRAMMER: J-P.KANIECKI, MIT-FEBRUARY 1979 */ /*RESTRICTION: N MUST BE AN INTEGER $\geq 0$ */ IF N < 0 THEN RETURN(ERROR), IF M > N THEN RETURN(0), IF M = 0 AND N = 0 THEN RETURN(1), IF M = N THEN RETURN ((2*N-1)!!), IF M = N-1 THEN RETURN (X*(2*N-1)!!), RETURN(RATSIMP(1/(N-M)*(X*(2*N-1)*Q(N-1,M,X)-(M+N-1)*Q(N-2,M,X)))) $
```
Usage of \( Q(N,M,X) \)

\[
\begin{align*}
(C1) & \quad Q(0,0,X); \quad 1 \\
(C2) & \quad Q(3,3,X); \quad 15 \\
(C3) & \quad Q(4,3,X); \quad 105 X \\
(C4) & \quad Q(6,2,X); \\
(D4) & \quad \frac{4^2 \left( 3465 X - 1890 X + 105 \right)}{8}
\end{align*}
\]

2.4.1.3 \( V(N,M) \)

A listing of this block and some examples are presented. The recursive formula used to generate the \( V_{n,m} \) coefficient is Equation (C-20) of Appendix C. The starting values for the recursion are given by Equation (C-22). The condition (C-25) has been included in this block in order to improve the computation time.

Listing of \( V(N,M) \)

\[
V(N,M) := \text{BLOCK[1],}
\]

\[
\begin{align*}
\text{/*THIS BLOCK COMPUTES THE COEFFICIENT V} & \quad V_{N,M} \\
\text{/*PROGRAMMER:} & \quad J-P. KANIECKI, MIT-FEBRUARY 1979 \\
\text{/*RESTRICTION: N MUST BE AN INTEGER >= 0}
\end{align*}
\]
IF N < 0 THEN RETURN(ERROR),
IF M > N THEN RETURN(0),
IF M = 0 AND N = 0 THEN RETURN(1),
IF INTEGERP((M+N)/2) = FALSE THEN RETURN(0),
IF M = N THEN RETURN(((2*N-1)!!)/((2*N)!)),
RETURN(((M-N+1)*V(N-2,M)/(M+N)))$

Use of $V(N,M)$

(C1) $V(0,0)$;
     (01) 0

(C2) $V(3,1)$;
     (02) 1
     (03) 8

(C3) $V(3,3)$;
     (03) 1
     (04) 48

(C4) $V(5,4)$;
     (04) 0

(C5) $V(9,1)$;
     (05) 7
     (06) 256

Now that all these fundamental blocks and functions have been presented, the generating functions and short periodic variations based on these previous blocks may be programmed in MACSYMA. The mean longitude will be designated by $L$ and the true longitude by $TL$.

2.4.2 Generating Functions

The blocks of the two closed form generating functions are presented. From the previous work of Zeis the following definitions are kept:
\[ c = p^2 + q^2 \]  
\[ x = (1 - h^2 - k^2)^{-1/2} \]  
\[ XN = n = \text{mean motion} \]

2.4.2.1 \( S(N) \)

This block constructs the closed form generating function expressed in the \( p \) and \( q \) elements using Equation (2-10).

Listing of \( S(N) \)

```plaintext
S(N):=BLOCK([]
    /*THIS BLOCK COMPUTES THE REAL PART OF THE CLOSED FORM GENERATING FUNCTION IN P AND Q FOR THE ZONAL HARMONIC J
    N
    */
    /*PROGRAMMER:
    J-P.KANIECKI,MIT-FEBRUARY 1979
    */
    /*RESTRICTIONS:N MUST BE AN INTEGER \( \geq 0 \)
    */
    IF N <= 0 THEN RETURN(ERROR),
    /*BLOCKS CALLED:VFUNCT1[N,R,M],
    SFUNCT1[N,M,S] (P,Q),
    HANSEN2[N,M,-1] (H,K)
    */
    S1(S):=VFUNCT1[N,S,0]*SFUNCT1[N,0,S] (P,Q),
    S2(S):=S1(S)*(TL-L)*HANSEN2[-N-1,-I,-1] (H,K),
    S3(I,S):=IF I = -S THEN 0
        ELSE HANSEN2[-N-1,-I,-1] (H,K)*%E^((I+S)*%I)*XL/((I+S)*%I),
    SSTAR(N):=SUM(S2(S),S,-N,N)+SUM(S1(S)*SUM(S3(I,S),I,-N+1,N-1),S,-N,N),
    RETURN(-MU*REAN/(AA(N+1)*XN)*J[N]*RATSIMP(REALPART(SSTAR(N))))
```

55
Usage of $S(N)$

(C1) $S(2)$:

\[ J \ 	ext{MU RE} \ ( (K \ Q \ - \ 2 \ H \ P \ Q \ - \ K \ P ) \ \sin(3 \ TL) \]

\[ + \ ( - \ H \ Q \ - \ 2 \ K \ P \ Q \ + \ H \ P ) \ \cos(3 \ TL) + (3 \ Q \ - \ 3 \ P ) \ \sin(2 TL) \]

\[ - \ 6 \ P \ Q \ \cos(2 \ TL) + (3 \ K \ Q \ + \ 6 \ H \ P \ Q \ - \ 3 \ K \ P + (C \ - \ 4 \ C \ + \ 1) \ K) \ \sin(TL) \]

\[ + \ (3 \ H \ Q \ - \ 6 \ K \ P \ Q \ - \ 3 \ H \ P + (-C \ + \ 4 \ C \ - \ 1) \ H) \ \cos(TL) \]

\[ + \ (C \ - \ 4 \ C \ + \ 1) \ TL + (-C \ + \ 4 \ C \ - \ 1) \ L) \ X / (A \ (2 \ C \ + \ 4 \ C \ + \ 2) \ X) \]

(C2) $S(3)$:

\[ J \ \text{MU RE} \ ( (6 \ H \ K \ Q \ + \ (9 \ K \ - \ 9 \ H \ ) \ P \ Q - 18 \ H \ K \ P \ Q \]

\[ + \ (3 \ H - 3 \ K ) \ P ) \ \sin(5 \ TL) + ((3 \ K \ - \ 3 \ H \ ) \ Q \ - 18 \ H \ K \ P \ Q \]

\[ + \ (9 \ H \ - \ 9 \ K \ ) \ P \ Q + 6 \ H \ K \ P \ ) \ \cos(5 \ TL) \]

\[ + \ (15 \ H \ Q \ + \ 45 \ K \ P \ Q \ - \ 45 \ H \ P \ Q \ - 15 \ K \ P ) \ \sin(4 \ TL) \]

\[ + \ (15 \ K \ Q \ - \ 45 \ H \ P \ Q \ - \ 45 \ K \ P \ Q + 15 \ H \ P \ ) \ \cos(4 \ TL) \]

\[ + \ ((30 \ K \ + \ 30 \ H \ + \ 60) \ P \ Q + (6 \ C \ - \ 18 \ C \ + \ 6) \ H \ K \ Q \]

\[ + \ (-10 \ K \ - \ 10 \ H \ - \ 20) \ P \ + ((3 \ C \ - \ 9 \ C \ + \ 3) \ K + (-3 \ C \ + \ 9 \ C \ - \ 3) \ H ) \]

\[ P \) \ \sin(3 TL) + ((10 \ K \ + \ 10 \ H \ + \ 20) \ Q \]

\[ + ((-30 \ K \ - \ 30 \ H \ - \ 60) \ P + (3 \ C \ - \ 9 \ C \ + \ 3) \ K + (-3 \ C \ + \ 9 \ C \ - \ 3) \ H ) \ Q \]

\[ + \ (-6 \ C \ + \ 18 \ C \ - \ 6) \ H \ K \ P \) \ \cos(3 \ TL) \]
\[
\begin{align*}
&+ (\cdot 30 \; H \; Q + 90 \; K \; P \; Q + (90 \; H \; P + (18 \; C - 54 \; C + 18) \; H) \; Q - 30 \; K \; P \\
&+ (18 \; C - 54 \; C + 18) \; K \; P) \; \sin(2 \; TL) + (30 \; K \; Q + 90 \; H \; P \; Q \\
&+ ((18 \; C - 54 \; C + 18) \; K - 30 \; K \; P) \; Q - 30 \; H \; P + (-18 \; C + 54 \; C - 18) \; H \; P) \\
&\cos(2 \; TL) + (-30 \; H \; K \; Q + (45 \; K - 45 \; H) \; P \; Q \\
&+ (90 \; H \; K \; P + (-18 \; C + 54 \; C - 18) \; H \; K) \; Q + (15 \; H - 15 \; K) \; P \\
&+ ((27 \; C - 81 \; C + 27) \; K + (9 \; C - 27 \; C + 9) \; H + 36 \; C - 108 \; C + 36) \; P) \\
&\sin(TL) + ((15 \; K - 15 \; H) \; Q + 90 \; H \; K \; P \; Q \\
&+ ((45 \; H - 45 \; K) \; P + (9 \; C - 27 \; C + 9) \; K + (27 \; C - 81 \; C + 27) \; H + 36 \; C \\
&- 108 \; C + 36) \; Q - 30 \; H \; K \; P + (-18 \; C + 54 \; C - 18) \; H \; K \; P) \; \cos(TL) \\
&+ ((-36 \; C + 108 \; C - 36) \; H \; Q + (36 \; C - 108 \; C + 36) \; K \; P) \; TL \\
&+ (36 \; C - 108 \; C + 36) \; H \; L \; Q + (-36 \; C + 108 \; C - 36) \; K \; L \; P) \; X \\
&/ (A \; (12 \; C + 36 \; C + 36 \; C + 12) \; XN) \\
\end{align*}
\]

2.4.2.2 \textbf{ST}(N)

This block uses directly the definition of the generating function (2-35) in the direction cosines of the inertial z-axis. The direction cosines (\(\alpha, \beta, \gamma\)) are represented in MACSYMA by (AL, BE, GA).
Listing of ST(N)

ST(N): = BLOCK([I],

/*THIS BLOCK COMPUTES THE REAL PART OF THE CLOSED FORM GENERATING
   FUNCTION USING DIRECTION COSINES FOR THE ZONAL HARMONIC J
   N */

/*PROGRAMMER:
   J-P. KANIECKI, MIT-MARCH 1979 */

/*RESTRICTIONS: N MUST BE AN INTEGER >= 0 */

IF N <= 0 THEN RETURN(ERROR),

/*BLOCKS AND FUNCTIONS CALLED: C(N,X,Y),
   SI(N,X,Y),
   V(N,M),
   Q(N,M,X),
   HANSEN2[N,M,-1] (H,K) */

/*DEFINITIONS:
   AL = -2*P/(1+P^2+Q^2),
   BE = 2*Q/(1+P^2+Q^2),
   GA = SQRT(1-AL^2-BE^2) */

ST1(M) := (IF M = 0 THEN 1 ELSE 2)*(C(M,AL,BE)+%I*SI(M,AL,BE)),
ST2(I,M) := IF I = M THEN 0 ELSE HANSEN2[-N-1,-I,-1] (H,K)*%E^((I-M)*%I)*TL)/((I-M)*%I),
STSTAR (N) := SUM(ST1(M)*V(N,M)*Q(N,M,GA)*((TL-L)*HANSEN2[-N-1,-M,-1] (H,K)+SUM(ST2(I,M),1,-N+1,N-1)),M,0,N),
RETURN (-MU*RE^N/(A^(N+1)*XN)*J[N]*RATSIMP(REALPART(STSTAR (N))))$
Usage of ST(N)

(C1) ST(2):

\[ J \mu \tau \epsilon \left( \left( (\beta - \alpha \kappa + 2 \alpha \lambda \eta \eta \sin(3 \tau) \right) + \left( (2 \alpha \lambda \eta \eta + (\alpha - \beta) \kappa \right) \cos(3 \tau) + \left( 3 \beta - 3 \alpha \right) \sin(2 \tau) \right) \right) \]

2.4.3 Short Periodic Variations

Two blocks are presented, one in the \( p \) and \( q \) elements, the other in the direction cosines \( (\alpha, \beta, \gamma) \) corresponding to the two generating functions. The blocks are obtained by programming directly Equation (2-37). The fact that there are only eleven non-zero \( (\bar{a}_i, \bar{a}_j) \) Poisson brackets has been used in order to improve the computational time. Several definitions which appear in the comments of these blocks have been made in order to get more compact analytical results. One will note that the result of each block can be expressed in two different ways; the result can be either a display of each \( \Delta a_i \):

\[ \Delta a_i = f_i(\tilde{\alpha}, \lambda, \eta) \quad (i = 1, \ldots, 6) \quad (2-41) \]

or a row vector whose six components are the right hand side of Equation (2-41) for \( i = 1, \ldots, 6 \).
The interest of the first representation is obviously to enable the user to see which parameters are determinant in the analytical formula. Unfortunately these formulas can be very complicated and getting physical insight just by looking at them is always a challenge! That is why the construction of graphical results is further emphasized and that is where the matrix representation is very useful in order to get plots as fast as possible. Furthermore, these two representations of the short periodic variation will be used for the other cases developed in this thesis.

2.4.3.1. **DELTAS(N,Z)**

This block computes the $\Delta a_i$ in the $p$ and $q$ elements, and therefore is based on $S(N)$.

**Listing of DELTAS(N,Z)**

```
DELTAS (N,Z):=BLOCK ([IS,DSDL,DSDT,DSOA,DSOH,DSOK,DSOP,DSDO,DSDL],
   /* This block computes the variations for the 6 equinoctial elements assuming that the disturbing force is only due to the zonal harmonic $J_N$ */
   /* Programmer: */
   /* J-P.Kaniecki, MIT-February 1979 */
   /* Restrictions: N must be an integer >= 0 */
```

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IF N <= 0 THEN RETURN(ERROR),

/*BLOCKS CALLED:S(N),
   POISSON(I,J,-1) */

/*LOCAL VARIABLES:
S=GENERATING FUNCTION FOR J,
DSDTL=PARTIAL DERIVATIVE OF S WITH RESPECT TO TL,
DTLDL=PARTIAL DERIVATIVE OF TL WITH RESPECT TO L,
DSDA=PARTIAL DERIVATIVE OF S WITH RESPECT TO A,
DSDH=PARTIAL DERIVATIVE OF S WITH RESPECT TO H,
DSDK=PARTIAL DERIVATIVE OF S WITH RESPECT TO K,
DSDP=PARTIAL DERIVATIVE OF S WITH RESPECT TO P,
DSDQ=PARTIAL DERIVATIVE OF S WITH RESPECT TO Q,
DSDL=PARTIAL DERIVATIVE OF S WITH RESPECT TO L, */

/*DEFINITIONS:
XN=SQRT(MU/A^3),
C=P^2+Q^2,
X=1/SQRT(1-H^2-K^2),
AR=(1+H*SIN(TL)+K*COS(TL))*X^2,
DTLDL=AR^2/X,
DTLDH=PARTIAL DERIVATIVE OF TL WITH RESPECT TO H
   =-(K+SIN(TL))*X^2-AR*COS(TL)-AR^2*K/(X+1),
DTL0K=PARTIAL DERIVATIVE OF TL WITH RESPECT TO K
   =(H+SIN(TL))*X^2+AR*SIN(TL)+AR^2*H/(X+1), */

S:S(N),
DSDTL:DIFF(S,TL),
DTLDL:AR^2/X,
GRADEF(X,H,H*X^3),
GRADEF(X,K,K*X^3),
GRADEF(C,P,2*P),
GRADEF(C,Q,2*Q),
DSDA:DIFF(S,A),
DSDH:DIFF(S,H)+DSDTL*DTLDH,
DSDK:DIFF(S,K)+DSDTL*DTL0K,
DSDP:DIFF(S,P),
DSDQ:DIFF(S,Q),
DSDL:DIFF(S,L)+DSDTL*DTLDL,

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/*DEFINITIONS OF THE VARIATIONS OF THE EQUINOCTIAL ELEMENTS*/

"DELTA-A":-POISSON(A,L,-1)*DSOL,
"DELTA-H":-POISSON(H,K,-1)*DSDK-POISSON(H,P,-1)*DSDP 
-POISSON(H,Q,-1)*DSDO-POISSON(H,L,-1)*DSDL,
"DELTA-K":-POISSON(K,H,-1)*DSDH-POISSON(K,P,-1)*DSOK 
-POISSON(K,Q,-1)*DSDP-POISSON(K,L,-1)*DSDL,
"DELTA-P":-POISSON(P,H,-1)*DSOH-POISSON(P,K,-1)*DSOK 
-POISSON(P,O,-1)*DSDP-POISSON(P,L,-1)*DSDL,
"DELTA-Q":-POISSON(Q,H,-1)*DSDH-POISSON(Q,K,-1)*DSOK 
-POISSON(Q,P,-1)*DSOK-POISSON(Q,L,-1)*DSDL,
"DELTA-L":-POISSON(L,A,-1)*DSDA-POISSON(L,H,-1)*DSDA 
-POISSON(L,K,-1)*DSDH-POISSON(L,P,-1)*DSDL 
-POISSON(L,O,-1)*DSDL-3*S/(XN*A^2),

/*DISPLAY OF THE RESULTS*/

IF Z = NO THEN RETURN(["DELTA-A", "DELTA-H", "DELTA-K", "DELTA-P", 
"DELTA-Q", "DELTA-L"]),
RETURN(DISPLAY(["DELTA-A", "DELTA-H", "DELTA-K", "DELTA-P", "DELTA-Q", 
"DELTA-L"]))$

Usage of DELTAS(N,Z)

(C1) DELTAS(2,YES);

DELTA-A = 2 (-----------------------------

A (2C + 4C + 2) XN

2
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2

A XN)
\[
\Delta H = \frac{(3 J + K \mu \text{RE} ((K Q - 2 H P Q - K P) \sin(3 TL))}{2}
\]
\[
+ \frac{(- H Q - 2 K P Q + H P) \cos(3 TL) + (3 Q - 3 P) \sin(2 TL)}{2}
\]
\[
- 6 P Q \cos(2 TL) + (3 K Q + 6 H P Q - 3 K P + (C - 4 C + 1) K) \sin(3 TL)
\]
\[
+ (3 H Q - 6 K P Q - 3 H P + (- C + 4 C - 1) H) \cos(3 TL)
\]
\[
+ (C - 4 C + 1) TL + (- C + 4 C - 1) L) X / (A (2 C + 4 C + 2) XN)
\]
\[
+ J D T L D K \mu \text{RE} ((- 3 (- H Q - 2 K P Q + H P) \sin(3 TL))}{2}
\]
\[
+ 3 (K Q - 2 H P Q - K P) \cos(3 TL) + 12 P Q \sin(2 TL)
\]
\[
+ 2 (3 Q - 3 P) \cos(2 TL) - (3 H Q - 6 K P Q - 3 H P)
\]
\[
+ (- C + 4 C - 1) H) \sin(3 TL) + (3 K Q + 6 H P Q - 3 K P)
\]
\[
+ (C - 4 C + 1) K) \cos(3 TL) + (C - 4 C + 1) X / (A (2 C + 4 C + 2) XN)
\]
\[
+ J \mu \text{RE} ((Q - P) \sin(3 TL) - 2 P Q \cos(3 TL))}{2}
\]
\[
+ (3 Q - 3 P + C - 4 C + 1) \sin(2 TL) - 6 P Q \cos(3 TL)) X
\]
\[
(A (2 C + 4 C + 2) XN)) / (A X XN) + (C + 1) K Q X
\]
\[
(J \mu \text{RE} ((2 K Q - 2 H P) \sin(3 TL) + (- 2 H Q - 2 K P) \cos(3 TL))}{2}
\]
\[
+ 6 Q \sin(2 TL) - 6 P \cos(2 TL) + (K (4 C Q - 8 Q) + 6 K Q + 6 H P) \sin(3 TL)
\]
\[
+ (H (8 Q - 4 C Q) + 6 H Q - 6 K P) \cos(3 TL) + (4 C Q - 8 Q) TL
\]
\[ + L (8 Q - 4 C P)) X / (A (2 C + 4 C + 2) XN)\]

\[ - J (8 C Q + 8 Q) R E ((K Q - 2 H P Q - K P) \sin(3 TL)) \]

\[ + (- H Q - 2 K P Q + H P) \cos(3 TL) + (3 Q - 3 P) \sin(2 TL)\]

\[ - 6 P Q \cos(2 TL) + (3 K Q + 6 H P Q - 3 K P + (C - 4 C + 1) K) \sin(TL)\]

\[ + (3 H Q - 6 K P Q - 3 H P + (- C + 4 C - 1) H) \cos(TL)\]

\[ + (C - 4 C + 1) TL + (- C + 4 C - 1) L X / (A (2 C + 4 C + 2) XN)\]

\[ / (2 A XN) + (C + 1) K P X (J MU RE ((- 2 H Q - 2 K P) \sin(3 TL)) \]

\[ + (2 H P - 2 K Q) \cos(3 TL) - 6 P \sin(2 TL) - 8 Q \cos(2 TL)\]

\[ + (6 H Q + K (4 C P - 8 P) - 6 K P) \sin(TL)\]

\[ + (- 6 K Q + H (8 P - 4 C P) - 6 H P) \cos(TL) + (4 C P - 8 P) TL\]

\[ + L (8 P - 4 C P)) X / (A (2 C + 4 C + 2) XN)\]

\[ - J (8 C Q + 8 P) R E ((K Q - 2 H P Q - K P) \sin(3 TL)) \]

\[ + (- H Q - 2 K P Q + H P) \cos(3 TL) + (3 Q - 3 P) \sin(2 TL)\]

\[ - 6 P Q \cos(2 TL) + (3 K Q + 6 H P Q - 3 K P + (C - 4 C + 1) K) \sin(TL)\]

\[ + (3 H Q - 6 K P Q - 3 H P + (- C + 4 C - 1) H) \cos(TL)\]

\[ + (C - 4 C + 1) TL + (- C + 4 C - 1) L X / (A (2 C + 4 C + 2) XN)\]

\[ J (- C + 4 C - 1) MU RE X \]

\[ / (2 A XN) - H \frac{-------------}{3} \]

\[ A (2 C + 4 C + 2) XN \]
+ J AR\textsuperscript{2} MU RE \cdot (- 3 (- H Q - 2 K P Q + H P ) \sin(3 TL) + 2
\begin{align*}
+ 3 (K Q - 2 H P Q - K P ) \cos(3 TL) + 12 P Q \sin(2 TL)
+ 2 (3 Q - 3 P ) \cos(2 TL) - (3 H Q - 6 K P Q - 3 H P
+ (- C + 4 C - 1) H) \sin(TL) + (3 K Q + 6 H P Q - 3 K P
+ (C - 4 C + 1) K) \cos(TL) + C - 4 C + 1) X /(A (2 C + 4 C + 2) XN))

/ (A (X + 1) XN)
\]DELTA-K = \\
\]DELTA-P = \\
\]DELTA-Q = \\
\]DELTA-L = \\
(D1) DONE

2.4.3.2 DELTAST(N,Z)

This block uses ST(N) and computes the short periodic variations in terms of the direction cosines of the inertial z-axis.

Listing of DELTAST(N,Z)

DELTAST(N,Z):=BLOCK([ST,DSSTL,DTLQL,DSDA,DSDH,DSDK,DSDP,DSDQ,DSDL],

/*THIS BLOCK COMPUTES THE VARIATIONS OF THE 6 EQUINOCTIAL ELEMENTS ASSUMING THAT THE DISTURBING FORCE IS ONLY DUE TO THE ZONAL HARMONIC J*/

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{*PROGRAMMER:
    J-P.KANIECKI,MIT-MAY 1979
*}/

/*RESTRICTIONS:N MUST BE AN INTEGER >= 0
*/

IF N <= 0 THEN RETURN(ERROR),

/*BLOCKS CALLED:ST(N),
    POISSON(I,J,-1)
*/

/*LOCAL VARIABLES:
ST=GENERATING FUNCTION FOR J,
    N
DSDTL=PARTIAL DERIVATIVE OF ST WITH
    RESPECT TO TL,
DTDL=PARTIAL DERIVATIVE OF TL WITH
    RESPECT TO L,
DSDA=PARTIAL DERIVATIVE OF ST WITH
    RESPECT TO A,
DSDH=PARTIAL DERIVATIVE OF ST WITH
    RESPECT TO H,
DSDK=PARTIAL DERIVATIVE OF ST WITH
    RESPECT TO K,
DSDP=PARTIAL DERIVATIVE OF ST WITH
    RESPECT TO P,
DSDQ=PARTIAL DERIVATIVE OF ST WITH
    RESPECT TO Q,
DSDL=PARTIAL DERIVATIVE OF ST WITH
    RESPECT TO L
*/

/*DEFINITIONS:
XN=SQRT(MU/A^3),
X=1/ SQRT(1-H^2-K^2),
AR=(1+H*SIN(TL)+K*COS(TL))*X^2,
DTDL=AR^2/X,
DTLDH=PARTIAL DERIVATIVE OF TL WITH
    RESPECT TO H
    =(K+SIN(TL))*X^2-AR*COS(TL)-AR*2K/(1+X),
DTLDK=PARTIAL DERIVATIVE OF TL WITH
    RESPECT TO K
    =(H+SIN(TL))*X^2+AR*SIN(TL)+AR^2H/(1+X),
DALDP=PARTIAL DERIVATIVE OF AL WITH
    RESPECT TO P
    =AL^2-1-GA,
*/
\[ \frac{\partial A}{\partial Q} = -A \cdot B, \]
\[ \frac{\partial \beta}{\partial P} = A \cdot B, \]
\[ \frac{\partial \beta}{\partial Q} = A + 1 - B \cdot A^2, \]
\[ \frac{\partial \gamma}{\partial P} = A \cdot (1 + \gamma), \]
\[ \frac{\partial \gamma}{\partial Q} = -B \cdot (1 + \gamma). \]

\[ \text{ST: ST(N),} \]
\[ \text{DSOTL: DIFF(ST, TL),} \]
\[ \text{DTOL: AR^2/X,} \]
\[ \text{GRADEF(X,H,H*X^3),} \]
\[ \text{GRADEF(X,K,K*X^3),} \]
\[ \text{GRADEF(AL,P,DALDP),} \]
\[ \text{GRADEF(AL,Q,DALOQ),} \]
\[ \text{GRADEF(BE,P,DBEOP),} \]
\[ \text{GRADEF(BE,Q,DBEDQ),} \]
\[ \text{GRADEF(GA,P,DGADP),} \]
\[ \text{GRADEF(GA,Q,DGADO),} \]
\[ \text{DSDA: DIFF(ST, A),} \]
\[ \text{DSOH: DIFF(ST,H) + DSOTL*DTLDH,} \]
\[ \text{DSOK: DIFF(ST,K) + DSOTL*DTLDK,} \]
\[ \text{DSOP: DIFF(ST,P),} \]
\[ \text{DSOQ: DIFF(ST,Q),} \]
\[ \text{DSOL: 0FF(ST,L) + OSOTL*DTLOL,} \]

\[ /* \text{DEFINITIONS OF THE VARIATIONS OF THE EQUINOCTIAL ELEMENTS} */ \]

"DELTA-A": -POISSON(A,L,-1) * DSOL, 
"DELTA-H": -POISSON(H,K,-1) * DSOK - POISSON(H,P,-1) * DSOP - POISSON(H,Q,-1) * DSOD, 
"DELTA-K": -POISSON(K,H,-1) * DSOH - POISSON(K,P,-1) * DSOK - POISSON(K,Q,-1) * DSOD, 
"DELTA-P": -POISSON(P,H,-1) * DSOP - POISSON(P,K,-1) * DSOK - POISSON(P,Q,-1) * DSOD, 
"DELTA-Q": -POISSON(Q,H,-1) * DSOD - POISSON(Q,K,-1) * DSOK - POISSON(Q,P,-1) * DSOD - POISSON(Q,L,-1) * DSOL, 
"NTA": -POISSON(L,A,-1) * DSDA - POISSON(L,H,-1) * DSOH - POISSON(L,K,-1) * DSOK - POISSON(L,P,-1) * DSOD - POISSON(L,Q,-1) * DSOQ - 3*ST/(XN*A^2), 

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/*Display of the Results*/

if Z = NO then return ("DELTA-A", "DELTA-H", "DELTA-K", "DELTA-P", "DELTA-Q", "DELTA-L");
return (display ("DELTA-A", "DELTA-H", "DELTA-K", "DELTA-P", "DELTA-Q", "DELTA-L"));


Usage of DELTAST(N, Z)

(C1) DELTAST(2, YES);

DELTA-A = ........;

DELTA-H = ........;

DELTA-K = -(3 J H MURE (((BE - AL) K + 2 AL BE H) SIN(3 TL)) +

2 2 2
2 AL BE K + (AL - BE) H) COS(3 TL) + (3 BE - 3 AL) SIN(2 TL) +

6 AL BE COS(2 TL) + ((6 GA + 3 BE - 3 AL - 2) K - 6 AL BE H) SIN(TL) +

(6 AL BE K + (-6 GA + 3 BE - 3 AL + 2) H) COS(TL) + (6 GA - 2) TL +

2 5 3
(2 - 6 GA) L) X / (8 A XN) + J DTLDH MURE 

2 2
(-3 (2 AL BE K + (AL - BE) H) SIN(3 TL) +

2 2
3 (((BE - AL) K + 2 AL BE H) COS(3 TL) - 12 AL BE SIN(2 TL) +

2 2
2 (3 BE - 3 AL) COS(2 TL) - (6 AL BE K + (-6 GA + 3 BE - 3 AL + 2) H) SIN(TL) +

2 2 2
((6 GA + 3 BE - 3 AL - 2) K - 6 AL BE H) COS(TL) + 6 GA - 2) X

2 5 3
(8 A XN) + J MURE (2 AL BE SIN(3 TL) + (AL - BE) COS(3 TL))

2

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\[-6 \text{AL BE SIN(TL)} + (-6 \text{GA} + 3 \text{BE} - 3 \text{AL} + 2) \text{COS(TL)} \times (8 \text{A XN}) \]
\[\frac{J \cos(2 - 6 \text{GA} \text{MU RE} X)^2}{(A X XN)} \frac{\text{XN}^2}{K (------------------------ \frac{3}{8 \text{A XN}}) - J (C + 1) \text{H MU Q RE}^2} \]
\[2 \frac{J \cos(6 \text{GA} + 3 \text{BE} - 3 \text{AL} - 2) \text{K} - 6 \text{AL BE H) COS(TL)} + 6 \text{GA} - 2) X^2}{(8 \text{A XN})} / (A (X + 1) XN) - J (C + 1) \text{H MU Q RE}^2 \]
\[\left( (2 \text{BE DBEDO} - 2 \text{AL DALDO}) \text{K} + 2 \text{AL DBEDO} H + 2 \text{BE DALDO H} \right) \text{SIN(3 TL)} \]
\[+ (2 \text{AL DBEDO} \text{K} + 2 \text{BE DALDO K} + (2 \text{AL DALDO} - 2 \text{BE DBEDO}) \text{H}) \text{COS(3 TL)} \]
\[+ (6 \text{BE DBEDO} - 6 \text{AL DALDO}) \text{SIN(2 TL)} + 6 \text{AL DBEDO COS(2 TL)} \]
\[+ 6 \text{BE DALDO COS(2 TL)} + ((12 \text{DGADO GA} + 6 \text{BE DBEDO} - 6 \text{AL DALDO}) \text{K} - 6 \text{AL DBEDO H} - 6 \text{BE DALDO H}) \text{SIN(TL)} \]
\[+ (6 \text{AL DBEDO K} + 6 \text{BE DALDO K} + (-12 \text{DGADO GA} + 6 \text{BE DBEDO} - 6 \text{AL DALDO}) \text{H}) \text{COS(TL)} + 12 \text{DGADO GA TL} - 12 \text{DGADO GA L}) X / (16 \text{A XN}) \]
\[2 \frac{- J (C + 1) \text{H MU P RE} \left( (2 \text{BE DBEDP} - 2 \text{AL DALDP}) \text{K} + 2 \text{AL DBEDP H} \right)^2}{2} \]
\[+ 2 \text{BE DALDP H) SIN(3 TL)} + (2 \text{AL DBEDP K} + 2 \text{BE DALDP K} \]
\[+ (2 \text{AL DALDP} - 2 \text{BE DBEDP}) \text{H} \text{COS(3 TL)} \]
\[+ (6 \text{BE DBEDP} - 6 \text{AL DALDP}) \text{SIN(2 TL)} + 6 \text{AL DBEDP COS(2 TL)} \]

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+ 6 BE DALOP COS(2 TL) + ((12 DGADP GA + 6 BE DBEOP - 6 AL DALOP) K
- 6 AL DBEOP H - 6 BE DALOP H) SIN(TL)
+ (6 AL DBEOP K + 6 BE DALOP K + (-12 DGADP GA + 6 BE DBEOP - 6 AL DALOP) H)
COS(TL) + 12 DGADP GA TL - 12 DGADP GA L) X /(16 A XN )

DELTA-P = .......
DELTA-Q = .......
DELTA-L = .......

(01) DONE

2.5 Plots of the Short Periodic Variations Versus Time

Because of the complexity of the formulas given by the blocks
DELTAS(N,Z) or DELTAST(N,Z) for the short periodic variations, the
simplest way to get good physical insight into what is going on is
to get numerical results for these variations for a particular set of
mean equinoctial elements (\(\overline{a}_0, \overline{h}_0, \overline{k}_0, \overline{p}_0, \overline{q}_0, \overline{\lambda}_0\)).

In this thesis, the variable is chosen to be the time \(t\). As
(\(\overline{a}_0, \overline{h}_0, \overline{k}_0, \overline{p}_0, \overline{q}_0\)) are the five slowly varying mean elements, it is
reasonable to assume that they can be taken as constants over at least
one period of the orbit. As for the mean longitude \(\overline{\lambda}\), it is assumed
to have the form

\[
\overline{\lambda} = \overline{\lambda}_0 + \overline{n}(t - t_0)
\] (2-42)
The initial time will arbitrarily be taken equal to zero in this thesis. Therefore

\[ \lambda = \lambda_0 + \bar{n}t \]  

(2-43)

In order to have numerical results in function of the time, the short periodic variations must have the following form.

\[ \Delta a_i = \delta_i^j \frac{\bar{a}_0, h_0, k_0, p_0, q_0, \lambda_0}{(t)} \ (i = 1, \ldots, 6) \]  

(2-44)

That is not the form of the results given by the blocks DELTAS(N, Z) or DELTAST(N, Z); therefore, several steps are still necessary. One must:

-- express \( \lambda \) in function of the time [Equation (2-43)]

-- express in terms of the six mean elements \( (\bar{a}_0, h_0, k_0, p_0, q_0, \lambda_0) \) all the variables defined inside the previous blocks (the partial derivatives are given in Appendices A and B)

-- plug in the numerical values for \( J_n, \mu, \) and \( R_e \)

-- express the true longitude in function of the time or, in other words, solve Kepler's equation.

Unfortunately, the form of Equation (2-44) cannot be achieved since there is no analytical solution for Kepler's equation. Therefore, Equation (2-44) will be replaced by the following system.
\[ \Delta a_i = \frac{\delta^i}{a_0, h_0, k_0, p_0, q_0, \lambda_0} (L, t) \]  \hspace{1cm} (2-45)

and

\[ L = L(\lambda_0, h_0, k_0, t) \]  \hspace{1cm} (2-46)

Because of this system, two more steps are required to have numerical results from the blocks DELTAS or DELTAST, and are presented next.

2.5.1 Short Periodic Variations as Functions of the Time and the True Longitude Only

These two blocks (corresponding to both generating functions) lead to the form of Equation (2-45) and obviously their results are absolutely identical.

2.5.1.1 DELTAS1(N,A0,H0,K0,P0,Q0,L0)

A listing of this block based on DELTAS(N,Z) is presented.

