POINT SOURCE OPTICAL PROPAGATION
IN A MULTIPLE SCATTERING MEDIUM

by

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ABSTRACT

The spatial and angular characteristics of point source propagation through a multiple scattering medium are investigated. A combined experimental and theoretical approach is taken, in which experimental data is used to indicate approximations that can be made to the general linear transport equation to simplify its solution.

Measurements of the received angular spectrum and the optical power transmission have been made in maritime fog at optical thicknesses, \( \tau \), less than 10. The source wavelength was 0.25 \( \mu \text{m} \). The data is compared with existing propagation theories. Among the many angular spectrum characteristics observed in the experiments, one of them, called "insensitivity", is invoked to simplify the transport equation. Insensitivity refers to the tendency of the scattered field to remain relatively constant in response to random fluctuations of the medium's extinction coefficient around its average. It is shown that the insensitivity property implies that the source term in the transport equation is negligibly small. Neglecting this term is called the strong multiple scatter (SMS) approximation, and allows application of the (measured or otherwise derived) boundary condition anywhere in the medium rather than necessarily at the source.

For isotropic scatter, the solution to the SMS transport equation is shown to be identical to the known exact solution when \( \tau \) is large. For anisotropic scatter, the large \( \tau \) limit of the solution is studied in detail using the spherical harmonics method. This asymptotic solution requires computation of the matrix exponential \( \exp[D \tau] \), where \( D \), called the channel matrix, has real, distinct and symmetric eigenvalues. Bounds on the dominant eigenvalue are obtained and simple algorithms for computing the left and right eigenvectors are developed.
The dominant eigenvalue term in the solution is shown to have a near-diffusion exponential decay rate and a broad angular spectrum. When subdominant eigenvalue terms in the solution are important, the angular spectrum is shown to be of two types, one corresponding to phase functions which are narrower than the applied boundary condition and one corresponding to broader phase functions. The first type of angular spectrum exhibits a uniform broadening of the initially narrow peak as $\tau$ increases. The second type retains a pronounced narrow peak, with broadening manifesting itself as a raising of the relatively flat pedestal on which the peak rests. The experimental angular spectrum is shown to evolve with $\tau$ in much the same way as the second, or broad phase function type, angular spectrum, indicating a significant amount of wide angle scatter for the fog phase function encountered experimentally.

Thesis Supervisor: Robert S. Kennedy

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CHAPTER I

INTRODUCTION AND BACKGROUND

Optical communication through the atmosphere has gained increasing interest in recent years. The information explosion has brought about the need for higher bandwidth communication channels which are not constrained to use cables. The continued development of highly collimated laser sources and other optical components has made it easier to use the optical carrier for point-to-point information transmission. The atmosphere has also been proposed as a medium for optical broadcast transmission using wide-angle or omnidirectional sources, as an alternative to conventional radio [1-9].

The atmosphere, however, is significantly different from a free space channel. It produces effects on optical radiation - due to multiple scattering, absorption and clear-air turbulence - which make the utilization of such a medium more difficult than free space. At present, the problem of propagation through turbulence is relatively well-understood [8]. However, propagation through a turbid atmosphere - resulting from severe weather conditions such as fog, haze rain or snow - is still not well understood.

Propagation through a scattering and absorbing medium involves interactions between photons and atmospheric constituents such as gas molecules and suspended aerosol particles. These interactions influence the communication receiver's power level in a number of ways (refer to
Figure 1.1):

1. There is an unrecoverable power loss due to absorption;
2. Light traveling along the line-of-sight (or on-axis) direction between the transmitter and receiver is redirected away from the receiver due to scattering. This effect manifests itself as a beamspread or decollimation in a laser beam, producing an effective power loss at a small fixed-diameter receiver;
3. The angular distribution of received light is broadened, due to the fact that some light is scattered back to the receiver from off-axis. This may introduce signal loss at the detector if the spatial distribution of the focussed power becomes broader than the detector diameter;
4. Pulse distortion is introduced by the increased pathlength travelled by the received scattered light; pulse to pulse variations are introduced by the non-stationarity of the atmosphere.

All of these effects, because they involve a loss of power or inefficient use of available power at the receiver, result in a degradation in communication system performance. It is obvious that in order to mitigate against this loss in performance, use must be made of the recoverable scattered power reaching the receiver. This, in turn, requires a quantitative understanding of the extent of the propagation effects. It can be shown [1,4] that the statistics of the received field in a predominantly scattering atmosphere are Gaussian. Hence, a complete
Figure 1.1: Effects of a Scattering/Absorbing Medium on Optical Radiation
channel description would involve knowledge of the space-time correlation function of the received field. However, for communication purposes, such a complete characterization is not necessary, and if it were available it would be too complex to be used in its entirety for communication system design. It is sufficient to characterize the channel in terms of certain simple parameters such as the medium's average transmission, the average beam spread, the average angular spread (width of the angular photon distribution or "angular spectrum") and the multipath time spread. This thesis is concerned with the development of such a description.

The outline of this chapter is as follows. Section 1.1 discusses various possible approaches to channel characterization, including both theoretical and experimental methods. Section 1.2 discusses the middle ultraviolet, which is the wavelength region of interest in this thesis. The motivation for using this spectral region, technology issues and propagation phenomenology are the subject of this section. Finally, Section 1.3 presents the organization of the rest of this thesis.
1.1. Approaches to Channel Characterization

In principle, the channel parameters of interest can be obtained from the linear transport equation [10]:

\[
\left[ \alpha(\mathbf{r},t) + \frac{1}{c} \frac{\partial}{\partial t} + \mathbf{\Omega} \cdot \nabla \right] p(\mathbf{r},\mathbf{\Omega},t) = \alpha_s(\mathbf{r},t) \int d\mathbf{\Omega}' \sigma(\mathbf{\Omega},\mathbf{\Omega}',\mathbf{r},t) p(\mathbf{r},\mathbf{\Omega}',t),
\]

(1.1)

where

- \( p(\mathbf{r},\mathbf{\Omega},t) \) is the probability density at time \( t \) for a photon not to be absorbed and to be at point \( \mathbf{r} \), going in the direction of the unit vector \( \mathbf{\Omega} \);
- \( \alpha(\mathbf{r},t) \) is the extinction coefficient in the medium;
- \( \alpha_s(\mathbf{r},t) \) is the scattering coefficient in the medium;
- \( \sigma(\mathbf{\Omega},\mathbf{\Omega}',\mathbf{r},t) \) is the single scatter phase function, normalized so that \( \int d\mathbf{\Omega}' \sigma(\mathbf{\Omega},\mathbf{\Omega}',\mathbf{r},t) = 1 \);
- \( c \) is the speed of light; and
- \( \nabla \) is the gradient operator.

The extinction coefficient, \( \alpha(\mathbf{r},t) \) is the fraction of power removed from a propagating beam at point \( \mathbf{r} \) and time \( t \) per unit distance traveled, due to both scattering and absorption. Similarly, the scattering coefficient is the fraction of power removed at \( (\mathbf{r},t) \) by scattering out of the beam, per unit distance traveled. The single scatter phase function is the conditional probability density at \( (\mathbf{r},t) \) for a photon to be changed from the direction \( \mathbf{\Omega} \) to the direction \( \mathbf{\Omega}' \) by a scattering encounter with an atmospheric particle.
While Eq. (1.1) has been written for the most general case, it is usually assumed (for simplicity) that $\alpha(\mathbf{r},t)$, $\alpha_s(\mathbf{r},t)$ and $\sigma(\Omega,\Omega',\mathbf{r},t)$ are not functions of position or time (i.e., the medium is a homogeneous distribution of scatterers). In this case they are denoted as $\alpha$, $\alpha_s$ and $\sigma(\Omega,\Omega')$. It is also assumed that $\sigma(\Omega,\Omega')$ is a function only of the angle between its two arguments. The latter assumption is made throughout this thesis. However, many of the important experimental and theoretical results in this thesis deal with inhomogeneous aspects of propagation, so the assumption of homogeneity is not always made. Where appropriate, homogeneity is assumed, and it will always be indicated clearly.

Although the transport equation embodies all of the propagation phenomena of interest, no general solution to it is available, even for the homogeneous medium. It has been solved in the case of isotropic scatter (both time independent [11] and time dependent [7]), for which

$$\sigma(\Omega,\Omega',\mathbf{r},t) = \frac{1}{4\pi}.$$  \hspace{1cm} (1.2)

It has also been solved when certain approximations have been made, such as the diffusion approximation [12-14], the single scatter approximation [15,16] and the small angle approximation [29]. However, theories based on such extreme approximations are strictly limited in their domain of applicability, and it is not apparent to what extent they help explain propagation in typical turbid atmospheric conditions.

1.1.1. **Numerical Techniques**

A number of numerical techniques exist for solving the transport
equation under more general conditions. One technique is the Neumann expansion or so-called scattering-order decomposition \[17,18\]. In this method, the photon density in the medium is considered to be partitioned into subgroups consisting of those photons which were scattered a particular number of times in propagating from the source to the receiver. Thus

\[
p(\bar{r}, \Omega, t) = \sum_{k=0}^{\infty} p(\bar{r}, \Omega, t | k) p(k)
\]  

(1.3)

where \( p(\bar{r}, \Omega, t | k) \) is the conditional space-angle density at time \( t \), given that the photon was scattered exactly \( k \) times, and \( p(k) \) is the probability mass function for the number of scattering events.

The motivation for the scattering order decomposition approach is that it divides the photon density into physically meaningful quantities - such as unscattered, single scattered, double scattered, etc. - and relates subsequent scattering orders to previous ones via integral transforms. Although these integrals are themselves numerically quite time consuming to compute, for some problems only a few orders may be required for convergence. And simple geometries often radically reduce the computational complexity, as in Zachor's point source problem \[19\].

A second powerful technique is the spherical harmonics method \[10,13,19\]. In this method, the photon probability density is expanded in terms of the spherical harmonic functions \( \phi_{km}(\Omega) \) \[20,21\]:

\[
p(\bar{r}, \Omega, t) = \sum_{k=0}^{\infty} \sum_{m=-k}^{k} \psi_{km}(\bar{r}, t) \phi_{km}(\Omega),
\]  

(1.4)
where \( \psi_{km}(\mathbf{r},t) \) are the space and time dependent expansion coefficients and the \( \Phi_{km}(\Omega) \) form a complete orthonormal set on the \( 4\pi \) sr sphere.

The motivation for the spherical harmonics method is the desire to separate spatio-temporal from angular characteristics of the photon density. It is easy to show [13] that a single product of functions which separates angular behavior from spatio-temporal behavior cannot be a unique solution of Eq. (1.1). Hence, the extension to an infinite series of such terms - to preserve the separation while still allowing for a solution - is a natural one. As in the case of scattering order decomposition, the general spherical harmonics method is computationally intensive, although reductions in required computer time are afforded by simple geometries or broad (not necessarily isotropic) single scatter phase functions.

Another generic approach to determining the channel parameters is Monte Carlo simulation [6,22-26]. In this approach, local scattering events between photons and atmospheric constituents are characterized statistically. Individual photons are put into the medium and, by computer generation of random variables, their interactions with atmospheric particles are simulated as they propagate toward the receiver (or exit) plane. By keeping track of exit position, angle and arrival time for a large enough number of photons, statistics of the required channel parameters can be obtained.

Although simulation can be used to solve a broad range of problems, including ones with complicated boundary conditions or a non-stationary atmosphere [23], its main limitation is that it requires enormous amounts of computer time to gain statistical regularity. Furthermore,
because simulation does not start with an equation for the quantities of interest, there is no possibility of developing any insight about the general nature of the solution unless large numbers of specific cases are considered, which even further increases the computational requirements. At best, then, simulation should be considered a supplement to other methods of solving the transport equation.

1.1.2. Combined Experimental/Theoretical Approach

Because of the limited usefulness of simulation, it is appropriate to focus on ways to make the numerical methods more efficient. For both of the numerical methods discussed above, it is apparent that improved computational efficiency may be obtained if simplifying assumptions are made. However, to be of practical use in real atmospheric propagation problems, these assumptions must be motivated not by convenience, but by actual experimental measurements. Thus, a combined experimental and theoretical approach to channel characterization is indicated.

A purely experimental approach, in which all of the relevant channel parameters are measured directly, would have as many disadvantages as a purely theoretical approach. In particular, it is limited by the difficulty of making such measurements, and the unpredictable and infrequent occurrence of the weather conditions of interest. Moreover, the wide variability of the channel parameters at different locations and at different wavelengths makes extrapolation from empirical formulas somewhat tenuous.
In the combined approach, a limited number of experimental measurements are made, and they are used to guide theoretical investigation. That is, they are used to indicate approximations to the transport equation which make it solvable for physical conditions of interest. There is no attempt to make measurements to exhaustively characterize the channel, but only to obtain rough size estimates of some of the parameters, so that regimes of behavior can be identified. The models developed from the resulting transport equation simplifications can then be used to determine parameter values that could not be measured, and to make predictions about physical conditions not encountered.

Two examples of transport equation approximations based on experimental observations are the multiple forward scatter (MFS) approximation \[27,28\] and the narrow angular spectrum approximation \[7\]. The MFS approximation consists of assuming that the phase function is sharply peaked in the forward direction, and that the only light which reaches the receiver is confined to a narrow cone around the line-of-sight axis. If the phase function includes some wide-angle scatter, the MFS approach truncates the phase function at a small angle. With these assumptions, the small angle approximation to the transport equation \[29\] can be made. Considering the homogeneous, time-independent transport equation for illustration,

\[
\mathbf{\Omega} \cdot \nabla p(\mathbf{r},\Omega) + \alpha p(\mathbf{r},\Omega) = \alpha_s \int d\Omega' \sigma(\Omega' \cdot \Omega') p(\mathbf{r},\Omega') , \tag{1.5}
\]
the small angle approximation is stated mathematically as

\[ \vec{\Omega} = (\overline{s}, 1), \quad |\overline{s}| \ll 1 \quad (1.6) \]

\[ \overline{s} = (s_x, s_y) \]

\[ \vec{\Omega} \cdot \vec{v} = \overline{s} \cdot \vec{v}_p + \frac{\partial}{\partial z} \quad (1.7) \]

\[ \vec{v}_p = \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} \]

\[ \sigma(\vec{\Omega} \cdot \vec{\Omega}') = \sigma(\overline{s} - \overline{s}') \quad (1.8) \]

where \( z \) is the space coordinate along the direction of the optical axis, and \( \overline{p} = (x, y) \) are the space coordinates transverse to the optical axis.

With these simplifications, the transport equation in Eq. (1.4) can be written as

\[ \overline{s} \cdot \vec{v}_p p(\overline{p}, \overline{s}, z) + \frac{\partial}{\partial z} p(\overline{p}, \overline{s}, z) + \alpha p(\overline{p}, \overline{s}, z) = \alpha_s \int_{-\infty}^{\infty} d\overline{s}' \sigma(\overline{s} - \overline{s}') p(\overline{p}, \overline{s}', z) \quad (1.9) \]

Since the integral term in Eq. (1.9) is now a two-dimensional convolution, the equation can readily be solved by Fourier transform techniques.

It should be pointed out that the MFS approximation has often
been made for reasons of convenience, rather than experimental evidence [1,29]. However, the detailed exploration of the MFS approach undertaken in reference [28] was motivated by persistent experimental evidence of little angular spread or multipath time spread [30,31,32], which evidence is consistent with narrow angle type scattering.

The MFS approximation is an indirect use of experimental data, because inferences about the scattering function are made based on the observations, and it is these inferences which are applied to the transport equation. A direct use of the experimental data is involved in the narrow angular spectrum approximation. As the name implies, this approximation makes use of the experimentally observed narrowness of the angular spectrum. Again considering the transport equation in Eq. (1.5), this approach assumes that in the integral, $p(\mathbf{r},\Omega')$ is impulsive in $\Omega'$, on the grounds that relative to the phase function it is concentrated in a very narrow angular region about some direction.

Mathematically, this assumption is

$$p(\mathbf{r},\Omega') = p(\mathbf{r}) \delta(\Omega' - \Omega_{\mathbf{r}}),$$

(1.10)

where $\Omega_{\mathbf{r}}$ is the space-dependent angle about which the angular spectrum is assumed to be concentrated, and

$$p(\mathbf{r}) = \int d\Omega \ p(\mathbf{r},\Omega)$$

(1.11)
This approximation reduces the transport equation to

\[ \mathbf{n} \cdot \nabla p(r, \mathbf{n}) + \alpha p(r, \mathbf{n}) = \alpha_s \sigma (\mathbf{n} \cdot \mathbf{n}_p) \ p(r) . \quad (1.12) \]

Kennedy has studied a more general form of Eq. (1.12), which includes time-dependence, and solved it for the case of isotropic scatter [7]. For isotropic scatter, his analysis is exact, i.e. no approximation is required to go from Eq. (1.4) to Eq. (1.12). Equation (1.12) has yet to be solved for other phase functions.

The multiple forward scatter and narrow angular spectrum approximation techniques described here are illustrative of the combined experimental/theoretical approach to channel characterization that appears most promising. That is the approach taken in this thesis. The thesis concentrates on the time-independent point source propagation problem. Experiments have been conducted in fog with cw sources at middle ultraviolet wavelengths. The motivation for using this wavelength region, the relation to visible wavelengths, and some of the basic phenomenology of optical scattering in the ultraviolet are presented in the next section.
1.2. The Middle Ultraviolet

1.2.1. Motivation and Technology Issues

If a scattered field is incident on an optical receiver, increasing the receiver field of view (FOV) increases the collected energy until the FOV is as wide as the expected angular spread. For the simplest kind of direct detection receiver, performance improves (error probability decreases) as collected energy increases, so that it will be desirable to have a relatively large FOV. In visible and near infrared systems, however, this has the disadvantage that the solar background is significant, and hence the system will be background limited in the daytime, exhibiting a much higher error probability than if it were quantum limited.

Because of the background noise problem, interest has developed in the middle ultraviolet (uv) portion of the wavelength spectrum (0.2 \( \mu \)m < \( \lambda \) < 0.3 \( \mu \)m). The spectral region below 0.29 \( \mu \)m is unique in that no solar radiation reaches the earth at these wavelengths, all of it being absorbed in the earth's upper ozone layer [33]. (The restriction to wavelengths larger than 0.2 \( \mu \)m is imposed by high molecular oxygen absorption below 0.2 \( \mu \)m [34,35].) Thus the middle ultraviolet affords the potential for a "solar blind" wide FOV scatter communication system, which is quantum limited.

The middle ultraviolet is not totally without problems. One problem is that there are no sources currently available which can be modulated at reasonable data rates (in excess of 1 MHz). There are
some good candidates, however, such as e-beam pumped fast phosphors [55, 56] and electro-optically modulated hollow cathode lasers [57-58].

A second problem is that there are no middle ultraviolet detectors that are truly solar blind. Photomultiplier tube cathodes such as CsTe and RbTe are partially solar blind. (At 0.4 μm their quantum efficiencies are down by 3-4 orders of magnitude from their peaks at approximately 0.2 μm.) However, both of these cathode materials have non-negligible response in the visible region of the spectrum. Cathode materials such as CsI, KBr and CuCl are truly solar blind, but their peak quantum yields are near 0.13 μm, with a very sharp drop before the 0.2-0.3 μm region is reached [36]. The consequences of these detector limitations are that the advantage of total solar blindness, which the middle ultraviolet potentially affords, cannot currently be achieved with a detector alone. Additional filtering is required. Efforts to develop middle ultraviolet filters which, in combination with available detectors, will produce completely solar blind systems are in progress [37].

1.2.2. Propagation Phenomenology

As stated in Section 1.1, this thesis is concerned with the propagation aspects of the communication problem, not the device aspects. It is appropriate, therefore, to consider the phenomenology associated with middle uv propagation, in particular quantities such as the extinction and scattering coefficients, α and α_s, and the single scatter phase function, σ(\hat{n} \cdot \hat{n}'). It will be seen as the
discussion proceeds, that the values of these quantities - for the types of situations of interest in this thesis - are such that many of the results derived in succeeding chapters are directly applicable to visible wavelengths as well.

The magnitude and character of these quantities depend upon the type of atmospheric particles with which photons interact. Besides rain and snow, there are two basic types of atmospheric constituents that are of importance at middle uv wavelengths: molecular constituents such as oxygen, ozone and other atmospheric gases; and aerosol constituents, consisting of suspended water droplets (fog) and dust particles (haze).

**Molecular Constituents**

The molecular gases, because their scattering cross sectional dimensions are much smaller than optical wavelengths, give rise to Rayleigh scatter [38]. The magnitude of Rayleigh scatter is proportional to \( \lambda^{-4} \), and hence at uv wavelengths it is much more significant than at visible wavelengths. The molecular gases are also absorbers of uv radiation. The most significant absorber is ozone \((O_3)\). Typically encountered fluctuations in ambient ozone levels can vary its absorption coefficient at \( \lambda = 0.25 \ \mu m \) from a level at which it is negligible to one at which it predominates, even over scattering losses.

Reilly [34] gives values of scattering and extinction coefficients in the middle uv, using the sea level Mid-Latitude Winter Supplement to the U.S. Standard Atmosphere, 1962 [39], as the source of molecular number densities. The ozone number density, used in Reilly's calculations was assumed fixed at \( 6.7 \times 10^{11} \) molecules/cm\(^3\) (slightly more than 2 parts
per hundred million [2 pphm]), and hence the resulting ozone absorption coefficient is fixed. To include fluctuations in the ozone absorption coefficient at \( \lambda = 0.25 \mu \text{m} \), the absorption contribution of ozone is calculated according to

\[
\alpha_{a03} = \alpha_{03} - \alpha_{s03} = 84.4 \frac{C_{03}}{T},
\]

where \( C_{03} \) is the concentration of ozone in pphm and \( T \) is the ambient temperature in °K. \( \alpha_a \) has units of km⁻¹. Equation (1.13) is a simple generalization of the calculations done by Sperry [40]. At the assumed value of \( T = 273°\text{K} \), \( \alpha_{a03} = 0.31 C_{03} \).

Table 1.1 shows typical values of scattering and extinction coefficients for three wavelengths in the middle uv and for various typical values of ozone concentration. The subscript \( M \) denotes that these coefficients are the part of the total scattering and extinction coefficients which are due to interactions with molecular constituents. Note from Table 1.1 that at \( \lambda = 0.2 \mu \text{m} \), \( \alpha_M \) is independent of \( C_{03} \) and completely dominated by absorption. This is the strong oxygen absorption band referred to earlier. At 0.3 \( \mu \text{m} \), \( \alpha_M \) is also independent of \( C_{03} \) and mostly dominated by scattering. However, at 0.25 \( \mu \text{m} \), where the ozone absorption cross-section nears its peak [48], the ozone concentration can shift the extinction mechanism from predominantly scattering to predominantly absorption.
Table 1.1
UV Scattering and Extinction Coefficients for Molecular Constituents

\[ \alpha_M = \alpha_{sM} + \alpha_{aM} \ (\text{km}^{-1}) \]

<table>
<thead>
<tr>
<th>(\lambda (\mu m))</th>
<th>(\alpha_{sM} \ (\text{km}^{-1}))</th>
<th>(C_{O_3} = 0.1 \text{ pphm})</th>
<th>(C_{O_3} = 0.5 \text{ pphm})</th>
<th>(C_{O_3} = 2 \text{ pphm})</th>
<th>(C_{O_3} = 10 \text{ pphm})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.954</td>
<td>8.12</td>
<td>8.12</td>
<td>8.12</td>
<td>8.12</td>
</tr>
<tr>
<td>0.25</td>
<td>0.338</td>
<td>0.407</td>
<td>0.533</td>
<td>0.966</td>
<td>3.48</td>
</tr>
<tr>
<td>0.3</td>
<td>0.153</td>
<td>0.176</td>
<td>0.176</td>
<td>0.176</td>
<td>0.176</td>
</tr>
</tbody>
</table>
A convenient parameter for measuring the relative amounts of scattering and absorption is the albedo

\[ \omega_M = \frac{\alpha_{SM}}{\alpha_M}, \]  

(1.14)

where again the subscript \( M \) stands for molecular. In terms of this quantity, then, the ozone concentration can vary \( \omega_M \) from very close to 1 (predominantly scattering) to very close to zero (predominantly absorption).

As stated earlier, scattering from molecular constituents is Rayleigh. This is a wide-angle scatter, with single scatter phase function [38]

\[ \sigma_{\text{Rayleigh}}(\mu) = \frac{3}{16 \pi} (1 + \mu^2) \]  

(1.15)

Here \( \mu = \overline{\mathbf{n}} \cdot \overline{\mathbf{n}}' \) is the cosine of the angle, \( \theta \), between the incident and scattered directions.

**Aerosol Constituents**

An aerosol is a suspension of small particles in a gas. There is an enormous variety of particles suspended in the atmosphere, from both natural and manmade sources. Some of these particles, such as dust and volcanic ash, do not absorb and retain water and are called nonhygroscopic; others, such as sea salt and many combustion products, absorb and retain varying amounts of water and are called...
hygroscopic. These hygroscopic particles act as condensation nuclei for water vapor, and hence can grow to many times their original size.

Haze aerosols consist of both hygroscopic and non-hygroscopic particles. Their particle diameters range from 0.01 to 10 μm, with a mean value around 0.1 μm. Fog aerosols consist exclusively of hygroscopic particles, with diameters ranging from 0.1 to 100 μm, and typical mean values of 5 to 10 μm [41]. As can be seen, there is an overlap in the particle size range between haze and fog. In fact, fog particles and the larger haze particles have a common origin in growth from condensation nuclei. However, the meteorological conditions necessary for fog development, its restriction to altitudes close to the ground, and the larger mean particle size, all distinguish it uniquely from haze [41].

Scattering by aerosol particles differs from molecular scattering in that aerosol particle sizes are comparable to the wavelength of light. Hence, there is no unique single scatter phase function that can be expressed analytically, as in the case of Rayleigh scatter. The details of the phase function differ radically for haze and fog, and even among different fog types. However, the phase function can be determined numerically from Mie scattering theory [34], given the particle size distribution and particle complex refractive index (and assuming that the particles are spherical). Alternatively, it can be measured directly. A number of these phase functions, for both fog and haze, at visible and uv wavelengths, have been published [32,42,43].

The scattering coefficient in a polydisperse aerosol, such as a fog, is given by
\[ \alpha_{SA} = N \int dr \ n(r) \ K_s(r,\lambda) , \]  

where \( n(r) \) is the normalized particle size distribution, \( N \) is the number density of particles, and \( K_s(r,\lambda) \) is the wavelength dependent scattering cross-section. (The subscript "A" denotes "aerosol".) The cross section, \( K_s(r,\lambda) \), is determined from the Mie theory, but in actual atmospheric conditions \( N \) and \( n(r) \) vary in complicated and unpredictable ways \([41,45,46]\). Therefore, it is most appropriate to consider \( \alpha_{SA} \) a free parameter in propagation modelling.

As in the case of molecular constituents, the extent of absorption is characterized by the albedo

\[ \omega_0 = \frac{\alpha_{SA}}{\alpha_A} , \]  

where \( \alpha_A \) is the extinction coefficient due to the aerosol component of the atmosphere. At ultraviolet wavelengths, this albedo is very close to 1. \([47]\).

For a composite atmosphere, consisting of both molecular and aerosol constituents, the phase function and albedo are weighted averages of those for the molecular and aerosol components:

\[ \sigma(\mu) = \frac{\alpha_{SM} \sigma_M(\mu) + \alpha_{SA} \sigma_A(\mu)}{\alpha_{SM} + \alpha_{SA}} \]  

\[ \omega_0 = \frac{\alpha_M \omega_M + \alpha_A \omega_A}{\alpha_M + \alpha_A} . \]
The composite scattering and extinction coefficients are given by

\[ \alpha_s = \alpha_{sM} + \alpha_{sA} \]  

\[ \alpha = \alpha_M + \alpha_A \]  

For theoretical calculations involving Mie-type scattering, it is advantageous to have an analytic expression for the aerosol function. In most situations involving a significant amount of multiple scattering, the actual form of the function is not crucial, as long as the peak value and width correspond to that of the actual phase function [6]. Hence, a variety of phase functions are used:

1. Gaussian form:

\[ \sigma_A(\theta) = \frac{1}{2\pi} \exp\left(-\frac{\theta^2}{2\theta_F^2}\right) \]  

where \( \theta \) is the angle between the incident and scattered directions, and \( \theta_F \) is the rms width of the phase function. When \( \theta_F \ll \pi \), the integral in the denominator in Eq. (1.22) is very close to \( \theta_F^2 \). Thus, this function is particularly simple in multiple forward scatter type calculations [28].

