A Viable Multivariable Adaptive Controller with Application to Autonomous Helicopters

by

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M.S., Boston University (1996)

Submitted to the Department of Mechanical Engineering in partial fulfillment of the requirements for the degree of

Doctor of Philosophy in Mechanical Engineering

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

September 2001

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Authorized Access

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Abstract

Autonomous helicopters carry out missions in inaccessible hazardous environments. Their performance capability in speed, maneuverability and trajectory tracking must be comparable, if not superior, to manned vehicles. Control laws that meet desired performance specifications are difficult to design because of the uncertain, fully-coupled nonlinear dynamics of autonomous helicopters. Uncertainties in helicopter parameters not only change the dynamics of the system but the trim inputs themselves. Many of the current control designs assume low speeds and neglect aerodynamics as well as uncertainties in system parameters. As a result, these linear and nonlinear controllers are adequate at best at hover, and therefore not viable in real plants, as the unknown trim conditions and dynamics result in severe performance degradation even at moderate speeds. For an autonomous vehicle control system to be viable, it must accommodate uncertainties in the trim conditions on-line, the effect of the aerodynamics, and parametric uncertainties, for various realistic maneuvers beyond hover.

This thesis presents a viable multivariable adaptive control design methodology that is applicable to the needs of uncertain plants and high bandwidth requirements. Control in the cases where the full and partial state variables are available for measurement are considered. A systematic control design procedure that fully accommodates the aerodynamics of the autonomous vehicle is developed. The principal features of the proposed controller are the following: The multivariable adaptive controller accommodates both parametric uncertainties and unknown trim conditions through on-line adjustment of appropriate control parameters. Closed-loop stability and robustness are demonstrated through the use of suitable Lyapunov functions. Judicious integration between linear robust control methods and online adaptive strategies is incorporated in the control design to maximize off-line information about the nominal conditions and on-line measurements. A two-step nonlinear optimization procedure is carried out to determine nominal trim states that allows the arbitrarily close con-
vergence to the global minima by making use of prior information available about sub-components of the trim states during a given maneuver.

The control design methodology is tested on a high fidelity simulation of a Draper Laboratory autonomous helicopter. Extensive simulation studies were carried out, beginning from a vehicle model that includes complete vehicle aerodynamics, gravitational and inertial effects, rotor dynamics, and rotor-vehicle interactions. Five different inputs including the roll-cyclic and pitch cyclic angles of the main rotor, the pedal command for the tail rotor, collective pitch angle for the main rotor, and the throttle are assumed to be present. The controller is evaluated at various operating conditions and maneuvers where aerodynamic nonlinearities and parametric uncertainties become dominant. Simulations are performed on a nonlinear longitudinal dynamics model to track step and sinusoidal changes in forward flight velocity, and a maneuver involving jumps over hurdles. Simulations on the full three dimensional nonlinear helicopter model are performed for vertical flight and coordinated turn maneuvers. These simulation studies established conclusively that the proposed adaptive controllers exhibited a satisfactory tracking performance even as the speeds and bandwidth requirements increased well beyond hover. Even as the parametric uncertainties and nonlinearities were increased by about 20% of their nominal values, the controllers continued to demonstrate satisfactory performance.

In summary, the new control structure together with a trim error estimate, controller parameter update laws and system augmentation for stable adaptation leads to a stable robust system with enhanced performance, thereby resulting in a viable multivariable adaptive controller for autonomous helicopters. As autonomous helicopters are among the most difficult plants to control, the newly developed control design methodology proposed here is applicable to a large class of highly coupled uncertain multivariable systems.

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Acknowledgments

I would like to express my sincere gratitude to my advisor Professor Anuradha Annaswamy. Her enthusiasm, dedication and rigor have contributed immensely to the success of this work. I have greatly benefited from her inspiration and guidance which have helped me mature as a researcher. I am deeply grateful to Dr. Rami Mangoubi for patiently helping me at virtually every stage of my research. His advice, support and close involvement with the research were invaluable. I would also like to thank Neil Adams, Brent Appleby, Marc McConley, George Schmidt, Eli Gai, and Paul Motyka from Draper Laboratory for their help and support.

I would like to thank Professors Eric Feron and Derek Rowell for their time and effort in serving on my thesis committee. Their comments and suggestions were very valuable and helped me look at my work from different perspectives.

I would also like to thank my colleagues Jean-Pierre Hathout and Aleksandar Kojic for their support and many stimulating discussions. Many thanks to the other students in the adaptive control laboratory, Chengyu Cao, Mehmet Yunt, Nhut Tan Ho, Jennifer Rumsey, Sungbae Park and Chayakorn Thanomsat for creating a pleasant collaborative atmosphere. I am grateful to my friends Prashant Sharma, Srinivasan Chakravarthi and Leo Wilson who have helped me greatly these past five years.

I am deeply grateful to my parents, Joseph and Nalini, and my sister, Deepti, for their love, encouragement and support.

This work was sponsored by Draper Laboratory University Research and Development Grant No. 902.
Contents

1 Introduction 11
   1.1 Current Status of Autonomous Vehicle control 14
   1.2 Viability of Adaptive Control 17
   1.3 Contributions of the Thesis 18
   1.4 Synopsis 20

2 Statement of the Problem 21
   2.1 Model of the Helicopter 21
      2.1.1 Vehicle Dynamics 21
      2.1.2 Aerodynamics 27
      2.1.3 Main Rotor Dynamics 29
      2.1.4 Tail Rotor Dynamics 33
      2.1.5 Engine Dynamics 35
      2.1.6 Quaternions 36
      2.1.7 Complete Dynamics 36
      2.1.8 Longitudinal Dynamics 37
   2.2 Problem Statement 39

3 Viable Adaptive Control Design: the Full State Feedback Case 41
   3.1 Pseudo-trim Condition 41
   3.2 Control Design when States are Accessible 42
      3.2.1 Reference Model 43
# List of Figures

1-1  Schematic of Helicopter Dynamics .............................. 12
2-1  Frontview of Helicopter ........................................ 22
2-2  Sideview of Helicopter .......................................... 23
2-3  Topview of Helicopter ........................................... 24
3-1  Linearized System and Reference Model ...................... 45
3-2  Dynamic Inversion ................................................ 48
3-3  Dynamic Inversion in Closed Loop .............................. 50
3-4  Dynamic Inversion with Parametric Uncertainties ........... 51
3-5  System with Integral Action ..................................... 53
3-6  Adaptive System for Full State Feedback ....................... 54
3-7  \( u \) for steps in forward flight : Random steps ............. 57
3-8  \( u \) for steps in forward flight : Adaptive control vs. DI ... 58
3-9  \( w \) for steps in forward flight : Adaptive control vs. DI ... 59
3-10 \( u \) for steps in forward flight with 5% noise ............... 61
3-11 \( w \) for steps in forward flight with 5% noise ............... 62
3-12 \( u \) for steps in forward flight against DI with integral action ... 63
3-13 \( w \) for steps in forward flight against DI with integral action ... 64
3-14 \( u \) for steps in forward flight with \( \dot{d} \) ....................... 65
3-15 \( w \) for steps in forward flight with \( \dot{d} \) ....................... 66
3-16 \( u \) for random steps in forward flight with \( \dot{d} \) against DI with integral action ................................. 67
3-17 $u$ for sinusoidal input for forward flight ........................................ 69
3-18 $U$ for complex maneuver in $U - W$ plane. Comparison against DI with integral action ................................................................. 71
3-19 $W$ for complex maneuver in $U - W$ plane. Comparison against DI with integral action ................................................................. 72
3-20 $U$ vs. $W$ for first cycle of complex maneuver in $U - W$ plane. Comparison against DI with integral action ................................................................. 73
3-21 $U$ vs. $W$ for first cycle of complex maneuver in $U - W$ plane. Blow up of maneuver region. Comparison against DI with integral action . 74
3-22 $U$ vs. $W$ for eighth cycle of complex maneuver in $U - W$ plane. Comparison against DI with integral action ................................................................. 75

4-1 Pole-placement Controller ................................................................. 81
4-2 Pole-placement Controller with Feedforward Gain ............................ 85
4-3 Adaptive Pole-placement Controller ................................................... 93
4-4 Gradient Stabilizer ........................................................................... 94
4-5 $w$ for steps in vertical flight for partial state feedback ......................... 98
4-6 $u$ for steps in vertical flight for partial state feedback ......................... 99
4-7 $q_i$ for turn with partial state feedback ............................................. 101
4-8 $X(ft)$ for turn with partial state feedback ......................................... 102
4-9 $X$ vs. $Y$ for turn with partial state feedback- First Cycle ................... 103
4-10 $X$ vs. $Y$ for turn with partial state feedback- Last Adaptive Cycle ... 104
4-11 $X$ vs. $Y$ for turn with partial state feedback- Adaptation Stopped ... 105
Chapter 1

Introduction

Autonomous vehicles are needed to carry out tasks in remote areas not easily accessible to humans. They may be used for applications that involve work in hazardous environments thereby eliminating the need for local human operators. For operations in which conventional vehicles which require human handling are expensive, they can serve as a cheaper substitute while also having the advantage of smaller size. Such vehicles need to be capable of achieving commanded objectives with minimum communication with the ground. On-board systems that can command and control the vehicle during its mission are hence required. The autonomous vehicle needs to have a performance comparable to manned vehicles in maneuverability and tracking of command signals for a broad range of mission objectives. The controller needs to keep the vehicle stable, reduce the effect of disturbances and noise, and account for changes in the environment or the vehicle. At the same time the control system needs to enable the vehicle meet performance objectives, such as, high speed flight, fast maneuvering and accurate command following.

Helicopter dynamics are highly nonlinear and fully-coupled. This can be seen from a schematic of the dynamics in Figure 1-1. The thrust calculation is a nonlinear function of all attitude variables. Unlike aircraft control where it is mainly a function of the angle of attack, in the case of helicopters it depends on all three attitude angles. Control authority is limited to just the variations in pitch of main rotor and tail rotor
Figure 1-1: Schematic of Helicopter Dynamics
blades and the throttle. Furthermore, the tail rotor needs to exactly cancel out the rotational torque due to the main rotor in order for the helicopter to maintain steady yaw angle. Autonomous helicopter control is thus complicated as lack of pilot action means that the controller has to generate these exact trim commands. For example, system uncertainties such as a change in the mass would need a change in the thrust to maintain the helicopter at hover.

The research presented in this thesis pertains to a control design method for autonomous helicopters. The highly nonlinear and cross-coupled dynamics of helicopters and stringent performance requirements for autonomous vehicle command following make this task difficult. The parametric uncertainties cause the trim to be unknown. At moderate speeds beyond hover, unknown trim conditions lead to severe performance degradation with linear and nonlinear controllers. Linear control designs exhibit either poor tracking performance or poor transient performance. Nonlinear control algorithms, on the other hand, are either not applicable here because of the complexity of the dynamics and lack of access to the state, or require very restrictive conditions to be satisfied. Moreover, many nonlinear methods are computationally expensive and require extensive calculations prior to implementation. It should also be pointed out that most existing control designs have been demonstrated only at low speeds near hover where aerodynamics are neglected or with near-zero parametric uncertainties. In this thesis, a new adaptive control design is developed which is capable of providing improved performance at high speeds without compromising on the robustness of the design. Aerodynamics cannot be neglected at these speeds and are included in the design procedure. The controller includes an adaptive estimate of the trim, as well as parameters that are updated online to account for uncertainties in the vehicle and the environment. The controller is applicable for a large class of highly coupled multivariable systems with uncertainties. A general design procedure that describes how the different components of the controllers can be determined, is presented. The effectiveness of this design in dealing with uncertainties in system parameters and unknown trim conditions for autonomous helicopters is demonstrated.
The design and simulations are carried out at speeds where aerodynamic nonlinearities become dominant.

1.1 Current Status of Autonomous Vehicle control

The design of a controller for autonomous helicopters needs to address several issues. The kinematics and aerodynamics of helicopter motion make the dynamic equations strongly nonlinear and complex. As a result, the underlying model has a large number of nonlinearly coupled states making the task of finding simple controllers very difficult. For example, even though most helicopter models assume lateral symmetry and therefore decoupling of the longitudinal and lateral dynamics [1], the motion of the rotor leads to considerable coupling of the longitudinal and lateral modes [2]. Furthermore, the parameters that are needed to generate the model are not accurately measurable. They may be affected by the environment (e.g. the aerodynamic constants) or by the helicopter (e.g. the effective rotor diameter, or lift curve slopes). The parameters can also change while the vehicle is in motion. Examples of these would be in-flight changes in the mass and moments of inertia, either due to delivery of payloads that are not accurately measurable, or as a result of fuel consumption. The dynamics of the vehicle also change because of ageing and wear. The unknown system parameters cause the trim conditions for the helicopter to be unknown. Quite often, the available models do not represent the complete dynamics of the helicopter despite their complexity. Unmodeled dynamics are almost always present and the controller needs to account for such unknown errors. Due to all of the above reasons, the requisite controller must accommodate the parametric uncertainties, unknown trim, the nonlinear dynamics, unmodeled dynamics, and any external or internal disturbances that may be present. While the above discussion is pertinent to any general helicopter problem, in the case of autonomous helicopters, these issues become even more important. The vehicle needs to sense and adjust to any changes in the vehicle dynamics or the environment while maintaining only minimal contact.
with the ground. The vehicle needs to be accurate in command following since pilot feedback and correction is not available. The trim error which is eliminated by the pilot in a manned vehicle by manual corrections, leads to a severe degradation in the performance when the helicopter is autonomous. Designs that accommodate these errors without sacrificing robustness need significant trade-offs with performance \[3\].

Helicopter control design has been carried out thus far mostly using linear methods. Control designs using Eigenstructure Assignment \[4\]–[9], use linearized helicopter models about hover or uniform forward flight trim conditions. The trim conditions are assumed to be accurately known and the modes are assumed to be decoupled. Controllers are then designed to provide stability while ensuring robustness to unmodeled dynamics, nonlinearities in the dynamics, parametric uncertainties and disturbances. For such controllers, the computations are performed off-line on a limited number of linearized time-invariant models representing the dynamics at different flight conditions. These require extensive gain-scheduling computations and are capable of only handling mild nonlinearities. Such designs cannot handle substantial changes in aircraft configuration, model inaccuracies or parametric uncertainties.

The modern robust control designs using $H_2$, $H_\infty$ \[3\], \[10\] and $\mu$-synthesis \[11\]–\[16\], have also been proposed for control of helicopters. These approaches attempted to remove the assumption that the modes were decoupled. These designs have better robustness properties. However, these methods also use linearization about hover and assume accurately known trim conditions to design the controllers. The computations are carried out off-line, as in eigenstructure assignment designs, and then gain-scheduled to generate a composite control law. The process of tuning the gains is long and expensive. Robustness to uncertainties in dynamics and disturbances (including trim) in measured outputs and control signals comes with a deterioration in command following. Performance becomes worse with higher parametric or model uncertainties.

In order to alleviate the problems associated with the above designs, Dynamic Inversion \[17\]–\[19\], was proposed. This involves inverting the dynamics of the plant using a model in the forward loop. This acts to simplify the forward loop linearized
dynamics thereby making feedback control easier. Robust linear control methods are then used to design the closed loop. Gain scheduling requirements are reduced significantly with this controller. However nonlinearities and uncertainties still cause degraded performance. Furthermore, non-minimum phase dynamics can only be approximately inverted in order to avoid unstable pole-zero cancellation. In the absence of exact knowledge of the linear dynamics, robustness to nonlinearities and disturbances is compromised. This is the case when parametric uncertainties are present [3].