```
DELTAS1(N,A0,H0,K0,P0,Q0,L0);=BLOCK([TEMPI, A, H, K, P, Q, C, MU, RE, XN, L, X, AR, DTLOH, DTLOK],

/*THIS BLOCK COMPUTES THE VARIATIONS OF THE 6 EQUINOCTIAL ELEMENTS DUE TO THE ZONAL HARMONIC J NUMERICALLY FROM THEIR INITIAL CONDITIONS(A0,H0,K0,P0,Q0,L0), THE FORMULAS ARE EXPRESSED IN FUNCTION OF THE TIME(T) AND THE TRUE LONGITUDE(TL). */
```

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/*PROGRAMMER:
J-P.KANIECKI,MIT-MAY 1979
*/

/*BLOCK CALLED: DELTAS(N,NO)
*/

/*LOCAL VARIABLES:
TEMP1=DELTAS(N,NO) EXPRESSED AS A ROW VECTOR,
A,H,K,P,Q=THE 5 SLOWLY VARIABLE EQUINOCTIAL ELEMENTS,
C=P^2+Q^2,
MU=GRAVITATIONAL CONSTANT FOR THE EARTH IN KM /S ,
RE=EQUATORIAL RADIUS OF THE EARTH IN KM,
XN=MEAN MOTION OF THE SATELLITE,
L=M.EAN LONGITUDE IN RADIANS EXPRESSED LINEARLY IN FUNCTION
OF THE TIME,
X=1/SQRT(1-H^2-K^2),
AR=(1+H*SIN(TL)+K*COS(TL))X^2, 

DTLDK=-----,

dK

dTL

DTLDH=-----,


dH
*/

TEMP1: DELTAS(N,NO),
J\[2]\:: 0.108265*10^-2,
J\[3]\:: -0.254503*10^-5,
J\[4]\:: -0.16715*10^-5,
A: AO,
H: HO,
K: K0,
P: PO,
Q: 00, C: P0^2+00^2,
MU: 398600.8,
RE: 6378.145,
XN: SQRT(398600.8/A0^3),
L: L0+631.3484/A0^3/2*T,
X: (1-K0^2-H0^2)^-(1/2),
AR: (1+H0*SIN(TL)+K0*COS(TL))/(1-H0^2-K0^2),

DTLDH: -K0*(H0*SIN(TL)+K0*COS(TL)+1)^2/((1-K0^2-H0^2)^2*(1+
(1-K0^2-H0^2)^2)-(1/2)))*COS(TL)/(1-H0^2-K0^2)/(1-H0*SIN(TL)
+K0*COS(TL)) - (K0+COS(TL))/(1-K0^2-H0^2),

DTLDH: H0*(H0*SIN(TL)+K0*COS(TL)+1)^2/((1-K0^2-H0^2)^2*(1+
(1-H0^2-K0^2)^2-(1/2)))*SIN(TL)/(1-K0^2-H0^2)/(1-H0*SIN(TL)
+K0*COS(TL)) + (H0+SIN(TL))/(1-H0^2-K0^2),

RETURN EV (TEMPI) $
2.5.1.2 \texttt{DELTAST1(N,A0,H0,K0,P0,Q0,L0)}

A listing of this block based on \texttt{DELTAST(N,Z)} is presented.

\texttt{DELTAST1(N,A0,H0,K0,P0,Q0,L0):=BLOCK(\{TEMP2,A,H,K,P,Q,RE,MU,C,XN,L,X, AR,DTLDH,DTLOK,AL,BE,GA,DALDP, DALDQ, DBEDP, DBEDQ, DGADP, DGADO\},}

/*THIS BLOCK COMPUTES THE VARIATIONS OF THE 6 EQUINOCTIAL ELEMENTS DUE TO THE ZONAL HARMONIC J NUMERICALLY FROM THEIR INITIAL N CONDITIONS(A0,H0,K0,P0,Q0,L0), THE FORMULAS ARE FUNCTIONS OF THE TIME(T) AND OF THE TRUE LONGITUDE(TL). *

/*PROGRAMMER: J-P. KANIECKI, MIT-JUNE 1979 *

/*BLOCK CALLED:DELTAST(N,NO), *

/*LOCAL VARIABLES: 
\texttt{.TEMP2=} \texttt{DELTAST(N,NO)} EXPRESSED AS A ROW VECTOR, A,H,K,P,Q=THE 5 SLOWLY VARIABLE EQUINOCTIAL ELEMENTS, C=P^2+Q^2, 3 2
MU=GRAVITATIONAL CONSTANT FOR THE EARTH IN KM /S , RE=EQUATORIAL RADIUS OF THE EARTH IN KM, XN=MEAN MOTION OF THE SATELLITE, L=TRUE LONGITUDE IN RADIANS EXPRESSED LINEARLY IN FUNCTION OF THE TIME(T), X=1/SQRT(1-H^2-K^2), AR=(1+H*SIN(TL)+K*COS(TL))*X^2, dTL DTLDH=-----, dH dTL DTLOK=-----, dK DTLDH=-----, dAL AL=-2*P/(1+P^2+Q^2), BE=2*Q/(1+P^2+Q^2), GA=(1-P^2-Q^2)/(1+P^2+Q^2), dAL DALDP=-----, dP dAL DALDQ=-----, dQ 

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dBE
DBEP=-----,
dP
dBE
DBEQ=-----,
dQ
dGA
DGAP=-----,
dP
dGA
DGADQ=-----,
dQ

* /

TEMP2: DELTAST (N, NO),
J[2]: 0.108256*10^(-2),
J[3]: -0.254503*10^(-5),
J[4]: -0.16715*10^(-5),
A: A0,
H: H0,
K: K0,
P: P0,
Q: Q0,
RE: 6378.145,
MU: 398600.8,
C: P0^2+Q0^2,
XN: SQRT (398600.8/A0A3),
L: L0+G31.3484/A0^3/2,
X: (1-K0^2-H0^2)^(-1/2),
AR: (1+H0*SIN(TL)+K0*COS(TL))*X^2,
DTLDH: -K0*(H0*SIN(TL)+K0*COS(TL)+1)^2/((1-K0^2-H0^2)^2* (1+ (1-K0^2-H0^2)^(-1/2)))-COS(TL)/((1-K0^2-H0^2)* (1+H0*SIN(TL)) +K0*COS(TL))-(K0+COS(TL))/(1-K0^2-H0^2),
DTLDK: H0*(H0*SIN(TL)+K0*COS(TL)+1)^2/((1-K0^2-H0^2)^2* (1+ (1-H0^2-K0^2)^(-1/2)))+SIN(TL)/((1-K0^2-H0^2)* (1+H0*SIN(TL)) +K0*COS(TL))+(H0+SIN(TL))/(1-K0^2-H0^2),
AL: -2*P0/(1+00^2+P0^2),
BE: 2*Q0/(1+P0^2+Q0^2),
GA: (1-P0^2-Q0^2)/(1+P0^2+Q0^2),
DALD: AL^2-1-GA,
DALDQ: -AL*BE,
DBEDP: AL*BE,
DBEDQ: GA+1-BE^2,
DGAP: AL*(1+GA),
DGADQ: -BE*(1+GA),
RETURN (EV (TEMP2))$
2.5.2 Numerical Solution of Kepler's Equation

There is no implicit Kepler's equation involving the true longitude, but this equation can be written in terms of the equinoctial elements $(\lambda,h,k)$ and of the eccentric longitude $F$ [Equation (A-2)].

\[ \lambda = F + h \cos F - k \sin F \]  
(2-47)

The numerical method used to obtain the eccentric longitude is the classical Newton's method, a listing of which is presented.

Listing of $\text{EL}(L,H,K,\text{EP})$

\begin{verbatim}
$\text{EL}(L,H,K,\text{EP})$: =BLOCK((F:L, ITERMAX:10, ITER:1),

    /* THIS BLOCK COMPUTES THE ECCENTRIC LONGITUDE SOLVING KEPLER'S EQUATION IN TERMS OF THE EQUINOCTIAL ELEMENTS BY NEWTON'S METHOD WITH AN ACCURACY EP. */

    /* PROGRAMMER:
        J-P. KANIECKI, MIT-APRIL 1979 */

    /* LOCAL VARIABLES:
        F = MEAN LONGITUDE,
        ITER:1 = INITIALIZATION FOR THE ITERATION,
        ITERMAX:10 = MAXIMUM NUMBER OF ITERATIONS, */

        DF(F,L,H,K):=EV(((F+H*\cos(F)-K*\sin(F)-L)/(1-H*\sin(F)-K*\cos(F)))*NUMER),
        IF ABS(DF(F,L,H,K)) > 1 THEN F:=DF(F,L,H,K)/ABS(DF(F,L,H,K)),
        FOR ITER WHILE (IF ABS(DF(F,L,H,K)) > \text{EP} THEN TRUE) DO
        (F:=DF(F,L,H,K),ITER:=ITER+1),
        IF ITER < ITERMAX THEN RETURN(F),

        /* IF THE NUMBER OF ITERATIONS > 10 THEN IT RETURNS AN ERROR MESSAGE */

        RETURN("ERROR TOO MANY ITERATIONS")$ 
\end{verbatim}

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In order to get the true longitude, the following relation between the eccentric and the true longitudes

\[
\tan \left( \frac{L-F}{2} \right) = \frac{k \sin F - h \cos F}{1 + \sqrt{1 - h^2 - k^2} - (k \cos F + h \sin F)}
\]

(2-48)

is used in the block TL(L,H,K,EP), the listing of which is presented.

**Listing of TL(L,H,K,EP)**

```plaintext
TL(L,H,K,EP):=BLOCK([F],

/* THIS BLOCK COMPUTES THE TRUE LONGITUDE FROM THE ECCENTRIC LONGITUDE WITH AN ACCURACY EP */

/* PROGRAMMER: J-P. KANIECKI, MIT-APRIL 1979 */

/* LOCAL VARIABLE: F = ECCENTRIC LONGITUDE WITH AN ACCURACY EP. */

F:EL(L,H,K,EP),
RETURN(EV(F+2*ATAN((K*SIN(F)-H*COS(F))/(1+SQRT(1-H^2-K^2)-(K*COS(F)+H*SIN(F))))),NUMER))$ 
```

2.5.3 **How to Use the Facility**

First one must load the file containing the fundamental blocks by using the MACSYMA command

