A Stepwise Methodology for the Calculation of Interlaminar Stresses in Transversely-Loaded Grooved Laminates

by

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Abstract

A stepwise methodology to assist in the development of a model to determine the interlaminar stress fields and the mechanisms that give rise to these stress fields in transversely-loaded grooved composite laminates was developed. The methodology consists of five steps, with each step representing a laminate configuration with an increase in complexity from the previous step. Models for each step were developed in the context of a general formulation proposed in the literature that was previously used to analyze the proposed problem for Step 1. This general formulation is based around an assumed stress approach, where unknown coefficients in the assumed stress shapes are solved via the Principle of Minimum Complementary Energy. Models derived from such an approach were designed to be both accurate in the results as well as more efficient in runtime as compared to other models (e.g. finite element models). Such models are particularly useful for preliminary design, where various laminate configurations need to be analyzed efficiently to find designs for more detailed analysis and further modification. The results from the models for Steps 1 through 3 were found to be in good agreement with results in the literature, when available, or finite element results when analyzing configurations with results not found in literature. Characteristic results were taken from the models for Steps 1 through 3 in order to determine modifications, as well as identifying phenomena in the stress distributions, that require particular attention in the formulation of subsequent models. Key controlling factors in the model for Steps 1 through 3 are identified from the results. Issues encountered in Step 4 with regard to representation of slanted dropoffs prevent further model development. The feasibility of the formulation of the Step 5 model, involving loaded edges, was established assuming a working Step 4 model. Modifications of Steps 4 and 5 of the initially-proposed stepwise methodology were developed such that models for the proposed problems can be developed using the general formulation established for previous steps. These modified steps allow for a closed-form solution within the context of the general formulation as well as identification of the mechanisms and laminate parameters that affect interlaminar stress fields in transversely-loaded grooved laminates. Recommendations for future work are made.

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Foreword

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Nomenclature

ROMAN SYMBOLS

a vector of unknown coefficients $a_j$

$a_j$ unknown stress coefficients in Region A

A Region A

b vector of unknown coefficients $b_j$

$b_j$ unknown stress coefficients in Region B

B Region B

c vector of interface stress functions $F_i$

$F_i$ stress function for $\sigma_{11}$ defined at the ith interface

G vector of interface stress functions $G_i$

$G_i$ stress function for $\sigma_{12}$ defined at the ith interface
\( h_{i1} \) horizontal vector that, multiplied by \( \mathbf{F} \) obtains the \( \sigma_{11} \) ply stresses in ply \( i \)

\( H \) variation of in-plane stress in the thickness direction

\( H_{in} \) weighting function applied to the stress function along the inner interface of a ply

\( H_{out} \) weighting function applied to the stress function along the outer interface of a ply

\( i \) \( i \)th ply or sublayer in a region as a superscript, \( i \)th interfacial stress function as a subscript

\( n_d \) number of dropped plies in a half-laminate from Region A to Region B

\( n_r \) number of plies in a region minus one, and number of interfaces in a region

\( S^m \) surface area of region \( M \)

\( S_{ijkl} \) components of the compliance tensor

\( S^*_{ijkl} \) reduced compliance tensor

\( t_{lam} \) total laminate thickness within a region

\( t_{i}^{\text{ply}} \) ply thickness of ply \( i \)

\( \tilde{u}_i \) prescribed displacements in the \( i \)-direction

\( V^m \) volume of region \( M \)

\( x_1 \) lengthwise coordinate

\( x_2 \) widthwise coordinate

\( x_3 \) global thickness coordinate

\( x_3^i \) local thickness coordinate of ply \( i \)

\( x_{left} \) left boundary of a region
$x_{\text{left}}$  right boundary of a region

**GREEK SYMBOLS**

$\lambda_j$ characteristic eigenvalues of a region

$\pi^*_C$ complementary energy

$\rho_{mn}$ vector of Lagrange Multipliers $\rho_{mn_j}$ on stress $mn$

$\rho_{mn_j}$ $j$th Lagrange Multiplier in a region on stress $mn$

$\sigma_{\text{comp}}$ contribution to the total stress by the complementary solution

$\sigma_{\text{farfield}}$ contribution to the total stress due to far-field stress solution

$\sigma_{\text{total}}$ total stress

$\Phi_j$ matrix of eigenvectors $\phi_j$

$\phi_j$ characteristic eigenvectors of a region

$\Psi_j$ matrix of eigenvectors $\psi_j$

$\omega_i$ Slant Angle of Ply $i$
Chapter 1

Introduction

The use of composite materials in structural components has increased significantly over the last several decades, particularly in aircraft and other applications within the aerospace industry. This is chiefly due to the fact that composite materials offer higher specific stiffness and strength over more traditional aerospace materials, and thus offer savings in weight and cost in aircraft and other applications [1]. Although many different manufacturing techniques are in use to produce composite structural components, the bulk of these manufactured components are layered in nature.

With an expansion of the usage of composite laminates in progressively more critical roles, there continues to be a need to predict the behavior of composite structures with greater efficiency. In particular, the behavior of laminated structures is far more complex than that of isotropic structures because of the presence of both anisotropy of the individual plies and large changes in material properties from ply to ply. These differences not only modify the stress field for a composite structural component versus an isotropic component, but the construction of a composite laminate presents greater challenges as there are modes in which a laminate can fail under loading that are not present for more traditional materials [2].

Of particular concern in composite laminates is the possibility of delamination, where individual plies separate. Such a separation greatly reduces the load-carrying capabilities of the laminate and can result in total failure. The separation of plies
generally occurs due to the presence of interlaminar stresses that arise due to the mismatch between properties of adjacent plies and the mismatch between properties of plies and the properties of the overall laminate [3, 4]. At interfaces between plies, there are no fibers present. Thus, the material response is predominantly governed by the matrix material, which is much weaker than the fibers used. The interlaminar stresses tend to peel the plies apart, in the case of positive interlaminar normal stress, to shear apart, in the case of interlaminar shear stress, or a combination thereof. In the design of laminated components, interlaminar stresses need to be carefully and accurately examined, as delamination can occur at loadings less than critical loadings associated with other modes of failure.

The focus of this work is a composite with a groove, loaded through the thickness within the groove. This configuration is motivated by a helically-grooved composite wing spar that can telescope in and out for storage. The grooves act as a track for bearings to run along to allow this telescoping motion [5]. Such a design could help both to optimize the performance of an aircraft in flight as well as to serve as a storage mechanism for the wings on the ground for use in UAVs and roadable aircraft [5, 6]. Though the use of composite materials in such a grooved configuration could improve the structural efficiency of the component versus other materials, the use of composite materials presents a challenge in that analysis, and behavior prediction of a grooved component is more difficult to execute. Furthermore, the presence of a groove running through the thickness of a composite component, coupled with transverse loading within the groove, may cause delaminations centered around the bottom surface of the groove and neighboring plies. Currently, no investigations exist with regard to the mechanisms involved in the rise of interlaminar stresses in such a structural configuration. Thus, there is a need to qualitatively and quantitatively predict the response of such a structural configuration with a particular concern on the analysis of the interlaminar stress field. With knowledge of the interlaminar stresses present, one could begin to identify configurations where delaminations are more likely to arise within the structure.

The objective of the work is a proposition for an approach to developing and
validating a model capable of estimating the interlaminar stress fields in transversely-loaded grooved laminates. The approach is based upon extending a known solution for a simple problem via a stepwise methodology. Every step in the methodology introduces an additional level of complexity in the proposed problem, which in turn requires an additional level of complexity in the model in order to obtain a solution. This methodology is extended until the problem of the transversely-loaded grooved laminate can be solved. Although the model can provide for a quantitative estimate of the magnitudes of the interlaminar stresses, the model is designed primarily to obtain a level of fidelity sufficient to be able to establish the overall variation of the interlaminar stresses in the laminate. Results from the model can be used to identify the relevant material and geometric parameters that control the variation of the stresses and provide insight to the mechanisms behind the rise of interlaminar stresses in such a structural configuration. A formulation general to all steps in the methodology is provided, as well as validation and results for the first three steps in the proposed approach. Finally, an examination of the issues encountered in the final two steps of the problem is provided, as well as a discussion as to the overall feasibility of the approach, particularly with regards to model formulation, in being able to obtain a solution for the problem of a transversely-loaded grooved laminate.

Although no work in the literature to date has dealt with interlaminar stresses in transversely-loaded grooved laminates, previous work has given an indication of the types of analyses that can be applied to other laminated structural configurations. In Chapter 2, previous pertinent works are summarized, as well as general works on the qualitative nature of interlaminar stresses. In Chapter 3, the overall objective and methodology is detailed, including how the problem, after reducing the complexity of the overall problem through a stepwise approach, can be idealized as a more general case of previously-analyzed structural configurations. The general formulation of the governing equations of the model to predict stresses in all steps, along with details of the methodology used to validate the model in the first three steps, are presented in Chapter 4. The overall details behind the applicability of the methodology to the previously-analyzed problem of an infinite-length laminate with a single dropoff un-
der tension are presented in Chapter 5. The extensions of the model to the second and third steps in the overall approach are presented in Chapters 6 through 7, respectively. Chapters 5 through 7 include results from finite element analyses for the purpose of comparison and validation of the model at each of the first three steps. In addition, intermediate results are presented, including a brief discussion of how the results influenced further model progression. The particular issues encountered in the formulation in the fourth step are discussed in Chapter 8. A discussion of the feasibility of the overall stepwise approach and formulation in estimating the interlaminar stresses for the problem of the transversely-loaded grooved laminate is presented in Chapter 9. Finally, the conclusions of the current work and recommendations for future work are presented in Chapter 10.
Chapter 2

Previous Work

The purpose of the current work is the creation of an approach in developing a model to estimate the interlaminar stresses in a transversely-loaded grooved laminate. As this structural configuration has not been analyzed in previous work, there exists no methodology specifically dealing with this problem. However, there is a sizable body of previous work that addresses interlaminar stresses in composite laminates. Thus, it is useful to examine the works previously presented in the analysis of interlaminar stresses in composites in order to obtain a better understanding of the issues inherent in interlaminar stress analysis, as well as to find suitable models that could be extended in scope to assist in the analysis of the current problem. In that vein, examination of analytical methods in the problem of contact with composite laminates is also undertaken, as the current problem is derived from a physical system where contact loads are being transferred from a bearing to the grooved surface in the laminate.

The focus of the first section of this chapter is to work towards an overview of the methods used to analyze the interlaminar stress field in composite laminates under various loadings and for various laminate geometries. In the second section, methodologies used to analyze the stress fields of composite laminates under contact loading are considered.
2.1 Analytical Methods for Interlaminar Stresses

The early work in the estimation of the interlaminar stress fields in composite laminates occurred in the later 1960s and the early 1970s, [e.g. 7-9, 12, 13]. In particular, three different approaches for the calculation of interlaminar stresses in composite laminates were developed in three works during that time period [7-9]. Discrepancies in the results of these three works prompted further investigation into the behavior of interlaminar stresses in composite laminates.

The bulk of works on the techniques for the analysis of interlaminar stresses have focused mainly on what has been referred to as the “free-edge problem.” In these, a loaded laminate, with geometry illustrated in Figure 2.1, is considered. The laminate, of arbitrary layup, is subject to either a uniform loading or displacement along its length, defined as the $x_1$-direction in Figure 2.1. The problem is defined such that there needs to be a resolution between the plane-stress solution presented by Classical Laminated Plate Theory (CLPT), with constant stresses in each ply, and the boundary condition that $\sigma_{22}$, the in-plane transverse stress, must equal zero at all points along the outer free edge of the laminate in the widthwise ($x_2$) direction. This is in contrast to the integral of these stresses equaling zero in an integral interest along the free edge as in CLPT [2]. Since good results are given in CLPT when invoking St. Venant’s principle in this manner, the complete solution needs to be such that the stresses equal values predicted by CLPT away from the free edge, but also equal zero at the free edge. Such a stress distribution results in a gradient of stress in moving along the $x_2$-direction. Such a gradient gives rise to interlaminar stresses due to considerations of equilibrium [3].

In all of the methods considered, all but a few share two underlying assumptions. The first is that the material properties of each layer can be treated as orthotropic and homogenous, ignoring the specifics of the microstructural interactions between different phases of the material. The second is that, referencing Figure 2.1, stresses are invariant in the lengthwise ($x_1$) direction.

A number of techniques were used in early work to allow for a closed-form solu-
Figure 2.1 Illustration of the “Free-Edge Problem” - a composite laminate under uniaxial tension/extension.
tion for the interlaminar stress fields of the free edge problem. These include finite differences [7], assumed displacement shapes [8], and finite elements [9]. The results from these three works showed a broad spectrum of discrepancies which inhibited the ability to identify mechanisms that give rise to interlaminar stresses in composite laminates. Three main discrepancies arose in the solutions obtained via the methodologies presented in the aforementioned works. The first discrepancy was the existence and behavior of the interlaminar normal stress, $\sigma_{33}$. The second discrepancy was the presence, or lack thereof, of a singularity in the interlaminar shear stress, $\sigma_{13}$, close to the free edge between ply interfaces. The third discrepancy relates the manner in which the geometric parameters of the laminate affect the interlaminar stress fields and the related lengthscales over which the interlaminar stresses act.

With such a broad spectrum of discrepancies between the results of the three initial methodologies, it had become difficult to characterize the stresses arising from the free-edge problem. Perhaps in an effort to try to gain insight into the problem in order to resolve the three aforementioned discrepancies, the bulk of significant works regarding interlaminar stress modeling that followed these three works focused on developing semi-analytical models rather than numerical models. Few subsequent numerical models significantly contributed to the understanding of interlaminar stresses until the work of Wang and Crossman in 1977 [10, 11].

Analytical models presented through the 1970s and into the early 1980s gave some insight into some of the issues previously mentioned, although each model had severe drawbacks that limited the applicability of such models. Despite these limitations, the results of these works began to resolve the discrepancies that arose from the work in the late 1960s and early 1970s. Work was presented based upon a number of methods, including heuristic methods [12], boundary-value methods [13–16], perturbation methods [17], stress potential methods [18, 19], and refinements to methods employed in earlier work [10, 11, 20].

The issues of the existence and sign of the interlaminar stresses in certain laminates were largely resolved due to reliance on less stringent assumptions on the behavior of the laminate stresses and displacements [10–15, 20]. Methodologies which allowed
for the stress fields to vary directly with laminate geometric and material parameters validated that the material properties of plies as well as ply thickness affected the behavior of interlaminar stresses as well as determining the lengthscales at which these stresses occur for a number of laminate configurations [10, 11, 13, 16–19].

Works which focused on understanding the nature of the presence and strength of the singularity at the free edge contributed to the conclusion that although a singularity does exist, it is of logarithmic order, and thus much weaker than other singularities present in analytical mechanics [10, 17–19, 21, 22]. In finite element methods, the effect of such a singularity could be mitigated via proper discretization of the elements close to the free surface [10, 11, 22]. Notable is that the issue of a singularity is only a theoretical one in works involving composite laminates, as there is no possible way that real laminates can carry an infinite stress. The singularity comes about in the discrete representation of the material properties moving through the thickness in a laminate. Upon crossing a ply interface, the material properties change instantaneously, and it is difficult to account for this using a stress that is continuous through the thickness of the model. In real laminates, the material properties transition continuously in moving from ply to ply with no discontinuous "jump" in the material properties. Thus, the singularity does not truly manifest itself in real laminates. With most modern delamination failure analysis based upon averaged stresses over some length, there is little practical application for the presence of the singularity in modern work.

Despite all the work in the modeling of solutions of interlaminar stress fields, the models investigated to this point in time, as reported in the literature into the early 1980s, remain limited in their applicability, their accuracy, or their computability. Many of the methods in the aforementioned work either obtained results that did not satisfy all stress boundary conditions [14, 15, 17] or required large runtimes in order to obtain details at the free edge [10, 11, 16]. Such drawbacks to the methodologies limited their applicability in analyzing a larger number of laminate configurations. Consideration of the literature suggests that there are two classes of analyses that have been conducted up to this point, with each class possessing some significant
weaknesses.

The first class of models in this consideration is the numerical methods that employ some finite difference or finite element scheme. These methods involve either a numerical solution of the differential equations of elasticity applied directly to the problem, or a formulation of the problem such that the equations of elasticity are applied in numeric form via a numerical representation of some variational principle, either minimization of laminate energy, complementary energy, or virtual work. The advantage of these methods is their applicability to a wide class of laminate configurations. However, these methods have a disadvantage in that, especially in the time when they were formulated, it was computationally expensive to obtain results, especially when concerning the analysis of a large number of laminate configurations. For example, in the work of Reference [7], the resultant finite-difference scheme led to the creation of a 1200-by-1200 linear system of equations. The resultant matrix itself was relatively sparse, although nonzero bands in the matrix contained approximately 60 to 80 entries each. The finite element system in the work of Reference [10] led to a mesh with over 27,000 elements for a symmetric half of an eight-ply laminate. Later work proposed a generalized system with each layer requiring 20 equations and 20 variables each [16].

The second class of models in this consideration involves semi-analytical methods. The term "semi-analytical" is applied due to the fact that the models cannot exactly solve the equations of elasticity under all required constraints, and thus, some sort of shape for either the stress or the displacement must be assumed a priori in order to obtain a closed-form solution. These models are less computationally expensive than the numerical methods, but are typically applicable to fewer laminate configurations than the numerical methods.

At this time in the literature, about the early 1980s, there was a need for the development of an accurate, efficient, and effective method for the calculation of interlaminar stresses. Efficiency deals with the need for the methodology to be computationally efficient in that the overall runtime required of the solution needs to be relatively small. Effectiveness deals with the need for the methodology to be applica-
ble to a large number of configurations rather than only a few idealized problems. In particular, this latter issue was a drawback of the semi-analytical methods in literature at this time. Up to this point in the literature, no model sufficiently embodied these three qualities, and thus, could not be considered to be accurate, efficient, and effective.

One of the more influential works in addressing these issues in the estimation of interlaminar stresses came in two works done by Kassapoglou and Lagace in 1986 and 1987 [23, 24]. They assumed stress shapes of layerwise products of exponentials along the length, and polynomials through the thickness that satisfied all differential and integral equations of equilibrium, as well as relevant boundary conditions at the free edge in what is termed the "Force-Balance Method." Unknown parameters in the stresses were to be determined via the minimization of complementary energy. The assumption that led to large simplification of expressions was that the decay parameters in the exponentials is the same for all layers. The resultant system of equations simplified to two algebraic, biquadratic expressions in terms of the decay parameters, even for thick laminates on the order of hundreds of plies. Despite its simplicity, the results from the method compared favorably to previous work [7, 10, 18].

Extensions to this approach are numerous. Direct extensions of the methodology applied to different laminate configurations include calculation of the interlaminar stress field for a laminate in bending [25], stresses arising through mismatches in thermal strains [26-28], and stresses in beamlike composite structures undergoing out-of-plane shear and bending [29]. The approaches in References [28] and [30] extended the formulation of the model to include terms that better account for the mismatches in Poisson’s Ratio and coefficients of mutual influence between adjacent plies. The resulting formulation increased the order of variability of stresses through the thickness and led to improvements in the estimation of the overall stress field at the cost of increased computation time [31].

Further extensions to the Force-Balance Method were made via reduction of the assumptions on the shapes of the stresses. Generally, the variation of the stresses was
assumed to be an unknown function of some parameter, and after minimizing the laminate complementary energy, the resultant equations could be defined as an eigenfunction problem with the variations of the stresses as variables. This eigenfunction problem would require the variations of the stresses to be a sum of exponential terms. This methodology was applied to both the free edge problem [32] and laminate with holes [33].

With the use of the approach based on the Force-Balance Method extended, Bhat and Lagace were able to determine the interlaminar stresses at the interface between laminates of different layups or material properties [34]. They developed a layerwise stress assumption that interpolated the stresses within a given ply based upon the values of the stresses at the interfaces of that ply. This avoided the assumption of the stress shapes along the length of the laminate and led to an eigenfunction solution with parameters set via the minimization of complementary energy. The method would reduce to the free-edge problem in a special case. This approach was later extended by Shim and Lagace to investigate laminates with internal and external ply dropoffs [35].

The Force-Balance Method and the works derived from the method have, to date, most directly dealt with the issues of model accuracy, efficiency, and effectiveness in the calculation of interlaminar stresses. The relative simplicity of the approaches coupled with the accurate results and efficient runtimes has not yet been matched in the literature. Although many other methods propose and model more complex mechanics using higher-order laminate and plate theories and obtain greater accuracy in the results, they have done so at a higher computational cost. Although issues of computational cost are less significant due to advances in computational technology, there still exists a need to develop and use efficient models with high solution accuracy and applicability over a wide variety of problems while keeping computational runtime as short as possible. This is particularly the case in preliminary design of structures, where a larger number of structural configurations need to be analyzed and judged. This is in contrast to more detailed and less efficient methods that are more applicable to detailed and final design.
As evidenced in the literature, a rather large number of methods for the calculation of interlaminar stresses exist. What is important to note is that out of all of the methods provided, no "perfectly analytical solution" exists. That is, no solution exists which gives a closed-form solution for the fields of interlaminar stresses while avoiding any prior assumption on either the fields of stress or displacement. Such a fact marks the overall difficulty in calculating interlaminar stresses, and thus the full three-dimensional stress field, in composite laminates.

Between the two classes of methods, the semi-analytical methods provide solutions with less computational time than those of the numerical methods. This makes them suitable for instances where multiple laminate configurations need to be analyzed efficiently, such as when investigating the influence of geometric and material parameters on the stresses in a set of laminate configurations, or for evaluation of laminate configuration in preliminary design. Another advantage of semi-analytical methods is that assumptions made for the stresses and/or displacements can be assessed in the solutions to the model more easily than numerical models. In contrast, numerical models typically offer a more robust range of laminate configurations that can be analyzed, whereas the semi-analytical methods generally can provide analysis for a certain subset of specialized problems. In general, by the definitions provided earlier, semi-analytical methods are more efficient and effective than their numerical counterparts. However, there is a tradeoff between the level of accuracy and detail present in a solution and the time needed to compute that solution for all models investigated. In working the solutions from semi-analytical methods, it is important to assess model accuracy via validation with either other demonstrated solutions in literature or results from previously-validated models.

No method presented in the literature exactly satisfies all requisite boundary conditions. A method that assumes displacement shapes cannot satisfy all traction-based and stress-based boundary conditions, whereas methods that assume stress shapes cannot satisfy all strain-based or displacement-based relationships. Despite this limitation, variational methods appear to be better at satisfying boundary conditions than non-variational methods. A comparison can be made by looking at the works
of Tang and Tang and Levy [14, 15] compared to those made by Kassapoglu and Lagace [23, 24]. In the former works, displacement shapes were assumed in a non-variational representation of the equations of elasticity. The resultant solutions in these works were deficient in that the stress solutions did not satisfy either continuity or the free-edge boundary conditions. In contrast, the latter works, where shapes of the stresses were assumed, were able to at least satisfy the displacement and strain boundary conditions in a weak (integral) sense over the entire laminate.

Based upon these observations, application of the Force-Balance Method, or a derivative of such, seems most appropriate in the current work for a number of reasons. One is that it is a stress-based method, and this allows more intuitive control over the manner in which the assumptions in the problem influence the interlaminar stress field. The second is that the method can exactly satisfy all stress-based boundary conditions while at least satisfying the strain constraints in a weak (integral) sense. A third is that the method is relatively efficient in obtaining accurate results in reduced time for a number of laminate configurations.

2.2 Contact Modeling

Analytical treatment of contact loading can be traced back via the work of Hertz on the contact between two general isotropic solids [36]. The work of Hertz was later extended by Timoshenko to beams by adding a combination of the equations Hertz derived to the normal modes of a beam [37]. Another extension of contact mechanics came about in the solution given by Willis for the contact between anisotropic bodies via Fourier Transforms [38]. Although the equations for relations between contact pressure and displacement varied from those of Hertz, the work done by Willis [38] indicated that the contact force of an anisotropic sphere contacting with an anisotropic surface was proportional to the indentation displacement raised to the power of 1.5. This value of 1.5 was also derived by Hertz in his work on isotropic bodies.

Although the equations of Hertz remain as a basis for a number of analyses involving contact, they are less applicable to the problem of the contact of laminated plates
due to several issues. One issue pertains to the overall thickness of laminated plates. In general, the stress fields that have been developed from Hertzian contact have been applied to problems involving flat surfaces that are semi-infinite in the thickness direction. This poses a problem in that laminated plates are relatively thin in that direction, and the presence of back surfaces so close to the loading can invalidate the assumption of a semi-infinite surface in the thickness direction. This can, in turn, lead to erroneous results. A second issue is the susceptibility of the laminate to plastic deformation through the thickness under either contact or impact loadings. Such a plastic deformation alters the elastic response of the laminate both during loading after the yield stress as well as in the unloading of the laminate. Such assumptions are not built into the Hertzian contact problem. Thus, the overall applicability of a Hertzian contact model to laminated composite plates is limited. Despite these limitations, Hertzian contact remains as a starting point and/or point of comparison for all analyses.

Among the earliest of the Hertzian-derived contact models for composite laminates is that proposed by Sun [39]. A finite element model was developed based on plate-bending elements with assumed displacements as well as a modified Hertzian contact expression. The primary focus of this work was on the means by which energy was transmitted through a relatively rigid impactor to a composite laminate, and on the amount of energy that was converted to either vibrational energy of the laminate or energy required for plastic deformation in the laminate. This work was extended via extensive testing to verify the power law of 1.5 as proposed and used, along with establishing a model to take into account contact during unloading and reloading of the laminate [40]. Further work using finite elements introduced a modified Hertzian contact model to better account for the effects of plasticity in the laminate during unloading [41]. These pivotal works demonstrated that, via some modification, the Hertzian contact laws were applicable to composite laminates undergoing small and primarily elastic deformations.

Other techniques, such as a semi-analytical approach to the Hertzian contact law on composite laminates, have been proposed, e.g. [42]. The displacement was assumed
to be represented by a discrete set of nodal displacements, and the formulation follows
that of a Rayleigh-Ritz method, where minimization of potential and kinetic energy
is used to find the forms of the unknown functions. The resultant equations are
integrated over time to obtain solutions for the contact force and plate displacement.

An important limitation of Hertzian theory applied to composite laminates comes
from the contact area and pressure distribution of a laminate in contact with a rigid
sphere. Typically, the use of Hertzian contact results in the prediction of a spherical
distribution of the stress over the contact area for small contact forces, with a maxi-
mum stress at the center of the contact area. However, for large contact forces that
induce large deformations over some critical value, the stress distribution is found to
be saddle-shaped, where the bulk of the pressure is concentrated at the ends of the
contact area, rather than the center. This was first noted through experimental work
[40], and numerous models have been proposed to account for this effect [e.g. 45-48].

Several models also attempt to estimate the response of composite laminates un-
der contact and impact without making assumptions from the Hertz model. These
methods utilize various approaches. One is the use of a new indentation law based
upon deflections predicted by Mindlin Plate Theory and developing an incremental so-
lution via Fourier Transformation [43, 44]. Another is the development a hybrid finite
element formulation interpolated through shape functions governed by 55 parameters
per ply as well as nodal displacements [45]. A further approach is the utilization
of interpolation functions and modal superposition under a higher-order layerwise
theory to obtain a three-dimensional representation of the stress field under various
contact loadings [46]. Additionally, there is the application of three-dimensional elas-
ticity theory to develop a contact law that is independent of stacking sequence in the
laminate [47].

Despite the differences in formulation and assumptions made from the expres-
sions derived by Hertz, similar results were obtained from model to model. The chief
conclusion is that although the specifics derived by Hertz may not be directly ap-
licable to composite laminates because of the assumptions required to obtain such,
the Hertzian results remain largely applicable to composite laminates by using proper
modification and experimentally verifying such. Such modification typically results in changing the general Hertzian Contact Law by obtaining values for coefficients for the power law via experimental measurement. This does not imply that the non-Hertzian models are in any way deficient, as the results from the models presented in this class either validated well with previous work or verified well with experimental results. In general, with proper application, there appears to be no distinct advantage to using a Hertzian or non-Hertzian model to estimate the response of a composite laminate under contact loading, except for the case of laminates under large contact loading and large deflection, where plastic behavior contributes significantly.

As with interlaminar stresses, the most significant differences in the results from models appear to occur between models that are semi-analytical or numerical in nature. The semi-analytical models again show an advantage in efficiency and effectiveness in obtaining relatively accurate results in less time, whereas the numerical models, though longer in runtime, obtain results that are more accurate.

An additional point should be made regarding the development of analytical solutions for contact in general. Although the body of work accomplished for composite laminates does not seem to explicitly suggest such, there does exist a number of classical problems of isotropic contact where an analytical solution to the problem is a linear superposition of many other problems, and such an extension can be made to contact cases involving composite laminates. For example, the solution proposed by Keer and Miller for a circular plate with a rigid indenter employs a superposition of an infinite-layer elastic solution with a bending solution from plate theory [48]. It would be good in the development of a robust model for the calculation of stresses in a laminate with a loaded groove to attempt to solve the general problem of a point load of arbitrary magnitude, direction, and location. Such a model, assuming linear elastic relations, would be able to make use of the principle of superposition to solve for any arbitrary loading in the groove as long as the general solution for the point load in the groove can be obtained. Thus, based on these concepts, linear elasticity and point loads within the groove are assumed for this work to utilize the principle of superposition for all possible loading distributions within the groove.
Chapter 3

Objectives and Overall Approach

The primary objective of this work is to develop an approach that will be used to create and validate a model to estimate the stress field, with particular emphasis on the variation of the interlaminar stresses, of a grooved composite laminate subject to transverse loading within the groove. As mentioned in the previous chapter, no approach yet exists to predict the stresses in such a structural configuration. However, a number of models exist that have been able to predict the interlaminar stresses, with varying degrees of accuracy and efficiency, in composite laminates with certain types of material discontinuities and stress-free edges of various shapes. The purpose of this chapter is to outline an initial approach of reducing the complexity of the general configuration of a transversely-loaded grooved laminate to a simpler problem that can be analyzed via the methods described in the previous chapter. The approach is then developed in a stepwise manner, where subsequent steps indicate an increase in problem complexity and an increase in the overall complexity of the model required to solve that problem.

3.1 Problem Statement and Objectives

The general problem of the transversely-loaded grooved laminate is taken from the helically-grooved composite wing spar, as shown in Figure 3.1. The fundamental shape of the spar is that of a layered composite tube composed of plies of varying
material properties. A helical groove is machined into the outer surface of the composite tube such that the ball bearings are allowed to run in the groove to facilitate the telescoping motion of the spar. The ball bearings will apply a contact load within the surface of the groove itself in order to pass the load of the spar from section to section. Such loading will give rise to stress fields within the tube.

As described in the previous chapter, previous works involving composite laminates have shown that interlaminar stresses arise from gradients in the in-plane stresses in order to satisfy equilibrium, e.g. [3]. In general, these gradients arise due to two effects. The first of these is the applied loading on the laminate, where the loading itself induces a gradient in the in-plane stress field. The second of these is the presence of free edges. This requires the stresses acting on the free edge to be equal to zero in order to satisfy pointwise equilibrium conditions. This transition from a nonzero in-plane stress within the laminate to a zero stress at the free edge of the laminate induces a gradient in the in-plane stresses. Such a gradient gives rise to a gradient in the out-of-plane interlaminar stresses. Previous work has shown that interlaminar stresses arising from applied loading are influenced by parameters that are different than those arising due to the issues involved with the presence of free surfaces [33]. In the current structural configuration being considered, both these effects are present. The transverse contact loading can induce a stress field that will give rise to in-plane stress gradients. Furthermore, the presence of free surfaces, represented here by the groove, can also give rise to gradients in the in-plane stress field.

An important goal of any model in predicting the interlaminar stresses, such as for the transversely-loaded grooved laminates herein, is to identify the mechanisms by which the variation and magnitude of the stresses are affected. For the particular configuration considered here, there are three key sources of these effects. The first is the laminate response to a transverse loading in absence of a groove. The second is the laminate response to transverse loading in the presence of the groove. The third is the laminate response to the presence of the details of the contact load transmitted from the ball bearing to the laminate [49]. Thus, the approach and the formulation must be able to provide for a means by which each effect can be observed and compared.
Figure 3.1 Illustration of a section of the general problem of the grooved composite spar.
without interference from other effects.

The general configuration of the grooved composite tube presents a problem that is complex to solve. The spar itself is a fully three-dimensional structure where all six independent stresses vary with the three directions. Furthermore, the helical nature of the groove in the tube gives the appearance of multiple grooves running along the length of the tube. This can lead to difficulties in separating how a single groove affects the interlaminar stress field, and thus leads to complications in the formulation in estimating the interlaminar stresses.

A stepwise methodology for developing a model to calculate the interlaminar stresses helps to facilitate both observing individual effects in the mechanisms giving rise to interlaminar stresses as well as simplifying the general problem in Figure 3.1 into a more tractable problem. The stepwise methodology, in looking from the final step to the initial step, provides a means to reduce the complexity of the proposed problem to a set of problems solvable within the context of a consistent formulation. In progressing through the steps in the methodology, additional complexity in the model is developed to handle the increase in problem complexity at each step until the final step is reached. In addition, it is possible to isolate the effects that contribute to interlaminar stresses in the laminate within the context of the stepwise methodology. If each effect is added separately as an individual step, then results from step to step can be compared in order to obtain an understanding of how one effect influences the distribution of interlaminar stresses within the laminate.

3.2 Configuration Reduction and Proposed Problem

Even with the stepwise methodology being considered, simplifications need to be made in order to reduce the complexity of the general problem illustrated in Figure 3.1 into a more tractable problem that provides for a closed-form semi-analytical solution. Several simplifications are made from the general configuration in Figure 3.1 in order to create a problem that gives rise to a tractable solution for the interlaminar stress field. The first is to transform the tube structure into a “quasi-two-dimensional”
(quasi-2D) composite laminate. In this plate, the widthwise ($x_2$-) direction of the laminate is assumed to be infinitely long and much larger than the length ($x_1$-) or thickness ($x_3$-) dimensions of the laminate. The purpose of this simplification is to reduce the variability of the overall stress field from three dimensions to two, as an infinitely-wide laminate will have no variation of the stress field in the $x_2$-direction. This change, however, does not constitute a plane stress assumption. The difference between the quasi-2D model and a plane stress assumption is that, while both models reduce the variation of stresses from three directions to two directions, the quasi-2D simplification allows for all six stresses to have non-zero values. As a result, the quasi-2D simplification allows a reduction of the variability of the problem while still allowing for non-zero interlaminar stress fields. Thus, only the derivative with respect to the $x_2$-direction is assumed to be zero.

The second simplification is that the helical groove in Figure 3.1 is represented as a single, perfectly semicircular groove running along the $x_2$-direction. This simplification isolates a single groove, and better allows the model to capture the effects of the groove and the loading within the groove without concerns that arise from multiple grooves interfering with each other. In addition, the problem is defined such that the laminate extends at some distance away from the groove along the length. In this way, the model can isolate effects due to the presence of the groove and loading, as those effects will eventually decay at some distance away from the groove and loading. This can be determined using the model, and then used in working applications back to the actual structural configuration.

Laminate symmetry with respect to the $x_1$- and $x_3$- directions reduces the overall complexity of the problem formulation. However, it is desired that the model derived from the proposed approach be applicable to nonsymmetric as well as midplane-symmetric laminates. Thus, the final proposed problem of the stepwise methodology will be nonsymmetric with respect to the midplane, although the complexity introduced with nonsymmetry will be introduced only in the final step. All other steps in the methodology will assume a geometric and material symmetry.

The curved surface of the groove poses a difficult problem during formulation.
This is due to the difficulty in defining a coordinate system that accurately captures the circular surface of the groove while allowing for a Cartesian coordinate system to be defined for the problem. A Cartesian coordinate system is desired as the stress boundary conditions and constraints within the laminate are simpler to describe in Cartesian coordinates everywhere but at the groove. As a result, the groove surface is assumed to be a piecewise, linearly-slanted surface for the initial approach. However, as described in later chapters, there is an inherent incompatibility in being able to represent the groove in any fashion with slanted surfaces and developing a consistent set of assumptions on the elastic behavior throughout the entire laminate. Resolution of this issue is discussed in the modified solution approach in Chapter 9.

One final reduction in the problem is that the distributed load within the groove transforms into a point load with an offset from the $x_3$-direction. While this loading does not capture the nature of the distributed load brought about by contact between the ball bearings and the laminate, the case of the contact load can be solved via the method of superposition. If the model can give a solution for a point load located within the groove, a solution to any arbitrary loading within the groove, including distributed loadings that arise from contact, can be obtained assuming the laminate follows a linearly elastic constitutive law.

These reductions in problem complexity define the proposed problem that the model developed from the stepwise methodology is to solve. The proposed problem is depicted in Figure 3.2.
Figure 3.2  Illustration of the proposed problem. (Note: This also serves as the final step (Step 5) in the initial proposed methodology.)
3.3 Reduction of Problem Complexity: Initial Step-wise Methodology

Even with the reductions in the complexity, as described in the previous section, used to define the reduced problem in Figure 3.2, it is not immediately obvious how the models and formulations described in the previous chapter can be directly applied to the proposed problem. None of the described models have analyzed such a structural configuration. The purpose of this section is to outline the initial stepwise methodology of further reducing the complexity of the proposed problem to a set of configurations of which the most simple can be addressed using existing models in the literature. Each successive step features a single physical change in the configuration of the laminate, and thus requires changes in model details from other steps. In total, five steps are proposed to reduce the complexity of the laminate configuration in the overall proposed problem to the laminate configuration as addressed via the method of References [35] and [50]. The problem configuration in that work was that of a symmetric laminate with one or more plies dropped subjected to tensile loading, bending, or a combination thereof.

The first step of the proposed methodology is the configuration of an infinite-length laminate under tension with multiple plies dropped (terminated at a free surface) at a single point along the length. Plies that are not dropped run the entire length of the laminate without any change in their geometric or material properties. Although the previous work reported in Chapter 2 considered laminates with outer and interior dropoffs, the focus of this work is only the case of a laminate with its outermost plies dropped. This will work towards the overall problem of the laminate with a groove. It has been shown that stress concentrations arise in continuous plies closest to the dropped plies, as stress is transferred from the dropped plies to continuous plies through a shear lag mechanism. This problem, defined as Step 1, was previously analyzed by Shim and Lagace using a stress-based Complementary Energy method [35, 50].

The second step in the overall direction of the complexity is to limit the infinite length of the dropped region in the laminate to one where the dropped region is of a
finite length. This step also introduces geometrical symmetry in the \(x_1\)-direction into the formulation. The laminate remains under tensile loading in this configuration. For dropped regions of sufficient length, the stress field in this problem should be the same as in Step 1. However, for dropped regions of shorter lengths, the stress concentration is expected to be greater, as the load needs to be transferred to the continuous plies at the dropoff over a shorter distance than in the problem of Step 1.

Step 3 involves a laminate with multiple locations where plies are dropped, as opposed to the one location in Steps 1 and 2. Stress concentration relationships in this laminate are expected to be more complex due to the presence of multiple dropoffs and potentially small dropped regions. However, the focus on this work is on the development of the model to the posed problem, and as such, this unique problem only serves as an intermediate step to transition between laminates with straight dropoffs and the grooved laminate in the overall proposed problem.

In Step 4, the multiple-dropoff laminate of Step 3 is taken and the dropoffs are slanted at an angle to discretize the free surfaces of the dropped plies to the shape of the semicircular groove desired in the proposed problem. The laminate is still loaded in tension. As mentioned in the previous section, the introduction of slanted dropoffs results in issues in developing a model based upon the Step 1 problem in References [35] and [50]. These are addressed in the consideration of Step 4 in Chapter 8.

In the final step, Step 5, the loading in Step 4 is altered from far-field tension to a transverse point load normal to the surface of the groove. In addition, the model is no longer considered symmetric about the laminate midplane. The changes required of the model in going from Step 4 to Step 5 are modifications of the boundary conditions as opposed to changes in the overall formulation of the model, as well as the introduction of higher-order terms in the formulation to account for the nonsymmetry in the laminate. Although accurate solutions for Step 4 were not obtained, the feasibility of the formulation in transitioning from the Step 4 model to the Step 5 model will be established in Chapter 9.
Figure 3.3 Illustrations of the posed problems for Steps 1 through 4 of the initial methodology.
A visual outline of the steps in the initial approach from Step 1 to Step 4 is presented in Figure 3.3. The Step 5 problem is the overall proposed problem as illustrated in Figure 3.2.

3.4 Overview of Complementary Energy Solution Methodology

The problem proposed in Step 1 has previously been analyzed via a stress-based method utilizing the principle of Minimum Complementary Energy. In general, the methodology entails assuming a set of statically-admissible stress shapes in terms of functions of unknown coefficients, and minimizing the total laminate Complementary Energy in order to obtain values for the unknown coefficients. Such a methodology has been used with good results for a number of problems beyond those of laminates with ply dropoffs, such as the free-edge problem [23, 24], laminates with circular cut-outs [33], laminates under bending [29], and laminates with material discontinuities [34], all of which made assumptions on the stresses on a ply-to-ply basis. Ideally, the initial solution methodology could provide a means that allows for analysis of all aspects of the proposed problem. However, issues in formulation prevent this. In particular, the formulation requires an assumption of statically admissible stress shapes, which is not possible in the context of the methodology presented in this section for Step 4 of the initial stepwise methodology. Modifications to the problems defined in the stepwise methodology are developed in Chapter 9. These modifications are developed such that the solution methodology proposed in this section and the formulation in Chapter 4 is applicable to all Steps in the modified methodology without issues in formulation. Both the initial and the modified stepwise methodologies focus on the development of models based around semi-analytical formulations. Although the problems defined in the initial and modified stepwise methodologies are different, the formulation required to develop models to solve the problems in the methodologies follow the same approach and share the same assumptions.

There are two distinct advantages in the use of such a method for the proposed problem. The first is that the use of a stress-based method in the formulation requires
some assumptions to be placed upon the stress shapes in order to develop a closed-
form solution to the proposed problems. This is in contrast to a displacement-based
solution, which instead imposes constraints on the shape of the displacements. Since
the ultimate goal of the model is to estimate the interlaminar stress fields in the pro-
posed problem, it is better that an assumed-stress model is used because assumptions
made on the stresses can more easily be assessed when validating the solution against
other models. A second advantage to the use of the Complementary Energy Method
is that it can be implemented as a semi-analytical method. This provides greater
efficiency than models based upon a discretized method (finite difference and finite
element), as discussed in the previous chapter.

The solution from such a proposed method will be more accurate if fewer as-
sumptions are initially placed upon the stresses [33, 34]. As a consequence, it would
improve the proposed model if the number of assumptions required to define the
stresses were limited. It can be shown that by assuming only the in-plane stresses
$\sigma_{11}$ and $\sigma_{12}$, all remaining stresses, including the interlaminar stresses, can be defined
through the application of the differential equations of equilibrium, as well as enforc-
ing the continuity of the in-plane strain $\varepsilon_{22}$ through the thickness of the laminate.
This is demonstrated in the next chapter. Here, it suffices to say that by assuming
only two stress shapes at the start of the analysis, there will be a greater degree of
accuracy in the overall solution than other methods that require more assumptions
on the stress or displacement.

One additional advantage that the proposed models derived from the solution
approaches is the ability to further discretize the solution via sublayering plies, as
shown in Figure 3.4. Sublayering a ply entails breaking down a ply into a number of
subplies of less thickness and with the same material properties as the parent ply. Any
number of subplies can be defined from the parent ply, and each one of those subplies
can be assigned any thicknesses as long as the sum of the thicknesses of the subplies
is equal to the thickness of the parent ply. The advantage of sublayering a ply into
subplies is that this provides a greater degree of variability in the overall stress field.
If a stress distribution is assumed through the thickness of a ply or subply, having a
larger number of layers to analyze allows the solution to represent a greater range of stress distributions through the thickness of the laminate. In addition, increasing the number of layers analyzed also increases the variability of the stresses in the thickness direction. Furthermore, discretizing a ply into sublayers allows different sublayers to terminate at different locations in $x_1$. Allowing the sublayers of a ply to terminate in differing locations allows for a better representation of the curved surface of the groove within a ply. These advantages are applicable for all models derived from the initial and revised methodologies, and the reasons for this are explained in the next chapter. This increase in variability should also increase the accuracy of the solution, and, for a model capable of analyzing the proposed problem, provide for a numerical assessment of the model via convergence studies using increasing numbers of subplies for a given problem.

Despite the advantages of the proposed methodology, no model derived from either the initial or revised methodologies will give an exact solution to the problem under consideration in each step. As mentioned in the previous chapter, stress-based models must relax some boundary conditions or constraints pertaining to displacement or strain in order to give a closed-form solution to the equations of elasticity. The proposed formulation cannot satisfy all in-plane strain continuity expressions through the thickness. Although the formulation defines $\sigma_{22}$ through the other stresses and application of the through-thickness continuity of $\epsilon_{22}$, the other in-plane strains, $\epsilon_{11}$ and $\epsilon_{12}$, cannot be defined continuously through the thickness, as doing so would overconstrain the system and result in a formulation that is not closed form. Thus, even for the models that were developed and validated in the initial solution methodology, results from the models will not give an exact analytical solution. However, the results from each step must be validated using the results from other models in order to develop the model from Step to Step. When analyzing the results from Step 4 of the initial stepwise methodology, the accuracy of the results are poor to the extent that further model development using the initial stepwise methodology as a guide is not considered.

The stepwise methodology allows for a robust, relatively accurate, and efficient
Figure 3.4 Representation of sublayering scheme within a laminate.
model to be developed in light of the fact that an exact solution cannot be obtained from this model. Model assessment is to be done through comparison and validation of the results against the results of finite element models for the problems outlined in Steps 2 and 3. As issues arose in the formulation of Step 4, model assessments of the Step 4 and Step 5 models of the initial methodology cannot be provided. These issues and their resolution are detailed in Chapters 8 and 9.

3.5 Overview of Validation

As the Step 1 problem has been previously analyzed using the same model as the proposed model, validation of Step 1 is with comparison to those results. It is expected that although the models and problems are exactly the same in both cases, the proposed model may have slightly different results than the previous model, as the two models are implemented in different computing environments and may have different built-in subroutines to numerically solve the equations derived for each model. However, the two sets of results are expected to compare very well, if not exactly. This step is thus important in assessing the numerical capability of the computing environment in which the current model is implemented.

Validation of models developed for Steps 2 through 3 are to be established by comparing the results from the models to results derived from ABAQUS®, a commercially-available finite element package, as no models exist in the literature that analyze the laminate configurations of Steps 2 through 3. ABAQUS is a displacement-based finite element package, which provides for a strong contrast to the proposed stress-based model. Results from both models are to be normalized and compared to each other to establish the validity of the proposed model. In addition, convergence studies using both models are conducted to assess how quickly the proposed model can converge on a solution, and this serves to validate the sublayering scheme proposed in the previous subsection. Validation for the models of Steps 4 and 5 is not established in the current work due to issues in results. This is addressed in Chapters 8 and 9.