Artificial neural network based controllers have been proposed to improve the performance of Dynamic Inversion and other linear controllers [20]–[26]. The mismatch in the nonlinear and linear dynamics is sought to be learnt using an artificial neural network online. This is expected to result in a cancellation of the nonlinearities in addition to the linear dynamics. The major disadvantage of this scheme is that the stability guarantees are local and limited to small regions of state space and parameter values. The initial state and neural network parameters need to be close to ideal values. For neural networks there is no method at present to guarantee such a bound on initial parameter error based on observed network output error. Uncertainties in parameters such as mass or moment of inertia have not been incorporated in the designs. These will severely affect stability as the trim state and ideal parameters will be very different from those based on nominal system parameter values.

Another nonlinear approach to this control problem uses differential flatness or feedback linearizability of an approximate nonlinear model of the dynamics [3], [27], [28]. This improves the tracking by an approximate cancellation of the nonlinear dynamics as compared to dynamic inversion where only the linear dynamics are cancelled. However, here too parametric uncertainties and ignored aerodynamics result in degraded performance and generate steady-state errors. Eliminating these errors requires larger control inputs and introduces undesirable transients [3]. Controllers using fuzzy logic have been proposed [3], [29], [30] to accommodate these uncertainties. However, the controller results in satisfactory performance only for operating condi-
tions that are close to hover. Yet another point to note is that currently stability results for closed-loop systems with such “soft-computing” based controllers such as fuzzy logic and genetic algorithms are not currently available.

Nonlinear controllers based on robust backstepping[31] designs for approximate helicopter models are proposed in [32], [33]. The advantage of this method is that it provides a Lyapunov function in its proof of stability which can be used for easy stability proofs of Hybrid systems used for overall guidance and control of the helicopter. The controlled system is robust to model approximation errors, uncertainties and disturbances. However, there is a trade-off between this robustness and performance.

In order to reduce the complexity of the motion planning problem for the autonomous helicopter, hierarchical hybrid control schemes have been proposed which incorporate continuous inner control layers and discrete logical decision making outer layers [34]–[38]. Inner layer controllers which have Lyapunov function based stability guarantee enable easier proof of stability for such hybrid systems.

1.2 Viability of Adaptive Control

For an autonomous vehicle control system to be viable, it must be capable of making on-line changes to accommodate issues such as a lack of pilot action which manifests itself through uncertainties in the trim commands. Furthermore, the effect of aerodynamics and uncertainties on the dynamics, for realistic maneuvers at speeds significantly greater than hover, need to be addressed.

Adaptive control has so far not been explored for this autonomous helicopter problem. An early attempt to design adaptive controllers for aircraft control was the quasi-adaptive control scheme proposed in (Athans, et al [39]). This relied on linear estimators to estimate parameters of the system and periodically updated robust linear controllers. Such a scheme has also been investigated in a helicopter control context in [40], [41]. This approach suffers from the drawback of having no theoretical stability guarantees. It is also constrained by the limitations of the robust control
methods that are part of the design process. While adaptive control schemes based on analytic design methods have been proposed in the aircraft and spacecraft control context, for example, [42]–[45], there is a lack of similar work on helicopter control. This thesis attempts to overcome this shortcoming and provides viable adaptive controllers for the challenging problem of autonomous helicopters. This controller can deal with the issues of system and environmental uncertainties and provide stable adaptive laws that result in greatly improved performance. The controller is viable because it provides a design procedure to address issues such as non-minimum phase zeros, unknown trim, singular high frequency gain, which arise in the case of the autonomous helicopter.

1.3 Contributions of the Thesis

This thesis develops a controller that adapts online to uncertainties in helicopter parameters and improves performance in high speed and high bandwidth maneuvers where cross-coupling of state dynamics and nonlinearities become important. The controller possesses several unique features which include (i) the inclusion of all aerodynamic components in the design model, (ii) the determination of a MIMO controller structure and parameter adaptation strategies that guarantee closed-loop stability through the existence of Lyapunov functions which help in easy integration of the controllers with proposed hybrid control methods, (iii) accommodation of unknown trim conditions that may be due to uncertainties and nonlinearities by determining an online estimate of the trim error, (iv) a systematic design procedure that enables the integration of linear robust control methods with adaptive control components, where the former is used to determine the starting values for the latter, (v) successful demonstration and improvement in performance using a high-fidelity nonlinear model of the autonomous helicopter, and (vi) the ability to transition into a learning controller where the adaptation is switched off after a prescribed set of command-following tasks have been completed.
The following are the specific contributions of this thesis:

- Adaptive controller design for case when states are accessible.
  - The requirements on plant structure in model reference adaptive control are relaxed thus reducing the order of the adaptive controller.
  - Effects of unknown trim are reduced by incorporating an adaptive estimate of the trim error.
  - The adaptive controller guarantees stability and robustness through a Lyapunov based stability proof.
  - Simulations results with the adaptive controller at high speeds show substantial improvement in performance of helicopter longitudinal dynamics with uncertainties and including aerodynamics.

- Design of adaptive control algorithm for case without full state access.
  - A computationally efficient procedure to obtain near global minima in the optimization procedure for the generation of trim conditions for the complete nominal nonlinear dynamics of the helicopter.
  - A systematic procedure for coprime matrix fraction decomposition needed for model reference adaptive control design.
  - An adaptive estimate of the trim error.
  - A controller design for case with non-minimum phase zeros. A procedure for obtaining a post-compensator which gives a minimum phase input-output equivalent system for the helicopter is provided. The controlled system using this modified output is stable with adaptation.
  - Precompensator to modify the system in a manner that provides an error measure for stable adaptation.
  - Adaptive laws that guarantee stability through a Lyapunov function
Simulation results with this controller for full authority control of high fidelity model which show substantial improvement in performance over other designs. Simulations are carried out with aerodynamics and substantial uncertainties in parameters.

1.4 Synopsis

The rest of the thesis is organized as follows. In chapter 2 a nonlinear model of the helicopter dynamics, which is used in chapters 3 and 4, is presented followed by a statement of the problem. In chapter 3 the adaptive controller for the case with full state access is developed. Then the simulation results obtained using this controller are presented. In chapter 4 the adaptive control design without full state access is presented. The simulation results with this controller are then presented and discussed. Conclusions and suggestions for future work are given in chapter 5.
Chapter 2

Statement of the Problem

In this chapter we introduce the problem of control of autonomous helicopters. First, we present a helicopter dynamics model developed at Draper Laboratory [1], [46]–[49], in section 2.1. The complete dynamics model is presented here. In section 2.1.8, a reduced longitudinal dynamics model is then presented. Finally in section 2.2 the problems faced in control of the autonomous helicopter because of uncertainties in its parameters are introduced.

2.1 Model of the Helicopter

2.1.1 Vehicle Dynamics

The helicopter model presented in [1], [46]–[49], is obtained by considering the fuselage of the helicopter as a rigid body. The fuselage is attached to the main rotor and tail rotor. The 6-DOF equations of the fuselage are derived from Newton’s second law. $\phi, \theta, \psi$, are the Euler angles representing the roll, pitch and yaw angles of the helicopter (Figures 2-1–2-3). The roll, pitch and yaw rates, $p, q, r$, are given by:

$$
\begin{bmatrix}
\dot{p} \\
\dot{q} \\
\dot{r}
\end{bmatrix} =
\begin{bmatrix}
I_{xx} & 0 & 0 \\
0 & I_{yy} & 0 \\
0 & 0 & I_{zz}
\end{bmatrix}^{-1}
\begin{bmatrix}
L \\
M \\
N
\end{bmatrix}
$$

(2.1)
Figure 2-1: Frontview of Helicopter
Figure 2-2: Sideview of Helicopter
Figure 2-3: Topview of Helicopter
where $L, M, N,$ are the Moments about the center of gravity, and $I_{xx}, I_{yy}, I_{zz},$ are the roll, pitch, and yaw axis moments of inertia. Similarly, the forward, sideways and downward body-fixed velocities, $u, v, w,$ are described by:

$$
\begin{bmatrix}
\dot{u} \\
\dot{v} \\
\dot{w}
\end{bmatrix} = \frac{1}{m} \begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix}
$$

(2.2)

where $X, Y, Z,$ are the forces about the center of gravity on the helicopter. $m$ is the mass of the helicopter and it is computed along with the weight, $W,$ by these relations.

$$
m = m_e + \lambda_f m_f. \quad (2.3)
$$

$$
W = mg. \quad (2.4)
$$

Here, $\lambda_f,$ is the fraction of fuel left in the helicopter, $m_e,$ is the mass without fuel, $m_f,$ is the fuel capacity and, $g,$ is the acceleration due to gravity. The forces and moments on about the center of gravity consist of components due to aerodynamics (A), main rotor (M), the tail rotor (T), the weight and the Euler coupling terms due to the rotation of the body-fixed coordinate axes. The helicopter orientation states can be identified from the schematic in Figures 2-1–2-3. By convention, when determining the Euler angles the following sequence is followed. Starting from the ground fixed reference frame:

1. Rotate about the $z$-axis, nose right (positive yaw $\psi$).

2. Rotate about the new $y$-axis, nose up (positive pitch $\theta$).

3. Rotate about the new $x$-axis, right side of main rotor down (positive roll $\phi$).

Conversely, if going from the body-fixed reference frame to the ground fixed reference frame, the sequence roll, pitch, yaw must be followed. The position of the helicopter in ground fixed coordinates can be found by integrating its velocity in the ground
frame. If the ground fixed velocity is $v^L$, then the helicopter position defined by, $x, y, z$, is given by:

$$
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{z}
\end{bmatrix} = V^L. 
$$

(2.5)

The transformation matrix to convert vectors from body fixed coordinates to ground fixed coordinates, $T^L_B$, is related to the Euler angles in the following equation.

$$
T^L_B = \begin{bmatrix}
\cos \psi \cos \theta & \cos \psi \sin \theta \sin \phi - \sin \psi \cos \phi \\
\sin \psi \cos \theta & \sin \psi \sin \theta \sin \phi + \cos \psi \cos \phi \\
-\sin \theta & \cos \theta \sin \phi \\
\cos \psi \sin \theta \cos \phi + \sin \psi \sin \phi & \sin \psi \sin \theta \cos \phi - \cos \psi \sin \phi \\
& \cos \theta \cos \phi
\end{bmatrix}.
$$

(2.6)

Thus the relation between the ground fixed velocity, $V^L$, and the body fixed velocities, $u, v, w$, is:

$$
V^L = T^L_B \begin{bmatrix}
u \\
v \\
w
\end{bmatrix},
$$

(2.7)

and the air relative velocities, $u_a, v_a, w_a$, are given by:

$$
\begin{bmatrix}
u_a \\
v_a \\
w_a
\end{bmatrix} = (T^L_B) (V^L - V^L_w).
$$

(2.8)
Here, $V_{w}^{L}$, is the wind velocity with respect to the ground frame of reference. The forces and moments can now be expressed as

$$
\begin{bmatrix}
X \\
Y \\
Z \\
L \\
M \\
N
\end{bmatrix} =
\begin{bmatrix}
X_A \\
Y_A \\
Z_A \\
L_A \\
M_A \\
N_A
\end{bmatrix} +
\begin{bmatrix}
X_M \\
Y_M \\
Z_M \\
L_T \\
M_T \\
N_T
\end{bmatrix} +
\begin{bmatrix}
0 \\
Y_T \\
0 \\
0 \\
0 \\
0
\end{bmatrix} +
\begin{bmatrix}
T_L^B \\
W
\end{bmatrix} \begin{bmatrix}
rv - qw \\
pw - ru \\
qu - pv
\end{bmatrix},
$$
(2.9)

Here the transformation matrix from body to ground fixed coordinates, $T_L^B = (T_B^L)^T$.

### 2.1.2 Aerodynamics

The aerodynamic forces and moments are caused by the drag forces on the fuselage, $X_{f, fus}, Y_{f, fus}$, and, $Z_{f, fus}$, and the horizontal tail, $Z_{f, ht}$ and, $Y_{f, vt}$ and the positions relative to the c.g. where they act on the fuselage.

$$
\begin{bmatrix}
X_A \\
Y_A \\
Z_A \\
L_A \\
M_A \\
N_A
\end{bmatrix} =
\begin{bmatrix}
X_{f, fus} \\
Y_{f, fus} + Y_{f, vt} \\
Z_{f, fus} + Z_{f, ht} \\
-Y_{f, fus}z_{fus} - Y_{f, vt}z_{vt} \\
X_{f, fus}z_{fus} - Z_{f, ht}x_{ht} \\
Y_{f, vt}x_{vt}
\end{bmatrix}.
$$
(2.12)

Here, $z_{fus}$ is the distance of fuselage center of pressure (c.p.) location below the center of gravity (c.g.), $z_{vt}$ is the distance of vertical tail c.p. forward of the c.g., $x_{ht}$ is the distance of horizontal tail c.p. forward of the c.g., and $x_{vt}$ is the distance of vertical tail c.p. forward of c.g. The drag forces, $X_{f, fus}, Y_{f, fus}, Z_{f, fus}$, are proportional to the square of the air-relative velocities in their respective directions.

$$
X_{f, fus} = \frac{1}{2} \rho X_{uu, fus}|u_a|u_a,
$$
(2.13)
\[ Y_{f,fus} = \frac{1}{2} \rho Y_{vv,fus} |v_a|v_a, \quad (2.14) \]
\[ Z_{f,fus} = \frac{1}{2} \rho Z_{ww,fus} |w_a|w_a. \quad (2.15) \]

\( \rho \) is the atmospheric density, \( X_{uu,fus} \) is the axial fuselage drag coefficient, \( Y_{vv,fus} \) is the lateral fuselage drag coefficient and, \( Z_{ww,fus} \) is the vertical fuselage drag coefficient. The drag, as expected, is always in the direction opposite to the motion of the helicopter relative to air. Hence, \( X_{uu,fus}, Y_{vv,fus}, Z_{ww,fus} \), are all negative constants.

The lift force, \( Z_{f,ht} \), and side force \( Y_{f,vt} \) due to the tail are calculated by the procedure described below.

The downwash station, \( d_{dw} \), describes the placement of the horizontal tail relative to the downwash air stream. A uniform downwash velocity profile is assumed and a factor \( \epsilon_{ht} \) is used to keep track of whether the horizontal tail is in the downwash stream. The downwash station is determined by the following expression:

\[
d_{dw} = \begin{cases} \frac{u_a(z_{ht} - z_{mr})}{v_{i,mr} - w_a} + x_{ht} - x_{mr} + R_{mr} & \text{if } v_{i,mr} > w_a \\ 0 & \text{otherwise} \end{cases}
\]  

(2.16)

Here, \( x_{mr} \) is the main rotor hub location forward of the c.g., \( z_{mr} \) is the main rotor hub location below the c.g., and \( R_{mr} \) is the main rotor radius. \( v_{i,mr} \) is the average downwash velocity and is obtained from the main rotor dynamics described in subsection 2.1.3. The air relative velocity of the tail, \( w_{a,ht} \), is given as:

\[
w_{a,ht} = w_a - \epsilon_{ht}v_{i,mr} - x_{ht}q,
\]

(2.17)

\[
\epsilon_{ht} = \begin{cases} 1 & \text{if } d_{dw} > 0 \text{ and } d_{dw} < R_{mr} \\ 0 & \text{otherwise} \end{cases}.
\]

(2.18)

The lift force, \( Z_{f,ht} \), is now calculated assuming constant lift coefficients until a stall force, \( Z_{f,ht,max} \), is reached.

\[
Z_{f,ht,max} = \frac{1}{2} \rho Z_{VV,ht,max} (u_a^2 + w_{a,ht}^2),
\]

(2.19)

\[
Z_{f,ht,max}^* = \frac{1}{2} \rho (Z_{uu,ht}|u_a|u_a + Z_{uw,ht}|u_a|w_{a,ht} + Z_{ww,ht}|w_{a,ht}|w_{a,ht}),
\]

(2.20)
Here, $Z_{V,V,ht,max}$ is the maximum horizontal tail lift coefficient, $Z_{uu,ht}$ is the trim horizontal tail lift coefficient, $Z_{uw,ht}$ is the horizontal tail lift coefficient due to angle of attack and, $Z_{ww,ht}$ is the horizontal tail lift coefficient due to downwash. The side force on the vertical tail, $Y_{f,vt}$, is computed similarly with the added assumption that the tail is always in the sidewalk air stream of the tail rotor.

\begin{equation}
Y_{f,vt} = \begin{cases} 
-\frac{1}{2} \rho (Y_{uu,ht}|u_a|u_a + Y_{uv,vt}|v_{a,vt}|u_a + Y_{vw,vt}|v_{a,vt}|u_a), \\
-\frac{1}{2} \rho Y_{V,V,vt,max}(u_a^2 + v_{a,vt}^2),
\end{cases}
\end{equation}

\begin{equation}
Y_{f,vt,\text{max}} = \begin{cases} 
Y_{f,vt} & Y_{f,vt} \leq Y_{f,vt,\text{max}} \\
 otherwise
\end{cases}
\end{equation}

$V_{i,vt}$ is the average sidewash velocity described in the section on tail rotor dynamics, $Y_{V,V,vt,max}$ is the maximum vertical tail lift coefficient, $Y_{uu,vt}$ is the trim vertical tail lift coefficient, $Y_{uv,vt}$ is the vertical tail lift coefficient due to sideslip angle and, $Y_{vw,vt}$, is the vertical tail lift coefficient due to sidewash.