```
LOADFILE(POT2,LISP,DSK,KANIEK)$
```
Then load the file containing the blocks developed in this section by typing

\[ \text{BATCH(ZONAL1,>,DSK,KANIEK);} \]

Note: ZONAL1 corresponds to the generating function in the \( p \) and \( q \) elements; if one wants to use the other form in the direction cosines, just replace ZONAL1 by ZONAL2.

At this point all the blocks are ready to execute, but as we are interested in plots just type for any set of mean equinoctial elements and for the \( n \)th desired harmonic

\[ \text{DELTAS1(N,A0,H0,K0,P0,Q0,L0)} \]

or alternatively, \( \text{DELTAST1(...)} \) if ZONAL2 is to be used.

It is highly recommended to use the dollar sign at the end of this command since the direct result is not really interesting in itself. Then by using jointly the MACSYMA commands \( \text{EXPAND} \) and \( \text{TRIGREDUCE} \) applied to the previous result, the formulas in TL and T will have not only a nicer form but a dramatically smaller size also. In consequence, the computational time for the points of the plots will be minimized

\[ \text{Z:EXPAND(TRIGREDUCE(EXPAND(%),TL))} \]
This result has been called Z (or anything else) for practical reasons since it is going to be needed again. For the same reasons, a function computing the true longitude in function of the time using the previous (A0, H0, K0, L0) can be defined with a desired accuracy EP, for example, F(T)

\[ F(T) = TL(L0 + 631.3484*A0^{-3/2}*T, H0, K0, EP) \]

In this function the true longitude has been replaced by \( \lambda_0 + \bar{n}t \) [Equation (2-43)], and EP is the desired accuracy for Kepler's solution.

Because of the structure of the blocks used, Z has the form of a row vector, and now will appear clearly the interest of this representation since Z[N] in MACSYMA represents the nth component of this vector. The six plots of the short periodic variations are given by the final unique command

\[
\text{FOR N:1 THRU 6 DO(PLOT2(SUBST(F(T),TL,Z[N]),T,Ti,Tf));}
\]

the order of the plots being \( \Delta a, \Delta h, \Delta k, \Delta p, \Delta q, \Delta \lambda \). The initial time \( T_i \) and the final time \( T_f \) for these plots must be taken as

\[ T_i = 0 \text{ s} \]
\[ T_f = 3.167823 \times 10^{-3} \pi A0^{3/2} \text{ s} \]

(Kepler's third law)
in order to have the short periodic variations over one period. They average to zero over such an interval.

2.5.4 Application to Real World Problems

The zonal short periodics of three real world satellites are plotted for different harmonics. These satellites are:

-- low altitude Earth observer
-- Atmospheric Explorer (AE-C)
-- International Sun Earth Explorer (ISEE)

It has been checked that both ZONAL1 and ZONAL2 give identical results. However, ZONAL2 executes slightly faster. Anyway, using either one or the other, the limitation of this utility is the same; they work till J_4 included. For higher harmonics, the LISP machine should be the tool to use. Another limitation due to the MC machine is its accuracy since it operates in a single precision mode at the level of the plots.

The exact orbital initial conditions used for the three test cases are:

Case 1 - Low altitude circular
\[ \bar{a}_0 = 6636.378 \text{ km} \]
\[ \bar{h}_0 = -0.0098178 \]
\[ \bar{k}_0 = 0.004008 \]
\[ \bar{p}_0 = 0.67373 \]
\[ \bar{q}_0 = -0.023467 \]
\[ \bar{\lambda}_0 = 1.6926 \text{ rad} \]

**Case 2 - Low altitude elliptic**

\[ \bar{a}_0 = 8514.4833 \text{ km} \]
\[ \bar{h}_0 = -0.02244518 \]
\[ \bar{k}_0 = -0.23685168 \]
\[ \bar{p}_0 = 0.67497841 \]
\[ \bar{q}_0 = -0.018305 \]
\[ \bar{\lambda}_0 = 1.757610889 \text{ rad} \]

**Case 3 - High eccentricity, high altitude**

\[ \bar{a}_0 = 70850 \text{ km} \]
\[ \bar{h}_0 = 0.6788752 \]
\[ \bar{k}_0 = 0.5755244 \]
\[ \bar{p}_0 = 0.1966544 \]
\[ \bar{q}_0 = 0.1679587 \]
\[ \bar{\lambda}_0 = 0.0 \text{ rad} \]

The plots of \( \Delta a, \ldots, \Delta q \), and \( \Delta \lambda \) for these three test cases follow for the \( J_2 \), \( J_3 \) and \( J_4 \) perturbations.

**Notes:** All times are in seconds.

For the third case, a few plots are blown up.
Figure 1. $\Delta a$ versus time for $J_2$ and Case 1

Figure 2. $\Delta h$ versus time for $J_2$ and Case 1
Figure 3. Δk versus time for J₂ and Case 1

Figure 4. Δp versus time for J₂ and Case 1
Figure 5. $\Delta q$ versus time for $J_2$ and Case 1

Figure 6. $\Delta \lambda$ versus time for $J_2$ and Case 1
Figure 7. $\Delta a$ versus time for $J_3$ and Case 1

Figure 8. $\Delta h$ versus time for $J_3$ and Case 1
Figure 9. $\Delta k$ versus time for $J_3$ and Case 1

Figure 10. $\Delta p$ versus time for $J_3$ and Case 1
Figure 11. $\Delta \eta$ versus time for $J_3$ and Case 1

Figure 12. $\Delta \lambda$ versus time for $J_3$ and Case 1
Figure 13. $\Delta a$ versus time for $J_4$ and Case 1

Figure 14. $\Delta h$ versus time for $J_4$ and Case 1
Figure 15. $\Delta k$ versus time for $J_4$ and Case 1

Figure 16. $\Delta p$ versus time for $J_4$ and Case 1
Figure 17. $\Delta q$ versus time for J₆ and Case 1

Figure 18. $\Delta \lambda$ versus time for J₄ and Case 1
Figure 19. Δa versus time for $J_2$ and Case 2

Figure 20. Δh versus time for $J_2$ and Case 2
Figure 21. \( \Delta k \) versus time for \( J_2 \) and Case 2

Figure 22. \( \Delta p \) versus time for \( J_2 \) and Case 2
Figure 23. \( \Delta q \) versus time for \( J_2 \) and Case 2

Figure 24. \( \Delta \lambda \) versus time for \( J_2 \) and Case 2
Figure 25. $\Delta a$ versus time for $J_3$ and Case 2

Figure 26. $\Delta h$ versus time for $J_3$ and Case 2
Figure 27. $\Delta k$ versus time for $J_3$ and Case 2

Figure 28. $\Delta \rho$ versus time for $J_3$ and Case 2
Figure 29. $\Delta q$ versus time for $J_3$ and Case 2

Figure 30. $\Delta \lambda$ versus time for $J_3$ and Case 2
Figure 31. $\Delta h$ versus time for $J_4$ and Case 2

Figure 32. $\Delta h$ versus time for $J_4$ and Case 2
Figure 33. $\Delta k$ versus time for $J_A$ and Case 2

Figure 34. $\Delta \rho$ versus time for $J_A$ and Case 2
1. $6E^{-6}$ seconds

Figure 35. $\Delta q$ versus time for $J_A$ and Case 2

$1.0E^{-6}$ seconds

Figure 36. $\Delta \lambda$ versus time for $J_A$ and Case 2

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Figure 37. $\Delta a$ versus time for $J_2$ and Case 3

Figure 38. $\Delta a$ versus time for $J_2$ and Case 3
Figure 30. $\Delta h$ versus time for $J_2$ and Case 3

Figure 40. $\Delta k$ versus time for $J_2$ and Case 3
Figure 41. Δη versus time for J2 and Case 3

Figure 42. Δq versus time for J2 and Case 3
Figure 43. $\Delta \lambda$ versus time for J$_2$ and Case 3

Figure 44. $\Delta \lambda$ versus time for J$_2$ and Case 3
Figure 45. Δa versus time for J₃ and Case 3

Figure 46. Δa versus time for J₃ and Case 3
Figure 47. $\Delta h$ versus time for $J_3$ and Case 3

Figure 48. $\Delta k$ versus time for $J_3$ and Case 3
Figure 49. $\Delta p$ versus time for $J_3$ and Case 3

Figure 50. $\Delta q$ versus time for $J_3$ and Case 3
Figure 51. $\Delta \lambda$ versus time for $J_3$ and Case 3

Figure 52. $\Delta a$ versus time for $J_4$ and Case 3
Figure 53. $\Delta a$ versus time for $J_4$ and Case 3

Figure 54. $\Delta h$ versus time for $J_4$ and Case 3
3.5E-7 seconds

Figure 55. $\Delta k$ versus time for $J_4$ and Case 3

2.5E-9 seconds

Figure 56. $\Delta p$ versus time for $J_4$ and Case 3
Figure 57. Δq versus time for J₄ and Case 3

Figure 58. Δλ versus time for J₄ and Case 3
3. **Tesseral m-daily Effects**

The m-daily effects include all the terms with a trigonometric argument depending on the Earth's rotation only, that is to say with t equal to zero and m not equal to zero. The "m-daily" name was originated by Kaula (Reference [2]), these terms repeat m times a day.

3.1 **Generating Function**

The generating function (1-46) for the m-daily effects of degree n due to the pair $C_{nm}$ and $S_{nm}$ reduce to

$$S_{nm} = \text{Real} \left\{ \frac{c}{a} \left( \frac{R_e}{a} \right)^n C_{nm}^* \left( \frac{j}{m \omega} \right) \exp(-j\omega t) \right\} \left( \sum_{s=-n}^{n} V_{ns}^m S_{2n}^{(m,s)} (p,q) Y_{0}^{-n-1,s} (h,k) \right) \quad (3-1)$$

Equation (3-1) presents several interesting features: first, there is only one summation and it is finite (this summation may be further truncated in terms of the eccentricity); in addition, all the Hansen coefficients in this equation have the subscript zero (it is to say that there exists a simple closed form representation of these coefficients). Finally, the ease of representing m-daily effects is particularly convenient given that these terms usually appear to be the most important short periodic terms after the $J_2$ zonal term.
3.2 Short Periodic Variations in the Equinoctial Elements

For m-daily effects, \( t \) equals zero, therefore equations (1-35) and (1-43) reduce to the particular form

\[
\Delta a_i = - \sum_{j=1}^{6} \left( \bar{a}_i, \bar{a}_j \right) \frac{\partial S_{nmso}}{\partial a_j} (i = 1, \ldots, 6)
\]

where \( \bar{a}_6 = \bar{\lambda} \) \hspace{1cm} (3-2)

As the coefficients of \( S_{nmso} \) no longer depend on \( t \), \( S_{nmso} \) can be replaced by \( S_{nm} \) defined in Equation (3-1). Therefore the formulas for \( \Delta a_i \) takes the final form

\[
\Delta a_i = - \sum_{j=1}^{6} \left( \bar{a}_i, \bar{a}_j \right) \frac{\partial S_{nm}}{\partial a_j} (i = 1, \ldots, 6) \hspace{1cm} (3-3)
\]

We can already anticipate before programming this equation on a computer that for the semimajor axis \( i = 1 \)

\[
\Delta a = 0 \hspace{1cm} (3-4)
\]

Since, as seen in Section 1, for \( i \) equal to 1 the only non zero Poisson bracket is \( \left( \bar{a}_1, \bar{\lambda} \right) \), then

\[
\Delta a = \left( \bar{\lambda}, \bar{a} \right) \frac{\partial S_{nm}}{\partial \bar{\lambda}} \hspace{1cm} (3-5)
\]
and since the generating function for the m-dailies (3-1) does not depend on the mean longitude

\[ \frac{\partial S_{nm}}{\partial \lambda} = 0 \quad (3-6) \]

3.3 MACSYMA Representation

This subsection will follow the same outline as the one dealing with the zonal harmonics. The programming of the generation function (3-1) and of the short periodic variations (3-3) is presented.

3.3.1 Fundamental Blocks

For the m-daily effects all the fundamental blocks have already been programmed by Zeis. These are:

-- VFUNCT1[N,R,M] which computes the rational number \( V_{n,r}^m \)
-- SFUNCT1[N,M,S](P,Q) which computes the function \( s_{2n}^{m,s} (p,q) \)
-- HANSEN2[N,M,-1](H,K) which computes the modified Hansen coefficient \( y_{0,m}^{n,m} (h,k) \) exactly
-- POISSON(I,J,-1) which computes the Poission bracket \( (i,j) \) of the equinoctial elements \( i \) and \( j \) exactly.
3.3.2 Generating Function

Direct programming of the generating function (3-1) gives the block SD(N,M) for the pair \( C_{nm} \) and \( S_{nm} \) (which are represented in MACSYMA by \( %C_{nm} \) and \( %S_{nm} \), and the Greenwich hour angle by \( \text{TH} \)).

Listing of SD(N,M)

```
SD(N,M):=BLOCK([],
/*THIS BLOCK COMPUTES THE REALPART OF THE CLOSED FORM
 GENERATING FUNCTION FOR THE M-DAILY OF DEGREE N.*/
/*PROGRAMMER:
 J-P.KANIECKI,MIT-JUNE 1979*/
/*RESTRICTIONS:M MUST BE AN INTEGER > 0,
 N MUST BE AN INTEGER >= M.*/
IF M<0 THEN RETURN (ERROR),
IF N<M THEN RETURN (ERROR),
/*BLOCKS CALLED:VFUNCT1[N,R,M],
 SFUNCT1[N,M,S](P,Q),
 HANSEN2[N,M,-1](H,K),*/
SDSTAR(N,M):=SUM(VFUNCT1[N,S,M]*SFUNCT1[N,M,S](P,Q)
 *HANSEN2[N-1,S,-1](P,Q),S,-N,N),
RETURN (FACTOR (MU*RE^N/A^(N+1) *REALPART ((%S[N,M]+%I*%C[N,M]))
 *%E^-(%I*%TH)/(M*E)*SDSTAR(N,M))))$ 114
```
Usage of SD(N,M)

(C1) SD(2,2);

\[ 2 \]

(O1) 3 MU RE (\( %C Q \sin(2\,\theta) + 2\, %S P Q \sin(2\,\theta) \))

\[ 2, 2 \]

2 - \( %C P \sin(2\,\theta) + %S Q \cos(2\,\theta) - 2\, %C P Q \cos(2\,\theta) \))

\[ 2, 2 \]

2 - %S P \cos(2\,\theta) \} X / (A (C + 1) \ WE)

\[ 2, 2 \]

(C2) SD(4,1);

(O2) - 15 MU (3 C Q - 15 C Q + 10 Q + 6 C P Q - 30 C P Q + 20 P Q

\[ 2 4 4 4 2 2 2 2 2 2 2 \]

3 2 2 2 2 2 2 4 4 4 3 2

- 6 C Q + 26 C Q - 21 C Q + 3 Q + 3 C P - 15 C P + 10 P - 6 C P

\[ 2 2 2 2 3 2 4 \]

+ 26 C P - 21 C P + 3 P - 4 C + 24 C - 24 C + 4) RE

\[ %S P \sin(2\,\theta) - %C P \sin(2\,\theta) - %C Q \cos(2\,\theta) - %S P \cos(2\,\theta) \]

\[ 4, 1 \]

\[ 4, 1 \]

\[ 4, 1 \]

7 5 4

X / (8 A (C + 1) \ WE)

3.3.3 Short Periodic Variations

As the m-daily short periodics are not functions of the longitudes but depend only on the rotation of the Earth, several partial derivatives in Equation (3-3) vanish.

Specially, as seen in Equation (3-4), we have

\[ \Delta a = 0 \] (3-7)
Taking advantage of this, a listing of the block DELTASD(N,M,Z) is presented.

Listing of DELTASD(N,M,Z)

DELTASD(N,M,Z):=BLOCK([[SD,DSDA,DSDP,DSDQ,DSDK,DSDH]],

/*THIS BLOCK COMPUTES THE VARIATIONS OF THE 6 EQUINOCTIAL
ELEMENTS ASSUMING THAT THE DISTURBING FORCE IS ONLY DUE TO
THE M-DAILY OF DEGREE N. */

/*PROGRAMMER:
J-P.KANIECKI,MIT-JUNE 1979 */

/*RESTRICTIONS:M MUST BE AN INTEGER > 0,
N MUST BE AN INTEGER > M. */

IF M<=0 THEN RETURN (ERROR),
IF N<M THEN RETURN (ERROR),

/*BLOCKS CALLED:SD(N,M),
POISSON(I,J,-1). */

/*LOCAL VARIABLES:
SD=GENERATING FUNCTION FOR THE M-DAILY OF DEGREE N,
DSDA=PARTIAL DERIVATIVE OF SD WITH
RESPECT TO A,
DSDH=PARTIAL DERIVATIVE OF SD WITH
RESPECT TO H,
DSDK=PARTIAL DERIVATIVE OF SD WITH
RESPECT TO K,
DSDP=PARTIAL DERIVATIVE OF SD WITH
RESPECT TO P,
DSDQ=PARTIAL DERIVATIVE OF SD WITH
RESPECT TO Q, */

/*DEFINITIONS:
XN=SQRT(MU/A^3),
C=P^2+Q^2,
X=1/SQRT(1-H^2-K^2), */
SD: SD(N,M),
GRADEF(X,H,H*X^3),
GRADEF(X,K,K*X^3),
GRADEF(C,P,2*P),
GRADEF(C,Q,2*Q),
DSDA: DIFF(SD,A),
DSDP: DIFF(SD,P),
DSDQ: DIFF(SD,Q),
DSDK: DIFF(SD,K),
DSDH: DIFF(SD,H),

/*DEFINITIONS OF THE VARIATIONS OF THE EQUINOCTIAL ELEMENTS. */

"DELTA-A": 0,
"DELTA-H": -POISSON(H,P,-1)*OSDP-POISSON(H,Q,-1)*OSDQ
-POISSON(H,K,-1)*OSDK,
"DELTA-K": -POISSON(K,P,-1)*OSDP-POISSON(K,Q,-1)*OSDQ
-POISSON(K,H,-1)*OSDH,
"DELTA-P": -POISSON(P,Q,-1)*OSDQ-POISSON(P,H,-1)*OSDH
-POISSON(P,K,-1)*OSDK,
"DELTA-Q": -POISSON(Q,P,-1)*OSDQ-POISSON(Q,H,-1)*OSDH
-POISSON(Q,K,-1)*OSDK,
"DELTA-L": -POISSON(L,P,-1)*OSDA-POISSON(L,Q,-1)*OSDQ-
-POISSON(L,Q,-1)*OSDQ-POISSON(L,H,-1)*OSDH-
-POISSON(L,K,-1)*OSDK,

/*DISPLAY OF THE RESULTS */

IF Z = NO THEN RETURN(["DELTA-A","DELTA-H","DELTA-K","DELTA-P", 
"DELTA-Q","DELTA-L"]),
RETURN(DISPLAY("DELTA-A", "DELTA-H", "DELTA-K", "DELTA-P", 
"DELTA-Q", "DELTA-L"))) $

Usage of DELTASD(N,M,Z)

(C1) DELTASD(2,2,YES);

DELTA-A = 0

2
DELTA-H = 9 K MU RE (XC + 2 %S P Q SIN(2 TH) + 2 %S P Q SIN(2 TH))
2 2 2 2
- %C P SIN(2 TH) + %S Q COS(2 TH) - 2 %C P Q COS(2 TH)
2 2 2 2
\[ \delta K = -3 H \mu \nabla (2 \% C \cos(2 \theta)) X / (A (C + 1) \nabla X n) \]

\[ + (C + 1) K Q X \left(3 \mu \mu \mu (2 \% C, 2 \% C, Q \sin(2 \theta)) + 2 \% S, P \sin(2 \theta) \right) \]

\[ - 2 \% S, P \cos(2 \theta) + 2 \% C \right) X / (A (C + 1) \nabla) \]

\[ + 2 \% S, Q \cos(2 \theta) - 2 \% C \right) X / (A (C + 1) \nabla) \]

\[ - 12 \mu \mu \mu \mu (2 \% C, 2 \% C, Q \sin(2 \theta)) + 2 \% S, P \mu \mu \mu \mu (2 \theta) \]

\[ - 2 \% C, P \sin(2 \theta) + 2 \% S, Q \cos(2 \theta) - 2 \% C \right) X / (A (C + 1) \nabla) \]

\[ - 2 \% S, P \cos(2 \theta) + 2 \% C \right) X / (A (C + 1) \nabla) \]

\[ - 12 \mu \mu \mu \mu \mu (2 \% C, 2 \% C, Q \sin(2 \theta)) + 2 \% S, P \mu \mu \mu \mu \mu (2 \theta) \]

\[ - 2 \% C, P \sin(2 \theta) + 2 \% S, Q \cos(2 \theta) - 2 \% C \right) X / (A (C + 1) \nabla) \]

\[ - 2 \% S, P \cos(2 \theta) + 2 \% C \right) X / (A (C + 1) \nabla) \]

\[ \delta K = -3 H \mu \nabla (2 \% C \cos(2 \theta)) X / (A (C + 1) \nabla X n) \]
\[
\delta P = (C + 1)^2 \times (3 \mu R \left( 2 \%C \right) Q \sin(2 \theta) + 2 \%S \left( P \sin(2 \theta) \right)) \frac{3 \times 3 \times 2}{2, 2, 2} \]

\[
+ 2 \%S \left( Q \cos(2 \theta) \right) - 2 \%C \left( P \cos(2 \theta) \right) \frac{2}{2, 2} \]

\[
- 12 \mu R \left( P \sin(2 \theta) \right) + 2 \%S \left( P \cos(2 \theta) \right) \frac{2}{2, 2} \]

\[
- \%C \left( P \cos(2 \theta) \right) + \%S \left( Q \sin(2 \theta) \right) - 2 \%C \left( P \sin(2 \theta) \right) \frac{2}{2, 2} \]

\[
- 2 \%C \left( Q \cos(2 \theta) \right) - 2 \%S \left( P \cos(2 \theta) \right) \frac{3 \times 3 \times 2}{2, 2} \]

\[
+ 2 \%S \left( Q \sin(2 \theta) \right) - 2 \%C \left( P \sin(2 \theta) \right) \frac{2}{2, 2} \]

\[
- 12 \mu R \left( P \cos(2 \theta) \right) \frac{2}{2, 2} \]

\[
- \%C \left( P \sin(2 \theta) \right) + \%S \left( Q \cos(2 \theta) \right) - 2 \%C \left( P \cos(2 \theta) \right) \frac{2}{2, 2} \]

\[
- \%S \left( P \cos(2 \theta) \right) \frac{3 \times 3 \times 2}{2, 2} \]

\[
+ 2 \%S \left( Q \sin(2 \theta) \right) - 2 \%C \left( P \sin(2 \theta) \right) \frac{3 \times 3 \times 2}{2, 2} \]

\[
- 12 \mu R \left( P \cos(2 \theta) \right) \frac{2}{2, 2} \]

\[
- \%S \left( P \sin(2 \theta) \right) + \%S \left( Q \cos(2 \theta) \right) - 2 \%C \left( P \cos(2 \theta) \right) \frac{2}{2, 2} \]

\[
- \%S \left( P \cos(2 \theta) \right) \frac{3 \times 3 \times 2}{2, 2} \]
- \(12 \mu Q \Re (\%C 2^2 Q \sin(2 \theta) + 2 \%S 2^2 P Q \sin(2 \theta) + 2, 2 \)

- \(\%C 2^2 P \sin(2 \theta) + \%S 2^2 P \cos(2 \theta) - 2 \%C 2^2 P Q \cos(2 \theta) + 2, 2 \)

- \(3^3 \%S 2^2 \cos(2 \theta) \times (A (C + 1) \Re)/(4 A X N) + 2, 2 \)

\[ \Delta L = 9 K \mu R \Re (\%C 2^2 Q \sin(2 \theta) + 2 \%S 2^2 P Q \sin(2 \theta) + 2, 2 \)

- \(\%C 2^2 P \sin(2 \theta) + \%S 2^2 P \cos(2 \theta) - 2 \%C 2^2 P Q \cos(2 \theta) + 2, 2 \)

- \(3^3 \%S 2^2 \cos(2 \theta) \times (A (C + 1) \Re)/(4 A X N) + 2, 2 \)

\[ \Delta Q = - (C + 1) X (3 \mu R \Re (2 \%S 2^2 Q \sin(2 \theta) - 2 \%C 2^2 P Q \sin(2 \theta) + 2, 2 \)

- \(2 \%C 2^2 Q \cos(2 \theta) - 2 \%S 2^2 P \cos(2 \theta) \times (A (C + 1) \Re)/(4 A X N) + 2, 2 \)

- \(12 \mu P \Re (\%C 2^2 Q \sin(2 \theta) + 2 \%S 2^2 P Q \sin(2 \theta) + 2, 2 \)

- \(\%C 2^2 P \sin(2 \theta) + \%S 2^2 P \cos(2 \theta) - 2 \%C 2^2 P Q \cos(2 \theta) + 2, 2 \)

- \(3^3 \%S 2^2 \cos(2 \theta) \times (A (C + 1) \Re)/(4 A X N) + 2, 2 \)
- $\frac{P \cos(2 \theta)}{(A (C + 1) W)} (X + 1) XN)$

+ $(C + 1) Q X (3 M R) - 2 - 2 \frac{P \sin(2 \theta)}{2, 2}$

+ $2 - 2 \frac{Q \sin(2 \theta) + 2 \frac{P \sin(2 \theta)}{2, 2}}{2, 2}$

+ $2 - 2 \frac{Q \cos(2 \theta) - 2 - 2 \frac{P \cos(2 \theta)}{2, 2}}{2, 2}$

- $2 - 2 \frac{P \sin(2 \theta) + 2 \frac{Q \sin(2 \theta)}{2, 2}}{2, 2}$

+ $(C + 1) Q X (3 M R) - 2 - 2 \frac{P \sin(2 \theta)}{2, 2}$

- $2 - 2 \frac{Q \sin(2 \theta) + 2 \frac{P \sin(2 \theta)}{2, 2}}{2, 2}$

- $2 - 2 \frac{Q \cos(2 \theta) - 2 - 2 \frac{P \cos(2 \theta)}{2, 2}}{2, 2}$

+ $(C + 1) Q X (3 M R) - 2 - 2 \frac{P \sin(2 \theta)}{2, 2}$

- $2 - 2 \frac{Q \sin(2 \theta) + 2 \frac{P \sin(2 \theta)}{2, 2}}{2, 2}$

+ $(C + 1) Q X (3 M R) - 2 - 2 \frac{P \sin(2 \theta)}{2, 2}$

- $2 - 2 \frac{Q \sin(2 \theta) + 2 \frac{P \sin(2 \theta)}{2, 2}}{2, 2}$

+ $(C + 1) Q X (3 M R) - 2 - 2 \frac{P \sin(2 \theta)}{2, 2}$

- $2 - 2 \frac{Q \sin(2 \theta) + 2 \frac{P \sin(2 \theta)}{2, 2}}{2, 2}$

+ $(C + 1) Q X (3 M R) - 2 - 2 \frac{P \sin(2 \theta)}{2, 2}$

- $2 - 2 \frac{Q \sin(2 \theta) + 2 \frac{P \sin(2 \theta)}{2, 2}}{2, 2}$

+ $(C + 1) Q X (3 M R) - 2 - 2 \frac{P \sin(2 \theta)}{2, 2}$

- $2 - 2 \frac{Q \sin(2 \theta) + 2 \frac{P \sin(2 \theta)}{2, 2}}{2, 2}$

+ $(C + 1) Q X (3 M R) - 2 - 2 \frac{P \sin(2 \theta)}{2, 2}$

- $2 - 2 \frac{Q \sin(2 \theta) + 2 \frac{P \sin(2 \theta)}{2, 2}}{2, 2}$

+ $(C + 1) Q X (3 M R) - 2 - 2 \frac{P \sin(2 \theta)}{2, 2}$

- $2 - 2 \frac{Q \sin(2 \theta) + 2 \frac{P \sin(2 \theta)}{2, 2}}{2, 2}$
3.4 Plots of the $m$-daily Effects Versus Time

The computational time of this process will be dramatically improved since the $m$-dailies do not depend on any longitudes. Therefore, no numerical solution of Kepler's equation for each point is required. The remaining fast variable is the Greenwich hour angle whose variation with time is given by

$$\theta = \omega_e t \quad (3-8)$$

In this case any given set of equinoctial mean elements will be assumed to have its slowly varying elements constant over one day.

By

-- expressing $\theta$ in function of the time [Equation (3-8)]
-- expressing in terms of the five given mean slowly varying elements ($\overline{a}_0, \overline{h}_0, \overline{k}_0, \overline{P}_0, \overline{q}_0$) all the variables defined inside the previous blocks
-- plugging in the numerical values for $C_{nm}, S_{nm}, \mu, R_e$ and $\omega_e$
it is then possible to obtain formulas for the short periodic variations which are functions of the time only, having the form

\[ \Delta a_i = \sum_{j=0}^{6} a_{ij} \cdot (t) \]  

for any set of mean equinoctial elements \( a_{ij} \) \((i = 1, \ldots, 6)\). Then plots of \( \Delta a_i \) versus time are obtained straightforwardly.

3.4.1 Short Periodic Variations as Functions of the Time Only

The listing of the block transforming the results of \( \text{DELTASD}(N,M,Z) \) into Equation (3-9) is now presented.

Listing of \( \text{DELTASD1}(N,M,A0,H0,K0,P0,Q0,\theta) \)

\begin{verbatim}
DELTASD1(N,M,A0,H0,K0,P0,Q0,\theta):=BLOCK([TEMP1,A,H,K,P,0,C,MU,RE,XN,X,aWETH9 /*THIS BLOCK COMPUTES THE VARIATIONS OF THE 6 EQUINOCTIAL ELEMENTS DUE TO THE M-DAILY OF DEGREE N NUMERICALLY FROM THEIR INITIAL CONDITIONS(A0,H0,K0,P0,eQ), THE FORMULAS ARE EXPRESSED IN FUNCTION OF THE TIME(T). */

/*PROGRAMMER: J-P.KANIECKI,MIT-JUNE 1979 */

/*BLOCK CALLED:DELTASD(N,M,No). */
\end{verbatim}
/* LOCAL VARIABLES:
TEMP1=DELTA50(N,M,NO) EXPRESSED AS A ROW VECTOR,
A,H,K,P,Q=THE 5 SLOWLY VARIABLE EQUINOCTIAL ELEMENTS,
C=P^2+Q^2,
MU=GRAVITATIONAL CONSTANT FOR THE EARTH IN KM/S^2,
RE=EQUATORIAL RADIUS OF THE EARTH IN KM,
XN=MEAN MOTION OF THE SATELLITE,
X=1/SQRT(1-H^2-K^2),
WE=ROTATION RATE OF THE EARTH IN RD/S,
TH=GREENWICH HOUR ANGLE IN RADIANS EXPRESSED LINEARLY
IN FUNCTION OF THE TIME(T).
*/

TEMP1: DELTA50(N,M,NO),
%C[2,2]: 0.1566511*10^-5,
%S[2,2]: -0.8869332*10^-6,
%C[2,1]: -0.1326733*10^-8,
%S[2,1]: -0.1374346*10^-7,
%C[3,1]: 0.2161875*10^-5,
%S[3,1]: 0.2571596*10^-6,
%C[3,2]: 0.3172142*10^-6,
%S[3,2]: -0.2078203*10^-6,
%C[3,3]: 0.1025055*10^-6,
%S[3,3]: 0.1949036*10^-6,
%C[4,1]: -0.5052256*10^-6,
%S[4,1]: -0.4199662*10^-6,
%C[4,2]: 0.7739965*10^-7,
%S[4,2]: 0.1515418*10^-6,
%C[4,3]: 0.5301404*10^-7,
%S[4,3]: -0.1277373*10^-7,
%C[4,4]: 0.6386659*10^-8,
A:A0,
H:H0,
K:K0,
P:P0,
Q:Q0,
C:P0^2+Q0^2,
MU:398600.8,
RE:6378.145,
XN:631.348/A0^(3/2),
X=1/SQRT(1-H0^2-K0^2),
WE:7.2921159*10^-5,
TH:7.2921159*10^-5*T,
RETURN(EV(TEMP1)))
3.4.2 How to Use the Facility

In order to obtain graphical results for any particular set of elements \((a_0, b_0, k, p_0, q_0)\) and for any \((n,m)\) field, the following sequence of MACSYMA commands is required:

```
LOADFILE(POT2,LISP,DSK,KANIEK)$
```

This calls the fundamental blocks. Then to call the blocks developed in this section

```
BATCH(MDAILY,>,DSK,KANIEK);
```

The plots will finally be given by

```
DELTASD1(N,M,A0,H0,K0,P0,Q0)$
```

```
Z: EXPAND(%)$
```

for \(N:1 \text{ THRU } 6 \) DO (PLOT2(Z[N],T,\(T_i\),\(T_f\));

The initial time \(T_i\) and the final time \(T_f\) must be taken as

\[
T_i = 0s \\
T_f = 86160s
\]

in order to have the short periodic variations over one day.
In this case, it must be emphasized that the short periodics will average to zero over one day but not over one period of the satellite.

3.4.3 Application to Real World Problems

Different m-daily short periodic variations of degree n are plotted for the three previous satellites over one day.

Plots have been obtained up to the (4,4) field without any limitation; higher cases should still be executed by MACSYMA but have not been considered in this thesis because of their negligible effects.

NOTES: The plots of $\Delta a$ are not presented since $\Delta a = 0$ [Equation (3-4)]. All times are in seconds.
Figure 59. $\Delta h$ versus time for the (2,1) m-daily and Case 1

Figure 60. $\Delta k$ versus time for the (2,1) n-daily and Case 1
Figure 61. $\Delta p$ versus time for the (2,1) m-daily and Case 1

Figure 62. $\Delta q$ versus time for the (2,1) m-daily and Case 1
Figure 63. $\Delta \lambda$ versus time for the (2,1) m-daily and Case 1

Figure 64. $\Delta h$ versus time for the (2,2) m-daily and Case 1
Figure 66. $\Delta k$ versus time for the (2,2) m-daily and Case 1

Figure 66. $\Delta p$ versus time for the (2,2) m-daily and Case 1
Figure 67. $\Delta g$ versus time for the (2,2) m-daily and Case 1

Figure 68. $\Delta \lambda$ versus time for the (2,2) m-daily and Case 1
Figure 69. $\Delta h$ versus time for the $(3,1)$ m-daily and Case 1.

Figure 70. $\Delta k$ versus time for the $(3,1)$ n-daily and Case 1.
Figure 71. Δp versus time for the (3,1) m-daily and Case 1

Figure 72. Δq versus time for the (3,1) m-daily and Case 1
Figure 73. $\Delta \lambda$ versus time for the (3,1) m-daily and Case 1

Figure 74. $\Delta h$ versus time for the (3,2) m-daily and Case 1
Figure 75. \( \Delta k \) versus time for the (3,2) m-daily and Case 1.

Figure 76. \( \Delta p \) versus time for the (3,2) m-daily and Case 1.
Figure 77. $\Delta q$ versus time for the (3,2) m-daily and Case 1

Figure 78. $\Delta \theta$ versus time for the (3,2) m-daily and Case 1
Figure 79. $\Delta h$ versus time for the (3,3) m-daily and Case 1

Figure 80. $\Delta k$ versus time for the (3,3) m-daily and Case 1
Figure 61. $\Delta \rho$ versus time for the (3,3) n-daily and Case 1

Figure 82. $\Delta q$ versus time for the (3,3) m-daily and Case 1
Figure 83. $\Delta \lambda$ versus time for the (3,3) m-daily and Case 1

Figure 84. $\Delta h$ versus time for the (4,1) m-daily and Case 1
Figure 85. $\Delta k$ versus time for the (1,1) m-daily and Case 1

Figure 86. $\Delta p$ versus time for the (4,1) m-daily and Case 1
Figure 37. $\Delta q$ versus time for the ($i,1$) n-daily and Case 1

Figure 38. $\Delta \lambda$ versus time for the ($i,1$) n-daily and Case 1
Figure 89. $\Delta k$ versus time for the (1,2) n-daily and Case 1

Figure 90. $\Delta k$ versus time for the (A,2) n-daily and Case 1
Figure 91. $\Delta p$ versus time for the (4,2) m-daily and Case 1

Figure 92. $\Delta q$ versus time for the (4,2) m-daily and Case 1
Figure 92. $\Delta \lambda$ versus time for the (4,2) m-daily and Case 1

Figure 94. $\Delta h$ versus time for the (4,3) m-daily and Case 1
Figure 95. $\Delta k$ versus time for the $(1,3)$ m-daily and Case 1

Figure 96. $\Delta p$ versus time for the $(4,3)$ m-daily and Case 1
Figure 97. $\Delta q$ versus time for the $(4,0)$ m-daily and Case 1

Figure 98. $\Delta \lambda$ versus time for the $(4,3)$ m-daily and Case 1
Figure 99. $\Delta t$ versus time for the (4,6) m-daily and Case 1

Figure 100. $\Delta k$ versus time for the (4,4) m-daily and Case 1
Figure 101. $\Delta p$ versus time for the $(4,4)$ m-daily and Case 1

Figure 102. $\Delta q$ versus time for the $(4,4)$ m-daily and Case 1
Figure 103. $\Delta \lambda$ versus time for the ($A$, $F$) $n$-daily and Case 1

Figure 104. $\Delta h$ versus time for the (2,1) $n$-daily and Case 2
Figure 105. $\Delta k$ versus time for the (2,1) m-daily and Case 2

Figure 106. $\Delta p$ versus time for the (2,1) m-daily and Case 2
Figure 197. $\Delta$ versus time for the (2,1) n-daily and Case 2

Figure 198. $\Delta$ versus time for the (2,1) n-daily and Case 2
Figure 109. $\Delta h$ versus time for the (2,2) n-daily and Case 2

Figure 110. $\Delta k$ versus time for the (2,2) n-daily and Case 2
Figure 111. $\Delta p$ versus time for the $(2,2) m$-daily and Case 2

Figure 112. $\Delta q$ versus time for the $(2,2) m$-daily and Case 2
Figure 113. $\Delta \lambda$ versus time for the (2,2) m-daily and Case 2

Figure 114. $\Delta h$ versus time for the (3,1) m-daily and Case 2
Figure 115. \( \Delta k \) versus time for the (3,1) n-daily and Case 2

Figure 116. \( \Delta p \) versus time for the (3,1) m-daily and Case 2
Figure 117. Δ\(n\) versus time for the (3,1) n-daily and Case 2

Figure 118. Δ\(\lambda\) versus time for the (3,1) n-daily and Case 2
Figure 119. $\Delta h$ versus time for the (3,2) m-daily and Case 2

Figure 120. $\Delta k$ versus time for the (3,2) m-daily and Case 2
Figure 121. $\Delta p$ versus time for the (3,2) m-daily and Case 2

Figure 122. $\Delta q$ versus time for the (3,2) m-daily and Case 2
Figure 123. $\Delta \lambda$ versus time for the (3,2) m-daily and Case 2

Figure 124. $\Delta h$ versus time for the (3,3) m-daily and Case 2
Figure 125. $\Delta k$ versus time for the $(3,3)$ $m$-daily and Case 2

Figure 126. $\Delta p$ versus time for the $(3,3)$ $m$-daily and Case 2
Figure 127. $\Delta q$ versus time for the (3,3) n-daily and Case 2

Figure 128. $\Delta \lambda$ versus time for the (3,3) n-daily and Case 2
Figure 130. $\Delta h$ versus time for the (4,1) m-daily and Case 2

Figure 130. $\Delta k$ versus time for the (4,1) m-daily and Case 2
Figure 101. \( \Delta p \) versus time for the \((r,1)\) n-daily and Case 2

Figure 102. \( \Delta q \) versus time for the \((r,1)\) n-daily and Case 2
Figure 133. $\Delta \lambda$ versus time for the $(\lambda,1)$ m-daily and Case 2

Figure 134. $\Delta h$ versus time for the $(\lambda,2)$ m-daily and Case 2
Figure 135. $\Delta k$ versus time for the (4,2) m-daily and Case 2

Figure 136. $\Delta p$ versus time for the (4,2) m-daily and Case 2
Figure 137. $\Delta q$ versus time for the $(\frac{1}{2},2) n$-daily and Case 2

Figure 138. $\Delta \lambda$ versus time for the $(\frac{1}{2},2) n$-daily and Case 2
Figure 10A. $\Delta h$ versus time for the (4,3) n-daily and Case 2

Figure 10B. $\Delta k$ versus time for the (4,3) n-daily and Case 2
Figure 141. \( \Delta p \) versus time for the (4,3) m-daily and Case 2

Figure 142. \( \Delta q \) versus time for the (4,3) m-daily and Case 2
Figure 143. $\Delta \lambda$ versus time for the (4,3) m-daily and Case 2

Figure 144. $\Delta h$ versus time for the (4,4) m-daily and Case 2
Figure 145. $\Delta t$ versus time for the $(1,6)$ n-daily and Case 2

Figure 146. $\Delta p$ versus time for the $(4,6)$ n-daily and Case 2
Figure 147. $\Delta q$ versus time for the $(\Lambda, \nu)$ $n$-daily and Case 2

Figure 148. $\Delta \lambda$ versus time for the $(\Lambda, \nu)$ $n$-daily and Case 2
Figure 149. $\Delta h$ versus time for the (2,1) n-daily and Case 3

Figure 150. $\Delta k$ versus time for the (2,1) n-daily and Case 3
Figure 151. \( \Delta p \) versus time for the (2,1) m-daily and Case 3

Figure 152. \( \Delta q \) versus time for the (2,1) n-daily and Case 3
Figure 153. $\Delta \lambda$ versus time for the (2,1) m-daily and Case 3

Figure 154. $\Delta h$ versus time for the (2,2) m-daily and Case 3
Figure 155. Δk versus time for the (2,2) m-daily and Case 3

Figure 156. Δp versus time for the (2,2) m-daily and Case 3
Figure 157. Δq versus time for the (2,2) n-daily and Case 3

Figure 158. Δλ versus time for the (2,2) n-daily and Case 3
Figure 159. $\Delta h$ versus time for the $(2,1)$ n-daily and Case 3

Figure 160. $\Delta k$ versus time for the $(3,1)$ n-daily and Case 1
Figure 161. $\Delta p$ versus time for the (3,1) n-daily and Case 3

Figure 162. $\Delta \omega$ versus time for the (3,1) n-daily and Case 3
Figure 1C3. $\Delta \lambda$ versus time for the (3,1) $m$-daily and Case 3

Figure 1C4. $\Delta h$ versus time for the (3,2) $m$-daily and Case 3
Figure 165. Δk versus time for the (3,2) m-daily and Case 3

Figure 166. Δp versus time for the (3,2) m-daily and Case 3
Figure 167. $\Delta q$ versus time for the (3,2) m-daily and Case 3

Figure 168. $\Delta \lambda$ versus time for the (3,2) m-daily and Case 3
**Figure 169.** \( \Delta h \) versus time for the \((3,3) \) \( m \)-daily and Case 3

**Figure 170.** \( \Delta k \) versus time for the \((3,3) \) \( m \)-daily and Case 3
Figure 171. $\Delta \rho$ versus time for the (3,3) n-daily and Case 3

Figure 172. $\Delta q$ versus time for the (3,3) m-daily and Case 3
Figure 173. $\Delta \lambda$ versus time for the (3,3) n-daily and Case 3

Figure 174. $\Delta h$ versus time for the (4,1) n-daily and Case 3
Figure 175. $\Delta k$ versus time for the $(\lambda,1)$ m-daily and Case 3

Figure 176. $\Delta p$ versus time for the $(\lambda,1)$ m-daily and Case 3
Figure 177. $\Delta \zeta$ versus time for the $(\zeta,1)$ m-daily and Case 3

Figure 178. $\Delta \lambda$ versus time for the $(4,1)$ m-daily and Case 3
Figure 179. Δh versus time for the (4,2) m-daily and Case 3

Figure 180. Δk versus time for the (4,2) m-daily and Case 3
Figure 131. $\Delta p$ versus time for the $(r,2)$ n-daily and Case 3

Figure 132. $\Delta q$ versus time for the $(4,2)$ m-daily and Case 3
Figure 104. $\Delta \lambda$ versus time for the $(i;i)$ n-daily and Case 3

Figure 104. $\Delta h$ versus time for the $(i;i)$ n-daily and Case 3
Figure 13E. $\Delta k$ versus time for the $(4,3)$ $n$-daily and Case 3

Figure 13F. $\Delta p$ versus time for the $(4,3)$ $m$-daily and Case 3
Figure 107. $\Delta q$ versus time for the $(4,3)$ m-daily and Case 3

Figure 108. $\Delta \lambda$ versus time for the $(4,3)$ m-daily and Case 3
4. **Tesseral Harmonic Short Periodics**

The tesseral short periodics include the terms with $m$ and $n$ both not equal to zero and exclude the resonant term, from Equation (1-12). Because of the restriction

$$t \lambda - m \omega \sim 0$$

a value $t_0$ of $t$ is defined

$$t_0 \sim m \frac{\omega}{\lambda}$$  \hspace{1cm} \text{(4-1)}$$

In the potential due to the harmonic of degree $n$ and order $m$, $t_0$ represents $m$ times the number of revolutions of the satellite around the Earth per day.

4.1 **Generating Function**

Presently there is no way to achieve a closed form for this generating function. Equation (1-45) for the tesseral short periodics of degree $n$ and order $m$ takes the following form

$$S_{nm} = \text{Real} \left\{ \frac{\mu}{\alpha} \left( \frac{R}{a} \right)^n c_{nm} \sum_{s=-n}^{n} \gamma_m \gamma_{s}^{(m,s)} (p,q) \right\}$$

$$\times \sum_{t=-\infty}^{+\infty} \gamma_{t-n-1,s} (h,k) \exp\left[ i(t\lambda-m\omega) \right] \frac{1}{j(tn-m\omega)}$$

$$\left\{ \begin{array}{l}
\text{and} \\
t \neq 0 \quad \text{and} \quad t \neq t_0
\end{array} \right\} \hspace{1cm} \text{(4-2)}$$
The only mechanism available to compress the computations is still truncation on the eccentricity for the Hansen coefficients and for the summation over t. Fortunately, preliminary numerical testing suggests that tesseral short periodics due to a low degree and order field are sufficient for low and medium altitude near circular orbits. However, for high eccentricity cases we must keep in mind the convergence problems of the Hansen coefficients.

4.2 Short Periodic Variations in the Equinoctial Elements

In this case, the expression for $\Delta \lambda$ becomes more complicated since the second term in the right hand side of Equation (1-43) depends on t.

For the tesseral short periodic effects Equations (1-35) and (1-43) become

4.2 Short Periodic Variations in the Equinoctial Elements

In this case, the expression for $\Delta \lambda$ becomes more complicated since the second term in the right hand side of Equation (1-43) depends on t.

For the tesseral short periodic effects Equations (1-35) and (1-43) become

$$\Delta a_i = - \sum_{j=1}^{6} (\bar{a}_i, \bar{a}_j) \frac{\partial \Omega_{nm}}{\partial \bar{a}_j} (i = 1, \ldots, 5) \quad (4-3a)$$

$$\Delta \lambda = - \sum_{j=1}^{6} (\bar{\lambda}, \bar{a}_j) \frac{\partial \Omega_{nm}}{\partial \bar{a}_j} - \frac{3u}{a^3} \left( \frac{e}{a^3} \right) \text{Real} \left\{ \sum_{s=-n}^{+n} \gamma_n s_n \left( \sum_{p,q} \gamma_{nm} s_{n,m,s} \right) \right\}$$

$$+ \sum_{t=n}^{+\infty} \frac{\gamma_{n-1}}{t} (h,k) \frac{\exp[j(t\bar{\lambda} - m\bar{\omega})]}{j(tn-m\bar{\omega})^2} \quad (4-3b)$$

$$(t \not= 0 \quad \text{and} \quad t \not= t_0)$$

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4.3 MACSYMA Representation

For the tesseral short periodics, no closed form of the generating function exists, therefore a truncation on the eccentricity is considered.

4.3.1 Fundamental Blocks

The fundamental blocks necessary for the tesseral short periodics have already been presented in Zeis' thesis (Reference [12]). These are:

-- VFUNCT1[N,R,M] computing the rational number $V_{n,s}^m$

-- SFUNCT1[N,M,S](P,Q) computing the function $S_{2n}^{m,s}(p,q)$

-- HANSEN1[T,N,M,MAXE](H,K) computing the modified Hansen coefficient $Y_{n,m}^n(h,k)$ with an expansion to order MAXE in terms of the eccentricity

-- POISSON(I,J,MACE) computing the Poisson bracket $(i,j)$ of the equinoctial elements $I$ and $J$ with an expansion through order MAXE of the eccentricity.

4.3.2 Generating Function

Only the Hansen coefficients of the generating function [Equation (4-2)] depend on the elements $h$ and $k$, or in other words, on the
eccentricity. Therefore, in order to obtain an expansion to a certain order of the eccentricity of the generating function, the expansion to the same order of the Hansen coefficients only is required.

A listing of the block STE(N,M,TØ,ORDER) and some examples are presented.

Listing of STE(N,M,TØ,ORDER)

STE(N,M,TØ,ORDER):=BLOCK(,)

/*THIS BLOCK COMPUTES THE REAL PART OF THE GENERATING FUNCTION FOR THE TESSERAL HARMONIC OF DEGREE N AND M FOR AN ORBIT TØ WHOSE PERIOD IS ---- TIMES A DAY WITH AN EXPANSION THROUGH M AN ORDER "ORDER" IN TERMS OF THE ECCENTRICITY. */

/*PROGRAMMER: J-P.KANIECKI,MIT-JUNE 1979 */

/*RESTRICTIONS:M MUST BE AN INTEGER > 0, N MUST BE AN INTEGER >= M, ORDER MUST BE AN INTEGER >= 0. */

IF M<=0 THEN RETURN (ERROR), IF N<M THEN RETURN (ERROR), IF ORDER<0 THEN RETURN (ERROR),

/*BLOCKS CALLED:HANSEN1(T,N,M,MAXE)(H,K), VFUNCT(N,R,M), SFUNCT(N,MS)(P,Q). */
Usage of $\text{STE}(N,M,T0,\text{ORDER})$

(C1) $\text{STE}(2,2,10,0)$:

\[
\begin{align*}
2 & \quad 3 \sin(2 \theta - 2 \lambda) \\
\text{(D1) } \mu \text{ re} & \quad \frac{3 \cos(2 \theta - 2 \lambda)}{(C + 1) (2 \mu - 2 \nu)} \\
2 & \quad 4
\end{align*}
\]

4.3.3 Short Periodic Variations

In order to obtain these variations expanded to a certain order in terms of the eccentricity, three conditions must be fulfilled:

-- the Poisson brackets of Equation (4-3) must be expanded to the same order of the eccentricity
the generating function must be expanded to the same order of the eccentricity for the partial derivatives \( \frac{\partial S_{nm}}{\partial a_j} \) appearing in Equation (4-3) when \( a_j \) are different from \( h \) and \( k \), but when \( a_j \) equals \( h \) or \( k \), the generating function must be expanded to this order + 1

-- the infinite summation over \( t \) in Equation (4-3b) must be replaced by \( \sum_{t=s-order}^{s+order} \) with the same restrictions (\( t \neq 0 \) and \( t \neq t_0 \)). As \( \psi_{t-s}^{-n-1}(h,k) \) is of order \( |s-t| \) in terms of the eccentricity, for an expansion to the given order of this infinite summation, the terms which are to be kept are those with subscript and superscript \( s \) such that

\[
|s - t| \leq \text{order}
\]

A listing of the block DELTASTE(N,M,T0,ORDER,Z) is now presented.

Listing of DELTASTE(N,M,T0,ORDER,Z)

DELTASTE(N,E,M,T0,ORDER,Z):-BLOCK(ESTE,STE1,DSDA,DSDK,DSDK,DSDP,DSDQ,DSDL), /*PROGRAMMER: J-P.KANIECKI,MIT-JUNE 1979
*/

/*REstrictions: M MUST be an integer > 0,
N MUST be an integer >= 0,"
ORDER MUST BE AN INTEGER $\geq 0$. 

IF $M \leq 0$ THEN RETURN (ERROR),
IF $N < M$ THEN RETURN (ERROR),
IF $\text{ORDER} < 0$ THEN RETURN (ERROR),

/* BLOCKS AND FUNCTION CALLED: STE($N, M, T0, \text{ORDER}$),
POISSON($I, J, M\text{AXE}$),
STE1($S, T, \text{ORDER}$).

/* LOCAL VARIABLES:
STE=GENERATING FUNCTION FOR THE TESSERAL HARMONIC
OF DEGREE $N$ AND $M$ EXPANDED THROUGH AN ORDER
"ORDER" IN TERMS OF THE ECCENTRICITY,
STE1=GENERATING FUNCTION FOR THE TESSERAL HARMONIC
OF DEGREE $N$ AND $M$ EXPANDED THROUGH AN ORDER
"ORDER+1" IN TERMS OF THE ECCENTRICITY,
DSDA=PARTIAL DERIVATIVE OF STE WITH
 RESPECT TO $A$,
DSDH=PARTIAL DERIVATIVE OF STE1 WITH
 RESPECT TO $H$,
DSDK=PARTIAL DERIVATIVE OF STE1 WITH
 RESPECT TO $K$,
DSDP=PARTIAL DERIVATIVE OF STE1 WITH
 RESPECT TO $P$,
DSDQ=PARTIAL DERIVATIVE OF STE1 WITH
 RESPECT TO $Q$,
DSDL=PARTIAL DERIVATIVE OF STE WITH
 RESPECT TO $L$.

/* DEFINITIONS:
$X_N=\sqrt{\mu/A^3}$,
$X=1/\sqrt{1-H^2-KA^2}$,
$C=P^2+QA^2$.

STE: STE($N, M, T0, \text{ORDER}$),
STE1: STE($N, M, T0, \text{ORDER+1}$),
GRADEF($X, H, H*X^3$),
GRADEF($X, K, K*X^3$),
GRADEF($C, P, 2*P$),
GRADEF($C, Q, 2*Q$),
DSDA: DIFF(STE, $A$),
DSDH: DIFF(STE1, $H$),
DSDK: DIFF(STE1, $K$),
DSDP: DIFF(STE1, $P$),
DSDQ: DIFF(STE, $Q$),
DSDL: DIFF(STE, $L$).
/*DEFINITIONS OF THE VARIATIONS OF THE EQUINOCTIAL ELEMENTS*/

"DELTA-A": -POISSON(A,L,ORDER)*DSOL,
"DELTA-H": -POISSON(H,K,ORDER)*DSDK-POISSON(H,P,ORDER)*DSOP
-POISSON(H,Q,ORDER)*DSDQ-POISSON(H,Q,ORDER)*DSOL,
"DELTA-K": -POISSON(K,H,ORDER)*DSDH-POISSON(K,P,ORDER)*DSOP
-POISSON(K,Q,ORDER)*DSDO-POISSON(K,Q,ORDER)*DSOL,
"DELTA-P": -POISSON(P,H,ORDER)*DSDP-POISSON(P,K,ORDER)*DSDK
-POISSON(P,Q,ORDER)*DSDP-POISSON(P,Q,ORDER)*DSOL,
"DELTA-Q": -POISSON(Q,H,ORDER)*DSDQ-POISSON(Q,K,ORDER)*DSDK
-POISSON(Q,P,ORDER)*DSDQ-POISSON(Q,P,ORDER)*DSOL,
"DELTA-L": -POISSON(L,A,ORDER)*DSLA-POISSON(L,H,ORDER)*DSOH
-POISSON(L,K,ORDER)*DSDH-POISSON(L,K,ORDER)*DSOH
-POISSON(L,Q,ORDER)*DSDO-3*MU*RE^N/A^(N+3)*REALPART((%C[N,M]
-%I*%S[N,M])*SUM(STE2(S)*SUM(STE1(S,T,ORDER)*T/(T*XN-M*WE)),
T,S-ORDER,S+ORDER),S,-N,N)),

/*DISPLAY OF THE RESULTS.*/

IF Z = NO THEN RETURN ("DELTA-A","DELTA-H","DELTA-K","DELTA-P",
"DELTA-Q","DELTA-L"),
RETURN(DISPLAY ("DELTA-A","DELTA-H","DELTA-K","DELTA-P","DELTA-Q",
"DELTA-L"))$
\[
\begin{align*}
12 & \quad Q \left( (3 P \cdot Q - Q) \sin(\theta + 2 L) + (3 P \cdot Q - P) \cos(\theta + 2 L) \right) \\
& \quad - \frac{3}{(C + 1) \left( -2 XN - WE \right)} \\
+ & \quad 3 \left( (3 P - 3 Q) \sin(\theta + 2 L) + 6 P \cdot Q \cos(\theta + 2 L) \right) \\
& \quad - \frac{2}{(C + 1) \left( -2 XN - WE \right)} \\
+ & \quad \frac{12 Q \left( P \sin(\theta - 2 L) - Q \cos(\theta - 2 L) \right)}{(C + 1) \left( 2 XN - WE \right)} \quad - \frac{3 \cos(\theta - 2 L)}{(C + 1) \left( 2 XN - WE \right)} \\
& \quad + \frac{3 (6 P \cdot Q \sin(\theta + 2 L) - (3 P - 3 Q) \cos(\theta + 2 L))}{(4 A \cdot XN)} \\
& \quad + \frac{12 Q \left( (3 P \cdot Q - P) \sin(\theta + 2 L) - (3 P \cdot Q - Q) \cos(\theta + 2 L) \right)}{(C + 1) \left( -2 XN - WE \right)} \\
& \quad - \frac{3 (2 (3 P \cdot Q - Q) \sin(\theta + 2 L) + 2 (3 P \cdot Q - P) \cos(\theta + 2 L))}{(C + 1) \left( -2 XN - WE \right)} \\
+ & \quad \frac{3 (2 P \sin(\theta - 2 L) - 2 Q \cos(\theta - 2 L))}{(C + 1) \left( -2 XN - WE \right)} \\
+ & \quad \frac{3 (2 (3 P \cdot Q - Q) \cos(\theta + 2 L) - 2 (3 P \cdot Q - P) \sin(\theta + 2 L))}{(C + 1) \left( -2 XN - WE \right)} \\
& \quad + \frac{2}{(C + 1) \left( 2 XN - WE \right)} \\
& \quad + \frac{3 (2 (3 P \cdot Q - Q) \sin(\theta + 2 L) + 2 (3 P \cdot Q - P) \cos(\theta + 2 L))}{(C + 1) \left( -2 XN - WE \right)} \\
& \quad + \frac{3 (2 P \sin(\theta - 2 L) - 2 Q \cos(\theta - 2 L))}{(C + 1) \left( -2 XN - WE \right)} \\
& \quad + \frac{2}{(C + 1) \left( 2 XN - WE \right)} \\
& \quad + \frac{3 (2 (3 P \cdot Q - Q) \cos(\theta + 2 L) - 2 (3 P \cdot Q - P) \sin(\theta + 2 L))}{(C + 1) \left( -2 XN - WE \right)}
\end{align*}
\]
4.4 Plots of the Tesseral Short Periodic Variations Versus Time

As for the m-daily effects, there is no need to solve Kepler's equation for this case. On the other hand, both fast variables $\bar{\lambda}$ and $\varnothing$ appear in these variations. Using Equations (2-43) and (3-8), these fast elements are given by

$$\bar{\lambda} = \bar{\lambda}_0 + \bar{\pi}t \quad (4-4)$$

and

$$\varnothing = \omega_0 t \quad (4-5)$$

For any given set of equinoctial mean elements $(\bar{a}_0, \bar{h}_0, \bar{k}_0, \bar{\pi}_0, \bar{\omega}_0, \bar{\lambda}_0)$, the five slow variables $(\bar{a}_0, \bar{h}_0, \bar{k}_0, \bar{\pi}_0, \bar{\omega}_0)$ are assumed to be constant over one day (or over one period of the satellite if the period is larger than one day).

It is possible to obtain formulas for the short periodic variations which are functions of the time only, having the form of Equation (2-44), by:
expressing $\Theta$ and $\lambda$ in function of the time [Equations (4-4) and (4-5)]

expressing in terms of the given mean elements ($a_0, \bar{h}_0, k_0, p_0, q_0, \lambda_0$) all the variables defined inside the previous blocks

plugging in the numerical values for $C_{nm}, S_{nm}, \mu, R_e$ and $\omega_e$

4.4.1 **Tesseral Short Periodics Variations as Functions of the Time Only**

The results of the block DELTASTE($N,M,T0,ORDER,Z$) can be transformed into Equation (2-44) by the use of the block DELTASTE1($N,M,T0,$ ORDER,$A0,P0,Q0,H0,K0,L0$) whose listing is now presented.

**Listing of DELTASTE1($N,M,T0,ORDER,A0,P0,Q0,H0,K0,L0$)**

DELTASTE1($N,M,T0,ORDER,A0,H0,K0,P0,Q0,H0,K0,L0$): BLOCK([TEMP1,A,H,K,P,Q,C,X, MU,RE,XN,WE,L,TH],

/* THIS BLOCK COMPUTES NUMERICALLY THE VARIATIONS OF THE 6 EQUINOCTIAL ELEMENTS DUE TO THE TESSERAL HARMONIC OF DEGREE N AND M WITH AN EXPANSION THROUGH ORDER "ORDER" IN TERMS OF THE T0 ECCENTRICITY FOR AN ORBIT WHOSE PERIOD IS ---- TIMES A DAY M FROM THEIR INITIAL CONDITIONS($A0,H0,K0,P0,Q0,H0$), THE FORMULAS ARE EXPRESSED IN FUNCTION OF THE TIME(T). */

/*PROGRAMMER: J-P.KANIECKI, MIT- JUNE 1979 */

/*BLOCK CALLED: DELTASTE($N,M,T0,ORDER,NO$). */
LOCAL VARIABLES:
TEMP1= DELTASTE(N, M, T0, ORDER) EXPRESSED AS A ROW VECTOR,
A, H, K, P, Q= THE 5 SLOWLY VARIABLE EQUINOCTIAL ELEMENTS,
C=P^2+Q^2
X=1/SORT(1-H^2-K^2),
MU=GRAVITATIONAL CONSTANT FOR THE EARTH IN KM/SEC
RE=EQUATORIAL RADIUS OF THE EARTH IN KM,
XN=MEAN MOTION OF THE SATELLITE,
WE=ROTATION RATE OF THE EARTH IN Radian/SEC,
L=MEAN LONGITUDE IN Radian EXPRESSED LINEARLY IN
FUNCTION OF THE TIME(T),
TH=GREENWICH HOUR ANGLE IN Radian EXPRESSED
LINEARLY IN FUNCTION OF THE TIME(T).
/*
TEMP1: DELTASTE(N, M, T0, ORDER, NO),
A: A0,
H: H0,
K: K0,
P: P0,
Q: Q0,
C: P0^2+Q0^2,
X: 1/SORT(1-H0^2-K0^2),
MU: 398600.8,
RE: 6378.145,
XN: 35484/A0^(3/2),
WE: 7.2921159*10^-5,
L: L0+T*631.3484/A0^(3/2),
TH: 7.2921159*10^-5*T,
%C[2,1]: -0.1326739*10^-8,
%S[2,1]: -0.1434936*10^-7,
%C[2,2]: -0.1566511*10^-5,
%S[2,2]: -0.1669325*10^-6,
%C[3,1]: 0.2161875*10^-5,
%S[3,1]: 0.2271506*10^-6,
%C[3,2]: 0.3172142*10^-5,
%S[3,2]: 0.3282203*10^-6,
%C[3,3]: 0.102555*10^-6,
%S[3,3]: 0.1051969*10^-6,
RETURN (EV(TEMP1)))$
4.4.2 How to Use the Facility

In a first step one needs to call the fundamental blocks and the blocks developed in this section by typing the following commands:

\[
\text{LOADFILE(POT2,LISP,DSK,KANIEK) } \$
\]
\[
\text{BATCH(TESSER,>,DSK,KANIEK);} \\
\]

At this point, all the blocks are ready to execute and in order to obtain the plots of the short periodic variations versus time for any set of mean elements \((A_0, W_0, K_0, P_0, Q_0, L_0)\) for a \((m,n)\) field and with an expansion to a given order in terms of the eccentricity, the following sequence of MACSYMA commands are needed:

\[
\text{DELTASTE1(N,M,T0,ORDER,A0,H0,K0,P0,Q0,L0) } \$
\]
\[
\text{%NUMER} \\
\text{Z:EXPAND(%) } \$
\]
\[
\text{FOR N:1 THRU 6 DO (PLOT2(Z[N],T_i,T_f));} \\
\]

The initial time \(T_i\) and the final time \(T_f\) must be taken as

\[
T_i = 0s \\
T_f = 86160s \\
\]

in order to have the short periodic variations over one day. Otherwise, for a representation over one period of the satellite, \(T_i\) and \(T_f\) must satisfy Kepler's third law.
\[ T_f - T_i = 3.167823 \times 10^{-3} \pi A \rho^{3/2} \]

4.4.3 Application to a Real World Problem

The results for the low altitude circular satellite only are presented, since the truncation process in terms of the eccentricity is really efficient for this particular case.

Unfortunately, the MC machine was unable to execute higher than the zeroth order in terms of the eccentricity and the LISP machine should be able to handle those higher order cases.

NOTES: All times are in seconds.

The plots are presented over one day and over one revolution of the satellite.
Figure 139. $\Delta a$ versus time for the (2,1) Tesseral and Case 1

Figure 190. $\Delta h$ versus time for the (2,1) Tesseral and Case 1
Figure 101. $\Delta k$ versus time for the (2,1) Tesseral and Case 1

Figure 192. $\Delta \rho$ versus time for the (2,1) Tesseral and Case 1
Figure 193. $\Delta q$ versus time for the (2,1) Tesseral and Case 1

Figure 194. $\Delta \lambda$ versus time for the (2,1) Tesseral and Case 1
Figure 195. $\Delta a$ versus time for the (2,1) Tesseral and Case 1

Figure 196. $\Delta h$ versus time for the (2,1) Tesseral and Case 1
Figure 197. $\Delta k$ versus time for the (2,1) Tesseral and Case 1

Figure 198. $\Delta p$ versus time for the (2,1) Tesseral and Case 1
Figure 199. $\Delta \lambda$ versus time for the $(2,1)$ Tesserol and Case 1

Figure 200. $\Delta \lambda$ versus time for the $(2,1)$ Tesserol and Case 1
Figure 201. \( \Delta a \) versus time for the (2,2) Tesseral and Case 1

Figure 202. \( \Delta h \) versus time for the (2,2) Tesseral and Case 1
Figure 203. $\Delta t$ versus time for the (2,2) Tesseral and Case 1

Figure 204. $\Delta p$ versus time for the (2,2) Tesseral and Case 1
Figure 205. Δq versus time for the (2,2) Tesseral and Case 1

Figure 206. Δλ versus time for the (2,2) Tesseral and Case 1
Figure 207. $\Delta a$ versus time for the (2,2) Tesseral and Case 1

Figure 208. $\Delta h$ versus time for the (2,2) Tesseral and Case 1
Figure 202. $\Delta k$ versus time for the (2,2) Tesseral and Case 1

Figure 210. $\Delta p$ versus time for the (2,2) Tesseral and Case 1
Figure 211. $\Delta q$ versus time for the $(2,2)$ Tesseral and Case 1

Figure 212. $\Delta \lambda$ versus time for the $(2,2)$ Tesseral and Case 1
Figure 213. \( \Delta a \) versus time for the (3,1) Tesseral and Case 1

Figure 214. \( \Delta h \) versus time for the (3,1) Tesseral and Case 1
Figure 215. $\Delta k$ versus time for the (3,1) TesserAl and Case 1

Figure 216. $\Delta \rho$ versus time for the (3,1) TesserAl and Case 1
Figure 217. $\Delta q$ versus time for the (3,1) Tesserai and Case 1

Figure 218. $\Delta \lambda$ versus time for the (3,1) Tesserai and Case 1
Figure 21. $\Delta a$ versus time for the $(3,1)$ Tesseral and Case 1

Figure 22. $\Delta h$ versus time for the $(3,1)$ Tesseral and Case 1
Figure 221. $\Delta k$ versus time for the (2,1) Tesseral and Case 1

Figure 222. $\Delta p$ versus time for the (3,1) Tesseral and Case 1
Figure 223. $\Delta q$ versus time for the (3,1) Tesserol and Case 1

Figure 224. $\Delta \lambda$ versus time for the (3,1) Tesserol and Case 1
Figure 225. $\Delta a$ versus time for the $(3,2)$ Tesseral and Case 1

Figure 226. $\Delta h$ versus time for the $(3,2)$ Tesseral and Case 1
Figure 227. Δk versus time for the (3,2) Tesseral and Case 1

Figure 228. Δp versus time for the (3,2) Tesseral and Case 1
Figure 229. $\Delta \phi$ versus time for the (3,2) Tesseral and Case 1

Figure 230. $\Delta \lambda$ versus time for the (3,2) Tesseral and Case 1
Figure 231. $\Delta a$ versus time for the (3,2) Tesseral and Case 1

Figure 232. $\Delta h$ versus time for the (3,2) Tesseral and Case 1
Figure 232. $\Delta t$ versus time for the $(3,2)$ Tesserai and Case 1

Figure 234. $\Delta p$ versus time for the $(3,2)$ Tesserai and Case 1
Figure 235. $\Delta q$ versus time for the (3,2) Tesseral and Case 1

Figure 236. $\Delta \lambda$ versus time for the (3,2) Tesseral and Case 1
Figure 237. $\Delta a$ versus time for the (3,3) Tesseral and Case 1

Figure 238. $\Delta h$ versus time for the (3,3) Tesseral and Case 1
Figure 239. $\Delta k$ versus time for the $(1,0)$ Tesserel and Case 1

Figure 240. $\Delta p$ versus time for the $(3,3)$ Tesserel and Case 1
Figure 241. $\Delta\varphi$ versus time for the (3,3) Tesseral and Case 1

Figure 242. $\Delta\lambda$ versus time for the (3,3) Tesseral and Case 1
Figure 243. $\Delta a$ versus time for the (2,2) Tesseract and Case 1

Figure 244. $\Delta h$ versus time for the (3,3) Tesseract and Case 1
Figure 245. $\Delta k$ versus time for the (3,3) Tesserol and Case 1

Figure 246. $\Delta p$ versus time for the (3,3) Tesserol and Case 1
Figure 247. $\Delta q$ versus time for the (3,3) Tesseral and Case 1

Figure 248. $\Delta \lambda$ versus time for the (3,3) Tesseral and Case 1
5. Third-Body Short Periodics: Restricted Case

The main assumption made in this section is that the motion of the third body is very slow relative to the motion of the satellite.

5.1 Third-Body Disturbing Potential in Terms of Equinoctial Variable

The total potential for the third body (Reference [9]) is given by

$$ U = \sum_{n=2}^{\infty} U_n $$

(5-1)

where

$$ U_n = \frac{\mu_3}{R_3} \left(\frac{r}{R_3}\right)^n P_n (\cos \psi) $$

(5-2)

where

- $\mu_3$ = gravitational constant of the third body
- $R_3$ = distance between the central body and the third body
- $r$ = distance between the central body and the satellite
- $\psi$ = angle between vectors $\vec{R}_3$ and $\vec{r}$
If \( \hat{r} \) = unit vector from the Earth to the satellite
and \( \hat{R}_3 \) = unit vector from the Earth to the third body
then \( \cos \psi = \hat{r} \cdot \hat{R}_3 \) \hspace{1cm} (5-3)

In the direct equinoctial frame \((f, g, \hat{w})\) these vectors are given by

\[
\hat{r} = \begin{pmatrix} \cos L \\ \sin L \\ 0 \end{pmatrix} \hspace{1cm} (5-4)
\]

\[
\hat{R}_3 = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \hspace{1cm} (5-5)
\]

where \( \alpha, \beta, \gamma \) can be obtained from \( \hat{R}_3 \) expressed in the Earth inertial frame and from Equation (1-2)

\[
\alpha = \hat{R}_3 \cdot \hat{f} \hspace{1cm} (5-6a)
\]

\[
\beta = \hat{R}_3 \cdot \hat{g} \hspace{1cm} (5-6b)
\]

\[
\gamma = \hat{R}_3 \cdot \hat{w} \hspace{1cm} (5-6c)
\]

Using Equations (5-4) and (5-5), then Equation (5-3) becomes

\[
\cos \psi = \alpha \cos L + \beta \sin L \hspace{1cm} (5-7)
\]
Substitution of Equation (5-7) into (5-2) leads to

\[ U_n = \frac{\mu_3}{R_3} \left( \frac{r}{R_3} \right)^n \sum_{\alpha \cos L + \beta \sin L} (5-8) \]

As the condition [Equation (2-20d)] \( \alpha^2 + \beta^2 + \gamma^2 = 1 \) still stands (since \( \hat{R}_3 \) is a unit vector), we can use directly the addition formula given by Equation (2-29) and then substitute Equation (2-31) in the expression of \( U_n \) which yields

\[ U_n = \frac{\mu_3}{R_3} \left( \frac{r}{R_3} \right)^n \sum_{m=0}^{n} K_{m} V_{nm} Q_{nm}(\gamma) (C_{m}^{\alpha \beta} \cos mL + S_{m}^{\alpha \beta} \sin mL) \]

\( (5-9) \)

It must be emphasized that the \( \alpha, \beta \) and \( \gamma \) used in Equation (5-9) are the components of \( \hat{R}_3 \) in the equinoctial frame.

Rearranging Equation (5-9)

\[ U_n = \text{Real} \left\{ \frac{\mu_3}{R_3} \left( \frac{a}{R_3} \right)^n \sum_{m=0}^{n} K_{m} V_{nm} Q_{nm}(\gamma) \left( \frac{r}{a} \right)^n \right\} \times (C_{m}^{\alpha \beta} + jS_{m}^{\alpha \beta}) \exp(-jmL) \]

\( (5-10) \)

Using the generating function for the non-singular Hansen coefficient (2-2) \( U_n \) becomes, in terms of the mean longitude
\[ U_n = \text{Real} \left( \frac{\mu_3}{R_3} \right) \left( \frac{a}{R_3} \right)^n \sum_{m=0}^{n} K_{1,m} V_{nm} Q_{nm}(\gamma) (C_m + jS_m) \]
\[ \times \sum_{t=-\infty}^{\infty} \gamma_{n,-m}^{(h,k)} \exp(jt\lambda) \]  

\[ (5-11) \]

5.2 Generating Function

The expression of \( U_n \) (5-11) corresponds to the form assumed in Equation (1-9) except that \( m \) should be taken to zero because there is no additional "fast" angular variable in Equation (5-11). Therefore we can obtain the short periodic generating function by applying Equation (1-36)

\[ S_n^* = \frac{\mu_3}{nR_3} \left( \frac{a}{R_3} \right)^n \sum_{m=0}^{n} K_{1,m} V_{nm} Q_{nm}(\gamma) (C_m + jS_m) \]
\[ \times \sum_{t=-\infty}^{\infty} \gamma_{n,-m}^{(h,k)} \frac{\exp(jt\lambda)}{jt} \]  

\[ (5-12) \]

It is possible to modify Equation (5-12) to obtain a closed form in the slowly varying elements, the averaged mean longitude and the averaged eccentric longitude.

Using relation (2-4):
\[
\sum_{t=-\infty}^{+\infty} Y_{t}^{n,-m} \frac{\exp(it\lambda)}{jt} = \int_{-\infty}^{\lambda} \left( \frac{r}{a} \right)^{n} \exp(-jmL) d\lambda = \lambda Y_{\infty}^{n,-m}
\] (5-13)

Introducing the eccentric longitude \( F \) and the change of variable (see Appendix A)

\[
d\lambda = \frac{r}{a} dF
\] (5-14)

The integral of Equation (5-13) can then be written as

\[
\int_{-\infty}^{\lambda} \left( \frac{r}{a} \right)^{n} \exp(-jmL) d\lambda = \int_{-n}^{F} \left( \frac{r}{a} \right)^{n+1} \exp(-jmL) dF
\] (5-15)

This last equation is very convenient since the integrand of its right hand side has a closed form representation in the eccentric longitude. Particularly here where \( n \) is positive

\[
\left( \frac{r}{a} \right)^{n} \exp(jsL) = \sum_{t=-n}^{n} W_{t}^{n,s} \exp(jtF)
\] (5-16)

\( n \) must be positive in order to have a finite summation in Equation (5-16) (see Reference [11], page 2-78). \( W_{t}^{n,s} \) is an additional special function of the eccentricity having a closed form representation, developed in Appendix D.
Substitution of Equation (5-16) into Equation (5-15) leads to

\[
\int_0^\lambda \left( \frac{r}{a} \right)^n \exp(-jmL)d\lambda = \sum_{t=-(n+1)}^{n+1} W_t^{n+1,-m} \frac{\exp(jtF)}{jt} + F W_0^{n+1,-m} \tag{5-17}
\]

Substitution of Equation (5-17) into Equation (5-13) leads to

\[
\sum_{t=\infty}^{t=-\infty} \gamma_{t,s} \exp(jt\lambda) = \sum_{t=-(n+1)}^{n+1} W_t^{n+1,-m} \frac{\exp(jtF)}{jt} + (F - \lambda) \gamma_0^{n,-m} \tag{5-18}
\]

where the identity

\[
\gamma_o^{n-1,s} = W_0^{n,s} \tag{5-19}
\]

has been used (see Appendix D).

Substitution of Equation (5-18) into Equation (5-12) gives the closed form representation of the third body short periodic generating function:
\[ S_n = \text{Re} \left[ \sum_{m=0}^{n} K_m V_{nm} Q_{nm}(y) \left( c_m^\alpha + j s_m^\beta \right) \right] \]

\[ x \left\{ \sum_{t=-(n+1)}^{n+1} \frac{W_{n+1,-m} \exp(jF)}{jt} + (F-\chi) \gamma_{0,m}^{n,-m}(h,k) \right\} \]

(5-20)

5.3 Short Periodic Variations in the Equinoctial Elements

As for the zonal short periodics, the achievement of a closed form generating function is at the expense of solving Kepler's equation in terms of the equinoctial elements in order here to compute the eccentric longitude.

For the third body short periodics, Equations (1-35) and (1-43) take the following form since \( m = 0 \).

\[ \Delta a_i = - \sum_{j=1}^{6} \langle \bar{a}_i, \bar{a}_j \rangle \frac{\partial S_n}{\partial a_j} (i = 1, \ldots, 5) \quad (5-21a) \]

\[ \Delta \lambda = - \sum_{j=1}^{6} \langle \bar{\lambda}, \bar{a}_j \rangle \frac{\partial S_n}{\partial a_j} - \frac{3 S_n}{n a^2} \quad (5-21b) \]

As \( S_n \) includes the eccentric longitude and as the eccentric longitude is a function of the three equinoctial elements \( h, k \) and \( \lambda \), it must be emphasized that for \( j = 2, 3 \) and 6:
\[ \frac{\partial S_n}{\partial n} \text{ must be replaced by } \frac{\partial S_n}{\partial h} + \frac{\partial S_n}{\partial F} \frac{\partial F}{\partial h} \]

\[ \frac{\partial S_n}{\partial k} \text{ must be replaced by } \frac{\partial S_n}{\partial k} + \frac{\partial S_n}{\partial F} \frac{\partial F}{\partial k} \]

\[ \frac{\partial S_n}{\partial \lambda} \text{ must be replaced by } \frac{\partial S_n}{\partial \lambda} + \frac{\partial S_n}{\partial F} \frac{\partial F}{\partial \lambda} \]

See Appendix A for the derivation of the partial derivatives of the eccentric longitude.

5.4 **MACSYMA Representation**

This representation is quite similar to the one for the zonals. In the generation function (5-20), the eccentric longitude appears instead of the true longitude, and the Hansen coefficients are replaced by the new special function \( W_{n+1,-m} \).

5.4.1 **Fundamental Blocks and Functions**

Five of them have already been described in Section 2.4.1. These are:

-- \( C(N,X,Y) \)

-- \( SI(N,X,Y) \)
The special function $W_n^s$ for $n \geq 0$ is the topic for Appendix D and the block computing this function uses Equation (D-36). A listing of this block and some examples are presented.

Listing of $W(T,N,S,X)$