2. Henyey-Greenstein form [44]:

\[ \sigma_A(\mu = \cos \theta) = \frac{1}{4\pi} \frac{1 - g^2}{(1 + g^2 - 2g\mu)^{3/2}} \]  

where $g$ is the average cosine of the phase function, defined by

$$g = 2\pi \int_{-1}^{1} \mu \sigma_A(\mu) \, d\mu. \quad (1.24)$$

3. Modified Henyey-Greenstein form [18]:

$$\sigma_A(\mu) = \frac{1-g^2}{4\pi} \left[ \frac{1}{(1+g^2-2g\mu)^{3/2}} + f \frac{0.5(3\mu^2-1)}{(1+g^2)^{3/2}} \right]. \quad (1.25)$$

In this expression, the first term is identical to the Henyey-Greenstein function, while the second term represents the backward peak in the phase function. The parameter $f$ is adjustable to match typical levels of the backward peak. The modified Henyey-Greenstein function is useful in calculations in which wide fields of view, and hence wide angle scatter, are important.

The relationship between $g$, used in the Henyey-Greenstein functions, and $\Theta_F$, used in the Gaussian function, is

$$g = \frac{1}{2} \left\{ \int_0^\pi \sin(2\Theta) \exp(-\Theta^2/2\Theta_F^2) \, d\Theta \right\}^{1/2} \quad (1.26)$$

$g$ is plotted vs. $\Theta_F$ in Fig. 1.2. As can be seen in the figure, for $\Theta_F \ll \pi$, $g$ is given approximately by
Figure 1.2: $g$ vs. $\Theta_F$ for a Gaussian Phase Function
\[ g \approx 1 - \frac{4}{3} \theta_F^2 + \frac{16}{25} \theta_F^3. \]  \hspace{1cm} (1.27)

As an illustration of how well these analytic phase functions represent real ones, as well as the approximate range of parameter values involved, consider the measured phase functions in Figures 1.3 and 1.4 [43]. These phase functions were obtained during radiational (inland) fogs. They are not meant to be representative of all types of fogs, but they do exhibit the pronounced forward peaking which is characteristic of scattering from aerosols with particle sizes much larger than the wavelength. Also evident in both phase functions is a slight backward peak, due to Rayleigh scatter.

Figures 1.5 and 1.6 show a comparison of composite forms of the three types of analytic phase functions with the measured phase functions from Figures 1.3 and 1.4. The values of \( g, f, \theta_F \) and \( \frac{S_A}{S_M} \) were chosen to best match the measured phase functions. The Henyey-Greenstein functions are reasonably good fits. It can be seen, however, that while the Gaussian function is capable of matching the narrow forward peak of the phase function, its wide angle behavior departs significantly from the actual phase function. Hence, it is generally limited to use in those calculations (eg. multiple forward scatter calculations) in which the forward peak is important and the wide angle behavior is ignored.

1.2.3. Application to Propagation at Visible Wavelengths

The utility of studying atmospheric propagation at ultraviolet wavelengths is increased if the propagation phenomena discovered can be
$\lambda = 250 \text{ nm}$

$12/1/77 \ k_{\text{scat(total)}} = 6.2 \ \text{km}^{-1}$

- Experimental
- Mie Scatter Theory + Rayleigh

Figure 1.3: Measured and Predicted Fog Phase Functions (after [43])
\( \lambda = 248.5 \) nm

- 9/2/77 6:25 AM
- 9/2/77 6:30 AM

--- MIE Scattering Theory, 6:30 AM

Figure 1.4: Measured and Predicted Fog Phase Functions (after [43])
Figure 1.5: Comparison of Actual and Analytic Phase Functions (12/1/77); $g = 0.9$, $f = 1.0$, $\theta_F = 0.1$, $\alpha_A/\alpha_M = 9$
Figure 1.6: Comparison of Actual and Analytic Phase Functions (9/2/77, 6:25 A.M.); $g = 0.9$, $f = 1.0$, $\theta_F = 0.1$, $\alpha_S / \alpha_M = 9$
related to what would be expected in the visible. It is the purpose of this section to identify the similarities and differences between the two wavelength regions, with respect to propagation through fog.

Since there is an inverse fourth power wavelength dependence for Rayleigh scatter, this is a comparatively small contribution at visible wavelengths. For example, at the Helium Neon wavelength $\lambda = 0.633 \, \mu m$, $\alpha_s = 0.034 \, km^{-1}$, as opposed to $\alpha_s = 0.338 \, km^{-1}$ at $\lambda = 0.25 \, \mu m$. (See Table 1.1.) Furthermore, there is virtually no absorption due to ozone at visible wavelengths. Above $\lambda = 0.3 \, \mu m$, the ozone cross-section is down by more than a factor of 20 from its peak at $\lambda = 0.255 \, \mu m$ [48]. Therefore, in those atmospheric conditions in which molecular scattering and absorption dominate in the uv (e.g. clear weather), the Rayleigh scatter at visible wavelengths is often small enough so that scattering from haze aerosols dominates. Thus, there is a major difference between uv and visible propagation.

For heavy multiple scattering through an aerosol such as fog, however, these differences are reduced significantly. Since, in the case of fog, the mean of the particle size distribution is around 5-10 \, \mu m, both uv and visible wavelengths are much smaller than the mean particle diameter. A very sharply peaked aerosol single scatter phase function results in both cases. Although, as discussed above, the degree of phase function peaking depends on fog type, this is due to the details of the fog particle size distribution, not a radical shift in the mean particle size, and should affect both uv and visible wavelengths uniformly. In addition, the albedo $\omega_{OA}$ is essentially one at both uv
and visible wavelengths [43]. Therefore, as long as interactions with aerosol constituents dominate over interactions with molecular constituents, the propagation behavior in the two wavelength regimes should be similar.

The scattering coefficient for fog, although it varies greatly with fog depth and type, is characteristically in the range $10 \text{ km}^{-1} < \alpha_{sFOG} < 40 \text{ km}^{-1}$. In addition, it is roughly constant with wavelength from 0.2 - 1.0 $\mu$m [41,49,50]. Therefore, using Table 1.1 as a comparison, it is evident that aerosol scattering in fog will never be totally dominant at $\lambda = 0.2 \mu$m, whereas it will always predominate at $\lambda = 0.3 \mu$m. At $\lambda = 0.25 \mu$m, the dominance of aerosol scattering or molecular absorption depends critically on the ozone concentration. For $C_{O_3} \sim 2$ pphm, scattering will dominate.

It has thus been shown that under certain conditions, the characteristics of propagation through fog will be similar at middle uv and visible wavelengths. In particular, this similarity occurs if the ozone concentration is low enough so that $\alpha_M$ is negligible compared to $\alpha_A$. Under these circumstances, Eqs. (1.18) through (1.21) become

\begin{align*}
\sigma(\mu) &= \sigma_A(\mu) \tag{1.28} \\
\omega_o &= \omega_{oA} \tag{1.29} \\
\alpha_s &= \alpha_{sA} \tag{1.30} \\
\alpha &= \alpha_A \tag{1.31}
\end{align*}
1.3. Thesis Organization

With the propagation phenomenology developed, it is now possible to proceed with the experimental and theoretical aspects of this thesis. Chapter II gives a description of the fog propagation experiments, and presents the experimental data. Essential features of the data are identified.

Chapter III introduces the results from three existing propagation theories -- and compares them with each other and the measured data. The extent to which each of these theories is applicable to propagation through fog is discussed.

Chapter IV makes use of one of the properties identified in Chapter 2 - the insensitivity property - in order to simplify the transport equation. This simplified transport equation, the so-called "strong multiple scatter" limit, is solved for distances far from the source. This asymptotic solution is shown to be the same as the known general solution for isotropic scatter [11] and for anisotropic scatter the asymptotic limit of the solution is explored in some detail. Comparisons are made with the data and with other theories.

Finally, Chapter V summarizes the major results of the thesis and gives an overall assessment of the asymptotic solution. Extensions to the non-asymptotic case are discussed.
CHAPTER II

EXPERIMENTS

This chapter discusses the experimental measurements which form the foundation of the channel modelling work in subsequent chapters. During the summers of 1978 and 1979, field trips were made to the cities of Lubec and Eastport, Maine, respectively. Both cities are located in the extreme eastern portion of Washington County, Maine, near the mouth of the Bay of Fundy (see Figure 2.1) and in the summer months have a relatively high frequency of fog occurrence [60,61].

The measurement program conducted during these field trips was directed primarily at characterizing the angular and spatial behavior of propagation through fog with cw point sources at a wavelength of 2537 Å. The use of this wavelength was dictated by the availability of a suitable source. The details of the angular distribution of received light were studied by scanning a narrow FOV receiver in angle. Cruder measurements of the angular spread were made by measuring the signal increase as the receiver FOV is opened. Path transmission measurements were made with both narrow FOV and wide FOV receivers.

Section 2.1 describes the specifics of the experimental setup, including site geometry, weather characterization and
Figure 2.1: Location of Lubec and Eastport, Maine
equipment details. Section 2.2 presents the results for the signal vs. FOV experiments and discusses essential features that can be abstracted from the data. Section 2.3 does the same for the angular scan data. The important "insensitivity" property, which is exploited in Chapter IV to simplify the transport equation, is also discussed in Section 2.3.
2.1. Experimental Setup

2.1.1. Site Geometry

The 1978 field experiments were conducted on the southeast side of Lubec, Maine, in an inlet called Bailey's Mistake (see Figure 2.2a). The receiver location was on land about 200 feet from the mean water line. Sources were located in two positions, one an island about 1000 feet offshore, and the other almost directly behind it on the opposite shore (one mile from the receiver). The terrain is shown in Figure 2.2h.

The 1979 field experiments were conducted in Deep Cove, Eastport, Maine at two locations: on the property of the Washington County Vocational Technical Institute (WCVTI), and at the Eastport Municipal Airport (see Figure 2.3a). For the WCVTI experiments, the receiver location was at the end of the WCVTI dock (see Figure 2.3b). Sources were located in a large building on the eastern end of the WCVTI parking lot, as well as on telephone poles at intervals closer to the dock. At the airport, the propagation path is of course horizontal. For airport experiments, both sources and receivers were located about five feet above the weathered asphalt surface of the runway.

The relevance of these details about site geometries is in assessing the degree to which boundary effects influenced the measurements, and in determining the comparability of the measurements taken at different locations. The discussion below will show that while the measurements are affected to some extent by the boundaries, the effects
Figure 2.2a: Bailey's Mistake Inlet

Figure 2.2b: Side View of Bailey's Mistake Propagation Link
Figure 2.3a: Overhead View of 2 Eastport Sites (WCVTI and Airport)
Figure 2.3b: Side View of WCVTI Propagation Link
are small.

Consider the boundary effects at the WCVTI field sight (see Figure 2.3b). For narrow FOV measurements (FOV < 20 mr), the boundary does not actually intersect the receiver FOV, but it acts as a partially absorbing/partially reflecting boundary for scattered photons. In general, this shifts the apparent source location toward the upper half plane and reduces the detected signal level. Even for the isotropic scatter case, however, in which the half plane absorbing boundary problem has been solved, this signal level reduction is small when any atmospheric absorption is present [9]. For the pronounced forward scatter typical in fogs, reflected light from the ground is very unlikely to reach the receiver even though multiple scattering is taking place. By the same argument, light absorbed by the ground is unlikely to have been a significant component of the received signal if the ground were not present. Therefore, narrow FOV measurements are not expected to be significantly affected by the boundary at the WCVTI site.

Wide FOV measurements at the WCVTI site and at the airport are affected by the ground. The change in detected signal level, however, is small, as indicated above, and is not a very strong function of range or FOV [9]. For the purpose of interpreting the experiments, the boundary is considered to reduce the detected power by a factor of two, corresponding to a reduction of the total scattering volume by the same factor.

At the Bailey's Mistake site, since the boundaries are even
farther from the line-of-sight path than at WCVTI, small FOV measurements should not be influenced at all (see Figure 2.2b). During wide FOV measurements in tenuous fog or clear weather, however, it was observed that the detected power level increased somewhat when the FOV was decreased. This was attributed to a combination of wide-angle reflections from the water surface and internal reflections on the inside of the FOV limiting tube on the wide FOV receiver (see Figures 2.4c-d). In heavier fog, the reflected power from the water's surface was extinguished significantly before reaching the receiver (relative to the power arriving within a small cone around the line-of-sight direction) and this effect was not observed.

Besides boundary effects, another factor influencing the comparability of the data of the various sites is absolute pathlength. The question is whether the data depends explicitly on the pathlength, \( L \), itself or only on the optical thickness, \( \tau = \alpha L \). In the former case, the data from the various sites is not comparable; in the latter it is. In Chapter 4, it is shown that for the point source problem, the transport equation can be written purely in terms of \( \tau \), with no explicit dependence on pathlength. Furthermore, the experimental data itself corroborates this lack of dependence on absolute pathlength [62]. The conclusion to be drawn from this is that as long as data from different sites are discussed at the same values of \( \tau \), they are comparable.

2.1.2. Weather Characterization

To relate theoretical work to the experimental data, it is
sufficient to know the extinction coefficient, the scattering coefficient and the single scatter phase function resulting from the weather conditions prevailing at the time of the experiments. The extinction coefficient can be measured easily. The scattering coefficient can then be inferred from the measurement of ozone concentration, since at middle uv wavelengths ozone is the primary absorber. In practical situations, of course, local fog density variations cause the scattering coefficient to be a random function of distance along the path between source and receiver, whereas the ozone concentration in a region is relatively stable if there are no ozone sources nearby. Therefore, an extinction measurement involving integration over the entire path is necessary, while a simple point measurement of ozone usually suffices if the path is not too long.

Unfortunately, there is no easy way to measure the single scatter phase function. Measuring the phase function directly in situ, or measuring the particle size distribution - from which the phase function can be determined by Mie calculations - is difficult and time consuming. However, the prevailing weather condition during both the Lubec and Eastport measurements was a "stable" maritime fog* which had been formed over water and was blown inland. Thus, while the phase function parameters are treated as free parameters in theoretical work in this thesis, the range of values identified in Section 1.2.2 for fog is applicable.

The procedure used to characterize the atmosphere during the

*This is contrasted with a "developing" fog, which is in the process of forming, and which undergoes dramatic changes in particle size distribution [41].
experiments is as follows: First, a qualitative weather type is noted (e.g., fog, clear), along with any other relevant descriptive information (e.g., high variability in conditions). Second, a point ozone concentration measurement is made at the receiver. Third, the average path extinction coefficient is determined.

The path extinction coefficient is derived by comparing the unscattered power received on a clear day with that received under low visibility conditions. The unextinguished (unscattered plus unabsorbed) power propagates according to the Bouguer-Lambert extinction law

\[
P_{\text{unextinguished}} = P_{\text{unextinguished}} \exp \left( \int_0^L \alpha(z) \, dz \right), \quad (2.1)
\]

where \( \alpha(z) \) is the extinction coefficient as a function of the distance, \( z \), along the line-of-sight path, and \( L \) is the pathlength. Thus the integrated optical thickness,

\[
\tau^* = \int_0^L \alpha(z) \, dz, \quad (2.2)
\]

is given by

\[
\tau^* = -\ln \left( \frac{P_{\text{unextinguished}} \text{(low visibility)}}{P_{\text{unextinguished}} \text{(clear)}} \right). \quad (2.3)
\]

(Note that \( \tau^* = \tau \) in the homogeneous case.)

\( \tau^*/L \) is the average extinction coefficient along the path.
A non-zero FOV receiver in general collects both scattered and unscattered light. To insure that predominantly unscattered light was collected during the extinction coefficient measurements, the receiver FOV was reduced to 1 mr, the smallest practicable value to avoid pointing problems. It can be argued theoretically (see Figure 3.5), and was observed experimentally, that use of a FOV this small is sufficient to collect predominantly unscattered light.

The ozone measurements were made with a Mast Development Company Model 724-5 electrochemical ozone meter. During both field trips, the ozone concentration was less than 2 pphm. (The ozone concentration at Lubec was virtually constant at 1 pphm. The concentration at Eastport was between 0.5 and 2.0 pphm.) Therefore, in accordance with Table 1.1 and the discussion in Section 1.2.3, the propagation phenomena observed in the experiments at $\lambda = 2537 \, \text{Å}$ are primarily due to aerosol scatter and would be observed at visible wavelengths as well. Because of the predominance of aerosol scatter, ozone absorption levels are omitted from the discussion of the experiments below, and only optical thickness is indicated.

2.1.3. Equipment Description

The source used in the Lubec and Eastport experiments is a mercury vapor germicidal lamp, which emits a nominal 1.4 watts of power at 2537 Å. The lamp is a 9" long by $\frac{1}{2}$" diameter cylindrical bulb which is mounted on an aluminum backing plate to distribute the lamp power over a $2\pi$ steradian solid angle.
Three types of receivers were used in the experiments: a small FOV receiver whose FOV is variable between 1 and 15 mr, a wide FOV receiver whose FOV is fixed at 17°, and a wide FOV receiver whose FOV can be changed from 22° to 84°. These three receivers are shown in Figure 2.4 (both 22° and 84° FOV configurations are shown for the Honeywell receiver), and detailed specification of the receiver systems are given in Table 2.1.
Figure 2.4a: Variable Narrow FOV Receiver

Figure 2.4b: Fixed Wide FOV Receiver

Figure 2.4c: Honeywell Receiver (Small FOV)

Figure 2.4d: Honeywell Receiver (Large FOV)

Figure 2.4: Measurement Receivers
### Table 2.1

**Receiver Specifications**

<table>
<thead>
<tr>
<th>Variable NFOV Receiver</th>
<th>Fixed WFOV Receiver</th>
<th>Variable WFOV Receiver (Honeywell)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Optical front end</strong></td>
<td>Reflecting telescope: - mirror reflectivity = 90% - focal length = 46&quot;</td>
<td>none</td>
</tr>
<tr>
<td><strong>Aperture</strong></td>
<td>4½&quot; diameter</td>
<td>4½&quot; diameter</td>
</tr>
<tr>
<td><strong>FOV</strong></td>
<td>Variable iris: 1-15 mr (full angle)</td>
<td>Controlled by length of pipe on front end; fixed at 17° (full angle)</td>
</tr>
<tr>
<td><strong>Filter</strong></td>
<td>Dielectric - center wavelength = 2537 Å - peak transmission = 7.3% - bandwidth = 81 Å</td>
<td>Solution of NiSO₄ and Cation X, 10% transmission at 2537 Å</td>
</tr>
<tr>
<td><strong>Photomultiplier tube (PMT)</strong></td>
<td>- Bialkali cathode - QE = 22% at 2537 Å - Gain = 2.5 x 10⁷ @-1400 V</td>
<td>- RbTe cathode - QE = 8.6% at 2537 Å - Gain = 2.6 x 10⁷ @ +2000 V</td>
</tr>
<tr>
<td><strong>Post Detection Processing</strong></td>
<td>Photon counting**</td>
<td>Photon counting**</td>
</tr>
</tbody>
</table>

*made smaller by using longer pipe.

**Princeton Applied Research, Model 1121 Amp/Disc and Model 1112 Counter/processor.

***Internal amp/disc and either Model 1112 counter or Fluke Model 1900 A multi-counter.
2.2. Experimental Results - Signal vs. FOV

As stated above, the experimental data collected falls naturally into two categories: exploration of the details of the angular spectrum by fine-grain angular scans of the received light field, and measurements of the increase in collected power as the receiver FOV is enlarged. The signal vs. FOV measurement is an integrated version of the angular scan, and therefore washes out much of the detail of the angular spectrum. However, it requires much less time to make the signal vs. FOV measurement, while significant features of the angular spectrum can be abstracted from it, and hence a larger volume of data using this technique has been obtained.

This section presents and discusses the signal vs. FOV data. The angular scan data - which exhibits the insensitivity property exploited theoretically in Chapter 4 - is the subject of the next section.

2.2.1. Angular Spreading

Table 2.2 is a summary of the small angle (< 15 mr) signal vs. FOV data taken at Eastport.* The table shows the amount by which the signal increased as the FOV was opened from 1 mr to the specified value. Figure 2.5 shows this data plotted for various ranges of \( \tau \). The curves in Figure 2.5 are drawn through the mean values of the data at each FOV.

*Although some small-angle data was obtained during the first field trip at Lubec, the data-taking procedure was slow and inaccurate. The results are at best of qualitative value, although the data is in general agreement with the more accurate Eastport data.
Table 2.2

Summary of UV Signal vs. FOV Data: Eastport, 1979†

<table>
<thead>
<tr>
<th>Date</th>
<th>Time</th>
<th>Optical Thickness</th>
<th>Multiplication of Mean Signal Level</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>5 mr</td>
</tr>
<tr>
<td>6/23</td>
<td>4:10 am</td>
<td>3.1-4.4</td>
<td>1.7-2.1</td>
</tr>
<tr>
<td></td>
<td>4:00</td>
<td>2.8</td>
<td>2.1</td>
</tr>
<tr>
<td></td>
<td>4:20</td>
<td>1.8-2.1</td>
<td>1.8-2.4</td>
</tr>
<tr>
<td></td>
<td>4:25</td>
<td>3.4-3.7</td>
<td>1.8-2.5</td>
</tr>
<tr>
<td></td>
<td>4:20</td>
<td>5.9 (?)</td>
<td>1.9</td>
</tr>
<tr>
<td></td>
<td>4:30</td>
<td>4.0</td>
<td>2.4</td>
</tr>
<tr>
<td></td>
<td>4:33</td>
<td>4.0</td>
<td>2.0</td>
</tr>
<tr>
<td></td>
<td>4:36-40</td>
<td>4.3</td>
<td>2.0</td>
</tr>
<tr>
<td></td>
<td>5:15</td>
<td>3.9</td>
<td>2.3</td>
</tr>
<tr>
<td></td>
<td>5:10</td>
<td>3.4</td>
<td>2.1</td>
</tr>
<tr>
<td></td>
<td>5:20</td>
<td>3.7</td>
<td>2.5</td>
</tr>
<tr>
<td></td>
<td>5:20</td>
<td>4.0</td>
<td>2.1</td>
</tr>
<tr>
<td>10:20 pm</td>
<td>6.3</td>
<td>2.8</td>
<td>4.9</td>
</tr>
<tr>
<td>11:40</td>
<td>5.6-6.1</td>
<td>2.4-3.9</td>
<td>3.4-5.6</td>
</tr>
<tr>
<td>11:45</td>
<td>6.2</td>
<td>3.1</td>
<td>5.2</td>
</tr>
<tr>
<td>7/1</td>
<td>12:20 am</td>
<td>***</td>
<td></td>
</tr>
<tr>
<td></td>
<td>12:23</td>
<td>5.2</td>
<td>2.4</td>
</tr>
<tr>
<td></td>
<td>12:30</td>
<td>4.9</td>
<td>2.1</td>
</tr>
<tr>
<td></td>
<td>3:30</td>
<td>4.8</td>
<td>2.9</td>
</tr>
<tr>
<td></td>
<td>3:40</td>
<td>4.4-4.7</td>
<td>2.8-4.1</td>
</tr>
<tr>
<td></td>
<td>3:50</td>
<td>4.6</td>
<td>3.7</td>
</tr>
<tr>
<td>11:10-15 pm</td>
<td>3.3</td>
<td>2.8</td>
<td>3.9</td>
</tr>
<tr>
<td>7/2</td>
<td>5:43 am</td>
<td>6.5</td>
<td>7.7</td>
</tr>
<tr>
<td></td>
<td>5:45-48</td>
<td>6.3</td>
<td>3.8</td>
</tr>
<tr>
<td>10:13 pm</td>
<td>4.3</td>
<td>1.8</td>
<td>3.2</td>
</tr>
<tr>
<td>11:50</td>
<td>***</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>11:53</td>
<td>6.4</td>
<td>2.3</td>
</tr>
<tr>
<td></td>
<td>11:55</td>
<td>6.8</td>
<td>3.0</td>
</tr>
<tr>
<td>7/3</td>
<td>12:00 am</td>
<td>7.1</td>
<td>2.7</td>
</tr>
<tr>
<td>1:35</td>
<td>***</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7/16</td>
<td>3:50 am</td>
<td>4.0-4.4</td>
<td>1.3-1.8</td>
</tr>
</tbody>
</table>

***Data obtained by weather too variable to make a reliable estimate of optical thickness.

†Pathlength = 0.43 km, FOV for unscattered measurement = 1.0 mr.
Figure 2.5: Small Angle Signal vs. FOV Characteristic
There are two salient features of this figure. First, all the curves have flattened out, or nearly flattened out, by about a 10 mr FOV. This indicates a narrow angular spectrum, since the rate of increase of light collected is small as the FOV is opened beyond the "knee" in the curve. While the FOV increases from 1 to 15 mr, corresponding to an increase in the solid angle subtended by the receiver cone by a factor of 225, the signal increases by only about a factor of 10.

The second noteworthy feature of Figure 2.5 is that most of the data are consistent with the conclusion that the angular spectrum broadens as the optical thickness increases. In general, the final height of the curve increases and the knee shifts to the right as \( \tau \) increases. The apparent exception to this conclusion is the inversion of the curves for \( 6.0 \leq \tau \leq 6.3 \) and \( 6.8 \leq \tau \leq 7.1 \). However, the large error bars on the \( 6.0 \leq \tau \leq 6.3 \) data overlap (or nearly overlap) the error bars for the \( 6.8 \leq \tau \leq 7.1 \) data. Indeed, if the measurements on 7/2/79 (at 5:45 - 5:48 A.M.) are not included in the data for the latter range, the two curves are quite close together, indicating a saturation effect as \( \tau \) increases beyond a certain value. Although there are a few other examples in Table 2.2 of data that are not consistent with the conclusion of angular spectrum broadening with increased optical thickness most of the data tends to support it. The data which deviates from this conclusion vary within expected experimental precision.

The conclusion is further strengthened by considering the large FOV data as well. Figure 2.6 shows the signal vs. FOV dependence for the FOV range 1 - 1500 mr. Observe that for \( \tau = 9 \) the wide FOV
Figure 2.6: Signal vs. FOV Characteristic (1-1500 mr)
receiver may collect more than 100 times more light than the 1 mr FOV receiver, although for $\tau = 5$ the increase in light collected in a wide FOV over that in 1 mr is only about a factor of 10.

2.2.2. Range Dependence

The explicit dependence of detected power on optical thickness is shown in Figure 2.7, for both the small and large field of view data. (The data points for small FOV data can be obtained from Table 2.2 and are omitted for clarity.) The vertical axis in the figure is the ratio of the power collected at a given optical thickness to that collected in clear weather, for the specified FOV. As can be seen from the figure, the range dependence for any single FOV is nearly exponential. The slope of the exponential curve is a function of FOV, ranging from 3.8 dB transmission loss per optical thickness (for FOV = 5 mr) to 1.25 dB per optical thickness (for FOV = 84°). Thus the data suggest a general exponential extinction law valid for a receiver processing both the scattered and unscattered signal components:

$$P_r \sim P_T e^{-f(\text{FOV})\tau},$$

(2.4)

where $f(\text{FOV}) \leq 1$ and is a monotonically decreasing function of FOV.

To determine an empirical form for $f(\text{FOV})$, consider Table 2.3, which summarizes the available data for transmission vs. FOV in terms of the slope of the transmission curve. The corresponding value of $f(\text{FOV})$ is shown for each FOV. Figure 2.8 is a plot of $f(\text{FOV})$ vs. FOV.
Figure 2.7: Optical Thickness Dependence of Transmission
(Note that the 1 mr plot is assumed to be unscattered signal so that \( f(\text{FOV}) = 1 \).)