### 2.1.3 Main Rotor Dynamics

The main rotor forces and moments are caused by the thrust $T_{mr}$ and the torque $\tau_{mr}$ generated by the rotor. These are, therefore, given by the following equations.

\begin{equation}
\begin{bmatrix}
X_M \\
Y_M \\
Z_M
\end{bmatrix} = 
\begin{bmatrix}
-T_{mr}a_1 \\
T_{mr}b_1 \\
-T_{mr}
\end{bmatrix}
\end{equation}
The forces are the components of the thrust along the \(x\), \(y\) and \(z\) axes. \(a_1\), the rotor disk pitch angle, is the effective angle of pitch measured from the helicopter fuselage axes at which the thrust acts. Similarly, \(b_1\) is the rotor disk roll angle and signifies the roll angle at which the thrust acts on the helicopter body. \(x_{mr}\) and \(z_{mr}\) are the main rotor hub location forward of the c.g. and below c.g. respectively. The moments are products of the forces and the distances at which they act from the c.g. The thrust, \(T_{mr}\), is calculated by the procedure described below. A few auxiliary variables related to the main rotor are first derived. The tip velocity of the rotor, \(v_{tip}\), equivalent flat plate area, \(F_{mr}\), rotor solidity, \(\sigma_{mr}\) and the aggregate blade velocities normal to the rotor disk plane, \(w_r\), and to the blade, \(w_b\), are first calculated.

\[
\begin{bmatrix}
L_M \\
M_M \\
N_M
\end{bmatrix} =
\begin{bmatrix}
-T_{mr} b_1 z_{mr} \\
T_{mr} x_{mr} - T_{mr} a_1 z_{mr} \\
-\tau_{mr}
\end{bmatrix}.
\]

\(2.27\)

\(v_{tip} = \Omega R_{mr}\),

\(2.28\)

\(F_{mr} = C_{D0,mr} R_{mr} b_{mr} c_{mr}\),

\(2.29\)

\(\sigma_{mr} = \frac{b_{mr} c_{mr}}{\pi R_{mr}}\),

\(2.30\)

\(w_r = w_a + a_1 u_a - b_1 v_a\),

\(2.31\)

\(w_b = w_r + \frac{2}{3} \Omega R_{mr} U_{col}\),

\(2.32\)

\(2.33\)

\(\Omega\), is the main rotor angular rate and \(U_{col}\) is the main rotor blade collective pitch angle. The constant parameters are, \(R_{mr}\), the main rotor radius, \(a_{mr}\), the lift curve slope of the main rotor blade, \(b_{mr}\) number of main rotor blades, \(c_{mr}\), chord of the main rotor blade, and, \(C_{D0,mr}\), the zero-lift drag coefficient of the main rotor blades. The effect of the ground is felt when the helicopter is close to the ground as in the case of landing or take-off. The downwash velocity of the air passing through the main rotor is reduced if the helicopter is close to the ground. The effect of the ground on
the thrust is expressed through a factor, $\eta_{GE}$, that changes the downwash velocity, $v_{i,mr}$ and thus the thrust, $T_{mr}$. The ground effect is dependent upon the height of the rotor hub above the ground, $h_{mr}$.

\[
\eta_{GE} = \frac{1}{1 + K_{GE} \left( \frac{2R_{mr}}{h_{mr}} \right)^2}.
\]

$K_{GE}$ is the ground effect parameter and $c_{zz} = \cos \theta \cos \phi$ is needed to account for the orientation of the helicopter. The thrust and the average downwash velocity are related through an implicit relationship expressed in the following equations.

\[
T_{mr} = \frac{1}{4} (w_b - v_{i,mr}) \Omega R_{mr}^2 \rho a_{mr} b_{mr} c_{mr},
\]

\[
v_{i,mr} = \frac{\eta_{GE} T_{mr}}{2 \rho \pi R_{mr}^2 \sqrt{u_a^2 + v_a^2 + (w_r - v_{i,mr})^2}}.
\]

The torque, $\tau_{mr}$ is now calculated from the induced power, $P_{i,mr}$, and the profile power, $P_{p,mr}$, dissipated by the air resistance.

\[
P_{i,mr} = T_{mr} (v_{i,mr} - w_r),
\]

\[
P_{p,mr} = \frac{\rho F_{mr} \Omega R_{mr}}{8 [\Omega^2 R_{mr}^2 + 4.6 (v_a^2 + v_d^2)]},
\]

\[
\tau_{mr} = \frac{P_{i,mr} + P_{p,mr}}{\Omega}.
\]

Now, a decoupled first-order dynamic model, is used to get the flapping states. These are, the rotor disk pitch angle, $a_1$, the rotor disk roll angle, $b_1$, and the corresponding flybar pitch and roll angles to which their dynamics are linked, $a_{1FB}$ and $b_{1FB}$. Some constant factors on which these dynamics depend are the Lock numbers for the flybar, $\gamma_{FB}$, main rotor Lock number, $\gamma$, the thrust coefficient, $C_T$, the coning angle of the

31
rotor tip path, \( a_0 \), and the ratio of the forward speed to the rotor tip speed, \( \mu \).

\[
\gamma_{FB} = \frac{\rho \alpha_{FB} c_{FB} s_{FB} R_{FB}^2}{I_{FB}},
\]

\[
\gamma = \frac{\rho \alpha_{mr} c_{mr} R_{mr}^4}{I_b},
\]

\[
C_T = \frac{T_{mr}}{\rho \pi R_{mr}^2 v_{tip}^2},
\]

\[
a_0 = \frac{2 \gamma C_T}{3 \alpha_{mr} \sigma_{mr}},
\]

\[
\mu = \frac{u_a}{v_{tip}}.
\]

Here, \( R_{FB} \) is the radius of the center of flybar paddle, \( a_{FB} \), is the lift curve slope of the flybar, \( c_{FB} \) is the chord of flybar paddle, \( s_{FB} \) is the span of flybar paddle, \( I_{FB} \), is the flapping inertia of flybar paddle and, \( I_b \) is the flapping inertia of a main rotor blade. The steady state flapping angles are related to the inputs, \( U_{pcyc} \), the cyclic pitch angle of the main rotor and, \( U_{rcyc} \), the cyclic roll angle.

\[
a_{1,ss} = \frac{\partial a_1}{\partial B_1} U_{pcyc} + \frac{\partial a_1}{\partial u} u_a + \frac{\partial a_1}{\partial v} v_a + K_{cyc}^{f_b} a_{1FB} - \frac{16}{\gamma \Omega} q,
\]

\[
b_{1,ss} = U_{rcyc} + \frac{\partial b_1}{\partial u} u_a + \frac{\partial b_1}{\partial v} v_a + K_{cyc}^{f_b} b_{1FB} - \frac{16}{\gamma \Omega} p,
\]

\[
a_{1FB,ss} = K_{cyc}^{f_b} U_{pcyc} - \frac{16}{\gamma_{FB} \Omega} q,
\]

\[
b_{1FB,ss} = K_{cyc}^{f_b} U_{rcyc} - \frac{16}{\gamma_{FB} \Omega} p.
\]

\( K_{cyc}^{f_b} \), is the main rotor cyclic pitch per flybar tip path deflection and, \( K_{cyc}^{f_b} \), is the flybar cyclic pitch per cyclic pitch control input. The changes in pitch sensitivity are reflected in these equations through \( \frac{\partial a_1}{\partial B_1} \), \( \frac{\partial a_1}{\partial u} \), \( \frac{\partial a_1}{\partial v} \), and the dihedral effect through, \( \frac{\partial b_1}{\partial u} \), \( \frac{\partial b_1}{\partial v} \), terms.

\[
\frac{\partial a_1}{\partial B_1} = \frac{2 + 3 \mu^2}{2 - \mu^2},
\]

\[
\frac{\partial a_1}{\partial u} = \frac{2}{v_{tip}} \left( \frac{8 |C_T|}{a_{mr} \sigma_{mr}} + \frac{2v_{tip}}{v_{tip}} \right),
\]
\[
\begin{align*}
\frac{\partial a_1}{\partial v} &= -\frac{4a_0}{3\Omega R_{mr}}, \\
\frac{\partial b_1}{\partial u} &= \frac{\partial a_1}{\partial v}, \\
\frac{\partial b_1}{\partial v} &= -\frac{\partial a_1}{\partial u}.
\end{align*}
\] (2.52)

Finally, the dynamics of the flapping angles are given by the following angles.

\[
\begin{align*}
\dot{a}_1 &= -\frac{\gamma\Omega}{16} (a_1 - a_{1,ss}), \\
\dot{b}_1 &= -\frac{\gamma\Omega}{16} (b_1 - b_{1,ss}), \\
\dot{a}_{1FB} &= -\frac{\gamma_{FB}\Omega}{16} (a_{1FB} - a_{1FB,ss}), \\
\dot{b}_{1FB} &= -\frac{\gamma_{FB}\Omega}{16} (b_{1FB} - b_{1FB,ss}).
\end{align*}
\] (2.55) (2.56) (2.57) (2.58)

2.1.4 Tail Rotor Dynamics

The forces and moments on the helicopter due to the tail rotor are caused by the tail rotor thrust, \(T_{tr}\), and the corresponding torque because of air resistance while generating this thrust, \(\tau_{tr}\).

\[
\begin{bmatrix}
Y_T \\
L_T \\
N_T
\end{bmatrix} =
\begin{bmatrix}
-T_{tr} \\
T_{tr}z_{tr} \\
-T_{tr}x_{tr}
\end{bmatrix}.
\] (2.59) (2.60)

Here, \(x_{tr}\) is the tail rotor hub location forward of c.g. and \(z_{tr}\) is the tail rotor hub location below c.g. Both these constants usually have negative values. Since, the tail rotor is connected to the main rotor, the tail rotor angular rate is just a multiple of the main rotor angular rate depending upon the gear ratio. If \(D_{tr}\) is the tail rotor turns per turn of main rotor, the tail rotor angular rate is given by:

\[
\Omega_{tr} = D_{tr}\Omega.
\] (2.61)
The tail rotor dynamics are similar to the main rotor dynamics. However, since there is no cyclic input on the tail rotor, there is no flapping motion. There is no ground effect either as the tail rotor sidewash is parallel to the ground and isn’t affected by any obstruction to its flow. Thus the tail rotor dynamics are simpler than the main rotor dynamics. The equivalent flat plate area, $F_{tr}$, and aggregate blade velocities normal to the rotor disk plane, $v_r$, and to the blade, $v_b$, are calculated as in the case of the main rotor. There are some differences from the main rotor case in order to accommodate the lever arm of the tail boom.

$$F_{tr} = C_{D0, tr} R_{tr} b_{tr} c_{tr},$$

(2.62)

$$v_r = v_a + r x_{tr} - p z_{tr},$$

(2.63)

$$v_b = v_r + \frac{2}{3} \Omega_{tr} R_{tr} U_{ped},$$

(2.64)

$$v_{i, tr} = \frac{T_{tr}}{2 \rho \pi R_{tr}^2 \sqrt{u_a^2 + w_a^2 + (v_r - v_{i, tr})^2}}.$$  

(2.67)

The input, $U_{ped}$, is the tail rotor blade collective pitch angle. The constant parameters here are, $R_{tr}$, the tail rotor radius, $a_{tr}$, the lift curve slope of the tail rotor blade, $b_{tr}$, number of tail rotor blades, $c_{tr}$, chord of the tail rotor blade, and, $C_{D0, tr}$, the zero-lift drag coefficient of the tail rotor blades. As with the main rotor, the thrust and the average sidewash velocity are related through an implicit relationship expressed in the following equations.

$$T_{tr} = \frac{1}{4} (v_b - v_{i, tr}) \Omega_{tr} R_{tr}^2 \rho a_{tr} b_{tr} c_{tr},$$

(2.66)

$$P_{i, tr} = T_{tr} (v_{i, tr} - v_r),$$

(2.68)
2.1.5 Engine Dynamics

The dynamics of the angular rate of the main rotor $\Omega$, and the fuel consumption rate $\dot{\lambda}_f$ are described here. If the gear reduction ratio of the main rotor is $D_e$, then the angular speed of the engine is

$$\Omega_e = \Omega D_e.$$  \hfill (2.71)

The power output is now given by the following relationship and is proportional to the engine throttle $U_{thr}$, the engine brake power $P_{bhp}$, efficiency of the engine $\eta$, and the ratio of the engine speed to the speed at the maximum output $\Omega_{max}$.

$$P_e = U_{thr} P_{bhp} \eta \left( \frac{\Omega_e}{\Omega_{max}} \right).$$  \hfill (2.72)

The engine torque $\tau_e$ is hence

$$\tau_e = \frac{P_e}{\Omega_e}. \hfill (2.73)$$

The fuel consumption is proportional to the power output of the engine.

$$\dot{\lambda}_f = -\frac{\dot{m}_{max} P_e}{m_f P_{bhp} \eta}. \hfill (2.74)$$

The angular acceleration of the main rotor is the total torque acting on the rotor shaft divided by the rotational inertia.

$$\dot{\Omega} = \frac{\tau_e D_e - \tau_{mr} - \tau_{tr} D_{tr}}{I_b b_{mr}}. \hfill (2.75)$$
2.1.6 Quaternion

The orientation of the helicopter is uniquely specified by the quaternions which are related to the Euler angles through the following relationship using simple trigonometry:

\[
\begin{bmatrix}
q_0 \\
q_1 \\
q_2 \\
q_3
\end{bmatrix} =
\begin{bmatrix}
\cos \frac{\phi}{2} \cos \frac{\theta}{2} \cos \frac{\psi}{2} + \sin \frac{\phi}{2} \sin \frac{\theta}{2} \sin \frac{\psi}{2} \\
\sin \frac{\phi}{2} \cos \frac{\theta}{2} \cos \frac{\psi}{2} - \cos \frac{\phi}{2} \sin \frac{\theta}{2} \sin \frac{\psi}{2} \\
\cos \frac{\phi}{2} \sin \frac{\theta}{2} \cos \frac{\psi}{2} + \sin \frac{\phi}{2} \cos \frac{\theta}{2} \sin \frac{\psi}{2} \\
\cos \frac{\phi}{2} \sin \frac{\theta}{2} \sin \frac{\psi}{2} - \sin \frac{\phi}{2} \cos \frac{\theta}{2} \cos \frac{\psi}{2}
\end{bmatrix}
\] (2.76)