```prolog
W(T,N,S,X) :- BLOCK([], N,S, T, X),
/*
   THIS BLOCK COMPUTES THE FUNCTION $W_T^N(X)$. */
/*PROGRMNER:
   J-P. KANIECKI, MIT- JUNE 1979 */
/*RESTRITIONS: N MUST BE AN INTEGER $\geq 0$
   IF $|S| > |T|$ THEN $N-|S|$ MUST BE $\geq 0$
   IF $|S| < |T|$ THEN $N-|T|$ MUST BE $\geq 0$ */
IF N<0 THEN RETURN (ERROR),
/* BLOCK CALLED: JACOB1(N,A,B)(Y). */
/* DEFINITION:
   BET=SQRT(H^2+K^2)/(1+1/SQRT(1-H^2-K^2)). */
```
RETURN IF ABS(S) >= ABS(T) THEN
  IF N-ABS(S) < 0 THEN ERROR
  ELSE (1/X)^N*(-BET*SQRT(H^2+K^2)/(K+%I*H))^ABS(T-S)
    *(N+S)!*(N-S)!/((N+T)!*(N-T)!)*(1-BET^2)^ABS(S)
    *JACOB1(N-ABS(S),ABS(T-S),ABS(T+S)) (X)
  ELSE IF N-ABS(T) < 0 THEN ERROR
  ELSE (1/X)^N*(-BET*SQRT(H^2+K^2)/(K+%I*H))^ABS(T-S)
    *(1-BET^2)^ABS(T)
    *JACOB1(N-ABS(T),ABS(T-S),ABS(T+S)) (X) )

Usage of \( W(T,N,S,X) \)

(C1) \( W(0,0,0,X) \):
(D1) \quad 1

(C2) \( W(1,2,3,X) \):
(D2) \quad \text{ERROR}

(C3) \( W(-2,2,1,X) \):
(D3) \quad \frac{3 \quad 2 \quad 2 \quad 3/2 \quad \text{BET} \quad (K + H)}{2 \quad 2 \quad 3 \quad 2 \quad (1 - \text{BET}) \quad (K + %I \ H) \quad X}

(C4) \( W(1,2,-1,X) \):
(D4) \quad \frac{2 \quad 2 \quad \text{BET} \quad (K + H) \quad (2 \, X + 1)}{2 \quad 2 \quad 2 \quad 2 \quad (1 - \text{BET}) \quad (K + %I \ H) \quad X}

(C5) \( W(-1,3,-2,X) \):
(D5) \quad \frac{2 \quad 2 \quad 5 \, \text{SQRT}(K + H) \quad (3 \, X - 1)}{2 \quad 2 \quad 3 \quad 2 \quad (1 - \text{BET}) \quad (K + %I \ H) \quad X}

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5.4.2 Generating Function

The eccentric longitude $F$ in the generating function (5-20) is represented by $EL$ and the direction cosines $(\alpha, \beta, \gamma)$ by $(AL, BE, GA)$ in MACSYMA. A listing of the block computing this generating function is now presented.

Listing of STB(N, THIRD)

```
STB(N, THIRD):=BLOCK((),

  /* THIS BLOCK Computes the real part of a closed form generating
     th
     function for the N harmonic of third body perturbations.
     IF THIRD = M THEN third body = moon,
     IF THIRD = S THEN third body = sun. */

  /* PROGRAMMER:          J-P. KANIECKI, MIT-JUNE 1979 */

  /* RESTRICTION: N must be an integer >= 2 */

  IF N<2 THEN RETURN (ERROR),

  /* Blocks and functions called: C(N,X,Y)
     SI(N,X,Y),
     W(T,N,S,X),
     V(N,M),
     Q(N,M,X),
     HANSEN2[N,M,-1](H,K). */