Table 2.3
Summary of Transmission Data

<table>
<thead>
<tr>
<th>FOV</th>
<th>Pathlength (km)</th>
<th>Slope of Transmission Curve</th>
<th>( f(\text{FOV}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 mr</td>
<td>0.43</td>
<td>-3.8 dB/τ</td>
<td>.87</td>
</tr>
<tr>
<td>10 mr</td>
<td>0.43</td>
<td>-3.7 dB/τ</td>
<td>.85</td>
</tr>
<tr>
<td>15 mr</td>
<td>0.43</td>
<td>-3.3 dB/τ</td>
<td>.76</td>
</tr>
<tr>
<td>17°</td>
<td>0.30</td>
<td>-2.5 dB/τ</td>
<td>.58</td>
</tr>
<tr>
<td>17°</td>
<td>1.6</td>
<td>-2.5 dB/τ</td>
<td>.58</td>
</tr>
<tr>
<td>22°</td>
<td>0.30</td>
<td>-1.5 dB/τ</td>
<td>.35</td>
</tr>
<tr>
<td>84°</td>
<td>0.43</td>
<td>-1.25 dB/τ</td>
<td>.29</td>
</tr>
</tbody>
</table>

With the limited data available, it is not possible to determine an analytical function for \( f(\text{FOV}) \) with great confidence. However, the data suggests that \( f(\text{FOV}) \) falls roughly logarithmically with FOV. Assuming this to be the case, an empirical formula for \( f(\text{FOV}) \) is

\[
 f(\text{FOV}) = 1 - k \ln(\text{FOV}), \quad 0.069 < k < 0.11 , \quad (2.5)
\]

where FOV is in milliradians (1 mr \( < \) FOV \( < \) 1500 mr).

Observe that this form for \( f(\text{FOV}) \) implies a signal vs. FOV characteristic in agreement with the measured one. Substituting Eq. (2.5) into Eq. (2.4), and assuming there is a maximum FOV, \( \text{FOV}_{\text{max}} \), at which the transmission at τ is greatest, produces a signal vs. FOV...
Figure 2.8: Experimental Values of $f(\text{FOV})$
characteristic

\[
\frac{T(\text{FOV})}{T(\text{FOV}_{\text{max}})} = \left( \frac{\text{FOV}}{\text{FOV}_{\text{max}}} \right)^{K_{\tau}}.
\]  

(2.6)

For \( \tau = 9 \), with \( K \approx 0.1 \), Eq. (2.6) displays the roughly linear signal vs. FOV relationship seen in Figure 2.6.

The data presented here is utilized in Chapter 3 to compare the relative applicability of various existing propagation theories. The data presented in the next section embodies the important "insensitivity property," which is the basis of the theoretical work in Chapter 4.
2.3. Experimental Results - Details of the Angular Spectrum

Five representative angular spectrum scans are shown in Figures 2.9 through 2.13. For comparison, a clear weather scan is shown in Figure 2.14. This latter scan is essentially the impulse response of the receiver optics. From these figures and the other available data, it is possible to abstract the following significant features:

1. There is a measurable unscattered signal peak on-axis for optical thicknesses below 10.
2. This unscattered peak disappears for optical thicknesses greater than about 10, and a uniform angular spectrum results, at least as far out as ±12 mr.
3. For optical thicknesses greater than 5, the angular spectrum magnitude is down by a factor of 10 between ±5 and ±10 mr. There is a tendency for the spectrum to broaden as the optical thickness increases, until it flattens out as described in (2).
4. The angular spectrum outside of ±10 mr is relatively uniform. This portion of the angular spectrum extends over a considerable range, at least as far out as 25°. After 5°, this portion of the angular spectrum slopes off at a rate of about 3 dB for each 5° - 10° in angle.
5. The angular spectrum for sufficiently off-axis angles varies very little over the course of a particular measurement, whereas the on-axis signal may fluctuate
Figure 2.9: Azimuth Scan (8/3/78), FOV = 16 mr
Figure 2.10: Elevation Scan (8/3/78), FOV = 1.35 mr
Figure 2.11: Elevation Scan (8/2/78), FOV = 1.35 mr

\[ \tau = 6.6 - 7.2 \]
$\tau = 6.8 - 7.8$

Figure 2.12: Elevation Scan (8/3/78), FOV = 1.35 mr
Figure 2.13: Elevation Scan (8/2/78), FOV = 1.35 mr
Figure 2.14: Clear Weather Elevation Scan (8/4/78),
FOV = 1.35 mr
a significant amount during the measurement.

This last property of the angular spectrum is the experimental evidence for the "insensitivity" property referred to above. Since this property has a fundamental role in the theoretical work in this thesis, it will now be considered in some depth. First, note that from property (1), the on-axis signal for $\tau < 10$ is dominated by the unscattered light. Furthermore, for sufficiently large angles off-axis, the receiver collects only scattered light. Therefore, property (5) essentially implies that the received unscattered light varies significantly, while the scattered light varies very little, over the course of a measurement.

The cause of the unscattered light fluctuations is the fluctuation in the integrated path extinction coefficient due to random variations in the local density of fog. These density fluctuations are evident visually to anyone who observes a fog and notes that it is patchy. Because of the motion of these fog patches across the path between the source and receiver, the local extinction coefficient is constantly changing, and depending on the wind speed these changes will occur on a time scale from a few seconds to many minutes.

Since the time required to make the angular scans was often as long as 15 minutes, most of them were expected to embody these fluctuations. In fact, the data taking procedure in some of the scans was designed specifically to minimize these effects. The procedure used was to make a very crude scan with only a few data points, and then return to fill in the missing points on a subsequent pass. This
way the complete range of angle could be scanned before the weather changed. Figure 2.9 is an example of such a scan. Another technique used instead of the crude scan approach was to repeatedly return to and sample the signal at the on-axis angle, in order to be able to later normalize the off-axis values to the value of the on-axis signal which occur nearest in time to them.

It was discovered, however, that there is surprisingly little variation in the signal off-axis, although the on-axis signal does exhibit the expected fluctuations. Apparently, while the unscattered signal is quite sensitive to the spatial inhomogeneities (fog patches) in the medium, the scattered signal is relatively insensitive to the same kinds of inhomogeneities. This property has been named the "insensitivity" property.

To quantify the extent of the insensitivity, consider the measure

\[ I = \frac{\% \text{ signal variation off-axis}}{\% \text{ signal variation on-axis}} \] (2.6)

A value of 1 for I means that the off-axis variations are as large as the on-axis variations. Any value less than 1 implies relative insensitivity. The on-axis variation is relatively easy to compute: It is simply the difference between the signal value of the highest data point and the signal value of the lowest data point, divided by the latter.

For the off-axis variation, some care is needed to obtain an appropriate value. In those scans which have a number of off-axis points that exhibited variation, the off-axis angle with the largest variation is
chosen to do the computation. However, note that in Figures 2.12 and 2.13 only one measurement was made at each off-axis angle. In these cases, the time sequencing of the points (indicated by the numbers and letters) is used. The off-axis variation is computed by comparing the values of the two off-axis points which are adjacent in time to the extreme (high and low) data points in the on-axis signal. Thus, in Figure 2.12, since points 16 and P are the low and high respectively of the on-axis data, the off-axis points 15 and 0 are compared. In Figure 2.13, the off-axis points 16 and 21 are compared.

Table 2.4 shows insensitivity values for the angular scans in Figures 2.9 through 2.13. Observe that in most cases, the off-axis (scattered) signal is less than one-third as sensitive to spatial inhomogeneities in the medium as is the on-axis (unscattered) signal.

<table>
<thead>
<tr>
<th>Figure No.</th>
<th>On-Axis Variation (%)</th>
<th>Off-Axis Variation (%)</th>
<th>I</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.9</td>
<td>300%</td>
<td>34%</td>
<td>.11</td>
</tr>
<tr>
<td>2.10</td>
<td>114%</td>
<td>37%</td>
<td>.32</td>
</tr>
<tr>
<td>2.11</td>
<td>129%</td>
<td>50%</td>
<td>.39</td>
</tr>
<tr>
<td>2.12</td>
<td>173%</td>
<td>22%</td>
<td>.13</td>
</tr>
<tr>
<td>2.13</td>
<td>100%</td>
<td>25%</td>
<td>.25</td>
</tr>
</tbody>
</table>
The physical origin of the insensitivity property is the fact that the scattered signal is composed of light which has traveled many different paths between source and receiver, whereas the unscattered signal consists of light which has travelled only the line-of-sight path. Figure 2.15 illustrates the difference.

Figure 2.15: Scattered vs. Unscattered Light Paths

The unscattered light signal is dependent solely on the random distribution of scatterers on the line-of-sight path, and hence is significantly affected by any variation in that distribution. The scattered light, however, since it is a composite of light traversing many paths, is not so subject to variations on any one path. In fact, since the extinction on some paths may be decreased while that on
others is increased, there is an averaging effect which reduces the overall fluctuation.

In light of this, it is worth noting that the value of I shown in Table 2.4 for the scan in Figure 2.9 is actually an overestimate. The reason is that the scan was done with a 16 mr FOV, so that the on-axis measurement contained a significant amount of scattered light. Hence, the averaging effect was occurring on-axis as well as off-axis, so that the ratio of the magnitude of the two fluctuation ranges was reduced.

It has thus been shown that the insensitivity property is a natural consequence of propagation in a spatially inhomogeneous medium. In principle, then, it can be derived from the general spatially inhomogeneous form of the transport equation. However, in this thesis, the alternative approach is followed, wherein the insensitivity property is used as a starting point, from which transport equation simplifications are then derived. This is taken up in Chapter 4.

As a prelude to this, Chapter 3 explores the results of existing propagation theories, based on single scatter, diffusion and multiple forward scatter, and compares them quantitatively with the experimental results. Specifically, these theories will be used to predict the angular and spatial characteristics identified in Section 2.2: the increase in the angular spreading as τ increases, and the FOV dependent exponential extinction. From a comparison of the data with the existing theories, a better understanding of their applicability - and necessary extensions and modifications - should arise.
CHAPTER III

COMPARISON OF DATA WITH EXISTING PROPAGATION THEORIES

This chapter discusses existing channel propagation theories and relates them to the experimental results presented in Chapter 2. As stated in Chapter 1, the transport equation has been solved in certain extreme cases, such as when the single scatter [15,16] or diffusion [12-14] approximations apply. However, it is not obvious to what extent they help to explain propagation in typical turbid atmospheric situations, when their rather limited validity conditions are not completely satisfied. One goal of the present chapter is to explore this question for the cw point source problem considered in this thesis.

In addition to the single scatter and diffusion extremes, the multiple forward scatter (MFS) theory, based on the narrow angle approximation [27-29], is considered. While this approximation does have foundation in experimental data (see Section 1.1.2), predictions from the theory have previously only been compared with data from measurements with highly collimated sources [28,32,67,68]. This chapter compares the theory in the context of a point source with data from the uv experiments.

Throughout this chapter, only the aspects of propagation which can be dealt with via the homogeneous form of the transport equation are considered. Thus, the starting point for developing all of the simplified theories presented below is the homogeneous, time-independent
transport equation (see Eq. (1.5)):

\[
\bar{\kappa} \cdot \nabla p(\vec{r}, \bar{\Omega}) + \alpha p(\vec{r}, \bar{\Omega}) = \alpha_s \int d\bar{\Omega}' \sigma(\bar{\Omega} \cdot \bar{\Omega}') p(\vec{r}, \bar{\Omega}')
\]  

(3.1)

The inhomogeneous aspects of propagation, such as the insensitivity property, are discussed in Chapter 4.

The outline of the chapter is as follows. Section 3.1 presents the results from the three propagation theories, with a brief discussion of the appropriate physical context of each of the theories. In keeping with the general approach in this thesis, these theories are discussed from the point of view of approximations to the transport equation. Section 3.2 applies these results to the uv experiments conducted at Lubec and Eastport, and discusses their applicability.
3.1. Results from Propagation Theories

In this section, the relevant theoretical expressions are presented for the single scatter, multiple forward scatter and diffusion approximations. For each theory, its generally accepted validity range is briefly discussed. The source/receiver configuration is that shown in Figure 3.1.

3.1.1. Single Scatter Theory

Single scatter refers to the condition in which photons traversing the medium between the source and receiver are scattered exactly once, if at all. Since the scattering process is random, no matter how thin the medium is there is at least some probability that photons will be scattered more than once. However, single scatter theory adequately describes propagation when the probability is very small that a photon will be scattered more than once between source and receiver.

The generally accepted rule of thumb for application of single scatter theory to a line of sight propagation path is that the optical thickness must be less than 0.1 [69]. This would appear to rule it out as a possible candidate for explaining propagation through fog on any path much longer than 10 meters. (Recall from Chapter 1 that the scattering coefficient in fog is typically greater than 10 km\(^{-1}\).) However, an alternative viewpoint for optical thickness larger than \(\tau = 0.1\) is that single scattering theory describes the first scattering order of the multiply scattered field. To get an exact solution, corrections for double scattering, triple scattering, etc. would have
Figure 3.1: Basic Source/Receiver Configuration
to be added. Thus a comparison of single scatter theory with the experimental data serves to determine exactly to what extent these correction terms are important.

Introducing the single scatter approximation into the transport equation is a simple matter. It is merely necessary to note that since the part of the light field being considered has been scattered at most once, all photons which are ultimately scattered into the direction of interest $\vec{\Omega}$ had to arrive at the scattering point directly from the source with no prior interactions. That is, Eq. (3.1) can be written as

$$\vec{\Omega} \cdot \nabla p_{ss}(\vec{r}, \vec{\Omega}) + \alpha p_{ss}(\vec{r}, \vec{\Omega}) = \alpha_s \int d\vec{\Omega}' \sigma(\vec{\Omega} \cdot \vec{\Omega}') p_u(\vec{r}, \vec{\Omega}')$$  \hspace{1cm} (3.2)$$

where $p_{ss}(\vec{r}, \vec{\Omega})$ denotes the single scattered portion of $p(\vec{r}, \vec{\Omega})$ and $p_u(\vec{r}, \vec{\Omega})$ denotes the unextinguished portion due to photons coming directly from the source.

It is important to note that $p(\vec{r}, \vec{\Omega})$, and hence $p_{ss}(\vec{r}, \vec{\Omega})$ and $p_u(\vec{r}, \vec{\Omega})$, are not probability densities. $p(\vec{r}, \vec{\Omega})$ is the steady state (time integrated) form of the probability density $p(\vec{r}, \vec{\Omega}, t)$ and has units sec m$^{-3}$ sr$^{-1}$. $cp(\vec{r}, \vec{\Omega})$ has units of m$^{-2}$ sr$^{-1}$ and $P_T cp(\vec{r}, \vec{\Omega})$, where $P_T$ is the transmitted power, is the specific intensity (or radiance) with units of watts m$^{-2}$ sr$^{-1}$.

The unextinguished density, $p_u(\vec{r}, \vec{\Omega})$, satisfies Eq. (3.1) with $\alpha_s = 0$ (no scattering):

$$\vec{\Omega} \cdot \nabla p_u(\vec{r}, \vec{\Omega}) + \alpha p_u(\vec{r}, \vec{\Omega}) = 0$$  \hspace{1cm} (3.3)$$
Integrating Eq. (3.3) yields

\[ p_u(\bar{r}, \Omega) = \frac{1}{c} \int_0^\infty dv \, p_o(\bar{r} - v\bar{\Omega}, \Omega) \, e^{-av}, \quad (3.4) \]

where \( p_o(\bar{r}, \Omega) \) is the space-angle probability density for the source.

For a source of radial extent \( \theta_T \) and half-angle extent \( \rho_T \) (as in Figure 3.1), \( p_o(\bar{r}, \Omega) \) is given by

\[ p_o(\bar{r} = (\bar{\rho}, z), \Omega) = \frac{\text{rect}(\theta/\theta_T)}{2\pi(1 - \cos \theta_T)} \frac{\text{circ}(\rho/\rho_T)}{\pi \rho_T^2} \delta(z) \]

where \( \bar{\rho} = (x, y) \) is the transverse space coordinate perpendicular to the direction of propagation, \( z \), and \( \text{rect}(\theta) \) and \( \text{circ}(\rho) \) are the one- and two-dimensional functions, respectively, shown in Figures 3.2.

Figure 3.2: One- and Two-Dimensional Circ Functions
Since the right-hand side of Eq. (3.2) does not depend on \( p_{ss}(\vec{r}, \Omega) \), it can be integrated similarly to yield

\[
p_{ss}(\vec{r}, \Omega) = \alpha_s \int_0^\infty d\nu \ e^{-\alpha \nu} \int d\Omega' \sigma(\Omega, \Omega') \ p_u(\vec{r} - \nu \vec{n}, \Omega') . \quad (3.6)
\]

with \( p_u(\vec{r}, \Omega) \) given by Eq. (3.4), Eq. (3.6) constitutes a complete solution for the single scattered space-angle density. The detected power, \( P_D \), is then simply the integral of \( p_{ss}(\vec{r}, \Omega) \) over the receiver aperture and detector FOV scaled up by the \( c \) times the transmitted power:

\[
P_D = c P_T \ 2\pi \int_0^{\rho_R} d\rho \rho^2 \int_{\Omega_{FOV}} d\Omega \ p_{ss}(L, \rho, \Omega) . \quad (3.7)
\]

For the purposes of this chapter, the integrals indicated in Eq. (3.6) were carried out in prolate spheroidal coordinates, following the method developed in references [16] and [34], and modified slightly in reference [70]. In this method, the source and receiver are considered to be located at the foci of an ellipsoid whose surface is defined by the intersection of the source and receiver cones. This approach is very efficient for the single scatter problem because ellipsoids inside the scattering volume are equitemporal surfaces. For some phase functions, closed form expressions are easily obtained using this approach [34]. The method relies on the fact that the source and receiver are confocal, but this is the only geometry considered here. (See Figure 3.1.)
3.1.2. **Diffusion Theory**

Diffusion theory is applicable in a very highly multiple scattering environment, which is the opposite extreme to that of single scatter. Physically, diffusion corresponds to the situation in which photons have been scattered so many times that they no longer have any preferred direction. The precise point at which atmospheric conditions are such that diffusion theory is applicable is still an open question. However, an estimate can be made for an isotropic point source as follows: Kennedy [9] has shown that for isotropic scatter with large single scatter albedo (> 0.9), diffusion theory gives essentially the same results as the exact transport equation solution, for optical thicknesses between 1 and 100. Bucher [6] has shown empirically via simulation that many isotropic scatter diffusion results can be applied to non-isotropic scatter situations as long as the diffusion thickness, 

\[ \tau_D = \tau(1 - g), \quad (3.8) \]

is equivalent in the two cases and is greater than 3. Here \( \tau \) is the optical thickness of the medium and \( g \) is the phase function's average cosine. (See Eq. (1.24).) \( \tau_D \) is thus the optical thickness scaled in inverse proportion to the light field's propensity to scatter in the forward direction.

Figure 3.3 shows the value of the crossover optical thickness, \( \tau_c \), required by Eq. (3.8) with \( \tau_D = 3 \) for diffusion theory to be applicable.
For an average cosine of $g = 0.9$, which was the best fit to the measured phase function in Figures 1.5 and 1.6, the crossover optical thickness is 30.

As in the single scatter case, this applicability criterion would appear to rule out diffusion theory as an explanation for the experimental observations presented in Chapter 2, since all the
measurements were made for $\tau \leq 10$. However, again as with single scatter, the alternative viewpoint of thinking of the light field as made up of various components, one of which is diffusion, is helpful. On this view, the received light may not be dominated by the diffusion component, but it is present and may manifest itself in ways which can be measured [68].

The angular scan in Figure 2.9 is an indication that this is in fact the case. Observe that the angular spectrum has a very narrow peak but that it becomes relatively flat at the larger angles. This uniformity in angle is precisely the kind of behavior that would be expected from a diffusive component. It is possible that the component of the field which is responsible for the narrow peak has died out at the larger angles, leaving the diffusion component to dominate.

The diffusion approximation can be made by introducing Fick's diffusion law into the transport equation, or by approximating the solution with two terms in a spherical harmonic expansion [10,13,19]. Both approaches produce the same result. The Fick's law approach has the advantage of simplicity and a physical interpretation, whereas the spherical harmonics approach provides an explicit formula for the diffusion constant. In what follows, the Fick's law approach is taken, and needed results from the spherical harmonics approach are cited.

For the purpose of this discussion, an integrated form of the transport equation is used. If the transport equation, Eq. (3.1), is integrated over all angles, the result is
Defining the absorption coefficient,

\[ \alpha_a = \alpha - \alpha_s \]  

Eq. (3.9) can be written simply as

\[ \nabla \cdot \mathcal{H}(\mathbf{r}) + \alpha_a h(\mathbf{r}) = 0. \]  

In order to introduce the diffusion approximation, the physical meaning of the functions \( \mathcal{H}(\mathbf{r}) \) and \( h(\mathbf{r}) \) must be identified. As a preliminary, note that the flux, or power per unit area, incident on a detector pointing in the direction \(-\mathbf{\Omega}\) is given by

\[ \Phi_{\Omega_{\text{FOV}}} (\mathbf{r}, \mathbf{\Omega}) = c \mathcal{P}_T \int_{\Omega_{\text{FOV}}} d\mathbf{\Omega}' \, p(\mathbf{r}, \mathbf{\Omega}') \, \mathbf{\Omega} \cdot \mathbf{\Omega}', \]  

where

\[ \mathcal{H}(\mathbf{r}) = \int d\mathbf{\Omega} \, \mathbf{\Omega} \, p(\mathbf{r}, \mathbf{\Omega}) \]  

and

\[ h(\mathbf{r}) = \int d\mathbf{\Omega} \, p(\mathbf{r}, \mathbf{\Omega}) \]
where $\Omega_{\text{FOV}}$ is the solid angle of the receiver's FOV and $c_{PT}$ is the required scaling constant to convert $p(\mathbf{r}, \Omega)$ to a power quantity. The $\Omega \cdot \Omega'$ in the integrand accounts for the projected area of the receiver surface, oriented in direction $-\Omega$, onto the direction of the incoming light density, $\Omega'$.

Now it is easy to show that, except for the scaling constant, $\mathbf{H}(\mathbf{r})$ is a vector whose direction, $\Omega_H$, is that of the maximum net flux across a surface at $\mathbf{r}$. Here net flux refers to the fact that the radiant flux can flow across the surface in two directions. Net flux is the difference between the two. The magnitude of $\mathbf{H}(\mathbf{r})$ will also be shown to be the magnitude of this maximum net flux.

To prove these statements, note that

$$
\mathbf{\Omega} \cdot \mathbf{H}(\mathbf{r}) = \mathbf{\Omega} \cdot \mathbf{H} |\mathbf{H}(\mathbf{r})| = \int \Omega' p(\mathbf{r}, \Omega') \mathbf{\Omega} \cdot \Omega' = \int_{\Omega^+} \Omega' p(\mathbf{r}, \Omega') \mathbf{\Omega} \cdot \Omega' \\
- \int_{\Omega^-} \Omega' p(\mathbf{r}, \Omega')(\mathbf{-\Omega}) \Omega' .
$$

(3.15)

Here, $\Omega^+$ refers to the hemisphere on the side of the surface which is in the direction of propagation, and $\Omega^-$ refers to the hemisphere on the other (back) side of that surface. Noting that the two quantities on the right hand side of Eq. (3.15) are proportional to the forward traveling and backward traveling fluxes respectively, it can be written

$$
\mathbf{\Omega} \cdot \mathbf{H} |\mathbf{H}(\mathbf{r})| = \frac{1}{c_{PT}} \left( \phi_+ (\mathbf{r}, \Omega) - \phi_- (\mathbf{r}, -\Omega) \right)
$$

(3.16)
Clearly, the net flux crossing the surface is maximized when \( \vec{\Omega} \cdot \vec{\Omega}_H = 1 \), or \( \vec{\Omega} = \vec{\Omega}_H \). Thus, \( |I(\vec{r})| \) is proportional to the magnitude of this maximum net flux.

To see the physical meaning of the function \( h(\vec{r}) \), consider the differential collecting area depicted in Figure 3.4. \( \hat{n} \) is the inward pointing normal to the surface. There are \( cP_Tp(\vec{r},\vec{\Omega})dA\hat{\Omega}\cdot\hat{n}d\vec{\Omega} \) watts of power incident on the projected surface element \( dA\hat{\Omega}\cdot\hat{n} \), and in time \( dt \) the total energy which crosses the surface is \( cP_Tp(\vec{r},\vec{\Omega})dA\hat{\Omega}\cdot\hat{n}d\vec{\Omega}dt \) joules. The energy fills up the volume \( dA\hat{\Omega}\cdot\hat{n}cdt \). Thus, the volume density of radian energy coming from the direction \( \vec{\Omega} \) is \( P_Tp(\vec{r},\vec{\Omega})d\vec{\Omega} \). Adding up the

Figure 3.4: Differential Collecting Area and Volume Swept Out in dt Seconds
contributions from all directions, the total volume density, denoted by \( u(\vec{r}) \), is

\[
u(\vec{r}) = P_T \int d\Omega \, p(\vec{r}, \vec{\Omega}) = P_T h(\vec{r}) \quad (3.17)
\]

Thus, \( h(\vec{r}) \) is proportional to the volume density of photons at point \( \vec{r} \).

With the physical meaning of \( \overline{H}(\vec{r}) \) and \( h(\vec{r}) \) understood, it is now possible to make the diffusion approximation. As in classical diffusion processes, the approximation states that the photon flux is away from areas of high photon concentration and toward areas of low concentration. More precisely, Fick's diffusion law, as applied to the case of diffusing photons, states that the net photon flux is proportional to the negative gradient of the photon volume density, or

\[
\overline{H}(\vec{r}) = -D \nabla h(\vec{r}). \quad (3.18)
\]

\( D \) is called the diffusion constant.

Substituting Eq. (3.18) into Eq. (3.13) yields

\[
\nabla^2 h(\vec{r}) - \frac{\alpha}{D} h(\vec{r}) = 0 \quad (3.19)
\]

which is the classical diffusion equation. Its point source Green's function is well known. For

\[
h_0(\vec{r}) = \delta(\vec{r}) \quad (3.20)
\]
-96-

\[ h(\mathbf{r}) = \frac{e^{-\kappa r}}{4\pi rc} \] \hspace{1cm} (3.21)

where \( r \) is the radial distance from the source and

\[ \kappa = \frac{\sqrt{\alpha_0/D}}{r} \] \hspace{1cm} (3.22)

Note that, as required by Eq. (3.17),

\[ P_T \int d\mathbf{r} h(\mathbf{r}) = U, \] \hspace{1cm} (3.23)

where \( U \) is the total unabsorbed radiant energy in all of space due to the source in Eq. (3.20).

From the spherical harmonics approach to the diffusion approximation, it can be shown that [13]

\[ D = \frac{1}{3\alpha(1 - g\omega_0)} \] \hspace{1cm} (3.24)

where \( g \) is the phase function average cosine and \( \omega_0 \) is the single scatter albedo. Also, the relation between \( p_D(\mathbf{r},\Omega) \), the diffusion approximation solution to the transport equation, and \( h(\mathbf{r}) \) is shown to be

\[ p_D(\mathbf{r},\Omega) = \frac{1}{4\pi} [h(\mathbf{r}) - 3D\Omega \cdot \nabla h(\mathbf{r})] \] \hspace{1cm} (3.25)

In order to give some physical meaning to the diffusion constant \( D \), note first that it has units of length. Note further that when \( \omega_0 \) is
close to 1 (highly scattering environment),

\[ \frac{L}{D} \approx 3\tau(1 - g), \quad (3.26) \]

where \( L \) is the distance between source and receiver. Thus, in accordance with Eq. (3.8),

\[ \frac{L}{D} \approx 3\tau_D . \quad (3.27) \]

Thus \( L/D \) itself is a kind of diffusion thickness.

Although it would be possible to use Eqs. (3.21) and (3.25) to produce a solution for any desired source distribution, only the point source solution will be of concern in Section 3.2. As in the single scatter case, the detectable power is simply the scaled integral of \( p(\vec{r},\Omega) \) over the area of the receiver aperture and detector FOV solid angle. (See Eq. (3.7).)