The transformation matrix \( T_B^L \), expressed in terms of the quaternions is now given by

\[
T_B^L =
\begin{bmatrix}
c_{xx} & c_{xy} & c_{xz} \\
c_{yx} & c_{yy} & c_{yz} \\
c_{zx} & c_{zy} & c_{zz}
\end{bmatrix}
= \begin{bmatrix}
1 - 2(e^2 + e^3) & 2(e_1e_2 - e_0e_3) & 2(e_1e_3 + e_0e_2) \\
2(e_1e_2 + e_0e_3) & 1 - 2(e^2 + e^3) & 2(e_2e_3 - e_0e_1) \\
2(e_1e_3 - e_0e_2) & 2(e_2e_3 + e_0e_1) & 1 - 2(e^1 + e^2)
\end{bmatrix}
\] (2.77)

The instantaneous orientation is obtained by integrating the quaternion derivatives. These are obtained through the following relation involving the angular rates.

\[
\begin{bmatrix}
\dot{q}_0 \\
\dot{q}_1 \\
\dot{q}_2 \\
\dot{q}_3
\end{bmatrix} = \frac{1}{2}
\begin{bmatrix}
0 & -p & -q & -r \\
p & 0 & r & -q \\
q & -r & 0 & p \\
r & q & -p & 0
\end{bmatrix}
\begin{bmatrix}
q_0 \\
q_1 \\
q_2 \\
q_3
\end{bmatrix}
\] (2.78)

2.1.7 Complete Dynamics

The complete helicopter is represented by Equations (2.1)–(2.78). A schematic of the complete system dynamics can be seen in Figure 1-1. The system can be expressed
as an equivalent block, $f$, in the following manner.

$$
\dot{X} = f(X, U; \Theta), \quad (2.79)
$$

where, $X$ is the state of the system, $U$ is the input to the system, and $\Theta$ is the constant parameter vector of the system.

\[
X = [e_0, e_1, e_2, e_3, u, v, w, p, q, r, x, y, z, \Omega, a_1, b_1, a_{1,FB}, b_{1,FB}, \lambda_f]^T, \quad (2.80)
\]

\[
U = [V_w^L, U_{rcyc}, U_{pcyc}, U_{ped}, U_{col}, U_{thr}]^T, \quad (2.81)
\]

\[
\Theta = [m_e, m_f, g, I_{xx}, I_{yy}, I_{zz}, x_{mr}, z_{mr}, R_{mr}, a_{mr}, b_{mr}, c_{mr}, C_{D0,mr}, I_b, K_{GE}, R_{FB}, a_{FB}, c_{FB}, s_{FB}, I_{FB}, K_{cyc}^{FB}, K_{cyc}^{FB}, \rho, x_{tr}, z_{tr}, D_{tr}, R_{tr}, a_{tr}, b_{tr}, c_{tr}, C_{D0,tr}, P_{bhp}, \eta, \Omega_{max}, \dot{\Omega}_{max}, D_e, z_{fus}, X_{uu,fus}, Y_{vu,fus}, Z_{wu,fus}, x_{ht}, z_{ht}, Z_{uu,ht}, Z_{uw,ht}, Z_{ww,ht}, Z_{VV,ht,max}, x_{vt}, z_{vt}, Y_{uu,vt}, Y_{uv,vt}, Y_{wu,vt}, Y_{ww,vt}, Y_{VV,vt,max}]. \quad (2.82)
\]

However, the practice, the states $a_1, b_1, a_{1,FB}, b_{1,FB}, \lambda_f$ are not measurable. The states $a_1, b_1$ are the effective pitch and roll angles made by the main rotor disk with the body fixed coordinates and can therefore not be measured as they represent the orientation of an imaginary disk formed through the revolution of the blades around the hub. Similarly, an exact measure of the fractional fuel capacity $\lambda_f$ is also usually not available. Therefore, the output of the system, $Y$, is given by

\[
Y = [e_0, e_1, e_2, e_3, u, v, w, p, q, r, x, y, z, \Omega] \quad (2.83)
\]

\[
(2.84)
\]

### 2.1.8 Longitudinal Dynamics

In many cases, the maneuvers to be executed are restricted to the x-z plane such as in forward flight or lift off. Here, it suffices to consider only the longitudinal dynamics.
In this reduced model, the lateral dynamics, tail rotor dynamics and engine dynamics due to lateral variables are removed, which leads to the longitudinal dynamics model described by the equations below.

\[
\begin{align*}
\dot{u} &= \frac{1}{m} \left[ -T_{mr} a_1 + \frac{\rho}{2} X_{uus, fus} |u| u - mg \sin \theta \right] - qw, \quad \text{(2.85)} \\
\dot{w} &= \frac{1}{m} \left[ -T_{mr} + \frac{\rho}{2} (Z_{ww, fus} + Z_{ww, ht}) |w| w + mg \cos \theta \right] - qy, \quad \text{(2.86)} \\
\dot{\theta} &= \frac{1}{I_{yy}} \left[ -T_{mr} x_{mr} + T_{mr} a_1 z_{mr} + \frac{\rho}{2} \left\{ (Z_{ww, fus} x_{fus} + Z_{ww, ht} x_{ht}) |w| w \right\} \right], \quad \text{(2.87)} \\
\dot{q} &= \frac{1}{L_{yy}} \left[ -T_{mr} x_{mr} + T_{mr} a_1 z_{mr} + \frac{\rho}{2} \left\{ (Z_{ww, fus} x_{fus} + Z_{ww, ht} x_{ht}) |w| w \right\} \right], \quad \text{(2.88)} \\
\dot{\dot{a}}_1 &= -\frac{\gamma \Omega}{16} \left( a_1 - \frac{\partial a_1}{\partial u} u - \frac{2}{3} a_{1, FB} - U_{pcyc} \right) - q, \quad \text{(2.89)} \\
\dot{a}_{1, FB} &= -\frac{\gamma F_{16}}{16} \left( a_{1, FB} - K_{cyc} U_{pcyc} \right) - q. \quad \text{(2.90)}
\end{align*}
\]

Here the assumption is made that there is no wind. The effect due to obstruction in the path of the downwash of the main rotor when the helicopter is close to the ground is also neglected. The thrust, \( T_{mr} \), is now given by equation (2.36) but the the lateral dynamics for the calculation of the average downwash velocity are now neglected giving the relation:

\[
v_{i, mr} = \frac{T_{mr}}{2 \rho \pi R_{mr}^2 \sqrt{u_a^2 + (w_r - v_{i, mr})^2}}, \quad \text{(2.91)}
\]

where, \( w_r \), the aggregate blade velocity normal to the rotor disk plane and, \( w_b \) the aggregate blade velocity normal to the blade are given by:

\[
\begin{align*}
w_r &= w + a_1 u, \quad \text{(2.92)} \\
w_b &= w_r + \frac{2}{3} \Omega R_{mr} U_{col}. \quad \text{(2.93)}
\end{align*}
\]
Thus, the states, $X$, inputs, $U$, parameters, $\Theta$, for the overall system as represented by Equations (2.79)-(2.82) here are:

$$X = [u, w, q, \theta, a_1, a_{1,FB}]^T, \quad (2.94)$$

$$U = [U_{pcyc}, U_{col}], \quad (2.95)$$

$$\Theta = [m, I_{yy}, \rho, m, g, Z_{ww, fus}, Z_{ww, ht}, X_{uu, fus}, \gamma, \Omega, \frac{\partial a_1}{\partial u},$$

$$\gamma_{FB}, K_{cyc}^{fb}, R_{mr}, a_{mr}, b_{mr}, c_{mr}]^T. \quad (2.96)$$

### 2.2 Problem Statement

For the system given by Equations (2.79)-(2.82) the objective is to find the input $U$, such that the output of the system, $Y$, tracks some commanded state $Y_c$, i.e. $Y \rightarrow Y_c$ while all other signals remain bounded even as there are uncertainties in the helicopter and environment. Now we make a few components below to tackle the complexity of the problem. First an adaptive scheme is developed for uncertain trim conditions. Since the system is highly nonlinear, the approach taken is to linearize the system about its trim conditions for a particular maneuver $X_e, U_e$. The trim conditions can be found by setting the state derivatives to zero or to their desired values for the trajectory, $\dot{X}_e$.

$$f(X_e, U_e, \Theta) = \dot{X}_e. \quad (2.97)$$

It is obvious that for the system, $f$, represented by Equations (2.1)-(2.78) determination of trim conditions for complex maneuvers is non-trivial. Sophisticated nonlinear minimization techniques need to be used to arrive at the trim conditions $X_e, U_e$ for complex maneuvers. Even with these techniques the minimization routine may settle on a local minima and many iterations are required to arrive at the global minima or the correct trim. The system parameters, $\Theta$, in practice are unknown. From equation (2.97) it can be seen that the trim will hence be also unknown. Even if
the parameters are identified online the problem of the nonlinear minimization on board remains. The adaptive control approach proposed in this thesis eliminates this difficulty by adapting the controller parameters and trim estimate online using just output error and input measures to accommodate unknown trim conditions and uncertainties in system parameters such that their effects are minimized.

Then for the case with full state access an adaptive controller needs to be developed that can relax the assumptions that have to be satisfied for state feedback adaptive control. In this case, the linearized system dynamics do not satisfy the conditions required for a simple state feedback adaptive controller to be implemented. In chapter 3 a design procedure for appropriate adaptive laws is developed such that these conditions can be relaxed. The stability of the new controller is shown through a Lyapunov function.

Now, for the case with partial state access, an adaptive controller is developed that can deal with the non-minimum phase nature of the helicopter along with the unknown trim conditions. The non-minimum phase input-output relationship complicates the design of an adaptive controller. This is so because the controller attempts to have desired closed-loop dynamics by canceling the changes in the dynamics brought about by the uncertainties. If there are nonminimum phase zeros, this results in the possibility of unstable pole-zero cancellations. Furthermore, the high frequency gains are singular and the relative degree of the transfer functions are greater than one making stable adaptation with output feedback infeasible. A design procedure is presented in chapter 4 which derives a stable adaptive controller for such systems. In both chapters simulations with the controller on the helicopter model presented in this chapter are shown and their effectiveness in improving performance while maintaining stability and robustness properties is demonstrated.
Chapter 3

Viable Adaptive Control Design: the Full State Feedback Case

The problem of controlling the complete system dynamics represented by equations (2.79)–(2.82) involves choosing a control input $U$ such that the state of the system, $X$, tracks a specified command, $X_c$, or the output $Y$ tracks a command output, $Y_c$. This task is made complicated by the fact that the system is significantly affected by changes in its parameters. Thus without accurate information on the parameters, the control system needs extensive tuning. The calculation of the trim condition from equation (2.97) for maneuvers involving several states takes substantial effort due to the highly nonlinear nature of the system. The trim is also a function of the system parameters and thus are unknown if the parameters are unknown. We describe below the approach taken in this thesis to design a controller for this complex problem.

3.1 Pseudo-trim Condition

Existing control strategies for aerial vehicle control involve linearization of the system about its trim conditions. An glance at the dynamics of the helicopter shows that the dynamics are highly nonlinear and interlaced. Thus nonlinear techniques for control
such as Feedback Linearization are not feasible. A major portion of the dynamics do not have a cascaded structure such as that required by techniques like Backstepping. The interlacing of the states in the dynamics means that such a cascaded design is not possible. A design based on the linearized model is therefore chosen. The true trim condition for a maneuver depends upon the parameters, \( \Theta \), of the helicopter. Since pilot action to achieve this trim in flight, \( X_e(\Theta), U_e(\Theta) \) is not available in the case of an autonomous helicopter we can only linearize the system about a pseudo-trim condition, \( X_0, U_0 \), which is the trim for nominal values of \( \Theta \). Thus,

\[
X_0 = X_e(\Theta_0),
\]

\[
U_0 = U_e(\Theta_0).
\]

Linearizing the plant in equation (2.79) about \( X_0, U_0 \) we get,

\[
\dot{x}_p = A_p(\Theta) x_p + B_p(\Theta) u + d_x(\Theta),
\]

\[
d_x(\Theta) = A_p(\Theta) (X_0 - X_e) + B_p(\Theta) (U_0 - U_e) + \dot{X}_e - \dot{X}_0.
\]

where \( x_p = X - X_0, u = U - U_0 \) and \( \dot{x}_p = \dot{X} - \dot{X}_0 \). It can be seen from this equation that an unknown constant disturbance \( d_x(\Theta) \) is now added because of the unknown trim conditions. The matrices \( A_p(\Theta), B_p(\Theta) \) are also affected by parametric uncertainties. An adaptive controller, to accommodate the parametric uncertainties and compensate for the unknown trim, is therefore considered now for control of this system. The objective is to design an input \( u \) to ensure robust stability and enhanced performance for the system in equations (3.3)–(3.4).

### 3.2 Control Design when States are Accessible

In the case when the entire state, \( X \) of the system given in equation (2.79), and therefore \( x_p \) in the system in equation (3.3), is available a state feedback design
is easiest to implement and gives good results. This utilizes the entire available information about the system. A low order controller design where the gains can be chosen by a simple design procedure like LQR or pole-placement would involve a choice of the input $u$ as

$$u = Q(Kx_p + r). \quad (3.5)$$

Here the feedback gain, $K$, can be chosen by using pole-placement or LQR techniques. $Q$ can be chosen to be an identity matrix of the same dimension as the input to this multivariable system.

### 3.2.1 Reference Model

A model reference adaptive control approach is chosen for this system since in general the system would need to track an input signal about an operating point. Choosing a reference model sets a target for the the controlled system which takes into account the dynamics of the system. Thus model reference adaptive control is less likely to have controller gains achieving very large values during adaptation. It should be noted that a model reference adaptive controller has been used successfully for retrofitting a reconfiguration module to an existing control law. In [50] the reconfiguration architecture is demonstrated using an F/A-18 and a generic transport nonlinear simulator. This need for controller gains to be within reasonable limits is important from the practical standpoint of the helicopter since control authority is limited and very high controller gains would lead to input saturation. The reference model provides a reference state for the system to track. The error between the plant state and reference model state is used to compute the adaptation laws for controller parameters $K, Q$. The reference model for the system in equation (3.3) is chosen to be of a similar form:

$$\dot{x}_m = A_m x_m + B_m r. \quad (3.6)$$
For perfect tracking of this reference model, there should exist values $K = K^*, Q = Q^*$ for the controller parameters, such that, the following conditions are satisfied, $\forall \Theta \in \Theta_c$, where $\Theta_c$ is a compact set in $\Theta$-space,

\[
B_p(\Theta)Q^* = B_m, \tag{3.7}
\]
\[
A_p(\Theta) + B_mK^* = A_m. \tag{3.8}
\]

This ensures that with a proper choice of the input command $r$ and the adaptation laws for the controller parameter, $K, Q$, the parameters converge to their ideal values and the plant in equation (3.3) tracks the reference model in equation (3.6) perfectly assuming that $d_x(\Theta)$ is zero. The complete system structure is given in Figure 3-1. It must be noted here that the equations (3.7)–(3.8), represent a greater restriction on the class of plants than controllability. $B_p(\Theta)$ and $B_m$ need to be invertible if they are square, or else they need to have many elements which are zero, in order for equations (3.7)–(3.8) to be satisfied. These conditions require that with $K = K^*$ and $Q = Q^*$ we match not just the eigenvalues of the plant but the eigenvectors too.

### 3.3 Other Fixed Controllers

In this section, we present three fixed controllers whose performance will be compared to the adaptive controller developed in this thesis. The first design is a linear LQ-controller which is designed using a model where the aerodynamics are neglected. As mentioned in chapter 1, a large class of controllers that are used at present come under this category. In [12], [15] LQ-controllers are implemented around hover for manned helicopters and were seen to provide decoupling of modes but with a tendency towards pilot-induced oscillations. The second control design is based on the idea of Dynamic Inversion. As mentioned in the introduction, flight control problems have been tackled using this approach in [17]–[19]. The idea here is to cancel out the plant dynamics using either a feedforward or a feedback controller so as to
Figure 3-1: Linearized System and Reference Model
improve its tracking performance. The third and final control design includes an integral action which is included for the purpose of eliminating tracking error and improving disturbance rejection properties over the frequencies of interest [3]. Since the action of unknown trim can be viewed as an input disturbance, a control design approach is quite appropriate. The performances of all these controllers is compared with the adaptive controller proposed in this thesis for unknown trim conditions and parametric uncertainties.