  /* Definitions:
     AL, BE, GA = the direction cosines of the third body
     with respect to the equinoctial frame */
```

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ASSUME \( (H^2 + K^2 > 0) \),

\[
STB1(M) := (\text{IF } M = 0 \text{ THEN } 1 \text{ ELSE } 2) \times (C(M, AL, BE) + \%I * SI(M, AL, BE)),
\]

\[
STB2(T, N, M) := \text{IF } T = 0 \text{ THEN } 0 \text{ ELSE } W(T, N+1, -M, X) \times \%I \times T \times EL \times (1 - \%T)
\]

\[
STB * (N) := \text{SUM} (V(N, M) \times 0(N, M, GA) \times STB1(M) \times (HANSEN2[N, -M, -1] + \%I \times SI(N, M, GA) \times STB2(T, N, M), T, -N-1, N+1), i, 0, N)
\]

\[
\text{RETURN\( ((\text{IF } T = 0 \text{ THEN } M \text{ ELSE } M) / (X^2 \times R3^N) \times A^N \times \text{REALPART}(STB * (N)) )) \}
\]

Usage of \( STB(N, \text{THIRD}) \)

(C1) \( STB(2, M) \):

\[
\begin{align*}
2 & 2 & 2 & 4 & 2 & 2 & 2 \\
3 & 3 & 4 & 2 & 2 & 4 \\
\cos(2 \text{ EL}) & (4 H K - 4 H K) & \sin(2 \text{ EL}) & (K - 6 H K + H) \\
\end{align*}
\]

\[
\frac{2}{2} 2 3 3 2 2 3/2 / (2 (1 - \%T) X) + 5 \%T (K + H)
\]

(D1) \( A \) \( \text{MUM} \) \( 3 ((AL - BE) (- \%T (K + H)) \)

\[
\begin{align*}
3 & 3 & 4 & 2 & 2 & 4 \\
\cos(2 \text{ EL}) & (4 H K - 4 H K) & \sin(2 \text{ EL}) & (K - 6 H K + H) \\
\end{align*}
\]

\[
\frac{2}{2} 2 3 3 2 2 3 / (K + H) + (K + H)
\]

\[
\text{SIN}(2 \text{ EL}) \times (3 X + 2) /
\]
\[
\begin{align*}
\text{BET} & \cdot (\cos(EL) - \sin(EL) \cdot K) (15 \times -3) \\
& + \frac{2}{4} \left(1 - \text{BET} \right) \cdot \sqrt{K + H} \cdot X \\
& + \frac{2}{3} \text{BET} \cdot \left( \sin(2EL) \cdot (K - H) + 2 \cos(2EL) \cdot H \cdot K \right) \\
& + \frac{2}{2} \left(1 - \text{BET} \right) \cdot (K + H) \cdot X \\
& + \frac{2}{3} \text{BET} \cdot \left( 2 \cos(2EL) \cdot H \cdot K - \sin(2EL) \cdot (K - H) \right) \\
& + \frac{2}{2} \left(1 - \text{BET} \right) \cdot (K + H) \cdot X \\
& + \frac{3}{2} \text{BET} \cdot \left( \frac{3/2}{2} \sin(3EL) \cdot (K - 3H \cdot K) \right) \cdot \cos(3EL) \cdot (3HK - H) \\
& + \frac{3}{2} \left(1 - \text{BET} \right) \cdot (K + H) \cdot X \\
& + \frac{3}{2} \text{BET} \cdot \left( \frac{3/2}{2} \cos(3EL) \cdot (3HK - H) \right) \cdot \sin(3EL) \cdot (K - 3H \cdot K) \\
& + \frac{3}{2} \left(1 - \text{BET} \right) \cdot (K + H) \cdot X \\
& + \frac{2}{3} \text{BET} \cdot \left( 3k + 3H + 2 \right) \cdot (EL - L) \\
& + \frac{3}{2} \left(1 - \text{BET} \right) \cdot \sqrt{R3 \cdot XN} \\
\end{align*}
\]
5.4.3 Short Periodic Variations

Direct programming of Equation (5-21) leads to this block, a listing of which is presented.

Listing of DELTASTB(N,THIRD,Z)

DELTASTB(N,THIRD,Z):=BLOCK([[ST,DEDL,DSDEL,DSOA,DSOH,DSOK,DSDP,DSOQ,DSOL]],

/*THIS BLOCK COMPUTES THE VARIATIONS OF THE 6 EQUINOCTIAL ELEMENTS ASSUMING THAT THE DISTURBING FORCE IS ONLY DUE TO THE N HARMONIC OF A THIRD BODY.
 IF THIRD = M THEN THIRD BODY = MOON,
 IF THIRD = S THEN THIRD BODY = SUN. */

/*PROGRAMMER:
 J-P. KANIECKI, MIT-JUNE 1979 */

/*RESTRICTION: N MUST BE AN INTEGER >= 2 */

IF N<2 THEN RETURN (ERROR),

/*BLOCKS CALLED: STB(N,THIRD),
 POISSON(I,J,-1). */

/*LOCAL VARIABLES:
 th
 ST=GENERATING FUNCTION FOR THE N HARMONIC OF THE THIRD BODY,
 DELDL=PARTIAL DERIVATIVE OF EL WITH RESPECT TO L,
 DSDEL=PARTIAL DERIVATIVE OF ST WITH RESPECT TO EL,
 DSOA=PARTIAL DERIVATIVE OF ST WITH RESPECT TO A,
 DSOH=PARTIAL DERIVATIVE OF ST WITH RESPECT TO H,
 DSOK=PARTIAL DERIVATIVE OF ST WITH RESPECT TO K, */
DSDP = PARTIAL DERIVATIVE OF ST WITH RESPECT TO P,
DSDQ = PARTIAL DERIVATIVE OF ST WITH RESPECT TO Q,
DSDL = PARTIAL DERIVATIVE OF ST WITH RESPECT TO L.

/* DEFINITIONS:
XN = SQRT(MU/A^3),
X = 1/SQRT(1-H^2-K^2),
C = P^2+Q^2,
DEDL = AR,
AR = 1/(1-H*SIN(EL)-K*COS(EL)),
DELDH = PARTIAL DERIVATIVE OF EL WITH RESPECT TO H
   = -AR*COS(EL),
DELDK = PARTIAL DERIVATIVE OF EL WITH RESPECT TO K
   = AR*SIN(EL),
DBETDH = PARTIAL DERIVATIVE OF BET WITH RESPECT TO H
   = H*BET*(X^2/(1+X)+1/(H^2+K^2)),
DBETDK = PARTIAL DERIVATIVE OF BET WITH RESPECT TO K
   = K*BET*(X^2/(1+X)+1/(H^2+K^2)),
DALDP = PARTIAL DERIVATIVE OF AL WITH RESPECT TO P
   = -2*(Q*BE+GA)/(1+P^2+Q^2),
DALDQ = PARTIAL DERIVATIVE OF AL WITH RESPECT TO Q
   = 2*P*BE/(1+P^2+Q^2),
DBEOP = PARTIAL DERIVATIVE OF BE WITH RESPECT TO P
   = 2*Q*AL/(1+P^2+Q^2),
DBEDQ = PARTIAL DERIVATIVE OF BE WITH RESPECT TO Q
   = -2*(P*AL-GA)/(1+P^2-Q^2),
DGADP = PARTIAL DERIVATIVE OF GA WITH RESPECT TO P
   = 2*AL/(1+P^2+Q^2),
DGADQ = PARTIAL DERIVATIVE OF GA WITH RESPECT TO Q
   = -2*BE/(1+P^2+Q^2).

ST: STB(N, THIRD),
GRADEF(X,H,H*X^3),
GRADEF(X,K,K*X^3),
GRADEF(BET,H,DBETDH),
GRADEF(BET,K,DBETDK),
GRADEF(AL,P,DALDP),
GRADEF(AL,Q,DALDQ),

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GRADEF (BE, P, DBEDP),
GRADEF (BE, Q, DBEDQ),
GRADEF (GA, P, DGADP),
GRADEF (GA, Q, DGADQ),
DELDL: AR,
DSDEL: DIFF (ST, EL),
DSDA: DIFF (ST, A),
DSDH: DIFF (ST, H) + DSDEL * DELDH,
DSDK: DIFF (ST, K) + DSDEL * DELDK,
DSDP: DIFF (ST, P),
DSOQ: DIFF (ST, Q),
DSOL: DIFF (ST, L) + DSDEL * DELDL.

/* DEFINITIONS OF THE VARIATIONS OF THE EQUINOCTIAL ELEMENTS */

"DELTA-A": - POISSON (A, L, -1) * DSDL,
"DELTA-H": - POISSON (H, K, -1) * DSDK - POISSON (H, P, -1) * DSDP
- POISSON (H, Q, -1) * DSDQ - POISSON (H, L, -1) * DSDL,
"DELTA-K": - POISSON (K, H, -1) * DSOK - POISSON (K, P, -1) * DSDP
- POISSON (K, Q, -1) * DSDQ - POISSON (K, L, -1) * DSDL,
"DELTA-P": - POISSON (P, H, -1) * DSDH - POISSON (P, K, -1) * DSOL
- POISSON (P, Q, -1) * DSDQ - POISSON (P, L, -1) * DSDL,
"DELTA-Q": - POISSON (Q, H, -1) * DSOH - POISSON (Q, K, -1) * DSDK
- POISSON (Q, P, -1) * DSDP - POISSON (Q, L, -1) * DSDL,
"DELTA-L": - POISSON (L, A, -1) * DSOL - POISSON (L, H, -1) * DSDH
- POISSON (L, K, -1) * DSDK - POISSON (L, P, -1) * DSDP
- POISSON (L, Q, -1) * DSDQ - 3 * ST / (XN * A^2),

/* DISPLAY OF THE RESULTS */

IF Z = NO THEN RETURN ("DELTA-A", "DELTA-H", "DELTA-K", "DELTA-P",
"DELTA-Q", "DELTA-L"),
RETURN (DISPLAY ("DELTA-A", "DELTA-H", "DELTA-K", "DELTA-P", "DELTA-Q",
"DELTA-L"));

5.5 Plots of the Short Periodic Variations Versus Time

The process is identical to the one developed in Section 2.4 for the zonal short periodics. But in this case only the eccentric longitude is needed, thus only the block EL (L, H, K, EP). In order to
express the short periodic variations as functions of the time and of
the eccentric longitude only, the block DELTASTB1 is used.

Listing of DELTASTB1(N,X0,Y0,Z0,A0,H0,K0,P0,Q0,L0,THIRD)

DELTASTB1(N,X0,Y0,Z0,A0,H0,K0,P0,Q0,L0,THIRD):-BLOCK([ITEM,A,H,K,P,Q,C,XN,
L,D,R3,MUM,MUS,AL,
BE,GA,X,BET,EX,
DBETOH,DBETOK,
DALDP,DALDO,DBEDP,
DBEDO,DGADO,DGADQ,
AR,DELDOH,DELDK],

/*THIS BLOCK COMPUTES THE VARIATIONS OF THE 6 EQUINOCTIAL
th
ELEMENTS NUMERICALLY FOR THE N Harmonic of a Third Body
FROM THEIR INITIAL CONDITIONS(A0,H0,K0,P0,Q0,L0) AND FROM
THE POSITION VECTOR OF THE THIRD BODY (X0,Y0,Z0) EXPRESSED
IN THE EARTH INERTIAL FRAME, THE FORMULAE ARE EXPRESSED IN
FUNCTION OF THE TIME(T) AND OF THE ECCENTRIC LONGITUDE(EL).
If THIRD = M THEN THIRD BODY = MOON,
If THIRD = S THEN THIRD BODY = SUN. */

/*PROGRAMMER:
J-P.KANIECKI,MIT-JUNE 1979 */

/*BLOCK CALLED:DELTASTB(N,THIRD,NO). */

/*LOCAL VARIABLES:
ITEM=DELTASTB(N,THIRD,NO) EXPRESSED AS A ROW VECTOR,
A,H,K,P,Q=THE 5 SLOWLY VARIABLE EQUINOCTIAL ELEMENTS,
C=P^2+Q^2,
XN=MEAN MOTION OF THE SATELLITE,
L=MEAN LONGITUDE IN RADIANS EXPRESSED LINEARLY IN
FUNCTION OF THE TIME T,
D=1+P^2+Q^2,
R3=MAGNITUDE OF THE POSITION VECTOR OF THE THIRD
BODY IN KM,
MUM=GRAVITATIONAL CONSTANT FOR THE MOON IN KM /S^2,
MUS=GRAVITATIONAL CONSTANT FOR THE SUN IN KM /S^2,
AL=(X0*(1-P^2+Q^2)+2*Y0*Q*P-2*P*Z0)/(D*R3).*/
BE = \frac{(2X0*P*Q+(1+P^2-Q^2)*Y0+2*Q*Z0)}{(D*R3)},
GA = \frac{(2P*X0-2Q*Y0+(1-P^2-Q^2)*Z0)}{(D*R3)},
X = \frac{1}{\text{SORT}(1-H^2-K^2)},
BET = \text{SORT}(H^2+K^2)/(1+1/X),
EX = X^2/(1+X)+1/(H^2+K^2),
dBET
dBETOH = \frac{dH}{dBET},
dBETDK = \frac{dK}{dBET},
DALP = \frac{dP}{dAL},
DALQ = \frac{dQ}{dAL},
DBEDP = \frac{dP}{dBED},
DBEDQ = \frac{dQ}{dBED},
DGADP = \frac{dP}{DGAD},
DGADQ = \frac{dQ}{DGAD},
AR = 1/(1-H\sin(EL)-K\cos(EL)),
DELH = \frac{dH}{dEL},
DELK = \frac{dK}{dEL},

TEM: DELTASTB \text{(N, THIRD, NO)},
A: A0,
H: H0,
K: K0,
P: P0,
Q: Q0,
C: P0^2+Q0^2,
XN: \text{SORT}(398608.8/A0^3),
L: L0+T*XN,
R3: \text{SORT}(X0^2+Y0^2+Z0^2),
MUM: 4902.778,
MUS: 0.13271545*10^12,
D: 1+P0^2+Q0^2,
AL: (X0*(1-P0^2+Q0^2)+2*Y0*P0*Q0-2*P0*Z0)/(D*R3),
BE: (X0*2*P0*Q0+(1+P0^2-Q0^2)*Y0+2*Q0*Z0)/(D*R3),
GA: (2P0*X0-2Q0*Y0+(1-P0^2-Q0^2)*Z0)/(D*R3),

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5.5.1 How to Use the Facility

As for the zonals, the following sequence of MACSYMA commands is required:

\begin{verbatim}
LOADFILE(POT2,LISP,DSK,KANIEK)$
BATCH(THIRD,>,DSK,KANIEK);
\end{verbatim}

for any set of mean elements \((a_0, h_0, \bar{k}_0, \bar{p}_0, \bar{q}_0, \bar{\lambda}_0)\), for the \(n\)th desired harmonic, for \((x_0, y_0, z_0)\) the position vector of the third body (see Appendix E) relative to the Earth inertial frame and finally for the moon, for example

\begin{verbatim}
DELTASTB1(N,X0,Y0,Z0,A0,H0,K0,P0,Q0,L0,THIRD)$
Z:EXPAND(%)$
F(T):=(L0+631.3484*A0^3/2*T,H0,K0,EP)$
\end{verbatim}

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where $EP$ is the desired accuracy for Kepler's solution. Finally, the plots are given by:

$$\text{FOR } N:1 \text{ THRU } 6 \text{ DO(PLOT2(SUBST(F(T),EL,Z[N]),T,T_i,T_f));}$$

The initial time $T_i$ and the final time $T_f$ must be taken as:

$$
T_i = 0s \\
T_f = 3.167823 \times 10^{-3} \pi A0^{3/2} \text{s}
$$

(Kepler's 3rd law)

in order to have the short periodic variations over one period of the satellite. They average to zero over such an interval.

5.5.2 Application to Real World Problems

The third body short periodic variations of the three previous satellites are plotted only for the 2nd harmonic ($P_2$ term) since the system was unable to execute for higher harmonics.

The plots of $\Delta a, \ldots, \Delta q$ and $\Delta \lambda$ for the three test cases follow for the $P_2$ term for the lunar perturbation on the 28th of June 1979 at 12:00 (G.M.T.).

NOTES: All times are in seconds.

For the 3rd case, a few plots are blown up.
Figure 249. $\Delta a$ versus time for the $P_2$ term and Case 1

Figure 250. $\Delta h$ versus time for the $P_2$ term and Case 1
Figure 251. $\Delta k$ versus time for the $P_2$ term and Case 1

Figure 252. $\Delta p$ versus time for the $P_2$ term and Case 1
Figure 253. $\Delta q$ versus time for the $P_2$ term and Case 1

Figure 254. $\Delta \lambda$ versus time for the $P_2$ term and Case 1
Figure 255. \( \Delta a \) versus time for the P\(_2\) term and Case 2

Figure 256. \( \Delta h \) versus time for the P\(_2\) term and Case 2
Figure 257. $\Delta K$ versus time for the $P_2$ term and Case 2

Figure 258. $\Delta p$ versus time for the $P_2$ term and Case 2
Figure 259. $\Delta q$ versus time for the P$_2$ term and Case 2

Figure 260. $\Delta \lambda$ versus time for the P$_2$ term and Case 2
Figure 261. $\Delta a$ versus time for the $P_2$ term and Case 3

Figure 262. $\Delta a$ versus time for the $P_2$ term and Case 3
Figure 263. \( \Delta h \) versus time for the \( P_2 \) term and Case 3

Figure 264. \( \Delta k \) versus time for the \( P_2 \) term and Case 3
Figure 265. $\Delta p$ versus time for the $P_2$ term and Case 3

Figure 266. $\Delta q$ versus time for the $P_2$ term and Case 3
Figure 267. $\Delta \lambda$ versus time for the $P_2$ term and Case 3
Conclusion

The main goal of this thesis was to generate analytical formulas for the short periodic variations due to some conservative forces by using MACSYMA. Thus, these results can be stored in a large scientific computer which would evaluate the formulas for each new set of input data without starting the computations all over again. Furthermore, this thesis emphasizes the ability of MACSYMA, even when the formulas involved are very complex, to give a good physical insight by using the plot package.

Unfortunately, to achieve those two goals, a compromise has been required, since getting as compact formulas as possible to feed the numerical computer and being able to obtain graphical results with MACSYMA for several different harmonics or n x m fields are somewhat contradictory. The formulas involved are so large that several MACSYMA simplifications (RATSIMP, FACTOR, RATCOEF, FACTORSUM, etc.) must be used in order to have compact results, but these commands are generally so powerful that they require a lot of core capacity as well as a lot of computational time. Therefore, using them intensively would result in a "core capacity exceeded" message from the machine or in a much too long computational time even for the lower harmonics or fields. In this thesis, such commands have been moderately used in order to obtain plots up to \( J_4 \) for the zonals, up to the \((4 \times 4)\) field (and even further) for the m-dailies, up to the \((3 \times 3)\)
field (circular case) for the Tesserals and up to the first non-zero harmonic term for the third body by using the MC machine only. The drawbacks of using too many simplification commands can be illustrated by two examples from Zeis' thesis:

-- for the (2 x 2) field, Zeis' block MDEL takes as long to give the analytical formulas for the m-dailies as it takes for the blocks presented in this thesis to construct the formulas and give the six plots of the m-daily variations
-- for the (2 x 2) field, circular case, Zeis' block TESDEL cannot execute: "core capacity exceeded."

Furthermore, the problems of core capacity and of long computational time are very accentuated by the fact that the MACSYMA system is overloaded much of the time. It is the author's advice for large computations in a reasonable amount of time to use the system between 6:00 and 9:00 A.M. (Eastern Standard Time).

The graphical results for the three real world problems provide a good rule of thumb as for the relative effects of the perturbations on the short periodic variations.* By examining the plots given in the previous sections, it can be seen that these effects, in order of importance for the three satellites and for each element are:

* These results are based on the Goddard Earth Model-9 (GEM9) gravitational coefficients.
1) \( \Delta a \)

<table>
<thead>
<tr>
<th>circular, low altitude case</th>
<th>eccentric, low altitude case (without the Tesserals)</th>
<th>high eccentricity and high altitude case (without the Tesserals)</th>
</tr>
</thead>
<tbody>
<tr>
<td>- ( J_2 )</td>
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<tr>
<td>- Tesseral (3,1)</td>
<td>- ( J_3 )</td>
<td>- moon</td>
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<tr>
<td>- Tesseral (3,3)</td>
<td>- ( J_4 )</td>
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</tr>
<tr>
<td>- Tesseral (2,2)</td>
<td>- moon</td>
<td>- ( J_4 )</td>
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</table>

2) \( \Delta h \)

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<tbody>
<tr>
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<td>- ( J_2 )</td>
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<tr>
<td>- Tesseral (2,2)</td>
<td>- m-daily (2,2)</td>
<td>- moon</td>
</tr>
<tr>
<td>- Tesseral (3,1)</td>
<td>- m-daily (3,3)</td>
<td>- ( J_3 )</td>
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<tr>
<td>- Tesseral (3,3)</td>
<td>- m-daily (4,3)</td>
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<tr>
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</table>
3) $\Delta k$

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<th>high eccentricity high altitude (without the Tesserals)</th>
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<tbody>
<tr>
<td>- $J_2$</td>
<td>- $J_2$</td>
<td>- $J_2$</td>
</tr>
<tr>
<td>- Tesseral (2,2)</td>
<td>- $J_3$</td>
<td>- moon</td>
</tr>
<tr>
<td>- Tesseral (3,1)</td>
<td>- $J_4$</td>
<td>- $J_3$</td>
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<tr>
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<td>- $J_4$</td>
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<tr>
<td>- Tesseral (3,2)</td>
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</table>

4) $\Delta p$

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<tr>
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<th>high eccentricity high altitude (without the Tesserals)</th>
</tr>
</thead>
<tbody>
<tr>
<td>- $J_2$</td>
<td>- $J_2$</td>
<td>- $J_2$</td>
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<tr>
<td>- m-daily (2,2)</td>
<td>- m-daily (2,2)</td>
<td>- moon</td>
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<tr>
<td>- m-daily (3,3)</td>
<td>- m-daily (3,3)</td>
<td>- $J_3$</td>
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<tr>
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</table>
4) \( \Delta p \) (continued)

<table>
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<tr>
<th>circular low altitude case</th>
<th>eccentric low altitude case (without the Tesserals)</th>
<th>high eccentricity high altitude (without the Tesserals)</th>
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</thead>
<tbody>
<tr>
<td>( - m\text{-daily } (3,1) )</td>
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<td>( - J_2 )</td>
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<tr>
<td>- ( \text{Tesseral } (2,2) )</td>
<td>( - J_4 )</td>
<td>- moon</td>
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<tr>
<td>- ( \text{Tesseral } (3,2) )</td>
<td>( - m\text{-daily } (2,2) )</td>
<td>( - J_3 )</td>
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<tr>
<td>- ( \text{m\text{-daily } (4,2)} )</td>
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<tr>
<td>- ( J_4 )</td>
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5) \( \Delta q \)

<table>
<thead>
<tr>
<th>circular low altitude case</th>
<th>eccentric low altitude case (without the Tesserals)</th>
<th>high eccentricity high altitude (without the Tesserals)</th>
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<td>( - J_2 )</td>
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<tr>
<td>- ( \text{m\text{-daily } (3,1)} )</td>
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<td>- ( \text{m\text{-daily } (4,2)} )</td>
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5) $\Delta q$ (continued)

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<tr>
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6) $\Delta \lambda$

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<tr>
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<th>high eccentricity high altitude (without the Tesserals)</th>
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<td>- moon</td>
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<td>- $J_4$</td>
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</table>
A few comments and advice should be useful for a serious MACSYMA user.

All the MACSYMA sessions which are presented in this thesis were prepared through EMACS. EMACS is a text editor which is not part of the MACSYMA system but which can be used advantageously in cooperation with it, especially by using a display terminal. All the blocks, functions and expressions saved on file by means of this text editor are written on disk in the same way they are typed, so that the files may be read without difficulty.

To load into MACSYMA the content of a file prepared with EMACS, the BATCH command must be used as it has already been described in this thesis. But this command causes the file to be printed completely which can take a very long time if the file is very large. The way to avoid this problem is to turn the flag TTYOFF[FALSE] to TRUE before loading the file; this will temporarily turn off the display and setting back TTYOFF to FALSE will cause the printing to be resumed. For example, type:

TTYOFF:TRUE$

BATCH(ZONAL,1,DSK,KANIEK);TTYOFF:FALSE;

Before using the PLOT2 command, theoretically declaring the time as a floating point number
and using the TRANSLATE command in order to create a LISP version of the functions defined in MACSYMA

TRANSLATE:TRUE$

should result in a gain of speed, but the experience of the author shows that the computational time for the plots were about the same, using or not using these commands. Anyway, it does not do any harm to use them, so their use is advised.

If one follows exactly the user's guides previously described for each case, the plots are automatically displayed on the paper (of a printing terminal) or on the screen (of a display terminal). If the user wants these plots nicely printed on the XGP, simply type before the PLOT2 command:

PLOTMODE(XGP,D)$

Then use the linefeed at the end of the display of a plot, or HARDCOPY( ); and the XGP will send a message when the plot is being printed.

In order to get analytical formulas to store in a big numerical computer, after having batched in the desired file just type:
DELTAx \text{(list of arguments)}$

where

\begin{align*}
    x &= S \text{ for ZONAL1} \\
    x &= ST \text{ for ZONAL2} \\
    x &= SD \text{ for MDAILY} \\
    x &= STE \text{ for TESSER} \\
    x &= STB \text{ for THIRDB}
\end{align*}

In the list of arguments the equinoctial elements (as well as the components of the third body for THIRDB) must have symbolic and not numerical values. For ZONAL1 or ZONAL2, TRIGREDUCE this previous expression:

\text{TRIGREDUCE(EXPAND(\%), TL)}$

For the other cases just EXPAND the previous expression.

Then in order to "fortranize" the resulting formula use the MACSYMA command

\text{FORTRAN(expression)};

At this point it must be emphasized that the fortranized expression is provided with single precision only, and Fortran routines must be applied to it in order to have double precision. After that
the Fortran computer should achieve the same plots as MACSYMA but with a greater accuracy and speed for each new set of input data. As for these inputs, it must be recalled that all distances must be expressed in kilometers, all angles in radians and the time in seconds.

It is hoped that, after having read this thesis and these comments, any user should be able, sitting at a terminal, to visualize directly in front of him the plots of the short periodic variations due to the main effects emphasized previously, and this for any set of equinoctial elements, for any case of direct orbit, and for any position of the third body.

Future works should include:

-- trying the system on the LISP machine which is at least as fast as the MC machine and which has a lot more core capacity

-- the generalization of the system by including the retrograde case (the author will work on it at the end of the summer of 1979)

-- the derivation of the second order effects due to the oblateness of the Earth $J_2$ in a more efficient and automated way than Zeis' method (the author will investigate a closed form generating function for the $J_2^2$ at the end of the summer of 1979)

-- the possibility of using ideal equinoctial elements which seem to lead to more compact final expressions and cause less intermediate expression swell than the equinoctial element set.
Finally, serious efforts should be devoted to the simplifications of the analytical formulas before they are fortranized. McClain already undertook the simplification of these formulas for the Tesseractals obtaining good results but without putting forward general rules for these simplifications.
Appendix A. Partial Derivatives of the Eccentric Longitude and of the True Longitude with Respect to the Equinoctial Elements $h, k, \lambda$.

A.1 Partial Derivatives of the Eccentric Longitude

In terms of the eccentric longitude $F$ the equation of orbit is given by

$$r = a(1 - h \sin F - k \cos F) \quad (A-1)$$

and Kepler's equation by

$$\lambda = F + h \cos F - k \sin F \quad (A-2)$$

Therefore the eccentric longitude is a function of $h, k$ and $\lambda$.

The six equinoctial elements are independent, thus differentiating Kepler's equation with respect to $\lambda$ yields

$$1 = \frac{\partial F}{\partial \lambda} - h \sin F \frac{\partial F}{\partial \lambda} - k \cos F \frac{\partial F}{\partial \lambda} \quad (A-3)$$

or

$$\frac{\partial F}{\partial \lambda} = \frac{1}{1 - h \sin F - k \cos F} \quad (A-4)$$
Substitution of Equation (A-1) into (A-4) leads to