3.1.3. **Multiple Forward Scatter Theory**

As stated in Chapter 1, making the small angle approximation to the transport equation results in valuable simplifications. In turbulence theory, where an approximation analogous to this originated as a means of converting the Helmholtz equation into the parabolic equation [71-74], this approximation is rigorously valid. The large spatial scales of turbulent refractive index eddies (10^{-3}-10 meters) insure that scattering
from them is very sharply forward directed [74].

For scattering in a turbid atmosphere, however, the ratio of the optical wavelength to the particle diameter is not nearly as large, at most on the order of 10 as opposed to the turbulence value of greater than 1000 (for visible wavelengths). Hence, while the phase function for aerosol scattering is predominantly forward directed, there is still a significant amount of wide-angle scatter. (See Figures 1.3 and 1.4.) Thus, in order to validly make the small angle approximation in transport theory, further conditions must be met which allow the phase function to be replaced by one that has no wide-angle scatter.

In principle, there are a number of physical conditions that could make the small-angle approximation valid. If the optical thickness in the medium is not very large (τ < 3), the probability of substantial wide-angle scatter reaching the receiver is quite small. Alternatively, if the channel geometry includes boundaries which absorb most photons scattered at wide angles, few of them will reach the receiver. Finally, if the receiver FOV is small enough so that only an insignificant number of wide-angle scattered photons can actually be detected [32], the medium will effectively be converted into one in which the narrow-angle approximation applies.

Observe that in all these physical situations, the wide-angle scattered light is considered absorbed, because there is an effective scattering angle, θₜ, beyond which scattered photons do not contribute to the received field. Thus, the small-angle approximation is made by scaling the absorption and scattering coefficients, and by replacing the
phase function by a scaled truncated one. This approach to solving the transport equation in physical situations in which the small-angle approximation is not rigorously valid, is called multiple forward scatter (MFS) theory. Defining the forward scatter efficiency,

$$\phi = 2\pi \int_0^{\theta_E} \sigma(\theta) \sin \theta d\theta ,$$

the new coefficients in the MFS theory become

$$\alpha_{a}^{' } = \alpha_{a} + \alpha_{s} (1 - \phi)$$
$$\alpha_{s}^{' } = \alpha_{s} \phi .$$

The new phase function becomes

$$\sigma_{T}(\theta) = \begin{cases} 
\frac{\sigma(\theta)}{\phi} & 0 \leq \theta \leq \theta_E \\
0 & 0 > \theta_E 
\end{cases}$$

Using these new coefficients and the scaled phase function, the small-angle approximation transport equation (Eq. (1.9)) can be written as

$$s \cdot \nabla p(\rho, s, z) + \frac{3}{3z} p(\rho, s, z) + \alpha p(\rho, s, z) = \alpha_{s}^{' } \int_{-\infty}^{\infty} ds' \sigma_{T}(s-s') p(\rho, s', z).$$
Defining the Fourier transform

\[ \mathcal{F}_T(q) = \int_{-\infty}^{\infty} ds \, \sigma_T(s) \exp(js \cdot q), \quad (3.33) \]

and using the boundary condition

\[ p(\rho, s, z = 0) = p_0(\rho, s), \quad (3.34) \]

the following solution can be obtained by Fourier transform techniques

\[ p(\rho, s, z = L) = \frac{1}{(2\pi)^4} \int dq_1 \int dq_2 \exp[-j(\overline{q}_1 \cdot \overline{\rho} + \overline{q}_2 \cdot \overline{s})] \cdot F_0(\overline{q}_1, \overline{q}_2 + \overline{a}_1 z) Q(\overline{q}_1, \overline{q}_2, z) \]

where

\[ F_0(\overline{q}_2, \overline{q}_2) = \int d\rho \int ds \, p_0(\rho, s) \exp[j(\overline{q}_1 \cdot \overline{\rho} + \overline{q}_2 \cdot \overline{s})] \quad (3.36) \]

and

\[ Q(\overline{q}_1, \overline{q}_2, z) = \exp \left\{ -\alpha_a L - \alpha_s \int_0^L dz \left[ 1 - \int \mathcal{F}_T(\overline{q}_2 + \overline{a}_1 (L-z)) \right] \right\}. \quad (3.37) \]

The integrals in Eqs. (3.35) through (3.37) are difficult to evaluate for a general phase function \( \sigma_T(s) \). However, by using a Gaussian-shaped phase function and Gaussian-shaped source distribution, receiver aperture and detector focal plane pinhole, Shapiro [27] has
obtained a closed form expression for the detected power. Assuming

\[ \sigma_T(s) = \frac{1}{2\pi\theta_F^2} \exp(-|s|^2/2\theta_F^2) \]  \hspace{1cm} (3.38)

where \( \theta_F \) is the rms forward scatter angle, the detected power in the far-field of a laser source for \( \alpha_s L >> 1 \) is given by:

\[ P_D = \left\{ \frac{P_{RFS} e^{-\alpha_s' L}}{1 + 2r_T^2/\rho_0^2} \right\} \left[ \frac{(k\rho_0)^2/8 \theta_{F0V}^2}{1 + (k\rho_0)^2/8\theta_{F0V}^2 + \rho_o^2/2\rho_R^2} \right]. \] \hspace{1cm} (3.39)

Here \( P_{RFS} \) is the received power for free-space propagation, given by

\[ P_{RFS} = P_T \left( \frac{k\rho_T \rho_R}{L} \right)^2, \] \hspace{1cm} (3.40)

\( \rho_o \) is the channel coherence length

\[ \rho_o = \sqrt{\frac{3}{\alpha_s' L \theta_F^2 k^2}} \] \hspace{1cm} (3.41)

and \( k = 2\pi/\lambda \).

The first factor in parentheses in Eq. (3.39) is the beamspread term. It consists of the natural diffraction limited far field spreading loss (\( P_{RFS} \)) with absorption added in, divided by a term which accounts for the atmospherically induced beamspread. This scattering beamspread is negligible when \( r_T << \rho_o \) (e.g., for a point source) and becomes important when \( r_T > \rho_o \) (e.g., highly collimated laser source). Physically, this latter condition corresponds to the situation in which different
parts of the source aperture are out of phase relative to one another as seen at the receiver.

The second factor in Eq. (3.39) represents loss in detected power due to angular spread. Note that, as expected, this term is unity for large enough FOV. The angular spread loss is negligible for $\rho_R \ll \rho_o$ and becomes important for $\rho_R \gtrsim \rho_o$. To see this, observe that when $\rho_R \ll \rho_o$, the factor can be written as

$$\text{angular spread loss} = \frac{\theta_{\text{FOV}}^2}{\theta_{\text{FOV}}^2 + \frac{4}{(k\rho_R)^2}}.$$  \hspace{1cm} (3.42)

For typical optical receiver radii $\rho_R$ on the order of $10^{-2}$ - $10^{-1}$ meters, this loss factor flattens out for $\theta_{\text{FOV}}$ around 1-10 μradians, which is much smaller than the minimum of 100 μradians needed in typical receiver systems to allow for ease in receiver aiming and to accommodate the focal plane blur circle due to lens aberration. For $\rho_R > \rho_o$, the angular spread factor becomes

$$\text{angular spread loss} = \frac{\theta_{\text{FOV}}^2}{\left[2\left(\frac{\lambda}{\pi\rho_o}\right)^2 + \theta_{\text{FOV}}^2\right]},$$  \hspace{1cm} (3.43)

which is significantly less than 1 for

$$\theta_{\text{FOV}} < \frac{\sqrt{\pi} \lambda}{\pi \rho_o}.$$  \hspace{1cm} (3.44)
In order to determine whether there is beamspread and angular spread loss, and if so how much, the size of the channel coherence length, \( \rho_0 \), must be ascertained. From Eq. (3.41), \( \rho_0 \) can be obtained only if the phase function's rms forward scatter angle, \( \theta_F \), is known. To specify this parameter, some strategy must be established for matching the Gaussian phase function in Eq. (3.38) with the truncated and scaled phase function in Eq. (3.31). This, in turn, involves the question of how the parameter \( \phi \) (or equivalently, \( \theta_E \)) is determined.

Consider first the choice of \( \phi \). Observe that as long as \( \theta_E \) is outside the forward peak of the phase function, the value of \( \phi \) is not very sensitive to the choice of \( \theta_E \). Nakai argues [28] that if the \( \theta_E \) value chosen is larger than the true \( \theta_E \) (and hence the \( \phi \) chosen is larger than the true \( \phi \)), the results of the theory will be insensitive to this difference because photons scattered outside the true \( \theta_E \) will not contribute to the received light. It appears that a value of \( \phi \) between 0.5 and 1.0 is most appropriate for phase functions in low visibility atmospheres [28,32,67].

To determine \( \theta_F \), Nakai further suggests that the Gaussian and truncated phase functions be matched at their peak values, since the forward peak of the phase function is such an essential feature of the MFS theory. Hence, equating Eqs. (3.31) and (3.38) at \( \theta = 0 \),

\[
\frac{\sigma_T(0)}{\phi} = \frac{1}{2\pi \theta_F^2},
\]

or
\[ \theta_F = \sqrt{\frac{\phi}{2\pi \sigma_T(0)}} . \quad (3.46) \]

Observe in this expression that, as stated above, the important parameter \( \theta_F \) is only mildly dependent on the choice of \( \phi \): For \( \phi \) between 0.5 and 1.0, \( \theta_F \) changes by only about 40%.

Whether or not there will be beamspread loss depends upon the source size, \( \rho_T \). For a Gaussian beam source [75]

\[ \rho_T = \frac{\lambda}{\pi \theta_T} . \quad (3.47) \]

Thus beamspread is significant when \( \rho_T > \rho_o \), or \( \theta_T < \lambda/(\pi \rho_o) \). For the point source considered in the next section, the channel induced beamspread is insignificant compared to the natural divergence of the source itself. The detected power in this case is given by

\[ P_D = P_{RFS} e^{-\alpha L} \frac{\theta^2_{FOV}}{8 \left( \frac{\theta^2_{FOV}}{(k\rho_o)^2} + \theta^2_{FOV} \right)} . \quad (3.48) \]
3.2. Comparison with UV Experiments

In this section, results from the three theories presented above are compared with each other and with the data from the uv experiments. Since the source is a hemispherical point source, $\theta_T = \pi/2$ for both the single scatter and MFS theories. For the diffusion theory, which was solved above for a spherical source, the collected power is simply multiplied by two, since the hemispherical source directs twice the power toward the receiver.

Section 3.2.1 compares the theoretical results with the signal vs. FOV characteristics of the data and Section 3.2.2 makes the comparison for the optical thickness dependence.

3.2.1. Signal vs. FOV Characteristics

Figure 3.5 shows measured and theoretical signal vs. FOV characteristics for two values of $\tau$. Assumed values for $g$ and $\omega_0$ are 0.85 and 1.0, respectively. For the MFS theory, the assumed values were $\theta_F = 100$ mr and $\phi = 0.75$. Only the scattered light is included in the plot, so the unscattered component (which was included in the actual signal vs. FOV measurements shown in Figure 2.6) has been subtracted out. In accordance with the discussion in Section 2.1.2, it was assumed for this purpose that the signal collected in the 1 mr FOV was entirely unscattered light.

A number of conclusions can be drawn from Figure 3.5. First, note that none of the theories adequately predict the small-angle behavior of the signal vs. FOV characteristics. The single scatter theory,
Figure 3.5: Measured and Theoretical Signal vs. FOV Curves
while it does have the same slope as the small-angle data, is more than an order of magnitude below the data. The MFS and diffusion theories have a steeper slope than the data and fall well below it, although the MFS curves are more than an order of magnitude higher than the diffusion curves in that region.

For the large FOV data, both diffusion and MFS do a better job of predicting the observed results, whereas the single scatter theory is still well below the data. The MFS appears to more accurately represent the flattening evident in the data.

The conclusion that diffusion contributes a negligible amount to the small FOV received signal, and that single scatter is insignificant for all fields of view, is insensitive to the specific assumptions made about the phase function. Figure 3.6 shows the variation in the two characteristics as the parameter \( g \) is varied. The optical thickness is 5. It is clear from these curves that diffusion depends only minimally on the value of \( g \). While the single scatter variations are more pronounced, the most extreme case is still well below the data.

The variation in the multiple forward scatter component is somewhat more complicated because of the dependence on two parameters, \( \theta_F \) and \( \phi \). Since it has been argued, however, that the most fundamental property of the phase function in the MFS theory is the height of the on-axis peak, \( \sigma_1(0) \), and since \( \theta_F \) and \( \phi \) are coupled via the value of this peak (see Eq. (3.46)), it is more appropriate to treat \( \sigma_1(0) \) and \( \phi \) as independent parameters of the theory, rather than \( \theta_F \) and \( \phi \). In terms of these parameters, the ratio of the scattered to the unscattered
Figure 3.6: Influence of $g$ on Single Scatter and Diffusion Predictions
component is given by

\[
\frac{P_D(\text{scattered})}{P_D(\text{unscattered})} = \frac{\theta^2 \exp(\tau \phi \omega_0)}{4 \frac{\tau \phi^2 \omega_0}{3 \pi \sigma_T(0)} + \theta^2} \tag{3.49}
\]

Observe that \( \phi \) may have a significant (exponential) effect on the final value of this ratio but that its influence on the breakpoint in the signal vs. FOV characteristic is only linear. Since it is only likely to change by a factor of 2 (from 0.5 to 1.0), it will not shift the knee in the curve very much. As will be discussed below, however, \( \sigma_T(0) \) may vary by orders of magnitude. Hence, its influence on the location of the knee in the curve may be profound. Although, in addition to \( \sigma_T(0) \) and \( \phi \) the ratio depends on the albedo \( \omega_0 \), the albedo will always be very close to 1, and hence its variations have only a minimal effect compared to that of the other two parameters.

Figure 3.7 shows the MFS signal vs. FOV characteristics for \( \sigma_T(0) = 11.75 \) and various values of \( \phi \). The value of \( \sigma_T(0) = 11.75 \) is the apparent peak value of the phase functions in Figures 1.5 and 1.6. Observe that, as indicated above, \( \phi \) influences the final value but has little effect on the knee in the curve. The curves are still very far below the data at the small fields of view.

There are a number of considerations, however, which indicate that the choice of \( \sigma_T(0) = 11.75 \) may be seriously in error. First, note from Figures 1.5 and 1.6 that the actual phase functions were not measured at the on-axis value. Practical considerations made it impossible
Figure 3.7: Influence of $\phi$ on MFS Signal vs. FOV;
$\tau = 5, \sigma_T(0) = 11.75$
to measure the phase function inside of a few degrees [42]. Hence the
to measure the phase function inside of a few degrees [42]. Hence the
value of 11.75 is actually an off-axis value, not the peak in the phase
value of 11.75 is actually an off-axis value, not the peak in the phase
function. Second, it is clear from Figures 1.3 and 1.4 that if the Mie
calculation can be believed, the steepness of the phase function near zero
calculation can be believed, the steepness of the phase function near zero
degrees implies a peak phase function value that may be an order of
degrees implies a peak phase function value that may be an order of
magnitude or more above the first data point. Finally, the MFS calculations
magnitude or more above the first data point. Finally, the MFS calculations
of Mooradian et al. [32] -- which were based on a maritime fog phase function for
of Mooradian et al. [32] -- which were based on a maritime fog phase function for
laser light propagation at $\lambda = 0.53 \, \mu m$ -- used a value of $\sigma_T(0) = 1200,$
laser light propagation at $\lambda = 0.53 \, \mu m$ -- used a value of $\sigma_T(0) = 1200,$
and reasonably good agreement between the calculations and experiments
and reasonably good agreement between the calculations and experiments
was obtained.
was obtained.

All of these facts indicate that values of $\sigma_T(0)$ orders of magnitude
All of these facts indicate that values of $\sigma_T(0)$ orders of magnitude
larger than the value used in Figure 3.7 may be appropriate. Figure 3.8
larger than the value used in Figure 3.7 may be appropriate. Figure 3.8
shows signal vs. FOV curves for $\sigma_T(0)$ values ranging from 11.75 to 10,000.
shows signal vs. FOV curves for $\sigma_T(0)$ values ranging from 11.75 to 10,000.
Obviously, variations of $\sigma_T(0)$ in this range have a drastic influence on
Obviously, variations of $\sigma_T(0)$ in this range have a drastic influence on
the knee of the curve. Comparing Figure 3.8 with Figure 3.6, it is clear
the knee of the curve. Comparing Figure 3.8 with Figure 3.6, it is clear
that there are choices of $\sigma_T(0)$ and $\phi$ which produce signal vs. FOV curves
that there are choices of $\sigma_T(0)$ and $\phi$ which produce signal vs. FOV curves
in better agreement with the data.
in better agreement with the data.

Figure 3.9 shows signal vs. FOV characteristics for $\sigma_T(0) = 4560$
Figure 3.9 shows signal vs. FOV characteristics for $\sigma_T(0) = 4560$
and $\phi = 0.6,$ along with the data from Figure 3.5. The $\sigma_T(0)$ value is
and $\phi = 0.6,$ along with the data from Figure 3.5. The $\sigma_T(0)$ value is
almost four times higher than the value used in reference [32] but $\phi$ is
almost four times higher than the value used in reference [32] but $\phi$ is
close to the value of 0.57 used there. The agreement for $\tau = 5$ is good.
close to the value of 0.57 used there. The agreement for $\tau = 5$ is good.
For $\tau = 9$, however, it is apparent that the data is increasing roughly
For $\tau = 9$, however, it is apparent that the data is increasing roughly
linearly with FOV over the entire range of angle, whereas the MFS curve
linearly with FOV over the entire range of angle, whereas the MFS curve
rises as the square of the FOV for small FOV and flattens out for large
rises as the square of the FOV for small FOV and flattens out for large
FOV.
Figure 3.8: Influence of $\sigma_T(0)$ on MFS Signal vs. FOV; $\tau = 5$; $\phi = 0.75$
Figure 3.9: MFS Signal vs. FOV; $\sigma_T(0) = 4560$, $\phi = 0.6$
3.2.2. **Range Dependence**

To further explore the applicability of the single scatter, multiple forward scatter and diffusion theories, the range dependence of the theories is now considered. Figure 3.10 shows transmission vs. optical thickness for the single scatter and diffusion theories, with full angle FOV as a parameter. Figure 3.11 compares single scatter and diffusion theories with the MFS theory for FOV = 100 mr and 1000 mr, and for $\sigma_T(0) = 4560$, $\phi = 0.6$.

The most important feature to note about these figures is that none of the theories exhibits the property, observed in the experiments, of a decreasing rate of range decay with increasing FOV. (Compare with Figure 2.7.) The single scattered light has virtually the same decay rate as the unscattered light. The diffuse light grows slightly with $\tau$ for all fields of view. The MFS light, while it has a decay rate slower than the unscattered light, $(\exp[-\tau(1 - \phi \omega)]$ vs. $\exp[-\tau])$, still depends only on $\phi$.

Recall, however, that in the discussion of the MFS theory in Section 3.1.3, it was stated that one of the ways in which an atmospheric channel with some wide angle scatter can be converted into an effective MFS channel is for the receiver to be small enough so that only an insignificant number of wide angle scattered photons can be detected. In this case, then, the effective truncation angle $\theta_E$ is equal to the receiver half angle FOV, and hence the value of $\phi$ depends on the FOV as follows

$$\phi = 2\pi \int_{0}^{\theta_{FOV}/2} d\theta \sigma_T(\theta) \sin \theta \quad (3.50)$$
Figure 3.10: Transmission of Single Scattered and Diffusion Light; $g = 0.85$, $\omega_0 = 1.0$
Figure 3.11: Single Scatter, Diffusion and MFS Transmission; 
\[ \sigma_T(0) = 4560, \phi = 0.6, g = 0.85, \omega_0 = 1.0 \]
Since $\sigma_T(\theta)$ also depends on $\phi$ (via Eqs. (3.38) and (3.46)), Eq. (3.50) must in general be solved numerically for the value of $\phi$ corresponding to the desired value of $\theta_{\text{FOV}}$. Figure 3.12 shows a plot of $\phi$ and the right hand side of Eq. (3.50) vs. $\phi$ for a peak phase function value of $\sigma_T(0) = 100$. The intersection points, along with those for other values of $\sigma_T(0)$, are shown in Table 3.1.

Table 3.1

<table>
<thead>
<tr>
<th>$\sigma_T(0)$</th>
<th>$\theta_{\text{FOV}}$ (mr)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0.004</td>
</tr>
<tr>
<td>10</td>
<td>0.0125</td>
</tr>
<tr>
<td>100</td>
<td>0.04</td>
</tr>
<tr>
<td>1000</td>
<td>0.085</td>
</tr>
</tbody>
</table>

For values of $\theta_{\text{FOV}}$ so small that $\phi \ll 1$, the right hand side of Eq. (3.50) can be integrated exactly, yielding the following relationship between $\phi$ and $\theta_{\text{FOV}}$:

$$\phi = \theta_{\text{FOV}} \sqrt{\frac{\pi \sigma_T(0)}{2}}$$

(3.51)

The dependence of $\phi$ on $\theta_{\text{FOV}}$ has a profound affect on the transmission characteristic. Figures 3.13 through 3.15 are MFS
Figure 3.12: Graphical Solution of Eq. (3.50); $\sigma_T(0) = 100$
transmission curves which incorporate the FOV dependence of $\Phi$. In Figures 3.13 and 3.14, for $\theta_{\text{FOV}}$ between 1 and 10 mr, the slope of the transmission curve is very close to that of the unscattered light, and the unscattered light dominates. As the FOV is opened, the MFS light dominates and the general behavior observed in the uv experiments is observed. The slope of the transmission curve is approximately $4.3(1 - \Phi_{\omega_0}) \, \text{db/}\tau$, where the $\Phi$ values are those in Table 3.1.

It appears that the curves in all of the figures underestimate the decay rate for $\theta_{\text{FOV}} = 1000$ mr, and in Figure 3.14, the decay rate is also underestimated for $\theta_{\text{FOV}} = 100$ mr. The reason is that the value of $\Phi$ used in these cases is 1.0, corresponding to 100% forward scatter efficiency. However, it is unlikely that $\Phi$ would ever reach this value even if the FOV is opened all the way. As long as there is a significant forward component in the phase function, the maximum value of $\Phi$, $\Phi_{\text{max}}$, will be limited by the phase function and not the FOV. The value of $\Phi$ will saturate at this maximum value if the FOV is opened far enough. The arguments in Section 3.1.3 which placed $\Phi$ between 0.5 and 1.0 now apply to the value of $\Phi_{\text{max}}$.

It should be noted that the changed strategy for choosing $\Phi$ will also have an impact on the shape of the signal vs. FOV characteristic. The extent to which the shape of this characteristic is changed depends upon the value of $\sigma_T(0)$. From Eq. (3.51), it is clear that the larger $\sigma_T(0)$ is, the smaller the value of $\theta_{\text{FOV}}$ at which $\Phi$ approaches $\Phi_{\text{max}}$, and hence saturates. Thus, for $\sigma_T(0) = 4560$ and $\Phi_{\text{max}} = 0.6$ (corresponding to the values in Figure 3.9), the signal vs. FOV curves are those shown
Figure 3.13: MFS Transmission With $\phi$ Dependent on FOV;
$\sigma_T(0) = 100$, $\omega_0 = 1.0$
Figure 3.14: MFS Transmission With $\phi$ Dependent on FOV; 
$a_T(0) = 4560$, $\omega_0 = 1.0$
Figure 3.15: MFS Transmission With $\phi$ Dependent on FOV;
$\sigma_T(0) = 10$, $\omega_o = 1.0$
in Figure 3.16. Observe that only the small FOV portion of the characteristic has undergone any significant change.

3.2.3. Summary of Results From Existing Theories

The comparisons made above between the data and the single scatter, multiple forward scatter and diffusion theories lead to a number of important conclusions. It has been shown that for both single scatter and diffusion, the theories do not predict the observed shape of the signal vs. FOV characteristic, and that for most fields of view the single scattered and diffusion light are insignificant compared to the unscattered light. Furthermore, neither theory exhibits the decreased rate of decay in the transmission curve as the FOV is increased.

The multiple forward scatter theory, on the other hand, exhibits qualitatively most of the features observed, given the appropriate choice for the parameter values and a strategy for choosing $\phi$ based on equating $\theta_E$ and $\theta_{FOV}/2$. In particular, the general shape of the signal vs. FOV characteristic can be reproduced, quite accurately for $\tau = 5$, and the FOV dependence of the transmission curve is predicted.

There are, however, two significant differences between the predictions from the MFS theory and the experimental measurements. The first difference is that the small FOV MFS transmission curves are well below the transmission curve for the unscattered light when the parameters are adjusted so that the slope of the curves matches the measured slope. The second difference is that the increase in the signal vs. FOV curve for the MFS theory is quadratic in the FOV for small FOV, whereas the
Figure 3.16: MFS Signal vs. FOV Characteristic With $\phi$ Dependent on FOV; $\sigma_f(0) = 4560$, $\omega_0 = 1.0$, $\phi_{\text{max}} = 0.6$
measured increased is closer to linear with FOV. These differences indicate that there is another propagation mechanism dominant at the small fields of view. The actual angular spectrum has more energy concentrated in the narrow peak, so that the scattered signal is comparable to, or dominates, the unscattered signal. In addition, the actual angular spectrum, in order to exhibit the linear increase of signal with FOV, must slope off faster than the MFS angular spectrum. It has been suggested [59] that the MFS theory might be modified to embody a faster slope in the angular spectrum by using a phase function which decays faster at small angles. This extension of the MFS theory has yet to be undertaken.

In the next chapter, another approximation to the transport equation is pursued. This approximation, based on the insensitivity property discussed in Chapter 2, is more directly tied to the uv experimental data than are the three theories discussed in the present chapter. Hence, the model developed from this approximation has the potential to exhibit a closer relationship to the observed behavior.
In summarizing the various properties of the angular spectrum in Chapter 2, it was observed that the scattered portion of the received field is much less sensitive to spatial inhomogeneities in the atmosphere than is the unscattered portion of the field. This phenomenon was called the insensitivity property. In the present chapter, this property is introduced in a quantitative way into the transport equation and solutions of the resulting simpler equation are explored.

The procedure is as follows: Since the insensitivity property is a statement about the spatially inhomogeneous character of the received field, it is necessary to start with the complete spatially inhomogeneous transport equation. Furthermore, since this property specifies a relation between the scattered and unscattered portions of the field, it is appropriate to separate these two components. For that purpose, it will be helpful to transform the transport equation into an integral equation. Upon examining the terms in the integral equation for the scattered field, it is shown that one of them behaves in a way which is inconsistent with the insensitivity property, and hence must be negligibly small. The result will be an equation for the scattered field which is identical to the original transport equation except that the source term is not present. The solution to this latter equation is explored for both
isotropic and non-isotropic scatter conditions. The isotropic scatter solution is shown to be identical, for large optical thicknesses, to the complete solution derived by Case [11]. For non-isotropic scatter, an asymptotic form of the solution is studied using the spherical harmonics method [10,13,19].
4.1. **Integral Equation for the Scattered Field**

The general spatially inhomogeneous, steady state, linear transport equation is given by:

\[
[a(\vec{r}) + \vec{n} \cdot \nabla] p(\vec{r}, \vec{n}) = \alpha_s(\vec{r}) \int d\vec{n}' \sigma(\vec{n} \cdot \vec{n}') p(\vec{r}, \vec{n}') + S(\vec{r}, \vec{n}) ,
\]

where \( S(\vec{r}, \vec{n}) \) is the source term. \( S(\vec{r}, \vec{n}) \) is given by

\[
S(\vec{r}, \vec{n}) = \frac{1}{c} p_0(\vec{r}, \vec{n}) ,
\]

where \( p_0(\vec{r}, \vec{n}) \) is the probability density in space and angle for the number of photons emitted per second by sources. \( \alpha(\vec{r}) \) and \( \alpha_s(\vec{r}) \) are the space-dependent extinction and scattering coefficients, respectively. Observe that the angular characteristics of the phase function are assumed to be constant throughout the medium: the inhomogeneity is lumped into the scattering coefficient \( \alpha_s(\vec{r}) \). This assumption is valid as long as the shape of the aerosol particle size distribution does not vary in the medium, i.e. as long as variations in the aerosol composition are solely variations in the overall number of scatterers, and not in their relative proportions.