3.3.1 (1) Control Design based on $H_2$ Design

For autonomous helicopters these LQ-controllers [10] are designed for the system without aerodynamics for the system given by

$$\dot{x} = Ax + Bu,$$  \hspace{1cm} (3.9)

with the input given by

$$u = -Kx.$$  \hspace{1cm} (3.10)

The results obtained, therefore, represent the best possible performance that is obtainable with no knowledge of the vehicle aerodynamics. The gravitational and inertial effects are however included in the control design. Comparisons of the adaptive controller developed in section 3.4 with the LQ-controller are also provided assuming knowledge of aerodynamics for the design of the LQ-controller.

3.3.2 (2) Control Design based on Dynamic Inversion

The main idea here is to choose the input $u$ to the system

$$\dot{x} = Ax + Bu,$$  \hspace{1cm} (3.11)
such that

\[ \dot{x} = \dot{x}_{\text{des}}. \quad (3.12) \]

Here \( \dot{x}_{\text{des}} \) are the desired values of the derivatives of the state. A representation of the DI control is shown in figure 3-2. This involves a choice of the input in the following manner.

\[ u = B^\dagger (\dot{x}_{\text{des}} - Ax_{\text{des}}), \quad (3.13) \]
\[ B^\dagger = (B^T B)^{-1} B^T. \quad (3.14) \]

This results in a relation between the state \( x \) and the new input \( v = \dot{x}_{\text{des}} \) given by

\[ \dot{x} = Ax + BB^\dagger (\dot{x}_{\text{des}} - Ax_{\text{des}}). \quad (3.15) \]

If the pseudo-inverse of the matrix \( B, B^\dagger \) is perfect, that is, \( BB^\dagger = I \), the system in equation (3.15) becomes equal to that in equation (3.12). This open loop cancellation of dynamics may result in instability if \( A \) is unstable. If the plant with DI has a new state \( x_n \) representing the plant and DI states we have

\[ \dot{x}_n = A_n x_n + B_n v, \quad (3.16) \]

given by

\[ \dot{x}_n = \begin{pmatrix} A & -BB^\dagger A \\ 0 & 0 \end{pmatrix} x_n + \begin{pmatrix} BB^\dagger \\ I \end{pmatrix} v. \quad (3.17) \]

If \( A \) is unstable, a stabilizing loop can be designed around \( A_n \) as in figure 3-3 with the input \( v \) given by

\[ v = Q (K x_n + r). \quad (3.18) \]
Figure 3-2: Dynamic Inversion
Q and K can be designed using linear control methods like pole-placement or LQR. A reference model as described in the previous section can now be chosen for adaptive control when there are parametric uncertainties.

For the helicopter with parametric uncertainties the system with DI is described by the following equations obtained through an analysis similar to that in section 3.1.

\[
\begin{align*}
\dot{x}_n &= \begin{pmatrix} A_p(\Theta) & -B_p(\Theta) B_p^\dagger(\Theta_0) A_p(\Theta_0) \\ 0 & 0 \end{pmatrix} x_n \\
&+ \begin{pmatrix} B_p(\Theta) B_p^\dagger(\Theta_0) \\ I \\ 0 \end{pmatrix} v + \begin{pmatrix} d_x(\Theta) \\ 0 \end{pmatrix}, \\
\dot{d}_x(\Theta) &= A_p(\Theta) (X_0 - X_e) + B_p(\Theta) (U_0 - U_e) + \dot{X}_e - \dot{X}_0.
\end{align*}
\]

Here \(X_0, U_0\) are the pseudo-trim conditions and \(X_e, U_e\) are the unknown real trim values for the plant with uncertainties. A representation of this system can be seen in figure 3-4. The adaptive controller can now be designed for this system as in the case without Dynamic Inversion.

It can be argued that a system with a DI controller may be affected to a smaller extent due to parametric uncertainties. This is because the dynamics of the system are almost cancelled by the input. The only reason for the final system not being close to that in equation (3.12) is because of imperfect pseudo-inverse of \(B, B^\dagger\), and parametric uncertainties \(\Theta\). Thus a comparison of the benefits of adaptation in a design that incorporates DI is warranted. The simulation results for this are described in section 3.5.

3.3.3 (3) Control using Integral Action

Adding an integrator in the input to the system improves tracking, especially at low frequencies. However, this increases the order of the system by one and can increase the rise time. If the rise time is reduced the transients increase and the system becomes less stable. We show a comparison of a controller with a Proportional-Integral (PI) compensator in the input against an adaptive controller without integral
Figure 3-3: Dynamic Inversion in Closed Loop
Figure 3-4: Dynamic Inversion with Parametric Uncertainties
action. The modified input to the system \( v \) here becomes

\[
v = \frac{(s + a)}{s} I u, \quad (3.21)
\]

where \( u \) is given by equation (3.5). Thus the input \( u \) is filtered through a PI compensator \( W_I(s) \). The structure of the closed loop system with integral action that we use is shown in figure 3-5.

### 3.4 Adaptive Control Design

For the system described in equations (3.3)-(3.4) an adaptive controller based on the controller mentioned in section 3.2 is now developed. The adaptive control input is chosen to be

\[
u = Q \left( K x_p + r + \hat{d} \right). \quad (3.22)
\]

Here the term \( \hat{d} \) is added to the input to compensate for the constant disturbance described by equation (3.4). \( K, Q \), and \( \hat{d} \) are adapted online. The structure of the adaptive system is shown in figure 3-6. The reference model is chosen as in equation (3.6). The error \( e \) is given as \( e = x_p - x_m \), and the adaptive laws are chosen as the following:

\[
\begin{align*}
\dot{K} &= -\Gamma_1 \left[ B_m^T P e x_p^T + \Gamma_{r1} K \right], \\
\dot{Q} &= -\Gamma_2 \left[ Q B_m^T P e u^T Q - \Gamma_{r2} Q \right], \\
\dot{\hat{d}} &= -\Gamma_3 \left[ B_m^T P e + \Gamma_{r3} \hat{d} \right].
\end{align*}
\]

Here \( \Gamma_{r1}, \Gamma_{r2}, \Gamma_{r3} > 0 \) are used for robustness of the control design. If the equation (3.7) is satisfied, as in the case of the helicopter uncertainties in mass \( m \) and moment of inertia, \( I_{yy} \), then provided the equation (3.8) is satisfied for a matrix \( \overline{A}_m \)
Figure 3-5: System with Integral Action
Figure 3-6: Adaptive System for Full State Feedback
close to \( A_m \),

\[
A_p(\Theta) + B_m \overline{K}^* = \overline{A}_m,
\]

(3.26)

the controlled system is stable given

\[
Q_0 = A_m^T P + PA_m, \quad Q_0 > 0
\]

(3.27)

\[
\Delta A_m(\Theta) = \overline{A}_m(\Theta) - A_m,
\]

(3.28)

\[
\Delta Q_0 = \Delta A_m^T P + P \Delta A_m,
\]

(3.29)

\[
Q_{on} = Q_0 - \Delta Q_0 > 0.
\]

(3.30)

When \( A_m \) is chosen as

\[
\sigma_{min}(A_m) > |\mu|_{max} n^2 \sum |\Delta \theta_i| \sigma_{max}(Q_0)
\]

\[
\sigma_{min}(Q_0),
\]

(3.31)

where, \( \sigma_{min}, \sigma_{max} \) represent the minimum and maximum singular values respectively, \( |\mu|_{max} \) represents the largest parameter absolute value, \( n \) is the order of the system, and \( \sum |\theta_i| \) is the sum of all parametric uncertainties, then equations (3.27)–(3.30) are satisfied.

**Theorem 3.1** For the system given by the dynamic equations (3.3)–(3.4), the input in equation (3.22), and adaptive laws in equations (3.23)–(3.25), and satisfying the conditions in equations (3.27)–(3.30), all signals are bounded and the system is stable.

Please see Appendix A for proof.

### 3.5 Numerical Studies

The controller developed in this chapter is simulated for the longitudinal nonlinear dynamics presented in section 2.1.8. A 2% increase in the mass \( m \) and moment of inertia \( I_{yy} \) is used as the uncertainty. These parameters are seen to have the worst
impact on the stability of the system and an increase in their values is seen to have the most effect. Three different tasks are performed and the results of the dynamic inversion controllers are compared against the adaptive controller. Adaptation is stopped after a time in each case based on the output error value to observe learning behavior of the controller.

### 3.5.1 Task 1: Track Step Changes in Forward Flight Velocity

The first simulation involves step changes in forward flight velocity between $28\text{ft/sec}$ and $40\text{ft/sec}$ for the helicopter. Controller (1) designed without the inclusion of aerodynamics is compared against the adaptive controller in figure 3-7. Random steps are taken subsequent to the stoppage of adaptation to test the learned behavior of the adaptive controller. It can be seen that controller (1) has a large steady state bias and is unable to take large steps in velocity. This is because the nonlinear part of the dynamics due to the aerodynamics are neglected in the design. This results in the instability of the system with large initial conditions as in the case of a large step. Furthermore, the worse performance at the lower speed of $28\text{ft/sec}$ is because the same LQ-weights are used for both designs at $28\text{ft/sec}$ and $40\text{ft/sec}$. Figures 3-8, 3-9 compare the adaptive controller without the $\hat{d}$ trim error term against controller (1) with aerodynamics included in the system design. The reference model is the nominal linearized dynamics with the LQ-controller. The starting values for the adaptive controller are the same as those for controller (1). Steady-state bias is reduced by as much as 95% in the adaptive case. Figures 3-10 and 3-11 demonstrate the robustness of the adaptive control design to sensor noise. White noise of magnitude equal to 5% of the state measures was added and the adaptive controller with trim error estimate $\hat{d}$ was simulated. The forward and downward velocities $u$ and $w$ now show the effect of noise but stability is maintained and improved performance is observed. The results are similar in the case with input disturbances. The robustness of the adaptive control design is thus demonstrated. This depends on the robustness terms, $\Gamma_{r1}, \Gamma_{r2}, \Gamma_{r3}$, in the adaptive laws. Larger values of $\Gamma_{ri}$ implies more robustness.
Adaptation stopped at 600s

![Graph showing time (sec) vs. u (ft/sec) with markers for LQ-Controller and Adaptive Reference, indicating blow-up and random steps.]

Figure 3-7: u for steps in forward flight: Random steps
Adaptation stopped at 320s

Figure 3-8: $u$ for steps in forward flight: Adaptive control vs. DI
Figure 3-9: $w$ for steps in forward flight: Adaptive control vs. DI
to disturbances and noise but a degradation in performance of the controller. For the simulation studies that correspond to figures 3-12, 3-13, integral action is added to controller (2). Dynamic inversion reduces the steady state bias compared to controller (1). The addition of control element (3) eliminates steady state error but increases transients for the dynamic inversion controller. For a control design which maintains a rise time of less than 5 seconds, transients of magnitude upto 10% of the step size and settling time greater than 40 seconds are introduced with integral action. In contrast, the adaptive controller proposed in section 3.4, even without an estimate of the trim error is seen to outperform this controller by having low steady state bias, fast rise time and no overshoot or transients after the initial adaptation. The initial transients introduced by the adaptation are of similar magnitude as those of the controller (2) with control element (3). These are eliminated in subsequent iterations of the maneuver as the controller parameter errors decrease. Finally, even after the adaptation is stopped the controlled system continues to show the learned performance.

Figures 3-14, 3-15 show the improvement in performance due to the addition of $\hat{d}$ term in the adaptive case for the same system. The steady state error in the adaptation is eliminated. It can be seen that the adaptive controller does not need integral action to eliminate constant disturbances.

Figure 3-16 shows training of the adaptive controller for a series of steps followed by stoppage of adaptation. Random steps are then taken in forward velocity with the same controller values. This shows that the controller gains and trim error estimate learned in the initial series of constant steps is sufficient to provide good performance for maneuvers of similar frequency content. This is because the adaptation enables the controller to minimize the state error for the particular maneuver. The controller gains are therefore values that make the adapted system similar to the reference model for these frequencies.
Figure 3-10: $u$ for steps in forward flight with 5% noise
Figure 3-11: $w$ for steps in forward flight with 5% noise
Figure 3-12: $u$ for steps in forward flight against DI with integral action
Figure 3-13: $w$ for steps in forward flight against DI with integral action
Adaptation stopped at 400s

Figure 3-14: $u$ for steps in forward flight with $\dot{d}$
Figure 3-15: $w$ for steps in forward flight with $d$
Adaptation stopped at 600s

Figure 3-16: $u$ for random steps in forward flight with $\dot{d}$ against DI with integral action
3.5.2 Task 2: Learning Over a Range of Frequencies

Here the controller is trained for an input which is a combination of two sinusoids. The learned controller is then tested for a sinusoidal input with a frequency in between the frequencies of the training set. The adaptation is stopped after the controlled system is simulated for the first signal with two sinusoids. Then the closed-loop is simulated for a sinusoidal input whose frequency lies in between the two earlier frequencies. The uncertainty is still 2% in the mass and moment of inertia. The adaptive controller is seen to outperform controller (2) for both steady state error and command following. This is shown in figure 3-17. The steady state bias is lower and it can be seen from this that the adaptive controller has the ability to perform as a learning controller. The closed-loop system is changed through adaptation to closely approximate the reference model for the range of frequencies over which it is trained. Thus, even though the input to the system is not persistently exciting (required for perfect identification of parameters [56]), and has only two frequency signals, the adaptation shows good performance for frequency content in the region around those two frequencies. Persistent excitation here would need half as many frequencies in the training input as the number of controller parameters (20).

3.5.3 Task 3: Complex Maneuver in Forward and Vertical Velocities

The simulation with the adaptive controller is now carried out to demonstrate the effectiveness of adaptation in reducing very large initial gains in a gain scheduled system. The objective here is jump over hurdles making a circle in the $U-W$, forward velocity – vertical velocity plane. That is, the helicopter stops over each hurdle and accelerates forward when going down. A gain scheduled controller is developed over 12 points of the maneuver. The adaptive controller is compared against controller (2) with control element (3). The system with controller (2) is seen to have very large initial transients. Eventually the integral action reduces the state error and shows
Adaptation stopped at 2160s

Trained over
\( \Omega = [0.01\text{Hz}, 0.1\text{Hz}] \)

Tested for signals in \( \Omega \)

Figure 3-17: \( u \) for sinusoidal input for forward flight
reasonable performance. But adaptation shows superior performance in reducing initial transients. It continues to match the state of the reference model more closely than the DI controller. It can thus be seen in figures 3-18–3-22 that adaptation is beneficial even for reduction of initial transients.

3.6 Chapter Summary

In this chapter, a viable adaptive controller is developed for helicopter when all states are measurable. A direct trim error estimate is added and a design procedure is presented which gives an adaptive controller for the case when matching conditions are not satisfied by the system dynamics. This controller allows a simple state feedback structure to be used for the adaptive controller. Through simulations it is demonstrated that the above adaptive controller outperforms linear control designs by learning a maneuver, eliminating steady state bias for trim error disturbance, and learning the frequency content of a maneuver and reducing initial transients.
Figure 3-18: $U$ for complex maneuver in $U - W$ plane. Comparison against DI with integral action
Figure 3-19: $W$ for complex maneuver in $U - W$ plane. Comparison against DI with integral action.
Figure 3-20: $U$ vs. $W$ for first cycle of complex maneuver in $U-W$ plane. Comparison against DI with integral action
Figure 3-21: $U$ vs. $W$ for first cycle of complex maneuver in $U - W$ plane. Blow up of maneuver region. Comparison against DI with integral action.
Figure 3-22: $U$ vs. $W$ for eighth cycle of complex maneuver in $U - W$ plane. Comparison against DI with integral action.
Chapter 4

Viable Adaptive Control Design:
the Partial State Feedback Case

In the previous chapter an adaptive control scheme was developed for the control of an autonomous helicopter when all states of the helicopter are available. In practice many states of the model given in equations (2.79)–(2.82) are not measurable. For example, the rotor disk pitch and roll angles, $a_1, b_1$, are the effective angles that the rotor disk makes with the helicopter due to the cyclic pitch movement of the blades in each rotation around the hub. As seen in section 2.1.3 these states and the flybar dynamics represented by $a_{1FB}, b_{1FB}$, depend upon parameters such as $a_{mr}, a_{FB}$, the lift curve slopes, that are not easily measurable and vary with operating conditions. Hence, measures of these states aren’t available. Sensors for other states in equation (2.80), may also be not available. Thus controllers that can accommodate the case without full state access need to be designed.