\[ \frac{\partial F}{\partial \lambda} = \frac{a}{r} \]  \hspace{1cm} (A-5)

Similarly, differentiating Kepler's equation with respect to \( h \) gives

\[ 0 = \frac{\partial F}{\partial h} + \cos F - h \sin F \frac{\partial F}{\partial h} - k \cos F \frac{\partial F}{\partial h} \] \hspace{1cm} (A-6a)

Substituting Equation (A-1) into (A-6a)

\[ \frac{\partial F}{\partial h} = -\frac{a}{r} \cos F \] \hspace{1cm} (A-6b)

Finally, differentiating Kepler's equation with respect to \( k \)

\[ 0 = \frac{\partial F}{\partial k} - h \sin F \frac{\partial F}{\partial k} - F \cos F \frac{\partial F}{\partial k} - k \cos F \frac{\partial F}{\partial k} \] \hspace{1cm} (A-7a)

or

\[ \frac{\partial F}{\partial k} = \frac{a}{r} \sin F \] \hspace{1cm} (A-7b)

A.2 Partial Derivatives of the True Longitude

In terms of the true longitude \( L \) the equation of orbit is given by

\[ r = \frac{a(1 - h^2 - k^2)}{1 + h \sin L + k \cos L} \] \hspace{1cm} (A-8)
Relative to the equinoctial frame the position vector is given by

\[ \mathbf{r} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \]  

(A-9)

where

\[ x = r \cos L \]  

(A-10a)

\[ y = r \sin L \]  

(A-10b)

From References [7] and [11] we have the following results:

For the time derivatives

\[ \dot{x} = \frac{-na}{\sqrt{1 - h^2 - k^2}} (h + \sin L) \]  

(A-11a)

\[ \dot{y} = \frac{na}{\sqrt{1 - h^2 - k^2}} (k + \cos L) \]  

(A-11b)

and for the partial derivatives

\[ \frac{\partial x}{\partial h} = -\frac{k\dot{x}}{n(1 + \sqrt{1 - h^2 - k^2})} + \frac{\dot{y}Y}{na\sqrt{1 - h^2 - k^2}} \]  

(A-12a)

\[ \frac{\partial y}{\partial h} = -\frac{k\dot{y}}{n(1 + \sqrt{1 - h^2 - k^2})} - \frac{\dot{x}X}{na\sqrt{1 - h^2 - k^2}} - a \]  

(A-12b)
\[
\begin{align*}
\frac{\partial x}{\partial k} &= \frac{hX}{n(1 + \sqrt{1 - h^2 - k^2})} + \frac{\dot{X}Y}{n\sqrt{1 - h^2 - k^2}} - a \quad (A-13a) \\
\frac{\partial y}{\partial k} &= \frac{hY}{n(1 + \sqrt{1 - h^2 - k^2})} - \frac{\dot{X}X}{n\sqrt{1 - h^2 - k^2}} \quad (A-13b) \\
\frac{\partial x}{\partial \lambda} &= \frac{X}{n} \quad (A-14a) \\
\frac{\partial y}{\partial \lambda} &= \frac{Y}{n} \quad (A-14b)
\end{align*}
\]

If we use the general notation \(\partial(h,k,\lambda)\) for \(\partial h\) or \(\partial k\) or \(\partial \lambda\), then, remembering that \(r\) depends on \(L\) (A-8), the differentiation of Equation (A-10) leads to

\[
\begin{align*}
\frac{\partial x}{\partial (h,k,\lambda)} &= \frac{\partial r}{\partial (h,k,\lambda)} \cos L + (-r \sin L + \frac{\partial r}{\partial L} \cos L) \frac{\partial L}{\partial (h,k,\lambda)} \quad (A-15a) \\
\frac{\partial y}{\partial (h,k,\lambda)} &= \frac{\partial r}{\partial (h,k,\lambda)} \sin L + (r \cos L + \frac{\partial r}{\partial L} \sin L) \frac{\partial L}{\partial (h,k,\lambda)} \quad (A-15b)
\end{align*}
\]

Then multiplying Equation (A-15a) by \(\sin L\) and Equation (A-15b) by \(\cos L\) and taking the difference gives
\[
\frac{r^2}{\partial} \frac{\partial L}{\partial (h,k,\lambda)} = X \frac{\partial Y}{\partial (h,k,\lambda)} - Y \frac{\partial X}{\partial (h,k,\lambda)} \quad (A-16)
\]

For \( \lambda \), substitution of Equation (A-14) into (A-16) leads to

\[
r^2 \frac{\partial L}{\partial \lambda} = \frac{\dot{X} Y - \dot{Y} X}{n} \quad (A-17)
\]

But the magnitude of the angular momentum vector is given by

\[
\| \vec{r} \times \vec{\dot{r}} \| = \dot{X} Y - \dot{Y} X = na^2 \sqrt{1 - h^2 - k^2} \quad (A-18)
\]

Substitution of this result in Equation (A-17) gives the result

\[
\frac{\partial L}{\partial \lambda} = \left( \frac{a}{r} \right)^2 \sqrt{1 - h^2 - k^2} \quad (A-19)
\]

Similarly, for \( h \), substitution of Equation (A-12) into (A-16) leads to

\[
r^2 \frac{\partial L}{\partial h} = -a X - \frac{r^2}{na \sqrt{1 - h^2 - k^2}} \frac{\dot{Y} - k}{n (1 + \sqrt{1 - h^2 - k^2})} \quad (A-20)
\]

Using the definitions of \( X, Y \) and of the angular momentum we have the result

\[
\frac{\partial L}{\partial h} = \frac{-(k + \cos L)}{1 - h^2 - k^2} - \frac{a}{r} \cos L - \frac{k \sqrt{1 - h^2 - k^2}}{1 + \sqrt{1 - h^2 - k^2}} \left( \frac{a}{r} \right)^2 \quad (A-21)
\]
Finally, for $k$, substitution of Equation (A-13) into (A-16) leads to

\[
\frac{r^2}{\hbar^2} \frac{\partial L}{\partial k} = a Y - \frac{r^2}{na\sqrt{1 - h^2 - k^2}} \dot{x} + \frac{h}{n(1 + \sqrt{1 - h^2 - k^2})} (\dot{Y} - \dot{X})
\]

\[(A-22)\]

Using the definitions of $Y$, $\dot{X}$ and of the angular momentum we have the final result

\[
\frac{\partial L}{\partial k} = \frac{h + \sin L}{1 - h^2 - k^2} + \frac{a}{r} \sin L + \frac{h\sqrt{1 - h^2 - k^2}}{1 + \sqrt{1 - h^2 - k^2}} \left(\frac{a}{r}\right)^2
\]

\[(A-23)\]
Appendix B. Derivation of the Partial Derivatives of $\alpha$, $\beta$ and $\gamma$ With Respect to $p$ and $q$

The direction cosines $(\alpha, \beta, \gamma)$ expressed in the direct equinoctial frame $(\hat{f}, \hat{g}, \hat{w})$ of an arbitrary unit vector $\hat{V}$ are given by

$$\alpha = \hat{f} \cdot \hat{V} \quad \text{(B-1a)}$$
$$\beta = \hat{g} \cdot \hat{V} \quad \text{(B-1b)}$$
$$\gamma = \hat{w} \cdot \hat{V} \quad \text{(B-1c)}$$

From Equations (1-2) the unit vectors $(\hat{f}, \hat{g}, \hat{w})$ expressed in the Earth inertial reference frame are given by

$$\hat{f} = \frac{1}{1 + p^2 + q^2} \begin{pmatrix} 1 - p^2 + q^2 \\ 2pq \\ -2p \end{pmatrix} \quad \text{(B-2a)}$$
$$\hat{g} = \frac{1}{1 + p^2 + q^2} \begin{pmatrix} 2pq \\ 1 + p^2 - q^2 \\ 2q \end{pmatrix} \quad \text{(B-2b)}$$
$$\hat{w} = \frac{1}{1 + p^2 + q^2} \begin{pmatrix} 2p \\ -2q \\ 1 - p^2 - q^2 \end{pmatrix} \quad \text{(B-2c)}$$

The arbitrary unit vector $\hat{V}$ is independent of the equinoctial elements $p$ and $q$ (for the applications of this thesis, $\hat{V}$ is either the inertial $\hat{z}$ axis in Section 2, or the unit position vector of the third body $\hat{R}_3$ in Section 5).
If the notation $\partial(p,q)$ is used for $\partial p$ and $\partial q$, it follows from Equation (B-1) that

\[
\frac{\partial x}{\partial (p,q)} = \frac{\partial f}{\partial (p,q)} \cdot \hat{V} \tag{B-3a}
\]
\[
\frac{\partial y}{\partial (p,q)} = \frac{\partial q}{\partial (p,q)} \cdot \hat{V} \tag{B-3b}
\]
\[
\frac{\partial z}{\partial (p,q)} = \frac{\partial w}{\partial (p,q)} \cdot \hat{V} \tag{B-3c}
\]

Taking the partial derivative of $\hat{f}$ [Equation (B-2a)] with respect to $p$ yields

\[
\frac{\hat{f}}{\partial p} = \frac{1}{(1 + p^2 + q^2)^2} \left( -2p(1 + p^2 + q^2) - 2p(1 - p^2 + q^2) \right) \tag{B-4}
\]

which simplifies to

\[
\frac{\hat{f}}{\partial p} = \frac{-2}{(1 + p^2 + q^2)^2} \left( -q + p^2q - q^3 \right) \tag{B-5a}
\]

Similarly

\[
\frac{\hat{f}}{\partial q} = \frac{2p}{(1 + p^2 + q^2)^2} \left( 1 + p^2 - q^2 \right) \tag{B-5b}
\]
\[ \frac{\partial \hat{g}}{\partial p} = \frac{2g}{(1 + p^2 + q^2)^2} \left( 1 - p^2 + q^2 \right) \]

\[ \frac{\partial \hat{g}}{\partial q} = \frac{-2}{(1 + p^2 + q^2)^2} \left( -p - p^3 + pq^2 \right) \]

\[ \frac{\partial \hat{w}}{\partial p} = \frac{2}{(1 + p^2 + q^2)^2} \left( 1 - p^2 + q^2 \right) \]

\[ \frac{\partial \hat{w}}{\partial q} = \frac{-2}{(1 + p^2 + q^2)^2} \left( -2p \right) \]

Using Equation (B-2) it can easily be shown that Equations (B-5) through (B-7) become

\[ \frac{\partial \hat{f}}{\partial p} = \frac{-2}{1 + p^2 + q^2} \left( \hat{g} + \hat{w} \right) \]

\[ \frac{\partial \hat{f}}{\partial q} = \frac{2p}{1 + p^2 + q^2} \hat{g} \]

\[ \frac{\partial \hat{g}}{\partial p} = \frac{2q}{1 + p^2 + q^2} \hat{f} \]

\[ \frac{\partial \hat{g}}{\partial q} = \frac{-2}{1 + p^2 + q^2} \left( \hat{p}f - \hat{w} \right) \]
\[
\frac{\partial^2 w}{\partial p^2} = \frac{2}{1 + p^2 + q^2} \hat{f}
\]  \hspace{1cm} (B-10a)

\[
\frac{\partial^2 w}{\partial q^2} = \frac{-2}{1 + p^2 + q^2} \hat{g}
\]  \hspace{1cm} (B-10b)

Substitution of Equations (B-8), (B-9) and (B-10) into Equation (B-3) leads to the final results

\[
\frac{\partial \alpha}{\partial p} = \frac{-2(q \beta + \gamma)}{1 + p^2 + q^2}
\]  \hspace{1cm} (B-11a)

\[
\frac{\partial \alpha}{\partial q} = \frac{2p \beta}{1 + p^2 + q^2}
\]  \hspace{1cm} (B-11b)

\[
\frac{\partial \beta}{\partial p} = \frac{2q \alpha}{1 + p^2 + q^2}
\]  \hspace{1cm} (B-12a)

\[
\frac{\partial \beta}{\partial q} = \frac{-2(p \alpha - \gamma)}{1 + p^2 + q^2}
\]  \hspace{1cm} (B-12b)

\[
\frac{\partial \gamma}{\partial p} = \frac{2 \alpha}{1 + p^2 + q^2}
\]  \hspace{1cm} (B-13a)

\[
\frac{\partial \gamma}{\partial q} = \frac{-2 \beta}{1 + p^2 + q^2}
\]  \hspace{1cm} (B-13b)
Appendix C. **Recursive Formulas for the Special Function** \( Q_{nm}(x) \) **and for the Coefficient** \( V_{n,m} \)

### C.1 \( Q_{nm}(x) \)

The definition of this special function is given by Equation (2-28)

\[
Q_{nm}(x) = (1 - x^2)^{-m/2} P_{nm}(x)
\]  

(C-1)

The standard recursions for the associated Legendre functions (Reference [16]) are

for fixed order and varying degree:

\[
(n-m) P_{nm}(x) = (2n-1) x P_{n-1,m}(x) - (m+n-1) P_{n-2,m}(x)
\]  

(C-2)

for fixed degree and varying order:

\[
P_{n,m+2}(x) - 2(m+1) x (1 - x^2)^{-1/2} P_{n,m+1}(x) =
\]

\[
= (m-n) (n+m+1) P_{n,m}(x)
\]  

(C-3)

Because of the definition of \( Q_{nm}(x) \), the recursive formula for fixed order for the \( Q_{nm} \) function is the same as for the \( P_{nm} \) function:
(n-m) Q_{nm}(x) = (2n-1) x Q_{n-1,m}(x) - (m+n-1) Q_{n-2,m}(x)

(C-4)

The transformation from \( P_{nm}(x) \) to \( Q_{nm}(x) \) depends on \( m \); therefore the recursive formula for \( Q_{nm}(x) \) will have a different form as Equation (C-3). Substitution of Equation (C-1) into (C-3) leads to

\[
(1-x^2)^{1+m/2} Q_{n,m+2}(x) - 2(m+1) x (1-x^2)^{m/2} Q_{n,m+1}(x) =
\]

\[
= (m-n)(n+m-1)(1-x^2)^{m/2} x Q_{n,m}(x)
\]

(C-5)

which simplifies to

\[
(m-n)(n+m+1) Q_{n,m}(x) = (1-x^2) Q_{n,m+2}(x)
\]

\[
- 2(m+1) x Q_{n,m+1}(x)
\]

(C-6)

The next step is to obtain starting values \( Q_{n,n}(x), Q_{n+1,n}(x), \) and \( Q_{n,n+1}(x) \) for these two recursive formulas. The associated Legendre functions can be expressed in terms of a hypergeometric series (Reference [16]) using Equation (C-1). \( Q_{nm}(x) \) can thus be written as

\[
Q_{nm}(x) = \frac{(n+m)!}{2^m m! (n-m)!} \, F(m-n,m+n+1; m+1; \frac{1-x}{2})
\]

(C-7)

For \( m = n \) Equation (C-7) becomes
\[ Q_{nn}(x) = \frac{(2n)!}{2^n n!} F(0,2n+1; n+1; \frac{1-x}{2}) \] (C-8)

With its first argument equal to zero, the hypergeometric series equals one. Therefore

\[ Q_{nm}(x) = \frac{(2n)!}{2^n n!} \] (C-9)

or in a more convenient way

\[ Q_{nn}(x) = (2n - 1)!! \] (C-10a)

with \( Q_{00}(x) = 1 \) (C-10b)

For \( n - m = 1 \), Equation (C-7) becomes

\[ Q_{n+1,n}(x) = \frac{(2n-1)!}{2^{n-1}(n-1)!} F(-1,2n,n;\frac{1-x}{2}) \] (C-11)

Expanding the hypergeometric series gives

\[ Q_{n+1,n}(x) = \frac{(2n-1)!}{2^{n-1}(n-1)!} \left[ 1 - (1 - x) + \frac{2(2n+1)}{n+1} \left( \frac{1-x}{2} \right)^2 + \ldots + \frac{(-1)^k 2(2n+1)\ldots(2n+k-1)}{(n+1)\ldots(n+k-1)} \left( \frac{1-x}{2} \right)^k + \ldots \right] \] (C-12)
which reduces to

\[ Q_{n+1,n}(x) = \frac{(2n + 2)!}{2^{n+1}(n+1)!} \times \]

or in a more convenient way

\[ Q_{n+1,n}(x) = x(2n + 1)!! \]  \hspace{1cm} (C-14)

As

\[ P_{nm}(x) = 0 \quad \text{for } m > n \]  \hspace{1cm} (C-15)

Therefore

\[ Q_{n,m}(x) = 0 \quad \text{for } m > n \]

and particularly

\[ Q_{n,n+1}(x) = 0 \]  \hspace{1cm} (C-17)

C.2 \hspace{1cm} V_{nm}

The definition of this coefficient is given by Equation (2-31)

\[ V_{n,m} = \frac{(n-m)!}{(n+m)!} Q_{n,m}(0) \]  \hspace{1cm} (C-18)
The recursions can easily be derived from the recursions of $Q_{nm}(x)$ setting $x$ equal to zero. In particular, the recursion with fixed order is obtained by substitution of Equation (C-17) into (C-4)

\[
(n-m) \frac{(n+m)!}{(n-m)!} V_{nm} = - (m + n - 1) \frac{(n - 2 + m)!}{(n - 2 - m)!} V_{n-2,m}
\]

(C-19)

which reduces to

\[
(n + m) V_{n,m} = (m - n + 1) V_{n-2,m}
\]

(C-20)

Because of Equation (C-15)

\[
V_{n,m} = 0 \quad \text{for } m > n
\]

(C-21)

The initial values for the recursion can be obtained from the definition of $V_{nm}$ and from the initial values for $Q_{nm}(x)$.

For $m = n$ using Equations (C-20) and (C-10)

\[
V_{nn} = \frac{(2n - 1)!!}{(2n)!}
\]

(C-22a)

with

\[
V_{00} = 1
\]

(C-22b)
Furthermore, using the following property

\[ P_{nm}(0) = 0 \quad \text{for } (n - m) \text{ odd} \quad (C-23) \]

Substitution of this result into (C-1) gives

\[ Q_{nm}(0) = 0 \quad \text{for } (n - m) \text{ odd} \quad (C-24) \]

and finally we have the interesting result

\[ V_{nm} = 0 \quad \text{for } (n - m) \text{ odd} \quad (C-25) \]
Appendix D. A Jacobi Polynomial Representation for the Special Function $W^n_s(t)$ for the Case Where $n$ is a Positive Integer

This special function has been introduced as the Fourier series coefficients in Equation (5-16).

$$\left(\frac{r}{a}\right)^n \exp(jsL) = \sum_{t=-n}^{n} W^n_s(t) \exp(jtF) \quad (D-1)$$

But

$$L = v + \Omega + \omega \quad (D-2a)$$
$$F = E + \Omega + \omega \quad (D-2b)$$

where

$E$ = eccentric anomaly
$v$ = true anomaly
$\Omega, \omega$ = Keplerian orbital elements

Equation (D-1) can be written in terms of anomalies:

$$\left(\frac{r}{a}\right)^n \exp(jsv) = \sum_{t=-n}^{n} W^n_s(t) \exp[j(t+s)(\Omega+\omega)] \exp(jtE) \quad (D-3)$$

Using the equinoctial elements $h$ and $k$ [Equations (1-1b) and (1-1c)]

Equation (D-3) becomes
\[
\left( \frac{r}{a} \right)^n \exp(jsv) = \sum_{t=-n}^{n} W_{t}^{n,s} \left( \frac{k+jh}{\sqrt{k^2+h^2}} \right)^{t-s} \exp(jtE)
\]  

(D-4)

Defining the new function

\[
W_{t}^{n,s} = \left( \frac{k+jh}{\sqrt{k^2+h^2}} \right)^{t-s} W_{t}^{n,s}
\]  

(D-5)

Equation (D-4) becomes

\[
\left( \frac{r}{a} \right)^n \exp(jsv) = \sum_{t=-n}^{n} W_{t}^{n,s} \exp(jtE)
\]  

(D-6)

The following definitions are made:

\[
\beta = \frac{e}{1 + \sqrt{1-e^2}} = \frac{\sqrt{h^2+k^2}}{1 + \sqrt{1-h^2-k^2}}
\]  

(D-7)

\[
x = \frac{1}{\sqrt{1-e^2}} = \frac{1 + \beta^2}{1 - \beta^2} = \frac{1}{\sqrt{1-h^2-k^2}}
\]  

(D-8)

\[
X = \exp(jv)
\]  

(D-9)

\[
Y = \exp(jE)
\]  

(D-10)

From classical two body mechanics, the equation of the ratio \( r/a \) is:
\[
\frac{r}{a} = 1 - e \cos E \quad \text{(D-11)}
\]

From Equation (D-7), we have:

\[
\frac{e}{2} = \frac{\beta}{1 + \beta^2} \quad \text{(D-12)}
\]

Therefore, using Equations (D-10) and (D-12) the Equation for \(r/a\) becomes

\[
\frac{r}{a} = \left[ \frac{1 + \beta^2}{1 + \beta^2} - \frac{\beta}{1 + \beta^2} (Y + Y^{-1}) \right] \quad \text{(D-13)}
\]

which simplifies to

\[
\frac{r}{a} = \frac{(1 - \beta Y) (1 - \beta Y^{-1})}{(1 + \beta^2)} \quad \text{(D-14)}
\]

Using the relation between the true and the eccentric anomaly

\[
\chi = Y \frac{(1 - \beta Y^{-1})}{(1 - \beta Y)} \quad \text{(D-15)}
\]

It follows from Equations (D-14) and (D-15) that

\[
\left( \frac{r}{a} \right)^n e^{j \psi} = \left( \frac{r}{a} \right)^n \chi^s = y^s \frac{(1 - \beta Y)^{n-s} (1 - \beta Y^{-1})^{n-s}}{(1 + \beta^2)^n} \quad \text{(D-16)}
\]
where \( n \) is a positive integer. For the purpose of this development the relation

\[
n \geq |s| \quad (D-17)
\]

is always satisfied. The next step is to expand the product

\[(1 - \beta Y)^{n-s} (1 - \beta Y^{-1})^{n-s} \]

of Equation (D-16). Since \( n \geq |s| \), the Binomial Theorem leads to

\[
(1 - \beta Y)^{n-s} (1 - \beta Y^{-1})^{n+s} = \sum_{k=0}^{n-s} \sum_{m=0}^{n+s} \binom{n-s}{k} \binom{n+s}{m} x (-\beta)^{k+m} \gamma^{k-m} \quad (D-18)
\]

or

\[
(1 - \beta Y)^{n-s} (1 - \beta Y^{-1})^{n+s} = \sum_{t=-n}^{n-s} \sum_{p=|t|}^{2n} \binom{n-s}{p+t} \binom{n+s}{p+t} x (-\beta)^{p+t} \gamma^t \quad (D-19)
\]

or

\[
(1 - \beta Y)^{n-s} (1 - \beta Y^{-1})^{n+s} = \sum_{t=-n}^{n} \sum_{i=0}^{2} (-1)^{|t-s|} n-|t-s| \binom{n-s}{i + (|t-s| + t-s)} \binom{n+s}{i + (|t-s|-t+s)} x \beta^{2i+|t-s|} \gamma^{t-s}
\]

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If

\[ \alpha = \frac{|t-s| + t - s}{2} \]  
(D-21a)

and

\[ \rho = \frac{|t-s| + s - t}{2} \]  
(D-21b)

It can then be written, using Pochhammer's notation

\[ \binom{n-s}{i+\alpha} = \binom{n-s}{\alpha} (-1)^i \frac{(\alpha - n + s)_i}{(\alpha + 1)_i} \]  
(D-22)

and

\[ \binom{n+s}{i+\rho} = \binom{n+s}{\rho} (-1)^i \frac{(\rho - n - s)_i}{(\rho + 1)_i} \]  
(D-23)

Multiplication of Equation (D-22) with (D-23) and noticing from Equation (D-21) that

\[ (\alpha + 1)_i (\rho + 1)_i = (|t| + 1)_i (i!) \]  
(D-24)

leads to

\[ \binom{n-s}{i+\alpha} \binom{n+s}{i+\rho} = \binom{n-s}{\alpha} \binom{n+s}{\rho} \frac{(\alpha-n+s)_i (\rho-n-s)_i}{(|t| + 1)_i i!} \]  
(D-25)
Expliciting $\alpha$ and $\rho$ and substituting Equation (D-25) into Equation (D-20), the product $(1 - \beta Y)^{n-s} (1 + \beta Y^{-1})^{n+s}$ can thus be expressed in terms of the hypergeometric series

$$(1 - \beta Y)^{n-s} (1 - \beta Y^{-1})^{n+s} = \sum_{t=-n}^{n} (-1)^{|t-s|} \binom{n-s}{|t-s|-t+s}$$

$$x \left( \begin{array}{c} n+s \\ \frac{|t-s|+t-s}{2} \end{array} \right) \beta^{|t-s|}$$

$$x F \left( \frac{|t-s|+t-s}{2} - n+s, \frac{|t-s|-t+s}{2} - n-s, |t-s|+1; \beta^2 \right)$$

Substituting Equation (D-26) into Equation (D-16), then equating Equation (D-16) to (D-6) yields the function $w_{t}^{n,s}$:

$$w_{t}^{n,s} = (1 + \beta^2) (-1)^{|t-s|} \binom{n-s}{\frac{|t-s|+t-s}{2}} \binom{n+s}{\frac{|t-s|-t+s}{2}} \beta^{|t-s|}$$

$$x F \left( \frac{|t-s|+t-s}{2} - n+s, \frac{|t-s|-t+s}{2} - n-s, |t-s|+1; \beta^2 \right)$$

(D-27)
Now we are interested in a Jacobi polynomial representation for $w^n_t$ and $W^n_t$. For that we apply the following linear transformations.

$$F(a, b, c, z) = \begin{cases} 
(1 - z)^{-a} F(a, c-b, c; \frac{z}{z-1}) & \text{(D-28a)} \\
(1 - z)^{-b} F(b, c-a, c; \frac{z}{z-1}) & \text{(D-28b)}
\end{cases}$$

to the hypergeometric series in Equation (D-27) which yields

$$F \left( \frac{|t-s|+t-s}{2} - n+s, \frac{|t-s|-t+s}{2} - n-s, |t-s|+1; \beta^2 \right) = \begin{cases} 
(1-\beta) \left( \frac{|t-s|+t-s}{2} \right) F \left( \frac{|t-s|+t-s}{2} - n+s, \frac{|t-s|+t-s}{2} + n+s+1, |t-s|+1; \frac{\beta^2}{\beta^2 - 1} \right) & \text{(D-29a)} \\
(1-\beta) \left( \frac{|t-s|-t+s}{2} \right) F \left( \frac{|t-s|-t+s}{2} - n-s, \frac{|t-s|-t+s}{2} + n-s+1, |t-s|+1; \frac{\beta^2}{\beta^2 - 1} \right) & \text{(D-29b)}
\end{cases}$$

Using the following relation between the hypergeometric sines and the Jacobi polynomials (Reference [14])

$$F(-m, a+b+m+1, a+1; z) = \frac{m!}{(a+1)^m} p^a, b (1 - 2z) \quad \text{(D-30)}$$
and noticing [Equation (D-8)] that

\[ 1 - \frac{2\beta^2}{\beta^2 - 1} = \frac{1 + \beta^2}{1 - \beta^2} = x \tag{D-31} \]

it follows that Equation (D-29) can be expressed as

\[
F \left[ \frac{|t-s|+t-s}{2} - n+s, \frac{|t-s|-t+s}{2} - n-s, |t-s|+1; \beta^2 \right] = \\
\frac{\begin{array}{c}
\begin{array}{c}
(1-\beta^2)^{n-s-\left(\frac{|t-s|+t-s}{2}\right)}
\end{array}
\frac{\begin{array}{c}
\begin{array}{c}
\left(\frac{\Gamma\left(\frac{|t-s|+t-s}{2}\right)\Gamma\left(\frac{|t-s|-t+s}{2}\right)}{(t-s)!}\right)
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
p|t-s|,t+s
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\frac{|t-s|+1}{n-s-\left(\frac{|t-s|+t-s}{2}\right)}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\left(\frac{\Gamma\left(\frac{|t-s|-t+s}{2}\right)\Gamma\left(\frac{|t-s|+t-s}{2}\right)}{(t-s)!}\right)
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
(x)
\end{array}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
(1-\beta^2)^{n+s-\left(\frac{|t-s|-t+s}{2}\right)}
\end{array}
\frac{\begin{array}{c}
\begin{array}{c}
\left(\frac{\Gamma\left(\frac{|t-s|-t+s}{2}\right)\Gamma\left(\frac{|t-s|+t-s}{2}\right)}{(t-s)!}\right)
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
p|t-s|,-t-s
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\frac{|t-s|+1}{n+s-\left(\frac{|t-s|-t+s}{2}\right)}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\left(\frac{\Gamma\left(\frac{|t-s|+t-s}{2}\right)\Gamma\left(\frac{|t-s|-t+s}{2}\right)}{(t-s)!}\right)
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
(x)
\end{array}
\end{array}
\end{array}
\end{array}
\] \tag{D-32a}

\[
\begin{array}{c}
\begin{array}{c}
(1-\beta^2)^{n+s-\left(\frac{|t-s|-t+s}{2}\right)}
\end{array}
\frac{\begin{array}{c}
\begin{array}{c}
\left(\frac{\Gamma\left(\frac{|t-s|-t+s}{2}\right)\Gamma\left(\frac{|t-s|+t-s}{2}\right)}{(t-s)!}\right)
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
p|t-s|,-t-s
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\frac{|t-s|+1}{n+s-\left(\frac{|t-s|-t+s}{2}\right)}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\left(\frac{\Gamma\left(\frac{|t-s|+t-s}{2}\right)\Gamma\left(\frac{|t-s|-t+s}{2}\right)}{(t-s)!}\right)
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
(x)
\end{array}
\end{array}
\end{array}
\] \tag{D-32b}

Since \( t+s \geq 0 \) in Equation (D-32a) and \( t+s \leq 0 \) in Equation (D-32b), considerations of the sign of \( (t-s) \) lead to the following simplifications of Equation (D-30).
\[
F \left[ \frac{|t-s|+t-s}{2} - n+s, \frac{|t-s|-t+s}{2} - n-s, |t-s|+1; \beta^2 \right] = \\
\begin{cases}
(1-\beta^2)^{n-t} \frac{(n-t)!}{(t-s+1)_n} p_{t-s,t+s} (x), & \text{for } t-s \geq 0, t+s \geq 0 \quad (D-33a) \\
(1-\beta^2)^{n-s} \frac{(n-s)!}{(s-t+1)_n} p_{s-t,s+t} (x), & \text{for } t-s \leq 0, t+s \geq 0 \quad (D-33b) \\
(1-\beta^2)^{n+s} \frac{(n+s)!}{(t-s+1)_n} p_{t-s,-(t+s)} (x), & \text{for } t-s \geq 0, t+s \leq 0 \quad (D-33c) \\
(1-\beta^2)^{n+t} \frac{(n+t)!}{(s-t+1)_n} p_{s-t,-(t+s)} (x), & \text{for } t-s \leq 0, t+s \leq 0 \quad (D-33d)
\end{cases}
\]

or, expliciting Pochhammer's notation

\[
F \left[ \frac{|t-s|+t-s}{2} - n+s, \frac{|t-s|-t+s}{2} - n-s, |t-s|+1; \beta^2 \right] = \\
\begin{cases}
(1-\beta^2)^{n-t} \frac{(n-t)!}{(t-s)_n} p_{t-s,t+s} (x), & \text{for } t \geq s \geq 0 \quad (D-34a) \\
(1-\beta^2)^{n-s} \frac{(n-s)!}{(s-t)_n} p_{s-t,s+t} (x), & \text{for } s \geq t \geq 0 \quad (D-34b) \\
(1-\beta^2)^{n+s} \frac{(n+s)!}{(t-s)_n} p_{t-s,-(t+s)} (x), & \text{for } s \leq -t \leq 0 \quad (D-34c) \\
(1-\beta^2)^{n+t} \frac{(n+t)!}{(s-t)_n} p_{s-t,-(t+s)} (x), & \text{for } t \leq s \leq 0 \quad (D-34d)
\end{cases}
\]

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Substitution of Equation (D-34) into (D-27) leads to

\[
\begin{align*}
\left(1-\beta^2\right)^{-t} P_{n-t}^{s-t,s+t+s}(x), (t \geq |s| \geq 0) \quad & (D-35a) \\
\frac{(n+s)!(n-s)!}{(n+t)!(n-t)!} \left(1-\beta^2\right)^{-s} P_{n-s}^{s-t,s+t+s}(x), (s \geq |t| \geq 0) \quad & (D-35b) \\
\frac{(n+s)!(n-s)!}{(n+t)!(n-t)!} \left(1-\beta^2\right)^{s} P_{n+s}^{s-t,-(t+s)}(x), (s \leq |t| \leq 0) \quad & (D-35c) \\
\left(1-\beta^2\right)^{t} P_{n+t}^{s-t,-(t+s)}(x), (t \leq |s| \leq 0) \quad & (D-35d)
\end{align*}
\]

which simplifies to

\[
\begin{align*}
\left(1-\beta^2\right)^{-|s|} P_{n-|s|}^{t-|s|,|t+s|}(x), (|s| \geq |t|) \quad & (D-36a) \\
\left(1-\beta^2\right)^{t} P_{n-|t|}^{t-|s|,|t+s|}(x), (|s| \leq |t|) \quad & (D-36b)
\end{align*}
\]
The relation (D-5) between \( W_{t}^{n,s} \) and \( W_{t}^{n,s} \) gives the final result

\[
W_{t}^{n,s} = x^{n(-\beta)}|t-s| \left( \frac{\sqrt{h^2+k^2}}{k+jh} \right) |t-s| \left\{ \begin{array}{l}
\frac{(n+s)!(n-s)!}{(n+t)-(n-t)!} (1-\beta^2)^{-\frac{s}{|s|}} \\
x p|t-s|, |t-s|(x), (|s|>|t|) \\
(1-\beta^2)^{-|t|} p|t-s|, |t+s|(x) \\
(|s|\leq|t|)
\end{array} \right. 
\]

(D-37a)

\[
W_{t}^{n,s} = x^{n(-\beta)}|t-s| \left( \frac{\sqrt{h^2+k^2}}{k+jh} \right) |t-s| 
\]

(D-37b)

A very interesting property can be derived by averaging Equation (D-1) with respect to the eccentric longitude.

\[
W_{0}^{n,s} = \frac{1}{2\pi} \int_{0}^{2\pi} \left( \frac{r}{a} \right)^n \exp(jtL) d\lambda 
\]

(D-38)

using the change of variable [Equation (5-14)]

\[
d\lambda = \left( \frac{a}{r} \right) d\lambda
\]

Equation (D-38) becomes

\[
W_{0}^{n,s} = \frac{1}{2\pi} \int_{0}^{2\pi} \left( \frac{r}{a} \right)^{n-1} \exp(jtL) d\lambda 
\]

(D-39)
The right hand side of Equation (D-39) represents the definition of the Hansen coefficient \( y_{n-1,s} \), therefore

\[
W_{0}^{n,s} = y_{0}^{n-1,s} \quad (D-40)
\]
Appendix E. Input Transformations

E.1 Conversion from Keplerian Elements to Equinoctial Elements

A MACSYMA block has been made in order to obtain the direct equinoctial elements \((a,h,k,p,q,\lambda)\) from the Keplerian elements \((a,e,\omega, i,\Omega,M)\). A listing of this block is presented.

Listing of `KEPEC(AX,E,PER,1,LON,M)`