Integrating Eq. (4.1) with respect to the pathlength variable in the direction of propagation to eliminate the spatial derivative yields

\[
p(\vec{r}, \vec{n}) = \int_0^\infty dv \alpha_s(\vec{r} - \vec{v} \vec{n}) \int d\vec{n}' \sigma(\vec{n} \cdot \vec{n}') p(\vec{r} - \vec{v} \vec{n}, \vec{n}') \exp[- \int_0^v \alpha(\vec{r} - \vec{v}' \vec{n}) dv'] + \frac{1}{c} \int_0^\infty dv p_0(\vec{r} - \vec{v} \vec{n}, \vec{n}) \exp[- \int_0^v \alpha(\vec{r} - \vec{v}' \vec{n}) dv'] .
\]

Observe that the second term in Eq. (4.3) represents the unextinguished light from the source. Thus, if \( p(\vec{r}, \vec{n}) \) is separated as

\[
p(\vec{r}, \vec{n}) = p_u(\vec{r}, \vec{n}) + p_s(\vec{r}, \vec{n}) ,
\]
where \( p_u(\vec{r}, \Omega) \) and \( p_s(\vec{r}, \Omega) \) are the unextinguished and scattered contributions respectively, the equation for the scattered contribution is

\[
p_s(\vec{r}, \Omega) = \int_0^\infty dv \, \alpha_s(\vec{r}-\vec{v\Omega}) \int d\Omega' \, \sigma(\Omega' \Omega)[p_s(\vec{r}-\vec{v\Omega}, \Omega')+p_u(\vec{r}-\vec{v\Omega}, \Omega')] \\
\cdot \exp \left[ -\int_0^\nu \alpha(\vec{r} - \nu' \Omega') d\nu' \right]
\]

(4.5)

with

\[
p_u(\vec{r}, \Omega) = \frac{1}{c} \int_0^\infty dv \, p_o(\vec{r}-\vec{v\Omega}, \Omega) \exp \left[ -\int_0^\nu \alpha(r-v' \Omega') d\nu' \right]
\]

(4.6)

Consider an isotropic point source

\[
p_o(\vec{r}, \Omega) = \frac{\delta(\vec{r})}{4\pi}.
\]

(4.7)

Substituting Eq. (4.7) into Eq. (4.6) and Eq. (4.6) into Eq. (4.5) gives the following integral equation for the scattered field:

\[
p_s(\vec{r}, \Omega) = \int_0^\infty dv \, \alpha_s(\vec{r}-\vec{v\Omega}) \int d\Omega' \, \sigma(\Omega' \Omega)p_s(\vec{r}-\vec{v\Omega}, \Omega')\exp \left[ -\int_0^\nu \alpha(\vec{r}-\nu' \Omega') d\nu' \right] \\
+ \frac{1}{4\pi c} \int_0^\infty dv \, \alpha_s(\vec{r}-\vec{v\Omega}) \frac{\sigma(\Omega + \vec{r}-\vec{v\Omega})}{|\vec{r}-\vec{v\Omega}|^2} \exp \left[ -\int_0^\nu \alpha(\vec{r}-\nu' \Omega') \frac{\vec{r}-\vec{v\Omega}}{|\vec{r}-\vec{v\Omega}|} d\nu' \right] \\
\cdot \exp \left[ -\int_0^\nu \alpha(\vec{r}-\nu'' \Omega') d\nu'' \right].
\]

(4.8)
4.2. **Character of the Single-Scatter Term**

The first term in Eq. (4.8) represents multiply-scattered light, i.e. light which had already been scattered at least once before being scattered into the direction $\overline{n}$. The second term in Eq. (4.8) represents single-scattered light. In this section, it is shown that, under some weak restrictions on the phase function, the single scatter term is negligibly small compared to the multiple scatter term when the insensitivity property holds.

To see this, consider the geometry depicted in Fig. 4.1, in which $\theta$, the receiver angle from the line of sight, is restricted to be non-zero but much less than 90°. This restriction will make it possible to easily evaluate the second term in Eq. (4.8). Referring to

![Figure 4.1: Small Receiver Angle Geometry](image)

The figure illustrates the geometry with the source, receiver, and various angles and distances involved.
Figure 4.1, observe that as the variable of integration, \( v \), increases from 0 to \( \infty \) in the single scatter term, there are 3 distinct regions:

a. \( v < |\vec{r}| = L \)

Thus

\[
|\vec{r} - v\vec{\Omega}| \approx L - v
\]

(4.9a)

and

\[
\vec{\Omega} \cdot \frac{\vec{r} - v\vec{\Omega}}{|\vec{r} - v\vec{\Omega}|} \approx \frac{1 - v/L}{\sqrt{\theta^2 + \left(1 - \frac{v}{L}\right)^2}}
\]

(4.9b)

b. \( v > |\vec{r}| = L \)

Thus

\[
|\vec{r} - v\vec{\Omega}| \approx v - L
\]

(4.10a)

\[
\vec{\Omega} \cdot \frac{\vec{r} - v\vec{\Omega}}{|\vec{r} - v\vec{\Omega}|} \approx -1
\]

(4.10b)

c. \( v = L \)

Thus

\[
|\vec{r} - v\vec{\Omega}| \approx L\theta
\]

(4.11a)

\[
\vec{\Omega} \cdot \frac{\vec{r} - v\vec{\Omega}}{|\vec{r} - v\vec{\Omega}|} \approx 0
\]

(4.11b)

Dividing the single scatter term up into 3 parts, corresponding to these 3 regions,
Here, $\varepsilon$ is small compared to $L$.

\begin{equation}
SS \approx \frac{1}{4\pi c} \int_0^{L-\varepsilon} dv \ \alpha_s((L-v)\bar{\alpha}) \frac{\sigma \left[ \frac{1 - v/L}{\sqrt{\theta^2 + \left(1 - v\right)^2}} \right]}{(L-v)^2} \exp \left[ -\int_0^{L-v} \alpha(\bar{r}-[v+v']\bar{\alpha}) dv' \right] \bigg( \int_0^{\varepsilon+\varepsilon} dv \ \alpha_s(\bar{r}=0) \alpha(0) \right)
\end{equation}

\begin{equation}
\cdot \exp \left[ -\int_0^{L} \alpha(\bar{r}-v''\bar{\alpha}) dv'' \right] + \int_{L-\varepsilon}^{L+\varepsilon} dv \ \alpha_s(\bar{r}=0) \alpha(0) \sigma(0) \int_0^L \exp \left[ -\int_0^{L-v} \alpha(\bar{r}-[v+v']\bar{\alpha}) dv' \right] \bigg) + \int_{L+\varepsilon}^{\infty} dv \ \alpha_s((L-v)\bar{\alpha}) \frac{\sigma(-1)}{(v-L)^2} \exp \left[ -\int_0^{L-v} \alpha(\bar{r}-[v+v']\bar{\alpha}) dv' \right] \bigg) \bigg) \bigg) \bigg)
\end{equation}

\begin{equation}
\text{(4.12)}
\end{equation}

Now observe that because of the sharply forward peaked character of the
phase function, \( \sigma(0) \) and \( \sigma(-1) \) are much smaller than \( \sigma(\mu) \) in the vicinity of \( \mu = 1 \). Hence the 2\textsuperscript{nd} and 3\textsuperscript{rd} terms in SS are very small compared to the first, and

\[
SS \approx \frac{\exp \left[ - \int_{0}^{L} \alpha(\mathbf{r} - \mathbf{v} \cdot \hat{\mathbf{n}}) d\mathbf{v} \right]}{4\pi c} \left[ \int_{0}^{L-\varepsilon} d\mathbf{v} \frac{\alpha_{s}[(L-\mathbf{v}) \cdot \hat{n}]}{(L-\mathbf{v})^2} \sigma \left[ \frac{1 - \mathbf{v}/L}{\sqrt{\theta^2 + (1 - \mathbf{v}/L)^2}} \right] \right] \tag{4.14}
\]

This term is virtually identical to the unscattered term in its dependence upon the spatially varying extinction coefficient, since the exponential term will dominate the linear dependence in the integrand. Thus, referring back to Eq. (4.8), the second term on the right hand side has a strong dependence on \( \alpha(\mathbf{r}) \), whereas the left hand side has only a very slight dependence on \( \alpha(\mathbf{r}) \). Ruling out any unusual kind of cancellation between the first and second terms on the right hand side, this implies that the single scatter term must be small.

It is a simple matter to extend this argument to large angles \( \theta \). Refer to Figure 4.2. Note that \( \theta_s \), the single scatter angle is the angle whose cosine is the argument of the phase function in the single scatter term in Eq. (4.8):

\[
\cos \theta_s = \frac{\mathbf{n} \cdot (\mathbf{r} - \mathbf{v} \mathbf{n})}{|\mathbf{r} - \mathbf{v} \mathbf{n}|} \tag{4.15}
\]
However, because of the narrowness of the phase function, only values of the integrand at small values of \( \nu (\nu < \nu_{\text{max}}) \) contribute significantly to the integral. The angle \( \theta_{s_{\text{max}}} \) in Figure 4.2 is roughly the width of the phase function.

For these small values of \( \nu \), the single scatter term is identical to the first term in Eq. (4.13), and thus have the same exponential dependence on the extinction coefficient. Therefore, the argument for neglecting it applies in this case as well.

As further evidence of the validity of this approximation,
consider the comparison in Figure 4.3. The figure shows normalized scattered power vs. receiver FOV as predicted by single scatter theory (for various values of phase function average cosine, g) and as actually measured. The optical thickness is 5. Observe that for even the least favorable single scatter curve (g = 0.95), most of the data points are at least a factor of 3 larger than the single scattered light. For the most favorable curve (g = 0.7), the difference is nearly an order of magnitude. Hence, the data is in accord with the assumption that the single scattered portion can be neglected.

The approximation introduced here is a consequence of the insensitivity property, and hence has been referred to as the "insensitivity" approximation [63]. However, a more appropriate name is the "strong multiple scatter" approximation, because it more accurately reflects the exact way in which the insensitivity property is used in simplifying the transport equation. It gives a better insight into which physical situations could be dealt with by the simplified equations: namely those situations in which the optical thickness is large enough, and therefore multiple scattering strong enough, so that single scattered light is a negligible part of the total scattered field. Henceforth, it will be denoted the "strong multiple scatter" approximation in this thesis.

Once the strong multiple scatter approximation has been made, Eq. (4.8) consists of two terms which depend mildly, if at all, on the spatial fluctuations in the extinction and scattering coefficients. Therefore, in addition to eliminating the last term, the equation can be simplified further by replacing the space-varying functions $\alpha(\mathbf{r})$ and $\alpha_s(\mathbf{r})$ by their space-averaged values, denoted by $\alpha$ and $\alpha_s$, respectively.
Data

\[ g = .95 \]
\[ g = .85 \]
\[ g = .7 \]

Single Scatter

Full Angle FOV (mr)
Equation (4.8) therefore reduces to

$$P_s(r,\Omega) = a_s \int_0^\infty d\nu \int d\Omega' \sigma(\Omega,\Omega') p_s(r',\Omega',\Omega') \exp(-a\nu). \tag{4.16}$$

Comparing Eq. (4.16) with Eq. (4.3), it is evident that Eq. (4.16) is identical to the complete transport equation except that it does not possess the source term. As discussed in succeeding sections of this chapter, this decoupling of the solution from the source gives one the freedom to apply the boundary condition anywhere in the medium, rather than necessarily at the source. On the other hand, the strong multiple scatter approximation still leaves the transport equation in quite a general form. In fact, it will be necessary to make further approximations below before Eq. (4.16) can be solved for anisotropic scatter.

Although the argument for the strong multiple scatter approximation presented in this section depended on assuming a phase function that was sharply forward peaked, this requirement can actually be relaxed. This is shown in the next section, in which the approximation is shown to be valid for large optical thicknesses even if the phase function is isotropic.
4.3. Isotropic Scatter

It was argued above that the strong multiple scatter approximation applies at sufficiently large optical thicknesses. Therefore, if the solution to the transport equation were available, it should be possible to show that it reduces to the solution of Eq. (4.16) at large optical thicknesses. A general analytical solution to the time independent, point source transport equation is available only in the case of isotropic scatter [11]:

\[ \sigma (\vec{n} \cdot \vec{n}') = \frac{1}{4\pi} \quad (4.17) \]

It will be shown in this section that in this isotropic scatter case, the solution to the transport equation is also a solution to Eq. (4.16) in the limit of large optical thickness.

More specifically, the proof will be in terms of the scattered photon density

\[ p_s (\vec{r}) = \int d\Omega \; p_s (\vec{r}, \vec{\Omega}) \quad (4.18) \]

It is shown in Section 12 of reference [11] that for isotropic scatter from a point source this is equivalent to a general proof in terms of \( p_s (\vec{r}, \vec{\Omega}) \) because \( p_s (\vec{r}, \vec{\Omega}) \) can be written as a function of \( p_s (\vec{r}) \). (It is possible to extend this equivalence to anisotropic scatter situations under some conditions [64].)

Substituting Eq. (4.17) into Eq. (4.16) and performing the
integration indicated in Eq. (4.18) yields

\[ p_s(\mathbf{r}) = \frac{\alpha_s}{4\pi} \int d\mathbf{\rho}' \quad p_s(\mathbf{r}-\mathbf{\rho}') \ e^{-\alpha|\mathbf{\rho}'|} \quad |\mathbf{\rho}'|^2 \quad , \quad (4.19) \]

The integration in Eq. (4.19) is a volume integration over all space.

Making the change of variables

\[ \mathbf{\bar{\rho}} = \alpha \mathbf{\bar{\rho}}' \quad (4.20) \]

and considering \( p_s(\mathbf{r}) \) in mean free path units

\[ p_s'(\tau = \alpha \mathbf{r}) \equiv p_s(\mathbf{r}) \quad , \quad (4.21) \]

Equation (4.19) becomes

\[ p_s'(\tau) = \frac{\omega_0}{4\pi} \int d\mathbf{\rho} \quad p_s'(\mathbf{\tau}+\mathbf{\rho}) \ e^{-\rho} \quad \rho^2 \quad (4.22) \]

With the spherical symmetry introduced by a point source, Eq. (4.22) becomes

\[ p_s'(\tau) = \frac{\omega_0}{4\pi} \int d\mathbf{\rho} \quad p_s'(|\mathbf{\tau}+\mathbf{\rho}|) \ e^{-\rho} \quad \rho^2 \quad , \quad (4.23) \]

where \( \tau \) is the optical range from the source. Here, \( \omega_0 \) is the albedo in the medium, and
In Section 14 of reference [11], it is shown that the general point source, isotropic scatter solution is of the form:

\[ p(T) = k e^{\omega_0 \tau} + \int_0^1 g(\mu) e^{\frac{-\tau}{\mu^2}} d\mu, \]  

(4.25)

where \( g(\mu) \) is a known function, \( k \) is a known constant, and \( \omega_0 \) is the solution of

\[ \omega_0 = \frac{\kappa_0}{\tanh^{-1}\kappa_0}. \]  

(4.26)

Values of \( \kappa_0 \) which satisfy Eq. (4.26) are between 0 and 1. Asymptotically, therefore, the first term in Eq. (4.25) dominates, since \( \mu \) in the integrand is also restricted to the range 0 - 1. Thus the large optical thickness solution to the general transport equation is

\[ p_s'(\tau) = \text{const} \times e^{\frac{-\kappa_0 \tau}{\tau}}. \]  

(4.27)

It must now be shown that if \( p_s'(\tau) \) of this form is substituted into the right hand side of Eq. (4.23), the left hand side will be of the same form. Substituting

\[ \rho \equiv \left| \frac{\rho}{p} \right|. \]  

(4.24)
\[ p'_s(\tau) = \frac{-\kappa_0 \tau}{\tau} \quad (4.28) \]

into the right hand side of Eq. (4.23),

\[
\text{RHS} = \frac{\omega_0}{2} \int_0^\infty d\rho \rho^2 \left[ \frac{1}{\sqrt{\tau^2 + \rho^2 - 2\tau \rho \mu}} \right] e^{-\rho} \int_0^1 d\mu \frac{p_s}{\rho^2} \left( \sqrt{\tau^2 + \rho^2 - 2\tau \rho \mu} \right) e^{-\rho} \]

\[
= \frac{\omega_0}{2} \int_0^\infty d\rho e^{-\rho} \left[ \frac{1}{\sqrt{\tau^2 + \rho^2 - 2\tau \rho \mu}} \right] e^{-\kappa_0 \sqrt{\tau^2 + \rho^2 - 2\tau \rho \mu}} \]

\[
= \frac{\omega_0}{2\kappa_0 \tau} \int_0^\infty d\rho \frac{e^{-\rho}}{\rho} \left[ e^{-\kappa_0 |\tau-\rho|} - e^{-\kappa_0 (\tau+\rho)} \right] \quad (4.29)
\]

Dividing up the interval of integration into 2 parts,

\[
\text{RHS} = \frac{\omega_0}{2\kappa_0 \tau} \left[ e^{-\kappa_0 \tau} \int_0^\tau d\rho \frac{e^{-\rho}}{\rho} \left( e^{\kappa_0 \rho} - e^{-\kappa_0 \rho} \right) + \int_0^\infty \left( e^{\kappa_0 \tau} - e^{-\kappa_0 \tau} \right) d\rho \frac{e^{-\rho (1+\kappa_0)}}{\rho} \right] \quad (4.30)
\]

It is easy to show - by expanding out the exponentials, performing the integrations and summing the series - that in the limit of large \( \tau \),

\[
\int_0^\tau d\rho \frac{e^{-\rho}}{\rho} \left( e^{\kappa_0 \rho} - e^{-\kappa_0 \rho} \right) = 2 \tanh^{-1} \kappa_0 . \quad (4.31)
\]

Hence, using Eq. (4.26), and substituting this result into Eq. (4.30),
\[
\text{RHS} \to \frac{e^{-\kappa_0 \tau}}{\tau} + \frac{\omega_0 e^{-\tau}}{2 \kappa_0 \tau} \int_0^\infty \frac{-\rho e^{(1+\kappa_0)}}{\rho} \, dp. \tag{4.32}
\]

Now the integral in Eq. (4.32) can be done by parts:

\[
\int_0^\infty \frac{-\rho e^{(1+\kappa_0)}}{\rho} \, dp = \frac{-\tau (1+\kappa_0)}{\tau (1+\kappa_0)} + \int_0^\infty \frac{dp e^{-\rho}}{\rho^2} \tag{4.33}
\]

The limit of the ratio of the two terms in Eq. (4.33), is given by

\[
\lim_{\tau \to \infty} \frac{-\tau (1+\kappa_0)}{\tau (1+\kappa_0)} = \lim_{\tau \to \infty} \frac{e^{-\tau (1+\kappa_0)}}{\tau (1+\kappa_0)} = \frac{1}{\tau} \tag{4.34}
\]

Therefore, the first term dominates, and

\[
\text{RHS} \to \frac{e^{-\kappa_0 \tau}}{\tau} + \frac{\omega_0 e^{-\tau}}{2 \kappa_0 \tau^2 (1+\kappa_0)} \tag{4.35}
\]

Since \( \kappa_0 < 1 \),

\[
\text{RHS} \to \frac{e^{-\kappa_0 \tau}}{\tau} \tag{4.36}
\]

as \( \tau \) becomes large, which was to be proven.

It has thus been shown that for the isotropic scatter case at least, the general solution to the point source transport equation is a solution to the simplified transport equation obtained via the strong multiple scatter approximation. This is proof that the requirement
of a peaked phase function in the previous section can be relaxed, and it provides further evidence that the critical parameter in determining the applicability of the strong multiple scatter approximation is the optical thickness.
4.4. **Anisotropic Scatter**

In the case of anisotropic scatter, no analytical solution to the general transport equation exists. This is equally true under the strong multiple scatter approximation. However, as is shown in this section, the simplifications introduced by this approximation afford significant reductions in numerical computation requirements, and provide relatively well-structured asymptotic solutions.

To better exploit the spherical symmetry of the point source problem, the coordinates \( \tau \) and \( \mu \), shown in Figure 4.4, are used in the present discussion. (It is worth noting that the general derivation of the strong multiple scatter approximation could not have been carried out in these coordinates, because the spatially random extinction and scattering coefficients - destroy the spherical symmetry.) Also, it will be more appropriate to go back to the integro-differential form of the transport equation for this analysis. In those coordinates,

\[
\tau = \alpha r \\
\mu = \cos \theta
\]

![Figure 4.4: Spherical Geometry Coordinates](image)

**Figure 4.4: Spherical Geometry Coordinates**
the spatially homogeneous transport equation at points away from the source becomes [65]

\[ \mu \frac{\partial p(\tau,\mu)}{\partial \tau} + \frac{1-\mu^2}{\tau} \frac{\partial p(\tau,\mu)}{\partial \mu} + p(\tau,\mu) = \omega_0 \int d\Omega' \sigma(\Omega' \Omega) p(\tau,\mu') \]  

(4.37)

As in Section 4.2, \( p(\tau,\mu) \) is divided into an unextinguished portion, \( p_u(\tau,\mu) \), satisfying

\[ \mu \frac{\partial p_u(\tau,\mu)}{\partial \tau} + \frac{1-\mu^2}{\tau} \frac{\partial p_u(\tau,\mu)}{\partial \mu} + p_u(\tau,\mu) = 0 \]  

(4.38)

and a scattered portion, \( p_s(\tau,\mu) \), satisfying

\[ \mu \frac{\partial p_s(\tau,\mu)}{\partial \tau} + \frac{1-\mu^2}{\tau} \frac{\partial p_s(\tau,\mu)}{\partial \mu} + p_s(\tau,\mu) = \omega_0 \int d\Omega' \sigma(\Omega' \Omega) p_s(\tau,\mu') + \omega_0 \int d\Omega' \sigma(\Omega' \Omega) p_u(\tau,\mu'). \]  

(4.39)

The strong multiple scatter assumption says that the last term in Eq. (4.39) is zero. Thus, \( p_s(\tau,\mu) \) satisfies

\[ \mu \frac{\partial p_s(\tau,\mu)}{\partial \tau} + \frac{1-\mu^2}{\tau} \frac{\partial p_s(\tau,\mu)}{\partial \mu} + p_s(\tau,\mu) = \omega_0 \int d\Omega' \sigma(\Omega' \Omega) p_s(\tau,\mu'). \]  

(4.40)

Observe, again, that the assumption has decoupled the scattered portion from the unscattered portion, and hence from the source.

The procedure used here to solve Eq. (4.40) is the spherical harmonics method. (See Section 1.1.) For the spherically symmetric
problem, the method reduces to expanding $p_s(\tau,\mu)$ in a series of Legendre polynomials,

$$p_s(\tau,\mu) = \sum_{\ell=0}^{\infty} \frac{2^{\ell+1}}{4\pi} \psi_{\ell}(\tau) P_{\ell}(\mu). \quad (4.41)$$

The scattering function is expanded similarly:

$$\sigma(\Omega' \Omega) = \sum_{\ell=0}^{\infty} \frac{2^{\ell+1}}{4\pi} f_{\ell} P_{\ell}(\Omega' \Omega'). \quad (4.42)$$

Using the orthogonality property of the Legendre polynomials [21]:

$$\int_{-1}^{1} d\mu \; P_{\ell}(\mu) P_{j}(\mu) = \frac{2}{2j+1} \delta_{\ell,j},$$

the coefficients $\psi_{j}(\tau)$ can be written in terms of the function $p_s(\tau,\mu)$ as

$$\psi_{j}(\tau) = 2\pi \int_{-1}^{1} d\mu \; p_s(\tau,\mu) P_{j}(\mu).$$

Substituting Eq. (4.42) and Eq. (4.41) into Eq. (4.40), multiplying by $P_{j}(\mu)$ and integrating over all $\mu$ yields a coupled infinite set of differential equations for the coefficients $\psi_{j}(\tau)$:

$$\frac{j}{2j+1} \frac{d}{d\tau} - \frac{j-1}{\tau} \psi_{j-1}(\tau) + \frac{j+1}{2j+1} \frac{d}{d\tau} + \frac{j+2}{\tau} \psi_{j+1}(\tau) + \epsilon_j \psi_{j}(\tau) = 0,$$

$$j = 0, 1, 2, \ldots \quad (4.43)$$
where

$$\epsilon_j = (1 - \omega_0 f_j)$$  \hspace{1cm} (4.44)

The recursion relations [21]

$$\left(\mu^2 - 1\right) \frac{dP_j(\mu)}{d\mu} = j[\mu P_j(\mu) - P_{j+1}(\mu)]$$  \hspace{1cm} (4.45)

and

$$\mu P_j(\mu) = \frac{1}{2^{j+1}} [(j+1) P_{j+1}(\mu) + \mu P_{j-1}(\mu)]$$  \hspace{1cm} (4.46)

have been used in obtaining Eq. (4.43) from Eq. (4.40).

In any actual computation, of course, only a finite number of terms in Eq. (4.41) are used. Thus, the infinite set of equations designated in Eq. (4.43) are truncated after N equations ($j_{\text{max}} = M = N-1$). If M is chosen to be odd, this is called an odd-order expansion, and an even number of equations are retained. If M is chosen to be even, this is called an even-order expansion, and an odd number of equations are retained. It can be shown [66] that even-order expansions introduce the need for additional assumptions over those required for odd-order expansions when boundary conditions are applied, and that any odd-order expansion is more accurate than the next succeeding even-order expansion. Hence, only odd-order expansions will be considered here.

The N equations corresponding to the truncated set can be
written compactly in vector notation. Defining

\[
\Psi(\tau) \equiv \begin{bmatrix}
\psi_0(\tau) \\
\psi_1(\tau) \\
\vdots \\
\psi_{N-1}(\tau)
\end{bmatrix},
\]

Equation (4.43) becomes

\[
\mathbf{A}\Psi'(\tau) = \left[\mathbf{B} + \frac{\mathbf{C}}{\tau}\right]\Psi(\tau)
\]

where

\[
\mathbf{A} = \begin{bmatrix}
0 & 1 & & & \\
1 & 0 & 2 & & \\
& 1 & 0 & 3 & \\
& & & \ddots & \\
& & & & 0 & M-1 \\
& & & & M-1 & 0 & M \\
& & & & M & 0 & \end{bmatrix},
\]

\[
N,
\] (4.47a)
\[ \beta_j = -(2j+1) \epsilon_j = -(2j+1)(1 - \omega_0 f_j), \quad (4.47c) \]

and

\[ \Sigma = \begin{bmatrix} 0 & -2.1 & \ldots & 0 \\ 0 & 0 & -3.2 & \ldots & 0 \\ 0 & 0 & 2.1 & 0 & -4.3 \\ 0 & 0 & 3.2 & 0 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & 4.3 & 0 & \ddots \\ 0 & 0 & \ddots & \ddots & \ddots \\ 0 & 0 & \ddots & \ddots & \ddots \\ 0 & 0 & \ddots & \ddots & \ddots \\ M(M-1) & 0 & \ldots & \ldots & 0 \\ M(M-1) & 0 & \ldots & \ldots & \ddots \\ \end{bmatrix} \quad (4.47d) \]
It is shown in Appendix A that the matrix $A$ is invertible. Hence Eq. (4.46) can be rewritten

$$\Psi'(\tau) = \left[ D + \frac{E}{\tau} \right] \Psi(\tau).$$  \hspace{1cm} (4.48)

where

$$D = A^{-1} \mathbb{B}$$

$$E = A^{-1} \mathbb{C}.$$  \hspace{1cm} (4.49)

$\Psi(\tau)$ must satisfy the boundary conditions

$$\Psi(\tau_0) = \Psi_{BC}$$

$$\Psi(\infty) = 0$$  \hspace{1cm} (4.50)

At this point it is worth pausing again to note the relationship between Eq. (4.48) and the corresponding equation for the complete transport equation. Since Eq. (4.40) differs from the complete transport equation only in the lack of the point source term (implied in Eq. (4.37)), the vector differential equation in Eq. (4.46) differs from that for the full transport equation by a constant vector whose only non-zero element is its first. In Eq. (4.48), this constant vector is the first column of $A^{-1}$. As discussed above, therefore, the solution of Eq. (4.48) is in fact quite general. The solution explored below introduces additional
approximations, however, and hence is not as general.