Nonlinear control techniques are inapplicable here since there is a high degree of coupling between states. For this reason techniques such as feedback linearization [51] and backstepping [52] cannot be used since they assume restrictions on the structure of the plant which are violated in this problem. Further, recently developed techniques like forwarding [52] too assume a similarly restrictive structure of the problem which is not available for the helicopter problem. Other techniques such as sliding
mode [53] need full state access, or have assumptions that restrict the plant zeros which may not be realistic in the case of autonomous helicopters. Therefore, instead of an approximate application of a nonlinear technique we design a linear controller that takes into account all coupling between the states. This has the added advantage of enabling easy integration of the adaptive controller with existing control architectures. Our focus is to redress the effects of parametric uncertainty and unknown trim conditions.

The problem considered for this chapter can be stated as follows. Suppose the plant to be controlled is given by the following equations

\[
\begin{align*}
\dot{X} &= f(X, U, \Theta), \\
Y &= g(X, U, \Theta).
\end{align*}
\]  

In the case of the helicopter the equation (4.2) reduces to

\[
Y = CX.
\]  

where \(C\) is a constant matrix selecting the states whose measures are available.

If \(Y_c\) is the desired value of \(Y\), the goal is to choose \(U\) such that the closed-loop signals remain bounded and

\[
\lim_{t \to \infty} \|Y(t) - Y_c(t)\|.
\]

As described in chapter 3, the controllers developed in this thesis are designed for a linearized version of this plant about various trim conditions. The system is extremely complicated for easy implementation of nonlinear control techniques. The major problem in control has been observed to be the changes in system parameters under varying environments and maneuvers. Moreover, the objective here is to provide a procedure for controller design which can easily be integrated with existing linear control designs and cancel effects of parametric uncertainties. For these reasons, a
model reference adaptive control approach is chosen. As described in section 3.1, the system is first linearized about pseudo-trim conditions, \( X_0, U_0 \) (chosen to be the trim values for the nominal plant for the maneuver), and the following linear plant is obtained

\[
\dot{x}_p = A_p(\Theta) x_p + B_p(\Theta) u + d_x(\Theta), \tag{4.4}
\]
\[
y_p = C_p(\Theta) x_p + D_p(\Theta) u + d_y(\Theta). \tag{4.5}
\]

Here, when \( X_e, U_e \) are the true trim values for the plant, the constant disturbances, \( d_x(\Theta), d_y(\Theta) \), are given by

\[
d_x(\Theta) = A_p(\Theta)(X_0 - X_e) + B_p(\Theta)(U_0 - U_e) + \dot{X}_c - \dot{X}_0, \tag{4.6}
\]
\[
d_y(\Theta) = C_p(\Theta)(X_0 - X_e) + D_p(\Theta)(U_0 - U_e). \tag{4.7}
\]

For the helicopter the linearized system reduces to,

\[
\dot{x}_p = A_p(\Theta) x_p + B_p(\Theta) u + d_x(\Theta), \tag{4.8}
\]
\[
y_p = C x_p + d_y(\Theta), \tag{4.9}
\]
\[
d_x(\Theta) = A_p(\Theta)(X_0 - X_e) + B_p(\Theta)(U_0 - U_e) + \dot{X}_c - \dot{X}_0, \tag{4.10}
\]
\[
d_y(\Theta) = C (X_0 - X_e). \tag{4.11}
\]

The control problem can now be restated, in terms of the linearized plant in equations (4.8)–(4.11), as follows: Choose the input \( u \) in (4.8) such that the closed-loop system has bounded solutions, and the output \( y_p \) follows its desired commanded value as closely as possible. The desired signal for \( y_p \) would be the output of the best design using linear control of the linearized plant (4.8)–(4.11) for nominal values of the system parameters \( \Theta = \Theta_0 \).
4.1 Controller Structure

For the design of a multivariable controller for this problem linear and robust control methods such as eigenstructure assignment, LQG, $H_\infty$, $\mu$-synthesis, etc. can be used. However, a pole-placement design is chosen here as it is more amenable for stable adaptation of controller parameters online.

The plant to be controlled is given by (3.3)-(4.11) with $u(t), y_p(t) \in \mathbb{R}^m$. For adaptive control, the system is chosen to have $m$-inputs and $m$-outputs. The corresponding transfer matrix between $u$ and $y_p$ is given by

$$y_p = W_p(s)[u + d_0] + d_1 \quad (4.12)$$

where $W_p(s)$ is

$$W_p(s) = C(sI - A(\Theta))^{-1}B(\Theta) \in \mathbb{R}_{p}^{m \times m}(s). \quad (4.13)$$

d_0 is the effective constant disturbance due to $d_x(\Theta)$ and $d_y(\Theta)$ which can be represented as an input disturbance and thereby cancelled out using a trim error estimate added to the input $u$. $d_1$ is the rest of these disturbances which remains in the form of an output noise. $\mathbb{R}_p(s)$ represents the ring of proper rational functions of a single variable, $s$, with coefficients in $\mathbb{R}$. The goal is find a controller that determines the input $u$ such that the plant together with the controller matches the transfer function $W_m(s)$ of a suitably chosen reference model. This strictly proper transfer matrix can be factored using the right coprime factorization [54] to give the following representation

$$W_p(s) = Z_p(s)R_p^{-1}(s). \quad (4.14)$$

The controller proposed is of a pole-placement type, whose structure needs to be carefully chosen. The design of such a controller is described below, where we
articulate how each of its components is chosen. The controller has a structure as shown in figure 4-1 with filters $F_1$ and $F_2$ in its feedback path and with transfer matrices $R_q^{-1} (s) Z_c (s)$ and $R_q^{-1} (s) Z_d (s)$ respectively, for the constant values of $\Theta_1$ and $\Theta_2 [55]$. The $m \times m$ matrix $R_q^{-1} (s)$ matrix is chosen as a diagonal matrix with elements on the diagonal equal to $1/r_q (s)$ where $r_q (s)$ is a hurwitz monic polynomial of degree $\nu - 1$. $\nu$ is the obsevability index of the minimal transfer matrix in equation (4.14). This choice of $R_q^{-1} (s)$ makes it row proper and commute with $Z_c (s)$ and $Z_d (s)$. $Z_c (s)$ and $Z_d (s)$ are determined by the input-output relation

$$
\omega_i (t) = \frac{s^{i-1}}{r_q (s)} u(t), \quad i = 1, ..., \nu - 1, \quad (4.15)
$$

$$
\omega_j (t) = \frac{s^{j-\nu}}{r_q (s)} y_p(t), \quad j = \nu, ..., 2\nu - 1. \quad (4.16)
$$

The $2m\nu$ dimensional vector $\omega$ and the $m \times 2m\nu$ matrix $\Theta_c$ are defined as

$$
\omega = [r, \omega_1^T, ..., \omega_{\nu-1}^T, \omega_\nu^T, ..., \omega_{2\nu-1}^T]^T, \quad (4.17)
$$

$$
\Theta_c = [I, C_1, ..., C_{\nu-1}, D_0, ..., D_{\nu-1}]. \quad (4.18)
$$

where $I, C_i, D_j$ are $m \times m$ matrices for $i = 1, ..., \nu - 1$, and $j = 0, ..., \nu - 1$. $C_i$ and $D_j$ correspond to the coefficients of $Z_c (s)$ and $Z_d (s)$ respectively. The control input to the plant can now be expressed as

$$
u(t) = \Theta_c(t) \omega(t). \quad (4.19)$$

For constant values of the parameter matrix $\Theta_c$, the closed loop transfer function is given by $W_d (s)$ where

$$
W_d (s) = Z_p (s) [(R_q (s) - Z_c (s)) R_p (s) - Z_d (s) Z_p (s)]^{-1} R_q (s). \quad (4.20)
$$

**Bezout Identity.** Let $Q (s), T (s) \in \mathbb{R}^{m \times m}$ (s) be right coprime with the transfer
Figure 4-1: Pole-placement Controller
matrix $T(s)Q^{-1}(s)$ strictly proper. Let $\nu$ be the observability index of the minimal transfer matrix $T(s)Q^{-1}(s)$. Then polynomial matrices $P(s), R(s) \in \mathbb{R}^{m \times m}(s)$ each having degree $\nu - 1$ exist such that

$$P(s)Q(s) + R(s)T(s) = M(s) \quad (4.21)$$

where $M(s)$ is any arbitrary polynomial matrix in $\mathbb{R}^{m \times m}(s)$ with column degrees $\partial_{c_j}[M(s)] \leq d_j + \nu - 1$ for $j = 1, \ldots, m$. Here $d_j$ represents the column degree of $Q(s)$.

The reader is referred to [56] for proof.

From the Bezout identity described above, as $R_p(s)$ and $Z_p(s)$ are coprime, it follows that the matrices $R_q(s) - Z_c(s)$ and $Z_d(s)$ of degree $\nu - 1$ exist such that the polynomial matrix within the brackets in equation (4.20) can be made equal to any arbitrary polynomial of degree $d_j + \nu - 1$ where $d_j$ is the column degree of $R_p(s)$. For this constant $\Theta_c$ the closed loop transfer function needs to match that of the reference model. Suppose the term within the brackets in equation (4.20) is chosen to cancel the unknown $Z_p(s)$. That is, if $Z_c(s)$ and $Z_d(s)$ are such that

$$[(R_q(s) - Z_c(s)) R_p(s) - Z_d(s) Z_p(s)] = R_m(s) Z_p(s), \quad (4.22)$$

and the closed loop system in equation (4.20) as

$$W_d(s) = R_m^{-1}(s) R_q(s) \quad (4.23)$$

where $R_m(s)$ is a matrix of stable monic polynomials whose degree equals $r + \nu - 1$ where $r$ is the minimum row relative degree of $W_p(s)$. That is, the closed-loop system can match any reference model whose transfer function $W_m(s) = R_m^{-1}(s) R_q(s)$. However, such a $W_m(s)$ cannot be realized due to the following reason. The Bezout Identity solution assumes that $R_q(s)$ is not a matrix of monic polynomials, which introduces the following restriction on the left side of equation (4.22), $(R_q(s) - Z_c(s)) R_p(s)$
has a higher degree than the term $Z_d(s)Z_p(s)$. Thus the highest degree coefficients on the right are equated by the highest degree coefficients of $(R_q(s) - Z_c(s))R_p(s)$. Further, as seen from equations (4.15)–(4.18), $Z_c(s)$ is of degree $\nu - 2$. The highest degree polynomials on the left of equation (4.22) are therefore in the term $R_q(s)R_p(s)$. Since $R_q(s)$ is a fixed constant polynomial matrix and $R_p(s)$ is unknown due to its dependence upon unknown parameters $\Theta$, this solution of the Bezout identity does not provide us with $Z_c(s)$ and $Z_d(s)$ that can satisfy equation (4.22). To provide sufficient degrees of freedom, another constant matrix $K_0$ is added in the the feedforward path to account for the highest degree coefficients of $Z_p(s)R_p^{-1}(s)$, or the high frequency gain $K_p$, resulting in the controller structure as in figure 4-2. The input, $u$, is now given by the following equations

\begin{align*}
\omega &= [r, \omega_1^T, ..., \omega_{\nu-1}^T, \omega_\nu^T, ..., \omega_{2\nu-1}^T]^T, \quad (4.24) \\
\Theta_c &= [K_0, C_1, ..., C_{\nu-1}, D_0, ..., D_{\nu-1}], \quad (4.25) \\
u(t) &= \Theta_c \omega(t). \quad (4.26)
\end{align*}

The closed loop system is given by,

\begin{equation}
W_{cl}(s) = Z_p(s)[(R_q(s) - Z_c(s))R_p(s) - Z_d(s)Z_p(s)]^{-1}R_q(s)K_0. \quad (4.27)
\end{equation}

Using the Bezout identity once again to pick $Z_c(s), Z_d(s)$ we need

\begin{equation}
[(R_q(s) - Z_c(s))R_p(s) - Z_d(s)Z_p(s)] = K_0R_m(s)Z_p(s), \quad (4.28)
\end{equation}

to get

\begin{equation}
W_{cl}(s) = R_m^{-1}(s)R_q(s). \quad (4.29)
\end{equation}

With this controller structure, the coefficients of the highest degree terms of $R_q(s)R_p(s)$ need to equal the highest degree terms of $K_0Z_p(s)$. This is achieved if $K_0 = K_p^{-1}$. 
This structure is therefore chosen for the structure of the pole-placement controller.

It needs to be noted here that for the left hand side term in equation (4.28), which is substituted into equation (4.27), $K_p$ needs to be non-singular and $Z_p(s)$ needs to be stably invertible to avoid unstable-pole zero cancellation. As $R_q(s)$ is known and stably invertible equation (4.28) can be further modified to be

\[
[(R_q(s) - Z_c(s)) R_p(s) - Z_d(s) Z_p(s)] = R_q(s) K_0 R_m(s) Z_p(s) .
\] (4.30)

This results in the closed loop system becoming

\[
W_c(s) = R^{-1}_1(s) = W_m(s) .
\] (4.31)

If $K_p$ is non-singular from [57] it is seen that $R^{-1}_m(s)$ can be equal to the decoupled stable Hermite normal form of $W_p(s)$, $H_p(s)$. The decoupled Hermite normal form, $H_p(s)$, is given by

\[
H_p(s) = \begin{bmatrix}
\pi^{-1}(s) & \cdots & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & \cdots & \pi^{-1}(s)
\end{bmatrix},
\] (4.32)

where $\pi (s)$ is a monic polynomial of degree 1, and $n_i$ is the minimum row relative degree of $W_p(s)$. In summary, with the controller structure as in Figure 4-2, and given by equations (4.24)–(4.26), it follows that $Z_c$ and $Z_d$ always exist such that equation (4.30) can be satisfied so that the closed-loop system has a transfer function given by 4.31.

In the next section we show how this control design can be implemented on an autonomous helicopter.
Figure 4-2: Pole-placement Controller with Feedforward Gain
4.2 Partial State Feedback Control in Helicopter

As mentioned in the introduction, the dynamic model of autonomous helicopters is given by equations (2.79)–(2.82), which is cast in the form of (4.1)–(4.3). These equations can then be linearized as in (4.8)–(4.11) where \( x_p = X - X_0 \), \( u = U - U_0 \) and the nominal trim values \( X_0 \) and \( U_0 \) are to be computed for each maneuver. The controller that we propose is chosen as described in section 4.1, where the coefficients of \( Z_e(s) \) and \( Z_d(s) \) are to be computed using nominal values of the plant parameters. These coefficients in turn serve as the starting values for the adaptive controller.