Each ply of all laminate configurations is assumed to be transversely isotropic, with
the plane of isotropy being perpendicular to the $z_1$-direction for a $0^\circ$ ply. The material properties for each ply is assumed to be the same in the $0^\circ$ direction. The material used represents graphite fibers in an epoxy matrix, and the values for the material properties are listed in Table 3.1. These material values were chosen as previous work done in [35] and [50] used these values, and thus, for accurate comparison, the same material properties are assumed for validation of the Step 1 model.
Table 3.1 Material properties of composite used in current work

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_L$</td>
<td>130. GPa</td>
</tr>
<tr>
<td>$E_T$</td>
<td>9.00 GPa</td>
</tr>
<tr>
<td>$E_Z$</td>
<td>9.00 GPa</td>
</tr>
<tr>
<td>$\nu_{LT}$</td>
<td>0.280</td>
</tr>
<tr>
<td>$\nu_{LZ}$</td>
<td>0.280</td>
</tr>
<tr>
<td>$\nu_{TZ}$</td>
<td>0.280</td>
</tr>
<tr>
<td>$G_{LT}$</td>
<td>4.80 GPa</td>
</tr>
<tr>
<td>$G_{LZ}$</td>
<td>4.80 GPa</td>
</tr>
<tr>
<td>$G_{TZ}$</td>
<td>3.51 GPa</td>
</tr>
<tr>
<td>$t_{ply}$</td>
<td>0.125 mm</td>
</tr>
</tbody>
</table>

*Note: Material property derived via other material properties and isotropic plane relations.*
Chapter 4

General Solution Procedure

In this chapter, a description is provided for the general formulation of the three models used to determine the stress fields in the problems defined in Steps 1 to 3 in the previous chapter. This formulation is applied to Steps 4 and 5 as well. However, issues in the specific formulation of Step 4 arise that prevent results from being obtained for Steps 4 and 5. This is discussed in detail in Chapters 8 and 9. Although each of the defined problems is different, thereby requiring differences in each model for their analysis, both the physical and mathematical formulations in each problem remain the same. The general base equations derived from the same physical concepts embodied in each problem are described in this chapter. The specifics of these equations as they apply to each problem are described in subsequent chapters. In addition, an overview is given with regard to the implementation of the models and the numerical concerns that arise from particular formulation. Finally, a brief summary is given of the manner in which the finite element validation models were constructed and analyzed.

4.1 Overall Assumptions

There are three assumptions of the proposed problem that hold true for all the problems of Steps 1 through 5. These assumptions lay the foundation of the formulation that follows, and simplifies the formulation to develop towards a closed-form
solution for Steps 1 through 3. The first assumption is that the partial derivative of all elastic fields in the laminate with respect to $x_2$, as defined in Figure 3.2, is equal to zero everywhere. One implication of this assumption is that the laminates under consideration are infinitely long in the $x_2$-direction. In addition, the influences that free surfaces normal to the $x_2$-direction have on stress field in the posed problems are neglected, since no such configuration could exist where the free surfaces could influence the stress fields and still meet the assumption that the partial derivatives of the stress in the $x_2$-direction are zero everywhere. Although any realistic laminate would have finite widths and have some influence from the free surfaces at ends of the laminate, this is outside the scope of the work presented here. The presented models are created with the intent of obtaining a determination of the key factors in the rise of interlaminar stresses for a large variety of laminate configurations. This is to be useful in preliminary design, as discussed previously. Although width effects would be important to consider in a detailed design, neglecting these effects here allows for the development of a more efficient model for preliminary design considerations.

The second assumption is that St. Venant’s Principle is imposed on the posed problems. The manifestation of this assumption is that the stress fields are defined for an infinite distance away from ply drops or grooves, and a far-field stress develops independently of either the details of the loading, or the presence of a dropoff or groove in the laminate. This allows, via the method of superposition, for a definition of the stresses where the overall stress field can be defined as a sum of the far-field loadings and another solution which, added to the far-field solution, allows for satisfaction of boundary conditions in the regions near the ply dropoffs or groove that the far-field stresses would not be able to satisfy on their own.

The third assumption is that each ply can be modeled on a macroscopic basis as a homogeneous specially-orthotropic material. Thus, the details of the fibers and the matrix are only captured via their effect on macroscopic material properties. Virtually all analyses of the stresses on the laminate level of structures of composite materials make use of this assumption.
4.2 Nomenclature and Geometric Definitions

The nomenclature used in developing the formulation for Steps 1 through 5 is subsequently presented. First, multiple "regions" that divide the laminate can be defined. This simplifies the formulation. In Steps 1 through 3, the laminate can be divided into two or more regions, where each region is modeled as a sublaminate with uniform thickness. Although each region defined in Steps 4 and 5 may not have uniform thickness throughout the region, an analogous formulation can be defined for those problems, and is detailed in Chapters 8 and 9. However, this formulation leads to issues in the definition of the stresses for Steps 4 and 5. For the remainder of this work, the region with the largest number of plies is referred to as "Region A," the region with the second-largest number of plies is referred to as "Region B," and this nomenclature continues until all regions under analysis are defined. Although Step 3 has more than two regions (labeled Region C, Region D, and etc.), Steps 1 and 2 have only two regions.

The coordinate in the direction of length, \( x_1 \), is defined as a global coordinate independent of region or ply number. An additional global coordinate, \( x_3 \), is defined through the thickness. The origin of \( x_1 \) and \( x_3 \) is along the dropoff bordered by Region A and B at the laminate midplane. This definition of the origin location allows for the simplification of the forms of the assumed stress shapes in Regions A and B. A representation of the region definitions and coordinate origin and directions is shown in Figure 4.1 for the case of Step 1.

The following nomenclature is used to label the plies and interfaces in each region. The plies in each region in one symmetric half of the laminate from the midplane are labeled in order from 1 to \( n_r \). Ply 1 is farthest from the midplane, and ply \( n_r \) has one of its interfaces at the midplane of the laminate. In addition, the interfaces in the associated symmetric half in a region are labeled interface 0 to interface \( n_r \), where "interface 0" is actually on the outer free surface of the region and "interface \( n_r \)" corresponds to the laminate midplane. Each region is defined as having a total of \((2n_r)^x\) plies, where the superscript \( x \) indicates the region of consideration. For the
Figure 4.1 Illustration of region and coordinate definitions for Step 1.
case of transition from Region A to Region B, the number of continuous plies running through the two regions is equal to $2n_r^B$, and the number of dropped plies from region to region is equal to $[2*(n_r^A-n_r^B)]$. These definitions are displayed in Figure 4.2.

In addition, each ply within a region is defined to have a local thickness coordinate, $x_3^i$, that is used for developing the ply stresses. The origin of this thickness coordinate is on the midplane of a given ply, and the coordinate varies from $+t_{ply}^i/2$ at the inner interface of the ply to $-t_{ply}^i/2$ at the outer interface of the ply. The inner interface of a ply is defined as the interface that lies closer to the laminate midplane, and the outer interface of a ply is defined as the interface that is farthest from the laminate midplane. This local-global coordinate system is displayed in Figure 4.3.

Tensorial notation is used for the formulation. A brief summary of the relevant aspects of the convention are as follows. One is that for tensors, Latin subscripts indicate a total of three dimensions and take values from 1 to 3. Greek subscripts indicate a total of two dimensions and take values from 1 to 2. A comma in expressions indicates a partial derivative with respect to a direction. Here, it is assumed that the $x_1^i$, $x_2^i$, and $x_3^i$-directions correspond to the 1-, 2-, and 3-directions, respectively. Within an expression, a repeated index variable within a single term is assumed to be summed over all possible values for that variable.

4.3 Formulation for Complementary Energy Procedure

The Complementary Energy Method requires an assumption be made on the shapes of the stresses in order to produce a solution. With these assumptions and nomenclature, suitable assumed stress shapes can be developed. The purpose of the stepwise methodology is to allow for the development of a model to determine the interlaminar stress fields that occur due to the presence of a transversely-loaded groove or, in Steps 1 to 4, some other feature of the laminate. Being able to isolate the response of the stress field due to the effects of the groove, dropoff, or loadings can provide better insight to the fundamental mechanisms at work, and how these mechanisms influence the interlaminar stress field. Thus, the stresses present in all cases,
Figure 4.2 Illustration of interface definitions for all posed problems.
Figure 4.3  Illustration of global $x_1$-coordinate and local $x_3^i$-coordinate system definitions and origins for ply $i$ for all posed problems.
\( \sigma_{total} \), are defined as a sum of some far-field solution (outlined in the second assumption mentioned in the previous section), \( \sigma_{far\,field} \), and a complementary solution, \( \sigma_{comp} \), that resolves the far-field solution with the local boundary conditions:

\[
\sigma_{total} = \sigma_{far\,field} + \sigma_{comp}
\]  \hspace{1cm} (4.1)

A visual representation of this methodology is in Figure 4.4.

Upon observation of the problems defined in Steps 1 through 3, the far-field stresses in each problem in each region can be computed rather easily. The problem of a laminate under tension in Steps 1 through 3 has far-field stresses derived from Classical Laminated Plate Theory (CLPT). This can be found in numerous references and is specifically given here. In the application of the Complementary Energy procedure in these steps, a solution is developed such that its contributions equal zero in the far-field, thereby allowing the far-field solution to be recovered at a sufficient distance from the groove or dropoffs. This solution also allows the stresses to satisfy boundary conditions at free surfaces in the laminate.

The addition of the complementary stress solution to the far-field stress solution to obtain the total stress field serves two purposes. The first is that the complementary solution is made to decay to zero in the far-field such that the far-field solution is recovered in the expression for the total stress. The second is that the addition of the complementary solution to the far-field solution allows the overall stress solution to satisfy all stress-based boundary conditions. Both the CLPT solutions for Steps 1 through 3 are constant within a given ply and are invariant with regard to the coordinates \( x_1, x_2, \) and \( x_3 \). As a result, the far-field solutions alone cannot satisfy the boundary conditions associated with free surfaces in Steps 1 through 3, as they are invariant along the length of a ply. However, the addition of a complementary solution, with a stress that can vary along the dimensions of the laminate, to the far-field solution allows for a total stress that can satisfy all stress-based boundary conditions.

Ply-by-ply stress equilibrium requires that for each ply in the laminate in all
Figure 4.4  Illustration of the superposition method to solve for the laminate stress fields.
regions:

\[ \sigma_{ijj} = 0 \quad (4.2) \]

As previously defined, the far-field stresses are constant within a given ply, and their variation in the \( x_1 \)- and \( x_3 \)- directions are equal to zero. Substitution of (4.1) into (4.2), and recalling that the far-field solutions are invariant with respect to the dimensional coordinates, results in:

\[ \sigma_{ij,j_{\text{comp}}} = 0 \quad (4.3) \]

Equation 4.3 is a consequence of having constant far-field solutions in Steps 1 through 4, as well as the fact that far-field solutions in Step 5 automatically satisfy equilibrium. Further invoking the assumption that the stresses are invariant in the \( x_2 \)-direction in Equation (4.3) yields three equations in five yet-unknown stresses \( \sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{23}, \) and \( \sigma_{33} \):

\[
\frac{d\sigma_{11_{\text{comp}}}}{dx_1} + \frac{d\sigma_{13_{\text{comp}}}}{dx_3} = 0 \quad (4.4a) \\
\frac{d\sigma_{12_{\text{comp}}}}{dx_1} + \frac{d\sigma_{23_{\text{comp}}}}{dx_3} = 0 \quad (4.4b) \\
\frac{d\sigma_{13_{\text{comp}}}}{dx_1} + \frac{d\sigma_{33_{\text{comp}}}}{dx_3} = 0 \quad (4.4c)
\]

Thus, defining the shape of the in-plane stresses \( \sigma_{11} \) and \( \sigma_{12} \) with regard to \( x_1 \) and \( x_3 \) results in expressions for the shape of all the interlaminar stresses in the model for all Steps, as noted in Chapter 3 regarding the Complementary Energy Method overview. To determine an expression for the widthwise stress, \( \sigma_{22} \), the linear constitutive law inherent in all plies in the model is invoked:

\[ \epsilon_{ij} = S_{ijkl}\sigma_{kl} \quad (4.5) \]
The case of substituting the value of 2 for i and j in Equation (4.5) yields an expression for $\sigma_{22}$:

$$
\sigma_{22} = -\left( \frac{\epsilon_{22}}{S_{222}} + \frac{S_{112}}{S_{222}} \sigma_{11} + \frac{S_{233}}{S_{222}} \sigma_{33} + \frac{S_{2212}}{S_{222}} \sigma_{12} \right)
$$

(4.6)

In summary, assumption of the shapes of $\sigma_{11}$ and $\sigma_{12}$ will define the stress shapes for all of the remaining stresses in all Steps. The interlaminar stresses are found through the equations of differential equilibrium in Equations (4.4a-c), and the width-wise stress can be found as a function of the other stresses as indicated by Equation (4.6).

Two further assumptions are made in defining the stresses for all Steps. The in-plane stresses at a point within a given ply are defined as a weighted linear combination of the stresses along the interfaces directly above and below that point in that ply. The weight given to each of the two interfacial stress functions in determining the ply stress is a function of the local ply thickness coordinate. This definition will allow for a closed-form solution when minimizing the laminate complementary energy. An additional assumption is that the forms of all the stress fields are functionally separate in $x_1$ and $x_3$. Each stress field in any ply in the laminate can be written in the form:

$$
\sigma_{comp} = \sum F_i(x_1) \ast H_i(x_3)
$$

where $F_i(x_1)$ is an interfacial stress function on the ith interface, and $H_i(x_3)$ is defined as the weighting function that acts on $F_i(x_1)$. This functionally-separate assumption allows for simplification when calculating the definite integrals of the stresses with respect to laminate volume required in the Complementary Energy Method, as the functional separation of variables in the stresses allow for decoupling of the variable terms when calculating the integrals.

The assumptions, applied to the formulation of all Steps, result in the definition of the variation of $\sigma_{11}$ in the $x_1$-direction along the ith interface via one interfacial stress function $F_i(x_1)$ and that for $\sigma_{12}$ along the ith interface via a second interfacial stress function $G_i(x_1)$. In addition, each ply has a set of weighting functions to interpolate
the interface stress functions to the ply stresses. These weighting functions are a function of the local thickness coordinate, as previously defined. As each ply has two interfaces, two weighting functions for each ply need to be developed, one for each interface. For ply $i$, the interpolation function that affects the stress function along the outer interface (interface $i-1$) is defined as $H_{out}^{i}(x_3^{i})$, and the interpolation function that affects the stress function along the inner interface (interface $i$) is defined as $H_{in}^{i}(x_3^{i})$.

To further simplify the resulting formulation, it is assumed that the value of the stress at a given point at a ply interface will equal the value of the interfacial stress function corresponding to the interface at that point. This assumption holds true only for stresses whose through-thickness variation is nonzero. In the case of a zero through-thickness stress variation, it is impossible to enforce the assumption of ply stresses interpolating from interface stresses within a ply unless the stress variations along the top and bottom interfaces are the same, and enforcing such a constraint is too limiting on the assumed behavior of the stresses in capturing any variation. Based upon the assumption that the stress in ply $i$ taken at points along an interface must equal the value of the interfacial stress function associated with that interface requires that, for the $H_{out}^{i}$ and the $H_{in}^{i}$:

\[
\begin{align*}
H_{out}(-t_i/2) &= H_{in}(+t_i/2) = 1 \\
H_{out}(+t_i/2) &= H_{in}(-t_i/2) = 0
\end{align*}
\] (4.7a)

These assumptions apply to the stress distributions in all Steps with nonzero through-thickness variation. Taking into account the aforementioned assumptions, the general forms for the in-plane complementary stresses in a ply for a particular region are:

\[
\begin{align*}
\sigma_{11_{comp}}^{i}(x_1, x_3^{i}) &= F_i(x_1) H_{in}(x_3^{i}) - F_{i-1}(x_1) H_{out}(x_3^{i}) \\
\sigma_{12_{comp}}^{i}(x_1, x_3^{i}) &= G_i(x_1) H_{in}(x_3^{i}) - G_{i-1}(x_1) H_{out}(x_3^{i})
\end{align*}
\] (4.8a)

These equations are general in nature for any region. However, the interfacial stress
functions, \( F_i(x_1) \) and \( G_i(x_1) \), for the in-ply stresses within each region are treated as independent functional variables from region to region. The superscript “\( i \)” on the local thickness coordinate indicates the local thickness coordinate associated with the \( i \)th ply.

Substituting the general form of the expressions for \( \sigma_{11} \) and \( \sigma_{12} \) in the differential equilibrium Equations (4.4a), (4.4b), and (4.4c) yields the forms of the interlaminar stresses within each region for all Steps:

\[
\begin{align*}
\sigma_{13_{\text{comp}}}^i(x_1, x_3) &= -F_i(x_1) \int_{\text{Region}} H_{\text{in}}(x_3)\,dx_3 + F'_{i-1}(x_1) \int_{\text{Region}} H_{\text{out}}(x_3)\,dx_3 \\
\sigma_{23_{\text{comp}}}^i(x_1, x_3) &= -G_i(x_1) \int_{\text{Region}} H_{\text{in}}(x_3)\,dx_3 + G'_{i-1}(x_1) \int_{\text{Region}} H_{\text{out}}(x_3)\,dx_3 \\
\sigma_{33_{\text{comp}}}^i(x_1, x_3) &= F''_i(x_1) \int_{\text{Region}} \int_{\text{Region}} H_{\text{in}}(x_3)dx_3^2 - F''_{i-1}(x_1) \int_{\text{Region}} \int_{\text{Region}} H_{\text{out}}(x_3)dx_3^2
\end{align*}
\]

(4.8c) \quad (4.8d) \quad (4.8e)

Note that a function with a prime (\('\)) denotes a derivative with respect to \( x_1 \). The constants of integration in each of the forms of the interlaminar stresses are defined on a ply-to-ply basis so as to satisfy through-thickness continuity of the interlaminar stresses from ply to ply.

One further adjustment is made to the definition of the stresses in Equations (4.8a-e). This is done by rewriting the ply stresses into matrix forms. Rewriting the ply stresses in matrix form has the advantage that matrix implementation in computing environments allows for a more efficient computation. Developing such a formulation in matrix form facilitates model implementation for all Steps. A vector (treated as a one-dimensional matrix) is defined for the interface stress functions \( F_i(x_1) \) for all of the stress functions within a given region from the top interface (interface 0) to the midplane (interface \( n_r \)):

\[
\mathbf{F} = \begin{bmatrix} F_0(x_1) & F_1(x_1) & F_2(x_1) & \ldots & F_{n_r}(x_1) \end{bmatrix}
\]

(4.9a)
A similar definition is made for the interfacial stress functions $G_i(x_1)$ into vector $G$. In order to develop a matrix form for the stresses, another vector containing the details of the interpolation functions, $H_{out}^i$ and the $H_{out}^i$, must also be defined for each ply in a region. The definitions of these vectors of the interpolation functions are such that, when the interpolation function vector of ply $i$ is multiplied by the stress function vector, the result equals the ply stresses for ply $i$, as given in Equations (4.8a-e). For the case of $\sigma_{11}$, in all Steps, this vector is defined by:

$$h_{11}^i = \begin{bmatrix} h_1^i & \ldots & h_{i-1}^i(x_3) & h_i^i(x_3) & \ldots & h_n^i(x_3) \end{bmatrix} = \begin{bmatrix} 0 & \ldots & -H_{out}^i(x_3) & H_{in}^i(x_3) & \ldots & 0 \end{bmatrix}$$

(4.9b)

The expression in this equation is defined on a ply-by-ply basis, whereas the definition of equation (4.9a) is general for any ply within a region. Multiplication of these two expressions yields a result that is equivalent to (4.8a):

$$\sigma_{11,comp}^i = h_{11}^i F$$

(4.10a)

Similar definitions for all Steps can be used for the other stresses, $\sigma_{12}, \sigma_{13}, \sigma_{23}, \sigma_{33}$, though each stress will have associated with it its own weighting interpolation vector $h^i$:

$$\sigma_{12,comp}^i = h_{12}^i G$$

(4.10b)

$$\sigma_{13,comp}^i = h_{13}^i F'$$

(4.10c)

$$\sigma_{23,comp}^i = h_{23}^i G'$$

(4.10d)

$$\sigma_{33,comp}^i = h_{33}^i F''$$

(4.10e)

In order to keep these definitions consistent with the definitions of the stresses in Equations (4.8b-e), the interpolation function vectors for $\sigma_{12}, \sigma_{13}, \sigma_{23}, \sigma_{33}$ must equal:
In order to obtain accurate results via the Complementary Energy Method, a statically-admissible set of stress functions must be defined. As mentioned previously, the constants of integration that arise from integrating the $H(x_3^i)$ functions in Equations (4.8a-e) are used to satisfy interlaminar stress continuity through the thickness of the laminate. In addition, laminate symmetry about the midplane in Steps 1 through 4 requires that the stresses follow other constraints. In order to satisfy integral equilibrium in a symmetric laminate, the distribution of $\sigma_{13}$ and $\sigma_{23}$ must be antisymmetric through the thickness, with the laminate midplane corresponding to the plane of antisymmetry. This requires that the interlaminar shear stresses must equal zero at the laminate midplane for Steps 1 through 4. Moreover, the top surface of each region is a free surface, and as such, all interlaminar stresses must equal zero on this top surface. Using the previously-defined stress shapes and definitions in Equations (4.8a-e) results in the following set of constraints:

$$F'_{n_r} = G'_{n_r} = F'_0 = G'_0 = F''_0 = 0 \quad (4.11)$$

In addition, each stress is defined as a sum of the far-field stress and a complementary stress, as defined in Equation (4.1). Far from the groove or dropoff, the complementary stress decays, and ultimately decays to zero at an infinite distance from the specific feature of a laminate, either the dropoff or groove. This imposes another set of constraints such that:

\[
\begin{align*}
  h^{i}_{12} &= \begin{bmatrix} 0 & \ldots & -H^i_{\text{out}}(x_3) & H^i_{\text{in}}(x_3) & \ldots & 0 \end{bmatrix} \quad (4.10f) \\
  h^{i}_{13} &= \int \begin{bmatrix} 0 & \ldots & H^i_{\text{out}}(x_3) & -H^i_{\text{in}}(x_3) & \ldots & 0 \end{bmatrix} dx_3 \quad (4.10g) \\
  h^{i}_{23} &= \int \begin{bmatrix} 0 & \ldots & H^i_{\text{out}}(x_3) & -H^i_{\text{in}}(x_3) & \ldots & 0 \end{bmatrix} dx_3 \quad (4.10h) \\
  h^{i}_{33} &= \int \int \begin{bmatrix} 0 & \ldots & -H^i_{\text{out}}(x_3) & H^i_{\text{in}}(x_3) & \ldots & 0 \end{bmatrix} dx_3^2 \quad (4.10i)
\end{align*}
\]
\[
\lim_{x_1 \to -\infty} F_i = F'_i = F''_i = G_i = G'_i = 0
\] (4.12)

In addition to the differential equations of equilibrium, the general forms of the stresses must satisfy force and moment integral equilibrium conditions for the stresses \(\sigma_{11}, \sigma_{12},\) and \(\sigma_{13}\). As the far-field loading, via the application of CLPT, automatically satisfies integral equilibrium of the applied loadings for Steps 1 through 3, there can be no contribution to the integral equilibrium expressions by the complementary part of the solutions. Thus, the integral equilibrium contributions from the complementary parts of \(\sigma_{11}, \sigma_{12},\) and \(\sigma_{13}\) must equal zero.

The general form of the expressions for the complementary part of \(\sigma_{13}\) in Steps 1 through 4 automatically satisfies force equilibrium due to the fact that \(\sigma_{13}\) must be antisymmetric throughout the length of the laminate and that \(\sigma_{13}\) equals zero in the far-field for all steps. This antisymmetric distribution automatically ensures that, regardless of the variation of \(\sigma_{13}\) through the thickness, so long as the distribution is antisymmetric, the integral will have to equal zero due to the fact that any contribution to the integral in one half of the laminate will be negated by an opposite contribution in the antisymmetric half of the laminate.

The in-plane stresses \(\sigma_{11}\) and \(\sigma_{12}\) must be symmetric about the midplane of the laminate though the thickness of the laminate in order to satisfy the symmetry conditions brought about by midplane symmetry in Steps 1 through 4. Due to this symmetry, these two in-plane stresses will always satisfy moment equilibrium conditions due to the fact that the moment contributions in one half of the laminate will always cancel with another, similar distribution in the symmetric half. Integral equilibrium conditions involving in-plane force balance require that:

\[
\int_{-t_{\text{lam}}/2}^{t_{\text{lam}}/2} \sigma_{1\alpha} dx_3 = 0
\] (4.13)

where the \(\alpha\) equals coefficients 1 and 2 for the case of \(\sigma_{11}\) and \(\sigma_{12}\). Substituting the definitions of the complementary stresses from equations (4.8a) and (4.8b) and converting the expression in equation (4.13) into the local thickness coordinate, equation
(4.13) becomes the following equation:

\begin{equation}
\sum_{j=1}^{n_r} F_j(x_1) \int_{-t_i/2}^{t_i/2} H_{in}(x_3) dx_3 - F_{j-1}(x_1) \int_{-t_i/2}^{t_i/2} H_{out}(x_3) dx_3 = 0 \tag{4.14a}
\end{equation}

\begin{equation}
\sum_{j=1}^{n_r} G_j(x_1) \int_{-t_i/2}^{t_i/2} H_{in}(x_3) dx_3 - G_{j-1}(x_1) \int_{-t_i/2}^{t_i/2} H_{out}(x_3) dx_3 = 0 \tag{4.14b}
\end{equation}

The transition from the global thickness coordinate in equation (4.13) to the local thickness coordinate in equations (4.14a) and (4.14b) requires that the integral through the thickness in equation (4.13) becomes a sum of integrals through all plies. As the stresses are defined on a ply-by-ply basis as a function of the local thickness coordinate in each ply, the integral as a function of the global thickness coordinate needs to be redefined in terms of the local thickness coordinates. To do so, the integral in terms of the global thickness is redefined as a sum of the integrals in each ply, taken as a function of the local thickness coordinate of the ply. The two expressions - the integral though the laminate thickness as a function of the global thickness coordinate and the sum of the integrals through the thickness of each ply as a function of the local thickness coordinate - are equivalent.

It is a key factor in determining the definitions of $H_{out}^i$ and $H_{in}^i$ for each step such that the above expressions are satisfied. The definitions of $H_{out}^i$ and $H_{in}^i$ can differ from step to step and is described for each step in subsequent chapters.

With the general forms of the stresses defined, the next step requires the application of the Principle of Minimum Complementary Energy. The principle states that when the Complementary Energy of a structure is calculated using an assumed stress shape with unknown variables, the values of the variables that best solve for the stress field will be those that minimize the Complementary Energy of the structure. The general definition for the laminate Complementary Energy is:

\begin{equation}
\pi_C^* = \sum_{m=1}^{All Regions} \frac{1}{2} \int_{V_m} \sigma_{ij} S_{ijkl} \sigma_{kl} dV + \int_{S_m} \sigma_{ij} n_j \tilde{u}_i dS \tag{4.15}
\end{equation}
where $V^m$ represents the volumetric regions of integration in Region $m$, $S^m$ represents the surface of Region $m$, and $\tilde{u}_i$ represents prescribed displacements. Minimizing the Complementary Energy requires that the first variation of the Complementary Energy with respect to all of the unknown functional variables be set equal to zero:

$$\delta \pi_c^* = 0 \quad (4.16)$$

Recall that the definition of each stress is the sum of a far-field stress and a complementary stress. Since the far-field stresses are invariant, the variation of the contribution of these stresses in the Complementary Energy must equal zero. Thus, only the complementary stress solution contributes to the process of minimizing the Complementary Energy. In addition, all the posed problems have no prescribed displacements. This, therefore, reduces the number of terms needed to be calculated in the energy. Thus, the Complementary Energy can be written as:

$$\pi_c^* = \sum_{m=1}^{\text{AllRegions}} \frac{1}{2} \int_{V^m} \sigma_{ijkl}^T S_{ijkl}^\text{comp} \sigma_{ijkl}^\text{comp} \, dV \quad (4.17)$$

One final simplification can be made to the above expression for the Complementary Energy. Because of the invariant nature of the far-field solutions, it is assumed here that the total strain $\epsilon_{22}$ is also constant and equal to the value predicted by application of the equations of elasticity to the far-field stresses. As a result, there is zero variation of the stresses with respect to $\epsilon_{22}$, and in the expression for the Complementary Energy in (4.17). Applying this assumption into equation (4.6) results in a new expression for $\sigma_{22}$ to be used when taking the variation of Complementary Energy:

$$\sigma_{22_{\text{total}}} = -\left(\frac{S_{1122}}{S_{2222}} \sigma_{11_{\text{total}}} + \frac{S_{2233}}{S_{2222}} \sigma_{33_{\text{total}}} + \frac{S_{2212}}{S_{2222}} \sigma_{12_{\text{total}}} \right) + C_1 \quad (4.18)$$

Substituting this expression for $\sigma_{22}$ into (4.17) reduces the complementary energy as a function of only five of the six independent stresses:
\[ \pi_C^* = \sum_{m=1}^{All Regions} \frac{1}{2} \int_{V_m} \sigma^*_{ijkl} S^*_{ijkl} \sigma^*_{klm} \, dV \]  
(4.19a)

where:

\[ \sigma^*_{comp} = \begin{bmatrix} \sigma_{11} & \sigma_{33} & \sigma_{13} & \sigma_{12} \end{bmatrix}^T \]  
(4.19b)

and:

\[ S^*_{ijkl} = S_{ijkl} - \frac{S_{22ij} S_{22kl}}{S_{2222}} \]  
(4.19c)

The matrix-based stress forms of Equations (4.10a-e) are substituted into Equation (4.19a) to obtain the expression for Complementary Energy for one region:

\[ \pi^*_c = \frac{1}{2} \int_{V_m} \left[ S^*_{1111} F^T h_{11}^T h_{11} F - S^*_{1313} F^T h_{13}^T h_{13} F' 
+ S^*_{3333} F''^T h_{33}^T h_{33} F'' + S^*_{1212} G^T h_{12}^T h_{12} G 
- S^*_{2223} G''^T h_{23}^T h_{23} G'' 
+ S^*_{1133} (F^T h_{11}^T h_{13} F' + F''^T h_{33}^T h_{11} F) 
+ S^*_{1112} (F^T h_{11}^T h_{12} G + G^T h_{12}^T h_{11} F) 
- S^*_{1323} (F^T h_{13}^T h_{13} G' + G''^T h_{13}^T h_{13} F') 
+ S^*_{1233} (G^T h_{12}^T h_{33} F'' + F'''^T h_{33}^T h_{12} G') \right] \, dV \]  
(4.20)

The expression for the Complementary Energy in Equation (4.20) is only written for one region. All other regions have the same form. As the energy in equation (4.20) is defined in terms of the functional variables \( F_i \) and \( G_i \), which are themselves functions of \( x_1 \), minimizing the energy requires defining the Generalized Euler-Lagrange equations for the expression of Complementary Energy given in (4.20). The Euler-Lagrange equations for the expression in equation (4.20) result in the expressions that are needed to be satisfied in order to minimize the Complementary Energy:
One final simplification is made to equation (4.20). As the laminate is symmetric about the midplane in Steps 1 through 4, only half of the laminate Complementary Energy needs to be evaluated, since the contribution to the Energy by the top half is equal to the contribution of the bottom half. Thus, the formulation can proceed by only considering the top half of the laminate and multiplying the total energy by two. Throughout the remainder of the formulation, only the top half of the laminate will be considered, and multiples of two representing the symmetry between the two halves of the laminate are included where appropriate.

The minimization of Equation (4.20) using the expressions in Equations (4.21a) and (4.21b) yields a system of coupled ordinary linear differential equations:

\[
\begin{align*}
\frac{d^2}{dx^2} \left( \frac{\delta \pi_C^*}{\delta F''} \right) & - \frac{d}{dx} \left( \frac{\delta \pi_C^*}{\delta F'} \right) + \frac{\delta \pi_C^*}{\delta F} = 0 \\
- \frac{d}{dx} \left( \frac{\delta \pi_C^*}{\delta G''} \right) & + \frac{\delta \pi_C^*}{\delta G} = 0
\end{align*}
\] (4.21a) (4.21b)

These systems of equations are applicable to each region. This system is in terms of the unknown stress functional variables \( F_i(x_1) \) and \( G_i(x_1) \) captured in the stress function matrices \( F \) and \( G \) expressed in Equation (4.9a). The terms in the matrices in equation (4.22) are a function of ply material properties, ply geometric properties, and the form of the interpolation functions of the interfacial stress functions in each ply. These terms, in integral form, evaluate to be:

\[
A_{11} = S^*_{333} \int_0^{t_{lam}/2} h_{33}^T h_{33} \, dx_3
\] (4.23a)
\[
B_{11} = -S_{1113}^* \int_0^{t_{lam}/2} h_{13}^T h_{13} d x_3 + S_{1133}^* \int_0^{t_{lam}/2} (h_{11}^T h_{33} + h_{33}^T h_{11}) d x_3
\]

(4.23b)

\[
B_{12} = -S_{1233}^* \int_0^{t_{lam}/2} (h_{12}^T h_{23} + h_{23}^T h_{12}) d x_3
\]

+ \[S_{2333}^* \int_0^{t_{lam}/2} (h_{12}^T h_{33} + h_{33}^T h_{12}) d x_3.
\]

(4.23c)

\[
B_{22} = -S_{2323}^* \int_0^{t_{lam}/2} h_{23}^T h_{23} d x_3
\]

(4.23d)

\[
C_{11} = S_{1111}^* \int_0^{t_{lam}/2} h_{11}^T h_{11} d x_3
\]

(4.23e)

\[
C_{12} = S_{1112}^* \int_0^{t_{lam}/2} (h_{11}^T h_{12} + h_{12}^T h_{11}) d x_3
\]

(4.23f)

\[
C_{22} = S_{1212}^* \int_0^{t_{lam}/2} h_{12}^T h_{12} d x_3
\]

(4.23g)

As minimization of the Complementary Energy results in the system of coupled linear differential equations in (4.22), the solution to this system of equations requires that the interfacial stress vectors be equal to a sum of exponential terms in \(x_1\). A general solution to the system of equations can be defined as:

\[
F(x_1) = \sum_{j=1}^{3*(n_r-1)} c_j \phi_j e^{\lambda_j x_1}
\]

(4.24a)

\[
G(x_1) = \sum_{j=1}^{3*(n_r-1)} c_j \psi_j e^{\lambda_j x_1}
\]

(4.24b)

Substitution of equations (4.24a) and (4.24b) into Equation (4.22) transforms the
differential equations to a generalized eigenvalue problem. The exponential solutions in equations (4.24a) and (4.24b) are given in terms of exponential coefficients $\lambda_j$, vectors $\phi_j$ and $\psi_j$, and unknown coefficients $c_j$. Substitution of these solutions into equation (4.22) and division by the exponential terms transforms the differential equations into an eigenvalue problem, with the $\lambda_j$ acting as eigenvalues, and the $\phi_j$ and $\psi_j$ acting as eigenvectors. This eigenvalue problem can be solved via a number of built-in methods in many computing environments.

Because of the assumption that the functions $F_i(x_1)$ and $G_i(x_1)$ are independent in each region, the above formulation can be applied to each region individually. However, the solution to the unknown coefficients $c_j$ requires an additional step to be solved. Once $c_j$ is solved for, Equations (4.24a) and (4.24b) can be substituted back into Equations (4.10a-e) to obtain the stress field in all plies for a given region. The solution for the $c_j$ is presented in the next section.

4.4 Stress Boundary Conditions and Constraints

There remain two problems left to solve at this point in the overall formulation for Steps 1 through 4. The first is that the minimization of Complementary Energy defined in the previous section generates a new set of variables, $c_j$, that need to be determined. The second problem is that no effort has yet been made to either enforce free-surface boundary conditions along the length of the laminate, nor have any constraints been imposed to ensure that $\sigma_{11}$, $\sigma_{12}$, and $\sigma_{13}$ remain continuous along the length of the laminate across different regions.

To solve the first problem, the laminate Complementary Energy is recalculated using the forms of the $F_i(x_1)$ and $G_i(x_1)$ defined in (4.24a) and (4.24b), and then re-minimized to solve for the numerical values of the $c_j$. The difference between this minimization of Complementary Energy and the previous minimization of Complementary Energy procedure in Section 4.3 is that the interfacial stress functions are now explicitly defined via Equations (4.24a) and (4.24b). In the previous minimization, the interfacial stress functions were assumed to be then-unknown expressions as
a function of $x_1$, and minimizing the Complementary Energy as described in Section 4.3 indicated that stresses functions in the form of exponential sums in $x_1$ would minimize the Complementary Energy of the laminate.

Henceforth in this chapter, the formulation generalizes the case of only two regions existing, and the unknown coefficients, the $c_j$, in equations (4.24a) and (4.24b) become $a_j$ and $b_j$ for Regions A and B, respectively. This generalization is directly applicable to Steps 1 and 2. This assumption is made in order to differentiate between the unknown coefficients that effect the stresses in Region A from the coefficients that effect the stresses in Region B. In addition, there are a total of $(2n_r^A)$ plies in the continuous region, and a total of $(2n_r^B)$ plies in the dropped region. The total number of continuous plies is thus $(2n_r^A)$, and the total number of dropped plies is equal to:

$$2 \ast n_d = 2(n_r^A - n_r^B).$$

(4.25)

Although Steps 3 through 5 require more than two regions, the following derivation is applicable to any location where plies are dropped or at the interface between different regions. The procedure by which solutions are determined for $c_j$ is applicable to Steps 3 through 5, although the resultant equations that come from the procedure are different. This is by extending the existence of two regions to multiple regions for Steps 3 through 5. Furthermore, consideration is provided only for half of the laminate, as the laminate is symmetric about the midplane.

Expressions involving the second problem of the boundary conditions at the free edge and the constraints across regions are presented. The problem of the free edge in dropped plies requires that all the $\sigma_{1a}$ stresses must equal zero at the dropoff. As previously defined, the dropoff occurs in all dropped plies at $x_1 = 0$. As there are a total of $n_d$ dropped plies moving from Region A to Region B in the half-laminate, it can be written, for the laminates in Steps 1 through 3 with non-slanted dropoffs, that:

$$\sigma^A_{11} \mid_{\text{total}} (x_1 = 0, x_3) = 0$$

(4.26a)
\[ \sigma_{12_{\text{total}}}^{A_i}(x_1 = 0, x_3) = 0 \]  \hspace{1cm} (4.26b)
\[ \sigma_{13_{\text{total}}}^{A_i}(x_1 = 0, x_3) = 0 \]  \hspace{1cm} (4.26c)

for plies 1 through \( n_d \). Substituting Equation (4.1) into the above expressions yields:

\[ -\sigma_{11_{\text{comp}}}^{A_i}(x_1 = 0, x_3) = \sigma_{11_{\text{far-field}}}^{A_i} \]  \hspace{1cm} (4.27a)
\[ -\sigma_{12_{\text{comp}}}^{A_i}(x_1 = 0, x_3) = \sigma_{12_{\text{far-field}}}^{A_i} \]  \hspace{1cm} (4.27b)
\[ -\sigma_{13_{\text{comp}}}^{A_i}(x_1 = 0, x_3) = \sigma_{13_{\text{far-field}}}^{A_i} \]  \hspace{1cm} (4.27c)

The superscript \( A \) denotes quantities that depend only on stresses within Region \( A \), and the superscript \( A_i \) denotes a quantity with regard to the \( i \)th ply in Region \( A \).

The free-surface constraints expressed in Equations (4.27a-c) are enforced over the entire free surface of a dropped ply by enforcement of the constraints at certain points through the thickness along the free surface for Steps 1 through 3. This can be done since the stress distributions through the thickness are assumed, and forcing the stress to equal zero at a number of points along the free surface will force the values of the stress to be equal to zero all along the surface. The order of the through-thickness variations of the stresses in a ply, determined via the \( H_{\text{out}}^i \) and \( H_{\text{in}}^i \), determines the number of points along the free surface of a dropped ply of which the conditions need to be enforced such that the stress through the thickness along the dropoff in a dropped ply is zero everywhere. For example, a stress that is constant through the thickness requires only that the particular stress be equal to zero at one point along the free surface to have the values of the stress equal to zero all along the surface. This is due to two reasons. The first is that the forms of the stress are functionally separate. Thus, through the thickness of a ply at a dropoff, the stresses are dependent only on the \( x_3 \)-coordinate. The second is that for a constant distribution dependent on a single parameter, knowledge of that distribution at a single point allows for the determination of that distribution everywhere, since the distribution is invariant. For higher-order distributions, more points are required to uniquely define the distribution. Two points are required for a linear distribution through the thickness, three
points are required for a quadratic distribution through the thickness, and so forth.

The stress continuity constraints in the laminate require that the $\sigma_{1a}$ stresses be continuous moving from Region A to Region B at the border between the regions, $x_1$ equal to 0, in the continuous plies. This requires that the difference in the stresses be equal to zero at $x_1$ equal to 0:

$$\sigma_{11\text{total}}^{A}(x_1 = 0, x_3) - \sigma_{11\text{total}}^{B}(x_1 = 0, x_3) = 0 \quad (4.28a)$$

$$\sigma_{12\text{total}}^{A}(x_1 = 0, x_3) - \sigma_{12\text{total}}^{B}(x_1 = 0, x_3) = 0 \quad (4.28b)$$

$$\sigma_{13\text{total}}^{A}(x_1 = 0, x_3) - \sigma_{13\text{total}}^{B}(x_1 = 0, x_3) = 0 \quad (4.28c)$$

for plies $(n_d + 1)$ through $n_r^A$ in Region A and plies 1 through $n_r^B$ in Region B. Note that ply $(n_d + 1)$ in Region A corresponds to ply 1 in Region B, ply $(n_d + 2)$ in Region A corresponds to ply 2 in Region B, and so forth. Substituting Equation (4.1) into Equations (4.28a), (4.28b), and (4.28c) results in:

$$-\sigma_{11\text{comp}}^{A}(x_1 = 0, x_3) + \sigma_{11\text{comp}}^{B}(x_1 = 0, x_3) = \sigma_{11\text{farfield}}^{A} - \sigma_{11\text{farfield}}^{B} \quad (4.29a)$$

$$-\sigma_{12\text{comp}}^{A}(x_1 = 0, x_3) + \sigma_{12\text{comp}}^{B}(x_1 = 0, x_3) = \sigma_{12\text{farfield}}^{A} - \sigma_{12\text{farfield}}^{B} \quad (4.29b)$$

$$-\sigma_{13\text{comp}}^{A}(x_1 = 0, x_3) + \sigma_{13\text{comp}}^{B}(x_1 = 0, x_3) = \sigma_{13\text{farfield}}^{A} - \sigma_{13\text{farfield}}^{B} \quad (4.29c)$$

The analysis requires that Complementary Energy be minimized in terms of the $a_j$ and $b_j$, as well as defining $a_j$ and $b_j$ in such a way that the free-surface boundary conditions and stress continuity conditions be satisfied. There are two procedures that can accomplish this. The formulation could reduce the number of independent variables that come from the constraint equations by substituting Equations (4.1), (4.10a-c), (4.24a) and (4.24b) into Equations (4.27a-c) and (4.29a-c), solving for a number of $a_j$ and $b_j$ in terms of the other $a_j$ and $b_j$, and then substituting these results into the equations for minimization of energy. However, this is a computationally inefficient process.
Instead, the current formulation redefines the Complementary Energy minimization problem into a Lagrangian Minimization, where the constraints are those required to enforce the required boundary conditions and constraints in Equations (4.27a-c) and (4.29a-c). The Lagrangian Coefficients are first defined as $\rho_{i}^{m}$, as related to each stress, $\sigma_{mn}$. The superscript in the coefficient, $i$, denotes that the Lagrangian coefficient multiplies a constraint equation that effects the $i$th ply in a region. Equations (4.27a-c) and (4.29a-c) are written in terms of these multipliers as:

$$\rho_{11}^{A}(\sigma_{11\text{comp}}^{A}(x_1 = 0, x_3) + \sigma_{11\text{farfield}}^{A}) = 0$$ (4.30a)
$$\rho_{12}^{A}(\sigma_{12\text{comp}}^{A}(x_1 = 0, x_3) + \sigma_{12\text{farfield}}^{A}) = 0$$ (4.30b)
$$\rho_{13}^{A}(\sigma_{13\text{comp}}^{A}(x_1 = 0, x_3) + \sigma_{13\text{farfield}}^{A}) = 0$$ (4.30c)

for plies 1 through $n_d$, and the stress continuity constraints can be written as:

$$\rho_{11}^{AB}(\sigma_{11\text{comp}}^{A}(x_1 = 0, x_3) - \sigma_{11\text{comp}}^{B}(x_1 = 0, x_3) + \sigma_{11\text{farfield}}^{A} - \sigma_{11\text{farfield}}^{B}) = 0$$ (4.30d)
$$\rho_{12}^{AB}(\sigma_{12\text{comp}}^{A}(x_1 = 0, x_3) - \sigma_{12\text{comp}}^{B}(x_1 = 0, x_3) + \sigma_{12\text{farfield}}^{A} - \sigma_{12\text{farfield}}^{B}) = 0$$ (4.30e)
$$\rho_{13}^{AB}(\sigma_{13\text{comp}}^{A}(x_1 = 0, x_3) - \sigma_{13\text{comp}}^{B}(x_1 = 0, x_3) + \sigma_{13\text{farfield}}^{A} - \sigma_{13\text{farfield}}^{B}) = 0$$ (4.30f)

for plies $(n_d + 1)$ through $n_r^A$ in Region A and plies 1 through $n_r^B$ in Region B. The superscript $AB$ denotes that the Lagrangian Multiplier operates on quantities in both Region A and Region B. As all the expressions in equations (4.30a-f) are equal to zero, the equations can be added to the expression for Complementary Energy without changing the value of the Complementary Energy. This yields:
The final step of the formulation requires the minimization of equation (4.31) with respect to all the remaining unknown variables. By substitution of Equations (4.10a-e), (4.24a), and (4.24b) into equation (4.31), an expression in unknown coefficients $a_j$ and $b_j$ and Lagrangian multipliers $\rho^i_{mn}$ is obtained. This is done by taking partial derivatives of the Complementary Energy in equation (4.31) for each independent variable in $a_j$, $b_j$, and $\rho^i_{mn}$, and setting those partial derivatives equal to zero. The solution for all of the $a_j$, $b_j$, and $\rho^i_{mn}$ that simultaneously solve for the partial derivative equations will define the values of the independent variables. The resultant system of equations, however, is particularly complex, even for laminates with only a few number of plies. In order to simplify the resulting formulation for the expressions in $a_j$, $b_j$, and $\rho^i_{mn}$, vectors of these unknowns are defined as:

$$
\pi^*_C = \sum_{\text{RegionA}} \frac{1}{2} \int_{V_i} \sigma^*_{ij\text{comp}} S^*_{ijkl\text{comp}} \sigma^*_kl dV
$$

$$
+ \rho^A_{11} (\sigma^A_{11\text{comp}} (x = 0, z) + \sigma^A_{11\text{farfield}})
+ \rho^A_{12} (\sigma^A_{12\text{comp}} (x = 0, z) + \sigma^A_{12\text{farfield}})
+ \rho^A_{13} (\sigma^A_{13\text{comp}} (x = 0, z) + \sigma^A_{13\text{farfield}})
+ \rho^{AB}_{11} (\sigma^A_{11\text{comp}} (x = 0, z) - \sigma^B_{11\text{comp}} (x = 0, z) + \sigma^A_{11\text{farfield}} - \sigma^B_{11\text{farfield}})
+ \rho^{AB}_{12} (\sigma^A_{12\text{comp}} (x = 0, z) - \sigma^B_{12\text{comp}} (x = 0, z) + \sigma^A_{12\text{farfield}} - \sigma^B_{12\text{farfield}})
+ \rho^{AB}_{13} (\sigma^A_{13\text{comp}} (x = 0, z) - \sigma^B_{13\text{comp}} (x = 0, z) + \sigma^A_{13\text{farfield}} - \sigma^B_{13\text{farfield}})
$$

(4.31)

$$
\mathbf{a} = \begin{bmatrix} a_1 & a_2 & \cdots & a_{3*(n^A-1)-1} & a_{3*(n^A-1)} \end{bmatrix}
$$

$$
\mathbf{b} = \begin{bmatrix} b_1 & b_2 & \cdots & b_{3*(n^B-1)-1} & a_{3*(n^B-1)} \end{bmatrix}
$$

$$
\mathbf{\rho}_{11} = \begin{bmatrix} \rho^A_{11} & \rho^A_{12} & \cdots & \rho^{A^*}_{11} & \rho^{AB}_{11} & \cdots & \rho^{AB^*}_{11} \end{bmatrix}
$$

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Using these definitions, minimization of Equation (4.31) with respect to \( a \), \( b \), and the Lagrangian multipliers yields a system of linear equations in matrix form:

\[
\begin{bmatrix}
\Omega^A & \Gamma_1^A T & \Gamma_2^A T & \Theta^A T \\
0 & \Omega^B & \Gamma_1^B T & \Gamma_2^B T & \Theta^B T \\
\Gamma_1^A & \Gamma_1^B & 0 & 0 & 0 \\
\Gamma_2^A & \Gamma_2^B & 0 & 0 & 0 \\
\Theta^A & \Theta^B & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
a \\ b \\ \rho_{11} \\ \rho_{12} \\ \rho_{13} \\
\end{bmatrix}
= \begin{bmatrix}
0 \\ 0 \\ \sigma^A_{11} - \sigma^B_{11} \\ \sigma^A_{12} - \sigma^B_{12} \\ \sigma^A_{13} - \sigma^B_{13} \\
\end{bmatrix}
\] (4.32)

This allows for the solution for the unknown coefficients \( a_j \) and \( b_j \). New terms are introduced in equation (4.32) relating to the coefficients of the independent variables \( a \), \( b \) and the Lagrangian Multipliers \( \rho_{mn} \) in the system of equations derived when minimizing the Complementary Energy in equation (4.31). The matrix in equation (4.32) is composed of submatrices \( \Omega^A \), \( \Omega^B \), \( \Gamma_1^A \), \( \Gamma_1^B \), \( \Gamma_2^A \), \( \Gamma_2^B \), \( \Theta^A \), and \( \Theta^B \). These submatrices define the coefficients of \( a \), \( b \) and \( \rho_{mn} \) that come as a result of minimizing the Complementary Energy in equation (4.31).