```macsyma
KEPEC(AX,E,PER,1,LON,M):=BLOCK(
    [A,H,K,P,Q,L],
    /*THIS BLOCK COMPUTES THE EQUINOCTIAL ORBIT ELEMENTS FROM THE KEPLERIAN ELEMENTS*/
    /*PROGRAMMER: J-P.KANIECKI,MIT-JULY 1979*/
    /*DEFINITIONS:
    AX=SEMI-MAJOR AXIS IN KILOMETERS,
    E=ECCENTRICITY,
    PER=ARGUMENT OF PERIGEE IN DEGREES,
    I=INCLINATION IN DEGREES,
    LON=LONGITUDE OF ASCENDING NODE IN DEGREES,
    M=MEAN ANOMALY IN DEGREES.*/
    /*LOCAL VARIABLES:
    A,H,K,P,Q,L=EQUINOCTIAL ELEMENTS,
    A IN KILOMETERS,
    L IN RADIANS.*/
    A:AX,
    H:EV(E*SIN((PER+LON)*%PI/180),NUMER),
    K:EV(E*COS((PER+LON)*%PI/180),NUMER),
    P:EV(TAN(I*%PI/360)*SIN(LON*%PI/180),NUMER),
    Q:EV(TAN(I*%PI/360)*COS(LON*%PI/180),NUMER),
    L:EV((M+PER+LON)*%PI/180,NUMER),
```
Components of the Position Vector of the Third Body With Respect to the Earth Inertial Frame

In order to know the position vector of the third body, one must use "The American Ephemeris and Nautical Almanac." For each day of the year, this book gives the rectangular coordinates \((X_0, Y_0, Z_0)\) of the sun. For the moon the right ascension \(\alpha\), the declination \(\delta\), and the geocentric distance \(a_0\) are given, the following transformation will lead to the rectangular coordinates of the moon

\[
\begin{pmatrix}
X_0 \\
Y_0 \\
Z_0
\end{pmatrix} = r
\begin{pmatrix}
\cos \alpha \cos \delta \\
\sin \alpha \cos \delta \\
\sin \delta
\end{pmatrix}
\]

where \(r = 6378.16\ a_0\)

A listing of the block computing \((Z_0, Y_0, Z_0)\) for the moon is presented.
Listing of RMOON(AL1,AL2,AL3,DEL1,DEL2,DEL3,A0)

RMOON(AL1,AL2,AL3,DEL1,DEL2,DEL3,A0):=BLOCK([[AL,D,X0,Y0,Z0)],

/* THIS BLOCK COMPUTES THE COMPONENTS OF THE POSITION VECTOR OF THE MOON WITH RESPECT TO THE EARTH INERTIAL FRAME */

/*PROGRAMMER: J-P.KANIECKI, MIT-JULY 1979 */

/*DEFINITIONS:
AL1-AL2-AL3=RIGHT ASCENSION IN H-M-S,
DEL1-DEL2-DEL3=DECLINATION IN DEG-M-S,
A0=GEOCENTRIC DISTANCE IN EARTH RADII */

/*LOCAL VARIABLES:
AL=RIGHT ASCENSION IN RADIANS,
D=DECLINATION IN RADIANS,
X0,Y0,Z0=COMPONENTS OF THE POSITION VECTOR OF THE MOON IN KILOMETERS */

AL:EV((3600*AL1+60*AL2+AL3)*%PI/43200,NUMER),
D:EV((DEL1+DEL2/60+DEL3/3600)*%PI/180,NUMER),
X0:6378.16*A0*COS(AL)*COS(D),
Y0:6378.16*A0*SIN(AL)*COS(D),
Z0:6378.16*A0*SIN(D),

/*DISPLAY OF THE RESULTS */

RETURN(DISPLAY(X0,Y0,Z0))$

Note: These two blocks presented in this appendix are stored on the File (INPUT,>,DSK,KANIEK). In order to load it just type:

BATCH(INPUT,>,DSK,KANIEK)
REFERENCES