The properties of the matrices \( A, D \) and \( E \) are presented in Appendix A. Note that since \( A \) is simply a matrix of numbers, the matrix \( E \) does not depend on the channel parameters. \( D \), on the other hand, depends through \( B \) on both the albedo, \( \omega_0 \), and the Legendre coefficients, \( f_j \), of the single scatter phase function. Hence \( D \) is appropriately called the channel matrix. It figures heavily in the asymptotic solution of Eq. (4.48) to be considered in the next section.

It is a simple matter to show that \( D \) and \( E \) do not commute. Because of this, no closed form solution to Eq. (4.48) exists. Observe, however, that the second term on the right hand side of Eq. (4.48) is inversely proportional to \( \tau \), and hence suggests a perturbation series solution of

\[
\overline{\psi}'(\tau) = \left[ D + \frac{\varepsilon E}{\tau} \right] \overline{\psi}(\tau) \tag{4.51}
\]

which has the form

\[
\overline{\psi}(\tau) = \sum_{n=0}^{\infty} \overline{\psi}_n(\tau) \varepsilon^n . \tag{4.52}
\]

Substituting Eq. (4.52) into Eq. (4.51) and equating like powers of \( \varepsilon \) yields the following set of differential equations for the \( \overline{\psi}_n(\tau) \):

\[
\overline{\psi}_0'(\tau) = D \overline{\psi}_0(\tau) \tag{4.53}
\]

\[
\overline{\psi}_n'(\tau) = D \overline{\psi}_n(\tau) + \frac{E}{\tau} \overline{\psi}_{n-1}(\tau) , \ n \geq 1
\]
Equations (4.53) must satisfy the boundary conditions

\[ \psi_0(\tau_0) = \psi_{BC} \]

\[ \psi_0(\infty) = 0 \]  \hspace{1cm} (4.54)

\[ \psi_n(\tau_0) = \psi_n(\infty) = 0, \quad n \geq 1 \]

The solution to the set of vector differential equations (4.53), with the boundary conditions Eqs. (4.54), can be generated sequentially as

\[ \psi_0(\tau) = \exp[D(\tau - \tau_0)] \psi_{BC} \]  \hspace{1cm} (4.55)

\[ \psi_n(\tau) = \int_{\tau_0}^{\tau} \frac{d\rho}{\rho} \exp[D(\tau - \rho)] \psi_{n-1}(\rho), \quad n \geq 1 \]

While Eqs. (4.55) in principle embody a complete solution to Eqs. (4.53), it becomes numerically quite tedious to compute even the first order term

\[ \psi_1(\tau) = \int_{\tau_0}^{\tau} \frac{d\rho}{\rho} \exp[D(\tau - \rho)] \exp[D(\rho - \tau_0)] \psi_0. \]  \hspace{1cm} (4.56)

Therefore, the discussion in this chapter concentrates exclusively on the asymptotic solution \( \psi_0(\tau) \). Extensions to the non-asymptotic case are discussed briefly in the next chapter.
4.5. **Asymptotic Solution for Anisotropic Scatter**

This section considers the solution for $\overline{\psi}(\tau)$ in the limit of $\tau$ large enough so that the asymptotic solution $\overline{\psi}_o(\tau)$ is valid. Written in terms of the matrix $R = -D^{-1}$ (see Eq. (A.13), Appendix A), this solution is

$$\overline{\psi}(\tau) = S \left[ \begin{array}{c} \frac{(\tau - \tau_o)}{\lambda_1} e^{-\frac{(\tau - \tau_o)}{\lambda_1}} \\ \vdots \\ \frac{(\tau - \tau_o)}{\lambda_N} e^{-\frac{(\tau - \tau_o)}{\lambda_N}} \end{array} \right] \overline{\psi}_{BC} , \quad (4.57)$$

Here $\lambda_i, 1 \leq i \leq N$, are the eigenvalues of $R$

$$S = \left[ \begin{array}{cccc} \overline{x}_1 & \overline{x}_2 & \cdots & \overline{x}_N \end{array} \right] \quad (4.58)$$

is the matrix whose columns, $\overline{x}_i$, are the right eigenvectors of $R$, and

$$Z = \left[ \begin{array}{c} Z_1^T \\ Z_2^T \\ \vdots \\ Z_N^T \end{array} \right] \quad (4.59)$$

is the matrix whose rows, $Z_i^T$, are the left eigenvectors of $R$, scaled so that

$$SZ = I , \quad (4.60)$$
where \( I \) is the identity matrix. The existence of \( S^{-1} = Z \), and hence the validity of the diagonal form written in Eq. (4.57), is guaranteed by the algorithms discussed in Appendix A. (See Sections A.2.1 and A.4.)

Writing out the matrix multiplications in Eq. (4.57) in detail, and substituting into Eq. (4.41), yields

\[
P_S(\tau, \mu) = \sum_{j=1}^{N} \sum_{\lambda=1}^{N} \sum_{m=1}^{N} \left( \frac{2\lambda-1}{4\pi} \right) P_{\lambda-1}(\mu) S_{\lambda j} Z_{jm}(\psi_{BC}) e^{-\frac{(\tau-\tau_0)}{\lambda_j}}
\]

(4.61)

where

\[
S_{\lambda j} = [S]_{\lambda j}
\]

(4.62)

\[
Z_{jm} = [Z]_{jm}
\]

This solution is explored in detail below. As a prelude to this, the application of boundary conditions and the validity range for Eq. (4.61) are considered.

4.5.1. Application of Boundary Conditions

The boundary condition vector, \( \psi_{BC} \), is determined from two conditions on \( p_s(\tau, \mu) \):

\[
p_s(\tau_0, \mu) = p_s(\mu)
\]

(4.63)

\[
p_s(\infty, \mu) = 0
\]
where \( p_{so}(\mu) \) is a known (measured or otherwise derived) angular spectrum at \( \tau = \tau_0 \). Note that since the condition at infinity must hold for all \( \mu \), it is equivalent to the condition at infinity on \( \bar{\psi} \).

Consider first the condition at \( \tau_0 \). Since \( S \) and \( Z \) are inverses,

\[
p_{so}(\mu) = \sum_{\ell=1}^{N} \left( \frac{2 \ell-1}{4 \pi} \right) p_{\ell-1}(\mu) (\psi_{BC})_{\ell-1} = \sum_{k=0}^{N-1} \left( \frac{2k+1}{4 \pi} \right) p_{k}(\mu) (\psi_{BC})_{k}.
\]

Thus, to meet the conditions at \( \tau_0 \) the elements of \( \bar{\psi}_{BC} \) must be the first \( N \) Legendre coefficients of \( p_{s}(\tau_0, \mu) \), given by

\[
(\psi_{BC})_{k} = 2\pi \int_{-1}^{1} d\mu \ p_{so}(\mu) \ p_{k}(\mu), \quad 1 \leq k \leq N
\]

To meet the boundary condition at \( \tau = \infty \), all terms in Eq. (4.61) which include negative eigenvalues, \( \lambda_{j} \), must be zero. It is shown in Appendix A that the eigenvalues of \( R \) are real, distinct, and symmetric about zero. Thus, there are \( N/2 \) negative eigenvalues. Numbering these eigenvalues in order of increasing value, starting with the most negative one, the boundary condition at infinity requires

\[
\sum_{\ell=1}^{N} \sum_{m=1}^{N} \left( \frac{2 \ell-1}{4 \pi} \right) p_{\ell-1}(\mu) S_{\ell j} Z_{jm}(\psi_{BC})_{m-1} = 0
\]

Since Eq. (4.66) must be valid for all angles \( \mu \), it reduces to
\[ S_{\lambda j} \sum_{m=1}^{N} Z_{jm}(\overline{\psi}_{BC})_{m-1} = 0, \quad \forall j, j \leq N/2, \forall \lambda \] (4.67)

Now, since there is no value of \( j \) for which \( S_{\lambda j} = 0 \) \( \forall \lambda \) (otherwise, one of the columns of \( S \) would be all zeros, which would mean it was not invertible), Eq. (4.67) becomes

\[ \sum_{m=1}^{N} Z_{jm}(\overline{\psi}_{BC})_{m-1} = 0, \quad \forall j, j \leq N/2 \] (4.68)

Equations (4.68) are \( N/2 \) equations for the \( N \) coefficients of \( \overline{\psi}_{BC} \), and Eqs. (4.65) consist of \( N \) equations for these coefficients, so the physical problem appears to be overdetermined. Note, however, that if \( N \) is large enough, the first \( N/2 \) Legendre coefficients will give a very good representation of \( p_{\mu}(\mu) \), and the increase in accuracy gained (or lost) by adding (or leaving out) the additional \( N/2 \) coefficients is negligible. Thus, one possible way to apply the boundary conditions is to set the first \( N/2 \) elements of \( \overline{\psi}_{BC} \) in accordance with Eqs. (4.65), and then solve Eqs. (4.68) for the remaining \( N/2 \) coefficients.

An alternative, but essentially equivalent, procedure is to compute all \( N \) elements of \( \overline{\psi}_{BC} \) by means of Eqs. (4.65), and then simply set to zero each term in the series in Eq. (4.61) which corresponds to a negative eigenvalue. This latter procedure is actually superior from a computational point of view, because the finite accuracy associated with solving the system in Eq. (4.68) ultimately leads to negative eigenvalue terms being admitted into the series, with a concommitant...
4.5.2. Validity Range of Asymptotic Solution

This section discusses the value of the optical thickness, \( \tau_{as} \), at which the asymptotic solution becomes valid. The most straightforward way to determine this crossover value of \( \tau \) is to compare the first and second terms on the right hand side of the differential equation for \( \bar{\psi} \) (Eq. (4.48)) and find the smallest value of \( \tau \) for which

\[
\left| \left| D \bar{\psi} \right| \right| >> \frac{\left| E \bar{\psi} \right|}{\tau} \tag{4.69}
\]

However, without a solution for \( \bar{\psi} \), evaluating the inequality in Eq. (4.69) is impossible. The procedure here will be to provide upper and lower bounds for the onset of validity.

The upper bound can be obtained by relating the asymptotic strong multiple scatter solution to diffusion theory. In Chapter 3, it was argued that diffusion theory is expected to be close to the exact transport equation solution when \( \tau > \tau_c \). (See Figure 3.3.) Thus, if the asymptotic solution in the strong multiple scatter approximation approaches the diffusion solution, and if the value of \( \tau \) at which this occurs, call it \( \tau_{upper} \), is greater than \( \tau_c \), then the asymptotic solution will be valid for all \( \tau \) greater than \( \tau_{upper} \). It is shown below that the dominant eigenvalue term in Eq. (4.61) (the term with the largest value of \( \lambda_j \)) is very close to the diffusion solution. Hence, \( \tau_{upper} \) is the value of \( \tau \) at which this term begins to dominate.
A simple approximate value for \( \tau_{\text{upper}} \) can be obtained by comparing the exponential portion of the largest two terms in Eq. (4.61). Denoting the largest eigenvalue by \( \lambda_N \) and the second largest by \( \lambda_{N-1} \), the criterion becomes

\[
\exp[-(\tau_{\text{upper}} - \tau_o)/\lambda_N] > \exp[-(\tau_{\text{upper}} - \tau_o)/\lambda_{N-1}] \tag{4.70}
\]

or

\[
\tau_{\text{upper}} - \tau_o > \frac{\lambda_N \lambda_{N-1}}{\lambda_N - \lambda_{N-1}}. \tag{4.71}
\]

Table 4.1 shows values of this bound for various values of \( g \) and \( \omega_0 \). Note that as the scattering gets increasingly forward directed

<table>
<thead>
<tr>
<th>( \omega_0 )</th>
<th>0.8</th>
<th>0.9</th>
<th>0.99</th>
<th>0.9975</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.8</td>
<td>2.7</td>
<td>4.3</td>
<td>20.8</td>
<td>46.1</td>
</tr>
<tr>
<td>0.9</td>
<td>2.3</td>
<td>3.4</td>
<td>17.5</td>
<td>34.0</td>
</tr>
<tr>
<td>0.99</td>
<td>1.8</td>
<td>2.8</td>
<td>9.9</td>
<td>13.1</td>
</tr>
</tbody>
</table>

\((g \rightarrow 1.0)\), or as absorption increases \((\omega_0 \rightarrow 0)\), the condition in Eq.
(4.71) for convergence to the diffusion limit gets increasingly hard to meet. This is consistent with previous observations about diffusion regime propagation [9]. At all values of $g$ shown in the table, however, the requirement that $\tau > \tau_c$, where $\tau_c$ is shown in Figure 3.3, is stricter than the requirement in Eq. (4.71). (This assumes that the boundary point $\tau_0$ is not taken to be too large.) Thus, the conclusion is that the onset of diffusion, $\tau = \tau_c$, is roughly the upper bound on the value of $\tau$ for which the asymptotic theory is valid. (However, see Section 4.6.2 below for a more accurate analysis.)

The lower bound on $\tau$ for the asymptotic solution to be valid can be obtained by assuming the solution to be valid and computing the value of $\tau$ below which this assumption is clearly violated. More specifically, the procedure is to use

$$\Psi(\tau) = \Psi_0(\tau) = \exp[D (\tau - \tau_0)] \psi_{BC}, \quad (4.72)$$

and find the smallest value of $\tau$ for which

$$\tau ||D \Psi_0(\tau) || > ||E \Psi_0||. \quad (4.73)$$

Table 4.2 shows values of $\tau - \tau_0$ above which the inequality in Eq. (4.73) is met. Figure 4.5 is a plot of the squares of the two sides in Eq. (4.73) for $g = 0.9$ and $\omega_0 = 0.9$. The boundary condition used to generate Figure 4.5 and Table 4.2 is the measured angular spectrum shown in Figure 2.9 ($\tau = 4.6 - 5.3$). The curve was extrapolated
Table 4.2

Lower Bound on Value of $\tau - \tau_0$ at which Asymptotic Solution Becomes Valid

<table>
<thead>
<tr>
<th>$\omega_0$</th>
<th>0.8</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.8</td>
<td>12</td>
<td>11</td>
</tr>
<tr>
<td>0.9</td>
<td>11</td>
<td>11</td>
</tr>
<tr>
<td>0.99</td>
<td>11</td>
<td>8.5</td>
</tr>
</tbody>
</table>

out to 180°, assuming the same rate of descent with angle as that in the vicinity of 25°. The values in Table 4.2 are fairly insensitive to changes in $g$ and $\omega_0$. At roughly an optical thickness of 10 beyond the boundary point, the necessary condition for the asymptotic solution to be valid is met.

In summary, this section has shown that the onset of validity for the asymptotic solution in the strong multiple scatter approximation is upper bounded by $\tau_c$, the crossover optical thickness for diffusion theory to be valid, and lower bounded by the values of optical thickness in Table 4.2. The next section discusses the asymptotic solution when $\tau$ is large enough that the term in the series in Eq. (4.61) corresponding to the largest eigenvalue dominates.

4.5.3. Dominant Eigenvalue Solution

Since the eigenvalues of $R = -D^{-1}$ are distinct, the exponential terms in Eq. (4.61) decay at different rates with $\tau$. Ultimately, a term
Figure 4.5: Comparison of Two Terms in Differential Equation for $\Psi$, $g = 0.9$, $\omega_0 = 0.9$, $\tau_0 = 5$
corresponding to the largest eigenvalue, \( \lambda_N \), will dominate. A crude estimate of the value of \( \tau \) at which this occurs was given above.

This section explores both the angular character and the range dependence of the dominate eigenvalue solution. The angular behavior can be derived from Eq. (4.61). When only the largest exponential term dominates,

\[
p_S(\tau, \mu) = \sum_{\ell=1}^{N} \sum_{m=1}^{\ell} \left( \frac{2\ell-1}{4\pi} \right) P_{\ell-1}(\mu) S_{\ell N} Z_{\ell m}(\psi) \exp\left[-(\tau-\tau_0)/\lambda_N \right]
\]  

(4.74)

Observe, first, that the shape of the angular spectrum does not depend on the optical thickness in this limit. This decoupling of range and angle is a common feature of solutions to the transport equation at extreme ranges, and has been observed before [76].

Since the shape of the angular spectrum no longer depends on range, it is as broad as it can ever become. It is easy to see that it is in fact quite broad. Appendix A shows that for \( \lambda_N \gg 0.5 \), the eigenvector element

\[
S_{j+1,N} = (\bar{x}_N)^{j+1} \approx \frac{(\bar{x}_N)_j}{2\lambda_N}
\]  

(4.75)

for \( j \) sufficiently greater than 1. Thus, the values of the \( S_{\ell N} \), \( \ell \leq 1 \leq N \), are rapidly decreasing after the first few elements, indicating that only the first few Legendre polynomials enter into the sum in Eq. (4.74). This is sufficient to guarantee a broad angular spectrum.
Bounds on the slope of the exponential range dependence of the dominant eigenvalue solution can be obtained from the bounds on \( \lambda_N \) derived in Appendix A. It was shown there that (see Eqs. (A.46) and (A.50b))

\[
\frac{\kappa}{1 + \frac{2}{\sqrt{5}} \sqrt{1 - \omega_0}} \leq \frac{1}{\lambda_N} \leq \frac{\kappa}{\sqrt{1 + \frac{4}{5} (1 - \omega_0)}}
\]

(4.76)

where

\[
\kappa = \sqrt{3(1 - \omega_0)(1 - \omega_0 g)}
\]

(4.77)

is identical to the diffusion theory rate of decay with optical thickness. (See Eqs. (3.22) and (3.24).) It is clear from Eq. (4.76) that the upper bound is smaller than \( \kappa \), which means that the dominant eigenvalue solution never decays faster than diffusion.

On physical grounds, it is obvious that an absolute lower bound to \( 1/\lambda_N \) must be the absorption rate of decay, \( (1 - \omega_0) \). The solution could never decay slower than the rate at which the solution for a purely absorbing medium would decay. The absorption lower bound can be obtained mathematically by a maximization procedure identical to that carried out in Section A.2.2 of Appendix A, except that the non-symmetrix matrix \( R = -D^{-1} \) is used for the maximization instead of the symmetric matrix \( T \).

The four bounds discussed here are listed in Table 4.3. They
are shown plotted and compared with the actual computed rate of decay in Figures 4.6 and 4.7. Observe from the figures that the actual decay rate is asymptotic to the absorption bound for a highly absorbing medium,

Table 4.3

Bounds on Decay Rate of Dominant Eigenvalue Solution

\( \kappa = \sqrt{3(1 - \omega_0)(1 - \omega_0 g)} \)

<table>
<thead>
<tr>
<th>Absorption Lower Bound</th>
<th>( 1 - \omega_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lower Bound</td>
<td>( \kappa \left[ 1 + \frac{2}{\sqrt{5}} \sqrt{1 - \omega_0 g} \right]^{-1} )</td>
</tr>
<tr>
<td>Upper Bound</td>
<td>( \kappa \left[ 1 + \frac{4}{5} (1 - \omega_0) \right]^{-\frac{1}{2}} )</td>
</tr>
<tr>
<td>Diffusion Upper Bound</td>
<td>( \kappa )</td>
</tr>
</tbody>
</table>

as expected. (Note that the lower bound may be smaller than the absorption bound for some choices of \( g \) and \( \omega_0 \).) For a highly scattering medium, the actual decay rate is asymptotic to the diffusion bound. Inspecting Table 4.3 shows that this trend continues as \( \omega_0 \) approaches 1, since both the lower and upper bounds converge to the diffusion bound in this limit.

The more forward directed the phase function (\( g = 0.99 \) vs. \( g = 0.9 \)), the larger is the discrepancy between the actual decay rate
Figure 4.6: Decay Rate For Dominant Eigenvalue Solution Compared With Bounds; Henyey-Greenstein Phase Function, $g = 0.9$
Figure 4.7: Decay Rate For Dominant Eigenvalue Solution Compared With Bounds; Henyey-Greenstein Phase Function, $g = 0.99$
and the diffusion bound. However, the actual decay rate is never less than a factor of 2 smaller than the diffusion decay rate, as can be seen by setting \( g = 1 \) in the lower bound in Table 4.3. For \( g = 1 \), the lower bound is minimized over all values of \( g \), and

\[
\frac{1}{\bar{\kappa}_N} \geq 0.53 \kappa ,
\]  

(4.78)

4.5.4. Dominant Eigenvalue Solution for the Special Case of Isotropic Scatter

It is instructive to compare the dominate eigenvalue term in the asymptotic solution with diffusion theory and with the exact solution, which is available for isotropic scatter. It was shown in Section 4.3 that the complete strong multiple scatter solution is identical to the exact transport equation solution, Eq. (4.25), in the limit of large \( \tau \). The strong multiple scatter solution converges to

\[
p_s(\tau) \sim \frac{e^{-\kappa_0 \tau}}{\tau}
\]  

(4.79)

where \( \kappa_0 \) satisfies

\[
\omega_0 = \frac{\kappa_0}{\tanh^{-1} \kappa_0}
\]  

(4.80)

In the asymptotic solution, for which the \( \bar{\psi}/\tau \) term is dropped in Eq. (4.48), the large \( \tau \) solution is
\[ p_s(\tau) \sim e^{-\gamma \tau}, \quad (4.81) \]

where \( \gamma = 1/\lambda_N \), and \( \lambda_N \) is the largest eigenvalue of

\[ R = \begin{bmatrix}
0 & \frac{1}{1 - \omega_0} \\
1/3 & 2/3 \\
2/5 & 3/5 \\
3/7 & 4/7 \\
& \ddots \\
& \cdots & \frac{N-1}{2N-3} \\
& \cdots & \frac{N-1}{2N-1} & 0
\end{bmatrix}. \quad (4.82) \]

Observe that one consequence of dropping the \( E \bar{\psi}/\tau \) term is that the dominant eigenvalue solution does not have an inverse \( \tau \) dependence as do both the exact solution and diffusion. (See Eq. (3.21).)

On the other hand, the exponential dependence of the dominant eigenvalue solution is identical to that of the exact solution for isotropic scatter. It can be shown [77] that as the order of the matrix \( R \) in Eq. (4.82) becomes large, the dominant eigenvalue approaches the positive root, \( \kappa_0 \), of Eq. (4.80). The coefficients for the exact and dominant eigenvalue solutions and for diffusion are plotted in Figure 4.8. As observed previously, the diffusion solution converges to the exact solution for large albedo. Diffusion differs by nearly a factor of 2 in
Figure 4.8: Comparison of Diffusion, Exact and Dominant Eigenvalue Extinction Coefficients
the exponent for a highly absorbing medium.

Note that for $\omega_0 = 1$, the coefficient is zero in all three cases. Thus the diffusion and the exact solution decay as $1/\tau$ in this extreme, whereas the dominant eigenvalue term of the asymptotic solution does not decay at all with range. In order to satisfy the boundary condition at infinity, therefore, this term would have to be forced to zero by appropriate choice of $\overline{\Phi}_{BC}$.

In summary, then, it has been shown in this and the last section that the dominant eigenvalue term of the asymptotic solution has many of the features of a diffusion solution. It has a broad angular spectrum. Its decay rate is never faster than diffusion, and is lower bounded by 0.53 times the diffusion rate. The dominant eigenvalue decay rate converges to the diffusion rate as $\omega_0$ increases, for all values of $g$. The convergence to diffusion occurs at larger values of $\omega_0$ as the phase function becomes more sharply forward peaked ($g \rightarrow 1$). However, even for the extreme case of isotropic scatter ($g \rightarrow 0$), Figure 4.8 shows that the dominant eigenvalue rate and the diffusion rate are essentially the same for $\omega_0 > 0.9$.

The dominant eigenvalue term in the asymptotic solution does not possess an inverse $\tau$ dependence. This differentiates it from both diffusion and the exact solution in the case of isotropic scatter, and it is due to the fact that the $\tau$ dependent coefficient $\Xi/\tau$ has been assumed negligible in the differential equation for $\overline{\Psi}(\tau)$. However, the dominant eigenvalue term has a decay rate identical to the exact solution for isotropic scatter, for any amount of absorption, while the diffusion decay
rate is \( \sqrt{3} \) times that of the exact solution for a purely absorbing medium.

Taking the large range limit of the asymptotic approximation to the strong multiple scatter transport equation can thus be considered an alternative means of arriving at a diffusion-like solution. There are a variety of approaches to diffusion in addition to the Fick's law approach followed in Chapter 2 [13,78,79], but a direct asymptotic approach such as the one presented in this Chapter is perhaps the most straightforward. The various approaches produce diffusion solutions which have the qualitative features discussed here, such as a broad angular spectrum and an exponential decay rate close to the Fick's law decay rate, but may differ in the exact value of the coefficient in the exponent. The dominant eigenvalue approach has the obviously desirable feature that its exponential decay is identical to that of the exact solution in the instance of isotropic scatter.

In the next section, numerical results are discussed for the asymptotic solution in the case in which the largest eigenvalue term is not dominant. The angular spectrum and range dependence of the solution are studied and the value of optical thickness at which the large eigenvalue term becomes dominant is determined.
4.6. Asymptotic Solution With Subdominant Eigenvalues

When the subdominant eigenvalues cannot be ignored, computation is necessary to obtain results from the asymptotic solution. The main computational task is the evaluation of the matrix exponential,

\[ \exp[D(\tau - \tau_0)] . \quad (4.83) \]

There are many ways to do this computation \[80,81\], and in the case of general \( D \) the eigenvalue decomposition method suggested in Eq. (4.57) is not recommended. The reason is that the matrix exponential is quite sensitive to inaccuracies in the eigenvectors of \( D \) when the condition number of \( D \) is high (when the elements of \( D \) vary over a very wide range), which may be the case for \( \omega_0 \) close to 1 (see Eqs. (A.13) and (A.14)). Standard matrix decomposition routines such as those provided by IMSL \[82\] require a high price in computer time for the needed accuracy. Also, the inversion of \( S \), the matrix of right eigenvectors, may be very slow, since \( S \) is nearly full.

Using the algorithms developed in Appendix A, however, the left and right eigenvectors of \( R = -D^{-1} \) can be computed with relative ease and accuracy. Hence, the inversion of \( S \) is unnecessary. Although the author has not conducted an exhaustive study of various alternative methods, a preliminary comparison of the algorithm in Appendix A with available IMSL routines showed the former to be substantially faster. It should be emphasized at this point that these conclusions are critically dependent
on the specific structure of $R$, and do not generalize.

4.6.1. Convergence

Before proceeding with a study of the angular and spatial behavior of the asymptotic solution, it is necessary to discuss the issue of convergence. The number of terms, $N$, required in the series in Eq. (4.61) depends on the width of the angular spectrum it must approximate, with the number of terms increasing as the angular spectrum gets narrower. An estimate of $N$ can be made by requiring that the highest order polynomial $P_N (\mu = \cos \theta)$ have a slope at least as large at $\mu = 1$ as does the angular spectrum that the series must represent.

The recurrence relation [21]

$$P'_N(1) = N + P'_{N-1}(1) \quad (4.84)$$

gives

$$P'_N(1) = \frac{N(N+1)}{2} \quad (4.85)$$

Thus, equating the two slopes yields

$$\frac{1}{\Delta \mu} = \frac{N(N+1)}{2} \quad (4.86)$$

Here $\Delta \mu$ is the $\mu$-width of the angular spectrum (assuming a linearly decaying angular spectrum), and is equal to $(\Delta \theta)^2$, where $\Delta \theta$ is the width
of the angular spectrum in radians. Assuming that $N \gg 1$, the requirement on $N$ is thus

$$N > \frac{\sqrt{2}}{\Delta \theta}$$

(4.87)

This is the familiar "gain-bandwidth product" expression which arises in the representation of a function by a series of orthogonal basis functions.

Since the measured angular spectra have widths on the order of $10 \text{ mr}$ (see Chapter 2), Eq. (4.87) states that the number of terms required to represent the angular spectrum accurately at the boundary point, $\tau_0$, is on the order of 100-200. This estimate is borne out by the trend shown in Figure 4.9. The figure shows a comparison of a normalized angular spectrum (based on a least squares fit to the data in Figure 2.9) with its Legendre polynomial expansion for different values of $N$. The approximate angular spectrum gets narrower, and the characteristic ringing (Gibbs phenomenon) is reduced, as the number of terms in the expansion increases. With 150 terms, the approximation appears to be reasonable. Since the angular spectrum broadens as the optical thickness increases beyond the boundary point, the number of terms required to represent the angular spectrum at the boundary is an upper bound on the number of terms needed for $\tau > \tau_0$.