As linear elasticity has been assumed in the formulation of Complementary Energy, the resultant expressions when minimizing the energy are also linear in \( a \), \( b \) and \( \rho_{mn} \). Thus, in terms of the Complementary Energy in equation (4.31), the submatrices can be defined in general form in terms of the complementary energy as:

\[
\Omega^A_{ij} = \frac{d\pi^*_C}{da_i da_j} \] (4.33a)

\[
\Omega^B_{ij} = \frac{d\pi^*_C}{db_i db_j} \] (4.33b)
\[ \Gamma^A_{1\, ij} = \frac{d\pi^*_C}{d\rho_{11}, \, da_j} \]  

(4.33c)

\[ \Gamma^A_{2\, ij} = \frac{d\pi^*_C}{d\rho_{12}, \, da_j} \]  

(4.33d)

\[ \Gamma^B_{1\, ij} = \frac{d\pi^*_C}{d\rho_{11}, \, db_j} \]  

(4.33e)

\[ \Gamma^B_{2\, ij} = \frac{d\pi^*_C}{d\rho_{12}, \, db_j} \]  

(4.33f)

\[ \Theta^A_{ij} = \frac{d\pi^*_C}{d\rho_{13}, \, da_j} \]  

(4.33g)

\[ \Theta^A_{ij} = \frac{d\pi^*_C}{d\rho_{13}, \, db_j} \]  

(4.33h)

To evaluate these terms with respect to laminate material and geometric parameters and the shapes of the assumed stresses in the thickness direction, the following matrix definitions are introduced:

\[ \Phi_{ij} = \phi_{ij} \]

\[ \Psi_{ij} = \psi_{ij} \]

\[ D\Phi_{ij} = \phi_{ij} \star \lambda_j \]

\[ D\Psi_{ij} = \psi_{ij} \star \lambda_j \]

\[ D^2\Phi_{ij} = \phi_{ij} \star \lambda_j^2 \]
where $\phi_{i,j}$ is the $i$th value of the $j$th eigenvector defined in (4.24a), $\psi_{i,j}$ is the $i$th value of the $j$th eigenvector defined in (4.24b), and $\lambda_j$ is the $j$th eigenvector of a region from equations (4.24a) and (4.24b). With these definitions in place, terms in the equations in equations (4.33a) and (4.33b) can be rewritten in terms of the specific material and geometric parameters of a region, the shape of the assumed stresses through the thickness, and the region's eigenvalues and eigenvectors:

$$\Omega_{ij}^A = \frac{2(e^{(\lambda_1^A+\lambda_2^A)\frac{1}{2}(x_{ij}^A-x_{right}^A)} - 1)}{\lambda_1^A + \lambda_2^A}$$

\[
\begin{align*}
&* \sum_{AllPlies} \int_0^{t_{ply}/2} S_{3333}^* (D^2 \Phi h_{33})^T (D^2 \Phi h_{33}) \\
&+ S_{1313}^* (D \Phi h_{13})^T (D^2 \Phi h_{33}) \\
&+ S_{2223}^* (D \Psi h_{23})^T (D \Psi h_{33}) \\
&+ S_{1133}^* ((\Phi h_{11})^T (D^2 \Phi h_{33}) + (D^2 \Phi h_{33})^T (\Phi h_{11})) \\
&+ S_{1323}^* ((D \Phi h_{13})^T (D \Psi h_{23}) + (D \Psi h_{23})^T (D \Phi h_{13})) \\
&+ S_{1333}^* ((\Psi h_{12})^T (D^2 \Phi h_{33}) + (D^2 \Phi h_{33})^T (\Psi h_{12})) \\
&+ S_{1323}^* (D \Psi h_{23})^T (D \Psi h_{23}) \\
&+ S_{1111}^* (\Phi h_{11})^T (\Phi h_{11}) \\
&+ S_{1112}^* ((\Phi h_{11})^T (\Psi h_{12}) + (\Psi h_{12})^T (\Phi h_{11})) \\
&+ S_{1212}^* (\Psi h_{12})^T (\Psi h_{12}) d\xi_{3ij}^A
\end{align*}
\]

The expression for $\Omega_{ij}^B$ can be written in a similar form, except that the expressions of $\Omega_{ij}^B$ are dependent on the eigenvalues and eigenvectors of Region B as well as the region boundaries of Region B in the lengthwise direction, $x_1$. For Steps 1 through 3, $\Omega_{ij}^B$ can be expressed as:
\[
\Omega^B_{ij} = \frac{2\left(e^{(\lambda^B_1 + \lambda^B_2)(x_{left}^B - x_{right}^B)} - 1\right)}{\lambda^B_1 + \lambda^B_2} 
\]

\[
* \left[ \sum_{AllPlies} \int_0^{t_{ply}/2} S_{333}^*(D^2\Phi_{h33})^T(D^2\Phi_{h33}) 
+ S_{1313}^*(D\Phi_{h13})^T(D^2\Phi_{h33}) 
+ S_{2323}^*(D\Psi_{h33})^T(D\Psi_{h33}) 
+ S_{1133}^*(((\Phi_{h11})^T(D^2\Phi_{h33}) + (D^2\Phi_{h33})^T(\Phi_{h11})) 
+ S_{1323}^*(((D\Phi_{h13})^T(D\Psi_{h23}) + (D\Psi_{h23})^T(D\Phi_{h13})) 
+ S_{2333}^*(((\Psi_{h12})^T(D^2\Phi_{h33}) + (D^2\Phi_{h33})^T(\Psi_{h12})) 
+ S_{2323}^*(D\Psi_{h23})^T(D\Psi_{h23}) 
+ S_{1111}^*(\Phi_{h11})^T(\Phi_{h11}) 
+ S_{1112}^*(((\Phi_{h11})^T(\Psi_{h12}) + (\Psi_{h12})^T(\Phi_{h11})) 
+ S_{1212}^*((\Psi_{h12})^T(\Psi_{h12})dx_{3ij}^B
\]

where \(x_{left}\) indicates the left boundary of a region, and \(x_{right}\) indicates the right boundary of a region. Although the \(\Omega\) terms can be defined generally for every step, the \(\Gamma\) and \(\Theta\) terms depend highly on the polynomial order in \(h\). This occurs since the terms in \(\Gamma\) and \(\Theta\) indicate how the unknown coefficients \(a_j\) and \(b_j\) must be constrained to satisfy free-surface conditions as well as stress continuity across regions. The expressions for \(\Gamma\) and \(\Theta\) are given for their respective steps in Chapters 5 through 7.

Substitution of these results and the solution of Equation (4.22) into the ply stresses defined in Equations (4.8a-e) yields five of the six stress fields in all regions of the laminate. Substitution of those results into equation (4.18) results in \(\sigma_{22}\) and fully defines the stress field in the laminate.
4.5 Implementation and Overall Numerical Issues

This formulation is applicable to all the problems of Steps 1 through 3. These applications of the formulation are described in Chapters 5 through 7 for each of these steps. Although the formulation is applied to Steps 4 and 5 in Chapters 8 and 9, issues arising in the formulation of Step 4 prevent the model from obtaining accurate results for Steps 4 and 5. The formulation for Steps 1 through 3 was implemented in Mathematica® as a number of subroutines called on by a main routine. The flow of the program is set such that the eigenvalue problem defined in Equation (4.22) is posed and solved. The linear equations, as expressed in Equation (4.32), are subsequently solved to obtain the overall solution to the formulation. Specialized subroutines built into the Mathematica computing environment were used without modification to solve both the eigenvalue and linear system problems as defined in this formulation. Any use of integrals, particularly those in equation (4.22), were numerically calculated via Gaussian Quadrature with an appropriate number of points to exactly calculate the integral for the maximum degree polynomials as found. Finally, subroutines were created to postprocess the solution of equation (4.32) and the stresses that arise from that solution. The routines for the models for Steps 1, 2, and 3 are included in Appendixes A and B.

One particular numerical concern about this formulation involves numerical stability. Along the diagonal of the matrix defined in Equation (4.32), a number of zeroes appear equal. This number is approximately equal to \((3n_{r}\nu)\). This constitutes between one-third and one-half of the total rank of the matrix, depending on laminate configuration. This tends to drive the matrix to being unstable and singular. This issue, inherent in many other numerical implementations of Lagrange Formulations, is referred to as “ill-conditioning” of the system. Any linear system defined as:

\[
A \ast x = b
\]

(4.35)

has an associated “condition number” which is defined as:
where $\|A\|$ is the infinity-norm of matrix $A$. The condition number gives an idea of the overall sensitivity of the solutions produced by the linear system with respect to changes in either matrix $A$ or vector $b$. The larger the number, the larger is the ratio between the changes in $x$ against changes in either $A$ or $b$. This is a particular concern in experimental-theoretical work where errors can appear in the acquisition of laboratory data. If such data were applied to some ill-conditioned models, the magnitude of the errors would multiply and variations in the model input could lead to large changes in the output. For a theoretical model such as the one herein, this does not pose an issue in this context.

However, a different issue arises when dealing with residual-based solution methods in the context of finite-number definitions in a computer. Residual-based methods for solving linear systems are the most computationally efficient from the family of solutions that solve linear systems. These methods typically require division of values. In an ill-conditioned system, the divisions are very difficult to perform in a finite-number setting, as, for an ill-conditioned system, the divisor in such operations tend to be very small. If a machine cannot accurately describe these small divisors, numerical error will appear in the results during this process. The error continues to grow in an interactive solution methodology, such as that employed in a residual method, until the residual is under tolerance. Because error is allowed to grow in subsequent iterations, the error may influence the final numerical result of the solution. Applied to the current formulation, the residual routine may give a solution that is numerically correct but results in stress distribution with large and rapid stress gradients.

No systematic methodology exists to improve the conditioning of an arbitrary system, and any attempts at improving conditioning typically result in multiplication of rows by scalars. However, there is no systematic way of knowing what row scalings are appropriate in reducing the condition number for a given linear system. Such determinations are typically made on a problem-to-problem basis. As the described
methodology is intended to analyze a wide variety of problems, which will result in a wide variety of linear systems and configurations from Equation (4.32), such methodology is beyond the scope of this work. Ultimately, the method with which ill-conditioning is dealt in the current formulation is to provide adequate validation to results obtained from the formulation as well as attempting to avoid problems that ill-condition the system beyond the ability of a computer to represent numbers internally.

4.6 Procedure for Validation via Finite-Element Analysis

The ABAQUS Finite Element package was used to provide results for the problems in Steps 2 and 3 in order to validate the current formulation. ABAQUS is a displacement-based analysis with numerous options available from the types of elements used to the level of mesh refinement desired for a given model.

The problems in Steps 2 and 3 were constructed in ABAQUS such that the dimensions of the model satisfied the two assumptions present in the current model. The thickness and material properties of each ply were input based upon the ply thickness and ply material properties used in the formulation. The length of the model was set such that, at the ends of the laminate, the difference in the stress values in neighboring elements was one percent or less. This length satisfies the requirement that far-field stresses are recovered in the far-field. In addition, the width of the model was set such that neighboring elements in the center of the model had a difference in stress values of less than one percent. The purpose of this is to create a model where results could be obtained in a region where the partial derivative of the stress values with respect to the width direction was nearly zero. As ABAQUS is limited in only being able to develop a two-dimensional model with all six independent stress fields for uniaxial laminates, full three-dimensional models with three-dimensional elements were developed to create a robust model that could analyze a large number of laminate configurations.

A convergence study was conducted focusing on the regions involving ply drops
and grooves. As per the recommendations given in the ABAQUS User’s Manual [51], quadratic continuum elements were used to mesh the model. In instances with laminates where ply dropoffs were modeled, hexahedral elements were used to mesh the model. In an effort to better mesh the curved surface of the groove, tetrahedral elements were used. The models were repeatedly run with increasingly finer meshes around the regions of interest until doubling the mesh density in the regions of interest yielded less than a five percent increase in the maximum stress. Further details of this for each step are outlined in Chapters 6 and 7.
Chapter 5

Step 1 Configuration:
Tensile-Loaded Laminate with a Symmetric Infinite-Length Single-Ply Dropoff

In this chapter, the specific formulation, implementation, validation, and results for the Step 1 problem outlined in Chapter 3 are described. The problem under consideration is that of a tensile-loaded laminate of infinite-length with plies dropped. The assumed stress shapes presented in Chapter 4 are specifically adapted to be able to estimate the stress field in the problem under consideration. Comparison between results of the current model and a similar model programmed in a differing computing environment is used to establish the validity of the current model. Finally, characteristic results of the model for a number of laminates are presented, along with a discussion of the results and of necessary adjustments that need to be made to the model in order to better predict the stress field in Steps 2 and 3.
5.1 Specific Formulation

The specific formulation for the Step 1 problem follows from the general Rayleigh-Ritz formulation presented in Chapter 4. The general formulation only requires that an assumption be made for the through-thickness variation of the in-plane stresses, the $H_{\text{out}}^{i}(x_3)$ and the $H_{\text{in}}^{i}(x_3)$ of Equations (4.8a) and (4.8b), to completely define the laminate stress fields. This arises since the expressions for the stresses are assumed to be functionally separate, and the variation of the stresses along the $x_1$-direction has been shown to be a sum of exponential terms for a linearly elastic problem in Equations (4.24a) and (4.24b).

As mentioned in Chapter 4, the model requires the definition of regions in the laminate that are differentiated from each other via changes in the laminate geometry as illustrated in Figure 4.1. The Step 1 problem requires the definition of only two regions. This is done in $x_1$ with one being the region where all plies exist, and one being the region after the dropoff where some plies terminate. The full-ply region is hereafter referred to as "Region A," and the dropped-ply region is hereafter referred to as "Region B."

One of the key factors in the accuracy and efficiency of the solution is the assumed shapes of the stress or displacement fields being determined. Not only must the assumed shapes be defined such that the shapes allow for a statically admissible field, but these must also allow for some degree of variability in the solution to capture the details of the stress or displacement fields possible in various solutions. However, allowing an assumed set of shapes to become overly variable can result in larger runtimes to obtain a solution.

The constraints on the through-thickness stress shapes are given in Equations (4.7a), (4.7b), (4.14a), and (4.14b). In particular, the constraints required for force balance in the $x_1$- and $x_2$-directions must be satisfied through the assumed through-thickness variation of the in-plane stresses. Because there is one constraint on each of the in-plane stresses $\sigma_{11}$ and $\sigma_{12}$, there must a minimum of one degree of variability of the in-plane stresses through the thickness of a ply or sublayer. If this does not occur,
the assumed stress shapes will not be statically admissible. As the through-thickness
variation of the in-plane stresses require one degree of variability through a ply or
sublayer, the in-plane stresses, at lowest polynomial order, must be constant through
the thickness. The simplest stress variations in $x_3$ that satisfy the aforementioned
constraints that allow for a constant though-thickness variation of the in-plane stresses
can be defined as:

$$H_{out}^i = H_{in}^i = \frac{1}{t_i} \quad (5.1)$$

Substituting Equation (5.1) into Equations (4.8a) and (4.8b) defines the stress
fields for each ply:

$$\sigma_{11}^i = \frac{1}{t_i} (F_{i-1}(x) - F_i(x)) \quad (5.2a)$$

$$\sigma_{12}^i = \frac{1}{t_i} (G_{i-1}(x) - G_i(x)) \quad (5.2b)$$

$$\sigma_{13}^i = F_{i}'(x) \left( \frac{x_3^i}{t_i} + \frac{1}{2} \right) - F_{i-1}'(x) \left( \frac{x_3^i}{t_i} - \frac{1}{2} \right) \quad (5.2c)$$

$$\sigma_{23}^i = G_{i}'(x) \left( \frac{x_3^i}{t_i} + \frac{1}{2} \right) - G_{i-1}'(x) \left( \frac{x_3^i}{t_i} - \frac{1}{2} \right) \quad (5.2d)$$

$$\sigma_{33}^i = \frac{F_{i-1}''(x) t^i}{2} \left( \frac{x_3^i}{t_i} - \frac{1}{2} \right)^2 - \frac{F_i''(x) t^i}{2} \left( \frac{x_3^i}{t_i} + \frac{1}{2} \right)^2 - \sum_{j=1}^{i-1} t^j F_j''(x) \quad (5.2e)$$

This result indicates that the interlaminar shear stresses have a linear distribution
through the thickness of each ply or sublayer, and the interlaminar normal stress has
a quadratic distribution through the thickness of each ply or sublayer. Using Equation
(5.1) in the stress definitions posed in Equations (4.10a-e) yields:

$$h_{11}^i = \begin{bmatrix} 0 & \ldots & \frac{1}{\nu} & -\frac{1}{\nu} & \ldots & 0 \end{bmatrix} \quad (5.3a)$$

$$h_{12}^i = \begin{bmatrix} 0 & \ldots & \frac{1}{\nu} & -\frac{1}{\nu} & \ldots & 0 \end{bmatrix} \quad (5.3b)$$
Substituting this result into the expression for Minimumization of Complementary Energy yields a system of equations in the form of Equation (4.22), with terms as defined by Equations (4.23a-g). The above expressions are applied to Region A and Region B individually, and each has its own solution, which entails having its own independent set of stress functions, eigenvalues, eigenvectors, and unknown coefficients. Rather than providing an explicit definition for the terms in Equations (4.23a-g) based upon the definitions in Equations (5.3a-e), the terms are left in integral form and numerically integrated in an effort to reduce model implementation complexity and model runtime.

The final step of the analysis is to reminimize the total Laminate Complementary Energy while enforcing the constraints on the stresses brought about by stress continuity of the $\sigma_{1i}$ stresses in continuous plies between the two regions, and the free-edge boundary conditions in the terminated plies. The constrained-minimization problem is given as Equation (4.31), and minimizing that expression yields Equation (4.32).

Completion of the specific formulation for Step 1 requires the definition of the $\Gamma$ and $\Theta$ vectors in Equation (4.32). These vectors represent the free-edge boundary conditions at dropped plies as well as stress continuity conditions from Region A to Region B. The boundary conditions and constraints affect $\sigma_{11}$, $\sigma_{12}$, and $\sigma_{13}$ in both regions. $\Gamma$ is made up of terms relating to the in-plane stresses, whereas $\Theta$ is made up of terms relating to the interlaminar shear stress.

The assumed stress shapes for the model for each ply are defined in Equations (5.2a-e). Both $\sigma_{11}$ and $\sigma_{12}$ have a constant distribution through the thickness of each ply. Thus, if $\sigma_{11}$ or $\sigma_{12}$ is enforced to equal zero at one point along the free surface of a ply, the value of the in-plane stress will equal zero everywhere along the free surface.
Thus, for dropped plies in Region A, free-edge boundary conditions for \( \sigma_{11} \) or \( \sigma_{12} \) can be satisfied by enforcing the condition at one point in each ply, which requires only one equation per ply. This implies there will be a total number of equations equal to the number of dropped plies required to enforce the free-edge boundary condition for each in-plane stress.

Similarly, in a continuous ply, if \( \sigma_{11} \) or \( \sigma_{12} \) within a ply is enforced to equal a constant value along a surface corresponding to the dropoff line at one point, the value of the in-plane stress will equal that constant value everywhere along the dropoff line. This implies there will be a total number of equations equal to the number of continuous plies required to enforce the stress continuity condition for each in-plane stress.

This implies that a number of equations equal to two times the total number of plies in the laminate are required to enforce the required constraints on the in-plane stresses. However, there are two redundant equations in this set, one equation representing terms related to \( \sigma_{11} \) and one equation representing terms related to \( \sigma_{12} \). The reason for this concerns force balance. The stress assumptions developed in Equations (5.2a-e) were developed with force-balance considerations built-in to the expressions for the stresses. The force balance-constraint relates the stresses through the thickness of the laminate at a given point. As stresses are developed on a ply-by-ply or layer-by-layer basis, this means that the behavior of the stress in the ply or layer closest to the laminate midplane is completely dependent on the behavior in the stresses in all other plies or layers. Thus, there is no need to generate a set of equations to enforce stress continuity in the ply or layer closest to the midplane, as force-balance considerations will already determine the behavior of that ply or layer. Thus, a number of equations equal to \([2*(n_{r}-1)]\) are sufficient to enforce free-surface conditions and stress continuity for the in-plane stresses.

It is chosen that the free-edge and stress continuity conditions be enforced at each ply or layer at \( x_3 \) equal to \( t_{\text{ply}}/2 \) to simplify the resultant expressions. Substituting the assumed stress shapes of Equation (5.2a-b) into Equation (4.32) yields the following definitions for the \( \Gamma \) vectors:

\[ \text{Definitions for the } \Gamma \text{ vectors:} \]
\[ \Gamma^A_{1ij} = \begin{cases} 
-1/t_{\text{ply}}^i \Phi^A_{ij} & i = 1 \\
1/t_{\text{ply}}^i \Phi^A_{i-1j} - 1/t_{\text{ply}}^i \Phi^A_{ij} & 2 < i < n_r^A - 1 
\end{cases} \] (5.4a)

\[ \Gamma^B_{1ij} = \begin{cases} 
0 & i \leq n_r^A - n_r^B \\
1/t_{\text{ply}}^k \Phi^B_{kj} & k = i - (n_r^A - n_r^B) = 1 \\
-1/t_{\text{ply}}^k \Phi^B_{k-1j} + 1/t_{\text{ply}}^k \Phi^B_{kj} & 2 < k = i - (n_r^A - n_r^B) < n_r^A 
\end{cases} \] (5.4b)

\[ \Gamma^A_{2ij} = \begin{cases} 
-1/t_{\text{ply}}^i \Psi^A_{ij} & i = 1 \\
1/t_{\text{ply}}^i \Psi^A_{i-1j} - 1/t_{\text{ply}}^i \Psi^A_{ij} & 2 < i < n_r^A - 1 
\end{cases} \] (5.4c)

\[ \Gamma^B_{2ij} = \begin{cases} 
0 & i \leq n_r^A - n_r^B \\
1/t_{\text{ply}}^k \Psi^B_{kj} & k = i - (n_r^A - n_r^B) = 1 \\
-1/t_{\text{ply}}^k \Psi^B_{k-1j} + 1/t_{\text{ply}}^k \Psi^B_{kj} & 2 < k = i - (n_r^A - n_r^B) < n_r^A 
\end{cases} \] (5.4d)

The interlaminar shear stress \( \sigma_{13} \) has a linear variation moving through the thickness of a ply or layer, and to enforce a specific variation of the stress along the dropoff line, two equations are needed for each ply or layer. This implies that a total number of equations equal to two times the number of plies in the laminate are required to satisfy all constraints on the interlaminar shear stress. However, there are constraints on the behavior of \( \sigma_{13} \) through the thickness of the entire laminate, which reduces the total number of equations required to enforce the free-edge and stress continuity conditions. The first is that the stress must equal zero at both the outer free surface of the laminate and at the laminate midplane. These two constraints have the effect of reducing the number of equations required to impose the free-edge and stress continuity equations by two. In addition, \( \sigma_{13} \) is enforced as being continuous
from ply to ply or layer to layer through the thickness of the laminate. There are a
total number of \((n_r^A-1)\) internal interfaces in the laminate, thus, there are \((n_r^A-1)\)additional constraints on the variability of \(\sigma_{13}\) through the thickness. Considerations
of stress continuity though the thickness has the effect of reducing the total number
of required equations for the free-edge and stress continuity expressions by \((n_r^A-1)\).
Thus, a total of \((n_r^A-1)\) equations are required.

Substituting the expression for \(\sigma_{13}\) in Equation (5.2c) into Equation (4.32) defines
\(\Theta\) as:

\[
\Theta^A_{ij} = \Phi^A_{ij} * \lambda_i^A \tag{5.4e}
\]

\[
\Theta^B_{ij} = \begin{cases} 
0 & i <= n_r^A - n_r^B \\
-\Phi^B_{kj} * \lambda_k^B & 1 < k = i - (n_r^A - n_r^B) < n_r^A
\end{cases} \tag{5.4f}
\]

The notation and symbols used in Equations (5.4a-f) follow similar definitions as the
general formulation in Chapter 4. The total number of plies in region A is equal to
\(n_r^A\), and the total number of plies in region B is equal to \(n_r^B\). \(\Phi\) is a matrix of
eigenvectors affecting the assumed forms of \(\sigma_{11}, \sigma_{13},\) and \(\sigma_{33}\), whereas \(\Psi\) is a matrix
of eigenvectors affecting the assumed forms of \(\sigma_{12}\) and \(\sigma_{23}\). The ith eigenvalue of a
region is represented as \(\lambda_i\). Subscripts “A” and “B” represent numerical values of
eigenvectors or eigenvalues inherent in Regions A or B, respectively. The superscripts
on \(t_{\text{ply}}\) indicate the thickness of the ith ply or sublayer in Region A.

Using these definitions for the \(\Gamma\) and \(\Theta\) vectors as well as Equations (4.34a) and
(4.34b) allow for the complete definition of all terms in Equation (4.32) for the Step
1 model.

### 5.2 Implementation

The implementation of this specific formulation in Mathematica follows the outline
for implementation of the general model as described in Section 4.4. Beyond the gen-
eral implementation of Section 4.4, one further refinement can be made to the overall
implementation of the model. The two main problems associated with Step 1 - the eigenvalue problem for the lengthwise variation of the stresses and the linear problem associated with the Lagrangian Minimumization of Laminate Complementary Energy - posses definite integrals through the thickness direction as part of setting up the overall problem. The method of Gaussian quadrature is utilized to numerically integrate these expressions while setting up the eigenvalue and linear problems. Upon substitution of the assumed stress shapes in Equation (4.10a-e) into the eigenvalue problem of Equation (4.22) and the linear problem of Equation (4.32), the thickness integrals integrate, at highest order, to quartic polynomials. These quartic polynomials come about upon terms which require the squaring of interlaminar normal stress terms. Squaring this expression that is quadratic in $x_3$ yields a result that is quartic in $x_3$. To obtain an exact solution for these integrals using Gaussian quadrature, a third-order quadrature rule needs to be implemented. Thus, definite integrals of the form:

$$\int_{-\frac{t}{2}}^{\frac{t}{2}} H(x_3^i) dz$$

where the integrand is always a quartic or lower-degree polynomial, can be numerically integrated exactly by computing:

$$\int_{-\frac{t}{2}}^{\frac{t}{2}} H(x_3^i) dz = \frac{t}{2} \left( \frac{5}{9} * H(-\sqrt{3/5}) + \frac{8}{9} * H(0) + \frac{5}{9} * H(\sqrt{3/5}) \right)$$  (5.4)

This numeric definition of the integral is implemented as a separate subroutine to be called upon when calculating the solution to Equations (4.22) and (4.32).

5.3 Validation

The validity of the Step 1 model described in Chapters 4 and 5 is established via comparison of the results of the current model to the results for the exact same
problems using other previously-validated models. Although few models exist that can accurately estimate the stress fields in the Step 1 problem, the works done by Shim and Lagace [35] and Shim [50] led to the development of a model for the problem of a laminate with external ply dropoffs. In terms of formulation, both the Step 1 model and the model proposed by Shim and Lagace [35] and Shim [50] are the same model.

There is, however, a difference in the two models in that both models were implemented in different computing environments. The Step 1 model, as previously mentioned, was implemented as a Mathematica routine, whereas the model developed by Shim was implemented in Matlab, a tensor-based computational environment. Both models implement different solution algorithms in solving the eigenvalue problem in Equation (4.22) and the linear problem in Equation (4.32). Thus, it is expected that, while the results of both models should be relatively close, there is allottable room for error due to the manner that the built-in routines solve Equations (4.22) and (4.32), as well as differences in the manner in which numbers are stored within a computer for each environment. As the routines to solve these problems are proprietary to each of their respective computing environments, a direct comparison of the solver codes is not possible. Only comparison of the results can be used to assess model validity. Thus, the following validation is targeted at the accuracy of the model running in Mathematica, as the model proposed and written by Shim in Matlab has been validated using previous work.

Two different laminates are considered for validation of the Step 1 model. The first laminate is an eight-ply unidirectional laminate with two of its outer plies dropped. A unidirectional laminate under tension will have values equal to zero for $\sigma_{12}$ and $\sigma_{23}$ everywhere within the laminate, and only $\sigma_{11}$, $\sigma_{13}$, $\sigma_{33}$, and $\sigma_{22}$ will be nonzero in the laminate. The reduction of the number of results to compare will allow for a more simple comparison between the two models. The second laminate under consideration is, as in the first configuration, an eight-ply laminate. In this case, the layup is that of a quasi-isotropic $[90/0/\pm45]_s$ laminate with the two outermost plies dropped in a symmetric fashion. The purpose of comparing the results for this problem is to validate the ability of the current model to estimate the stresses in non-unidirectional
laminates, as well as assess the ability of the model to specifically estimate $\sigma_{12}$ and $\sigma_{23}$. These problems are illustrated in Figure 5.1.

One note should be made about laminate notation concerning laminates with dropped plies. In all of the Step 1 problems that can be defined, the structure of the laminate changes after the dropoff. Thus, there needs to be a modification of traditional laminate layup notation that can define the layup the laminate has in the full-ply region (Region A) and the new laminate configuration within the region after the plies are dropped (Region B). To provide for this, the notation of a subscript "D" is used to indicate the plies that are dropped at the dropoff line. As an example, the notation $[0_2D/0_2]_S$ indicates a symmetric unidirectional laminate with its outer two plies dropped in a symmetric fashion. This laminate configuration corresponds to the topmost figure in Figure 5.1. The second validation case corresponds to a $[[90/0]_D/\pm 45]_S$ laminate, the bottommost figure in Figure 5.1.

It should be noted that the emphasis on the comparison between the results of the two models focuses mainly on the distribution of the stresses rather than the magnitudes of the stresses. This is due to the intent and design of the current model. The current model is designed to be as efficient as possible in estimating the stress fields within composite laminates, which is useful in preliminary design in assessing competing design configurations. Although it would be beneficial to obtain a high accuracy of the solutions, this should not be done at significant cost to the efficiency of the model. As reported in previous work, there is a tradeoff between the accuracy of the results generated by analytical and numerical models and the overall runtime required to obtain those results. One of the purposes of the models is to identify the locations where interlaminar stresses are likely to arise as well as what factors affect the rise of interlaminar stresses. This is in contrast to a different model, such as finite element models, in determining the point-by-point numerical values of stresses. As a result, some loss in accuracy in the values of the stresses is acceptable as long as this loss in accuracy does not effect the distribution of the stresses throughout the laminate. Thus, emphasis in the validation will be placed upon stress distribution rather than magnitude, as the stress distributions are considered more important in
Figure 5.1 Illustration of laminate configurations analyzed for validation of the Step 1 model: (top) uniaxial laminate, and (bottom) quasi-isotropic laminate.
achieving the overall goals of the model. However, comparisons of stress values, particularly maximum and minimum stress values, will be noted to indicate discrepancies between the results from the two models.

All results reported are assumed to be in the global laminate axes defined in Chapter 4. The material properties for each ply are the same as described in Table 3.1.

5.3.1 Unidirectional ([02\text{D}/02]_s) Laminate

A comparison of the results for the in-plane stress $\sigma_{11}$ on the outer surface in the $x_1$-direction of the outermost continuous ply between the Step 1 model and the model by Shim [50] is presented in Figure 5.2. The stress results are normalized by the values of $\sigma_{11}$ in the far-field of Region A, which correspond to CLPT values, defined in the figure as $\sigma_0$. For the case of the unidirectional laminate, this value is equal to the value of the applied load divided by the cross-sectional area. This particular interface was chosen because the results in Reference [50] indicated that the largest gradients present in the stress fields could be seen along this line. If there is a significant difference in the results between the two models, it is expected that the differences would be most significantly manifested along this line. The length parameter is normalized by the overall laminate thickness.

In general, the results compare well. Similar distributions can be seen in the current model and the Shim model. It is noted that both results reach their peak value just before the dropoff at $x_1$ equal to 0. The current model has a maximum value for $\sigma_{11}$ of $2.22\sigma_0$, and the Shim model has a slightly higher value of the peak stress at approximately $2.31\sigma_0$. In addition, there are larger gradients in the stress distribution for the Shim model, as it converges to far-field stress values more quickly in the continuous region than the current model. In both Region A and Region B, the results from the Shim model approach 20% of the far-field stress values in Region A and 1% of the far-field stress values in Region B within a length equal to a laminate thickness from the dropoff line. The results from the current model approach 23% of far-field values in Region A and 1% of far-field values in Region B over the same
Figure 5.2  Results for $\sigma_{11}$ along the outer surface of the outermost continuous ply in a $[0_{2D}/0_{2}]_s$ laminate under uniform tension.
length. In addition, the results from the Shim model indicate that the stresses equal 101% of CLPT values at a distance equal to 2.85 laminate thicknesses away from the dropoff in Region A, and at a distance equal to 1.11 laminate thicknesses away from the dropoff in Region B. In contrast, the current model predicts these values at 3.71 laminate thicknesses away in Region A and 1.11 laminate thicknesses away in Region B. Although the decay rates are similar in the results from both models in Region B, the results from the Shim model approach far-field values more quickly than the current model.

A further note about validation can be seen in the far-field values of the stresses in Region A and Region B. Half of the total plies in the laminate are dropped at the dropoff, and thus the continuous plies in Region B need to carry twice the load compared to plies in Region A in order to equilibrate the laminate with the applied loading. Thus, both models converge to a value of $2.0\sigma_o$ in Region B, twice the value of the far-field stresses in Region A.

Comparison of the two models for the estimation of the interlaminar shear stress $\sigma_{13}$ along the interface between the outermost continuous ply and the innermost dropped ply is presented in Figure 5.3. The results of the current model compare very well with those from the Shim model except with regards to the value of the maximum stress, located at $x_1/t_{lam}$ equal to -0.11. The current model suggests a smaller maximum stress at this location equal to $0.26\sigma_o$, whereas the Shim model predicts a maximum stress magnitude of $0.27\sigma_o$. This difference is relatively small (less than 5%) and should not indicate the models compare unfavorably to each other, especially considering the nearly matching distributions away from the location of the maximum stress. There is also further evidence that the results from the Shim model decay more quickly than the results from the current model. Both models predict a far-field stress equal to zero. The Shim model predicts a total stress equal to 1% of the average applied stress at a distance of 2.6 laminate thicknesses away from the dropoff. The current model predicts this stress at a distance of 3.2 laminate thicknesses away from the dropoff. A further point of validation can be demonstrated in the current model by observation that $\sigma_{13}$ decays to a value of zero at the dropoff.
This is indicative of the fact that, along the line where stresses are being considered, the ply interface becomes a free surface after the dropoff, and as such, \( \sigma_{13} \) is required to equal zero along this surface.

Comparison of the two models for the estimation of the interlaminar normal stress \( \sigma_{33} \) along the interface between the outermost continuous ply and the innermost dropped ply is presented in Figure 5.4. Similar to the results of \( \sigma_{13} \) presented previously, the two models compare very well except at the location where the stresses increase to maximum value at the dropoff. The difference in the maximum values of the stress is significant between the two models. The current model predicts a maximum stress of 0.26\( \sigma_o \) compared to 0.27\( \sigma_o \), the value predicted by the Shim model. Despite this difference in maximum value, the distribution of the stresses compare favorably away from the dropoff. In particular, both models predict a crossover from compressive stress to tensile stress around the point of \( x_l/t_{lam} \) being equal to -0.35. In addition, both models indicate a stress of zero in Region B, as the line on which stresses were taken lies upon a free surface after the dropoff. The fact that the current model predicts zero stresses here indicates that the implementation of boundary conditions and constraints in the model is correct. Similar to the other stresses considered for comparison of results of the unidirectional laminate, the results from the Shim model decay more quickly than the results from the current model. The Shim model predicts that the value of the interlaminar normal stress will equal 1% of the average applied stress at a distance of 1.22 laminate thicknesses away from the dropoff. The current model predicts this value of stress at a distance of 1.40 laminate thicknesses away from the dropoff.

Although further results can be presented for the purposes of validation, the comparisons of the results from the two models for \( \sigma_{11}, \sigma_{13}, \) and \( \sigma_{33} \) should give a sufficient indication of the validity of the current model. There are a number reasons for this. First, the in-plane and interlaminar shear stresses, \( \sigma_{12} \) and \( \sigma_{23} \), are zero everywhere for this laminate, and thus, comparison between the two models is trivial and suggests very little about the validity of the current model. Second, results for \( \sigma_{22} \) are omitted because of the dependence of \( \sigma_{22} \) on the other stresses as suggested in Equation (4.6).
Figure 5.3  Results for $\sigma_{13}$ along the outer surface of the outermost continuous ply in a $[0_{2D}/0_2]_s$ laminate under uniform tension.
Figure 5.4 Results for $\sigma_{33}$ along the outer surface of the outermost continuous ply in a $[0_{2D}/0_2]_s$ laminate under uniform tension.
All of the other stresses compare favorably with each other between the two models, and this suggests that, if plotted, $\sigma_{22}$ would compare favorably, as well. Third, results of the stresses through the thickness of the laminate are not included in the current validation. This is due to the fact that the results in the other plies are related to the results presented due to consideration of differential equilibrium. Finally, results taken along other interfaces in the laminate can provide additional information to the overall behavior of the stresses in the laminate, but it is not necessary for validation. This is due to the dependence of the results through the thickness of the laminate due to concerns of force-balance. As the distributions along the outermost continuous interface are similar, it can be inferred that these distributions will be similar from model to model. Although comparison of the results in other plies have been completed and compared favorably, the results of the comparison are similar to the presented validation and are not included.

5.3.2 Quasi-Isotropic ($[[90/0]_D/\pm45]_s$) Laminate

The validation procedure for comparing the results of the quasi-isotropic laminate with ply dropoffs is similar to the validation when comparing the uniaxial laminate. The main difference between the two laminates is that, because of the non-unidirectional layup of this laminate, $\sigma_{12}$ and $\sigma_{23}$ are nonzero in the laminate. This provides additional results that can be compared between the two models.

As with the previous model, the stress results are normalized. However, the stress results are normalized via the far-field stresses in the uniaxial problem, rather than the quasi-isotropic problem. The current configuration is defined such that the far-field CLPT stresses in each ply in Region A are different from the far-field stresses in the uniaxial problem. The normalizing stress in the current problem is equal to the tensile load per unit length divided by the total thickness of the laminate, which is also equal to the far-field value of the axial stresses in the unidirectional configuration. Selecting this normalizing stress allows comparison between the two results in a later section, as the two laminates in the posed problems differ only in their layups and give an indication on how layup affects the stress fields. The length parameter continues to
be normalized by the overall laminate thickness, and the plots represent the stresses along the outer surface of the outermost continuous ply.

The results from the Shim model and the current model for the in-plane stress $\sigma_{11}$ for the quasi-isotropic layup in the outermost continuous ply is plotted in Figure 5.5. As per the comparison of the results in the unidirectional laminate, the stresses are expected to be largest in magnitude, and thus, comparison of results in this ply is used for validation. The far-field values of $\sigma_{11}$ equal $0.62\sigma_o$ in Region A and $2.00\sigma_o$ in Region B for this ply. The far-field values of $\sigma_{12}$ equal $0.42\sigma_o$ in Region A and $0.84\sigma_o$ in Region B. All other far-field values of the stress are equal to zero.

The behavior of the stress distribution in both models compare well, although the current model exhibits less of a stress gradient in $\sigma_{11}$ moving along the length than in the model by Shim. The current model predicts that $\sigma_{11}$ will equal 101% of the far-field values in the ply at a distance equal to 2.4 laminate thicknesses away from the dropoff in Region A, and 0.4 laminate thicknesses away from the dropoff in Region B. In contrast, the results from the model developed by Shim predict these locations at a distance of 2.16 laminate thicknesses away from the dropoff in Region A, and 0.29 laminate thicknesses away from the dropoff in Region B. The magnitude of the maximum stress, as well, is smaller in the current model, as in the case of the unidirectional laminate. The maximum stresses, located at a distance less than 1% of a laminate thickness away from the dropoff in both models, are approximately equal to $2.22\sigma_o$ for the current model and $2.31\sigma_{11}$ for the Shim model.

Comparison of the results from the two models for the in-plane stress $\sigma_{12}$ is presented in Figure 5.6. Unlike previous comparisons of the stresses from both models, the current model predicts a larger value of the maximum stress compared to the results of the Shim model. The stresses rise from their far-field values in Region A to a peak before the dropoff, then begin to decay to their values in Region B. The current model predicts a maximum value of the stress equal to $1.17\sigma_o$, whereas the Shim model predicts a maximum value of the stress equal to $1.16\sigma_o$. Both models predict the maximum stress at a location equal to 0.11 laminate thicknesses away from the dropoff in Region A. Like previous comparisons, the results of the Shim model
Figure 5.5  Results for $\sigma_{11}$ along the outer surface of the outermost continuous ply in a $[[90/0]_{D}/\pm 45]_{s}$ laminate under uniform tension.
decay to their far-field values more quickly than the current models. The results from
the Shim model indicate the stress equals a value equal to 101% of far-field values
at a distance of 1.40 laminate thicknesses away from the dropoff and 0.30 laminate
thicknesses away in Region B. In contrast, the current model predicts these locations
to be 2.60 laminate thicknesses away from the dropoff in Region A, and 0.37 laminate
thicknesses away from the dropoff in Region B. The behavior of the stress compares
very well in both models, although, as in the results for $\sigma_{11}$, the current model tends
to exhibit smaller stress gradients.

The comparison of results for the interlaminar stresses $\sigma_{13}$ and $\sigma_{33}$ is presented in
Figures 5.7 and 5.8, respectively. Similar to previous results, the stress distributions
compare favorably in both models. The current model exhibits less of a stress gradient
than in the Shim model as well as indicating a smaller value in maximum stress. The
maximum values of the stresses as predicted using the results of the current model are
equal to $0.57\sigma_o$ for $\sigma_{13}$ and $0.40\sigma_o$ for $\sigma_{33}$. The results from the Shim model predict
larger values of the maximum stresses, with $\sigma_{13}$ having a maximum value of $0.62\sigma_o$
and $\sigma_{33}$ having a maximum value of $0.46\sigma_o$. Both models predict the maximum stress
occurring in the same location. This is located at $x_1$ equal to -0.09$t_{lam}$ for $\sigma_{13}$ and
the dropoff ($x_1$ equal to 0) for $\sigma_{33}$. The stresses decay more quickly in the results
of the Shim model compared to the results of the current model. As both stresses
have a far-field value equal to zero, comparison will be done by locating the point
where the stresses equal $0.01\sigma_o$ in both models for both stresses. For $\sigma_{13}$, this location
corresponds to $x_1$ equal to -1.88$t_{lam}$ for the Shim model and -2.0$t_{lam}$ for the current
model. For $\sigma_{33}$, this location corresponds to $x_1$ equal to -1.24$t_{lam}$ for the Shim model
and -1.3$t_{lam}$ for the current model. The distributions of these stresses is similar to
that for the unidirectional laminate, although the magnitudes of the stresses tend of
be larger in the quasi-isotropic problem.

Results for the two models for the $\sigma_{23}$ interlaminar stresses are presented in Figure
5.9. The results between the two models compare very favorably, especially with
regard to the transitions of the stress from negative to positive and back to negative
along the length. The stress has two crossover points where the stress goes from
Figure 5.6  Results for $\sigma_{12}$ along the outer surface of the outermost continuous ply in a $[[90/0]_D/\pm45]_s$ laminate under uniform tension.
Figure 5.7  Results for $\sigma_{13}$ along the outer surface of the outermost continuous ply in a $[[90/0]_o/\pm45]_s$ laminate under uniform tension.
Figure 5.8  Results for $\sigma_{33}$ along the outer surface of the outermost continuous ply in a $[[90/0]_D/\pm 45]_s$ laminate under uniform tension.
negative to positive or positive to negative, and both models predict these points at $x_1/t_{\text{lam}}$ equal to -0.66 and -0.08. The maximum absolute value of the stress as predicted by the Shim model is equal to $-0.19\sigma_0$, whereas the current model predicts a value of $-0.13\sigma_0$. Both models predict the location of the maximum absolute value of the stress at the dropoff. The results of both models suggest that the stress takes on a value of $0.01\sigma_0$ at a distance from the dropoff equal to 0.44 laminate thicknesses away from the dropoff.