4.2.1 Determination of Nominal Trim Values \( X_0, U_0 \)

In order to linearize the system dynamics of the helicopter in equations (2.79)–(2.82) we need to first determine accurate values of the trim conditions \( X_0 \) and \( U_0 \) for nominal values of the parameters. That is, we need to find \( X_0, U_0 \) such that

\[
f(X_0, U_0, \Theta_0) = \dot{X}_e
\]

where \( \Theta_0 \) denotes the nominal value of \( \Theta \). We also note that equation (4.33) represents 19 highly coupled nonlinear equations, and hence an explicit determination of the solutions \( X_0 \) and \( U_0 \) is near impossible. Optimization schemes need to be used to find a solution to this equation. Linear methods like Simplex are seen to converge to a local minima from almost all starting values. Therefore, nonlinear methods such as Simulated Annealing or Genetic Algorithms need to be used to find the trim. However, these methods are computationally expensive.

A simpler way of solving this problem is now presented which exploits insight into the nature of the helicopter dynamics and consists of a two stage optimization procedure for accurate trim determination. Often a part of the overall state that includes the attitude angles and angular rates have, either, a small value for most maneuvers, or values that can be determined reasonably accurately. Defining
\( X_a = [\phi, \theta, \psi, p, q, r]^T \), we fix \( X_a = X_{ac} \), and use a Nelder-Mead simplex search to determine the remaining component \( X_b \) of \( X_0 \) and \( U_0 \). Denoting the resulting values that this simplex search leads to as, \( X_1 \) and \( U_1 \), in the second stage of the nonlinear optimization, we begin with \( X_1 \) and \( U_1 \), and carry out a simplex search in the overall \((X,U)\) space to result in the final trim determination of \((X_0^*, U_0^*)\). This procedure is outlined below:

Step 1: Define \( X_{ac} = [\phi_c, \theta_c, \psi_c, p_c, q_c, r_c]^T \), Carry out a Nelder-Mead simplex search on \( X_b, U_0 \) so that they converge to values \( X_{b1}, U_1 \) which give

\[
X_1 = \begin{bmatrix}
X_{ac} \\
X_{b1}
\end{bmatrix}
\]

such that \( X_1, U_1 \) satisfy (4.33) with a sufficiently small error.

Step 2: Starting with \([X_1, U_1]^T\), carry out a Nelder-Mead simplex search in the entire \((X, U)\) space, without constraining \( X_1 \), so that

\[
\begin{bmatrix}
X_1 \\
U_1
\end{bmatrix} \rightarrow \begin{bmatrix}
X_2 \\
U_2
\end{bmatrix}.
\]

However, \( \begin{bmatrix} X_2 \\ U_2 \end{bmatrix} \) was observed to be arbitrarily close to the global minimum \( \begin{bmatrix} X_0^* \\ U_0^* \end{bmatrix} \) of the overall nonlinear optimization problem.

The above two-step procedure has the potential to converge to the global minimum mainly because of the prior information available about the trim values of a sub-component of the state variables and inputs. This information is almost always available even in the most complex maneuvers, and therefore the above procedure is a valuable step in the overall design procedure of the viable controller proposed in this thesis. In all simulations presented in this chapter this method was used for the determination of the nominal trim for all maneuvers and resulted in significant reduction in computational time.
4.2.2 Coprime Matrix Fraction Decomposition

The first step in the development of the pole-placement controller is the computation of $Z_p(s)$ and $R_p(s)$ defined as in (4.14), that are coprime. Often in many models this may not be available. In the case of the given helicopter, the plant dynamics are in a state space form as in equations (4.8)–(4.11). Diagonalizing the numerator matrix of $W_p(s)$ and separating out the poles from the transmission zeros is very sensitive to numerical errors. For the helicopter, therefore, the algorithm suggested in [58] for right coprime matrix fraction decomposition is used. The algorithm is as outlined below. For details of the procedure please see appendix B.

1. Form Selector matrices $S_a, S_{ld}, S_{ih}$ using pseudo-controllability indices.

2. With $A_c(\Theta)$, the controllable canonical form of $A(\Theta)$, get $R_{pre}$, the matrix column form of the coefficients of $R_p(s)$,

\[
A_{cc} = A_c S_a, \tag{4.34}
\]
\[
R_{pre} = S_{ld} - S_{ih} A_{cc}. \tag{4.35}
\]

3. Find $Z_p(s)$ using the following equations

\[
N(s) = Z_p(s) R_{pad}(s), \tag{4.36}
\]
\[
\sum_{j=0}^{i} Z_{pj} R_{pad_{i-j}} = N_i, \quad i = 1, \ldots, n. \tag{4.37}
\]

where $R_{pad}(s)$ is the adjoint of $R_p(s)$ and $n$ is the order of the plant.

This algorithm is found to give a reasonable accurate representation $Z_p(s) R_p^{-1}(s)$ of the system for simulations of the helicopter model.
4.2.3 Non-singular High Frequency Gain

Starting with the coprime factors $Z_p(s)$ and $R_p(s)$, the next step is to find $Z_c(s)$ and $Z_d(s)$ that are the solutions of equation (4.30). We note that a necessary requirement for finding $Z_c(s)$ and $Z_d(s)$ is the nonsingularity of $K_p$. In the case of the helicopter, the relative degree of some columns of $W_p(s)$ is higher than others. That is, there are some elements of the input vector $u$ which have lower relative degree transfer functions to all outputs when compared to the other transfer functions. This results in the high frequency gain matrix $K_p$ to have the columns corresponding to these input elements being identically zero. Therefore, $K_p$ is not invertible. This problem can be resolved by filtering these input elements through stable filters of appropriate degree. A pre-compensator of the form

$$W_c(s) = \begin{bmatrix} \frac{1}{\pi_p} & 0 & \ldots & 0 \\ 0 & \ddots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & \frac{1}{\pi_p^m} \end{bmatrix},$$

(4.38)

is selected, where $\pi_p$ is a monic polynomial of degree 1 and $k_i$ are equal to the maximum of the minimum column relative degree of the matrix minus the minimum column relative degree of the column $i$. The new input to the system $v$ is given by

$$v = W_c(s)u.$$  

(4.39)

This changes the new transfer function of the plant to the following:

$$\overline{W}_p = W_p(s)W_c(s).$$

(4.40)

Now this augmented plant for which the adaptive controller is designed has a high frequency gain $\overline{K}_p$ which is obviously different from $K_p$. By choosing this $W_c(s)$ it is possible to have $\overline{K}_p$ invertible. Thus $\overline{K}_p^{-1} = \overline{K}_0$ is also non-singular which is needed
for stable adaptation.

### 4.2.4 Minimum Phase Plant

In addition to $K_p$ being nonsingular, for the existence of $Z_c(s), Z_d(s)$ in equation (4.30) leading to $W_p(s) = H_p(s)$ without unstable pole-zero cancelltions, we need the transmission zeros, i.e. the roots of $\text{det } Z_p(s)$ to be stable. This implies that $Z_p(s)R_p^{-1}(s)$ is minimum phase. For the helicopter the states that are measurable are

$$Y = \begin{bmatrix} e_0, e_1, e_2, e_3, u, v, w, p, q, r, x, y, z, \Omega \end{bmatrix}^T,$$

(4.41)

and the controlled inputs are

$$U = \begin{bmatrix} U_{rcyc}, U_{pcyc}, U_{ped}, U_{col}, U_{col} \end{bmatrix}^T.$$

(4.42)

With the inputs given in equation (4.42) any set of 5 outputs in equation (4.41) is seen to lead to a plant with unstable transmission zeros. The natural outputs corresponding to these inputs are $p, q, r, w, \Omega$ which are chosen so that the resultant $W_p(s)$ is close to a diagonal matrix. The plant therefore is nearly decoupled. As other outputs become available, the problem of non-minimum phase system can be overcome for the adaptive control design by the use of a linear combination of the other available states with these outputs.

A rectangular $Z_p(s)$ is available in the representation of the system where the number of rows of $Z_p(s)$ is greater than the number of columns. A constant post-compensator matrix $T_c$ is now introduced such that $TZ_p(s)$ is square and has stable transmission zeros. A search needs to be performed for this $T$ such that a stable $TZ_p(s)$ available for the entire range of system parameter values $\Theta \in \Theta_s$. $\Theta_s$ is a compact set of possible parameter values. The output of this modified system $z_p$ is
thus given as

$$z_p(t) = Ty_p(t). \quad (4.43)$$

Such a $T$ is available because the transfer functions from the inputs to each of the measured states is different and there is some cross-coupling of the modes through the introduction of this post-compensator. In addition some modes which are un-observable in $p, q, r, w, \Omega$ are observable in the other state measurements. Hence the addition of these state measures to the output helps in removing the unstable zeros.

### 4.2.5 Helicopter Relative Degree

For the helicopter model it is seen that the relative degree of the plant after the modifications in sections 4.2.3 and 4.2.4 is 1 or 2. This enables the adaptive control structure to be of much lower order. For the adaptive controller designed a relative degree 1 plant needs $m \times (2m\nu + 1)$ controller parameters, $\Theta_c$, and a $2m\nu$ element controller state vector $\omega$ defined in equations (4.15)–(4.16). With the simplified input and adaptation rules for a plant with relative degree 2, $\omega$ increases to a $4m\nu$ element vector as described in section 4.3. A filtered measure of $\omega$ through a stable first order filter is needed. However, if an augmented error approach (required for systems with relative degree three or more) is used this controller state vector, $\omega$, needs to be filtered through $H_p(s)$. This increases the size of the adaptive controller state vector to $(2m + 1) \times 2m\nu$ for this relative degree plant. For the complete helicopter dynamics $m = 5$ and $\nu = 4$. Thus the controller states needed for the augmented error approach are 440. The special design described in the next section reduces the controller states to a vector of 80 elements.

The complete system is now represented by the following equation:

$$z_p(s) = TZ_p(s) R_p^{-1}(s) W_c(s)(u(s) + d_0(\Theta)) + Td_1(\Theta). \quad (4.44)$$
Here \( Z_p(s) R_p^{-1}(s) \) is the coprime matrix fraction decomposition of the state space model in equation (4.8)–(4.9). The output in equation (4.9) is assumed to have all available states. \( d_x(\Theta) \) is the constant disturbance term from equation (4.11) caused by unknown trim conditions and \( I \) is an identity matrix of the same dimension as the length of \( d_x(\Theta) \).

### 4.3 Adaptive Control

The adaptive controller is now designed for the partial state feedback case for the helicopter. The system is described by equations (4.8)–(4.11), and with the addition of the precompensator and postcompensator the transfer function changes to the representation in equation (4.44). An adaptive controller structure based on the structure in figure 4-2 is now chosen for the helicopter. This is shown in figure 4-3.

For adaptation to the constant disturbance a new term is added to the input described by equations (4.24)–(4.26). The new input \( u \) is now described by

\[
\omega = \begin{bmatrix} \omega_0, r, \omega_1^T, ..., \omega_{\nu-1}^T, \omega_\nu^T, ..., \omega_{2\nu-1}^T \end{bmatrix}^T, \\
\Theta_c = \begin{bmatrix} I \hat{d}, K_0, C_1, ..., C_{\nu-1}, D_0, ..., D_{\nu-1} \end{bmatrix}, \\
u(t) = \Theta_c \omega(t). 
\]

\( \omega_0 \) is a vector of unit magnitude elements of the same size as \( r \). \( \hat{d} \) is an estimate of constant disturbance and \( I \) is an identity matrix of size equal to \( K_0 \). \( \hat{d} \) is added to cancel the constant disturbance term and thereby reduce the error between the reference model and the plant. The nonminimal representation of the state \( \omega_i, \omega_j \) is determined by equations (4.15)–(4.16). The reference model and plant are now

\[
z_m(s) = H_p(s) r(s), \\
z_p(s) = H_p(s) [r(t) + K_p \Phi(t) \omega(t)] + T d_1(\Theta), \\
\Phi(t) = \Theta_c(t) - \Theta_c^*. 
\]
Figure 4-3: Adaptive Pole-placement Controller
Figure 4-4: Gradient Stabilizer

\[ \frac{1}{s + a} \]
where $H_p(s)$ is the Hermite normal form of the plant in equation (4.44). $\Theta_c^*$ is the ideal value of the controller parameters $\Theta_c$ for the particular system parameter value $\Theta$. That is, when $\Theta_c(t) \equiv \Theta_c^*$, the closed loop transfer function satisfies $W_p(s) \equiv H_p(s)$ and the constant disturbance in the output $d_0(\Theta)$ is exactly cancelled. The initial value of $K_0$ is the inverse of the high frequency gain, $K_p^{-1}$, for the plant in equation (4.44) with nominal values for system parameters $\Theta$. The initial value of $\dot{d}$ is chosen as zero. The initial values of the rest of the controller parameters $\Theta_c$ are found by solving the following equation for $Z_c(s), Z_d(s)$ with the same plant parameters $\Theta$.

$$
[(R_q(s) - Z_c(s)) R_p(s) - Z_d(s) Z_p(s)] = R_q(s) K_0 H_p^{-1}(s) Z_p(s). \tag{4.51}
$$

The error here, $e_1(t) = z_p(t) - z_m(t)$ is found to be

$$
e_1(t) = H_p(s) K_p \Phi(t) \omega(t) + T d_1(\Theta). \tag{4.52}
$$

For stable adaptation the transfer function from $K_p \Phi(t) \omega(t)$ to $e_1$ needs to be Strictly Positive Real (SPR). If $H_p(s)$ is relative degree 2 as in the case of the helicopter a further modification needs to be made for stable adaptation. A gradient stabilizer is added as in figure 4-4 with a transfer function

$$
W_a(s) = \frac{1}{s + a}, \tag{4.53}
$$

where $a$ is a positive integer [56]. The input and error equations now become

$$
u(t) = \dot{\Theta}_c(t) \bar{\omega}(t) + \Theta_c(t) \omega(t), \tag{4.54}
$$

$$
\bar{\omega}(t) = W_a(s) \omega, \tag{4.55}
$$

$$
e_1(t) = H_p(s) W_a^{-1}(s) K_p \Phi(t) \omega(t). \tag{4.56}
$$
$W_a(s)$ can be chosen such that $H_p(s)W_a(s)$ is SPR. The following adaptation law is now chosen for stable adaptation:

$$\dot{\Theta}_c = -Pe_1\bar{\Theta}_c^T + \Gamma_r \Theta_c, \quad \Gamma_r > 0$$

(4.57)

$$P = \Gamma^{-1}. \quad \Gamma$$

(4.58)

Here $K_p$ needs to satisfy

$$\Gamma K_p + K_p^T \Gamma = Q_0 > 0, \quad \forall \Theta \in \Theta_s.$$

(4.59)

$\Gamma$ is chosen for robustness of the design to the trim disturbance $d_1(\Theta)$ which is not cancelled out, nonlinearities in the system, noise and disturbances other than those due to unknown trim.

**Theorem 4.1** For the error given in equation (4.56), given $H_p(s)W_a(s)$ is SPR, $K_p$ satisfies equation (4.59), and $\bar{\Theta}$ is represented by equation (4.55), the adaptation law for $\dot{\Phi} = \dot{\Theta}_c$ in equation (4.57)-(4.58), guarantees that all signals are bounded.

The reader is referred to [56] for proof.

### 4.4 Numerical Studies

The controller developed in the previous sections is now simulated for the full helicopter dynamics presented in section 2.1. The full three degree of freedom system is used for the simulations with a 20% uncertainty in the mass. The states that are not accessible are the rotor states $a_1, b_1, a_{1,FB}, b_{1,FB}$ and fractional fuel capacity $\lambda_f$. Two different tasks involving vertical flight and coordinated turn maneuvers are performed and the results of the pole-placement controller are compared against the adaptive controller. Adaptation is stopped after some time to observe learning. It needs to be noted that, the controller design uses knowledge of the aerodynamics for the linearization. At these speeds a design neglecting aerodynamics has inadequate robustness properties and simulations fail because of instability.
4.4.1 Vertical flight

The first task performed involves steps in vertical velocity. The helicopter is rising upwards and the velocity varies between 5 ft/sec and 10 ft/sec. Figures 4-5–4-6 show the performance in the relevant states downward velocity, $w$, and forward velocity, $u$. The controller has 96 parameters with the throttle kept constant for a 4-input 4-output system. The adaptation is seen to outperform the constant controller in terms of steady state error and transients. In the forward velocity the transients are seen to be lower than the linear controller. Learning behavior is observed for the maneuver as the adaptation is switched off after just two complete cycles. The behavior for the next step is seen to be as good as in the last adaptive cycle. The adaptive controller with trim error estimate eliminates steady state bias. The increased initial transients are also removed in the second adaptive cycle.