All of the numerical results presented below were obtained with $N = 150$. Instead of using all 75 eigenvectors, however, (corresponding to the 75 negative eigenvalues), it was found that 40 eigenvectors were sufficient for convergence.
Figure 4.9: Convergence of Legendre Expansion to Boundary Point Angular Spectrum
4.6.2. Angular Spectrum

Figures 4.10 through 4.13 show the evolution, with distance from the boundary, of the normalized angular spectrum, for different combinations of $g$ and $\omega_0$. A number of important common features are evident in these figures. First, note that in all cases, the optical distance from the boundary point must be greater than 10 before significant spreading occurs. The regime in which spreading becomes important is the regime in which the medium is filtering out the higher order angular frequency components, and thus in all of the curves, the overshoot and ringing associated with these components is reduced as $\tau$ increases. All of the angular spectra exhibit convergence to a dominant eigenvalue solution in the limit of large distance from the boundary point. This solution is very broad in angle, ranging in half-width from 30 degrees to more than 90 degrees.

The angular spectra fall into two distinct groups: those resulting from a relatively broad phase function ($g = 0.9$), and those resulting from a sharply peaked phase function ($g = 0.99$). (The general features exhibited by the former group are also characteristic of angular spectra for $g < 0.9$, so the two groups are collectively exhaustive.) The angular spectra in the broad phase function group display the following pattern as distance from the boundary is increased: As observed above, they remain virtually identical to the boundary point angular spectra until at least $\tau - \tau_0 > 10$. After that, broadening occurs, by an elevation of the angular spectrum tail at wide angles, but a pronounced narrow peak remains. (See the $\tau - \tau_0 = 20$ curve in Figures 4.10 and 4.11.) As the optical thickness is
Figure 4.10: Evolution of Angular Spectrum; $g = 0.9, \omega_0 = 0.9$
Figure 4.11: Evolution of Angular Spectrum; 
\( g = 0.9, \omega_0 = 0.99 \)
Figure 4.12: Evolution of Angular Spectrum; \( g = 0.99, \omega_0 = 0.9 \)
Figure 4.13: Evolution of Angular Spectrum; $g = 0.99$, $\omega_0 = 0.99$
increased further, enough light is converted from the narrow forward peak into the scattered field that the peak becomes submerged. For the two cases considered, the narrow peak is fully submerged by an optical distance of 25 or 30 from the boundary point. Note that the transition between the two regimes is quite rapid, taking place within about 5-10 optical thicknesses. Figure 4.14 shows this transition in more detail for $\omega_0 = 0.9$.

The second category of angular spectrum, resulting from a sharply peaked phase function, undergoes a qualitatively different kind of broadening as optical thickness increases. Observe in Figures 4.12 and 4.13 that there is no distinction between the broadened portion of the angular spectrum and the narrow on-axis peak. The peak simply broadens uniformly with increasing $\tau$. Note further that the peak remains narrow much further from the boundary point in this category of angular spectrum than in the broad phase function category. The dominant eigenvalue extreme is not reached until $\tau - \tau_0$ is over 100.

The underlying explanation for the qualitatively different character of the two types of angular spectra is that the atmosphere characterized by a broad phase function has a high probability of producing scattered light at angles larger than the width of the boundary point peak, whereas an atmosphere with a narrow peak in the phase function has a high probability of scattering photons at angles within the initial peak. The outside-the-peak/inside-the-peak distinction gives rise to the discrete change in character between the $g = 0.9$ and $g = 0.99$ angular spectra. The curves in Figure 4.14 show that a medium with a
Figure 4.14: Transition From Peaked to Broad Angular Spectrum; $g = 0.9, \omega_0 = 0.9$
phase function broader than the angular spectrum width very rapidly produces an angular spectrum which is broad. This appears to be strong evidence for the assumption commonly made in multiple forward scatter theory that the consistently observed narrowness of the angular spectrum implies that the phase function is narrow.

The effect of differences in phase function peaking and in albedo can be quantified in terms of 3 basic features of the angular spectrum shape: the half-width at a given $\tau - \tau_0$, the ratio of the angular spectrum height at $90^\circ$ to that at $0^\circ$ in the dominant eigenvalue limit, and the value of $\tau$, $\tau_{DOM}$, at which the maximum eigenvalue term is dominant. These quantities are shown in Tables 4.4 through 4.6. It is

<table>
<thead>
<tr>
<th>$\omega_0$</th>
<th>0.8</th>
<th>0.9</th>
<th>0.99</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.8</td>
<td>$1^\circ$</td>
<td>$2^\circ$</td>
<td>$4^\circ$</td>
</tr>
<tr>
<td>0.9</td>
<td>$2^\circ$</td>
<td>$8^\circ$</td>
<td>$5^\circ$</td>
</tr>
<tr>
<td>0.99</td>
<td>$35^\circ$</td>
<td>$60^\circ$</td>
<td>$5^\circ$</td>
</tr>
</tbody>
</table>

clear from the tables that, as has been consistently observed to be the case, the angular spectrum tends to be narrower with increasing values of $g$ (more phase function peaking) and decreasing values of $\omega_0$ (more absorption).
Table 4.5

Values of $\frac{p(\tau_{DOM}, \theta = 90^\circ)}{p(\tau_{DOM}, \theta = 0^\circ)}$

<table>
<thead>
<tr>
<th>$\omega_0$</th>
<th>0.8</th>
<th>0.9</th>
<th>0.99</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.8</td>
<td>0.15</td>
<td>0.09</td>
<td>0.02</td>
</tr>
<tr>
<td>0.9</td>
<td>0.3</td>
<td>0.2</td>
<td>0.07</td>
</tr>
<tr>
<td>0.99</td>
<td>0.7</td>
<td>0.63</td>
<td>0.51</td>
</tr>
</tbody>
</table>

Table 4.6

Values of $\tau_{DOM}$

<table>
<thead>
<tr>
<th>$\omega_0$</th>
<th>0.8</th>
<th>0.9</th>
<th>0.99</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.8</td>
<td>50</td>
<td>40</td>
<td>100</td>
</tr>
<tr>
<td>0.9</td>
<td>40</td>
<td>35</td>
<td>100</td>
</tr>
<tr>
<td>0.99</td>
<td>35</td>
<td>35</td>
<td>100</td>
</tr>
</tbody>
</table>

Referring back to the discussion in Section 4.5.2 of the upper bound on the onset of validity of the asymptotic solution, note that the values of $\tau_{DOM}$ in Table 4.6 are much larger than the estimates in Table 4.1. In fact, for $g = 0.8$ and $g = 0.9$, $\tau_{DOM}$ is larger than the rule-of-thumb crossover optical thickness, $\tau_c$, for the onset of validity of diffusion. Therefore, in accordance with the discussion in Section 4.5.2, it is $\tau_{DOM}$ and not $\tau_c$ which serves as the appropriate upper bound on the
validity region in those cases.

4.6.2. Range Dependence

Figures 4.15 and 4.16 show transmission curves versus optical distance from the boundary point. The positioning of the curves relative to the unscattered transmission curve was determined as follows: The experimental data shows that at a 1 mr FOV, the unscattered component is about a factor of 10 above the scattered component at the boundary point, \( \tau = \tau_0 \). Thus, the 1 mr scattered signal transmission is set at \( \exp(-5)/10 \), or .0067, at \( \tau_0 = 5 \).

The transmission curves exhibit two noteworthy features. The first is that at some distance from the boundary point, a distance which decreases as the receiver FOV is opened wider, the decay rate of the transmitted scattered light changes to the dominant eigenvalue (near diffusion) decay rate. The crossover point is roughly the value of \( \tau - \tau_0 \) in Figures 4.10 through 4.13 at which significant off-axis light appears. Note that in the case of \( g = 0.9 \) (Figure 4.15), the transition is sharper than for \( g = 0.99 \) (Figure 4.16). This corresponds to the distinction between the sharply peaked and less sharply peaked phase functions discussed above. For the less peaked phase function, a dominant narrow peak was evident for some distance from the boundary, but was rapidly submerged as the wide-angle scattered light increased. The transmission curves in Figures 4.15 and 4.16 indicate that the narrow angular spectrum peak and the wide-angle scattered light decay with range at their own characteristic rates: the narrow peak decays at the same
Figure 4.15: Optical Thickness Dependence of Transmission; $g = 0.9, \omega_0 = 0.9, \tau_0 = 5$
Figure 4.16: Optical Thickness Dependence of Transmission;
\( g = 0.99, \omega_0 = 0.9, \tau_0 = 5 \)
rate as the unscattered light and the wide-angle scattered light decays at the diffusion rate. Since in the case $g = 0.99$ there is no sharp transition between a dominant narrow peak and a broad angular spectrum (see Figures 4.12 and 4.13), but merely a uniform broadening of the narrow peak, the transition region in the transmission curve is correspondingly less sharp.

The second noteworthy features in Figures 4.15 and 4.16 is that the slope of the transmission curve in the vicinity of the boundary appears to be independent of the receiver FOV and very close to that of the unscattered light. This is a disturbing conclusion in two respects. First, it contradicts the experimental data, which predicts a FOV dependent transmission slope. Second, it is physically unrealistic, in that it implies that the low angular frequency modes of the received field (detected by the wide FOV receiver) are decaying as rapidly as the high angular frequency modes. But if this were the case, the low angular frequency modes could never become dominant, implying that a narrow angular spectrum would last indefinitely.

The source of this apparent contradiction is numerical inaccuracy. Note that the curves in Figures 4.15 and 4.16 do not accurately represent one important feature of the boundary condition at $\tau = \tau_0$, namely the linear increase in signal with FOV. It can be seen from the figures that the increase in signal from the 1 to 15 mr curve, for example, is more than two orders of magnitude, rather than the experimentally observed increase of slightly more than 10. This quadratic increase at the narrow fields of view means that although
there is a relatively narrow spike in the computed angular spectrum, it falls off with angle much less rapidly than that actually encountered experimentally. Therefore, the highest frequency modes are not present in the solution, and those modes that are present decay at essentially the same rate with $\tau$ as does the lowest frequency mode.

The inability of the asymptotic solution to match the boundary condition is due to the fact that the empirical angular spectrum does not fall linearly with angle, as implied in the convergence arguments of Section 4.6.1. In fact, Eq. (4.92) below shows that the slope of the angular spectrum is much steeper than linear between 1 and 10 mr. Thus many more than 100 to 200 terms would be required in the Legendre series to accurately represent this narrow peak.

In principle, more terms can be added, but numerical inaccuracies begin to cause the solution to diverge when more than 200 terms are included, even when full available machine precision is used. Therefore, the accuracy of the computations could not be improved substantially.

4.6.4. Comparison with Experimental Data

This section considers the relation between the asymptotic solution and the experimental data with respect to the evolution of the angular spectrum with optical thickness. Since, as discussed above, numerical problems limit the conclusions one can draw about the range dependence of the transmission characteristic, the comparison is not made for transmission.

There is an inherent difficulty in comparing the results of the
asymptotic solution with the experimental data thus far obtained. It has been shown (see Section 4.5.2) that the asymptotic solution cannot be valid for optical thicknesses within $\tau = 10$ of the boundary point, yet the experimental data is limited to optical thicknesses less than 10. Thus, to make comparisons, the experimental data must be extrapolated out beyond $\tau = 10$. This can be done using the empirical formula developed in Chapter 2 for the signal vs. FOV characteristic.

Recall from Section 2.2.2 that for small FOV, the signal vs. FOV characteristic was shown to be of the form

$$\frac{T(FOV)}{T_{max}} = \left[ \frac{FOV}{FOV_{max}} \right]^{k\tau},$$

(4.88)

where $T_{max}$ is the transmission in the saturation region (see Figure 2.6) and $FOV_{max}$ is the FOV at which saturation begins. The constant $k$ is approximately 0.1. In order to incorporate the saturation effect in the signal vs. FOV expression, the convenient functional form

$$\frac{T(FOV)}{T_{max}} = \left\{ 1 - \exp\left[ -\left( \frac{FOV}{FOV_{max}} \right)^{k\tau} \right] \right\}$$

(4.89)

is assumed in the following discussion.

To derive an angular spectrum from Eq. (4.89), the signal vs. FOV characteristic is related to the angular spectrum as

$$T(FOV) = 2\pi \int_{0}^{\frac{FOV}{2}} \, d\theta \sin \theta \, p_N(\tau, \theta),$$

(4.90)
where $p_N(\tau, \tau)$ is the normalized angular spectrum

$$p_N(\tau, \theta) = \frac{p_S(\tau, \theta)}{p_{\text{clear}}^{(\text{FOV})}}$$  \hspace{1cm} (4.91)

Substituting Eq. (4.89) into Eq. (4.90) and differentiating with respect to FOV gives the angular spectrum shape

$$p_S(\tau, \theta) = \frac{\left(\frac{2\theta}{\text{FOV}_{\text{max}}}\right)^{k\tau - 1} \exp\left[-\left(\frac{2\theta}{\text{FOV}_{\text{max}}}\right)^{k\tau}\right]}{\sin \theta}.$$  \hspace{1cm} (4.92)

Two observations are in order. First, Eq. (4.92) cannot be strictly valid at $\theta = 0$ because the expression blows up there for $\tau < 20$. Second, for $k = 0.1$, $\tau = 20$ implies a quadratic signal vs. FOV curve (uniform angular spectrum) at small angles. Since $\tau > 20$ would mean an angular spectrum growing with increasing angle, Eq. (4.92) can only be valid for values of $\tau \leq 20$.

Figures 4.17 through 4.19 show the evolution of the angular spectrum in Eq. (4.92) as $\tau$ increases, for three choices of FOV$_{\text{max}}$. Observe that all of the angular spectra begin quite narrow and that the smallest value of FOV$_{\text{max}}$ corresponds to the narrowest angular spectrum, as one would expect. Observe also that at $\tau = 20$ all of the angular spectra are uniform until the saturation region is reached.

Comparing Figures 4.17 through 4.19 with Figures 4.10 through 4.14, there is a striking similarity between the experimental curves.
Figure 4.17: Evolution of Empirical Angular Spectrum; FOV\textsubscript{max} = 500 mr
Figure 4.18: Evolution of Empirical Angular Spectrum; FOV$_{\text{max}}$ = 1000 mr
Figure 4.19: Evolution of Empirical Angular Spectrum; FOV\textsubscript{max} = 2000 mr
and the theoretical angular spectra of the broad phase function type. (See Section 4.6.1.) In both cases, a narrow peak remains dominant. Angular spectrum broadening manifests itself as an increase in the height of the wide-angle tail, rather than the uniform broadening corresponding to the narrow phase function type angular spectrum. In accordance with the discussion in Section 4.6.1, this type of broadening indicates that the phase function applicable in the fog propagation experiments discussed in Chapter 2 is much broader than the boundary point angular spectrum.

The qualitative similarities observed here between the extrapolated experimental data and the asymptotic solution should be reflected in the transmission curve. However, as discussed above, the characteristics of the transmission curve cannot be represented without more numerical accuracy than is available, so it must remain an open question whether the asymptotic solution actually exhibits the FOV dependent transmission curve slope.

In the next Chapter, the major results of this thesis are summarized. The value of the strong multiple scatter theory in the asymptotic limit is assessed, and extensions to the non-asymptotic case are discussed.
CHAPTER V

CONCLUSIONS

This thesis has investigated methods of solving the transport equation for point source optical propagation through a multiple scattering medium. The goal of the thesis has been to closely link theoretical studies of propagation channel parameters with available experimental data. This link is not merely a comparison between theoretical results and experiments, although this is important. In the approach taken here, the experimental data has been placed at the foundation of theoretical investigation as a guide to possible avenues of study. The data has provided the justification for making approximations to the transport equation which make it easier to solve.

The middle ultraviolet wavelength, $\lambda = 0.25 \, \mu m$, has been the wavelength of primary interest in this thesis. The motivation for studying propagation at this wavelength is the desirability of achieving daytime quantum limited operation with a wide FOV scatter communication system, which is possible because the middle UV is completely solar blind. Although the results obtained in this research are therefore directly applicable to propagation in the ultraviolet, there is good reason to believe that they apply to propagation at visible wavelengths as well. The weather type considered exclusively here has been fog, which has mean particle dimensions on the order of 5-10 $\mu m$. Since this particle size is much larger than both visible and ultraviolet wavelengths, and
much larger than the difference in their wavelengths, propagation affected by scattering from fog aerosols should be similar in the two wavelength regions.

In the course of this research, experimental data was gathered during two field trips to the coast of Maine. In varying thicknesses of maritime fog, measurements were made of the angular spectrum and the optical power transmission received from a cw point source. Based on a study of this data, a number of features of propagation from a point source have been identified. The angular spectrum is narrow for optical thicknesses less than 10, roughly $\pm 10$ mr in width. The rapid dropoff in the angular spectrum is reflected in the signal vs. FOV characteristic, which exhibits a linear or slower increase with FOV. A prominent unscattered signal peak is evident in the angular spectrum until $\tau = 10$, after which it is submerged. The angular spectrum broadens as $\tau$ increases and becomes flat over a $\pm 10$ mr angular region for $\tau \geq 10$.

The range dependence of the transmission characteristic in the optical thickness region 0-10 is essentially exponential, with a decay rate that is dependent on the FOV. The data indicates that the decay rate is a decreasing logarithmic function of FOV. As shown in Chapter 2, this type of decay rate dependence is consistent with the observed linear increase in the signal with increasing FOV.

The third major feature of the experimental data is that it exhibits what has been termed the insensitivity property of the scattered field. That is, the sensitivity of the off-axis (scattered)
portion of the received light to a spatially inhomogeneous distribution of scatterers in the medium is much lower than the sensitivity of the on-axis (unscattered) portion. The insensitivity property is a natural consequence of the fact that the scattered light is an average over many different paths, whereas the unscattered light is affected by the distribution of scatterers on only one path. This property has been of fundamental importance in this thesis in that it provided the initial experimental evidence which led to the strong multiple scatter approximation to the transport equation.

In Chapter 3, three existing propagation theories -- single scatter, diffusion and multiple forward scatter theory -- were compared with each other and with the uv experimental data. It was found that single scatter theory and diffusion theory could be ruled out as possible explanations for the behavior observed in the experiments, because in almost all crucial respects they did not correspond to the data. This was not entirely unexpected, because the single scatter and diffusion theories are strictly valid in optical thickness regimes which are far removed from the $2 < \tau < 10$ region covered in the experiments. However, the comparison served the useful purpose of explicitly delineating a range of optical thickness into which neither of these two theories can be extrapolated without significant deviation from observed results. Hitherto, it was unclear to what extent diffusion and single scatter theory would be in error as a result of such extrapolation.

In contrast to single scatter and diffusion, the multiple forward scatter theory is promising in its ability to reflect the basic
characteristics observed in the experiments. It has been shown that by appropriate choice of the phase function forward scatter efficiency, \( \phi \), and peak value, \( \sigma_T(0) \), the measured signal vs. FOV characteristic can be reproduced quite closely. In addition, if \( \phi \) is considered a function of FOV, the FOV dependent transmission slope can be reproduced as well.

As discussed in Section 3.2.2, incorporating a FOV dependence into \( \phi \) requires a view of the "channel" as a combination of atmosphere plus receiver, in which changes in the receiver actually change the channel characteristics through \( \phi \). This is a departure from the traditional (and perhaps more satisfying) view of the channel as consisting solely of the atmosphere, which independently affects propagation, and the receiver as a detection and processing device which merely operates on the output of this channel. However, this departure has been shown to be necessary in order to insure the validity of making the small-angle approximation to the transport equation when dealing with phase functions which exhibit a significant amount of wide-angle scatter.

Despite the promise of the multiple forward scatter theory, it is premature to judge its ultimate applicability until actual values of the phase function parameters, particularly the size of the peak, \( \sigma_T(0) \), can be ascertained at the same time as angular spectrum and transmission measurements are made. The ranges of parameter values chosen in Chapter 3 to compare the MFS theory with the experimental data appear to be reasonable based on measured fog phase functions that have been published [32,42]. However, a definitive comparison must await in situ measurements of these parameters.
Although there are potentially many ways in which the experimental data presented in Chapter 2 could be used to simplify the transport equation, Chapter 4 deals with only one such approach, that based on the insensitivity property. Since the insensitivity property differentiates the scattered and unscattered light on the basis of sensitivity to random spatial variations in the medium's extinction coefficient, the transport equation was separated into scattered and unscattered parts, $p_u(r,\Omega)$ and $p_s(r,\Omega)$. Then the scattered portion was studied in detail.

It was shown that the single scatter term in the integral equation for $p_s(r,\Omega)$ has an exponential dependence on the path integrated extinction coefficient, which is the same dependence as that of $p_u(r,\Omega)$. This exponential fluctuation is not consistent with the observed insensitivity, and hence the single scatter term must be negligibly small.

Neglecting this single scatter term in the transport equation is called the strong multiple scatter approximation. While the approximation was motivated originally by inferences from the insensitivity property, there is direct support for it in the experimental data. Comparisons between the data and single scatter theory show that the amount of received single scattered light is small compared to the amount of multiply-scattered light. The divergence between the data and single scatter theory grows with optical thickness, hence strengthening the strong multiple scatter approximation as $\tau$ increases. For isotropic scatter, it was proven in Chapter 4 that the strong multiple scatter solution converges identically to the known exact solution as $\tau$ becomes large.

The strong multiple scatter transport equation is identical to
the exact transport equation except that it does not include a source term. That is, the strong multiple scatter transport equation applies at ranges far enough from the source that the source boundary condition no longer affects the solution. Thus, the boundary condition, whether measured or derived by an alternative method, can be applied anywhere in the medium, and the unscattered term can simply be added in to obtain the complete (scattered plus unscattered) solution.

Since the strong multiple scatter equation is simply the homogeneous transport equation, it is still quite general. In this thesis, only the asymptotic solution has been studied. This solution was obtained by making the further approximation that the $\frac{E \Psi}{\tau}$ term in the differential equation for the Legendre coefficient vector, $\Psi$, is small compared to $D \Psi$. This approximation results in a matrix exponential solution for $\Psi$. The approximation is reasonable for large $\tau$, but its validity is not guaranteed by the data. In fact, it was shown in Section 4.5.2 for a specific case that the asymptotic solution cannot be valid in the optical thickness region covered by the data. The onset of validity of the asymptotic solution was shown to be bounded by $10 \leq \tau - \tau_0 \leq \max (\tau_{DOM}, \tau_c)$ where $\tau_{DOM}$ is the value of $\tau$ at which the largest eigenvalue term in the asymptotic solution becomes dominant, and $\tau_c$ is the crossover optical thickness for the validity of diffusion.

The asymptotic solution has been studied in detail, both in the limit of $\tau > \tau_{DOM}$ and for values of $\tau$ at which subdominant eigenvalue terms are important. The dominant eigenvalue limit has been shown to
give a broad angular spectrum whose shape is independent of optical thickness. Bounds on the largest eigenvalue of $D$ have been obtained, which show that the exponential decay rate of the dominant eigenvalue solution is always slower than that in diffusion theory, but never more than a factor of two slower. Because of the similarity to diffusion in both its angular spectrum and range dependence, the dominant eigenvalue solution may be considered an alternative form of diffusion.

For isotropic scatter, the exponential decay rate of the dominant eigenvalue solution is identical with that of the exact solution. The decay rate of diffusion is a factor of $\sqrt{3}$ higher for a completely absorbing medium, but converges to the exact (and dominant eigenvalue) decay rate for albedos close to 1.0. Unlike both the exact solution and diffusion, the dominant eigenvalue solution does not have an inverse $\tau$ dependence in the isotropic scatter extreme. This is a consequence of neglecting the inverse $\tau$ dependent coefficient, $E/\tau$, in the differential equation for $\Psi$.

When the subdominant eigenvalue terms are significant, the matrix exponential must be computed. Exploiting the unique structure of the matrix $D$ (or more specifically, its negative inverse $R = -D^{-1}$), simple algorithms for computing its left and right eigenvectors have been developed. Using these algorithms, the eigenvalue decomposition method was used to compute the matrix exponential.

The number of terms required in the Legendre series is determined by the width of the angular spectrum that it must approximate. It was shown that for a boundary point angular spectrum which linearly decayed to zero in 10 mr, 100-200 terms in the series suffices, with the required
number of terms decreasing as the distance from the boundary point increases. However, it was found that significantly more terms are needed to adequately represent an angular spectrum which has a steeper than linear slope near zero degrees.

The angular spectrum predicted by the asymptotic solution was found to be of two distinct types. The first results from a phase function which is much broader than the boundary point angular spectrum, and the second type results from a phase function which is narrower than the boundary angular spectrum. In the former case, the predicted angular spectrum evolves with $\tau$ by retaining a dominant narrow peak which is the same width as that at the boundary. Broadening manifests itself as an increase in the height of the relatively flat pedestal upon which the dominant peak rests. In the latter type of angular spectrum, broadening manifests itself as a uniform spreading of the angular spectrum peak, due to the fact that most of the scattering is taking place at angles smaller than, or comparable to, the width of the peak. In this latter case, since scattering was so sharply forward directed, the angular spectrum stayed relatively narrow out to optical thicknesses as large as 100.

In comparisons between the results predicted by the asymptotic solution and empirical expressions developed from the experimental measurements, it was found that the experimental angular spectrum evolves with optical thickness in much the same was as does the broad phase function type angular spectrum in the asymptotic solution. The conclusion is that the phase function for the fog aerosol encountered
in the experiments had significant scatter probability outside of 10 mr, which was the width of the angular spectrum used as the boundary condition in the computations.

The comparison between the predicted and measured transmission curves showed significant differences. The slope of the predicted transmission curve has no FOV dependence, while the measured slope does. This was shown to be a consequence of the inability of the Legendre series to adequately represent the steep dropoff in the angular spectrum at the boundary point, when less than 200 terms were used. When more terms were added, numerical inaccuracies in computing the eigenvector elements caused the series to diverge.

In summary, then, the asymptotic solution embodies important features of the experimentally observed angular spectrum. It allows one to apply a measured or otherwise computed boundary condition at a convenient point in the medium, and to predict the evolution of the angular spectrum as light propagates away from that point. Due to numerical inaccuracies, the eigenvalue decomposition method of computing the solution causes errors in representing a very narrow boundary angular spectrum, and hence the solution does not predict correctly those phenomena which depend critically upon doing so.

Other methods of obtaining the solution, or increased machine precision, must be used when more than 200 terms are required. One obvious alternative method is to directly integrate the asymptotic differential equation numerically. However, since $D$ has growing modes, direct integration results in a growing solution unless the boundary
condition forces them to zero. No matter how this is done, computation of the eigenvalues and at least half the left eigenvectors is required. (See Section 4.5.1.)

Application of the boundary condition at infinity is in fact the major obstacle, not only to extending the asymptotic solution to higher orders, but also to obtaining the general solution. With the $\tau$ dependent term, $D + \frac{E}{\tau}$, in the differential equation, growing modes will be present and they will change with $\tau$. Short of computing the eigenvalues and eigenvectors at each step in the integration, there is no obvious way to eliminate these unwanted modes. Developing an efficient way to apply the boundary condition at infinity therefore appears to be the most productive avenue of further research.
APPENDIX A

PROPERTIES OF COEFFICIENT MATRICES IN DIFFERENTIAL EQUATION FOR $\tilde{\psi}(\tau)$

This appendix presents the properties of the matrices $A$, $D$ and $E$ defined in Eqs. (4.47) and (4.49). The existence of $A^{-1}$ is first proven and its form given. Then algorithms for computing the eigenvalues and eigenvectors of $D$ and $E$ are developed.