Both models show good agreement in the calculation of the stresses. This is an expected result, as both models are identical, though implemented in different computing environments. This difference in the computational environments leads to the differences in the results between both models. In general, the current model underpredicts the maximum values of the stresses and exhibits somewhat smaller stress gradients when compared to the results from the Shim model. This is attributable to how both models solve the eigenvalue problem in Equation (4.22) and the final linear problem in Equation (4.32). Comparing the numerical results of both models indicates that both models compute the exact same eigenvalues and eigenvectors to within one-tenthousandth of a percent of accuracy. Despite this, the differences in the results of the two models from the eigenvalue problem propagate to the linear problem in Equation (4.32). This propagated error is supplemented by the errors introduced in solving the linear problem. These errors come about in both how each computing environment represents numbers internally as well as the differing methods used to solve both the eigenvalue and linear problems. Despite these differences, the current model matches well the behavior of the Shim model, and the numerical differences described appear to have little effect on the overall behavior of the stresses in the laminate.

### 5.4 Results

Characteristic results observed in the model for a number of laminate configurations are presented in this section. Although these results are intermediate in that
Figure 5.9 Results for $\sigma_{23}$ along the outer surface of the outermost continuous ply in a $[[90/0]_D/\pm 45]_s$ laminate under uniform tension.
they do not give solutions to the full posed problem in Chapter 3, they can provide insight into the characteristic behavior of the model, which will be useful as the model is built up according to the stepwise methodology. The focus of this section is the presentation of certain results in order to obtain a better understanding of what mechanisms could potentially be present in laminates with terminated plies, and how future models can be formulated and implemented in order to better capture these mechanisms.

5.4.1 Stress Concentrations at the Dropoff

Observation of the results obtained for the purpose of validation indicates that stress concentrations occur in the outermost continuous ply of the laminate. This is observed in all of the stresses for both the laminates considered in the validation. These phenomena are a consequence of a shear lag mechanism arising due to the presence of terminated plies. The tensile load induces a $\sigma_{11}$ in order to equilibrate the laminate with the applied load, since terminated plies cannot continue carrying this load at the dropoff. The load thus must be transferred to neighboring plies such that the terminated ply can satisfy the free-edge condition at the dropoff. This requires a negative gradient of $\sigma_{11}$ along the length of a terminated ply to initiate before the dropoff. In order to satisfy the differential equations of equilibrium, a positive gradient of $\sigma_{13}$ must arise through the thickness to balance the negative gradient of $\sigma_{11}$ along the length. Thus, the mechanism in which axial load is transferred from dropped plies to continuous plies is through gradients in the interlaminar shear stresses through the thickness. These lead to stress concentrations in the regions below the dropoff.

Notable is that the magnitude of the stress concentrations are larger in the quasi-isotropic laminates. This is caused by a combination of two factors. First, the mismatch between the ply properties in the quasi-isotropic laminate is greater than the unidirectional laminate, where material mismatch does not exist. Mismatches in material properties of the plies of a laminate give rise to gradients in the in-plane stresses. This in turn leads to gradients in the interlaminar stresses. The second factor in the greater stress concentrations present in the quasi-isotropic laminate is that the pri-
mary load-carrying ply in the quasi-isotropic laminate, the 0° ply, is dropped. This requires a larger load transfer to the continuous ±45° plies. In contrast, the dropped plies in the unidirectional laminate carry less of the applied load, and a smaller stress gradient is needed to allow for the transfer of loading from the dropped plies to the continuous plies. As more load is required to be transferred to the continuous plies in the quasi-isotropic laminate, larger stress concentrations arising from larger stress gradients are expected to occur compared to the unidirectional laminate.

The size of the region in which the stresses are magnified is on the order of a ply thickness. Results for $\sigma_{11}$ in all plies for both the unidirectional and quasi-isotropic laminates are presented in Figures 5.10 and 5.11. The four plies in Region A are labeled from 1 to 4 in order of distance from the laminate midplane, where Ply 1 corresponds to the outermost ply, and Ply 4 corresponds to the ply closest to the midplane. These definitions are taken from the general ply and interface numbering illustrated in Figure 4.3.

From these results, it is observed that the only ply that experiences a sharp positive gradient in the $\sigma_{11}$ stresses is Ply 3 for both laminates. The stresses in all other plies tend to either gradually rise or decay in order to fulfill the free-edge conditions in terminated plies or the far-field stresses in continuous plies. These results indicate that the ply closest to the free surface in Region B carries the bulk of the stresses that are transferred from the dropped plies.

In the uniaxial configuration, Ply 4 carries the majority of the load in the laminate away from the dropoff in Region A. In this configuration, all plies have a far-field stress equal to CLPT, defined in this work as $\sigma_o$. However, plies 1 and 4 decay moving away from the dropoff more slowly than plies 2 and 3. The stresses in plies 1 and 4 come within 1% of far-field values at a distance of 4.5 laminate thicknesses away from the dropoff. In contrast, the stresses in plies 2 and 3 come within 1% of far-field values at a distance of 3.7 laminate thicknesses away from the dropoff.

Plies 1 and 4 are subject to similar boundary conditions within the model. The presence of the free edge on the outer surface of the laminate requires that $\sigma_{13}$ equal zero on the outer surface. Similarly, symmetry conditions require that $\sigma_{13}$ equal zero.
on the inner interface of Ply 4. As the variation of $\sigma_{13}$ is tied to the variation of $\sigma_{11}$ due to considerations of differential equilibrium, the boundary conditions on $\sigma_{13}$ will have an effect on the distribution of $\sigma_{11}$. As the stress distributions in plies 1 and 4 are governed by similar boundary conditions, the stress distributions, especially in regard to the decay rates of the stresses, should have some degree of similarity. The reason that greater values of $\sigma_{11}$ are found in Ply 4 away from the dropoff is a consequence of the slow decay rates in Ply 4. These slow decay rates are due to the presence of the laminate midplane, on which boundary conditions related to laminate symmetry are imposed. The boundary conditions enforce distributions on $\sigma_{13}$ through the thickness of Ply 4, which affects the distribution and decay rates of $\sigma_{11}$ along the length of Ply 4. The elevated stresses at the dropoff coupled with the slow decay within the ply allows for Ply 4 to carry most of the load in the laminate at distances greater than two ply thicknesses away from the dropoff in Region A. This effect is likely numerical and a consequence of Ply 4 being effected by the boundary conditions enforced at the laminate midplane.

In addition, there is a small decay just before the dropoff in Ply 4 present in both the uniaxial and quasi-isotropic configurations. In both configurations, the local minimum of this decay is located at $x_1$ equal to $-0.05t_{\text{lam}}$ with a value equal to $1.76\sigma_o$. This decay is unexpected and is likely more attributable to the numerical aspects of the formulation than being indicative of any sort of physical phenomenon. This decay comes as a result of the behavior of the other plies, particularly Ply 3. Since integral equilibrium of the $\sigma_{11}$ stresses is balanced against the applied load through the thickness, any positive variation of the stresses in one ply needs to be offset by a negative variation of the stresses in other plies in order to maintain integral equilibrium. The region before the dropoff where Ply 4 exhibits a decay in $\sigma_{11}$ correlates to the same location where Ply 3 experiences the largest gradients in $\sigma_{11}$. The reasoning behind this small decay having the same minimum value in both configurations is likely due to the fact that the far-field stresses in Plies 3 and 4 are equal to $2.0\sigma_o$ in Region B in both configurations. Thus, the behavior of Ply 4 can be attributed to the behavior of the other plies, particularly Ply 3.
Figure 5.10  Results for $\sigma_{11}$ for all plies in a $[0_{2D}/0_2]$$_s$ laminate under uniform tension.
Figure 5.11 Results for $\sigma_{11}$ for all plies in a $[[90/0]_{D}/\pm45]$$_s$ laminate under uniform tension.
5.4.2 Influence of Model Discretization on Stress Concentrations

In order to obtain a better understanding of the lengthscales involved in the regions where the stress concentrations in $\sigma_{11}$ are located, sublayering of the laminate is utilized. A sublayering scheme is applied to the uniaxial $[0_{2D}/0_2]_s$ laminate where each ply is subdivided into multiple sublayers. The sublayering scheme requires that each ply be represented by a number of sublayers with the same material properties as the parent ply. Each sublayer has its own thickness, and the sum of the thicknesses of the sublayers derived from a ply equals the parent ply thickness. Each sublayer is then treated as a ply in the formulations outlined in Chapters 4 and 5. The sublayering increases the number of terms in the solution for the variation of the stresses along the $x_1$-direction as well as allowing for greater variability of the stresses through the thickness. Results for $\sigma_{11}$ on the outer surface of Ply 3 for the uniaxial laminate with plies being represented by two and four sublayers each is presented in Figure 5.12, along with the base solution with no sublayering.

The results for $\sigma_{11}$ from all the models exhibit the same characteristics in that they all reach a maximum stress in Region A just before the dropoff. However, an observable trend among the results is that as the discretization of the laminate increases, the stresses increase throughout the outermost continuous layer. In the non-discretized case, the maximum value of $\sigma_{11}$ is equal to 2.22$\sigma_o$. The model with a discretization of two sublayers per ply has a maximum value of the stress equal to 2.59$\sigma_o$, and the model with a discretization of four sublayers per ply has a maximum value of the stress equal to 3.07$\sigma_o$. This occurs since the stresses present in each ply or sublayer are averages through a particular ply or sublayer. In the formulation of the model, it was assumed that the in-plane stresses would be constant through the thickness of a ply or sublayer. This assumption limits the variability of the stresses in the model versus the actual stress distribution that would be present in the laminate in the physical world. Whatever details the stress distribution would exhibit are “averaged out” in such a way that, through a ply or sublayer, the in-plane stresses are constant
Figure 5.12  Results for $\sigma_{11}$ in Ply 3 in a $[0_{2D}/0_s]_s$ laminate for different discretizations.
through the thickness. Thus, particular details of the actual stress distribution will be lost when assuming a through-thickness variation of the stresses.

The above effect can be limited through either a more variable assumption of the distribution of the in-plane stresses or by increasing the level of discretization in the model by moving from plies to sublayers of plies. The sublayering method allows the stresses to be averaged over a smaller thickness than in the case of the non-discretized laminate. This allows for the model to better capture the details of the through-thickness and lengthwise variations of the stress fields at the cost of increased computational runtime. In the limiting case, a model with an infinitesimal sublayer thickness would be able to give the best results possible for the model.

The averaging behavior of the model is the reason that the the $\sigma_{11}$ stresses in the model increase on the outermost continuous ply or layer as the level of discretization increases. Increasing levels of discretization lead to the length in the thickness direction over which stresses are averaged to decrease. At the outermost layer, decreasing the averaging length causes the outermost layer to average more of the stress concentration in the regions near the dropoff and thereby average less of the stresses closer to the far-field values away from the dropoff. Thus, the increase in the magnitudes of the stress with increasing discretization in Figure 5.12 is a consequence of the outermost layer averaging stresses that are more associated with the stress concentrations due to the dropoff.

Sublayering also has the effect of improving the accuracy of the distribution of the stresses. This can be observed by comparing the location of the maximum stress in Ply 3. The maximum stress in each configuration is located at $x_1$ equal to $-0.02t_{\text{lam}}$ in the model without sublayering, $x_1$ equal to $-0.01t_{\text{lam}}$ in the model with two sublayers representing each ply, and $x_1$ equal to $-0.005t_{\text{lam}}$ in the model with four sublayers representing each ply. It is expected that the location of the maximum stress in Ply 3 is at the dropoff line, $x_1$ equal to 0. Thus, increasing the discretization of the model via sublayering has the effect of improving the model accuracy with regards to the overall distribution of stresses along the length of the laminate.

Sublayering also has the effect of modifying the decay rates of the stresses in the
model. This can be observed by comparing the locations where the stress in Ply 3 equals 101% of the far-field stress. For the non-sublayered model, the location of this value of the stress is at 3.71 laminate thickness away from the dropoff in Region A and 1.1 laminate thicknesses away from the dropoff in Region B. In contrast, the stresses in the model with two sublayers representing each ply are located at 3.1 laminate thicknesses away in Region A and 1.2 laminate thicknesses away in Region B. For the model representing each ply with four sublayers, the locations in ply 3 where the stresses are 101% of far-field values are 2.5 laminate thicknesses away in Region A and 1.3 laminate thicknesses away in Region B. In the uniaxial laminate configuration, discretization of the plies leads to more rapid decay of $\sigma_{11}$ in Region A and less rapid decay of $\sigma_{11}$ in Region B.

### 5.5 Discussion

The purpose of the presentation of results for the Step 1 model is not to obtain a better understanding of the capability of the the Step 1 model in estimating the stress fields in that problem. The objective of the current work is to develop an approach that will ultimately allow for the creation of a model for the estimation of the interlaminar stress fields in the problem of a transversely-loaded grooved laminate. However, intermediate results can give an indication of issues that occur in modeling the problems in the stepwise methodology and the manner in which these issues can be resolved in models of later steps.

In addition, the results from each Step can give an indication as to how to set up the model for the Step 5 problem, as well as subsequent steps. If there are features in the stress fields for a given Step that require adjustments to either the implementation of the model or the sublayering methodologies employed, these features can assist in obtaining accurate results in later models. The purpose of this section is to explain how the results obtained from the Step 1 model influence the development of the Step 2 and Step 3 models to obtain more accurate results.

The initial results for the $\sigma_{11}$, $\sigma_{13}$, and $\sigma_{33}$ stresses indicate that the continuous
ply closest to the dropoff experiences the highest stress concentrations present in the laminate. Although stress concentrations may arise in some stresses close to the laminate midplane due to boundary condition considerations, the largest magnitude of the stresses occurs in the outermost continuous ply in Region A. The region of this stress concentration is on the order of a ply thickness away from the dropoff, with increasing sublayering indicating that the region is on the order of a fraction of the ply thickness in both the $x_1$- and $x_3$-directions. In addition, the stress decays appear to be exponential in $x_1$ along the length of the laminate, and $x_3$ within the thickness of the laminate. This is consistent with the literature on the work of interlaminar stresses and stress concentrations in the presence of free surfaces [7, 23, 33, 35].

Validation indicates that accurate stress distributions are obtained using the current model. However, there is some disagreement on the magnitudes of the stresses, particularly those in the regions near the dropoff where stress concentrations develop. Although the focus of this work is to develop a model that efficiently captures the behavior of the stresses for a large number of laminates, applying a sublayering scheme to the laminate can result in a more accurate solution. Although the sublayering schemes result in longer runtimes for the model, the results coming from the improved accuracy can give a better indication of the behavior of the stresses.

Three main points can be made from the results presented. First, stress concentrations will develop in regions close to free surfaces, which, in the current model, corresponds to dropped plies. Second, these stress concentrations will decay at an exponential rate moving away from the free surfaces. Third, sublayering can assist in observing features of stress fields by reducing the effect of how stresses are averaged within a ply or sublayer. Based upon these three points, it would seem prudent that for plies closest to the dropped plies, a refined sublayering scheme should be used in order to accurately assess the details in the stress distribution where the stresses concentrate. These stress concentrations appear to be likely candidates for areas where delamination could occur, and obtaining a relatively accurate stress distribution, even for preliminary design, would be practical. In contrast, there is little variation of the stresses away from the dropoff, even at distances of less than a ply thickness away.
Thus, results from the next series of models could be optimized by sublayering the ply or plies closest to the free surfaces, which are the likely regions of interest in the laminate. In contrast, the plies away from the dropoff can be left without sublayering, as the variation of the stresses in these plies are small compared to the regions closer to the dropoff. This scheme provides for a balance between faster computational time and improved accuracy in the regions of interest. Such a scheme is analogous to mesh refinement in finite element analyses, where the mesh is refined in only the regions of interest to obtain a more accurate solution while keeping computational time relatively low.
Chapter 6

Step 2 Configuration:
Tensile-Loaded Laminate with a Symmetric Finite-Length Single-Ply Dropoff

In this chapter, the specific formulation derived from the general formulation in Chapter 4 as applied to Step 2 is described. Step 2 corresponds to the problem of a single ply dropoff that is now of a finite length. This is in contrast to Step 1, where the dropoff is of infinite length. The effects that the changes in Step 2 have on the formulation as compared with Step 1 are presented in the first section. Issues in the implementation of the Step 2 model and the resolution of these issues are presented in the second section. Validation is subsequently addressed via comparison of the results of the Step 2 model with finite element results. Characteristic results for the model are presented in the fourth section, followed by a closing discussion of the influence of aspects of the results within the Step 2 model on subsequent models.
6.1 Specific Formulation

The two primary differences between the Step 1 problem and the Step 2 problem are the presence of a finite-length region after the dropoff as well as a plane of symmetry in the laminate normal to the $x_1$-direction. This can be seen in Figure 3.3. In Step 1, both Regions A and B are assumed to be sufficiently long (being infinite in length from a mathematical perspective) such that far-field solutions for the stresses can be recovered in the overall solution. In Step 2, Region A is still assumed to have this property. However, Region B, where the ply has been dropped, is now defined such that it is not sufficiently long in order to recover far-field stresses. Only two regions need to be defined in order to analyze the entire laminate, as in Step 1. Modeling the undropped region as Region A and half of the dropped region as Region B is sufficient to obtain results for the entire laminate due the symmetry of the laminate about $x_1$ and the laminate midplane ($x_3$-axis). This configuration is shown in Figure 6.1.

Although the changes in the length of the dropped region and the presence of an additional plane of symmetry change the formulation, it does not require a change in the assumed stress shapes for $H_{\text{out}}^i(x_3)$ and $H_{\text{in}}^i(x_3)$ as used in Step 1. This produces a result that satisfies the stress-based boundary conditions at the free surface of terminated plies. Thus, the stress shapes assumed in Step 1 as Equations (5.2a-e) and (5.3a-e) can be assumed for the Step 2 problem without any loss in accuracy.

The primary change resulting from the finite length for Region B occurs in the limits of integration when calculating the laminate complementary energy from the complementary energy density as expressed in Equations (4.20) and (4.31) for the general formulation. In the former equation, the integration with respect to $x_1$ is applied only on the then-unknown set of functional variables $F_i(x_1)$. Direct integration of the functional variables is impossible as the functional variables are, at that point in the formulation, still an unknown function of $x_1$. However, the limits of integration do not influence the formulation of the problem at this point. This is due to the aspects of the minimization performed on the integral of energy in order to obtain the eigenvalue problem in Equation (4.22). In order to perform the minimization of

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the complementary energy, partial derivatives with respect to the functions \( F_i(x_1) \) need to be taken and set to zero. This requires that the integral, taken with respect to the direction \( x_1 \), of these partial derivatives be equal to zero. In order for this to be true in the general sense, the partial derivatives of the complementary energy must equal zero independent of the integration in \( x_1 \). Thus, the limits of integration do not factor in the formulation in Equation (4.20), and thus the eigenvalue problem remains unchanged from Step 1 to Step 2.

The integration along the length direction, \( x_1 \), in Equation (4.31) is defined in terms of known stress functions, and the result of that integration is shown in Equations (4.34a) and (4.34b). In the latter of these equations, there are exponential terms dependent on \( x_{\text{left}}^B \) and \( x_{\text{right}}^B \). These terms correspond to both the left and right geometric boundaries of region B in \( x_1 \) as well as the limits of integration of the Complementary Energy Density Integrals in Equations (4.20) and (4.31) for Region B in the \( x_1 \)-direction. In particular, the definition of \( \Omega^B \), which gives the expressions required to minimize the Complementary Energy in Region B, will change in value moving from Step 1 to Step 2. For reference, the definition of \( \Omega^B \), taken from Chapter 4, is as follows:
\[ \Omega_{ij}^B = \frac{2(e^{(\lambda_1^B + \lambda_2^B) \cdot (x_{left}^B - x_{right}^B)} - 1)}{\lambda_1^B + \lambda_2^B} \]

\[ * \sum_{AllPlies} \int_{0}^{t_{pli}/2} S_{3333}^* (D^2 \Phi h_{33})^T (D^2 \Phi h_{33}) \]

\[ + S_{1313}^* (D \Phi h_{13})^T (D^2 \Phi h_{33}) \]

\[ + S_{2323}^* (D \Psi h_{33})^T (D \Psi h_{33}) \]

\[ + S_{1133}^* ((\Phi h_{11})^T (D^2 \Phi h_{33}) + (D^2 \Phi h_{33})^T (\Phi h_{11})) \]

\[ + S_{1323}^* ((D \Phi h_{13})^T (D \Psi h_{23}) + (D \Psi h_{23})^T (D \Phi h_{13})) \]

\[ + S_{1333}^* ((\Phi h_{12})^T (D^2 \Phi h_{33}) + (D^2 \Phi h_{33})^T (\Phi h_{12})) \]

\[ + S_{2323}^* (D \Psi h_{23})^T (D \Psi h_{23}) \]

\[ + S_{1111}^* (\Phi h_{11})^T (\Phi h_{11}) \]

\[ + S_{1112}^* ((\Phi h_{11})^T (\Psi h_{12}) + (\Psi h_{12})^T (\Phi h_{11})) \]

\[ + S_{1212}^* (\Psi h_{12})^T (\Psi h_{12}) dx_{3ij}^{1B} \]

The initial term of this expression is dependent on the boundaries of Region B in \( x_1, x_{left}^B \) and \( x_{right}^B \). The location of the dropoff is defined as \( x_1 \) equal to zero for Steps 1 and 2. As this dropoff coincides with the left boundary of the region, \( x_{left}^B \) is equal to 0. In Step 1, Region B is of infinite length, and thus, \( x_{right}^B \) is equal to infinity in Step 1. Substituting these values into Equation (4.34b), the value of the initial quantity in \( \Omega^B \) is equal to:

\[ \frac{2(-1)}{\lambda_1^B + \lambda_2^B} \]

In Step 2, the value of \( x_{right}^B \) is no longer infinite, but equal to some finite value equal to the half-length of Region B. This is in contrast to the value of infinity in Step 1 due to Region B being of infinite length. Substituting the finite value of \( x_{right}^B \) in the initial term in Equation (4.34b) for the Step 2 problem results in this term becoming:
For the Step 1 model, the value of $x_{\text{right}}^B$ is equal to infinity, as Step 1 models a dropoff of infinite length. This change from Step 1 is required for the model to obtain results for the Step 2 problem.

The inclusion of another plane of symmetry at the right end of Region B, which corresponds to $x_1$ equal to $x_{\text{right}}^B$, adds an additional set of boundary conditions to the overall formulation. A free-body diagram of a portion of the laminate is displayed in Figure 6.1. This planar portion of the laminate is cut through the $x_1$-$x_2$-plane at an arbitrary $x_3$-location to apply force balance. By considering the force balance in the $x_1$-direction, it can be shown that the integral of the interlaminar shear stress, $\sigma_{13}$, along the length of the laminate must equal zero. An additional constraint on this stress is that $\sigma_{13}$ must be continuous along the $x_1$-direction of the laminate at every location.

The satisfaction of both of these conditions requires one of two possible sets of constraints. The first is that $\sigma_{13}$ equals zero everywhere along the length of the laminate. As the variations of the interlaminar shear stress and interlaminar normal stress are related via the equations of differential equilibrium, this constraint would also require that $\sigma_{33}$ be equal to zero everywhere. By similar arguments invoking differential equilibrium constraints in the in-plane and interlaminar shear stresses, $\sigma_{11}$ would be constant throughout the laminate. This solution corresponds to the CLPT solution, which is inadequate to calculate the stress distributions in proximity to the dropoff.

The alternate means of enforcing the symmetry constraint is that $\sigma_{13}$ be antisymmetric in the $x_1$-direction of the laminate. This antisymmetric condition requires that:

$$\sigma_{13}(x_1 = x_{\text{right}}^B, x_3) = 0$$

at the plane of symmetry for all locations through the thickness, i.e., at all values of
Figure 6.1 Illustration of force equilibrium in the $x_1$-direction for Step 2.
This constraint can be written as a Lagrangian constraint as the free-surface and stress continuity constraints were in Equations (4.31) in the general formulation:

\[
\rho_{13}^{B_i}(\sigma_{13}(x_1 = x_{\text{right}}^B, x_3)) = 0
\]  
(6.2)

The variable \(\rho_{13}^{B_i}\) in Equation (6.2) indicates a Lagrangian multiplier acting as a constraint on the values of \(\sigma_{13}\). The superscript "B" on the Lagrangian multiplier indicates a constraint equation being enforced in the ith ply or sublayer in Region B.

In order to ensure that this condition is satisfied at all locations in \(x_3\) through the thickness of a ply, the constraint must be imposed at two points per ply or sublayer. The reasoning for this comes from the fact that \(\sigma_{13}\) is linear in \(x_3\). As a consequence of this, the stress \(\sigma_{13}\) has two degrees of variability through each ply or sublayer. By enforcing that \(\sigma_{13}\) equal to zero at two points through the thickness of a ply or sublayer, the value of \(\sigma_{13}\) will equal zero everywhere through the thickness. This ensures that the value of this stress is equal to zero through the thickness at \(x_1\) equal to \(x_{\text{right}}^B\) for a given ply. As \(\sigma_{13}\) is required to equal zero at a number of points equal to two times the number of plies or sublayers in Region B, this implies that a number of constraint equations equal to \((2 \cdot n_i^B)\) are required to satisfy the symmetry conditions in \(x_1\).

A number of these equations are redundant due to the fact that there are additional constraints on the variability of \(\sigma_{13}\) through the thickness of the laminate outside of the constraints due to symmetry imposed on \(\sigma_{13}\). The first constraint is that \(\sigma_{13}\) must equal zero on the outer surface of the laminate, as the outer surface of the laminate corresponds to a free surface. The second constraint is that \(\sigma_{13}\) must equal zero on the laminate midplane so as to satisfy both symmetry conditions and stress continuity through the thickness. A third set of constraints requires that \(\sigma_{13}\) be continuous through the thickness of the laminate from ply to ply. As there are \((n_i^B - 1)\) interfaces on which this constraint is enforced, stress continuity through the thickness imposes a number of constraints equal to \((n_i^B - 1)\) on the overall distribution of \(\sigma_{13}\) through the thickness. In total, there are \((n_i^B + 1)\) total constraints on the through-thickness

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variation of $\sigma_{13}$. These constraints reduce the overall number of equations required to enforce the symmetry boundary condition on $\sigma_{13}$.

In total, the $(2*n_r^B)$ constraint equations required to enforce the symmetry boundary condition are reduced by a total of $(n_r^B + 1)$ equations. Thus, only $(n_r^B - 1)$ equations are required to enforce the symmetry boundary condition. The excess $(n_r^B + 1)$ equations are redundant and can be defined as a linear combination of the $(n_r^B - 1)$ unique equations used to enforce the symmetry boundary condition and the set of $(n_r^B + 1)$ constraints on $\sigma_{13}$ outlined in the previous paragraph.

The symmetry boundary condition in Region B also effects the interlaminar shear stress $\sigma_{23}$ in the exact same fashion in that $\sigma_{23}$ must equal zero at the plane of $x_1$-symmetry. The exact same arguments used to define the $(n_r^B - 1)$ constraint equations for $\sigma_{13}$ are equally applicable to $\sigma_{23}$. An additional set of Lagrangian constraints is developed in order to satisfy symmetry of $\sigma_{23}$:

$$\rho_{23}^B \left( \sigma_{23}(x_1 = x_{right}^B, x_3) \right) = 0 \quad (6.3)$$

These Lagrangian constraints are added to the final general expression for complementary energy of Equation (4.31) to obtain the the laminate complementary energy for the Step 2 solution:
\[ \pi_C^* = \sum_{\text{Region } A} \int_{V_i} \sigma_{ijk\text{comp}}^{T^*} S_{ijkl\text{comp}}^{T^*} \sigma_{kl}^* dV \]
\[ + \rho_{11}^A (\sigma_{11\text{comp}}^A (x = 0, z) + \sigma_{11\text{farfield}}^A) \]
\[ + \rho_{12}^A (\sigma_{12\text{comp}}^A (x = 0, z) + \sigma_{12\text{farfield}}^A) \]
\[ + \rho_{13}^A (\sigma_{13\text{comp}}^A (x = 0, z) + \sigma_{13\text{farfield}}^A) \]
\[ + \rho_{11}^{AB} (\sigma_{11\text{comp}}^A (x = 0, z) - \sigma_{11\text{comp}}^B (x = 0, z) + \sigma_{11\text{farfield}}^A - \sigma_{11\text{farfield}}^B) \]
\[ + \rho_{12}^{AB} (\sigma_{12\text{comp}}^A (x = 0, z) - \sigma_{12\text{comp}}^B (x = 0, z) + \sigma_{12\text{farfield}}^A - \sigma_{12\text{farfield}}^B) \]
\[ + \rho_{13}^{AB} (\sigma_{13\text{comp}}^A (x = 0, z) - \sigma_{13\text{comp}}^B (x = 0, z) + \sigma_{13\text{farfield}}^A - \sigma_{13\text{farfield}}^B) \]
\[ + \rho_{13}^B (\sigma_{13} (x_1 = x_{right}, x_3)) \]
\[ + \rho_{23}^B (\sigma_{23} (x_1 = x_{right}, x_3)) = 0 \]
(6.4)

Minimization of this expression yields a linear system similar to the general form of Equation (4.32):

\[
\begin{bmatrix}
\Omega^A & 0 & \Gamma_1^A T & \Gamma_2^A T & \Theta^A T & 0 \\
0 & \Omega^B & \Gamma_1^B T & \Gamma_2^B T & \Theta^B T & \Delta B^T \\
\Gamma_1^A & \Gamma_1^B & 0 & 0 & 0 & 0 \\
\Gamma_2^A & \Gamma_2^B & 0 & 0 & 0 & 0 \\
\Theta^A & \Theta^B & 0 & 0 & 0 & 0 \\
0 & \Delta B & 0 & 0 & 0 & 0 \\
0 & ZB & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
a \\ b \\ \rho_{11} \\ \rho_{12} \\ \rho_{13} \\ \rho_B \\ \rho_{23}
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
\sigma_{11}^{AFarfield} - \sigma_{11}^{BFarfield} \\
\sigma_{12}^{AFarfield} - \sigma_{12}^{BFarfield} \\
\sigma_{13}^{AFarfield} - \sigma_{13}^{BFarfield} \\
\rho_B \\
\rho_{23}
\end{bmatrix}
\]
(6.5)

where:

\[ \Delta_{ij}^B = -\lambda_j^B \phi_{ij}^B \ast e^{(-\lambda_j^B x_{right})} \]
(6.6a)
Overall, the change from the Step 1 problem to the Step 2 problem requires two changes in the formulation. The first is that the limits of integration in Region B change, as Region B is not considered to have infinite length, but instead has a finite length with finite boundaries to the region. The second is that the inclusion of a plane of symmetry in the $x_1$-direction requires that the interlaminar shear stresses equal zero at all points along this plane of symmetry.

6.2 Implementation

No changes need to be made as to how the implementation Equation (4.34b) is defined. Subroutines to generate the additional constraints corresponding to the symmetry about $x_1$ were implemented. In addition, the subroutine by which the final linear problem in Equation (6.5) is assembled was modified from the implementation of the final linear problem in Step 1, defined in Equation (4.32). This modification makes the adjustments necessary to adjust the final equation representing the Step 1 problem (Equation (4.32)) to the final equation representing the Step 2 problem (Equation (6.5)).

The presence of the aforementioned boundary conditions of symmetry are required to produce as exact a solution as possible within the framework of the general formulation presented in Chapter 4. However, the inclusion of these constraints causes the system to be ill-conditioned. The inclusion of the expression in Equations (6.6a) and (6.6b) tend to create rows in Equation (6.5) that are nearly equal to zero for all entries. These rows push the matrix expression in Equation (6.5) to being singular, and as a result, this ill-conditions the system and gives rise to numerical issues in the solution of Equation (6.5).

Although a numerical solution can be obtained for Equation (6.5), the solution can only be obtained with two to three significant digits due to the conditioning issues. The results from the ill-conditioned system have stresses with unexpectedly
large gradients along the length, particularly at the dropoff and nearby the dropoff in Region A. This comes as a result of numerical issues propagating into the solution to Equation (6.5), which, in turn, gives results with unexpectedly high gradients.

The primary objective of the current work is the development of a methodology which lends itself to the creation of a model that emphasizes efficiency and accuracy in developing solutions to the posed problems. The results from a model that implements the symmetry boundary conditions have large gradients in the results that do not appear in other works with regard to the calculation of interlaminar stresses. Thus, in order to condition the system and remove the large gradients in the solution, the symmetry conditions are not implemented. Although this will lead to results that must have some degree of error due to a lack of an exact match of stress boundary conditions on a point-by-point basis, that does not preclude validation compared with finite element results. If a model can obtain accurate results, then that model would be considered within the purview of this work.

As discussed in Chapter 2, no model exists where the results are perfectly accurate with respect to all physical considerations of laminates with dropoffs. Validation is established when the results of a model compare well to the results of an accepted model, even if differences in the results of the two models appear. As long as the stress distributions from the current model compare favorably to the results from the finite element model, the validity of the current model is established. Thus, despite the errors that do exist in the results of the Step 2 model, the implementation of the Step 2 model and the results obtained from it are considered for the purposes of validation despite being without the symmetry boundary conditions discussed in the previous section.

6.3 Validation

Validation of the implementation of the Step 2 model is done via comparison of the results from the model to the results of a finite element model performed in ABAQUS for the same problem [51]. In the previous chapter, the validity of the Step
1 model was established for a uniaxial laminate and quasi-isotropic laminate using the results obtained through the model implemented by Shim [50]. This had been validated against finite element results in the literature and produced in ABAQUS. There are only a few changes in the current implementation of Step 2 as compared to the Step 1 model, which deal with the handling of the finite length of Region B as opposed to an infinite length of this region in Step 1. Thus, validation should center on the effects of this finite region on the stresses in both Region A and Region B. For laminates with a Region B of sufficient length, CLPT values of stresses should be recovered. In such cases, the analysis from the Step 2 model must match that of the Step 1 model for that particular configuration. Thus, in order to validate the changes in implementation, comparison of results for a laminate with a Region B that is sufficiently short must be made. This length of Region B is defined such that CLPT values of stress are not recovered in the region before the plane of symmetry. In the previous chapter, results were presented indicating the locations where stresses were equal to within 1% of far-field values to indicate the decay rates of stresses in plies. For this chapter, it is defined that stresses have decayed to far-field values when the values of the stresses are less than within 1% of far-field values.

The proposed problem used for validation is that of a uniaxial laminate with two external plies symmetrically dropped, \([0_{2D}/0_2]_s\) with the dropoff location defined as \(x_1\) equal to zero. The half-length chosen for Region B is a ply thickness, \(t_{\text{ply}}\). The material properties used in both models are in Table 3.1. Based upon the criteria developed in Section 4.6, a laminate was implemented in ABAQUS with a total half-length of 3.125 mm and a width of 2.50 mm. A mesh of 9,984 elements was applied to the laminate, with mesh density increasing both at the dropoff and at the plane of symmetry. Quadratic tetrahedral continuum elements were used to model the entire laminate in the finite element analysis [51]. The Step 2 model was discretized with four sublayers representing each ply. It was found that this sublayering scheme led to both convergent values within the Step 2 model, as well as the best comparison between the current model and the finite element results. A figure of the meshing used for this configuration is given in Figure 6.2.
Figure 6.2  Meshing of finite element model for validation in Step 2: (top) meshing of total model, and (bottom) meshing around dropoff regions.
Figure 6.3  Results for $\sigma_{11}$ along the outer surface of the outermost continuous ply in a $[0_{2D}/0_{2}]_s$ laminate under uniform tension for the Step 2 problem.
Results of the two models for the longitudinal stress, $\sigma_{11}$, for the aforementioned problem are displayed in Figure 6.3. In general, the distribution of stresses for the two models compare reasonably well except in a region within a ply thickness of the dropoff in Region A. There is a more rapid decay in the stress in the Finite Element model within that region than in the stress in the Step 2 model. In contrast, there is less rapid decay in Region B for the finite element model. At a distance of a quarter of a ply thickness from the dropoff, the Finite Element model predicts a stress equal to 1.93 times the far-field value of the longitudinal stress, whereas the Step 2 model predicts a value of the stress equal to 2.95 times the far-field stress.

This discrepancy can be attributed to the limitations in ABAQUS in using quadratic elements. Although each three-dimensional quadratic element utilizes 20 nodes to input into the analysis, ABAQUS will only return results along the eight corners of the element. Thus, ABAQUS must perform a linear interpolation between nodes in order to develop the element stresses, as there are not sufficient points along the edge of an element to develop a quadratic distribution. It is believed that additional node data would allow for a piecewise quadratic distribution of the interpolation of the element stresses, which would give the results for the finite element model a more curved nature in the areas around the dropoff in Region A as shown for the Step 2 model.

In addition, issues with regard to the degree of continuity of the stresses from the finite element results give rise to the stress gradients in the results from the finite element model. Although the distribution of the stress and its first derivative can be represented as continuous using quadratic elements, the second derivatives of the stress must be discontinuous from element to element. This is due to the use of an assumed quadratic distribution of the stresses. This requires the second derivative of the distribution of the stress to be constant from element to element. In contrast, the Step 2 model assumes an exponential variation of the stress along the length. This results in a distribution with an infinite number of continuous derivatives. The discontinuous changes in the second derivative of the distribution of the stress can give rise to the stress gradients seen in the results of the finite element model. Although
finer meshing can be used to reduce the severity of the effect of having a discontinuous second derivative of the stress, this gives rise to a tradeoff between the accuracy of the finite element results and the runtime needed to obtain these results.

The maximum value of $\sigma_{11}$ predicted via the Step 2 model is equal to $3.27\sigma_0$ just before the dropoff at $x_1$ equal to $-0.002t_{\text{lam}}$, whereas the finite element model predicts a maximum value equal to $3.81\sigma_0$ at the dropoff ($x_1=0$). The current model has its stress values decay to $101\%$ of those of CLPT in Region A at $x_1/t_{\text{lam}}$ equal to $-2.6$, whereas the finite element results suggest that the values of the stress decays to $101\%$ of that of CLPT in Region A at $x_1/t_{\text{lam}}$ equal to $-1.7$. In general, the finite element model exhibits more rapid decay in the quantity of the stress compared to the Step 2 model, and indicates a maximum stresses that is $15\%$ greater than the value predicted in the Step 2 model. Despite this, there is favorable comparision in the overall distribution and variation of the stress along the length, as both models predict similar behavior of the stress in both Regions A and B. Overall, the Step 2 model predicts a longitudinal stress with slower variation along the length than the Finite Element model. At the plane of symmetry, which corresponds to $x_1/t_{\text{lam}}$ equal to $0.125$, the Step 2 model predicts a value of the stress equal to $2.19\sigma_0$, and the finite element model predicts a value of the stress equal to $3.30\sigma_0$. This indicates a slower decay of the stresses in Region B for the Finite Element model compared to the Step 2 model. This slower decay can be attributable to the larger stress magnitude of the stress concentration in proximity to the dropoff for the finite element model, which requires more stress to be transferred in order to decay to far-field values as compared to the Step 2 model. For reference, the CLPT value of $\sigma_{11}$ in Region A is equal to the applied load per unit length divided by the thickness of the laminate, $1.00\sigma_0$.

The results for $\sigma_{13}$ are displayed in Figure 6.4. The results for the current model indicate that the stress along the outermost interface of the outermost ply is equal to zero in Region B. This is a physical reality, as that interface corresponds to a free surface. Since the finite element model is displacement-based, it cannot accurately represent all stress-based boundary conditions. This is evidenced by the nonzero stress existing along the free surface in Region B for the finite element solution. In addition,
Figure 6.4  Results for $\sigma_{13}$ along the outer surface of the outermost continuous ply in a $[0_{2D}/0_{2}]_s$ laminate under uniform tension for the Step 2 problem.
there are significant gradients present in the finite element results within the order of a fifth of a ply thickness in Region A. Close to the dropoff, the values of $\sigma_{13}$ from the finite element model decay and then quickly grow at the dropoff before decaying again in Region B. Similar to the gradients seen in the results for $\sigma_{11}$, these stress gradients arise due to the element size chosen for convergence as well as the limitations in using quadratically-varying elements to mesh the laminate.

The value of $\sigma_{13}$ in the finite element model is equal to $0.29\sigma_0$ at a distance of $x_1/t_{\text{lam}}$ equal to 0.025. The value of the stress, moving along the length of the laminate, equals $0.26\sigma_0$ at a distance of $x_1/t_{\text{lam}}$ equal to 0.0125 and $0.30\sigma_0$ at the dropoff. Such behavior is unexpected for the stresses along the length of the laminate.

The current model gives a maximum value of stress at $x_1/t_{\text{lam}}$ equal to -0.027 with a value of $0.45\sigma_0$. The finite element model gives the maximum stress at $x_1/t_{\text{lam}}$ equal to -0.033 with a value of $0.30\sigma_0$. The current model has the $\sigma_{13}$ stress decay to 1% of $\sigma_0$ at $x_1/t_{\text{lam}}$ equal to -2.95, whereas the finite element results suggest that the stresses decay to 1% of $\sigma_0$ at $x_1/t_{\text{lam}}$ equal to -2.79. The values of the stress from both models in proximity to the dropoff show some differences likely due to the unexpected behavior of the stress from the finite element models in this region. Despite this, the overall distribution of the stresses compare well between the two models.

The results for $\sigma_{33}$ are displayed in Figure 6.5. As with the interlaminar shear stresses, the Step 2 model produces a stress with value of zero along the free surface of the outermost ply for the interlaminar normal stress. This satisfies the physical boundary condition imposed on the laminate configuration. The finite element results have a nonzero value for the stresses along this surface. For both models with $x_1$ equal to zero (the location of the dropoff), a maximum value of $0.69\sigma_0$ for the Step 2 model and $0.56\sigma_0$ for the finite element model is predicted. Both models predict a maximum value of stress at the dropoff, due to element size and the discontinuity of the second derivative of the stress distribution. Both models also show a sign change in $\sigma_{33}$ at $x_1/t_{\text{lam}}$ equal to $-0.32t_{\text{lam}}$. In general, the results of the two models compare well in Region A.

The discrepancies between the two models can largely be attributed to the differ-
Figure 6.5 Results for $\sigma_{33}$ along the outer surface of the outermost continuous ply in a $[0_{2D}/0_2]$$_s$ laminate under uniform tension for the Step 2 problem.
ences in formulation. The Step 2 model is a stress-based model and thus has different constraints relaxed than the finite element model, which is displacement-based. In particular, the current model cannot represent the strain continuity constraints across plies or layers, whereas the finite element model cannot enforce free-surface boundary conditions. Despite these discrepancies, the general distributions of the stresses between the two models compare favorably. Although the current model tends to suggest slower decay rates for the stresses moving away from the dropoffs in Region A and more rapid decay of the stresses in Region B, the distributions in both models appear to be the same away from the dropoff. At the dropoff, however, large gradients appear in the finite element results for \( \sigma_{11} \) and \( \sigma_{13} \). This is likely due to a combination of low-level interpolation in the elements of the analysis and the inability of the finite element model to represent the free-surface boundary conditions. Although the formulation in both models are different, the distribution in Region B of \( \sigma_{11} \) is similar for both models, which seems to suggest that, even without the symmetry conditions being implemented, the current model can still predict the stress distribution for small lengths of Region B. Favorable comparison between the results for the stress distribution of the two models indicates that the current model is valid for the prediction of the distribution of the stress fields in the Step 2 problem, particularly in regard to the interlaminar stresses.

The purpose of comparison of the results of the two models is to demonstrate similarity of the stress distributions in the problem of a laminate with a dropoff of finite length. Although results taken along different interfaces may give more insight into the distribution of the stresses throughout the laminate, such results add little to the current validation. The first reason for this is that the results from both models obey both the differential and integral expressions of equilibrium, and, as a result, the variation of the stresses along a different interface would be subject to this constraint. As both models give results that are similar in distribution along the interface of the outermost continuous ply, and both models obey equilibrium, the results for the stresses taken along different interfaces would be similar. In addition, the Step 1 model, upon which the Step 2 model was built, was validated with the results taken
from the model proposed by Shim [50]. The model from Reference [50], in turn, was also validated against finite element results using ABAQUS. As there are few changes from the Step 1 model, the Step 2 model would also validate well compared to further results from ABAQUS by means of validation to the Shim model for the Step 1 problem.

6.4 Results

Results for unidirectional laminates with finite and infinite values of the length of Region B are considered. The purpose of this is to obtain a better understanding of the effects of changing the length of the finite region. The conclusions drawn from these results affect the approach to obtain more accurate results in the models of subsequent steps. In addition, the $x_1$-symmetry is examined. Since all boundary conditions were unable to be satisfied in a numerical sense because of conditioning issues as explained in Section 6.2, an examination of the stresses, particularly the $\sigma_{13}$ stresses, can give some indication as to how much loss in accuracy results from not implementing the boundary conditions associated with this symmetry.

6.4.1 Stress Concentrations arising from Finite Length Issues

Plots of the distribution of $\sigma_{11}$ along the length of the outermost ply in a $[0_{2D}/0_2]_s$ laminate are given in Figure 6.6. The plots in the figure represent a uniaxial laminate configuration without any sublayering. Each ply in the results presented is represented by one layer. The results are provided for three finite region half-lengths corresponding to one-quarter, one-tenth, and one-hundredth of the laminate thickness in Region A. These particular regions lengths were chosen so as allow for as much difference as possible in the results so as to allow better assessment of the effects of changing the finite region length on the magnitude, decay, and distribution of the stresses present. In addition, results for Region B of infinite length, corresponding to the Step 1 model, are also provided for comparison. Results for plots along the length in Region B are only plotted up to the plane of symmetry, as the results on
Figure 6.6  Results for $\sigma_{11}$ along the outer surface of the outermost continuous ply in a $[0_{2D}/0_{2}]_{s}$ laminate under uniform tension for the Step 2 problem for various half-lengths of Region B.
the other half of the plane of symmetry follow from the current results, and applying
the appropriate symmetric transformation. The location of the dropoff is defined as $x_1$ equal to zero.

The stress distributions for $\sigma_{11}$ for all cases are similar in that they all reach the
maximum value of stress near the dropoff in region A. For the case of a Region B of
infinite length, the location of the maximum stress is at $x_1/t_{\text{lam}}$ equal to -0.010.
The maximum stress for the other cases is located at $x_1/t_{\text{lam}}$ equal to -0.013 for the
cases where the Region B half-length is equal to $t_{\text{lam}}/4$ and to $t_{\text{lam}}/10$. The maximum
stress for the case where the Region B half-length is equal to $t_{\text{lam}}/100$ is located at
$x_1/t_{\text{lam}}$ equal to -0.014.

The maximum value of stress for $\sigma_{11}$ generally increases as the half-length of the
dropped region (Region B) decreases. This is expected due to physical considera-
tions. As the finite region length distance decreases, so does the distance between
the dropoffs within each of the symmetric regions of the laminate. Moving along the
length of the laminate, as the half-length of the dropped region decreases, the distance
over which the ply and its stresses are “picked up” in the other half of the laminate
decreases. This requires that, after stresses are transferred from dropped plies to con-
tinuous plies at the dropoff, the stresses must transfer back over a shorter distance,
as additional plies are picked up soon afterwards. This pick-up of plies soon after
the dropoff has the effect of quick transfer of the stresses from the continuous plies
to the picked-up plies, which requires gradients in the stress. Thus, as the Region B
half-length decreases, the dropoffs have the effect of increasing the magnitude of the
stresses on the other dropoff in the symmetric half of the laminate. In the limiting
case where the dropoff distance becomes infinitesimal, the ply dropoff and pick-up
resembles a crack in the laminate. Although the model does not have the capability
of all the details required in modeling a crack, the stress results for laminates with
decreasing dropoff lengths indicate an increase in the values of the stress, which is
also predicted by models with capabilities to more accurately model cracks. The lo-
cation of the peak value of the stress occurs farther away from the dropoff in Region
A as the length of the dropped region decreases. This is indicative of the presence of
the dropoff in the symmetric half of the laminate influencing the stresses around the dropoff in the results considered.