4.4.2 Coordinated Turn

In this maneuver, the helicopter moves from a coordinated turn of $2.5^\circ/sec$ to $5^\circ/sec$ with a forward velocity of $5ft/sec$. A 20% uncertainty in the mass is added to the system. The results presented are for the case when 20% of the nonlinearities are added to the simulated system. The controller has 200 parameters. In this case too the adaptive controller is seen to outperform the linear pole-placement robust design. In the quaternion $q_1$ corresponding to the roll angle the adaptive controller shows similar improvements as in previous maneuvers. The steady state error is reduced and the initial transients are reduced with time. Moreover, after stoppage of adaptation the learned values of controller parameters continue to show good performance for the maneuver. In this maneuver, over a period of 30 seconds the linear controller is seen to result in a 6.5 feet error in the displacement of the helicopter from the nominal designed model. The helicopter travels about 45 feet in the X-direction during this period. The adaptive controller reduces the error to less than 3 feet in the first cycle and to around 2 feet in the second cycle. After stopping adaptation the
Adaptation stopped at 80s

Figure 4-5: $w$ for steps in vertical flight for partial state feedback
Adaptation stopped at 80s

Figure 4-6: $u$ for steps in vertical flight for partial state feedback
controller continues to demonstrate the learned performance. For longer durations and for more aggressive maneuvers, gain-scheduling similar to that in chapter 3 is needed. Figures 4-7–4-11 show this case.

4.5 Chapter Summary

In this chapter a viable adaptive controller is developed for helicopter when only a partial state of the helicopter is measurable. A design procedure is presented which develops an adaptive controller with an adaptive trim error estimate, for this design problem where non-minimum phase behavior is observed, high frequency gains are nonsingular and the plant has a relative degree greater than unity. The effectiveness of this controller is demonstrated through simulations where the adaptive controller outperforms linear control designs by learning a maneuver, eliminating steady state bias for trim error disturbance, and learning the frequency content of a maneuver and reducing initial transients. However, the number of controller parameters increases from 90 in the full state feedback case to 250 in the partial state access case. Furthermore, the partial state access case requires 80 controller states. This increases the online computation requirement.
Adaptation stopped at 60s

Figure 4-7: \( q_1 \) for turn with partial state feedback
Figure 4-8: $X$(ft) for turn with partial state feedback
Figure 4-9: $X$ vs. $Y$ for turn with partial state feedback- First Cycle
Figure 4-10: $X$ vs. $Y$ for turn with partial state feedback- Last Adaptive Cycle
Figure 4-11: $X$ vs. $Y$ for turn with partial state feedback- Adaptation Stopped
Chapter 5

Summary and Concluding Remarks

This thesis provides a design procedure for viable adaptive controllers for a large class of nonlinear fully-coupled systems with parametric uncertainties. This controller is demonstrated for the control of an autonomous helicopter. First, a design procedure is developed for the case when the complete state of the helicopter is accessible. A trim error estimate is added to the adaptive controller and the requirement of “matching conditions” for stable adaptation is relaxed. The controller is demonstrated through simulations for the nonlinear longitudinal dynamics with uncertainties in the mass and moment of inertia. Maneuvers involving trajectory tracking, steady state bias and complex maneuvers are simulated for high velocities and show considerable improvement with adaptation. Existing control designs are mostly for helicopters at hover and neglect aerodynamics or have very low robustness to uncertainties. Their performance is severely degraded at moderate speeds with uncertainties. The adaptive controller is stable, robust and shows vast improvement in performance over other designs.

A design procedure is presented for the case with partial state access. The adaptive controller includes the trim error estimate developed in the previous design. The non-minimum phase behavior of the helicopter is accounted for through the design of a post compensator. The controller includes a pre-compensator and gradient stabilizer for stable adaptation with the helicopter having singular high frequency gain
and a relative degree greater than one for the maneuvers in consideration. Procedures for the easy determination of nominal trim conditions and for coprime matrix fraction decomposition are also presented. Two maneuvers are demonstrated through simulations for this controller which include a coordinated turn and vertical flight. Improvements similar to those in the full-state access case is observed for uncertainties upto 20% in the mass. Similar or better results are seen for other parametric uncertainties.

Thus a viable multivariable adaptive controller is proposed and demonstrated in this thesis which enables the design of stable, robust control systems for autonomous helicopters capable of high speed and high bandwidth maneuvers.

5.1 Future Work

The design procedure presented in this thesis should provide a framework for further simulations for other aggressive maneuvers on the helicopter model. Implementation issues on a real helicopter would require work on integration of the adaptive controller with existing linear robust or hybrid control architectures used for autonomous vehicles.

Lower order controllers can be explored to reduce the number of controller parameters for the case without full state access. Integration with learning controllers and approximate nonlinear methods can be explored to further improve performance and stability properties for aggressive maneuvers.

The adaptive controller presented in this thesis can also be used for similar nonlinear systems. Underwater autonomous vehicles can be a potential application since drag forces are significant even at low velocities for those vehicles. Applications in other non-minimum phase systems such as certain robots and electro-mechanical devices can also be explored.
Appendix A

Proofs

Theorem 3.1 For the system given by the dynamic equations (3.3)–(3.4), the input in equation (3.22), and adaptive laws in equations (3.23)–(3.25), and satisfying the conditions in equations (3.27)–(3.30), all signals are bounded and the system is stable.

Proof:

Given the equation of the system (2.79) linearized about \((X_0, U_0)\) to give equations (3.3)–(3.4), we have, with the reference model in equation (3.6), the error \(e = x_p - x_m\) is given by

\[
\dot{e} = A_m e + B_m \Psi u + B_m \Phi x_p + B_m \Upsilon + \Delta A_m x_p + d_{x_r}(\Theta) \quad (A.1)
\]

where

\[
\Phi = K - \overline{K}^* 
\]

\[
\Psi = Q - Q^* 
\]

\[
\Upsilon = \dot{d} - \dot{\bar{d}}^* 
\]

\[
d_{x_r}(\Theta) = d_x(\Theta) - \dot{\bar{d}}^*. \quad (A.5)
\]

Here \(\overline{K}^*\) and \(Q^*\) are given by equations (3.7) and (3.26), respectively, and \(\dot{\bar{d}}^*\) is the part of \(d_x(\Theta)\) which can be cancelled using the trim error estimate \(\dot{\bar{d}}\). This depends
upon the terms in $B_p(\Theta)$. It has been assumed that (3.7) is satisfied. This is the case with the helicopter. In case this is not satisfied, adaptive laws with bounded controller parameters can be used to arrive at a similar stability result. $\Delta A_m$ is given by equation (3.28).

Choosing a Lyapunov function candidate

\[ V = e^T P e + Tr \left( \frac{1}{\Gamma_1} \Phi^T \Phi + \frac{1}{\Gamma_2} \Psi^T \Psi + \frac{1}{\Gamma_3} T^T T \right) \]  

(A.6)

we get, by defining the error as above and with the controller as in the theorem, the derivative of the Lyapunov function as

\[
\dot{V} = -e^T Q_0 e + 2 e^T P B_m \Phi x_p + 2 e^T P B_m \Psi u - 2 Tr \left( x_p e^T P B_m \Phi \right) \\
-2 Tr \left( u e^T P B_m \Psi \right) + e^T P \Delta A_m x_p + x_p P \Delta A_m e^T \\
+2 \frac{\Gamma_1}{\Gamma_1} Tr \left( \Phi^T K^\ast \right) + 2 \frac{\Gamma_2}{\Gamma_2} Tr \left( \Psi Q^{*-1} \right) + 2 \frac{\Gamma_3}{\Gamma_3} Tr \left( \Upsilon d^\ast \right) \\
-2 \frac{\Gamma_1}{\Gamma_1} Tr \left( \Phi^T \Phi \right) - 2 \frac{\Gamma_2}{\Gamma_2} Tr \left( \Psi^T \Psi \right) - 2 \frac{\Gamma_3}{\Gamma_3} Tr \left( \Upsilon^T \Upsilon \right) + d_{x_r} (\Theta). \]  

(A.7)

This leads to the condition that

\[
\dot{V} \leq -e^T Q_0 e + 2 e^T P \Delta A_m x_m \\
+2 \frac{\Gamma_1}{\Gamma_1} Tr \left( \Phi^T K^\ast \right) + 2 \frac{\Gamma_2}{\Gamma_2} Tr \left( \Psi Q^{*-1} \right) + 2 \frac{\Gamma_3}{\Gamma_3} Tr \left( \Upsilon d^\ast \right) \\
-2 \frac{\Gamma_1}{\Gamma_1} Tr \left( \Phi^T \Phi \right) - 2 \frac{\Gamma_2}{\Gamma_2} Tr \left( \Psi^T \Psi \right) - 2 \frac{\Gamma_3}{\Gamma_3} Tr \left( \Upsilon^T \Upsilon \right) + d_{x_r} (\Theta). \]  

(A.8)

For some time $t > t_0$ the sum of the first two terms in (A.8) becomes negative and for some large finite values of $\Psi$, $\Phi$ and, $\Upsilon$ the sum of the remaining terms is negative. That is

\[
-e^T Q_0 e + 2 e^T P \Delta A_m x_m \leq 0 \]  

(A.9)
for some $t > t_0$, and

$$2 \frac{\Gamma_{r1}}{\Gamma_1} \text{Tr} \left( \Phi^T K^* \right) + 2 \frac{\Gamma_{r2}}{\Gamma_2} \text{Tr} \left( \Psi Q^* \right) + 2 \frac{\Gamma_{r3}}{\Gamma_3} \text{Tr} \left( \Upsilon \dot{r} \right)$$

$$-2 \frac{\Gamma_{r1}}{\Gamma_1} \text{Tr} \left( \Phi^T \Phi \right) - 2 \frac{\Gamma_{r2}}{\Gamma_2} \text{Tr} \left( \Psi^T \Psi \right) - 2 \frac{\Gamma_{r3}}{\Gamma_3} \text{Tr} \left( \Upsilon^T \Upsilon \right) + d_{xx} (\Theta) \leq 0 \quad \text{(A.10)}$$

for some large $(\Psi, \Phi, \Upsilon)$

Thus from La Salle’s theorem it is ensured that the system is stable and all signals are bounded.
Appendix B

Coprime Matrix Fraction Description

The following algorithm describes the method used to arrive at a coprime matrix fraction description (MFD) for the linear system described in equations (4.8), (4.9). This method described in [58] is seen to provide sufficiently accurate coprime MFD for low errors in the solution of equation (4.30).

The following definitions describe admissible pseudo-controllability indices, which are needed for the algorithm to find the coprime MFD. Let a linear dynamic system with $m$-inputs, $m$-outputs, and $n$-states be described in the state space form by

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du.
\end{align*}
\]

**Definition**: The set of individual controllability indices, $\{\alpha_i\}$, $1 \leq i \leq m$, is defined by

\[
\alpha_i = \text{rank} \begin{bmatrix} b_i & A b_i & \ldots & A^{n-1} b_i \end{bmatrix}
\]

where $b_i$ is the $i^{th}$ column of the matrix $B$. 

111
Definition: The set of pseudo-controllability indices \( \{ \mu_i \} \), \( 1 \leq i \leq m \), is any set of numbers satisfying

\[
1 \leq \mu_i \leq n - m + 1, \quad \text{and} \quad \sum_{i=1}^{m} \mu_i = n.
\]

Definition: The set of pseudo-controllability indices, \( \{ \mu_i \} \), \( 1 \leq i \leq m \), is admissible if

\[
\text{rank} \begin{bmatrix} b_1 & A b_1 & \cdots & A^{\mu_1 - 1} b_2 & \cdots & A^{\mu_m - 1} \end{bmatrix} = n. \tag{B.3}
\]

Crate diagrams are now formed for all admissible pseudo-controllability indices or nice indices. The crate's column entries correspond to columns of the controllability matrix,

\[
Q_c = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix},
\]

associated with a particular input. An admissible set of pseudo-controllability indices could be the as in the table below.

<table>
<thead>
<tr>
<th>Case</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pseudo-Controllability Indices</td>
<td>{3,3,1}</td>
<td>{3,1,3}</td>
<td>{1,3,3}</td>
</tr>
</tbody>
</table>

The crate diagrams corresponding to these sets of nice indices would be as given below.

From the crate diagrams several related "selector vectors" are generated:
1. By omitting the first row of, say the center diagram, corresponding to the indices \{3,3,1\}, the vector \(v_i\) is created by selecting the non-blank elements row-wise:

\[
v_i = [1 \ 0 \ 1 \ 1 \ 1 \ 0 \ 0]^T
\]

2. From \(v_i\) the binary complement is formed, and denoted as \(v_a\):

\[
v_i = [0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 1]^T
\]

3. By considering the blank elements to be zeros, \(v_{hi}\) is formed in like manner, but with row 1 included:

\[
v_{hi} = [1 \ 1 \ 1 \ 1 \ 0 \ 1 \ 1 \ 0 \ 1 \ 0 \ 0 \ 0]^T
\]

4. Finally, \(v_{ld}\) is formed by again including the first row, but now taking the blank elements of the diagram to be unit valued, and finally taking the binary complement, leading to:

\[
v_{ld} = [0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 1]^T
\]

The best pseudo-controllability indices are now selected as the set which gives the lowest condition number for the matrix in equation (B.3). For these the above “selector vectors” are used to then get “selector matrices” by selecting rows corresponding to ones from identity matrices of sizes equal to the vectors. The “selector matrices” for the indices \{3,3,1\} are:

\[
S_i = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]
The transformation matrix $T_c$ is now chosen as $T_c = Q_cS_{li}$ and the controllable form of the system is given by:

$$A_c = T_c^{-1}AT_c$$
$$B_c = T_c^{-1}B$$
$$C_c = CT_c$$
$$D_c = D.$$ 

Now the following steps are followed to get the coprime MFD of the system.

1. Find $A_{cc}$ from $A_c$ using $S_a$ as,

$$A_{cc} = A_cS_a.$$ 

2. Find the denominator polynomial matrix $D_s(s)$ from its column form $D_{rc}$ from
Using $S_{ld}$ and $S_{li}$.

$$D_{rc} = S_{ld} - S_{li} A_{cc}.$$  

Here $D_{rc} = [D_0 \ldots D_k]^T$ and $k = \mu_m = \max\{\mu_i\}$. Matrices $D_i$ correspond to the coefficients of the $s^i$ term in $D(s)$.

3. Find $T_d(s)$ from $D(s)$.

$$T_d(s) = \text{adj} D(s).$$

4. Find the transfer function matrix from the state space representation.

$$W(s) d(s) = C_c (sI - A_c)^{-1} B_c + D_c.$$

Here $W(s)$ is the matrix of numerators of transfer functions and $d(s)$ is the characteristic equation.

5. From $T_d(s)$ and $W(s)$ solve the following equation:

$$[N_0 \ldots N_k] \begin{bmatrix} T_{d0} & \ldots & T_{dn} \\ \vdots & \ddots & \vdots \\ T_{d0} & \ldots & T_{d(n-k)} \end{bmatrix} = [W_0 \ldots W_n].$$

Here $T_{di}$, $N_i$ and $W_i$ are corresponding real number submatrices in the polynomial matrices $T_d(s)$, $N(s)$ and $W(s)$.

6. From $N_r = [N_0 \ldots N_k]$ get $N(s)$.

$N(s)$ and $D(s)$ represent the Right Coprime Matrix Fraction Description of the system in equations (B.1) and (B.2).
Bibliography