A.1. The Inverse of $A$

From Eq. (4.47a), $A$ is given by

$$A = \begin{bmatrix}
0 & 1 & & & \\
1 & 0 & 2 & & \\
& 2 & \ddots & 3 \\
& & \ddots & \ddots & \\
& & & N-2 & 0 \\
& & & & N-1 \\
& & & \end{bmatrix} \quad (A.1)$$

To show that $A$ is non-singular, consider computing the polynomial

$$D_N = \det(A_{N\times N} - \lambda I_{N\times N}). \quad (A.2)$$
Defining

\[ D_K = \det(A_{KXK} - \lambda I_{KXK}), \quad K \leq N \quad (A.3) \]

where

\[
A_{KXK} = \begin{bmatrix}
0 & N-K+1 & & \\
N-K+1 & 0 & N-K+2 & \\
& N-K+2 & 0 & \\
& & & \ddots & \ddots & \ddots & \\
& & & & 0 & N-2 & \\
& & & & N-2 & 0 & N-1 & \\
& & & & & N-1 & 0 & \\
\end{bmatrix}
\]

(A.4)

the recursion formula for the characteristic polynomial is

\[ D_K = -\lambda D_{K-1} - (N-K+1)^2 D_{K-2} \quad (A.5) \]

Observe that for any even value of \( K \), Eq. (A.5) is a polynomial in even powers of \( \lambda \). Since only even values of \( N \) are considered in this thesis (see Section 4.4), this property applies to \( D_N \). Therefore, if zero is a root of the characteristic polynomial, it must be a double root. However, it can be shown that all roots of \( D_n \) are real and distinct [51], since it is a special case of a general class of
tri-diagonal (also called Jacobi) matrices. Therefore, zero cannot be a root of $D_n$, and hence $A$ is non-singular.

It is now shown that $A^{-1}$ is of the form:

$$
\begin{bmatrix}
0 & 1 & 0 & -\frac{2}{3} & 0 & \frac{2.4}{3.5} & 0 & -\frac{2.4.6}{3.5.7} & \cdots & (-1)^{\frac{N}{2}-1} \frac{2.4.6\cdots(N-2)}{3.5\cdots(N-1)} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & -\frac{1.4}{3.5} & 0 & -\frac{1.4.6}{3.5.7} & \cdots & (-1)^{\frac{N}{2}} \frac{1.4.6.8\cdots(N-2)}{3.5.7\cdots(N-1)} \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & -\frac{1.6}{5.7} & \cdots & (-1)^{\frac{N}{2}-1} \frac{1.6.8\cdots(N-2)}{5.7\cdots(N-1)} \\
0 & 0 & 0 & -\frac{(N-2)}{(N-3)(N-1)} \\
0 & 0 & 0 & \frac{1}{N-1} \\
0 & 0 & 0 & 0
\end{bmatrix}
$$

(A.6)

Only the elements to the right of the diagonal are shown, since $A^{-1}$ is symmetric. The algorithm for generating $A^{-1}$ is as follows:

1. All diagonal elements are zero, and $A^{-1}$ is symmetric.
2. All even numbered rows have a zero as the first element to the right of the diagonal.
3. The $(2j+1)^{st}$ row, $0 \leq j \leq \frac{N}{2} - 1$, has $\frac{1}{j}$ as the first element to the right of the diagonal.
4. Considering only elements to the right of the diagonal, the
element in the \( K^{th} \) column is \( \frac{(K-2)}{(K-1)} \) times the element in the \((K-2)^{nd}\) column.

To demonstrate that Eq. (A.6) is in fact the form of \( \mathbf{A}^{-1} \), consider the \((i,j)^{th}\) element of the product of \( \mathbf{A} \) and \( \mathbf{A}^{-1} \):

\[
P_{ij} \equiv [\mathbf{A} \cdot \mathbf{A}^{-1}]_{ij} = (i-1) \mathbf{A}^{-1}_{i-1,j} + i \mathbf{A}^{-1}_{i+1,j} \quad (A.7)
\]

Now, for \( \mathbf{A}^{-1} \) to be of the asserted form, \( Q \), it is necessary and sufficient to show that

\[
P_{ij} = 1 \quad i = j \quad (A.8)
\]

\[P_{ij} = 0 \quad i \neq j\]

a. \( i = j \): In this case,

\[
P_{ii} = (i-1) \mathbf{A}^{-1}_{i-1,i} + i \mathbf{A}^{-1}_{i+1,i} = 1 \quad (A.9)
\]

But for \( Q \),

\[
Q^{-1}_{i-1,i} = \frac{1}{i-1} \quad \text{i even}
\]

\[
Q^{-1}_{i+1,i} = 0 \quad (A.10)
\]

\[
Q^{-1}_{i-1,i} = 0 \quad \text{i odd}
\]

\[
Q^{-1}_{i+1,i} = \frac{1}{i}
\]

Therefore \( Q \) satisfies Eq. (A.9).
b. if $i \neq j$:

In this case,

$$p_{ij} = (i-1) A^{-1}_{i-1,j} + i A^{-1}_{i+1,j} = 0 \quad (A.11)$$

But for the $j^{th}$ column, elements of alternate rows of $Q$ are related by

$$Q_{i+1,j} = -Q_{i-1,j} \left(\frac{i-1}{i}\right) \quad (A.12)$$

Therefore $Q$ satisfies Eq. (A.11), which proves that $A^{-1} = Q$.

A.2. Eigenvalues and Eigenvectors of $D$

Although it is possible to write $D$ down, given the form of $A^{-1}$, computation of eigenvalues and eigenvectors is complicated because the matrix is about one-third full. Thus, in this section, the matrix $B = -D^{-1}$, given by

$$B = -B^{-1}A = \begin{bmatrix}
0 & -\frac{1}{\beta_0} \\
-\frac{1}{\beta_1} & 0 & -\frac{2}{\beta_1} \\
-\frac{2}{\beta_2} & 0 & -\frac{3}{\beta_2} \\
& \ddots & \ddots \\
& & & 0 & -\frac{N-1}{\beta_{N-2}} \\
& & & & -\frac{N-1}{\beta_{N-1}} \\
& & & & & 0
\end{bmatrix} \quad (A.13)$$
will be considered. Recall that (see Eq. (4.47c)\)

$$\beta_j = -(2j + 1)(1 - \omega_0 f_j), \quad (A.14)$$

where $\omega_0$ is the single scatter albedo. The eigenvalues of $D$ are simply the negative inverse of the eigenvalues of $R$, and the eigenvectors of $D$ and $R$ are the same.

Even for $R$ it is not possible to obtain a closed form expression for the eigenvalues. The only general statement that can be made about the eigenvalues is that they are real, distinct, and symmetric about the origin. (See Section A.1 Arscott's Theorem [51] and its consequences apply here as well.) However, the tri-diagonal form has the advantage that relatively simple numerical transformations are required to obtain the eigenvalues accurately [52]. And because of the symmetry of the eigenvalues, only half of them need to be computed. Furthermore, once the eigenvalues are known, a very simple algorithm can be developed to compute the eigenvector elements. Finally, $R$ can be shown to be similar to a symmetric matrix identical in form to $R$. This property makes it possible to obtain fairly useful upper and lower bounds on the dominant eigenvalue.

A.2.1. Algorithm for the Eigenvector Elements

For simplicity in developing the algorithm, the following notation will be used:
Denoting the eigenvector of $R$ by $X_i$, and assuming that the $i^{th}$ eigenvalue is known, the equation for the eigenvector elements is

$$RX_i = \lambda_i X_i.$$  \hspace{1cm} (A.17)

Writing these equations out explicitly

$$a_1(X_i)_2 = \lambda_i (X_i)_1$$

$$b_1(X_i)_1 + a_2(X_i)_3 = \lambda_i (X_i)_2$$
\[ b_2(x_i)_2 + a_3(x_i)_4 = \lambda_i(x_i)_3 \]  
\[ \vdots \]
\[ b_{K-1}(x_i)_{K-1} + a_K(x_i)_{K+1} = \lambda_i(x_i)_K \]
\[ b_{N-1}(x_i)_{N-1} = \lambda_i(x_i)_N \]

Observe that starting with \((x_i)_1\), all of the other eigenvector elements can be computed sequentially as

\[ (x_i)_2 = \frac{\lambda_i}{a_1} (x_i)_1 \]  
\[ (x_i)_K = \frac{\lambda_i (x_i)_{K-1} - b_{K-2}(x_i)_{K-2}}{a_{K-1}}, \quad 3 \leq K \leq N \]

and the \(N^{th}\) element must also satisfy

\[ (x_i)_N = \frac{b_{N-1}(x_i)_{N-1}}{\lambda_i} \]

This sequential computation requires only about \(2N\) basic operations per eigenvector.

However, there is a potential problem with this approach. Since succeeding eigenvector elements are dependent on the values of all past elements, it is possible for error to accumulate. In particular, significant computational error may be introduced when the two terms in the numerator in Eq. (A.19) are very close to each other.
To see that it is possible for those two terms to be close, consider computing the element \((\bar{x}_1)_3\) corresponding to the dominant eigenvalue \(\lambda_1\):

\[
(\bar{x}_1)_3 = \frac{\lambda_1^2}{a_1} - b_1.
\] 

(A.21)

Here, \((\bar{x}_1)_1\) has been assumed to be 1. From Eq. (A.16),

\[
a_1 = \frac{1}{1 - \omega_0}
\]

\[
a_2 = \frac{2}{3} \frac{1}{1 - \omega_0 g}
\]

(A.22)

\[
b_1 = \frac{1}{3} \frac{1}{1 - \omega_0 g}
\]

where

\[
g = f_1 = 2\pi \int_{-1}^{1} d\mu \sigma(\mu) P_1(\mu)
\]

(A.23)

\[
= 2\pi \int_{-1}^{1} d\mu \mu \sigma(\mu)
\]

is the average cosine of the phase function, \(\sigma(\mu)\) (see Eq. (1.24)). Hence

\[
(\bar{x}_1)_3 = \frac{3}{2} \left[ \lambda_1^2 (1-\omega_0)(1-\omega_0 g) - \frac{1}{3} \right].
\] 

(A.24)
But it is shown below that the dominant eigenvalue may be close to

$$\lambda \approx \frac{1}{\sqrt{3(1-\omega_0)(1-\omega_0g)}}.$$ \hspace{1cm} (A.25)

Therefore, significant error may be introduced in the computation of $(\overline{x}_1)_3$, and this error will be propagated downward.

To avoid this problem, the eigenvalue elements are best computed in reverse order, starting with the $N^{th}$ element. Thus

$$(\overline{x}_i)_{N-1} = \frac{\lambda_i(\overline{x}_i)_N}{b_{N-1}}$$ \hspace{1cm} (A.26)

$$(\overline{x}_i)_K = \frac{\lambda_i(\overline{x}_i)_{K+1} - a_{K+1}(\overline{x}_i)_{K+2}}{b_{K}} \hspace{1cm} 1 \leq K \leq N-2$$

and the first element must also satisfy

$$(\overline{x}_i)_1 = \frac{a_1(\overline{x}_i)_{2}}{\lambda_i}.$$ \hspace{1cm} (A.27)

The advantage of computing the elements this way is that all the matrix elements for large values of $K$ are very nearly equal, and are approximately 0.5. The reason for this is that the phase function Legendre coefficients, $f_j$, tend to zero as the order increases. Hence the matrix elements $a_j$ and $b_j$ are given by
\[ a_j \approx \frac{j}{2j-1} \approx \frac{1}{2} \quad j \gg 1 \]  
(A.28)

\[ b_j \approx \frac{j}{2j+1} \approx \frac{1}{2} \quad j \gg 1 \]

Now computing \((\bar{x}_i)_{N-2}\) from Eq. (A.26), assuming \((\bar{x}_i)_N = 1\),

\[ (\bar{x}_i)_{N-2} = 4\lambda_i^2 - 1 \]  
(A.29)

Observe that the two terms in Eq. (A.29) are not close as long as \(\lambda_i \gg \frac{1}{2}\). This condition on \(\lambda_i\) certainly holds for the dominant eigenvalue, for typical choices of \(\omega_0\) and \(g\). And it will also hold for a significant number of the subdominant eigenvalues.

In fact, as long as \(K\) remains large enough so that the approximations in Eq. (A.28) hold true, subsequent eigenvector elements can be computed trivially:

\[ (\bar{x}_i)_{N-3} = 2\lambda_i (2\lambda_i)^2 - 2\lambda_i \approx (2\lambda_i)^3 \]  
(A.30)

\[ (\bar{x}_i)_{N-4} = 2\lambda_i (2\lambda_i)^3 - (2\lambda_i)^2 \approx (2\lambda_i)^4. \]

And in general

\[ (\bar{x}_i)_{N-m} \approx (2\lambda_i)^m. \]  
(A.31)
Equation (A.31) will hold until \( m \) gets large enough that the values of \( a_j \) and \( b_j \) are influenced significantly by the values of the \( f_j \). Then the same problem of error accumulation that occurred in the forward computation of the elements will become evident. However, there are now significantly fewer elements left to compute, and hence the total accumulated error after computation of the \( N \) elements is smaller.

A.2.2. Bounds on the Dominant Eigenvalue

As a preliminary to bounding the dominant eigenvalue of \( D \), it will be shown that \( R = -D^{-1} \) is similar to a symmetric matrix with the same tri-diagonal form. In fact, the transformation matrix is diagonal.

Define

\[
M = \begin{bmatrix}
m_1 \\
m_2 \\
\ddots \\
m_N
\end{bmatrix}
\]  \hspace{1cm} (A.32)

It will be shown that \( m_i \) can be found such that \( I = M R M^{-1} \) is of the form
For Eq. (A.33) to be satisfied, \( y_i \) must be given by

\[
\gamma_i = (M B M^{-1})_{i,i+1} = (M B M^{-1})_{i+1,i}
\]

(A.34)

or, referring to Eq. (A.15),

\[
\frac{m_i}{m_{i+1}} a_i = \frac{m_{i+1}}{m_i} b_i
\]

(A.35)

Thus, starting with \( m_1 = 1 \),

\[
m_i = \sqrt[k]{\prod_{i=1}^{k-1} \left( \frac{a_i}{b_i} \right)}
\]

(A.36)

are the required values of the \( m_i \). And therefore super- and sub-diagonal elements are thus related to the \( a_i \) and \( b_i \) by
The symmetric matrix $T$, which has the same eigenvalues as $R$, will now be used to bound the dominant eigenvalue, $\lambda_{\text{max}}$. It will be clear as the discussion proceeds what advantage is gained by symmetrizing $R$.

**Upper Bound**

Let $R_i$ and $T_j$ denote the sum of the absolute values of the entries in the $i^{\text{th}}$ row of $T$ and the sum of the absolute values of the entries in the $j^{\text{th}}$ column of $T$ respectively. Let

$$R = \max_i R_i$$

$$T = \max_j T_j$$

Then the dominant eigenvalue is bounded by [53]

$$\lambda_{\text{max}} \leq \min (R, T) .$$

(A.39)

Since $T$ is symmetric, $R = T$, and thus the bound simplifies to

$$\lambda_{\text{max}} \leq R .$$

(A.40)

Since all the elements of $T$ are positive, and since therefore
\[ \gamma_1 + \gamma_2 \geq \gamma_1 \quad (A.41) \]

\[ \gamma_{N-2} + \gamma_{N-1} \geq \gamma_{N-1}, \]

only rows 2 through \( N-1 \) need be considered in the maximization (see Eq. (A.33)). Thus

\[ \gamma_{\text{max}} \leq \max_{1 \leq i \leq N-2} (\gamma_i + \gamma_{i+1}) . \quad (A.42) \]

From Eqs. (A.16) and (A.37),

\[ \gamma_i = \sqrt{\frac{i^2}{(2i-1)(2i+1)(1-\omega_0 f_{i-1})(1-\omega_0 f_{i})}} \quad (A.43) \]

\[ \gamma_{i+1} = \sqrt{\frac{(i+1)^2}{(2i+1)(2i+3)(1-\omega_0 f_{i})(1-\omega_0 f_{i+1})}} . \]

Both \( \gamma_i \) and \( \gamma_{i+1} \) are maximized for \( i = 1 \), for typical values of \( \omega_0 \). The reason is that as \( i \) goes from 1 to \( N-2 \),

\[ \frac{1}{4} \leq \frac{i^2}{(2i-1)(2i+1)} \leq \frac{1}{3} \quad (A.44) \]

and

\[ \frac{1}{4} \leq \frac{(i+1)^2}{(2i+1)(2i+3)} \leq \frac{4}{15} , \]

a very small variation. Thus, when \( \omega_0 \) is close to 1, the \( i \) dependence of \( \gamma_i \) and \( \gamma_{i+1} \) is essentially determined by the values of the \( f_i \).
But for typical phase functions, the $f_i$ are non-increasing functions of $i$. Thus the smallest $i$ maximizes $f_i$, and hence $(1 - \omega f_i)^{-1}$.

For $i = 1$, then,

$$\lambda_{\text{max}} \leq \gamma_1 + \gamma_2$$

$$\leq \frac{1}{\sqrt{3(1-\omega_0 f_0)(1-\omega_0 f_1)}} + \frac{2}{\sqrt{15}} \frac{1}{\sqrt{(1-\omega_0 f_1)(1-\omega_0 f_2)}}$$

(A.44)

Using the fact that

$$f_0 = 2\pi \int_{-1}^{1} d\mu \sigma(\mu) = 1,$$  \(\text{(A.45)}\)

and (see Eq. (A.23)) $f_1 = g$, and the monotonicity of the $f_i$ to replace $f_2$ in Eq. (A.44) by $f_1$, the upper bound becomes

$$\lambda_{\text{max}} \leq \frac{1}{\sqrt{3(1-\omega_0)(1-\omega_0 g)}} + \frac{2}{\sqrt{15}} \frac{1}{(1-\omega_0 g)}$$

(A.46)

A looser, but simpler, bound comes from replacing $f_1$ by $f_0$ in the second term in Eq. (A.44):

$$\lambda_{\text{max}} \leq \frac{1.89}{\sqrt{3(1-\omega_0)(1-\omega_0 g)}}.$$  \(\text{(A.47)}\)
Lower Bound

Since $\mathcal{I}$ is symmetric, $\lambda_{\text{max}}$ is given identically by [54]

$$\lambda_{\text{max}} = ||I||$$  \hspace{1cm} (A.48)

where

$$||I|| = \max_{x_i} ||I x_i||.$$ \hspace{1cm} (A.49)

and $|| \cdot ||$ denotes the Euclidean norm. For all other $x_j \neq x_i$,

$$\lambda_{\text{max}} = ||I|| \geq ||I x_j||.$$ \hspace{1cm} (A.50)

The maximization indicated in Eq. (A.49) is difficult to carry out in general. However, a vector $x_j$ which emphasizes the larger elements of $\mathcal{I}$ will produce a relatively large lower bound in Eq. (A.50).

For simplicity, consider only vectors $x_j$ with a single non-zero element. This will give a vector $tx_j$ which is one of the columns of $\mathcal{I}$. Clearly, the second column of $\mathcal{I}$ is the appropriate one, since as shown above, it has the largest elements. Thus,
\[
\lambda_{\text{max}} > \| \gamma_2 \| = \sqrt{\gamma_1^2 + \gamma_2^2}
\]

\[
= \sqrt{\frac{1}{3(1-\omega_0)(1-\omega_0 g)} + \frac{4}{15} \frac{1}{(1-\omega_0 g)(1-\omega_0 f_2)}}
\]

\[
= \frac{1}{\sqrt{3(1-\omega_0)(1-\omega_0 g)}} \sqrt{1 + \frac{4}{5} \frac{(1-\omega_0)}{(1-\omega_0 f_2)}}
\]

Two looser, but simpler, bounds are

\[
\lambda_{\text{max}} > \frac{1}{\sqrt{3(1-\omega_0)(1-\omega_0 g)}} \sqrt{1 + \frac{4}{5} \frac{(1-\omega_0)}{(1-\omega_0 f_2)}}
\]

and

\[
\lambda_{\text{max}} > \frac{1}{\sqrt{3(1-\omega_0)(1-\omega_0 g)}}
\]

A.3. Eigenvalues and Eigenvectors of $E$

The matrix $E$ is given by

\[
E = A^{-1} C,
\]

where $A^{-1}$ is the matrix in Eq. (A.6) and from Eq. (4.47d), $C$ is
Although $E$ can be computed easily enough from the known matrices $\tilde{A}^{-1}$ and $\tilde{C}$, it is very complicated to write down in its complete generality. Thus, only the properties needed to obtain its eigenvalues and eigenvectors will be presented here.

First observe that $E$ has a very regular pattern of zeroes. Since all even numbered rows of $\tilde{A}^{-1}$ have zeroes to the right of the diagonal, while $\tilde{C}$ has only one non-zero element above the diagonal in each column, all even numbered rows of $E$ have zeroes to the right of the diagonal. Furthermore, since all even numbered columns of $\tilde{C}$ have non-zero elements only in odd numbered rows, while all non-zero elements of $\tilde{A}^{-1}$ to the right of the diagonal are in even numbered columns, all even numbered columns of $E$ have zeroes above the diagonal.

A similar argument can be made for the odd numbered rows and columns of $E$. Since all the odd numbered rows of $\tilde{A}^{-1}$ have zeroes to
the left of the diagonal, while $C$ has only one element below the diagonal in each column, all odd numbered rows of $E$ will have zeroes to the left of the diagonal. Also, since all odd numbered columns of $C$ have non-zero elements only in even numbered rows, while all non-zero elements of $A^{-1}$ to the left of the diagonal are in odd numbered columns, all odd numbered columns of $E$ will have zeroes below the diagonal.

The structure of this pattern of zeroes is summarized in Eq. (A.54):

\[
E = \begin{bmatrix}
  x & x & & \\
  x & & x & \\
  & x & & x \\
  & & & \ddots \\
  & & & x \\
  & & & x
\end{bmatrix}, \quad (A.54)
\]

where $x$ denotes diagonal elements and lines denote strings of zeroes.
A.3.1. Eigenvalues of $E$

The structure of $E$ makes obtaining its eigenvalues a trivial matter. Observe that if the determinant of $E$ is computed by alternating between expanding its subdeterminants by minors of the first row and first column, the determinant of $E$ is simply the product of the diagonal elements. Hence, the elements of the diagonal are the eigenvalues.

To see what these diagonal elements are, note that $E_{ii}$ is given by

$$E_{ii} = (A^{-1}C)_{ii} = i(i-1)\left(\frac{1}{A_{i,i+1}} - \frac{1}{A_{i,i-1}}\right).$$  \hspace{1cm} (A.55)

However (see Eq. (A.6)),

$$A^{-1}_{i,i+1} = 0 \quad \text{i even}$$

$$A^{-1}_{i,i-1} = \frac{i}{i-1} \quad \text{i odd}$$ \hspace{1cm} (A.56)

Therefore,

$$E_{ii} = -i \quad \text{i even}$$

$$E_{ii} = i-1 \quad \text{i odd}$$ \hspace{1cm} (A.57)
Thus, the diagonal elements, and hence the eigenvalues are given by

\[ \lambda = 0, \pm 2, \pm 4, \ldots, \pm N-2, -N. \] (A.58)

### A.3.2. Eigenvectors of \( E \)

The structure of \( E \) shown in Eq. (A.54) also affords a simple algorithm for computing the eigenvector elements. The equation for the eigenvector elements is

\[
\begin{bmatrix}
(X_i)_1 \\
(X_i)_2 \\
\vdots \\
(X_i)_N
\end{bmatrix}
= \lambda_i
\begin{bmatrix}
(X_i)_1 \\
(X_i)_2 \\
\vdots \\
(X_i)_N
\end{bmatrix}
\]

(A.59)

Observe that there is only one element in the \((N-1)^{st}\) row of \( E \). Thus, the \((N-1)^{st}\) equation for the eigenvector elements is \( (\lambda_i \text{ is assumed known}) \)
(N-2)(\overline{x}_i)_{N-1} = \lambda_i (\overline{x}_i)_{N-1} \quad (A.60)

If \lambda_i = N-2, then Eq. (A.60) does not determine \((\overline{x}_i)_{N-1}\) and it is arbitrary. If \lambda_i \neq N-2, then \((\overline{x}_i)_{N-1} = 0.

Now consider the (N-3)rd equation:

\[(N-4)(\overline{x}_i)_{N-3} + e_{N-3,N-1}(\overline{x}_i)_{N-1} = \lambda_i (\overline{x}_i)_{N-3} \quad (A.61)\]

If \((\overline{x}_i)_{N-1} = 0, then

\[(N-4)(\overline{x}_i)_{N-3} = \lambda_i (\overline{x}_i)_{N-3} , \quad (A.62)\]

and \((\overline{x}_i)_{N-3} = 0 unless \lambda_i = N-4. In that case \((\overline{x}_i)_{N-3}\) is arbitrary. If \((\overline{x}_i)_{N-1} \neq 0, then \((\overline{x}_i)_{N-3}\) can be computed in terms of \((\overline{x}_i)_{N-1}\).

This procedure repeats itself for all preceding odd numbered eigenvector elements. Either they are all zero, or one of them - the one corresponding to the row containing \lambda_i on its diagonal - becomes arbitrary. Once this happens, all subsequent odd numbered eigenvector elements can be computed sequentially from the ones that are known.

As for the even numbered eigenvector elements, it is easy to see that if the eigenvalue lies on an odd numbered row of \(E\), then all the even numbered eigenvector elements are zero. Simply compute the even numbered elements starting with the second equation:
-2(\overline{X}_i)_2 = \lambda_i (\overline{X}_i)_2 \quad (A.63) \\

Since \lambda_i lies on an odd numbered row, \lambda_i \neq -2 and thus (\overline{X}_i)_2 = 0. The next even numbered equation is

\[ E_{4,2}(\overline{X}_i)_2 - 4(\overline{X}_i)_4 = \lambda_i (\overline{X}_i)_4 , \quad (A.64) \]

and hence (\overline{X}_i)_4 = 0. Again, this procedure repeats itself for all subsequent even numbered eigenvector elements.

When the eigenvalue \lambda_i lies on an even numbered row, the procedure is directly parallel to that outlined above.

The following summarizes the procedure for computing the eigenvector elements of \( E_{ij} \):

- \( \lambda_i \geq 0 \) (i odd)
  1. All even numbered elements of \( X_i \) are zero.
  2. \( (\overline{X}_i)_j = 0, \ j > i \).
  3. \( (\overline{X}_i)_i \) is arbitrary.
  4. \( (\overline{X}_i)_j = \frac{1}{2} \sum_{k=1}^{i-j} \sum_{j=1}^{i-j} \sum_{j=1}^{i-j} E_{j,j+2k} (\overline{X}_i)_{j+2k} , j < i \).

- \( \lambda_i < 0 \) (i even)
  1. All odd numbered elements of \( X_i \) are zero.
  2. \( (\overline{X}_i)_j = 0, \ j < i \)
  3. \( (\overline{X}_i)_i \) is arbitrary.
  4. \( (\overline{X}_i)_j = \frac{1}{2} \sum_{k=1}^{j-i} \sum_{j=1}^{j-i} \sum_{j=1}^{j-i} E_{j,j-2k} (\overline{X}_i)_{j-2k} , j > i \).
A.4. Left Eigenvectors of \( D \) and \( E \)

In computations requiring functions of the matrices \( D \) and \( E \), the inverse of the matrix

\[
S = \begin{bmatrix}
X_1 & X_2 & \cdots & X_N \\
\end{bmatrix}
\]

is required. In general, this inversion is numerically very time consuming and often inaccurate for large matrices. However, the rows of \( S^{-1} \) are the left eigenvectors of the respective matrices. In the case of both \( D \) and \( E \), these left eigenvectors are easily computed by means of algorithms directly parallel to those developed, and hence inversion of \( S \) is not necessary. Because of the similarity of the algorithms for the right and left eigenvectors, algorithms for the latter will not be presented here.
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BIOGRAPHICAL NOTE

Warren Steven Ross was born in New York City, New York on June 27, 1953. He graduated from Roosevelt High School in Yonkers, New York in June, 1971.

Warren attended M.I.T. as an undergraduate and received a B.S. degree in Ocean Engineering in September, 1975. As an undergraduate, he entered the Ocean Engineering Department's Cooperative Program and gained work experience at Westinghouse Electric Corporation's Oceanic Division in Annapolis, Maryland, where he worked on acoustic propagation and sonar signal processing.

In 1975, he became a graduate student in the Department of Electrical Engineering and Computer Science at M.I.T. He has worked as a Teaching Assistant and as a Research Assistant in the Department. In September 1977, he received his M.S. degree in Electrical Engineering and Computer Science. He has been working toward his Ph.D. degree since then.

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