The maximum values of stress for $\sigma_{11}$ are 2.22$\sigma_0$ in the infinite (Step 1) case, 2.25$\sigma_0$ in the case where the half-length is equal to $t_{\text{lam}}/4$, 2.31$\sigma_0$ in the case where the half-length is equal to $t_{\text{lam}}/10$, and 2.35$\sigma_0$ in the case where the half-length is equal to $t_{\text{lam}}/100$. The relative difference in the maximum value of $\sigma_{11}$ between the case of infinite length and the case with a Region B half-length equal to $t_{\text{lam}}/100$ is about 6%. Although not shown in Figure 6.6, all the cases represented in the figure decay to one percent of CLPT values at values of $x_i/t_{\text{lam}}$ between -3.6 and -3.7 in Region A, with cases with shorter Region B half-lengths decaying slightly more quickly (-3.67 for the case of $t_{\text{lam}}/100$ as compared to -3.71 for the infinite case). The result that decay rates are slightly larger for the cases of shorter lengths of Region B is unexpected due to the location of the maximum stress being located farther from the dropoff in cases with shorter lengths of Region B. Examination of the results indicates that the value of $\sigma_{11}$ for cases with shorter lengths of Region B is larger at distances away from the dropoff up to $x_i/t_{\text{lam}}$ equal to -2.0. It is only farther from this location where the results for cases of shorter lengths of Region B decay more quickly relative to far-field ply stress values than those for cases with longer lengths of Region B. It is unlikely in the physical behavior of the stresses that the decay rates would be smaller in one region (-2.0 < $x_i/t_{\text{lam}}$ < 0.0) compared to another region ($x_i/t_{\text{lam}}$ < -2.0). Thus, the differences in decay rates is likely due to a numerical representation of the solutions and does not represent the physical reality of the model.

Plots of the distributions of $\sigma_{13}$ in the $[0_{2D}/0_2]$ , laminate are shown in Figure 6.7. The magnified view of the stress distributions in the region corresponding to the maximum stresses, for $x_i/t_{\text{lam}}$ from -0.10 to -0.30, is presented in Figure 6.8. All of the plots compare very well in that there are few notable differences among the results. All results decay to less than one percent of CLPT $\sigma_{11}$ values at $x_i/t_{\text{lam}}$ equal to -3.0. The maximum values of the stresses are 0.260$\sigma_0$ for the infinite case, 0.265$\sigma_0$ for Region B with a half-length of $t_{\text{lam}}/4$, 0.269$\sigma_0$ for Region B with a half-length of $t_{\text{lam}}/10$, and 0.282$\sigma_0$ for Region B with a half-length of $t_{\text{lam}}/100$. The relative
Figure 6.7  Results for $\sigma_{13}$ along the outer surface of the outermost continuous ply in a $[0_{2D}/0_{2}]_s$ laminate under uniform tension for the Step 2 problem for various half-lengths of Region B.
Figure 6.8 Results from Figure 6.7 magnified in the region $x_1/t_{\text{lam}}$ from -0.10 to -0.30.
percentage difference between the cases with infinite length and with half-length equal to \( t_{\text{lam}}/100 \) in the aforementioned \( \sigma_{11} \) plots and the \( \sigma_{13} \) plots is greater in the latter case. The relative difference in the maximum value of \( \sigma_{13} \) between the infinite case and that for the \( t_{\text{lam}}/100 \) is about 8%, slightly larger than that in the case of \( \sigma_{11} \), where the difference is approximately 6%.

The distributions of \( \sigma_{33} \) for these cases are shown in Figure 6.9. A magnified view of the stress distributions in the region around the dropoff, out to \( x_1/t_{\text{lam}} \) equal to -0.10, is shown in Figure 6.10. Similar to the plots for \( \sigma_{13} \), there is little difference from plot to plot in the results. All plots decay to 0.01\( \sigma_0 \) at \( x_1/t_{\text{lam}} \) equal to -1.53, and there is a change in the sign of the stress in all plots at \( x_1/t_{\text{lam}} \) equal to -0.32. The maximum stress for the infinite case is equal to 0.260\( \sigma_0 \). For the finite-length cases, the maximum stress values are equal to 0.262\( \sigma_0 \) for Region B with a half-length of \( t_{\text{lam}}/4 \), 0.270\( \sigma_0 \) for Region B with a half-length of \( t_{\text{lam}}/10 \), and 0.317\( \sigma_0 \) for Region B with a half-length of \( t_{\text{lam}}/100 \). The relative difference between the infinite case and that with a half-length of \( t_{\text{lam}}/100 \) is equal to approximately 22%. Results for \( \sigma_{12} \) and \( \sigma_{23} \) are omitted because these stresses are equal to zero for all cases in the laminate.

In general, reducing the length of Region B has the effect of increasing the value of the stresses along the length of the laminate without changing the shape of overall distribution. The relative change in the magnitude of the stresses is largest when comparing results for the interlaminar stresses. All plots for each stress analyzed demonstrate changes in decay rates of less than 3% for all cases considered, and in the case of \( \sigma_{33} \), had the same location where the sign of the stress changed. The maximum stress values were located at the same location for all stresses except for \( \sigma_{11} \), where the location of the maximum stress was located at a slightly greater distance (a 1% relative difference among all cases considered) from the dropoff for cases with Region B of shorter length. Thus, for cases of Region B with a half-length up to 1% of a laminate thickness, the distribution of the stresses remain relatively similar compared to that of the infinite case except for \( \sigma_{11} \). Only the magnitudes of the stresses appear to be affected by the change in half-length of Region B, and the relative changes in the magnitude of the stresses are greater for the interlaminar
Figure 6.9  Results for $\sigma_{33}$ along the outer surface of the outermost continuous ply in a $[0_{2D}/0_2]_s$ laminate under uniform tension for the Step 2 problem for various half-lengths.
Figure 6.10  Results from Figure 6.9 magnified in the region $x_1/t_{\text{lam}}$ from -0.10 to 0.00.
stresses. The reasoning behind the larger gradients of the stresses in cases with shorter Region B length is due to the shorter distance over which load is transferred from dropped plies to continuous plies. As the length of the dropped region decreases, the distance in which plies and their stresses are picked up after the dropoff decreases. This leads to stresses being transferred to continuous plies at the first dropoff and then transferred quickly away from these plies as additional plies are picked up. This leads to larger gradients in the in-plane stresses. These in-plane stress gradients lead to larger magnitudes of the interlaminar stress gradients.

Plots of the distribution of $\sigma_{11}$ through the thickness, $x_3$, at the dropoff of a $[0_{2D}/0_2]_s$ laminate are given in Figure 6.11. Each ply is represented as two sublayers, and each sublayer has a thickness equal to half of the total thickness of the parent ply. This sublayering is done in order to obtain a result that better displays the variation of the stress through the thickness compared to the non-sublayered model. For the results of $\sigma_{11}$ and $\sigma_{13}$, the results for Plies 3 and 4 are shown, as the stresses in the other plies are equal to zero due to those plies corresponding to free surfaces at the dropoff. Results for Plies 1 through 4 are shown for $\sigma_{33}$. As in the previous sets of results, results are provided in the figure for finite region half-lengths corresponding to one-quarter, one-tenth, and one-hundredth of the laminate thickness in Region A, as well as for the case of an infinite length.

It is difficult to compare the distribution of $\sigma_{11}$ through the thickness, the $x_3$-direction, as that stress is piecewise-linear in that direction. Ply 4 corresponds to values of $x_3/t_{\text{lam}}$ from 0 to 0.125, and Ply 3 corresponds to values of $x_3/t_{\text{lam}}$ from 0.125 to 0.250. Thus, little distinction can be made among the results for the in-plane stresses for the Step 2 problems except by comparing the magnitudes of the stresses. In general, having a smaller finite length results in having a larger stress concentration on the outermost continuous ply. However, considering the sublayer which makes up the inner half of the outermost continuous ply, the case where the Region B half-length is equal to $t_{\text{lam}}/100$ has the smallest value of the stress $\sigma_{11}$ within this layer for all cases compared. In contrast, the values of the stress within this layer for the other three cases considered are very similar. The value of this stress in this region
Figure 6.11 Results for $\sigma_{11}$ through the thickness of a $[0_{2D}/0_2]_s$ laminate at the dropoff for various half-lengths of Region B.
for the case of $t_{\text{lam}}/100$ is equal to $1.52 \sigma_o$, whereas the value of the stresses in the other cases are equal to $1.70$ to $1.72 \sigma_o$.

The behavior of this stress shows that in moving from the case of infinite length to the case of half-length of $t_{\text{lam}}/10$ yields a slight increase in the stresses. However, moving from the case with half-length of $t_{\text{lam}}/10$ to the case with half-length of $t_{\text{lam}}/100$ yields a sharp drop in the stresses. This phenomenon can be attributed to two factors. First, the results from the previous subsection indicated that decreasing the length of Region B has the effect of increasing the stress concentration in the outermost continuous ply at the dropoff. Thus, as the length of Region B decreases, the greater the stress concentrations in the sublayers that comprise the outermost continuous ply. As a consequence, the stresses in other plies must decrease in order for the overall stress distribution to satisfy integral equilibrium conditions. Second, as the length of Region B gets smaller, the distance over which the load is transferred from the dropped plies to the continuous plies also decreases due to $x_1$-symmetry. As a consequence, the thickness through which the load is carried from the dropped region should also decrease. For laminates with a half-length of Region B on the order of a sublayer thickness or less, the load from the dropped plies is almost exclusively carried by the outermost layer, as there is not enough distance for the stresses to propagate deeper into the laminate compared to laminates with larger half-lengths of Region B. Representing each ply as more than one sublayer allows for a greater degree of variability of the stress though the thickness of each ply, as explained in Section 3.4. Representing each ply with more than one sublayer has the characteristic that the main fraction of the load is transferred primarily to the outermost sublayer of the outermost continuous ply. These results are not presented here.

There is an unexpected result in the sublayer closest to the laminate midplane. In this sublayer, the results indicate that as the length of the finite region after the dropoff decreases, the value of the stress $\sigma_{11}$ decreases in moving from the infinite case to the cases where the Region B half-length is equal to $t_{\text{lam}}/4$ and $t_{\text{lam}}/10$. Further decreasing the half-length of Region B from $t_{\text{lam}}/10$ to $t_{\text{lam}}/100$ results in an increase in the stresses in this sublayer. Also, the value of this stress in this sublayer is
greater than in the other sublayers which model the laminate except for the outermost sublayer. The value of $\sigma_{11}$ in this sublayer is equal to $1.95\sigma_o$ for the case of infinite length of Region B, $1.90\sigma_o$ for the case of half-length of Region B equal to $t_{\text{lam}}/4$, $1.88\sigma_o$ for the case of half-length of Region B equal to $t_{\text{lam}}/10$, and $1.96\sigma_o$ for the case of half-length of Region B equal to $t_{\text{lam}}/100$. The behavior of the stress concentration in the sublayer can be justified through the proximity of the sublayer to the laminate midplane. At the midplane, there is a boundary condition enforced on $\sigma_{13}$ via laminate symmetry in the $x_3$-direction. In particular, the variation of $\sigma_{13}$ through the thickness near the laminate midplane is equal and opposite to the variation of $\sigma_{11}$ along the length in this location. This relation is established through the differential equations of equilibrium outlined in Equation (4.2). Thus, the boundary condition on $\sigma_{13}$ at this location influences the variation of $\sigma_{11}$. In addition, $\sigma_{11}$ is assumed constant through the thickness of a ply or sublayer, and $\sigma_{13}$ has an assumed linear distribution through the thickness of a ply or sublayer. This limitation on the variation of the stresses has the effect of preventing certain stress variations in both $\sigma_{11}$ and $\sigma_{13}$ in both the length and thickness directions. Thus, the behavior of the stresses as indicated in the results of the Step 2 model may be different than the actual behavior of the laminate due to the limitations on the variation of the stresses required to develop the current model.

The stress concentration in $\sigma_{11}$ in the innermost sublayer thus arises due to two factors in the model. The first is the constraint that $\sigma_{13}$ must equal zero at the laminate midplane, and via the equations of differential equilibrium, this constraint will influence the distribution of $\sigma_{11}$. The second is the limitation on the distribution of $\sigma_{11}$ and $\sigma_{13}$ through each ply or sublayer through the thickness, which prevents higher-order variation of the stresses that may be needed to better capture the realities of the problem. Thus, the stress concentration in this innermost sublayer is a numerical phenomenon brought about by the assumptions and constraints of the formulation, but does not represent physical reality.

Results for $\sigma_{13}$ through the thickness at the dropoff are plotted in Figure 6.12. Plies 3 and 4 are located at the same $x_3$-locations as for $\sigma_{11}$. The results have sim-
Figure 6.12 Results for $\sigma_{13}$ through the thickness of a $[0_{2D}/0_2]_s$ laminate at the dropoff for various half-lengths of Region B.
ilar distributions from case to case. The stresses in all cases equal zero at both the laminate midplane \( (x_3 \text{ equal to 0}) \) and at the outer free edge of the continuous region \( (x_3/t_{\text{lam}} \text{ equal to 0.25}) \). The magnitudes of the stress are greatest in Ply 3, which consists of the two outermost sublayers. This corresponds to values of \( x_3/t_{\text{lam}} \) greater than 0.125 and less than 0.25. In contrast, the sublayers corresponding to Ply 4, which corresponds to values of \( x_3/t_{\text{lam}} \) less than 0.125, have smaller values. This is indicative of the shear lag mechanism which transfers the load from the dropped plies to the continuous plies. This transfer of the loading from the dropped plies gives rise to a concentration of \( \sigma_{11} \) in Ply 3, which, in turn via the relationship between \( \sigma_{11} \) and \( \sigma_{13} \) through the differential equations of equilibrium, gives rise to a stress concentration in \( \sigma_{13} \) in Ply 3.

The magnitude of the maximum values of the stress in Ply 3 through the thickness occur at the midline, corresponding to \( x_3/t_{\text{lam}} \) equal to 0.1875. The behavior of the values of the stress corresponding to the case of infinite length and the cases where the Region B has a finite half-length equal to \( t_{\text{lam}}/4 \) and to \( t_{\text{lam}}/10 \) indicates that the values of the stress at the dropoff line decrease as the length of Region B decreases. In contrast, continuing to decrease the length of Region B beyond this point leads to a larger stress concentration along the midline of Ply 3. This behavior can be explained as a combination of two effects. The first is that the location of maximum stress along the length of the outermost Ply 3 is farther from the dropoff as the length of Region B is decreased. Thus, the load is transferred from the dropped plies to the continuous plies at a distance farther from the dropoff as the length of Region B decreases. The decrease in the values in the through-thickness distribution of the stresses is due to this effect. The stresses at the dropoff are smaller in magnitude for laminates with Region B lengths that are shorter because more of the load is transferred at distances farther from the dropoff. Thus, the stress concentrations at the dropoff are smaller for these laminate configurations, as the load from the continuous plies is transferred at distances farther from the dropoff.

The second effect is that as Region B half-lengths decrease, the dropoffs from the symmetric sides of the laminate come closer together. Thus, the length over which
load can be transferred from the dropped plies to the continuous plies also decreases. As a result, the stress concentrations, particularly in $\sigma_{11}$, must increase in order to transfer the load carried by the dropped plies before the dropped plies terminate at the free surface. Thus, shortening the length of Region B will have the effect of increasing the magnitude of the stress concentrations of $\sigma_{11}$, which in turn, increases the magnitude of $\sigma_{13}$.

These two effects, due to the Region B length becoming smaller, gives rise to the variation of the value of the $\sigma_{13}$ through the thickness of Ply 3. For half-lengths of Region B larger than 75% of a ply thickness, decreasing Region B length leads to a decrease in the magnitude of $\sigma_{13}$ at the dropoff, as more of the load in the dropped plies is transferred at distances farther from the dropoff. For smaller lengths of Region B, the magnitude of $\sigma_{13}$ increases because the magnitude of the load that must be transferred from the dropped plies to the continuous plies is greater considering both dropoffs. This results in greater stress concentrations in the laminate at the dropoff.

Stress values for $\sigma_{33}$ through the thickness at the dropoff are plotted in Figure 6.13. The values corresponding to Plies 3 and 4 are the same as in the results for $\sigma_{11}$ and $\sigma_{13}$. Ply 2 corresponds to values of $x_3/t_{lam}$ from 0.25 to 0.375, and Ply 1 corresponds to values of $x_3/t_{lam}$ from 0.375 to 0.50. All plots have maximum stress values at $x_3/t_{lam}$ equal to 0.21, just before the free surface of the outermost continuous ply. The values of the stress are equal to $0.81\sigma_o$ in the case of infinite length, $0.82\sigma_o$ in the case of half-length of Region B equal to $t_{lam}/4$, $0.85\sigma_o$ in the case of half-length of Region B equal to $t_{lam}/10$, and $0.99\sigma_o$ in the case of half-length of Region B equal to $t_{lam}/100$. This trend is similar to previous cases where decreasing the length of Region B results in increases in the value of the stress concentrations in the laminate. The distributions of $\sigma_{11}$, $\sigma_{13}$, and $\sigma_{33}$ are related through the equations of differential equilibrium. For decreased lengths of Region B, stress concentrations develop in $\sigma_{11}$ in the continuous plies. Through the equations of differential equilibrium, this also tends to have stress concentrations arise in both $\sigma_{13}$ and $\sigma_{33}$.

The general trend of a decrease in finite region lengths leading to larger stress gradients and concentrations in looking at results along the length applies to the
Figure 6.13  Results for $\sigma_{33}$ through the thickness of a $[0_{2D}/0_2]_s$ laminate at the dropoff for various half-lengths of Region B.
results through the thickness. However, the interlaminar shear stress at the dropoff may not necessarily increase, as load is being transferred from the dropped plies to continuous plies at a distance farther from the dropoff. There are also issues with a stress concentration developing in the innermost sublayer in $\sigma_{11}$. However, this phenomenon is related to the numerical aspects of the formulation as well as to the assumed through-thickness distribution of the stresses, and is not related to the physical reality of the laminate.

6.4.2 Stress Results at the Plane of Symmetry

As all stress boundary conditions could not be satisfied in the implementation of the Step 2 model, there is a need to examine the effect of not satisfying these boundary conditions on the accuracy of the solution. Of particular concern is the boundary condition requiring that $\sigma_{13}$ be equal to zero everywhere along the plane of $x_1$-symmetry at $x_1$ equal to $x_{\text{right}}^B$. As described in Section 6.2, an explicit enforcement of this condition on $\sigma_{13}$ was relaxed, as enforcing the constraint caused the resulting formulation to become ill-conditioned and give erroneous results. The results from the current model, where the constraint on $\sigma_{13}$ is not enforced, better validates using the results from finite element analysis compared to the model with the constraint on $\sigma_{13}$ enforced. As a consequence of not enforcing this constraint in the current model, $\sigma_{13}$ will not be continuous across the plane of symmetry in $x_1$.

The results presented in the current subsection are examined to indicate how well the stress results obey the physical reality of the constraint on $\sigma_{13}$ without that constraint being explicitly enforced. The values of the interlaminar shear stress in Region B will equal zero through the thickness in the far-field, provided the length of Region B is sufficiently long. This is due to the laminate being loaded in tension, and far-field values of the stresses can be obtained via the results of CLPT. As results from CLPT give a value of $\sigma_{13}$ equal to zero, the far-field values of $\sigma_{13}$ are also equal to zero in the current model. Thus, as the length of Region B increases, the more likely that the value of $\sigma_{13}$ along the boundary corresponding to the plane of $x_1$-symmetry will equal zero. Thus, for a laminate with a length of Region B sufficiently long, the
constraint on $\sigma_{13}$ will be satisfied via the decay of the interlaminar stress rather than the enforcement of a boundary condition.

An examination of $\sigma_{13}$ at the plane of symmetry for the Step 2 problem may give an indication of how well the results from the Step 2 model satisfy the boundary condition at the plane of symmetry without that boundary condition being explicitly enforced. A statically-admissible stress field at the plane of $x_1$-symmetry would be equal to zero everywhere through the thickness as a result of the symmetry boundary condition. This distribution would also correspond with the Step 1 problem, where the far-field values of zero stress in $\sigma_{13}$ are obtained. Comparison of the results from the cases where Region B has a finite length against the case of infinite case can give an indication of the static-admissibility of the results from the Step 2 model. The closer the values of $\sigma_{13}$ along the plane of symmetry are to zero for the finite length cases, the greater the degree of the static admissibility of the results.

Results for $\sigma_{13}$ versus the thickness direction for the case of infinite length and for several cases of finite length are shown in Figure 6.14. Each plot in the figure is taken at the location of the plane of $x_1$-symmetry for each case. This is at a different value of $x_1$ for each case as the location of the plane of symmetry in the Step 2 problem is located at the $x_1$-boundary of Region B that does not correspond to the dropoff line at $x_1$ equal to zero. This location corresponds to the other $x_1$-boundary of Region B, which is located at $x_1$ equal to the half-length of Region B for each case. As the length of Region B changes, so do the boundaries of the region, and thus, so does the location of this plane of symmetry.

All results corresponding to cases of finite lengths of Region B indicate a stress concentration centered at the midline of Ply 3, corresponding to $x_3/t_{\text{lam}}$ equal to 0.1875. The value of the stress at this location is equal to 0.324$\sigma_o$ for the case of half-length of Region B equal to $t_{\text{lam}}/100$, 0.111$\sigma_o$ for the case of half-length of Region B equal to $t_{\text{lam}}/10$, and 0.0400$\sigma_o$ for the case of half-length of Region B equal to $t_{\text{lam}}/4$. This is in contrast to the value of zero for the case of infinite length. The value of $\sigma_{13}$ is smaller at the plane of symmetry than at the dropoff ($x_1/t_{\text{lam}}$ equal to zero) for all cases upon comparison with the results in Figure 6.12. Thus, moving along the
Figure 6.14 Results for $\sigma_{13}$ through the thickness of a $[0_{2D}/0_2]_s$ laminate at the plane of symmetry for various Region B half-lengths.
length in Region B from the dropoff to the plane of symmetry, the stresses will decay. The ratio of the magnitude of the maximum value of $\sigma_{13}$ at the plane of symmetry compared to the maximum value of the stress at the dropoff is equal to 13% in the case of half-length of Region B equal to $t_{\text{lam}}/4$, 61% in the case of half-length of Region B equal to $t_{\text{lam}}/10$, and 86% in the case of half-length of Region B equal to $t_{\text{lam}}/100$.

The results from the midline of Ply 3 indicate that the stress concentration in $\sigma_{13}$ decay quickly in moving from the dropoff along the length in Region B. More than 60% of the value of the stress is shed in moving from the dropoff along a length equal to a ply thickness for the cases considered. This result is indicative of the fact that the stresses decay quickly moving away from the dropoff, and thus, decay closer to a value of zero away from the dropoff. Although the value of the stresses do not equal zero at the plane of symmetry in any finite-length case considered, the rate at which they decay indicates that, for half-lengths of Region B on the order of one to two plies, the results resemble results that would require the stress to equal zero on the plane of symmetry. Thus, although the models do not explicitly enforce the constraints imposed by symmetry, the decay of stresses moving along the length in Region B indicate that the results decay to values close to zero for lengths of Region B on the order of one to two ply thicknesses.

There is further support on the idea that the Step 2 model produces results that are accurate with regard to the stresses equaling zero along the plane of symmetry. Away from the dropoff, the values of the stress $\sigma_{13}$ in Ply 4 is nearly zero for all cases considered. At the plane of symmetry, the maximum value of this stress in Ply 4 for each case is equal to $0.0302\sigma_o$ for the case of half-length of Region B equal to $t_{\text{lam}}/100$, $0.0509\sigma_o$ for the case of half-length of Region B equal to $t_{\text{lam}}/10$, and $0.0269\sigma_o$ for the case of half-length of Region B equal to $t_{\text{lam}}/4$. The small values of the stresses in Ply 4 at the plane of symmetry suggest that, even without explicit enforcement of the constraints imposed by symmetry, the stress distributions in Ply 4 resemble those that would come as a results of enforcing the symmetry condition.
6.5 Discussion

The changes brought about by the transition of the model from Step 1 to Step 2 involve setting a finite length to Region B as well as imposing a symmetric boundary condition within Region B. Although the former constraint was implemented, the latter was relaxed due to numerical concerns. Comparison of results with finite element models indicate that the stress distributions in Region A and Region B in the current model compare well with those in a model that can represent these constraints. Furthermore, the results in Figures 6.12 and 6.14 show that the values of the interlaminar shear stresses, the stresses on which the symmetry boundary condition is imposed, decay quickly away from the dropoff in Region B. This suggests that, for cases with a Region B half-length on the order of a ply thickness or greater, the symmetry boundary condition will have less of an effect on the values of the stresses. Furthermore, although the stress boundary condition cannot be satisfied exactly on a point-by-point basis, the integral of this stress, \( \sigma_{13} \), through the thickness will equal zero at the plane of symmetry as a consequence of force balance. Thus, even without all constraints enforced in the current model, the comparison of results with finite element results, along with the rapid decay of the interlaminar shear stresses in Region B, indicate that the model can still obtain accurate stress distributions. As the focus of this work is on the development of a model to efficiently and accurately analyze a large number of laminate configurations with regard to obtaining an understanding on the mechanisms in the context of stress distribution, the current model is appropriate for continued development.

In general, shortening the length of Region B has the effect of increasing the stresses seen in the plies closest to the free surface at the dropoff. Along the length of these plies, there were only smaller changes in the distribution of the stress. However, when looking through the thickness of the laminate at the dropoff, more noticeable changes in both the magnitude and distribution of the stresses were present, particularly in the vicinity of the laminate midplane. This appears to indicate that the depth at which load is carried from dropped plies to continuous plies at the dropoff.
is dependent on the length of the region after the dropoff, or more accurately, the
distance between locations where plies are dropped. A smaller region length means
that the plane of $x_1$-symmetry is closer to the dropoff, and as a consequence, the
dropoffs in symmetric halves of the laminate are closer. As model development pro-
gresses, special attention must be paid to the results through the thickness, especially
in dropped regions. The results of the Step 2 model indicate that changes in the
stress distribution through the thickness can occur with differing region lengths after
the dropoff. Thus, if mechanisms are to be identified with regard to changing region
lengths, special attention should be given as to the effect of changing region length on
the stress distribution through the thickness of the laminate at the dropoff and in the
dropped region. Thus, special consideration for the through-thickness distributions
are made in the development of Step 3. This is further discussed in the results of
Chapter 7.
Chapter 7

Step 3 Configuration:
Tensile-Loaded Laminate with
Symmetric Multiple-Ply Dropoffs

The specific formulation, implementation, validation, and results for the Step 3 problem as presented in Chapter 3 are described in the current chapter. The particular problem under consideration is that of a symmetric laminate with ply dropoffs located at multiple locations along the length of the laminate. The changes required in the formulation and implementation as compared with the Step 1 and 2 models are described. Validation via comparison of results from the current model with finite element results is then presented. Finally, characteristic results for the current model are presented for a number of laminates, as well as a discussion of the results of the model.

7.1 Specific Formulation

The major complication in the Step 3 model from the Step 2 model is that of the presence of "intermediate" regions that are bounded on both sides in the length direction by ply dropoffs. These intermediate regions are brought about via the presence of multiple dropoffs in the laminate. The two-region model of Steps 1 and 2 is
insufficient to represent the laminate, as there are multiple locations where plies are
dropped. As a consequence, there are multiple locations where stress concentrations
can arise. In order to be able to accurately represent these multiple locations where
stresses can concentrate, the formulation must be modified so as to be capable of
analyzing multiple regions in the laminate, particularly those bounded by dropoffs.
As per the definitions illustrated in Figure 4.1, the region that corresponds with no
plies yet dropped is referred to as Region A. As each ply dropoff is transversed along
the length of the laminate, new regions are defined in alphabetical order, with the
region after the first dropoff defined as Region B, the region after the second dropoff
defined as Region C, etc. The final region contains only the plies that are continuous
throughout the laminate. For a laminate with a total number of dropoffs equal to
\( n_{\text{drop}} \), there will be a total number of \( (n_{\text{drop}} - 1) \) intermediate regions present along
with Region A (prior to any dropoffs) and the final region with only the continuous
plies. As with Step 2, only a quarter of the laminate is considered for analysis, as the
results from this quarter define the stress fields throughout the entire laminate via
arguments of symmetry.

The stresses that are assumed for an intermediate region must allow for stress
concentrations at two dropoffs that define the lengthwise boundaries of that region.
The definition of the interfacial stress functions and the weighting functions \( H(x_3) \)
that define the ply stresses from the interface stresses are maintained from previous
Steps in the intermediate regions. Thus, the system of equations given as Equation
(4.22) are applicable for the intermediate regions:

\[
\begin{bmatrix}
A_{11} & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
F \\
G
\end{bmatrix}'''
+ \begin{bmatrix}
B_{11} & B_{12} \\
B_{12} & B_{22}
\end{bmatrix}
\begin{bmatrix}
F \\
G
\end{bmatrix}''
+ \begin{bmatrix}
C_{11} & C_{12} \\
C_{12} & C_{22}
\end{bmatrix}
\begin{bmatrix}
F \\
G
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

(4.22)

The solution for these equations applied to Regions A and B for Steps 1 and 2 have
been given as Equations (4.24a) and (4.24b):
These results continue to define the assumed stress shapes for Region A and the last region of a Step 3 laminate configuration. In order to derive stress shapes for the intermediate regions, it is noted that Equations (4.24a) and (4.24b) are a non-general solution to (4.22) and are applicable only to a laminate with two regions. The general solution of Equation (4.22) is of the form:

\[ F(x_1) = \sum_{j=1}^{3*(n_r-1)} c_j \phi_j e^{\lambda_j x_1} \]  
\[ G(x_1) = \sum_{j=1}^{3*(n_r-1)} c_j \psi_j e^{\lambda_j x_1} \]

where \( C \) is an arbitrary constant. As the homogeneous differential equations in Equation (4.22) are linear in \( F(x_1) \), \( G(x_1) \), and their derivatives, a sum of solutions that is derived from the general forms in (7.1a) and (7.1b) is also itself a solution to (4.22). This principle is utilized in order to develop stress states for intermediate regions that have stress concentrations at both of the dropoffs that make up the boundaries of the intermediate regions.

In order to develop a stress shape that allows for stress concentrations at both dropoffs in an intermediate region, two sets of exponential terms must be used. One set must peak at the left dropoff and decay with positive \( x_1 \), and one set must peak at the right dropoff and decay with negative \( x_1 \). This definition is similar to the stress shapes assumed for Regions A and B, where the stresses peak at the dropoff and decay moving away from the dropoff (moving in the negative \( x_1 \)-direction in Region A and the positive \( x_1 \)-direction in Region B.) This assumption can be written in the form:
These stress forms introduce two different sets of unknown stress coefficients, \( c_{j}^{\text{left}} \) and \( c_{j}^{\text{right}} \), with each set of coefficients corresponding to the exponential terms that peak at the left or right dropoffs, respectively. The arbitrary constants \( C_{\text{left}} \) and \( C_{\text{right}} \) are defined at \( x_{\text{left}}^{i} \) and \( x_{\text{right}}^{i} \), the left and right geometric boundaries of the \( i \)th intermediate region in the \( x_{1} \)-direction. This assumption simplifies later formulation. The assumed stress shapes in the intermediate regions can thus be written in the forms of equations (5.2a-e):

\[
\begin{align*}
\sigma_{11}^{i} &= \frac{1}{t_{i}}(F_{i-1}(x_{1}) - F_{i}(x_{1})) \quad (5.2a) \\
\sigma_{12}^{i} &= \frac{1}{t_{i}}(G_{i-1}(x_{1}) - G_{i}(x_{1})) \quad (5.2b) \\
\sigma_{13}^{i} &= F_{i}'(x_{1})\left(\frac{x_{3}^{i}}{t_{i}} + \frac{1}{2}\right) - F_{i-1}'(x_{1})\left(\frac{x_{3}^{i}}{t_{i}} - \frac{1}{2}\right) \quad (5.2c) \\
\sigma_{23}^{i} &= G_{i}'(x_{1})\left(\frac{x_{3}^{i}}{t_{i}} + \frac{1}{2}\right) - G_{i-1}'(x_{1})\left(\frac{x_{3}^{i}}{t_{i}} - \frac{1}{2}\right) \quad (5.2d) \\
\sigma_{33}^{i} &= \frac{F_{i}''(x_{1})t_{i}^{3}}{2}\left(\frac{x_{3}^{i}}{t_{i}} - \frac{1}{2}\right)^{2} - \frac{F_{i}''(x_{1})t_{i}^{3}}{2}\left(\frac{x_{3}^{i}}{t_{i}} + \frac{1}{2}\right)^{2} - \sum_{j=1}^{i-1} t_{j}t_{j}F_{j}''(x_{1}) \quad (5.2e)
\end{align*}
\]

with the lengthwise variation of the interficial stress functions for the \( k \)th intermediate region in the \( i \)th ply defined as:

\[
F_{i}(x_{1}) = \sum_{j=1}^{3(n_{r}t_{i}-1)} c_{j}^{\text{left}} \phi_{j}^{k} e^{-\lambda_{j}(x_{1}-x_{k}^{l})} + \sum_{j=1}^{3(n_{r}-1)} c_{j}^{\text{right}} \phi_{j}^{k} e^{\lambda_{j}(x_{1}-x_{i}^{k})} \quad (7.3a)
\]
The inclusion of additional regions beyond Region A and B, as well as the modifications on the stresses shapes required to allow for stress analysis in these regions, modify the manner by which the unknown stress coefficients are calculated. For the Step 1 and Step 2 problems, these unknown coefficients, defined as $a_j$ and $b_j$, are solved for by substituting the assumed stress shapes into the expression of Complementary Energy, adding Lagrange multiplier terms that allow for satisfaction of free-surface boundary conditions as well as enforcement of stress continuity across regions, and minimizing the resulting expression. This results in a linear system of equations in terms of the unknown stress coefficients and Lagrange multipliers in Equation (4.32):

\[
\begin{bmatrix}
\Omega^A & 0 & \Gamma_1^{AT} & \Gamma_2^{AT} & \Theta^A^T \\
0 & \Omega^B & \Gamma_1^{BT} & \Gamma_2^{BT} & \Theta^B^T \\
\Gamma_1^A & \Gamma_1^B & 0 & 0 & 0 \\
\Gamma_2^A & \Gamma_2^B & 0 & 0 & 0 \\
\Theta^A & \Theta^B & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
a \\
b \\
\rho_{11} \\
\rho_{12} \\
\rho_{13} \\
\end{bmatrix}
=
\begin{bmatrix}
0 \\
0 \\
\sigma_{11}^{A_{\text{Farfield}}} - \sigma_{11}^{B_{\text{Farfield}}} \\
\sigma_{12}^{A_{\text{Farfield}}} - \sigma_{12}^{B_{\text{Farfield}}} \\
\sigma_{13}^{A_{\text{Farfield}}} - \sigma_{13}^{B_{\text{Farfield}}} \\
\end{bmatrix}
\] (4.32)

The general procedure to find the unknown stress coefficients in Step 3 follows that of the procedure to find the stress coefficients in Steps 1 and 2. There are two differences in the formulation to find these stress coefficients between Step 3 and Steps 1 and 2. The first is that the assumed stress shapes for the intermediate regions, Equations (7.2a) and (7.2b), add additional terms to the total laminate complementary energy to be minimized, as well as how the stress continuity constraints are defined across region boundaries. The second is that the presence of multiple regions requires more constraint equations so that stress continuity is satisfied throughout the length of the entire laminate.

At each dropoff along the length in the quarter of the laminate analyzed, free-edge boundary conditions and stress continuity constraints must be enforced. For
plies terminated at the dropoff, the stresses in the length-direction must be equal to zero. As the total stress fields considered are defined as a sum of a far-field stress and complementary stress, these constraints can be defined in a similar form to Equations (4.26a-c). For the ith ply or sublayer along the jth dropoff:

\[
-\sigma_{11_{\text{comp}}}^{\text{LeftRegion}i} (x_1 = x_{\text{right}}^{\text{LeftRegion}i}, x_3) = \sigma_{11_{\text{far field}}}^{\text{LeftRegion}i} \tag{7.4a}
\]

\[
-\sigma_{12_{\text{comp}}}^{\text{LeftRegion}i} (x_1 = x_{\text{right}}^{\text{LeftRegion}i}, x_3) = \sigma_{12_{\text{far field}}}^{\text{LeftRegion}i} \tag{7.4b}
\]

\[
-\sigma_{13_{\text{comp}}}^{\text{LeftRegion}i} (x_1 = x_{\text{right}}^{\text{LeftRegion}i}, x_3) = \sigma_{13_{\text{far field}}}^{\text{LeftRegion}i} \tag{7.4c}
\]

where the superscript “LeftRegion” denotes the region in the negative \(x_1\)-direction of the dropoff.

In continuous plies across a dropoff, the lengthwise stress must be continuous. As dropoffs coincide with region boundaries in the model, this constraint requires that the lengthwise stresses on both regions bordering the dropoff must be equal at the dropoff. Slight reordering of this basic constraint results in the equations:

\[
-\sigma_{11_{\text{comp}}}^{\text{LeftRegion}i} (x_1 = x_{\text{right}}^{\text{LeftRegion}i}, x_3) + \sigma_{11_{\text{comp}}}^{\text{RightRegion}i} (x_1 = x_{\text{left}}^{\text{RightRegion}i}, x_3) = \sigma_{11_{\text{far field}}}^{\text{LeftRegion}i} - \sigma_{11_{\text{far field}}}^{\text{RightRegion}i} \tag{7.5a}
\]

\[
-\sigma_{12_{\text{comp}}}^{\text{LeftRegion}i} (x_1 = x_{\text{right}}^{\text{LeftRegion}i}, x_3) + \sigma_{12_{\text{comp}}}^{\text{RightRegion}i} (x_1 = x_{\text{left}}^{\text{RightRegion}i}, x_3) = \sigma_{12_{\text{far field}}}^{\text{LeftRegion}i} - \sigma_{12_{\text{far field}}}^{\text{RightRegion}i} \tag{7.5b}
\]

\[
-\sigma_{13_{\text{comp}}}^{\text{LeftRegion}i} (x_1 = x_{\text{right}}^{\text{LeftRegion}i}, x_3) + \sigma_{13_{\text{comp}}}^{\text{RightRegion}i} (x_1 = x_{\text{left}}^{\text{RightRegion}i}, x_3) = \sigma_{13_{\text{far field}}}^{\text{LeftRegion}i} - \sigma_{13_{\text{far field}}}^{\text{RightRegion}i} \tag{7.5c}
\]

where a superscript “RightRegion” denotes the region in the positive \(x_1\)-direction of the dropoff.
The above constraints are then defined as Lagrange Multiplier terms. These terms are added to the total laminate complementary energy, allowing for a minimization of complementary energy under the constraints and boundary conditions imposed by Equations (7.4a-c) and (7.5a-c). The free-surface boundary conditions in Equations (7.4a-c) are defined in Lagrangian terms as:

\[ \rho_{11}^{\text{Left Region}} (\sigma_{11}^{\text{Left Region}} (x_1 = x_{\text{right}}, x_3) + \sigma_{11}^{\text{Left Region}} \right) = 0 \]  \hspace{1cm} (7.6a)

\[ \rho_{12}^{\text{Left Region}} (\sigma_{12}^{\text{Left Region}} (x_1 = x_{\text{right}}, x_3) + \sigma_{12}^{\text{Left Region}} \right) = 0 \]  \hspace{1cm} (7.6b)

\[ \rho_{13}^{\text{Left Region}} (\sigma_{13}^{\text{Left Region}} (x_1 = x_{\text{right}}, x_3) + \sigma_{13}^{\text{Left Region}} \right) = 0 \]  \hspace{1cm} (7.6c)

and the stress continuity constraints in Equations (7.5a-c) are defined in Lagrangian terms as:

\[ \rho_{11}^{\text{Left Region}} (\sigma_{11}^{\text{Left Region}} (x_1 = x_{\text{left}}, x_3) - \sigma_{11}^{\text{Right Region}} \right) = 0 \]  \hspace{1cm} (7.7a)

\[ \rho_{12}^{\text{Left Region}} (\sigma_{12}^{\text{Left Region}} (x_1 = x_{\text{left}}, x_3) + \sigma_{12}^{\text{Right Region}} \right) = 0 \]  \hspace{1cm} (7.7b)

\[ \rho_{13}^{\text{Left Region}} (\sigma_{13}^{\text{Left Region}} (x_1 = x_{\text{left}}, x_3) + \sigma_{13}^{\text{Right Region}} \right) = 0 \]  \hspace{1cm} (7.7c)

The coefficients \( \rho_{ij} \) correspond to the Lagrangian coefficients defined for the constraints in Equations (7.4a-c) and (7.5a-c) on \( \sigma_{ij} \). One set of the Lagrangian Multiplier terms in Equations (7.6a-c) and (7.7a-c) are defined and added to the complementary energy for each dropoff present in the quarter of the laminate analyzed.

Substituting Equations (7.6a-c) and (7.7a-c) into the expression of complementary
energy yields a new expression for the complementary energy in the Step 3 problem. The following formulation assumes that the problem under consideration is a laminate with three regions, labeled Regions A, B, and C. This configuration corresponds to a laminate with two dropoffs along the quarter-laminate analyzed by the formulation. Laminate configurations with more regions and dropoffs can be modeled by the formulation through additional regions and the eigenvalues, eigenvectors, and unknown stress coefficients associated with the additional regions. However, this number of three regions was chosen so as to simplify the formulation and its presentation, as well as to allow for identification of patterns in the formulation that come about by adding additional regions and dropoffs. The complementary energy of a Step 3 configuration with three regions is equal to:

\[ \pi_C^* = \sum_{\text{Region } B} \frac{1}{2} \int_V \sigma_{ij}^{\text{comp}} S_{ijkl}^{\text{comp}} \sigma_{kl}^* dV \]

\[ + P(A, B) + P(B, C) \] (7.8a)

\[ P(\text{LeftRegion, RightRegion}) = \]

\[ \rho_{11}^L (\sigma_{11}^{\text{Left}} (x_1 = x_{\text{LeftRegion}}, x_3) + \sigma_{11}^{\text{farfield}}) \]

\[ + \rho_{12}^L (\sigma_{12}^{\text{Left}} (x_1 = x_{\text{LeftRegion}}, x_3) + \sigma_{12}^{\text{farfield}}) \]

\[ + \rho_{13}^L (\sigma_{13}^{\text{Left}} (x_1 = x_{\text{LeftRegion}}, x_3) + \sigma_{13}^{\text{farfield}}) \]

\[ + \rho_{11}^R (\sigma_{11}^{\text{Right}} (x_1 = x_{\text{RightRegion}}, x_3) - \sigma_{11}^{\text{comp}} (x_1 = x_{\text{RightRegion}}, x_3) ... \]

\[ + \sigma_{11}^{\text{farfield}} - \sigma_{11}^{\text{farfield}} \]

\[ + \rho_{12}^R (\sigma_{12}^{\text{Right}} (x_1 = x_{\text{RightRegion}}, x_3) - \sigma_{12}^{\text{comp}} (x_1 = x_{\text{RightRegion}}, x_3) ... \]

\[ + \sigma_{12}^{\text{farfield}} - \sigma_{12}^{\text{farfield}} \]

\[ + \rho_{13}^R (\sigma_{13}^{\text{Right}} (x_1 = x_{\text{RightRegion}}, x_3) - \sigma_{13}^{\text{comp}} (x_1 = x_{\text{RightRegion}}, x_3) ... \]

\[ + \sigma_{13}^{\text{farfield}} - \sigma_{13}^{\text{farfield}} \] (7.8b)

Minimumization of this expression requires taking the partial derivatives of Equa-
tions (7.8a) and (7.8b) with respect to the independent variables (the unknown stress coefficients and the Lagrange Multipliers) and setting the resultant equations equal to zero. Doing so results in a system of linear equations in terms of the stress coefficients and Lagrange Multipliers:

\[
\begin{bmatrix}
\Omega^A & \text{sym} \\
0 & \Omega^B & \text{sym} \\
0 & \Omega^{BC} & -\Omega^B & \text{sym} \\
0 & 0 & 0 & \Omega^C & \text{sym} \\
\Gamma_1^A & \Gamma_1^B & \Gamma_1^{B_{left}} & 0 & 0 & \text{sym} \\
\Gamma_2^A & \Gamma_2^B & \Gamma_2^{B_{left}} & 0 & 0 & 0 & \text{sym} \\
\Theta^A & \Theta^B_{left} & \Theta^B_{right} & 0 & 0 & 0 & 0 & \text{sym} \\
0 & -\Gamma_1^{B_{right}} & -\Gamma_1^{B_{left}} & \Gamma_1^C & 0 & 0 & 0 & 0 & \text{sym} \\
0 & -\Gamma_2^{B_{right}} & -\Gamma_2^{B_{left}} & \Gamma_2^C & 0 & 0 & 0 & 0 & 0 & \text{sym} \\
0 & -\Theta^{B_{right}} & -\Theta^{B_{left}} & \Theta^C & 0 & 0 & 0 & 0 & 0 & 0 & \text{sym}
\end{bmatrix}
\]

(7.9a)

The terms in Equation (7.9a) have the following definitions:
\[ \Omega_{ij}^{R} = \frac{2\left(e^{(\lambda_{1}^{R}+\lambda_{2}^{R})\left(x_{right}^{R}-x_{left}^{R}\right)} - 1\right)}{\lambda_{1}^{R} + \lambda_{2}^{R}} \]

\[ \times \left[ \sum_{RllPlies} \int_{0}^{t_{ply/2}} S_{3333}^{*}(D^{2}\Phi_{33})^{T}(D^{2}\Phi_{33}) \right. \\
+ S_{1313}^{*}(D\Phi_{13})^{T}(D^{2}\Phi_{33}) \\
+ S_{2323}^{*}(D\Psi_{33})^{T}(D\Psi_{33}) \\
+ S_{1133}^{*}((\Phi_{11})^{T}(D^{2}\Phi_{33}) + (D^{2}\Phi_{33})^{T}(\Phi_{11})) \\
+ S_{1333}^{*}((D\Phi_{13})^{T}(D\Psi_{23}) + (D\Psi_{23})^{T}(D\Phi_{13})) \\
+ S_{1233}^{*}((\Psi_{12})^{T}(D^{2}\Phi_{33}) + (D^{2}\Psi_{33})^{T}(\Psi_{12})) \\
+ S_{2323}^{*}(D\Psi_{23})^{T}(D\Psi_{23}) \\
+ S_{1111}^{*}(\Phi_{11})^{T}(\Phi_{11}) \\
+ S_{1112}^{*}((\Phi_{11})^{T}(\Psi_{12}) + (\Psi_{12})^{T}(\Phi_{11})) \\
+ S_{1212}^{*}(\Psi_{12})^{T}(\Psi_{12})dx_{31ij}^{R} \]

\[ \Omega_{ij}^{RC} = rC(i, j) \]

\[ \times \left[ \sum_{RllPlies} \int_{0}^{t_{ply/2}} S_{3333}^{*}(D^{2}\Phi_{33})^{T}(D^{2}\Phi_{33}) \right. \\
+ S_{1313}^{*}(D\Phi_{13})^{T}(D^{2}\Phi_{33}) \\
+ S_{2323}^{*}(D\Psi_{33})^{T}(D\Psi_{33}) \\
+ S_{1133}^{*}((\Phi_{11})^{T}(D^{2}\Phi_{33}) + (D^{2}\Phi_{33})^{T}(\Phi_{11})) \\
+ S_{1333}^{*}((D\Phi_{13})^{T}(D\Psi_{23}) + (D\Psi_{23})^{T}(D\Phi_{13})) \\
+ S_{1233}^{*}((\Psi_{12})^{T}(D^{2}\Phi_{33}) + (D^{2}\Psi_{33})^{T}(\Psi_{12})) \\
+ S_{2323}^{*}(D\Psi_{23})^{T}(D\Psi_{23}) \\
+ S_{1111}^{*}(\Phi_{11})^{T}(\Phi_{11}) \\
+ S_{1112}^{*}((\Phi_{11})^{T}(\Psi_{12}) + (\Psi_{12})^{T}(\Phi_{11})) \\
+ S_{1212}^{*}(\Psi_{12})^{T}(\Psi_{12})dx_{31ij}^{R} \]

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\[ rC(i, j) = \begin{cases} 
\frac{e^{(x^i_{\text{left}} - x^i_{\text{right}})t_i} \times (x^i_{\text{right}} - x^i_{\text{left}})}{\lambda_j - \lambda_i} & \text{if } i = j \\
\frac{-1}{t^{i}_{\text{ply}}} \times \Phi^A_{ij} & \text{otherwise} 
\end{cases} \]  
(7.9d)

\[ \Gamma^A_{1\ ij} = \begin{cases} 
-1/t^{i}_{\text{ply}} \times \Phi^A_{ij} & i = 1 \\
1/t^{i}_{\text{ply}} \times \Phi^A_{i-1\ j} - 1/t^{i}_{\text{ply}} \times \Phi^A_{ij} & 2 < i < n^A_r - 1 
\end{cases} \]  
(7.9e)

\[ \Gamma^A_{2\ ij} = \begin{cases} 
-1/t^{i}_{\text{ply}} \times \Psi^A_{ij} & i = 1 \\
1/t^{i}_{\text{ply}} \times \Psi^A_{i-1\ j} - 1/t^{i}_{\text{ply}} \times \Psi^A_{ij} & 2 < i < n^A_r - 1 
\end{cases} \]  
(7.9g)

\[ \Theta^A_{ij} = \Phi^A_{ij} \times \lambda^A_i \]  
(7.9g)

\[ \Gamma^B_{1\ ij} = \begin{cases} 
0 & i <= n^A_r - n^B_r \\
1/t^{k}_{\text{ply}} \times \Phi^B_{kj} & k = i - (n^A_r - n^B_r) = 1 \\
-1/t^{k}_{\text{ply}} \times \Phi^B_{k-1\ j} + 1/t^{k}_{\text{ply}} \times \Phi^B_{kj} & 2 < k = i - (n^A_r - n^B_r) < n^A_r 
\end{cases} \]  
(7.9f)

\[ \Gamma^B_{2\ ij} = \begin{cases} 
0 & i <= n^A_r - n^B_r \\
1/t^{k}_{\text{ply}} \times \Psi^B_{kj} & k = i - (n^A_r - n^B_r) = 1 \\
-1/t^{k}_{\text{ply}} \times \Psi^B_{k-1\ j} + 1/t^{k}_{\text{ply}} \times \Psi^B_{kj} & 2 < k = i - (n^A_r - n^B_r) < n^A_r 
\end{cases} \]  
(7.9g)

\[ \Theta^B_{ij} = \begin{cases} 
0 & i <= n^A_r - n^B_r \\
-\Phi^B_{kj} \times \lambda^B_k & 1 < k = i - (n^A_r - n^B_r) < n^A_r 
\end{cases} \]  
(7.9h)

\[ \Gamma^B_{1\ ij} = \Gamma^B_{1\ ij} \times e^{(x^i_{\text{left}} - x^i_{\text{right}})t_i} \]  
(7.9i)
Equations (7.9a-n) define the final linear problem that must be solved in order to obtain the stress fields in the Step 3 problem.

### 7.2 Implementation

The primary difference in the implementation of the Step 3 model as compared to the implementation of the Step 2 model is the manner in which Equation (7.9a) is assembled. Subroutines were developed that allow for the inclusion of additional terms from the final linear system of Step 2, Equation (6.5), that results in the final linear system of Step 3, Equation (7.9a). In addition, a modified version of the plotting subroutine used to generate figures of stress fields was developed such that the
general forms of the stresses in Equations (7.3a) and (7.3b) could be represented and accurately plotted in Mathematica. No further modifications to the implementation from the Step 2 model were required for the implementation of the Step 3 model.

7.3 Validation

Validation of the implementation of the Step 3 model is done via comparison of results from the current model and results from a finite element model in ABAQUS. The primary difference in the laminate configuration from Step 2 to Step 3 is the presence of multiple dropoff locations along the length of the laminate. For dropoffs that are sufficiently distant, the effects on the stress due to a dropoff will not affect the stresses in proximity to other dropoffs. Far-field stresses are recovered in Regions in laminate configurations with enough separation between dropoffs, and can be analyzed via the Step 1 model. For dropoffs that are sufficiently close, the effects of one dropoff influence the stresses around other dropoffs. The ability to analyze the influence of dropoffs in proximity to other dropoffs is the major difference in Step 3 compared to Steps 1 and 2. This effect forms the basis of the validation, as it is required that the current model be able to accurately analyze the influence of dropoffs on the stress distribution in proximity to other dropoffs.

The laminate configuration under consideration is that of a uniaxial $[0_D/0_D/0_2]_s$ laminate under tension with ply properties given in Table 3.1. As the purpose of this section is to assess the capacity of the current model to estimate the stress distributions in laminate configurations with dropoffs in close proximity, the two dropoffs in the $[0_D/0_D/0_2]_s$ laminate under consideration are positioned as such that far-field stresses are not recovered in the dropped regions. For this analysis, the length of Region B and the half-length of C are equal to a ply thickness. As the laminate under consideration has eight plies, this distance is equivalent to $t_{ply}$. The first ply is dropped at $x_1/t_{lam}$ equal to zero. As Region B has a length of $t_{ply}$, the second dropoff is located at $x_1/t_{lam}$ equal to 0.125. Stress results are taken along the outermost interface of the outermost ply (Ply 3.) This interface is chosen since
the largest stress concentrations and stress gradients in the results are present along this interface. This allows for the best comparison of the manner in which the stress gradients from the presence of the dropoffs give rise to the interlaminar stresses. This laminate configuration is illustrated in Figure 7.1.

Based upon the criteria developed in Section 4.6, a finite element model was implemented in ABAQUS with a laminate half-length of 5.0 mm and a width of 2.0 mm. One symmetric half of the total laminate was modeled in ABAQUS, as modeling a symmetric half with appropriate symmetry boundary conditions yielded identical results as the non-symmetric model with the benefit of being more computationally efficient. Three-dimensional continuum quadratic stress elements were used to model the laminate. Finer mesh density is imposed in Regions B and C, with elements on the order of 0.005 mm in length, as well as part of Region A. Finer mesh density is used in these regions to better capture the details of the expected stress concentrations in these regions. A total of 4,190 elements were used to mesh the half-laminate. The mesh used for this analysis is shown in Figure 7.2.

Results for the longitudinal stress $\sigma_{11}$ for both models are shown in Figure 7.3. The two results show agreement in terms of the distribution of the stresses. Both models show the stress as growing from the far-field in Region A to reach a maximum value at the second dropoff, $x_1/t_{\text{lam}}$ equal to 0.125. As in previous results, the value of $\sigma_{11}$ in the far-field of Region A is defined as $\sigma_0$. This value, for the uniaxial laminate under consideration, is also equal to the applied tensile stress on the laminate. The maximum value of the stress predicted in the current model is equal to $3.03\sigma_0$, and the maximum value of the stress predicted in the finite element model is equal to $3.87\sigma_0$. As in the validation of the Step 2 model in Section 6.3, there is a sharp gradient in the stresses in the finite element model. This occurs at $x_1/t_{\text{lam}}$ equal to -0.0375, which corresponds to a quarter of a ply thickness away from the first dropoff in Region A. This result is unexpected and is a result of the limitations of the continuum elements used in ABAQUS and their interpolation, as previously discussed in Chapter 6.

The results in the Finite Element model decay more quickly to far-field values compared to the current model. The value of $\sigma_{11}$ decays to $1.01\sigma_0$ at $x_1/t_{\text{lam}}$ equal to
Figure 7.1 Configuration of the \([0_D/0_D/0_2]_s\) laminate analyzed for validation of the Step 3 model.
Figure 7.2  Meshing of finite element model for validation in Step 3: (top) meshing of total model, and (bottom) meshing around dropoff regions.
Figure 7.3  Results for $\sigma_{11}$ along the outer surface of the outermost continuous ply in a $[0_D/0_D/0_2]$, laminate under uniform tension with Region B length and Region C half-length equal to a ply thickness.
-1.6 in the finite element model, whereas the current model decays to the same value at \( x_1/t_{lam} \) equal to -2.6.

Results for the interlaminar shear stress, \( \sigma_{13} \), are shown in Figure 7.4. The stress distributions from the results of both models show good agreement. Both models predict a maximum value of the stress after the first dropoff. The finite element model, in Region B, shows unexpected stress gradients from element to element. These gradients occur at \( x_1/t_{lam} \) equal to 0.0375 and 0.125. The latter of these locations corresponds to the location of the second dropoff in the laminate. Furthermore, the finite element results suggest a nonzero value of the stress for \( x_1/t_{lam} \) between 0.125 and 0.25, which corresponds to the free surface of Region C. This is not possible in reality. In contrast, the current model predicts the correct result of zero stresses along this length. The differences in the results appear due to differences in the formulation between the two models. In particular, the current model is based upon an assumed-stress formulation, whereas the finite element model is based on an assumed-displacement formulation. Nonzero stress values appear in Region C for the finite element results, as assumed-displacement formulations cannot satisfy all boundary conditions placed on the stresses.

The maximum value of the stress predicted by the current model is equal to \( 0.472\sigma_o \), and this value of the stress is located at \( x_1/t_{lam} \) equal to 0.021. The maximum value of the stress predicted by the finite element model is equal to \( 0.318\sigma_o \), located at \( x_1/t_{lam} \) equal to 0.051. Although the current model predicts a larger maximum value of the stresses, the location of the maximum value indicates good agreement as to the distribution of the stress between the models. The presence of larger stress concentrations in the results of the current model compared to the finite element model is due to the requirement that the integral of the stress along the interface considered must balance with the applied tensile load on the laminate. As the value of the stress in the results of the Step 3 model is nonzero along a shorter length than the finite element results, the stress concentrations must be larger along the length in the Step 3 model in order to equilibrate the applied far-field load.

Both models show good agreement in the decay rates of the stresses away from
Figure 7.4 Results for $\sigma_{13}$ along the outer surface of the outermost continuous ply in a $[0_D/0_D/0_2]_s$ laminate under uniform tension with Region B length and Region C half-length equal to a ply thickness.
the dropoffs. Both models predict that the value of $\sigma_{13}$ equals 0.01$\sigma_0$ at a location corresponding to $x_1/t_{\text{lam}}$ equal to -2.6. With the two exceptions of the stress gradients in Region B and the nonzero values of the stress predicted along the free surface in Region C in the finite element solution, the stress distributions of the two results compare favorably.

Results for $\sigma_{33}$ are shown in Figure 7.5. The stress distribution results of the two models again compare favorably. The major difference is again that the results from the finite element model indicate a nonzero stress along the free surface of Region C between $x_1/t_{\text{lam}}$ equal to 0.125 and 0.250. This does not satisfy the actual boundary condition imposed and comes as a result of the displacement-based formulation of the finite element models. Both models predict a maximum value of the stress at the second dropoff, $x_1/t_{\text{lam}}$ equal to 0.125. The current model predicts a value of the stress here of 0.750$\sigma_0$, whereas the finite element model predicts a value of 0.4649$\sigma_0$. The stresses decay more quickly in the Step 3 model than in the finite element model. The location where the value of the stress is equal to -0.01$\sigma_0$ is at $x_1/t_{\text{lam}}$ equal to -1.2 in the Step 3 model and $x_1/t_{\text{lam}}$ equal to -1.4 in the finite element model. Both models indicate a transition from tensile stress to compressive stress in Region A, although the location where the stress transition occurs is different in the two models. The Step 3 models predicts this crossover at $x_1/t_{\text{lam}}$ equal to -0.058, whereas the finite element model predicts this crossover at $x_1/t_{\text{lam}}$ equal to -0.31.

The results for the stresses along the outer interface of Ply 3 indicate good agreement in the stress distributions of the two models. Results for $\sigma_{12}$ and $\sigma_{23}$ are omitted in the current analysis, as the values of these stresses are equal to zero at all locations in the laminate for the given configuration. Results for $\sigma_{22}$ are omitted, as $\sigma_{22}$ is a linear cobination of the stresses considered in this section. As the results of the stresses in other interfaces and plies are tied to the presented results via the integral equations of equilibrium, the results for other plies and interfaces show good agreement in the distribution as well and are omitted from the analysis. Good comparison of the stress distributions between the two models indicate the results of the Step 3 model are valid and suitable for further analysis.
Figure 7.5  Results for $\sigma_{33}$ along the outer surface of the outermost continuous ply in a $[0_D/0_D/0_2]_s$ laminate under uniform tension with Region B length and Region C half-length equal to a ply thickness.
7.4 Results

Results for the stresses are presented for a number of laminate configurations for the Step 3 problem. The purpose of presenting these results is to indicate the effects of modeling laminates with multiple dropoff locations within the context of the general formulation in Chapter 4. Results for the ply stresses for increasing length between dropoffs are presented. Although additional results from Step 3 suggest additional consideration of issues in laminate modeling, these issues that arise from these additional results are not novel from the current discussion and the discussions found in Chapters 5 and 6 for the Step 1 and Step 2 models. As a result, only one set of results are presented to outline the new issues brought about via the changes in Step 3.

The first set of results presented are those for a uniaxial \([0_D/0_D/0_2]_s\) laminate with ply material properties given in Table 3.1. Region A is considered to have semi-infinite length, and Region C has a half-length equal to 1.5 times the laminate thickness. This particular value was chosen such that all cases considered have stresses decay to within 1% of far-field values in Region C. The cases considered in this configuration are those of Region B length set equal to one laminate thickness, one-half of a laminate thickness, one-quarter of a laminate thickness, and one-eighth of a laminate thickness. For all configurations considered, the location of the first dropoff is defined as \(x_1\) equal to zero. The location of the second dropoff in each case is at \(x_1/t_{lam}\) equal to 1.0 in the case of Region B length equal to \(t_{lam}\), at \(x_1/t_{lam}\) equal to 0.5 in the case of Region B length equal to \(t_{lam}/2\), at \(x_1/t_{lam}\) equal to 0.25 in the case of Region B length equal to \(t_{lam}/4\), and at \(x_1/t_{lam}\) equal to 0.125 in the case of Region B length equal to \(t_{lam}/8\). The laminate configuration considered is similar to the configuration considered in Section 7.3 (Figure 7.1), except that the values of the lengths of Regions B and C differ in the current results as compared to the configuration used for validation. In addition, the current configuration treats the length of Region B as variable, whereas the laminate configuration used for validation does not. Results are plotted for the nonzero stresses along the outermost interface of the outermost continuous ply (Ply 3.)
Figure 7.6 Results for $\sigma_{11}$ along the outer surface of the outermost continuous ply in a $[0_D/0_D/0_2]_s$ laminate under uniform tension for various Region B lengths.
Results for $\sigma_{11}$ in the aforementioned configurations are displayed in Figure 7.6. The value of the in-plane stress, for all cases considered, grows until the stress reaches a maximum value, and then decays to far-field values in Region C. The maximum value and location of the maximum value of the stress differs from case to case. The maximum value of the stress in the case of Region B length equal to $t_{\text{lam}}$ is equal to $2.46\sigma_o$, and is located at $x_1/t_{\text{lam}}$ equal to 0.98. The maximum value of the stress in the case of Region B length equal to $t_{\text{lam}}/2$ is equal to $2.12\sigma_o$, and is located at $x_1/t_{\text{lam}}$ equal to 0.68. The maximum value of the stress in the case of Region B length equal to $t_{\text{lam}}/4$ is equal to $2.33\sigma_o$, and is located at $x_1/t_{\text{lam}}$ equal to 0.44. The maximum value of the stress in the case of Region B length equal to $t_{\text{lam}}/8$ is equal to $2.43\sigma_o$, and is located at at the second dropoff in this case ($x_1/t_{\text{lam}} = 0.25$).

In general, the location of the maximum value of stress occurs earlier in the laminate as the length of Region B is decreased. This is a consequence of the shear lag mechanism in which load is transferred from dropped plies to continuous plies at each dropoff. In the case of Region B length equal to $t_{\text{lam}}$, there is enough distance between dropoffs such that the effects that each dropoff has on the stress distribution remains isolated from each other. There is no stress concentration at the first dropoff in Ply 3, as Ply 2, with results not presented here, carries most of the load from Ply 1 at the dropoff. Thus, the stress concentration occurring near the second dropoff in Region B for the case of Region B length equal to one laminate thickness comes as a consequence of the transfer of load from Ply 2, which is dropped at the second dropoff, to Ply 3, the outermost continuous ply.

As the length of Region B becomes smaller, the distance over which load can be transferred from Ply 1 to Ply 2 also becomes smaller. Upstream (in the negative $x_1$-direction) of the first dropoff, load transfer begins from Ply 1 to the other plies such that all the load from Ply 1 is transferred before the first dropoff. In a similar vein, load transfer from Ply 2 to the continuous plies begins upstream of the second dropoff. With the two dropoffs close to each other, the upstream effects that come as a result of the presence of the second dropoff occupies the same regions as the upstream effects that come as a result of the presence of the first dropoff. Thus, the
lengthscales of the effects of the presence of dropoffs in combination with shortening distance between dropoffs has the effect of both dropoffs influencing the stress fields in proximity to the first dropoff.

As the length between dropoffs decreases, the location of the maximum value of the stresses occurs closer to the first dropoff. In addition, going from the case of Region B length equal to $t_{\text{lam}}/2$ to the case of Region B equal to $t_{\text{lam}}/8$ indicates larger stress concentrations present in Ply 3. These two results are indicative of the manner in which load is transferred from Ply 1 before the first dropoff. The length of Ply 2 after the first dropoff becomes shorter for smaller Region B lengths. Thus, only partial load transfer from Ply 1 to Ply 2 can be achieved, as Ply 2 also needs to transfer its load to the continuous plies before the second dropoff. As a result, although load transfer from Ply 1 to Ply 2 occurs before the first dropoff, load transfer from Ply 2 to 3 must also occur in this region. Thus, for progressively shorter Region B lengths, Ply 1 transfers a larger quantity of the load to Ply 3 in the region before the first dropoff, as any load transferred to Ply 2 must be again transferred before the second dropoff. Thus, for cases of short Region B lengths, Ply 3 carries more of the load transferred from Ply 1 before the first dropoff. This can be observed in Figure 7.7, where the values of the stresses in Ply 2 decrease in proximity to the first dropoff as the length of Region B decreases.

The shortening of the Region B length has the effect of reducing the rate of decay of the stresses in the far-field of Region A. The stress decays to 101% of far-field values at a location of $x_1/t_{\text{lam}}$ equal to -3.2 in the case of Region B length equal to $t_{\text{lam}}$, $x_1/t_{\text{lam}}$ equal to -3.4 in the case of Region B length equal to $t_{\text{lam}}/2$, and $x_1/t_{\text{lam}}$ equal to -3.5 in the cases of Region B length equal to $t_{\text{lam}}/4$ and $t_{\text{lam}}/8$. This reduction of the decay rates in Region A is tied to the larger values of the stresses at the first dropoff. Larger values of the stress require a longer distance away from the dropoff in order for the stresses to decay to some specified value.

Results of the interlaminar shear stress $\sigma_{13}$ for the cases of variable Region B lengths are shown in Figure 7.8. All results presented indicate a stress concentration located at the first dropoff. The value of the stress at the first dropoff is equal to
Results for $\sigma_{11}$ along the outer surface of the innermost dropped ply in a $[0_D/0_D/0_2]_s$ laminate under uniform tension for various Region B lengths.

Figure 7.7
Figure 7.8 Results for $\sigma_{11}$ along the outer surface of the outermost continuous ply in a $[0_D/0_D/0_2]_s$ laminate under uniform tension for various Region B lengths.
0.176\sigma_0 in the case of Region B length equal to \(t_{\text{lam}}\), 0.225\sigma_0 in the case of Region B length equal to \(t_{\text{lam}}/2\), 0.469\sigma_0 in the case of Region B length equal to \(t_{\text{lam}}/4\), and 0.444\sigma_0 in the case of Region B length equal to \(t_{\text{lam}}/8\). Stress concentrations also occur in proximity to the second dropoff for the cases of Region B length equal to \(t_{\text{lam}}\) and \(t_{\text{lam}}/2\). The values of the stresses here are equal to 0.256\sigma_0 and 0.235\sigma_0 respectively for the two cases. This stress concentration is located at \(x_1/t_{\text{lam}}\) equal to 0.87 for the case of Region B length equal to \(t_{\text{lam}}\) and 1.23 for the case of Region B length equal to \(t_{\text{lam}}/2\).

The presence of stress concentrations at the dropoffs in the cases considered comes as a consequence of load transfer from dropped plies. Load transfer arising due to ply termination causes gradients in the in-plane stress along the length, which in turn gives rise to stress gradients through the thickness at the dropoff. Of particular interest in the results is the manner by which these stress gradients are manifested for cases of variable Region B length. In the two cases with larger Region B lengths as considered, there are two distinct stress concentrations in \(\sigma_{13}\), which correlate to locations just before the two dropoffs in these cases. However, for the cases of Region B length equal to \(t_{\text{lam}}/4\) and \(t_{\text{lam}}/8\), only one stress concentration is observable, and this stress concentration occurs at the first dropoff. The two dropoffs are in such close proximity to each other that they effect the stress distributions so that they act in a manner similar to that with only a single dropoff present.

In all cases considered, load transfer from dropped plies occurs in regions upstream of the dropoffs. This load transfer induces an in-plane stress gradient along the length of continuous plies. This gives rise to an interlaminar stress gradient through the thickness. As the regions where load transfer occurs before the first and second dropoffs overlap in cases with reduced Region B lengths, stress gradients occurring from load transfer from Ply 1 and load transfer from Ply 2 are in closer proximity to the first dropoff. Thus, the gradients in the interlaminar shear stresses that are seen in the cases with reduced Region B lengths are a sum of gradients arising from the load transfer from Ply 1 and the load transfer from Ply 2. In the extreme case, this overlap can act as a single dropoff, similar to the stress distributions seen for the cases of
Region B length equal to $t_{\text{lam}}/4$ and $t_{\text{lam}}/8$.

The smaller value of the stress concentration seen in the case of the Region B length equal to $t_{\text{lam}}/8$ compared to the case of the Region B length equal to $t_{\text{lam}}/4$ is indicative of the change in the effects on the stress distributions due to the presence of dropoffs. For the value of Region B length equal to $t_{\text{lam}}/8$, the net effect of the two dropoffs is that of a single dropoff with double the total number of plies dropped. In this configuration, the stress distribution is similar to that of a single dropoff with two plies dropped, and the quantities and distribution of the stress are similar to those of a $[0_D/0_D/0_2]_s$ laminate. The quantities of the stress are smaller for this configuration, as each dropoff does not individually affect the stress distributions in proximity to the other dropoff. This is in contrast to the results from the case where Region B length equals $t_{\text{lam}}/4$, where the influence of the effects of the two dropoffs leads to higher stress concentrations at the first dropoff.

As in the results for $\sigma_{11}$, the decay rates of $\sigma_{13}$ are tied to the value of the stress concentrations in each case. The stress decays to a value equal to $0.01\sigma_o$ at a location of $x_1/t_{\text{lam}}$ equal to -2.5 for the case of the Region B length equal to $t_{\text{lam}}$, $x_1/t_{\text{lam}}$ equal to -2.7 for the case of the Region B length equal to $t_{\text{lam}}/2$, $x_1/t_{\text{lam}}$ equal to -2.8 for the case of the Region B length equal to $t_{\text{lam}}/4$, and $x_1/t_{\text{lam}}$ equal to -2.7 for the case of the Region B length equal to $t_{\text{lam}}/8$. As observed in the results of $\sigma_{11}$, the larger the magnitude of the stress concentrations present in the ply, the greater the distance required for the stresses to decay to some specified value.

Results for the interlaminar normal stress $\sigma_{33}$ for each case are shown in Figure 7.9. All locations of the maximum absolute value of the stress are concurrent with the second dropoff in each case considered. The values of the stress at the second dropoff are equal to 0.46007 $\sigma_o$ for the case of the Region B length equal to $t_{\text{lam}}$, 0.101 $\sigma_o$ for the case of Region B length equal to $t_{\text{lam}}/2$, -0.305 $\sigma_o$ for the case of Region B length equal to $t_{\text{lam}}/4$, and 0.570 $\sigma_o$ for the case of Region B length equal to $t_{\text{lam}}/8$.

The stress distributions differ significantly from case to case. In the cases of the Region B length equal to $t_{\text{lam}}$ and $t_{\text{lam}}/8$, the stresses have a maximum positive value at the second dropoff. This is a consequence of the largest stress gradients in Ply 3
Figure 7.9 Results for $\sigma_{33}$ along the outer surface of the outermost continuous ply in a $[0_D/0_D/0_2]_s$ laminate under uniform tension for various Region B lengths.
being present at the second dropoff. In the case of the Region B length equal to $t_{\text{lam}}$, there is also a smaller stress concentration at the first dropoff due to the transfer of load from Ply 1 to the continuous plies, which gives rise to gradients in the in-plane and interlaminar shear and normal stress distributions via expressions of differential equilibrium. In the cases where Region B length equals $t_{\text{lam}}/2$ and $t_{\text{lam}}/4$, there is a decay of the value of the stresses in a region just prior to the second dropoff on the order of half of a ply thickness. In the case of the Region B length equal to $t_{\text{lam}}/4$, this decay leads to a compressive interlaminar normal stress at the second dropoff.

The decay rates of $\sigma_{33}$ are related to the maximum value of the stress in the ply. The location in the ply where the absolute value of the stress equals 1% of $\sigma_0$ is at $x_1/t_{\text{lam}}$ equal to -1.1 for the cases of the Region B length equal to $t_{\text{lam}}$, $t_{\text{lam}}/2$, and $t_{\text{lam}}/4$, and at $x_1/t_{\text{lam}}$ equal to -1.2 for the cases of the Region B length equal to $t_{\text{lam}}/8$.

The results indicate the effects that dropoff proximity has on the stresses. For cases with dropoffs sufficiently far apart (on the order of a laminate thickness or longer) the effects of one dropoff on the stresses in proximity to the other dropoff is small. The stress concentrations seen at these lengthscales fall within 3% of those of the stress concentrations seen in the cases with an infinite distance between dropoffs. For cases with dropoffs in close proximity to each other (on the order of a distance of a ply thickness), the results resemble that of stresses influenced by a single dropoff. Outside of these bounds, the presence of a dropoff influences the stress fields around the region of other dropoffs.

Results for $\sigma_{13}$ indicate that stress concentrations in the interlaminar shear stress arise in both continuous plies, Ply 3 and Ply 4. For Region B lengths of a quarter of a laminate thickness or shorter, more load is being carried by Ply 4 as a result of ply drops. Thus, for laminates with ply dropoffs in close proximity, stress concentrations in continuous plies other than the outermost continuous ply should be carefully considered.

Further results regarding the interlaminar stresses in Ply 4, as well as all other plies, are presented in order to give a better indication of the nature of the stress
concentrations in continuous plies other than the outermost continuous ply in Step 3. The laminate configuration under consideration is \([0_D/0_D/0_2]_s\), with Regions B length and Region C half-length set equal to a half of a laminate thickness. In this configuration, the two dropoffs are located at \(x_i/t_{\text{lam}}\) equal to 0 and 0.5. This configuration is similar to the configuration used in validation (Figure 7.1), except that the length of Region B and half-length of Region C is different in the current configuration. Although this laminate configuration is different than in the results previously presented, the laminate configuration under consideration is concerned with the distance between dropoffs and how changing the distance affects the magnitude and distribution of interlaminar stresses in the laminate. Thus, the results for this configuration, as well as the previous configuration with Region C half-length held constant, give indication of the influence of dropoff spacing on the stress distributions in the laminate.

Results for the interlaminar shear stress along ply interfaces are presented in Figure 7.10. In this configuration, there are stress concentrations at both dropoffs. However, the interface at which the largest concentrations are seen changes from dropoff to dropoff. At the first dropoff at \(x_i/t_{\text{lam}}\) equal to zero, the interface between Plies 2 and 3 exhibits the largest stress concentration. This interface corresponds to the outer interface of the outermost continuous ply. In previous Steps, this interface also had the largest stress gradients of all interfaces in the laminate. The maximum value of the interlaminar shear stress at this dropoff is equal to 0.226\(\sigma_o\). However, the largest stress concentration in the laminate in this configuration occurs at the interface between Plies 3 and 4 at the second dropoff \((x_i/t_{\text{lam}}\) equal to 0.5). The magnitude of the stress concentration is equal to -0.343\(\sigma_o\). This interface corresponds to the inner interface of the outermost continuous ply as well as the outer interface of the innermost continuous ply. This behavior has not been observed in the results of previous Steps.

The larger stress concentration present is due to a larger magnitude of the load being transferred at the second dropoff compared to the first dropoff. At the first dropoff, Ply 1 is dropped, reducing the number of continuous plies from 4 to 3 moving
Figure 7.10  Results for $\sigma_{13}$ along all ply interfaces in a \([0_{D}/0_{D}/0_{2}]_s\) laminate under uniform tension with Region B length and Region C half-length equal to half a laminate thickness.
in the $x_1$-direction. Thus, the load carried by Ply 1 needs to be transferred to Plies 2 through 4 at the first dropoff. At the second dropoff, Ply 2 is dropped. However, the load transferred at the second dropoff from Ply 2 is larger than the load transferred from Ply 1 at the first dropoff. This is due to the fact that, between the first and second dropoffs, Ply 2 is carrying additional load due to load transfer from Ply 1 at the first dropoff. Thus, later dropoffs require the transfer of more load to the continuous plies. This gives rise to larger gradients in the in-plane stress, which in turn, gives rise to gradients in the interlaminar stresses via the expressions for differential equilibrium.

Results for the interlaminar normal stress along ply interfaces are presented in Figure 7.11. As with the interlaminar shear stresses, there are stress concentrations occurring at both dropoffs. The magnitude of the concentration at the first dropoff is equal to $0.13994 \sigma_n$, and lies on the interface between Plies 2 and 3. The magnitude of the concentration at the second dropoff is equal to $0.583\sigma_n$, and lies on the interface between Plies 3 and 4. Large gradients in the in-plane stresses at the second dropoff lead to large gradients in the interlaminar shear stress in proximity to the dropoff. Similarly, large gradients in the interlaminar shear stress lead to larger gradients in the interlaminar normal stress in order for the stress fields to satisfy the expressions of differential equilibrium.

The results indicate that stress concentrations arise at dropoffs, but the largest stress concentrations observed in the laminate may no longer be limited to the behavior of the outermost continuous ply. These effects are tied to the ratio of the length of the dropoff region to the ply thickness. At one extreme, where there is an infinite distance separating ply dropoffs, this ratio is infinite. In these cases of larger length of the dropped region, the effects of each dropoff on the stress distribution is isolated, and the dropoffs act in a similar fashion to the isolated dropoffs seen in the Step 1 problem. At the other extreme, where there is a ratio of dropoff distance to ply thickness equal to zero or nearly zero, the dropoffs act as a single dropoff with double the number of plies dropped. For the intermediate cases, on the order a ratio between 1 to 8, the effects of each dropoff influences the stress distributions around other dropoffs without having the effect of the two dropoffs acting as a single dropoff.
Figure 7.11  Results for $\sigma_{33}$ along all ply interfaces in a $[0_D/0_D/0_2]_s$ laminate under uniform tension with Regions B and C length equal to half a ply thickness.
It is within these intermediate cases where the largest stress concentrations observed in the results may no longer occur in the outermost continuous ply or sublayer, but in other continuous plies or sublayers. The results, particularly those for the interlaminar shear and normal stresses, indicate that significant stress concentrations in Step 3 can occur in numerous continuous plies outside of the outermost continuous ply.

### 7.5 Discussion

The new issue arising from the results is that of load transfer into continuous plies for laminates with multiple dropoffs. As each ply is dropped along the length of the laminate, the load carried by that ply must be transferred to continuous plies so that the free-surface boundary condition at the dropoff can be satisfied. For Steps 1 and 2, it was found that the bulk of this load was transferred only to the outermost continuous ply. In the presented results for Step 3, results indicate that stress concentrations occur at the interface between Plies 3 and 4. This indicates that stress concentrations arise in both Ply 3, the outermost continuous ply, and Ply 4. The lengthscale factor that gives indication of the likelihood that the largest stress concentrations in continuous plies or sublayers other than the outermost ply or sublayer is the ratio of the distance between dropoffs divided by the ply thickness. In looking at the ply stress results for the interlaminar stresses, the largest stress concentrations occur in dropoffs in later regions. The large stress concentrations that occur in later dropoffs are due to the larger gradients in the stress required to transfer the load from dropped plies to continuous plies.

Particular care needs to be made in modeling the stresses in continuous plies adjacent to the outermost continuous ply in laminates with multiple dropoffs. As these continuous plies are likely locations for stress concentrations to occur within the laminate, more accurate results for continuous plies are required. This can be achieved by sublayering continuous plies outside of the outermost continuous ply. In the discussion in Section 5.5, it was recommended that, for laminates with a single dropoff, further modeling should sublayer the outermost continuous ply to order to
obtain more accurate results for the stress concentrations in this ply. For plies with multiple dropoffs, sublayering of other continuous plies should be worked so that stress concentrations at all dropoffs are accurately modeled. The sublayering of multiple continuous plies will allow for more accurate representation of the stress fields in the laminate, a better understanding of the behavior of the stresses at these multiple stress concentrations, and a better ability to assess which interfaces and locations in the laminate are likely to undergo failure.
Chapter 8

Step 4 Configuration:
Tensile-Loaded Laminate with a Groove

The specific formulation of the Step 4 model as well as the issues encountered in both the formulation and implementation are presented in this chapter. These issues prevent the completed Step 4 model from providing accurate results for the stress fields in the Step 4 problem as described in Chapter 3. The changes in formulation from Step 3 to Step 4 is presented in the first section. An examination of these issues is presented in the next section in describing the reasons that the issues in formulation prevent the use of the model in the development of an accurate solution for the magnitude and distribution of the stress fields in the problem of a laminate with slanted dropoffs under tension. These issues are further discussed in Section 8.3, and form the basis for the changes that are required to the Stepwise Methodology described in Section 3.3 to analyze the interlaminar stress fields in grooved composite laminates. These changes are detailed in Chapter 9.
Figure 8.1  Representation of the definition of the slant angle, $\omega_i$, in a dropped ply in the Step 4 Problem.
8.1 Specific Formulation

The change in the physical representation of the laminate in moving from Step 3 to Step 4 is that dropped plies are made to terminate along a surface that is not parallel to the thickness direction, $x_3$. Dropped plies and/or sublayers are terminated with certain dropoff angles such that the slanted surfaces of the dropped plies or layers construct a piecewise-linear representation of the curved groove in the general problem described in Chapter 3. This is in contrast to Steps 1 through 3, where any terminated plies or sublayers terminate along a surface that is parallel to the $x_3$-direction. It is defined that, for ply or sublayer $i$, the angle that the slanted free-surface makes with the $x_3$-axis is equal to $\omega_i$. A representation of this definition is shown as Figure 8.1.

The primary adjustment required of the formulation from Step 3 in terms of implementing slanted dropoffs is that of the enforcement of the boundary conditions along the slanted free surface of dropped plies. The assumed forms of the stresses used in Steps 1 through 3 are insufficient for this purpose. In Steps 1 through 3, ply or sublayer stresses are interpolated from the stress values at the interfaces of the ply or sublayer. The values of the interfacial stress functions used for the interpolation at a given $x_1$-location are those corresponding to that $x_1$-location. For example, in ply/sublayer $i$ bounded by interfaces $i$-$i'1$ and $i$, the stresses at a location of $x_1$ equal to $X_0$ are interpolated from the interfacial stress values of $F_{i-1}(x_1 = X_0)$ and $F_i(x_1 = X_0)$.

This interpolation presents two issues when considering plies or sublayers with slanted dropoffs. The first is that the interface terminates within that ply or sublayer at different locations in $x_1$. Thus, there is a triangular region located after one interface terminates in plies or sublayers with slanted dropoffs where the interpolation of interfacial stress functions is not possible, since an interface no longer exists in this region. This region is illustrated in Figure 8.2. The second issue with regard to the interpolation of ply stresses from Steps 1 through 3 involves variation of the stresses along the dropoff, and the manner in which the free-surface boundary conditions are
Figure 8.2  Illustration of region within a dropped ply in Step 4 in which the outer interface terminates but the inner interface remains.
implemented in the model. Based on the general formulation outlined in Chapter 4, all stress shapes are assumed to have a polynomial distribution in the thickness \((x_3)\) direction. This assumption, as well as the assumption that the stress field is functionally separate in \(x_1\) and \(x_3\), leads to the result that the ply stresses and interfacial stress functions are exponential in \(x_1\). For Steps 1 through 3, the orientation of the free surface at the dropoff is parallel to the \(x_3\)-direction. The variation of the stresses along the dropoff in Steps 1 through 3 is the same as the variation of the assumed variation of the stresses in the \(x_3\)-direction. Thus, the variation of the stresses along the dropoff is polynomial in \(x_3\) for Steps 1 through 3.

The enforcement of the free-surface boundary conditions in Steps 1 through 3 requires that the stress be set to equal zero at a number of points along the dropoff of a ply or sublayer. If the number of points where this condition is enforced is equal to the variability of the distribution of the stresses along the dropoff, then the stresses will equal zero everywhere along the dropoff. As the surface of the dropoff is parallel to the \(x_3\)-direction, this boundary condition is enforced on \(\sigma_{11}, \sigma_{12},\) and \(\sigma_{13}\). These three stresses are assumed to have a polynomial distribution through the thickness of a ply, and, thus, they have a polynomial distribution along the dropoff. As a consequence, the enforcement of the free-surface boundary conditions requires that the stresses be equal to zero at a finite number of points. By enforcing the boundary condition at a number of points equal to the degree of variability of the stress in the through-thickness direction, the boundary condition is enforced everywhere along the free surface of a ply or sublayer.

In contrast, assuming that the interpolation of the ply stresses from Steps 1 through 3 were to be used in Step 4, the variation of the stresses along the dropoff would be exponential for the Step 4 problem. This is due to the fact that the dropoff is no longer parallel to the \(x_3\)-direction as it is in Steps 1 through 3. Under the assumption of a functionally separate form for the stresses in the laminate, the variation of all stresses is exponential in the \(x_1\)-direction and polynomial in the \(x_3\)-direction. As the dropoff no longer remains parallel to the \(x_3\)-direction, the exponential variation of the stresses in the \(x_1\)-direction will appear in the variation of the stresses along the
dropoff. Thus, the variation of the stresses along a slanted dropoff will be exponential.

An exponential variation of stresses along a slanted dropoff poses an issue in that the exponential variation of the stresses cannot be enforced to equal zero everywhere along a slanted dropoff. An exponential variation can be represented as an infinite series of polynomial terms, which in turn possesses an infinite number of degrees of variability. This requires imposition of one of two set of constraints. The first set of constraints would require that the stress be forced to equal zero at an infinite number of points along the dropoff, which is impossible to do in a computational environment. The second set of contraints would require all stresses in the shaded region in Figure 8.2 to be equal to zero. This would result in stress distributions equivalent to the Step 3 problem.

These two issues prevent the use of the stress interpolations used in Steps 1 through 3 in dropped plies for the Step 4 problem. Thus, a new definition of the stress shapes are required for the analysis of Step 4. The new interpolation avoids the issues that arise in using interpolations from Steps 1 through 3 with regard to the ply interfaces terminating at different $x_1$-locations, as well as avoiding exponential variation of the stresses along the slanted dropoff. The manner in which this is accomplished is by changing which values of the interfacial stress functions interpolate to the ply stresses at a given point. In Steps 1 through 3, the ply stresses are interpolated from the values of the interfacial stress functions directly above and below, in the $x_3$-direction, a given point within a ply. Thus, a line that connects the interpolation values for a given point in the ply is parallel to the $x_3$-direction. An illustration of this “interpolation line” is shown in Figure 8.3.

The resolution of the issues involved in using an interpolation line for Step 4 requires that the interpolation line no longer be parallel to the $x_3$-direction, as it is in Steps 1 through 3. Instead, the interpolation line is angled in such a manner that it is parallel to the dropoff line with the dropoff angle, $\omega_i$, within a given ply or sublayer. This interpolation scheme requires that laminates containing plies with differing dropoff angles have differing interpolation lines from ply to ply. Thus for Step 4, the assumed stress shapes may be different from ply to ply. However, for
Figure 8.3  Illustration of definitions of local length coordinates and interpolation lines in Steps 1-3 and in Step 4.
non-dropped plies, the interpolation of the stresses will remain unchanged from Steps 1 through 3, as there is no need to change the assumed stress shapes in continuous plies, since these plies are not subject to the boundary conditions imposed by slanted dropoffs.

One additional change is made to the assumed forms of the ply stresses in terminated plies. A change in the local coordinate systems of dropped plies is introduced so as to reduce the complexity of the formulation. The local through-thickness direction, $x_3^i$, remains the same as in previous steps. The local thickness coordinate still varies from $-t_{ply}^i/2$ to $+t_{ply}^i/2$, and the coordinate varies based upon the through-thickness location of a point in question. The local length coordinate is defined as $x_1^i$. The coordinate is defined in each dropped ply such that the location at which both interfaces of the ply terminate corresponds to $x_1^i$ equal to zero. The difference between the local length coordinate, $x_1^i$, and the global length coordinate, $x_1$, is that the local length coordinate is defined such that the two interfaces of a terminated ply occur at the same value of $x_1^i$ for both interfaces. For each terminated ply, this coordinate varies from a value of 0 at the dropoff to a value of negative infinity in the far-field. Points within a ply corresponding to a single $x_1^i$-location will lie on a line slanted at an angle equivalent to the slant angle of the dropoff. The origin of this local coordinate system, where $x_1^i$ and $x_3^i$ equal zero, is located along the midpoint of the slanted free surface. These definitions are shown in Figure 8.3. The equations that map the local coordinate system used in Step 4 to the global/local coordinate system defined in Figure 4.3 are:

\[
x_1^i = x_1 - (x_3^i + t^i/2)\tan(\omega_i)
\]

\[
x_3^i(\text{Step} 4) = x_3^i(\text{Steps} 1 - 3)
\]

\[
x_3^i = t^i/2 + \frac{\sum_{j=1}^{i-1} t_{n,1-j} - x_3}{\cos(\omega_i)}
\]

where $\omega_i$ is equal to $0^o$ for all plies in Steps 1 through 3. With this change in
the interpolation of the interfacial stress functions, as well as the definition of a new coordinate system, the ply stresses in dropped plies for Step 4 retain the same form as they have in previous steps. The difference in the forms of the stresses between this step and previous steps is that the stresses are now defined in the local length coordinate, $x_1^i$:

$$\sigma_{11}^i = \frac{1}{t_i}(F_{i-1}(x_1^i) - F_i(x_1^i)) \quad (8.2a)$$

$$\sigma_{12}^i = \frac{1}{t_i}(G_{i-1}(x_1^i) - G_i(x_1^i)) \quad (8.2b)$$

$$\sigma_{13}^i = F'_i(x_1^i)(\frac{x_3^i}{t_i} + \frac{1}{2}) - F'_{i-1}(x_1^i)(\frac{x_3^i}{t_i} - \frac{1}{2}) \quad (8.2c)$$

$$\sigma_{23}^i = G'_i(x_1^i)(\frac{x_3^i}{t_i} + \frac{1}{2}) - G'_{i-1}(x_1^i)(\frac{x_3^i}{t_i} - \frac{1}{2}) \quad (8.2d)$$

$$\sigma_{33}^i = \frac{F''_{i-1}(x_1^i)t_i}{2}((\frac{x_3^i}{t_i} - \frac{1}{2})^2 - \frac{F''_i(x_1^i)t_i}{2}((\frac{x_3^i}{t_i} + \frac{1}{2})^2 - \sum_{j=1}^{i-1} t^i F''_j(x_1^i)) \quad (8.2e)$$

$$F(x_1^i) = \sum_{j=1}^{3(n_r-1)} c_j \phi_j e^{\nu_j x_1^i} \quad (8.3a)$$

$$G(x_1^i) = \sum_{j=1}^{3(n_r-1)} c_j \psi_j e^{\nu_j x_1^i} \quad (8.3b)$$

These assumed forms for the stresses in terms of the local thickness coordinate have no effect on the calculation of laminate complementary energy compared to the formulation in Steps 1 through 3. The reason for this is that the assumed shapes are used only for dropped plies, and the contribution of the stresses in the dropped ply to the overall laminate complementary energy is equivalent in Step 4 as it is in previous steps. The stress shapes in Equations (8.2a-e) are of similar form compared to the assumed stress shapes in Equations (5.2a-e), the stress shapes used for Steps 1 through 3. The difference between the two sets of stress shapes is that the former are defined in the local length and thickness coordinates. The limits of integration
Figure 8.4 Illustration of stresses normal and tangent to slanted free surfaces of dropped plies.
for plies subject to the stress shapes in Equations (8.2a–e) are the same as plies subject to the stress shapes in Equations (5.2a–e). As a result, the contribution to the complementary energy of a ply with a slanted dropoff in Step 4 is equivalent to the contribution of a ply with a flat dropoff in Steps 1 through 3. Thus, the eigenvalue problem in Equation (4.22) remains applicable to the Step 4 problem.

The difference in the current step from previous steps is how the free-surface boundary conditions are defined along the length of the dropoff. At the free surface, the stresses normal to the free surface must be equal to zero. The stresses normal to the free surface along a slanted dropoff are different from those in the dropoffs of Steps 1 through 3. In the latter case, these stresses are \( \sigma_{11}, \sigma_{13}, \) and \( \sigma_{33} \). The three stresses normal to the free surface in Step 4 are defined as \( \sigma_{nn}, \sigma_{n2}, \) and \( \sigma_{nt} \). This is illustrated in Figure 8.4. These three stresses are defined through the expressions for rotation of the stress tensor as:

\[
\sigma_{nn} = \sigma_{11}\cos^2(\omega_i) + \sigma_{33}\sin^2(\omega_i) + 2\sigma_{13}\cos(\omega_i)\sin(\omega_i) \quad (8.4a)
\]

\[
\sigma_{n2} = \sigma_{12}\cos(\omega_i) - \sigma_{23}\sin(\omega_i) \quad (8.4b)
\]

\[
\sigma_{nt} = \sigma_{13}(\cos^2(\omega_i) - \sin^2(\omega_i)) + (\sigma_{33} - \sigma_{11})\cos(\omega_i)\sin(\omega_i) \quad (8.4c)
\]

For slant angles equal to zero degrees, \( \sigma_{nn}, \sigma_{n2}, \) and \( \sigma_{nt} \) are equivalent to \( \sigma_{11}, \sigma_{12}, \) and \( \sigma_{13} \), respectively.

The stress boundary conditions at the dropoff are thus defined in terms of these stresses and the local coordinates as:

\[
\sigma_{nn}(x_1^i = 0, x_3^i) = 0 \quad (8.5a)
\]
Equations (8.4a) and (8.4c) define a quantity that is quadratic in $x_3^i$ and independent of $x_1^i$. Thus, these constraints must be enforced at three locations along the dropoff of each dropped ply in order to set $\sigma_{nn}$ and $\sigma_{nt}$ equal to zero everywhere. In contrast, Equation (8.4b) defines a quantity that is linear in $x_3^i$ and independent of $x_1^i$. Thus, the constraint must be enforced at two locations along the dropoff of each dropped ply so that $\sigma_{n2}$ is equal to zero everywhere along the dropoff.

### 8.2 Implementation and Results

The changes required in implementing the Step 4 model from the Step 3 model involve modification of the definitions of the boundary conditions at the slanted dropoffs. This involves creation of a new subroutine to modify the $\Gamma$ and $\Omega$ vectors in Equation (7.9a) to represent the free-surface boundary conditions at the dropoff. With the assumed stress shapes for terminated plies, as expressed in Equations (8.2a-e) and (8.3a-b), there is no change in the manner in which the laminate complementary energy is calculated as compared to Step 3. Thus, no further modifications need to be made to the implementation, as there are no other changes in the formulation from Step 3 to Step 4.

The results from the model as implemented are contrary to the expected distributions of the stress in the laminate. As they are inaccurate to the point of irrelevance in the context of this work, full sets of results are therefore not presented herein. However, characteristics of these results, particularly those which indicate the issues in the model, are discussed. These characteristics form the basis of the changes to the stepwise methodology as discussed in Chapter 9.

As a baseline problem to examine the issues with regard to the results, a $[0_D/0_3]_s$
Figure 8.5  Illustration of baseline configuration considered in Step 4.
laminate with ply properties as in Table 3.1 is considered. The half-length of Region B is equal to a ply thickness, or equivalently, one-eighth of the thickness of the laminate. The dropped plies are assumed to have a slant angle equal to \( \omega_1 \). This baseline configuration is shown in Figure 8.5. Results are considered for a number of values of the slant angle, \( \omega_1 \), in order to show the issues involved in adjusting the slant angle of a dropped ply.

A comparison of the results for \( \sigma_{11} \) from the cases where the slant angle of the dropped ply is equal to 0° and the case where the slant angle of the dropped ply is equal to 1° indicates severe numerical issues with the Step 4 model. The characteristics of the results for \( \sigma_{11} \) are similar as for the results for \( \sigma_{13} \) and \( \sigma_{33} \). The results are significantly different between the two cases of slant angles. The case where the slant angle is equal to 0° corresponds to the Step 2 problem with a single ply dropped. It is expected that the results from the case with a slant angle equal to 0° should be comparable to the results from the case with a slant angle equal to 1°, as there is little physical change in the laminate geometry.

The results for the case of a slant angle equal to 0° are indeed consistent with previous results in that the largest concentrations in \( \sigma_{11} \), \( \sigma_{13} \), and \( \sigma_{33} \) occur in the outermost continuous ply. In this laminate configuration, the outermost continuous ply corresponds to Ply 2. However, the distribution of the stresses in the case with a slant angle of 1° is significantly different compared to the case with a slant angle of 0°. The chief difference is that the stress concentrations in the laminate occur in the ply closest to the laminate midplane, Ply 4. For the case of a slant angle of 1°, the stresses \( \sigma_{11} \) and \( \sigma_{33} \) decay in Region A along the length of Ply 2, whereas they are expected to grow along the length in that region. Furthermore, the value of the stress at the dropoff in the Step 2 results is equal to 1.42\( \sigma_o \), whereas the value of the stress at the dropoff in the Step 4 results is equal to -1.63\( \times 10^{-27} \sigma_o \). As a result of the difference in the two values, comparison of the stress distribution for both models on a figure is impossible. The value of the \( \sigma_{11} \) stress in Step 4, being so far removed from the value of the stress in Step 2, indicates severe numerical issues in the model. The results for the case of slant angle equal to 1° compare poorly to the case of slant
angle equal to 0°. This presents an issue in that the two laminate configurations are similar to each other in terms of geometry, and thus, should have stress distributions that are similar.

In an effort to alleviate numerical issues in the Step 4 model, a change in the through-thickness variation of the stresses was assumed and implemented. It was assumed that the in-plane stresses have a linear distribution through the thickness of a ply or sublayer. This, in turn, required the assumption that the interlaminar shear stresses have a quadratic distribution through the thickness of a ply or sublayer, and that the interlaminar normal stresses had a cubic distribution through the thickness of a ply. As issues are present in the results of the model with these assumptions on the stresses, the explicit forms of the assumed stress shapes are not given. The results for $\sigma_{11}$ of the model with assumed linear distribution through the thickness for the aforementioned case are shown as Figure 8.6.

Although the results are more well-behaved numerically, there still are significant issues in the results. As in the case of the Step 4 model with an assumed constant in-plane distribution through the thickness of a ply or sublayer, the results from the linear in-plane model in Region A decay at the dropoff in the outermost continuous ply of the laminate, Ply 2. Further issues are observed when comparing the results of the case with a slant angle of 1° to a case with a more significant slant angle chosen to be 45°. Specifically, the results from the two cases are identical in $\sigma_{11}$. Although not included here, the results for $\sigma_{13}$ and $\sigma_{33}$ are also identical for the cases of slant angle considered. As further examined, this shows that the numerical solutions of both cases are the same up to eight significant digits. This presents an issue in that the results suggest that adjusting the slant angle of the dropped ply has no effect on the solution in Step 4. The overall description of the geometry between the two cases is similar. However, the significant physical change in the slant angle indicates that differences should exist when comparing the results of the two cases. The results being identical in the two cases is indicative of issues in the model to accurately represent the effects of the angle of the slanted free surface of dropped plies.

There are significant issues in the results of Step 4 based upon the formulation de-
Figure 8.6 Results along the outermost interface of Ply 2 in a $[0_D/0_3]_s$ laminate for various slant angles using an assumed linear distribution through the thickness for the in-plane stresses.
scribed in Section 8.1. For the assumption of constant through-the-thickness in-plane stresses, significant numerical issues manifest in the results that give rise to stress concentrations that are orders of magnitude larger than expected, on the order of $10^{30} \sigma_0$, in the laminate. These stress concentrations arise from the significant numerical issues present in the model. Although the results from the linear through-the-thickness in-plane stresses result in a more well-behaved system, the stress distributions seen in the results are contrary to expectations. Although numerical issues with the model can be used to justify the results seen in the constant in-plane stress model, the lack of numerical issues present in the linear in-plane stress model suggest that there are other issues present in the formulation and implementation of the Step 4 model outside of those influencing the numerical characteristics of the formulation.

8.3 Discussion

The results from the Step 4 model go against the expectations of the distributions of the stress fields in the problem of a laminate with slanted dropoffs under tension. Comparison of the results of the Step 2 problem to a Step 4 problem with nearly the same configuration indicate a significant change in the distribution of the stresses, whereas there should be little difference between the results. In addition, there is no change in the results with significant changes in the value of the slant angle of dropped plies. This suggests that the results from the Step 4 model are inaccurate and unsuitable for the use in identification of the mechanisms that give rise to the distribution of interlaminar stresses in the Step 4 problem. The source of these issues is tied to the assumptions on the stresses made in the model.

The cause of these issues lies in the static inadmissibility of the stress shapes in the laminate. The assumed through-thickness distributions in Steps 1 through 3, Equations (5.2a-e), automatically satisfy integral equilibrium constraints through the thickness of the laminate. With this assumed stress shape, the forms of the through-thickness integrals of the stress are independent of the stress distribution in the length direction, $x_1$. This is due to the assumption that the stresses are functionally separate
in $x_1$ and $x_3$, as well as the fact that integral equilibrium is enforced in the thickness direction. As a result, the variation of the stresses along the length does not factor into the satisfaction of integral equilibrium conditions, and satisfaction of the integral equilibrium conditions through the thickness at one value of $x_1$ allows the stress fields to satisfy integral equilibrium at all $x_1$-locations. Thus, the assumed forms of the stress shapes automatically satisfy integral equilibrium everywhere for Steps 1 through 3 without the need to impose the force-balance constraint later in an explicit manner.

In contrast, the assumed forms for the stresses in dropped plies in Step 4, Equations (8.2a-e) and (8.3a-b), do not automatically satisfy integral equilibrium conditions. This is due to the stress shapes being defined in the local length coordinate, $x_1^i$, as opposed to being defined in the global length coordinate, $x_1$. Thus, when taking the integral of the stresses through the thickness of dropped plies in Step 4 for force-balance considerations, terms related to the lengthwise variation of the stresses appear in the integral. As the lengthwise variation of the stresses are exponential along the length of the laminate, exponential terms which depend on $x_1^i$ will appear in the form of the integral representing the force-balance constraint. Thus, the form of the integral of the stresses through the thickness of the laminate vary as the $x_1$-location of the integral is changed. As a consequence, the integral equilibrium conditions are not automatically satisfied everywhere via the assumed form of the stress shapes.

Without being able to enforce integral equilibrium via the assumption of stress shapes in Chapter 4, the integral equilibrium constraints must be satisfied in another manner, such as in the manner of the free-edge and stress continuity expressions. This requires an introduction of Lagrangian multipliers in the expression for the complementary energy of the laminate. However, there is no form which allows the integral equilibrium constraints to be expressed everywhere along the laminate, which prevents the use of Lagrangian multipliers to express this constraint. The through-thickness variation of the stress in a ply or sublayer results in an expression that depends on the lengthwise variation of the stresses. For all laminates considered under the general formulation in Chapter 4, this lengthwise variation is exponential. Recall that exponential variations require an infinite number of constraint equations in order
to impose a specific distribution that comes as a result of boundary conditions. As the expression for integral equilibrium depends on exponential terms in the length coordinate, $x_1$, the integral equilibrium constraints must be enforced at an infinite number of points in order to ensure that the stresses satisfy integral equilibrium everywhere in the laminate. This is a computational impossibility. As a result, there is no means by which integral equilibrium can be enforced everywhere in the laminate in the Step 4 problem with the assumed stress shapes of Equations (8.2a-e) and (8.3a-b).

The assumed stress shapes in Equations (8.2a-e) and (8.3a-b) are statically inadmissible for Step 4, and therefore, the results from such an assumption lead to significant inaccuracy. There is a need to redefine the stress shapes in order to further develop a model that can be used to solve the Step 4 problem. However, there are inherent issues in the general formulation presented in Chapter 4 that limit the applicability of the formulation to the Step 4 problem. The issues stem from the lengthwise variation of the stresses appearing in the equations of integral equilibrium in the Step 4 problem. Considering the shaded region of a ply with a slanted dropoff, as illustrated in Figure 8.2, the limits of the integral through the thickness of the ply are a function of the length, $x_1$, when considering integral equilibrium through the thickness. Assuming the general form of the assumed stresses in Equations (4.8a-e), the expression for the through-thickness integral of the stresses in the shaded region will, with one exception, have terms that depend on the interfacial stress functions $F_i(x_1)$ and $G_i(x_1)$. No change in the local ply coordinate system or global laminate coordinate system will effect the appearance of these terms in the expression for integral equilibrium.

There is one manner in which the through-thickness integral of the stresses in a ply with a slanted dropoff can be independent of the functions $F_i(x_1)$ and $G_i(x_1)$. This would require that these functions, which correspond to the ply stresses in the dropped plies, be set equal to a specific function of $x_1$ that depends on the ply thickness and slant angle. The variation in $x_1$ that appears in the expressions of integral equilibrium due to the limits of the integral being dependent on $x_1$ can be made to-cancel out with
the aforementioned specific functional forms of $F_i(x_1)$ and $G_i(x_1)$. The issue in using this assumption is that the variation of the stresses along the slanted free surface of the ply will be dependent on exponential terms in $x_1$. As previously mentioned, it is impossible to enforce the free-edge boundary conditions if there is an exponential variation of the stresses along the free surface.

Although the assumed stress shapes in Equations (8.2a-c) and (8.3a-b) allow for the free-surface conditions to be satisfied at the dropoff, these assumed forms prevent the stress fields from satisfying force-balance conditions through the thickness at every lengthwise location in the laminate. There is thus an inherent incompatibility in the general formulation in that integral equilibrium and free-surface boundary conditions cannot be satisfied simultaneously unless the free surfaces corresponding to the dropoffs in the laminate are parallel to the through-thickness direction. If the free surfaces are not parallel, terms relating to the variation of the stresses in $x_1$ will appear in these expressions regardless of how the stresses are defined. As this variation is exponential in $x_1$, exponential terms will appear in the equations expressing these constraints.

The presence of exponential variations in either of these constraints makes the constraint impossible to enforce at all applicable points in the laminate. Thus, for laminates with slanted dropoffs, the general formulation as presented in Chapter 4 requires an assumption of a statically inadmissible stress field. This has significance when using a Rayleigh-Ritz method to estimate the stress fields, as the method requires an assumed stress field that is statically admissible. Without such an assumed stress field, the results will be less accurate. With the current assumptions on the stresses, it was found that the results from the model were inaccurate and unsuitable for future work or analysis. Thus, there is a need to reexamine the general solution methodology presented in Chapter 3 due to the inapplicability of developing the Step 5 model from the Step 4 model because of inaccuracy in the latter. This requires that either a new general formulation be developed, or that the problems defined in the stepwise methodology be modified such that the general formulation provides accurate results. The latter approach requires defining problems with laminate con-
figurations with non-slanted dropoffs. Such modifications to the overall approach are discussed in the next chapter.
Chapter 9

Assessment and Discussion of Overall Approach

Issues observed in the results of Step 4 prevent the model from being developed further to obtain results for the Step 5 problem of a grooved laminate under transverse loading, as defined in Figure 3.2. As a consequence, the stepwise methodology proposed in Chapter 3 cannot be used in the development of such a model. Thus, the development of a model that can accurately and efficiently analyze the general problem of a grooved laminate under transverse loading requires modification of either the problems defined by the stepwise methodology or the general formulational framework of Chapter 4.

In order to assess which manner would best allow for the development of an appropriate model to estimate the stress fields in transversely-loaded grooved laminates, an examination of the formulation of Step 5 in the proposed solution methodology as described in Chapter 3 is conducted so that any additional issues that may exist with the methodology as pertaining to model formulation are discovered. Based upon the examination, modifications are made to the problems defined by the stepwise methodology while maintaining the same general formulation of Chapter 4. These modifications define a new stepwise methodology that both allows for the problems in all steps to be formulated in the manner defined in Chapter 4, and still provide insight to the mechanisms that give rise to interlaminar stresses in transversely-loaded
grooved laminates. An examination of the feasibility with regard to the formulation of these proposed changes to the stepwise methodology is then presented.

9.1 Feasibility of Step 5 Items of Modification

The feasibility of introducing the items needed to progress from a working Step 4 to Step 5 is considered. The chief assumptions made in this section is that the Step 4 model was successfully implemented, and that accurate results were therefore obtained from the model. This implies that further model development would be appropriate. Thus, if no issues were present in formulating the items of modification in transitioning from Step 4 to Step 5, then the only issues in the stepwise methodology come about due to the implementation of the Step 4 model, as Steps 1 through 3 of the methodology presented no issues in the formulation and implementation that prohibited the model from obtaining valid results. It is therefore important to consider these items of modification.

There are two items that are changed in the physical representation of the laminate in moving from Step 4 to Step 5. The first is that the laminate is no longer symmetric with respect to the \( x_1 \)-axis. The second is that the laminate is no longer under tension, but rather under a loading transverse to the laminate applied at the ply dropoffs.

The additional complexity required of the formulation in transitioning from analysis of midplane-symmetric laminates to that of nonsymmetric laminates comes from considerations of force-balance in the laminate. In the general formulation in Chapter 4, force-balance and moment-balance considerations are taken into account when developing the stress shapes defined in Equations (4.8a-e). The general form of the stress shapes satisfies force balance in the \( x_3 \)-direction and moment balance in all directions.

Force balance in the \( x_3 \)-direction for Steps 1 through 3 come as a result of three factors. The first factor is the symmetry of the laminates under consideration along the midplane. The second factor is that \( \sigma_{13} \) equals zero at the outer free surfaces of the laminate. The third factor is that \( \sigma_{13} \) is continuous in the \( x_3 \)-direction. As a result
of these three factors, the distribution of $\sigma_{13}$ is antisymmetric in the $x_3$-direction. This requires that this stress equals zero at the laminate midplane. As a result of having an antisymmetric distribution of $\sigma_{13}$ through the thickness of the laminate, the sum contribution of $\sigma_{13}$ to force-balance in the $x_3$-direction equals zero. Thus, by requiring that $\sigma_{13}$ equals zero at the laminate midplane, the contribution of $\sigma_{13}$ to $x_3$-force balance is made to equal zero. With no applied loads on the laminate in the transverse direction, this allows the laminate to satisfy force-balance in the $x_3$-direction.

Moment balance considerations are satisfied in Steps 1 through 3 via the symmetric and antisymmetric distributions of the stresses through the thickness of the laminate. Of the six independent stresses in the laminate, $\sigma_{11}$, $\sigma_{12}$, and $\sigma_{33}$ are required to be symmetric through the thickness of the laminate in order to satisfy symmetry conditions. The interlaminar shear stresses $\sigma_{13}$ and $\sigma_{23}$ are required to have an antisymmetric distribution through the thickness of the laminate in order to satisfy laminate symmetry conditions. As a result of having a symmetric or antisymmetric distribution of the stresses through the thickness of the laminate, the contributions of all stresses to moment balance expressions is equal to zero. This occurs because the contribution to the moment created by the stresses in a symmetric half of the laminate is balanced out against the contribution to the moment by the stresses in the other half. This is true both for stresses with a symmetric distribution and stresses with an antisymmetric distribution. Thus, due to symmetry of the laminate, all moment balance considerations are satisfied regardless of the specific stress distribution in a symmetric half of the laminate.

Four of the six integral equilibrium expressions are satisfied as a consequence of laminate symmetry, required symmetric distributions in the in-plane and interlaminar normal stresses, as well as antisymmetric distributions in the interlaminar shear stresses. The two remaining integral equilibrium expressions, which pertain to force-balance in the $x_1$- and $x_2$-directions, are satisfied via the specific stress shapes chosen in Equations (5.2a) and (5.2b). Thus, force-balance considerations in the $x_1$- and $x_2$-directions are not automatically satisfied as a consequence of laminate symmetry, but
via assumption of the shapes of the stresses.

An additional important point involves the assumed variability of the in-plane stresses through the thickness of the laminate. In the previous model, the in-plane stresses $\sigma_{11}$ and $\sigma_{12}$ were assumed to have constant distribution through the thickness of a ply. As a consequence, the distribution of both these stresses have one degree of variability in each ply. Force balance considerations in the $x_1$- and $x_2$-directions constitute a single constraint on each of the stresses. Force balance in the $x_1$-direction constrains the distribution of $\sigma_{11}$ through the thickness of the laminate, and force balance in the $x_2$-direction constrains the distribution of $\sigma_{12}$ through the thickness of the laminate. Assuming that the interpolation of the ply stresses from the interfacial stress functions is the same from ply to ply, the inclusion of $N$ constraints on the through-thickness distribution of the stresses requires that the stresses have a distribution with $N$ degrees of variability through the thickness of a ply. If stresses are assumed with fewer than $N$ degrees of variability, the resultant stress field cannot satisfy all constraints placed upon its through-thickness distribution. In previous models, one constraint was enforced on each of the in-plane stresses, $\sigma_{11}$ and $\sigma_{12}$, with regard to force balance in the $x_1$- and $x_2$-directions, respectively. As both of these stresses require one degree of variability through the thickness of a ply to satisfy integral equilibrium conditions, both of these stresses could be assumed constant through the thickness of a ply and still be able to satisfy all constraints imposed on the through-thickness distribution.

In the nonsymmetric case, as occurs in the transition to Step 5, it can be shown that $\sigma_{11}$ is constrained by three expressions related to force-balance in the $x_1$- and $x_3$-directions, as well as moment balance in the $x_2$-direction. Force balance considerations in the $x_3$-direction affect $\sigma_{11}$ due to the fact that the force balance consideration affects the distribution of $\sigma_{13}$ through the thickness of a ply, which in turn, affects the distribution of $\sigma_{11}$ due to considerations of differential equilibrium. Similarly, $\sigma_{12}$ is constrained by three expressions related to force-balance in the $x_2$-direction as well as moment balance in the $x_1$- and $x_3$-directions. Thus, when assuming the shapes of the in-plane stresses, it is required that the in-plane stresses be assumed with a
through-thickness distribution with three degrees of variability for each stress, as each stress has imposed on it three constraints on the through-thickness distribution. Thus, \( \sigma_{11} \) and \( \sigma_{12} \) must be, at minimum order, quadratic through the thickness of a ply. In turn, via the equations of differential equilibrium, the interlaminar shear stresses \( \sigma_{13} \) and \( \sigma_{23} \) must be cubic through the thickness, and the interlaminar normal stress \( \sigma_{33} \) must be quartic through the thickness. Although the specific stress shapes in Equations (5.2a-e) are insufficient to satisfy all constraints placed upon the stresses when considering non-midplane-symmetric laminates, the general forms of the stresses defined in Equations (4.8a-e), in which ply stresses are defined as an interpolation of the interface stresses, is still applicable as long as the specific interpolation of the stresses are of sufficiently high order.

One further constraint is imposed on the stresses in order to model non-symmetric laminates. A symmetric half of a laminate as well as a nonsymmetric whole of a laminate are subject to the same boundary conditions with two exceptions. The first is that additional constraints relating to force and moment balance need to be explicitly enforced in the nonsymmetric laminate. This is achieved via a higher-order assumption of the interpolation of the stresses, as previously discussed. The second is that \( \sigma_{33} \) must equal zero along both surfaces of the laminate in the \( x_3 \)-direction. In a symmetric half of a laminate, \( \sigma_{33} \) is required to equal zero only at the outer free surface of the laminate, as the inner surface of the half-laminate corresponds to the laminate midplane, on which there is no constraint imposed on \( \sigma_{33} \). Thus, for non-symmetric laminates, there needs to be a manner in which \( \sigma_{33} \) can be made equal to zero on both surfaces. This is done simply by imposing constraints on the interfacial stress functions similar to Equation (4.11). Although this may change the values of the Laminate Complementary Energy of the general formulation in Chapter 4, there is no reason to expect that, after minimizing the energy, the resultant expressions for the interfacial stress functions will be of different form than that of Equation (4.22). Thus, with modifications to both the through-thickness distribution of the in-plane stresses as well as modifications to the interfacial stress functions, the general formulation of Chapter 4 remains applicable to nonsymmetric laminates.
The second physical item of modification in going from the Step 4 problem to the Step 5 problem is that the applied load is transverse to the ply dropoffs. This requires two modification parts to the general formulation in Chapter 4. The first is that the far-field contribution to the stresses as defined in Equation (4.1) changes from the case of a laminate under tension to that of a laminate loaded transverse to the dropoff surfaces. For the nonsymmetric laminates analyzed in Step 5, the applied loading on the surfaces of plies at the dropoff resolves to a net equipollent transverse load applied at the plane of $x_3$-symmetry. An illustration of the resolution of the applied load along a set of dropoff locations to an equipollent force is shown in Figure 9.1. Supports for the Step 5 laminate are not specified due to the fact that the manner in which the laminate is supported influences the magnitude and distribution of the far-field stresses. Far-field stress solutions can be developed by taking the general governing differential equations for the bending of laminated cylindrical plates, e.g., [52], and solving the equations subject to the boundary conditions at the plate supports and the applied equipollent load. Thus, the manner by which the far-field stresses are determined for the transversely-loaded grooved laminate are different than the manner in which far-field stresses are computed in Steps 1 through 4, which rely on a solution from Classical Laminated Plate Theory.

The second part of the item of modification involves a change in boundary conditions at the surface of dropped plies. The surfaces of dropped plies are no longer free surfaces, as they were in Steps 1 through 4, but loaded surfaces. All possible loads applied to the surface of dropped plies can be decomposed to combinations of $\sigma_{11}$, $\sigma_{12}$, and $\sigma_{13}$ as applied at the surface of the dropoff. This decomposition of the load at the surface of a dropped ply is illustrated in Figure 9.2. For dropped plies subject to load transverse to the dropped surface, the boundary conditions require that the stresses in the ply at the dropoff equilibrate the applied stresses due to loading. Unloaded dropped plies are still subject to free-surface boundary conditions at dropoffs, which require that $\sigma_{11}$, $\sigma_{12}$, and $\sigma_{13}$ equal zero at the dropoff. The difference in the boundary conditions in loaded and unloaded dropped plies is only that of requiring $\sigma_{11}$, $\sigma_{12}$, and $\sigma_{13}$ equal a nonzero value at the dropoff, whereas in an unloaded surface,
$F_i = \text{Applied Load}$

Figure 9.1  Resolution of loading applied transverse to a set of dropoff surfaces into equipollent load.
Figure 9.2  Illustration of resolution of applied load at a point on a ply surface into independent stress components $\sigma_{11}$ and $\sigma_{13}$. 
the stresses are required to equal zero.

An additional note regarding stress boundary conditions involves sublayering laminates subject to transverse point loads. In the current framework, the applied transverse load is treated as a distributed load along the surface of a dropped ply. The distributed load can approximate a point load as the thickness of the loaded ply becomes thinner. In the limiting case of a ply with an infinitesimal thickness, the distributed load on the ply resolves to a point load.

Within the context of the current formulation, the manner in which the surface on which the applied load becomes smaller requires sublayering of the loaded ply. A loaded ply is discretized into a number of sublayers of differing thicknesses, with particular regard to the thickness of the sublayer on which the point load is applied. In order to best represent a point load applied to a ply, the sublayering discretization must be such that the location at which the point load is applied corresponds to a sublayer with small thickness. The point load is then assumed to be a distributed load applied on the surface of the small sublayer with magnitude such that the total loading on the sublayer is equivalent in magnitude and direction to the original point load. Decreasing the thickness of the sublayer on which the point load is applied improves the approximation of the point load as a distributed load.

The two items of modification in the configuration of the Step 5 problem from the Step 4 problem require several changes in the formulation. Upon examination of these modifications, no issues are present in formulating the Step 5 model assuming that the Step 4 model produces acceptable results. Although issues in the stepwise methodology remain due to the implementation of slanted dropoffs in Step 4, no changes made in Step 5 aggravate the overall issues with the stepwise methodology.

9.2 Proposed Changes to Approach

The observations from the considerations as presented and discussed in the previous section suggest that the only issues in formulation that prevent the use of the semi-analytic model proposed in Chapter 4 involve the viability of the model to ac-
count for stress-free surfaces that are slanted without assuming stress shapes that do not satisfy integral equilibrium conditions. The results from the model with this statically inadmissible stress field give rise to errors in the results that prevent further development of the model to Step 5.

All other complexities in the model within the stepwise methodology outside of Step 4 are implementable. Accurate results were obtained for Steps 1 through 3, and the changes in formulation required to develop a model for Step 5 are feasible. Thus, the only issue involved in formulation comes in Step 4, and the only issue present in the formulation and implementation of Step 4 comes in implementing slanted dropoffs in the model. As previously discussed in Chapter 8, there is no means within the general formulation given in Chapter 4 in which all constraints in the stresses can be met. Thus, the results from the Step 4 model are inaccurate, and this prevents the development of the Step 5 model from the Step 4 model.

There are two alternatives that can be pursued in the development of a workable model to analyze interlaminar stresses in grooved laminates within the current framework. The first is that the stepwise methodology from Chapter 3 remains unchanged, and an alternate formulation than that of Chapter 4 be implemented for each of the five steps in the methodology. The second is that the problems of the stepwise methodology be redefined such that the general formulation of Chapter 4 remain applicable to the new stepwise methodology. Given the accuracy of the results in Steps 1 through 3 within the context of the formulation in Chapter 4, this work considers the approach where the stepwise methodology is redefined so as to allow analysis for all steps via the general formulation of Chapter 4.

The main issue encountered in the formulation of all steps in the stepwise methodology is that there is no way to represent slanted dropoffs while assuming a statically admissible stress field. Thus, modification of the problems in the stepwise methodology would require redefining problems such that no slanted dropoffs are present. The chief issue involved with this modification is the representation of the groove within the laminate. The groove is a curved surface, which was discretized to a piecewise linear surface in the original methodology. This results in the presence of slanted dropoffs.
within the laminate. As slanted dropoffs cannot be modeled accurately, the groove must be represented as a number of non-slanted dropoffs, as in the multiple dropoff case in Step 3. This approximation of the curved nature of the groove is physically less accurate than that of the representation of the groove as a sequence of slanted dropoffs.

If a non-slanted sequence of dropoffs is to represent the groove, then the ability of the formulation to obtain relevant results for the general case of a grooved laminate must be established with this representation. The development of a model via the stepwise formulation, as previously established in this work, focuses on the development of a model with results that are accurate with regard to distribution of the stresses in the laminate. Such a model would be useful for preliminary sizing and design of such a grooved laminate. Although a model with the capacity to estimate both stress magnitudes and distributions in the laminate is desired, the focus of the model centers on the stress distributions and the mechanisms that give rise to those distributions. Thus, losses in accuracy with regard to the magnitude of the stress values in the laminate are acceptable as long as results indicate an accurate distribution of the stress compared to the results from other validated models.

The mechanism which gives rise to interlaminar stress fields in composite laminates is that of gradients in the in-plane stresses. Gradients in in-plane stresses give rise to gradients in interlaminar shear stresses, which, in turn, give rise to gradients in the interlaminar normal stress. It has been previously established in Reference [49] that there are three sources of in-plane stress gradients present in the problem of a transversely-contact-loaded grooved laminate. The first is that the far-field solution of a laminate under transverse loading will give rise to gradients in the in-plane stress $\sigma_{11}$. The second is that the presence of the groove imposes boundary conditions on the far-field stresses, and gradients in the in-plane stress field will arise due to the requirement that far-field values of the stress must recover from the imposed values of the stress at the groove. The third is that the specifics of the contact loading in the groove will differ from that of a pure transverse load, and the boundary conditions imposed via the specifics of the loading at the grooved surface will give rise to gradients
in the in-plane stresses as the stresses decay to far-field values. A model that represents the groove as a sequence of dropoffs must be able to account for the variation of interlaminar stresses due to gradients from these three effects.

The current formulation, regardless of how the groove is represented, is able to capture the mechanism that gives rise to interlaminar stresses due to gradients in the far-field stresses. In the formulation, the total stress field is represented as a sum of the far-field stresses and the complementary stresses, as defined in Equation (4.1). The interlaminar shear stresses are defined in terms of the assumed in-plane stresses, and this definition comes about via enforcement of the differential equations of equilibrium. Thus, as long as the far-field values and distributions for the in-plane stresses can be found, the manner in which the interlaminar shear stresses are defined captures the gradients in the in-plane stresses in the far-field. The interlaminar normal stress is defined via the differential equations of equilibrium, and thus the gradients in the interlaminar shear stresses will give rise to a gradient in the definition of the interlaminar normal stress. Thus, the in-plane stress gradients in the far-field will give rise to interlaminar stress gradients due to the manner in which the interlaminar stresses are defined from the assumed in-plane stresses via the equations of differential equilibrium.

The presence of the grooved surface in the laminate gives rise to further stress gradients in the in-plane stresses, which, in turn, give rise to interlaminar stress gradients. In the current formulation, the total stress field for each stress is given as the sum of a far-field stress and a complementary stress. The complementary stress field modifies the total stress field such that appropriate boundary conditions are met at ply dropoffs, and the complementary stress field decays to a value of zero at a sufficient distance from the dropoffs such that far-field values of the stresses are recovered. Thus, the manner by which the groove surface influences stress gradients comes about via the manner that the complementary stresses modify the total stresses such that the total stresses satisfy the boundary conditions imposed by the groove surface. The in-plane stress gradients that come as a result of the presence of the grooved surface are captured within the complementary parts of the in-plane stress.
solutions. As the complementary part of the interlaminar shear stresses are defined from the complementary part of the in-plane stresses via the differential equations of equilibrium, the gradients in the in-plane complementary stresses give rise to gradients in the interlaminar shear complementary stresses. Similarly, the gradients in the interlaminar shear complementary stresses give rise to gradients in the interlaminar normal complementary stresses. Thus, although the representation of the groove surface as a sequence of flat dropoffs may not yield fully accurate results compared to other representations of the groove, the mechanisms by which interlaminar stresses arise due to the presence of the groove are present in the general formulation with dropoffs. Thus, the results from such a model will accurately capture the mechanisms of influence that the groove surface has on the interlaminar stresses.

The influence of the specifics of the contact loading on the interlaminar stresses is captured in the current model in a manner similar to the presence of the groove surface. The specifics of the loading impose boundary conditions on the stress fields in certain plies at the dropoff. These plies are subject to a loading at the surface of the dropoff that can be resolved into components in \( \sigma_{11}, \sigma_{12}, \) and \( \sigma_{13} \). To equilibrate the ply at the dropoff, the ply stresses must equal the applied loading. This boundary condition at the dropoff influences the complementary stress parts of the in-plane stresses, and as a result, gives rise to gradients in the in-plane complementary stresses. This, in turn, causes gradients in the complementary parts of the interlaminar shear and normal stresses. Thus, the formulation provides a means to assess the specifics of the manner by which the contact loading influences the interlaminar stress fields via imposition of appropriate boundary conditions on loaded plies or sublayers.

Though the current set of loadings and laminate configurations can approximate the loaded groove, an additional level of complexity is introduced to the model such that the loaded groove can be more accurately represented by a sequence of loaded dropoffs. Loadings within the sequence of dropoffs, which represent the groove, can only be applied to the dropoff surfaces of plies. In order to expand the variety of loadings that are able to be represented in the formulation, an additional level of complexity in the external load is introduced. This complexity allows for the formula-
tion to analyze laminates with intermediate regions subject to transverse loading on the outer surfaces of the region. The solution of laminates subject to these transverse loadings, when superimposed to the solution of laminates subject to loaded dropoffs, allow for a more accurate representation of the grooved laminate subject to transverse contact loading. A representation of a typical laminate subject to both transverse loading on intermediate regions as well as loading along the dropped surface of plies is shown in Figure 9.3.

With the additional level of complexity added to the model, the formulation can better assess the overall effects which give rise to interlaminar stresses in the grooved laminate with transverse contact loads via representation of the groove as sequences of dropoffs and contact loads as a superposition of transverse loads and loads applied to dropped ply surfaces. The three mechanisms that give rise to interlaminar stresses are each represented separately in the model. Thus, the formulation, with modifications to account for the transverse loadings, is capable of accurately representing the mechanisms that give rise to interlaminar stresses in the transversely-loaded grooved laminate. This formulation is therefore capable of results that accurately represent the distribution of interlaminar stresses.

The original stepwise methodology can therefore be modified such that the current model can accurately represent the stress distributions in the laminate with transverse contact loads. As no issues in the formulation or results in Steps 1 through 3 of the original stepwise methodology were manifested, these Steps are preserved in the revised stepwise methodology. Steps 4 and 5 are modified from the original methodology. Step 4 of the revised methodology is made to represent the problem of a symmetric laminate with multiple dropoffs under transverse loading applied to its intermediate regions. Step 5 is made to represent the problem of an unsymmetric laminate with multiple dropoffs under both transverse loading and loading applied to the dropoff edges of dropped plies. Illustrations of the revised Step 4 and Step 5 problems, under the overall stepwise methodology, are shown in Figure 9.4.
Figure 9.3  Representation of laminate with combined transverse and dropoff-surface loading.
Figure 9.4 Illustration of Steps 4 and 5 of the revised stepwise methodology.
9.3 Feasibility of Proposed Changes

In modified Steps 4 and 5 of the overall stepwise methodology, three total changes to the physical laminate are made. The first is that transverse loads on intermediate regions are considered. The second is that the laminate is considered to be non-symmetric about the $x_3$-axis. The third is that load is applied to the dropped surfaces. The second and third changes have already been considered previously in this chapter. In order to establish the feasibility of the formulation using the modified steps, the feasibility of the formulation of laminates under transverse loading within the context of this modified general formulation must be established.

The modified Step 4 problem is considered, as this is the step in which the transverse loading is introduced within the methodology. In this step, a laminate with multiple finite length dropoffs have regions loaded transverse to the length of the laminate. The laminate remains symmetric in geometry and loading. As this loading configuration has zero resultant force and moment, it does not contribute to the far-field stresses. This configuration is shown as revised Step 4 in Figure 9.4.

Two changes are required of the stress shapes from Step 3 to revised Step 4. The first is a modification of the far-field stresses. As there is no net force applied to the laminate in this configuration, the values of the far-field stresses will equal zero for all stresses. This is in contrast to Steps 1 through 3, where the laminate was loaded in tension, and far-field stress values were set equal to solutions from CLPT. The second change of the stresses requires that the values of $\sigma_{33}$ on the outer surfaces of intermediate regions equilibrate the applied transverse loading. The assumed form of $\sigma_{33}$ within a region that satisfies this constraint can be written in a similar form to Equation (5.2e), the assumed shape of $\sigma_{33}$ in Steps 1 through 3:

$$\sigma_{33}^i = \frac{F''_{i-1}(x)t^i}{2}(x_3^i + \frac{1}{2})^2 + \frac{F''_i(x)t^i}{2}(x_3^i - \frac{1}{2})^2 + p_{Region}^{Region}(x_1) - \sum_{j=1}^{i-1} t^j F''_j(x)$$

where $p_{Region}^{Region}(x_1)$ represents the transverse loading distribution in the region. All other
stress shapes remain unchanged, as the addition $p^{\text{Region}}(x_1)$ does not appear in the expressions of $\sigma_{11}$ and $\sigma_{13}$ due to the fact that the partial derivatives taken in $x_3$ in the differential equations of equilibrium will cause the function, $p^{\text{Region}}(x_1)$, to vanish when considering stresses other than $\sigma_{33}$.

These two changes to the stress function shapes modify the expressions that result from minimizing the laminate complementary energy. Minimizing the laminate complementary energy with these assumed stress shapes yield a set of differential equations similar to Equation (4.22):

$$
\begin{bmatrix}
A_{11} & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\mathbf{F} \\
\mathbf{G}
\end{bmatrix}'''' +
\begin{bmatrix}
B_{11} & B_{12} \\
B_{12} & B_{22}
\end{bmatrix}
\begin{bmatrix}
\mathbf{F} \\
\mathbf{G}
\end{bmatrix}'' +
\begin{bmatrix}
C_{11} & C_{12} \\
C_{12} & C_{22}
\end{bmatrix}
\begin{bmatrix}
\mathbf{F} \\
\mathbf{G}
\end{bmatrix} =
\begin{bmatrix}
\mathbf{R}(x_1) \\
0
\end{bmatrix}
$$

(9.2)

where $\mathbf{R}(x_1)$ is a novel addition from models in previous Steps. It is a function of the laminate geometry, material properties, and $p^{\text{Region}}(x_1)$. As this section only considers the feasibility of the formulation of the modified Step 4 problem, and not the specific formulation, the general form of $\mathbf{R}(x_1)$ is not considered.

The solution to the above system of equations requires a homogeneous and particular solution. The homogeneous solution is found by solving Equation (9.2) with $\mathbf{R}(x_1)$ equal to zero. The solution of this system is identical to the solution obtained from the system above in previous Steps expressed in Equation (4.22) - a sum of exponentials with still-unknown stress coefficients. To obtain a particular solution, it is noted that the polynomial order of $\mathbf{R}(x_1)$ is proportional to the polynomial order of $p(x_1)$. To reduce the complexity of the formulation, $p^{\text{Region}}(x_1)$ is assumed constant within a region. Thus, $\mathbf{R}(x_1)$ will be constant and independent of $x_1$, which allows for much greater simplicity in the formulation. With this assumption, the particular solutions to the system in Equation (9.2) are:

$$
\begin{bmatrix}
\mathbf{F} \\
\mathbf{G}
\end{bmatrix} =
\begin{bmatrix}
C_{11} & C_{12} \\
C_{12} & C_{22}
\end{bmatrix}^{-1}
\begin{bmatrix}
\mathbf{R}(x_1) \\
0
\end{bmatrix}
$$

(9.3)
These solutions are added to the homogeneous (exponential sum) solutions derived from the general formulation.

The solution of the linear system that determines the unknown stress coefficients also changes and is of the form:

\[
\begin{bmatrix}
\Omega^A & 0 & \Gamma_1^{A^T} & \Gamma_2^{A^T} & \Theta^{A^T} \\
0 & \Omega^B & \Gamma_1^{B^T} & \Gamma_2^{B^T} & \Theta^{B^T}
\end{bmatrix}
\begin{bmatrix}
a \\ b \\
\rho_{11} \\
\rho_{12} \\
\rho_{13}
\end{bmatrix}
= 
\begin{bmatrix}
C_1 \\
C_2
\end{bmatrix}
\]

(9.4)

where \( C_1 \) and \( C_2 \) are modifications from previous models, and are a function of the system eigenvalues, eigenvectors, ply geometries, and the transverse loadings, \( p_{\text{Region}}(x_1) \). All terms in this system, including the \( C_1 \) and \( C_2 \) terms, are constant regardless of the shape of \( p_{\text{Region}}(x_1) \). Thus, the resultant system is linear and can be solved via a number of techniques used to solve linear systems. General forms for \( C_1 \) and \( C_2 \) are not considered, as this is only a consideration of feasibility.

Though changes are required in both the shapes of the assumed stresses and the equations representing the minimization of Laminate Complementary Energy, the formulation for laminates under transverse loading follows that of the general formulation in Chapter 4. The shapes of the assumed stresses in Steps 4 and 5 of the modified stepwise methodology are statically admissible, in contrast to the stress shapes in Steps 4 and 5 of the previous stepwise methodology. As a result of this, formulation and implementation of Steps 4 and 5 of the modified stepwise methodology are feasible within the context of the general formulation in Chapter 4. Thus, the modified stepwise methodology offers improvements over the previous methodology in that no issues appear in the formulation of the modified methodology. Further work in developing a model to estimate the interlaminar stress fields in transversely-loaded groove laminates using the general formulation of Chapter 4 should follow along the lines of the modified methodology for Steps 4 and 5.
Implementation of the changes required for the modified Step 4 and Step 5 problems requires that new stress shapes be developed such that the stresses can satisfy all equilibrium constraints. In addition, terms relating to quantities in the problem of transverse loading, the $R(x_1)$ in Equation (9.2) and the $C_1$ and $C_2$ in Equation (9.4), must be implemented via new subroutines. These expressions can be feasibly implemented in the context of the general formulation of Chapter 4. All that remains is to work through the details of each of these expressions and implement them. As the general formulation of Chapter 4 is preserved for the Step 4 and Step 5 models, a large number of subroutines from Steps 1 through 3 can be used to assemble the key equations, Equations (9.2) and Equation (9.4), of the modified Step 4 and Step 5 models.
A stepwise methodology to assist in the development of a model to determine the interlaminar stresses in transversely-loaded grooved laminates was presented. This methodology was developed in increasing steps of complexity starting with a problem and its known solution (laminates with infinite-length dropoffs) until a final problem that can be used to assess the mechanisms that give rise to the interlaminar stress fields in transversely-loaded grooved laminates was reached. The model formulation used to analyze the problems in the stepwise methodology is based on an assumed stress approach, where unknown stress parameters are found via minimization of laminate complementary energy. Of key importance in these models is an approach that was both accurate in results and efficient in terms of computational time. Such a model is useful for stress sizing and assessment in configurations for preliminary design. Particular conclusions from this work are as follows:

1. The general formulation in Chapter 4 leads to the development of a model with results that compare well with finite element analyses for laminate configurations with non-slanted dropoffs (Steps 1 through 3 in the initial stepwise methodology). The models developed from the general formulation are both accurate in their results and more efficient in their runtime when compared to
the results from finite element analyses.

2. The general formulation in Chapter 4 allows for the discretization of plies via sublayering, where a single ply is modeled as several subplies with the same material properties. The thickness of each subply is smaller than that of the parent ply, and the sum of the subply thicknesses is made to equal the total thickness of the parent ply. Sublayering plies in a laminate configuration leads to greater variability of the stresses along the length of a ply as well as through the thickness of a laminate. Sublayering also reduces the distances over which stress values are averaged through the thickness of the laminate. These two effects lead to results that are more accurate with respect to the physical behavior of the laminate at the cost of an increase in runtime.

3. Models derived from the general formulation in Chapter 4 have the potential for numerical issues, which may lead to significant issues in the stress results from the model. These numerical issues come as a result of implementation of a Lagragian Minimization problem to solve for the unknown stress coefficients of the linear systems in models of the form of Equation (4.32). As a consequence of this, symmetry boundary conditions that arise when dealing with laminate configurations with a single and multiple finite-length dropoffs cannot be fully implemented within the context of the general formulation due to numerical issues. Despite this, accurate results can be obtained for models of laminates with finite-length dropoffs without implementing all boundary conditions required of symmetry in the laminate. For Steps 1 through 3, numerical issues do not adversely affect the results in a manner that invalidates the models.

4. The largest stress concentrations found in laminates with infinite-length dropoffs arise in either the outermost continuous ply or the outermost continuous sublayer of a model.

5. Larger in-plane stress gradients appear in laminates with finite-length dropoffs as the length of the dropped region decreases. This is due to the fact that as the
dropped region length decreases, the distances between dropoffs also decreases. Thus, the distance in which stresses transferred to continuous plies are again picked up after the dropoff decreases, since moving along the length of the laminate, stresses are transferred to continuous plies at the first dropoff and then transferred quickly away from these plies as additional plies are picked up. This leads to larger gradients in the interlaminar stresses in such configurations.

6. In-plane stresses are transferred from dropped plies to continuous plies at distances farther from the dropoff as region length decreases. This is due to the effects of one dropoff in a symmetric half of a the laminate having a greater influence on the stresses around the dropoff in the other half of the laminate.

7. In analysis of a symmetric quarter of a laminate, the location of the largest stress concentrations in a laminate with multiple dropoffs may appear in plies or sublayers outside of the outermost continuous ply or sublayer. This is due to the influence of dropoffs on the stress distribution within proximity to other dropoffs in a symmetric quarter of the laminate. This is in contrast to the results in laminates with infinite-length dropoffs.

8. A governing factor in the behavior of stress distributions in laminates with multiple finite-length dropoffs is the ratio of the length of the dropped region divided by ply thickness. For values of this ratio below 1, the stress behaves in a manner similar to that of being influenced by a single dropoff with double the number of plies dropped. For values of this ratio larger than 8, the effects of each dropoff are isolated from each other, and the results look similar to those of the distributions of two infinite-length configurations appended together. For values of this ratio between 1 and 8, the effects of a dropoff influence the distribution of the stress around the other dropoff. Such configurations lead to larger concentrations than in cases with the dropoffs sufficiently close together or further apart.

9. The initially-proposed stepwise methodology does not provide a means to accu-
rately assess the magnitudes and distributions of the stress fields in problems with slanted dropoffs. This is due to an inability of developing a statically-admissible assumption of the stress shapes for laminate configurations with slanted dropoff edges (Steps 4 and 5 in the initial stepwise methodology) within the context of the general formulation of Chapter 4. This is due to the appearance of exponential variations in the expressions relating to integral equilibrium through the thickness of the laminate, as well as the free-edge boundary conditions at the dropped edge. This prevents the implementation of Steps 4 and 5 of the initial stepwise methodology.

10. Assuming that the Step 4 model is implemented, the items of change required in transitioning from the Step 4 model to the Step 5 model are feasible. These items of change allow the model to account for laminate configurations that are not symmetric about the laminate midplane as well as laminate configurations with loaded dropped edges.

11. A new stepwise methodology with revisions to Steps 4 and 5 is proposed that is able to capture the mechanisms that give rise to gradients in the in-plane stresses and is feasible within the context of the general formulation. These gradients in the in-plane stresses give rise to gradients in the interlaminar stresses via the expressions for differential equilibrium. The main feature of the changes in the initial stepwise methodology to the new stepwise methodology is that the latter avoids analysis of laminate configurations with slanted dropoffs.

12. Further modification of the problems in the form of transversely-loaded regions is proposed to allow for better representation of the problem of a transversely-loaded groove.

Based upon the work completed and the observations and conclusions made, the following recommendations are made:

1. Work regarding the specific details of the formulation of the models for the revised Steps 4 and 5 is needed. Although the formulation of these steps is fea-
sible, further derivation of quantities appearing in key equations are needed for implementation of the models. In addition, these implemented models require validation with other methodologies to establish the accuracy of the results of the models for the revised Step 4 and Step 5 problems.

2. Parametric studies should be conducted with regard to the revised Step 5 model in order to better ascertain which material and geometric parameters contribute to the magnitude and distribution of interlaminar stresses, as well as which mechanisms give rise to interlaminar stress concentrations around contact-loaded grooves in composite laminates.

3. Improvements in the numerical stability of the model may be needed to analyze more complex laminate configurations. This can be accomplished by two means. The first is that the Lagrangian Minimization in solving for the unknown stress coefficients be replaced by a back-substitution scheme where dependent stress coefficients are identified and eliminated via expressions for stress continuity across regions as well as free-surface boundary conditions. The second is that a conditioning scheme for the general formulation be developed such that models derived from the general formulation are more well-conditioned, and thus, more numerically stable.

4. Investigation of delamination of the transversely-loaded grooved laminate is needed. The results from models from the stepwise methodology may be used to this end in concert with an appropriate failure theory to predict the onset of delamination in a laminate. Such a theoretical investigation of delamination should be supplemented via comparison with results from appropriate experimental testing.
References


Appendix A

Source Code Listing for Analysis of Steps 1 and 2

The source code of the program for analysis of the Step 1 and Step 2 problems of a laminate with a single infinite-length (Step 1) or finite length (Step 2) dropoff is listed in this appendix. The general structure for running this script in Mathematica is as follows. The input of laminate geometric and material properties is first read. The code subsequently compiles and stores into memory all subroutines required for analysis. The main script, which calls on these subroutines, is then run until a solution to the eigenvalues, eigenvectors, and unknown stress coefficients is found. The main script then assembles these values to find the solution to the ply stresses. Finally, the ply stresses are plotted. All code is written for Mathematica Release 7.x, although accurate results for the code have been obtained on distributions 8.0 and greater.
BEGIN LAMINATE CONFIGURATION INITIALIZATION

(* The following is a list of the material parameters of plies used in the current work. Each line acts as its own set of material properties. The format of the variable is \{El, Et, \nu_\perp, It, \nu_\parallel, Glt, Gt, PlyAngle\}. *)

AS4 = \{130, 9.0, 0.28, 0.28, 4.8, 3.51, 0\};
AS90 = \{130, 9.0, 0.28, 0.28, 4.8, 3.51, 90\};
ASp45 = \{130, 9.0, 0.28, 0.28, 4.8, 3.51, 45\};
ASm45 = \{130, 9.0, 0.28, 0.28, 4.8, 3.51, -45\};
(* Measure of Ply/Sublayer Thickness. Entered as a variable, as all problems considered for the work have had equal ply or sublayer thickness. However, the current code is robust enough to support ply/sublayers with differing thickness. *)

\[ \text{Thk} = .125; \]

(* The Layup Variable Denotes the laminate layups in all regions. Each entry in the matrix below denotes an Independent region. Layup is a matrix of submatrices, where each submatrix describes the layup in a single region. The format for each submatrix is a number of rows equaling the number of plies/sublayers in a region, and each row is denoted as \{Ply/SublayerThickness, Ply/SublayerMaterialParameters\}. Externally dropped plies are indicated via reduction of the number of plies from Region to Region. *)

\[ \text{Layup} = \left[ \begin{array}{ccc}
\text{Thk} & \text{AS4} \\
\text{Thk} & \text{AS4} \\
\text{Thk} & \text{AS4} \\
\text{Thk} & \text{AS4}
\end{array} \right] \left[ \begin{array}{c}
\text{Thk} \\
\text{AS4}
\end{array} \right]; \]

(* Internal variable for number of regions *)

\[ \text{NRegions} = \text{Length[Transpose[Layup]]}; \]

(* Internal variable for number of dropoffs *)

\[ \text{NDrops} = \text{NRegions} - 1; \]

(* Internal variable for total laminate thickness; used mainly for nomalization of results. *)

\[ \text{LT} = 2 \times \text{Extract[Total[Extract[Layup, \{1, 1\}], 1], 1]}; \]

(* Vector of absolute region lengths. It is assumed Region A is infinite in Length. The number of entries in RLengths must equal NRegions - 1. *)

\[ \text{RLengths} = \{\infty\}; \]

(* Vector that stores region boundaries in the global x_1-system. This initializes the vector. *)

\[ \text{Len} = \{0\}; \]

(* Operates on Len and RLengths to obtain and store Region Boundary information. *)

\[ \text{For}[i = 1, i \leq \text{NDrops}, i++, \]
\[ \text{If}[i == 1, \]
Len = Append[Len, Extract[RLengths, i]]; 
,
Len = Append[Len, Extract[RLengths, i] + Extract[Len, i]]; 
]; 
];

(*
BEGIN SUBROUTINE DEFINITION AND COMPILATION

(* Rotates ply material properties in planar form for use in calculating CLPT solution. Foris a switch that denotes either forward rotation to calculate CLPT A-matrices or to rotate back into the ply coordinate frame to calculate ply stresses. *)

TwoDElas[{El, Et, vlt, vtz, Glt, Gtz, th_}, For_]:= 
If[True,
theta = (-1 + 2 * KroneckerDelta[For, 1]) * th/180 * \pi;

c = Cos[theta]; 
s = Sin[theta]; 
vtl = vlt * Et/El;
Div = (1 - vlt * vtl);
e1111 = El/Div;
e2222 = Et/Div;
e1122 = vlt * Et/Div;
e1212 = Glt;

e1212 = Glt;

\[
\begin{pmatrix}
Q1111 \\
Q2222 \\
Q1122 \\
Q1212 \\
Q1112 \\
Q1222
\end{pmatrix}
= \begin{pmatrix}
c^4 & s^4 & 2 * c^2 * s^2 & 4 * c^2 * s^2 \\
s^4 & c^4 & 2 * c^2 * s^2 & 4 * c^2 * s^2 \\
c^2 * s^2 & c^2 * s^2 & c^4 + s^4 & -4 * c^2 * s^2 \\
c^2 * s^2 & c^2 * s^2 & -2 * c^2 * s^2 & (c^2 - s^2)^2 \\
s * c^3 & -s^3 * c & (c * s^3 - c^3 * s) & 2 * (c * s^3 - c^3 * s) \\
s^3 * c & -s * c^3 & c^3 * s - c * s^3 & 2 * (c^3 * s - c * s^3)
\end{pmatrix}
\begin{pmatrix}
e1111 \\
e2222 \\
e1122 \\
e1212
\end{pmatrix}
\]
(* Calculates full Material Compliance Matrix rotated by angle th. *)

Compliance[\{\text{El, Et, }\nu lt, \nu tz, \text{Glt, Gtz, th}\}] :=

\[
\text{If}\left[\text{True,}
\theta = \frac{-\text{th}}{180} \cdot \pi; \\
c = \cos[\theta]; \\
s = \sin[\theta];
\right]
\]

\[
K = \begin{pmatrix}
c^2 & s^2 & 0 & 0 & 0 & 2c_s \\
s^2 & c^2 & 0 & 0 & 0 & -2c_s \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & c & s & 0 \\
0 & 0 & 0 & -1s & c & 0 \\
-cs & cs & 0 & 0 & 0 & c^2 - s^2
\end{pmatrix};
\]

\[
\text{IK} = \text{Inverse}[K];
\]

\[
\text{Return Transpose}[IK].
\]

\[
\begin{pmatrix}
1/\text{El} & -\nu lt/\text{El} & -\nu lt/\text{El} & 0 & 0 & 0 \\
-\nu lt/\text{El} & 1/\text{Et} & -\nu tz/\text{Et} & 0 & 0 & 0 \\
-\nu lt/\text{El} & -\nu tz/\text{Et} & 1/\text{Et} & 0 & 0 & 0 \\
0 & 0 & 0 & 1/\text{Gtz} & 0 & 0 \\
0 & 0 & 0 & 0 & 1/\text{Glt} & 0 \\
0 & 0 & 0 & 0 & 0 & 1/\text{Glt}
\end{pmatrix}.\text{IK};
\]

\[
\]

(* Calculates material matrices relevant to CLPT analyses. Let_{-} denotes what material matrix (A,B,or D) to return. *)

273
LPTM[PlyA_, Let.]:= 
If[True,
Matrix = {{}, {}};
Matrix = 0 * Transpose[{Range[3]}];
Matrix = PadRight[Matrix, {3, 3}];
TotalT = Total[Extract[Transpose[PlyA], 1]]; 
zupper = TotalT;
ErrorFlag = False;
For[i = 1, i \leq \text{Length}[PlyA], i++,
zlower = zupper - Extract[PlyA, {i, 1}];
If[Let===A, 
Multi = zupper - zlower;,
If[Let===D, 
Multi = 1/3 * (zupper^3 - zlower^3);,
If[Let===B, 
Multi = 1/2 * (zupper^2 - zlower^2);,
If[ErrorFlag == False, 
Print["Error - unspecified Matrix for CLPT. Enter in letter A, B, or D."]; 
ErrorFlag = True;];
Multi = 0;
];
];
];
];
Adde = Multi * TwoDElas[Extract[PlyA, {i, 2}], 0];
Matrix = Matrix + Adde;
zupper = zlower;
];
Return[Matrix];
];
(*Calculates h11 matrices as denoted in Equation 5.3a for ply CPly at local x.3-coordinate GPoint. *)

S11F[Nn, CPly, Ply, GPoint.]:=
If[True,
  TopTerm = {1/Extract[Ply, {CPly, 1}]};
  BottomTerm = {-1/Extract[Ply, {CPly, 1}]};
  If[CPly == 1,
    Return[BottomTerm];
  ,
    If[CPly == Nn,
      Return[TopTerm];
    ,
      Return[Join[TopTerm, BottomTerm]];]
  ];
];

(*Calculates h12 matrices as denoted in Equation 5.3b for ply CPly at local x.3-coordinate GPoint. *)

S12F[Nn, CPly, Ply, GPoint.]:=
If[True,
  TopTerm = {1/Extract[Ply, {CPly, 1}]};
  BottomTerm = {-1/Extract[Ply, {CPly, 1}]};
  If[CPly == 1,
    Return[BottomTerm];
  ,
    If[CPly == Nn,
      Return[TopTerm];
    ,
      Return[Join[TopTerm, BottomTerm]];]
  ];
];
If[CPly == Nn, 
Return[TopTerm]; 
, 
Return[Join[TopTerm, BottomTerm]]; ]; ]; ]; (*Calculates h13 matrices as denoted in Equation 5.3c for ply CPly at local x3-coordinate GPoint.*)

S13F[Nn_, CPly_, Ply_, GPoint_] := If[True, 
TopTerm = {-(GPoint/Extract[Ply, {CPly, 1}] - 1/2)}; 
BottomTerm = {(GPoint/Extract[Ply, {CPly, 1}] + 1/2)}; 
If[CPly == 1, 
Return[BottomTerm]; , 
If[CPly == Nn, 
Return[TopTerm]; , 
Return[Join[TopTerm, BottomTerm]]; ]; ]; ]; (*Calculates h23 matrices as denoted in Equation 5.3d for ply CPly at local x3-coordinate GPoint. *)
S23F[Nn_, CPly_, Ply_, GPoint_] :=
If[True,
TopTerm = {-(GPoint/Extract[Ply, {CPly, 1}]) - 1/2};
BottomTerm = {(GPoint/Extract[Ply, {CPly, 1}]) + 1/2};
If[CPly == 1,
Return[BottomTerm];
, If[CPly == Nn,
Return[TopTerm];
, Return[Join[TopTerm, BottomTerm]];]
];
(*Calculates h33 matrices as denoted in Equation 5.3e for ply CPly at local x_3-coordinate GPoint.*)
S33F[Nn_, CPly_, Ply_, GPoint_] :=
If[True,
Outvec = 0 * Range[Nn - 1];
For[k = 1, k ≤ 1 * (CPly - 1), k++,
Outvec = ReplacePart[Outvec, -(Extract[Ply, {k, 1}] + Extract[Ply, {k + 1, 1}])/2, k];
];
If[CPly ≠ 1,
*(GPoint/Extract[Ply, {CPly, 1}] - 1/2)^2, CPly - 1];
];
If[CPly ≠ Nn,
Outvec = ReplacePart[Outvec, Extract[Outvec, CPly] - Extract[Ply, {CPly, 1}]/2 * (GPoint/Extract[Ply, {CPly, 1} + 1/2, CPly]; ]; Return[Outvec]; ];

(* Sets up the eigenfunction problem in Equation 4.22 and solves for the eigenvalues and eigenvectors. *)

Eigenfunction[PlyA.]:=

If[True,
Nn = Length[PlyA];

(* Implements Gaussian Quadrature tables for integration of h-matrices in x_3 *)

GP = \{-\sqrt{\left(3 + 2\sqrt{6}/5\right)/7}, -\sqrt{\left(3 - 2\sqrt{6}/5\right)/7}, \sqrt{\left(3 - 2\sqrt{6}/5\right)/7},\}

GW = \{ (18 - \sqrt{30})/36, (18 + \sqrt{30})/36, (18 + \sqrt{30})/36, (18 - \sqrt{30})/36 \};

WGP = {};

WGW = {};

(* Components of matrices in the eigenfunction problem *)

E0 = Table[0, \{2 * Nn - 2\}, \{2 * Nn - 2\}];

E1 = E0;

E2 = E0;

E3 = E0;

E4 = E0;

E5 = E0;

E6 = E0;

E7 = E0;

E8 = E0;

(* Further modifies Gaussian Quadrature tables for integration of h-matrices in x_3. * )
This code handles plies or sublayers of differing thicknesses. *)

\[
\text{For}\{i = 1, i \leq Nn, i++.,} \\
WGP = \text{Append}[WGP, \text{Extract}[\text{PlyA}, \{i, 1\}] / 2 \times GP]; \\
WGW = \text{Append}[WGW, \text{Extract}[\text{PlyA}, \{i, 1\}] / 2 \times GW]; \\
\]

(* Calculates reduced compliance matrix terms. *)

\[
\text{For}\{i = 1, i \leq Nn, i++.,} \\
\text{CompI} = \text{Compliance}[\text{Extract}[\text{PlyA}, \{i, 2\}]]; \\
\text{RComp} = \\
\{\text{Extract}[\text{CompI}, 1, 1], \text{Extract}[\text{CompI}, 1, 3], 0, 0, \text{Extract}[\text{CompI}, 1, 6]\}, \\
\{\text{Extract}[\text{CompI}, 1, 3], \text{Extract}[\text{CompI}, 3, 3], 0, 0, \text{Extract}[\text{CompI}, 3, 6]\}, \\
\{0, 0, \text{Extract}[\text{CompI}, 4, 4], \text{Extract}[\text{CompI}, 4, 5], 0\}, \\
\{0, 0, \text{Extract}[\text{CompI}, 4, 5], \text{Extract}[\text{CompI}, 5, 5], 0\}, \\
\{\text{Extract}[\text{CompI}, 1, 6], \text{Extract}[\text{CompI}, 2, 6], 0, 0, \text{Extract}[\text{CompI}, 6, 6]\} - \\
1/\text{Extract}[\text{CompI}, \{2, 2\}] * \\
\{\text{Extract}[\text{CompI}, 1, 2]^2, \text{Extract}[\text{CompI}, 1, 2] \text{Extract}[\text{CompI}, 3, 2], 0, 0, \\
\text{Extract}[\text{CompI}, 1, 2] \text{Extract}[\text{CompI}, 6, 2]\}, \\
\{\text{Extract}[\text{CompI}, 1, 2] \text{Extract}[\text{CompI}, 3, 2], \text{Extract}[\text{CompI}, 3, 2]^2, 0, 0, \\
\text{Extract}[\text{CompI}, 3, 2] \text{Extract}[\text{CompI}, 6, 2]\}, \\
\{0, 0, \text{Extract}[\text{CompI}, 4, 2]^2, \text{Extract}[\text{CompI}, 4, 2] \text{Extract}[\text{CompI}, 5, 2], 0\}, \\
\{0, 0, \text{Extract}[\text{CompI}, 4, 2] \text{Extract}[\text{CompI}, 5, 2], \text{Extract}[\text{CompI}, 5, 2]^2, 0\}, \\
\{\text{Extract}[\text{CompI}, 1, 2] \text{Extract}[\text{CompI}, 6, 2], \text{Extract}[\text{CompI}, 3, 2] \text{Extract}[\text{CompI}, 6, 2], \\
0, 0, \text{Extract}[\text{CompI}, 6, 2]^2\}; \\
(* Assembles of matrix components for the eigenfunction problem. *)

\[
\text{Fill} = i - 2; \\
\text{For}\{j = 1, j \leq 4, j++.,} \\
\text{CurrS11} = \text{S11F}[\text{Nn}, i, \text{PlyA}, \text{Extract}[WGP, \{i, j\}]]; \\
\text{CurrS12} = \text{S12F}[\text{Nn}, i, \text{PlyA}, \text{Extract}[WGP, \{i, j\}]]; \\
\text{CurrS13} = \text{S13F}[\text{Nn}, i, \text{PlyA}, \text{Extract}[WGP, \{i, j\}]]; \\
\]
CurrS23 = S23F[Nn, i, PlyA, Extract[WGP, {i, j}]];
CurrS33 = S33F[Nn, i, PlyA, Extract[WGP, {i, j}]];
S11j = Join[0 * Range[Fill], CurrS11, 0 * Range[Nn - 1 - Length[CurrS11] - If[Fill > 0, Fill, 0]], 0 * Range[Nn - 1]];
S12j = Join[0 * Range[Nn - 1], 0 * Range[Fill], CurrS12, 0 * Range[Nn - 1 - Length[CurrS12] - If[Fill > 0, Fill, 0]]];
S13j = Join[0 * Range[Fill], CurrS13, 0 * Range[Nn - 1 - Length[CurrS13] - If[Fill > 0, Fill, 0]], 0 * Range[Nn - 1]];
S23j = Join[0 * Range[Nn - 1], 0 * Range[Fill], CurrS23, 0 * Range[Nn - 1 - Length[CurrS23] - If[Fill > 0, Fill, 0]]];
S33j = Join[CurrS33, 0 * Range[Nn - 1]];
E0 = E0 + 2 * Extract[WGW, {i, j}] * Transpose[{S33j}].{S33j} * Extract[RComp, {2, 2}];
E1 = E1 - 2 * Extract[WGW, {i, j}] * Transpose[{S13j}].{S13j} * Extract[RComp, {3, 3}];
E2 = E2 - 2 * Extract[WGW, {i, j}] * Transpose[{S23j}].{S23j} * Extract[RComp, {4, 4}];
E3 = E3 + 2 * Extract[WGW, {i, j}] * (Transpose[{S33j}].{S11j} + Transpose[{S11j}].{S33j}) * Extract[RComp, {1, 2}];
E4 = E4 - 2 * Extract[WGW, {i, j}] * (Transpose[{S13j}].{S23j} + Transpose[{S23j}].{S13j}) * Extract[RComp, {3, 4}];
E5 = E5 + 2 * Extract[WGW, {i, j}] * (Transpose[{S12j}].{S33j} + Transpose[{S33j}].{S12j}) * Extract[RComp, {2, 5}];
E6 = E6 + 2 * Extract[WGW, {i, j}] * Transpose[{S11j}].{S11j} * Extract[RComp, {1, 1}];
E7 = E7 + 2 * Extract[WGW, {i, j}] * Transpose[{S12j}].{S12j} * Extract[RComp, {5, 5}];
E8 = E8 + 2 * Extract[WGW, {i, j}] * (Transpose[{S11j}].{S12j} + Transpose[{S12j}].{S11j}) * Extract[RComp, {1, 5}];

(* Assembles matrices in the eigenfunction problem. *)
AA = E0;
BB = E1 + E2 + E3 + E4 + E5;
CC = E6 + E7 + E8;

(* Calculates eigenvalues and eigenvectors of the eigenfunction problem. *)
EMat1 = Transpose[Join[Transpose[Join[-CC, Table[0, {2 * Nn - 2}, {2 * Nn - 2}]]],
            Transpose[Join[Table[0, {2 * Nn - 2}, {2 * Nn - 2}], AA]]]];
EMat2 = Transpose[Join[Transpose[Join[BB, AA]], Transpose[Join[AA,
            Table[0, {2 * Nn - 2}, {2 * Nn - 2}]]]]];
EValues2 = Eigenvalues[{EMat1, EMat2}];
EVectors = Eigenvectors[{EMat1, EMat2}];
EValues = 0 * EValues2;
For[i = 1, i <= Length[EValues2], i++,
    EValues = ReplacePart[EValues, (Extract[EValues2, i])^1/2, i];
];

(* Formats eigenvalue and eigenvector data for use later in the code. *)
Outmat = {{}, {}};
Outmat = 0 * Transpose[{Range[Length[EValues] + 1]}];
Outmat = PadRight[Outmat, {Length[EValues] + 1, Length[EValues] + 1}];
Outmat = ReplacePart[Outmat, EValues, 1];
For[i = 1, i < Length[EValues2], i++,
    Outmat = ReplacePart[Outmat, Extract[EVectors, i + 1], i + 1];
];
Return[Outmat];

(* Extracts eigenvalue data from matrices generated in Eigenfunction. *)
ValExtract[Region_] :=
Return[Extract[Transpose[Drop[Transpose[Drop[Region, {2, Length[Region]}]],
    {3 * (Length[Region] - 1)/4 + 1, Length[Region] - 1}]], 1];
(* Extracts eigenvector data from matrices generated in Eigenfunction. *)

VecExtract[Region_] :=
If[True,
  UnNorm = Transpose[Drop[Transpose[Drop[Drop[Region, {1, 1}], {3*(Length[Region] - 1)/4 + 1, Length[Region] - 1}]], {(Length[Region] - 1)/2 + 1, Length[Region] - 1}]],
  NormEigen = Table[Extract[UnNorm, I]/Norm[Extract[UnNorm, I], 2],
    {I, 1, Length[UnNorm]}];
  Return[NormEigen];
];

(* Calculates derivative of exponential terms in Equations 4.24a and 4.24b using eigenvalue and eigenvector data. *)

Deriv[EValue_, EVec_] :=
If[True,
  DerivMatrix = Table[Extract[EVec, {p, q}] * Extract[EValue, q], {p, 1, Length[EValue]/3},
    {q, 1, Length[EValue]}];
  Return[DerivMatrix];
];

(* Assembles Omega matrix for Region Region. *)

Omega[Region_, PlyA_, LVec_] :=
If[True,
  Nn = Length[PlyA];
  (* Implements Gaussian Quadrature tables for integration of h-matrices in x.3. *)
}
\[
GP = \left\{ -\sqrt{\left(3 + 2\sqrt{6/5}\right)/7}, -\sqrt{\left(3 - 2\sqrt{6/5}\right)/7}, \sqrt{\left(3 - 2\sqrt{6/5}\right)/7}, \sqrt{\left(3 + 2\sqrt{6/5}\right)/7} \right\};
\]

\[
GW = \left\{ \frac{18 - \sqrt{30}}{36}, \frac{18 + \sqrt{30}}{36}, \frac{18 + \sqrt{30}}{36}, \frac{18 - \sqrt{30}}{36} \right\};
\]

WGP = \{\};

WG W = \{\};

(* Extracts eigenvalue and eigenvector entries for current region. *)

OEValue = ValExtract[Region];

OEVector = VecExtract[Region];

(* Separates eigenvector terms to terms acting on interfacial stress functions \( F(x_1) \)
and those acting on \( G(x_1) \). *)

EVecF = Drop[Transpose[OEVector], {Length[Transpose[OEVector]]/2 + 1, Length[Transpose[OEVector]]}];

EVecG = Drop[Transpose[OEVector], {1, Length[Transpose[OEVector]]/2}];

(* Calculates derivatives of \( F(x_1) \) and \( G(x_1) \) with respect to \( x_1 \). *)

EVecFPrime = Deriv[OEValue, EVecF];

EVecFDPrime = Deriv[OEValue, EVecFPrime];

EVecGPrime = Deriv[OEValue, EVecG];

(* Initializes components of Omega matrix. *)

E0 = Table[0, {3 * Nn - 3}, {3 * Nn - 3}];

E1 = E0;

E2 = E0;

E3 = E0;

E4 = E0;

E5 = E0;

E6 = E0;

E7 = E0;

E8 = E0;

(* Initializes output matrix. *)
OmegaOut = Table[0, {3 * Nn - 3}, {3 * Nn - 3}];

(* Modifies Gaussian Quadrature Terms for varying ply/sublayer thicknesses. *)
For[i = 1, i <= Nn, i++]
WGP = Append[WGP, Extract[PlyA, {i, 1}]/2 * GP];
WGW = Append[WGW, Extract[PlyA, {i, 1}]/2 * GW];
);

(* Calculates reduced compliance matrix *)
For[i = 1, i <= Nn, i++]
CompI = Compliance[Extract[PlyA, {i, 2}]];
RComp =
{Extract[CompI, 1, 1], Extract[CompI, 1, 3], 0, 0, Extract[CompI, 1, 6]},
{Extract[CompI, 1, 3], Extract[CompI, 3, 3], 0, 0, Extract[CompI, 3, 6]},
{0, 0, Extract[CompI, 4, 4], Extract[CompI, 4, 5], 0},
{0, 0, Extract[CompI, 4, 5], Extract[CompI, 5, 5], 0},
{Extract[CompI, 1, 6], Extract[CompI, 2, 6], 0, 0, Extract[CompI, 6, 6]}-
1/Extract[CompI, {2, 2}]*

{Extract[CompI, 1, 2]^2, Extract[CompI, 1, 2] Extract[CompI, 3, 2], 0, 0,
Extract[CompI, 1, 2] Extract[CompI, 6, 2]},
{Extract[CompI, 1, 2] Extract[CompI, 3, 2], Extract[CompI, 3, 2]^2, 0, 0,
Extract[CompI, 3, 2] Extract[CompI, 6, 2]},
{0, 0, Extract[CompI, 4, 2]^2, Extract[CompI, 4, 2] Extract[CompI, 5, 2], 0},
{0, 0, Extract[CompI, 4, 2] Extract[CompI, 5, 2], Extract[CompI, 5, 2]^2, 0},
{Extract[CompI, 1, 2] Extract[CompI, 6, 2], Extract[CompI, 3, 2] Extract[CompI, 6, 2],
0, 0, Extract[CompI, 6, 2]^2};

(* Calculates output matrix components *)
Fill = i - 2;
For[j = 1, j <= 4, j++]
CurrS11 = S11F[Nn, i, PlyA, Extract[WGP, {i, j}]];
CurrS12 = S12F[Nn, i, PlyA, Extract[WGP, {i, j}]];

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CurrS13 = S13F[Nn, i, PlyA, Extract[WGP, \{i, j\}]];
CurrS23 = S23F[Nn, i, PlyA, Extract[WGP, \{i, j\}]];
CurrS33 = S33F[Nn, i, PlyA, Extract[WGP, \{i, j\}]];
s11j = {Join[0 * Range[Fill], CurrS11, 0 * Range[Nn - 1 - Length[CurrS11] - If[Fill > 0, Fill, 0]]];
s12j = {Join[0 * Range[Fill], CurrS12, 0 * Range[Nn - 1 - Length[CurrS12] - If[Fill > 0, Fill, 0]]];
s13j = {Join[0 * Range[Fill], CurrS13, 0 * Range[Nn - 1 - Length[CurrS13] - If[Fill > 0, Fill, 0]]];
s23j = {Join[0 * Range[Fill], CurrS23, 0 * Range[Nn - 1 - Length[CurrS23] - If[Fill > 0, Fill, 0]]];
s33j = {CurrS33};
E0 = E0 + 2 * Extract[WGW, \{i, j\}] * Transpose[s33j.EVecFDPrime].s33j.EVecFDPrime
   *Extract[RComp, \{2, 2\}];
E1 = E1 + 2 * Extract[WGW, \{i, j\}] * Transpose[s13j.EVecFPrime].s13j.EVecFPrime
   *Extract[RComp, \{3, 3\}];
E2 = E2 + 2 * Extract[WGW, \{i, j\}] * Transpose[s23j.EVecGPrime].s23j.EVecGPrime
   *Extract[RComp, \{4, 4\}];
E3 = E3 + 2 * Extract[WGW, \{i, j\}] * (Transpose[s33j.EVecFDPrime].s11j.EVecF
   + Transpose[s11j.EVecF].s33j.EVecFDPrime) * Extract[RComp, \{1, 2\}];
E4 = E4 + 2 * Extract[WGW, \{i, j\}] * (Transpose[s13j.EVecFPrime].s23j.EVecGPrime
   + Transpose[s23j.EVecGPrime].s13j.EVecFPrime) * Extract[RComp, \{3, 4\}];
E5 = E5 + 2 * Extract[WGW, \{i, j\}] * (Transpose[s33j.EVecFDPrime].s12j.EVecG
   + Transpose[s12j.EVecG].s33j.EVecFDPrime) * Extract[RComp, \{2, 5\}];
E6 = E6 + 2 * Extract[WGW, \{i, j\}] * Transpose[s11j.EVecF].s11j.EVecF
   *Extract[RComp, \{1, 1\}];
E7 = E7 + 2 * Extract[WGW, \{i, j\}] * Transpose[s12j.EVecG].s12j.EVecG
   *Extract[RComp, \{5, 5\}];
E8 = E8 + 2 * Extract[WGW, \{i, j\}] * (Transpose[s11j.EVecF].s12j.EVecG
   *Extract[RComp, \{1, 5\}];
\text{Transpose[sl2j.EVecG].sllj.EVecF} \ast \text{Extract[RComp, \{1, 5\}]};

\text{OmegaOut} = \text{OmegaOut} + E0 + E1 + E2 + E3 + E4 + E5 + E6 + E7 + E8;

\text{FinalOmegaOut} = \text{Table[Extract[OmegaOut, \{I, J\}]} \ast

\left( 1 - e^{(\text{Extract[LVec,1] - Extract[LVec,2]})\ast(\text{Extract[OEValue,l] + Extract[OEValue,j]})} \right)

/\left(\text{Extract[OEValue, I] + Extract[OEValue, J]}, \{I, 1, \text{Length}[OmegaOut]\}, \{J, 1, \text{Length}[OmegaOut]\}\right);

\text{Return[FinalOmegaOut];}

(* \text{Calculates values used for enforcing free-surface and stress-continuity across dropoffs in Sigma}_{11} \text{ for Ply CPly at local x}_{3}\text{-coordinate zval. Terms here are assembled into matrix Gamma^A. *})

S11BC[EVecF, OEValue, Ply, LVec, zval, CPly] := 

\text{If[True,}

\text{t = Extract[Ply, \{CPly, 1\}]};

\text{If[CPly == 1,}

\text{Vec1l = -1/t \ast (Extract[EVecF, \{CPly\}]));}

,\text{If[CPly == \text{Length}[Ply],}

\text{Vec1l = 1/t \ast (Extract[EVecF, \{CPly - 1\}]));}

,\text{Vec1l = 1/t \ast (Extract[EVecF, \{CPly - 1\}]) - 1/t \ast (Extract[EVecF, \{CPly\}]));}

\text{OutVec1l = Table[Extract[Vec1l, j] \ast e^{Extract[OEValue,j]}*0, \{i, 1, 1\}, \{j, 1, 3 \ast \text{Length}[Ply] - 3\};}

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Return[OutVec11];
]
(*---------------------------------------------*)
(* Calculates values used for enforcing free-surface and stress-continuity across dropoffs in
Sigma_12 for Ply CPly at local x_3-coordinate zval. Terms here are assembled into matrix
Gamma`B. *)
S12BC[EVecG_, EValue_, Ply_, LVec., zval, CPly.]:= If[True,
  t = Extract[Ply, {CPly, 1}];
  If[CPly == 1,
    Vec12 = -1/t * (Extract[EVecG, {CPly}]);
  ,
  If[CPly == Length[Ply],
    Vec12 = 1/t * (Extract[EVecG, {CPly - 1}]);
  ,
    Vec12 = 1/t * (Extract[EVecG, {CPly - 1}]) - 1/t * (Extract[EVecG, {CPly}]);
  ];
  ];
OutVec12 = Table [Extract[Vec12, j] * e^Extract[EValue, j]*0., {i, 1, 1},
  {j, 1, 3 * Length[Ply] - 3}];
Return[OutVec12];
]
(*---------------------------------------------*)
(* Calculates values used for enforcing free-surface and stress-continuity across dropoffs in
Sigma_13 for Ply CPly at local x_3-coordinate zval. Terms here are assembled into matrix
Theta. *)
S13BC[EVecF_, EValue_, Ply_, LVec., zval, CPly.]:= If[True,
\[ t = \text{Extract}[\text{Ply}, \{\text{C Ply}, 1\}]; \]

\[
\text{If}[\text{C Ply} == 1, \\
\text{Vec13} = (\text{Extract}[[\text{E VecF}, \{\text{C Ply}\}]] \ast (z\text{val} / t + 1/2)); \\
, \\
\text{If}[\text{C Ply} == \text{Length}[\text{Ply}], \\
\text{Vec13} = -(\text{Extract}[[\text{E VecF}, \{\text{C Ply} - 1\}]] \ast (z\text{val} / t - 1/2)); \\
, \\
\text{Vec13} = (\text{Extract}[[\text{E VecF}, \{\text{C Ply}\}]] \ast (z\text{val} / t + 1/2) - (\text{Extract}[[\text{E VecF}, \{\text{C Ply} - 1\}]] \ast (z\text{val} / t - 1/2)); \\
]; \\
]; \\
\text{OutVec13} = \text{Table}[(\text{Extract}[\text{Vec13}, j] \ast \text{E Value}[j] \ast 0 \ast \text{Extract}[\text{E Value}, j], \{j, 1, 3 * \text{Length}[\text{Ply}] - 3\}]; \\
\text{Return}[\text{OutVec13}]; \\
]$

(* Assembles Gamma and Theta Matricies for region Region in the negative \(x_1\)-direction of a dropoff. *)

\text{GammaThetaMat}[\text{Region}, \text{PlyA}, \text{LVec}]:=

\text{If}[\text{True}, \\
(* Initalize Output matrices. *) \\
\text{GaAF} = \{\text{Null}\}; \\
\text{GaAG} = \{\text{Null}\}; \\
\text{ThA} = \{\text{Null}\}; \\
(* Extract Region Eigenvalues and Eigenvectors. *) \\
\text{GTEValue} = \text{ValExtract}[\text{Region}]; \\
\text{GTEVector} = \text{VecExtract}[\text{Region}]; \\
(* Separate Eigenvector terms to those acting on \(F(x_1)\) and \(G(x_1)\). *)

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EVecF = Drop[Transpose[GTEVector], {Length[Transpose[GTEVector]]}/2 + 1, Length[Transpose[GTEVector]]];

EVecG = Drop[Transpose[GTEVector], {1, Length[Transpose[GTEVector]]}/2];

(* Assembles Gamma and Theta Matrices. *)

For[i = 1, i ≤ Length[PlyA], i++,
GaAF = Join[GaAF, S11BC[EVecF, GTEValue, PlyA, LVec, 1/2 * Extract[PlyA, {i, 1}], i]]; 
GaAG = Join[GaAG, S12BC[EVecG, GTEValue, PlyA, LVec, 1/2 * Extract[PlyA, {i, 1}], i]]; 
ThCo = 1; 
ThA = Join[ThA, ThCo * S13BC[EVecF, GTEValue, PlyA, LVec, 0, i]]; 
]

(* "Cleans up" initialization of output matrices. *)

GaAF = Drop[GaAF, 1];
GaAG = Drop[GaAG, 1];
ThA = Drop[ThA, 1];
GaAF = Take[GaAF, {1, Length[GaAF] - 1}];
GaAG = Take[GaAG, {1, Length[GaAG] - 1}];
ThA = Take[ThA, {1, Length[ThA] - 1}];
Return[{GaAF, GaAG, ThA}];

(* Assembles Gamma and Theta Matrices for region Region- in the positive x-1-direction of a dropoff. *)

GammaThetaMatCut[Region_, PlyB_, LVec_, PlyA_]:=
If[True,
(* Initialize Output matrices. *)
GaAF = {Null};
GaAG = {Null};
ThA = {Null};

(* Extract Region Eigenvalues and Eigenvectors. *)
GTEValue = ValExtract[Region];
GTEVector = VecExtract[Region];

(* Seperate Eigenvector terms to those acting on F(x-1) and G (x - 1). *)
EVecF = Drop[Transpose[GTEVector], {Length[Transpose[GTEVector]]/2 + 1, Length[Transpose[GTEVector]]}];
EVecG = Drop[Transpose[GTEVector], {1, Length[Transpose[GTEVector]]/2}];

(* Assembles Gamma and Theta Matricies. *)
For[i = 1, i < Length[PlyA], i++,
  If[i > Length[PlyA] - Length[PlyB],
    k = i - (Length[PlyA] - Length[PlyB]);
    GaAF = Join[GaAF, S11BC[EVecF, GTEValue, PlyB, LVec, 1/2 * Extract[PlyB, {k, 1}], k]];
    GaAG = Join[GaAG, S12BC[EVecG, GTEValue, PlyB, LVec, 1/2 * Extract[PlyB, {k, 1}], k]];
    ThCoC = 1;
    ThA = Join[ThA, ThCoC * S13BC[EVecF, GTEValue, PlyB, LVec, 0, k]];
    ,
    GTFill = Table[0, {FillI, 1, 1}, {FilIJ, 1, 3 * Length[PlyB] - 3}];
    GaAF = Join[GaAF, GTFill];
    GaAG = Join[GaAG, GTFill];
    ThA = Join[ThA, GTFill];
  ];
];

(* "Cleans up" initialization of output matrices. *)
GaAF = Drop[GaAF, 1];
GaAG = Drop[GaAG, 1];
ThA = Drop[ThA, 1];
GaAF = Take[GaAF, {1, Length[GaAF] - 1}];

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GaAG = Take[GaAG, {1, Length[GaAG] - 1}];
ThA = Take[-ThA, {1, Length[ThA] - 1}];
Return[-1 * {GaAF, GaAG, ThA}];
(* Assembles the matrix used in the final linear system in Equations 4.32 for Step 1 and 6.5 for Step 2. The manner in which the matrix is assembled is by assembling all independent columns in the matrix and filling the rest of the matrix, as the matrix is always symmetric. *)
MainAssembly[OmSystem_, GammaSystem_, ThetaSystem_, Layup_] :=
If[True,
(* Defines local variables for number of regions and number of dropoffs present in laminate. *)
NRegions = Length[Transpose[Layup]];
NDrops = NRegions - 1;
(* Initializes matrix of independent columns in matrix. *)
ANumber = 2 * NRegions - 2;
ABCRow = 3 * NDrops;
(* Initializes independent-column matrix. *)
AComp = 0 * Transpose[{Range[ANumber]}];
AComp = PadRight[AComp, {1, ANumber}];
(* Counter used for Boundary-condition matrix assembly. *)
BCCounter = 1;
(* Assembles Omega matrices into output matrix. *)
For[i = 1, i <= ANumber, i++,
ATemp = Table[0, {U, 1, 1}, {V, 1, 3 * Length[Extract[Layup, {1, Ceiling[(i + 1)/2]}]] - 3}];
For[j = 1, j <= ANumber, j++,
If[i == j,
AMult = If[j ≠ 1 && Mod[j, 2] == 1, -1, 1];
ATemp = Join[ATemp, AMult*Extract[OmSystem, {1, Ceiling[(j + 1)/2]}]];,
OFill = Table[0, {m, 1, 3*Length[Extract[Layup, {1, Ceiling[(j + 1)/2]}]] - 3},
{n, 1, (3*Length[Extract[Layup, {1, Ceiling[(i + 1)/2]}]] - 3)}];
ATemp = Join[ATemp, OFill];

(* Assembles Gamma and Theta matrices into output matrix. *)
BCMultTable = Table[0, {J, 1, NDrops}];
If[i == 1,
BCMultTable = ReplacePart[BCMultTable, 1, i];,
If[i == ANumber,
BCMultTable = ReplacePart[BCMultTable, 1, NDrops];,
BCMultTable = ReplacePart[BCMultTable, 1, Ceiling[(i - 1)/2]];,
BCMultTable = ReplacePart[BCMultTable, 1, Ceiling[(i + 1)/2]]];
BCTemp = {Null};
For[j = 1, j ≤ NDrops, j++,
If[Extract[BCMultTable, j] == 1,
BCTemp = Join[BCTemp, Extract[GammaSystem, {1, BCCounter}]]];
BCTemp = Join[BCTemp, Extract[GammaSystem, {2, BCCounter}]]];
BCTemp = Join[BCTemp, Extract[ThetaSystem, {1, BCCounter}]];
BCCounter++;,
BCFill = Table[0, {J, 1, (3*Length[Extract[Layup, {1, j}]] - 3)}],
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\{J, 1, (3 \times \text{Length}[\text{Extract}[\text{Layup}, \{1, \text{Ceiling}[(i + 1)/2]\}] - 3)]\};

\text{BCTemp} = \text{Join}[\text{BCTemp}, \text{BCFill}];

\text{BCTemp} = \text{Drop}[\text{BCTemp}, 1];

(* Cleans up initialization of independent-column matrix. *)

\text{ATemp} = \text{Join}[\text{ATemp}, \text{BCTemp}];

\text{ATemp} = \text{Drop}[\text{ATemp}, 1];

\text{AComp} = \text{ReplacePart}[\text{AComp}, \text{ATemp}, \{1, i\}];

(* Assembles symmetric half of output matrix from previously-calculated terms.*)

\text{ANonSym} = \text{Transpose}[\text{Extract}[\text{AComp}, \{1, 1\}]];

\text{For}[\text{Ai} = 2, \text{Ai} \leq \text{ANumber}, \text{Ai}++]

\text{ANonSym} = \text{Join}[\text{ANonSym}, \text{Transpose}[\text{Extract}[\text{AComp}, \{1, \text{Ai}\}]]];

\text{OmTotalLength} = 0;

\text{For}[\text{OmLi} = 1, \text{OmLi} \leq \text{NRegions}, \text{OmLi}++]

\text{OmTotalLength} = \text{OmTotalLength} + (2 - \text{KroneckerDelta}[1, \text{OmLi}] - \text{KroneckerDelta}[, \text{NRegions}, \text{OmLi}]) \times \text{Length}[\text{Extract}[\text{OmSystem}, \{1, \text{OmLi}\}]];

\text{AMatrix} = \text{Table}[
\text{If}[\text{I} \leq \text{OmTotalLength}, \text{Extract}[\text{ANonSym}, \{\text{I}, \text{J}\}],
\text{If}[\text{J} \leq \text{OmTotalLength}, \text{Extract}[\text{ANonSym}, \{\text{J}, \text{I}\}],
0]], \{\text{I}, 1, \text{Length}[	ext{Transpose}[	ext{ANonSym}]]\}, \{\text{J}, 1, \text{Length}[	ext{Transpose}[	ext{ANonSym}]]\}];

\text{Return}[\text{AMatrix}];

(*-----------------------------------------------*)
(* Calculates and assembles Right-Hand Vector in Equation 4.22, as well as far-field stresses from CLPT. *)

BVector[OmSystem_, Layup_]:= If[True,
(* Defines local variables for number of regions and number of dropoffs present in laminate. *)
NRegions = Length[Transpose[Layup]];
NDrops = NRegions - 1;
(* Initializes output vector. *)
Nn = Sum[(2 - KroneckerDelta[1, i] - KroneckerDelta[NRegions, i]) * Length[Extract[OmSystem, {1, i}]], {i, NRegions}];
BB = Table[0, {i, 1, Nn}, {j, 1, 1}];
(* Far-field load per unit length on the laminate. This follows CLPT conventions in the form \{P_{-11}, P_{-22}, P_{12}\}. *)
Load = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix};
(* Calculates CLPT solution. *)
CLPTSystem = 0 * Transpose[{Range[NRegions]}];
CLPTSystem = PadRight[RegESystem, {1, NRegions}];
For[CLPTi = 1, CLPTi \leq NRegions, CLPTi++,
(* Calculates Ply Strains. *)
AMat = 2 * LPTM[Extract[Layup, {1, CLPTi}], A];
\epsilon Strain = LinearSolve[AMat, Load];
(* Calculates ply/sublayer stresses. *)
\sigma Temp = Table[0, \{Q, 1, Length[Extract[Layup, {1, CLPTi}]]\}, \{X, 1, 3\}];
For[CLPTj = 1, CLPTj \leq Length[Extract[Layup, {1, CLPTi}]], CLPTj++,
CLPTSol = TwoDElas[Extract[Extract[Layup, {1, CLPTi}], {CLPTj, 2}], 1].\epsilon Strain;
\sigma Temp = ReplacePart[\sigma Temp, Extract[CLPTSol, \{1, 1\}], \{CLPTj, 1\}];
\(\sigma\text{Temp} = \text{ReplacePart}[\sigma\text{Temp}, \text{Extract}[\text{CLPTSol}, \{2, 1\}], \{\text{CLPTj}, 2\}];\)
\(\sigma\text{Temp} = \text{ReplacePart}[\sigma\text{Temp}, \text{Extract}[\text{CLPTSol}, \{3, 1\}], \{\text{CLPTj}, 3\}];\)
\)
\(\text{CLPTSystem} = \text{ReplacePart}[\text{CLPTSystem}, \sigma\text{Temp}, \{1, \text{CLPTi}\}];\)
\)
(* Assembles output vector. *)
\)
\(\text{For}[B\text{Filli} = 1, B\text{Filli} \leq \text{NDrops}, B\text{Filli}++,\)
\(\text{LDiff} = (\text{Length}[\text{Extract}[\text{Layup}, \{1, B\text{Filli}\}]] - \text{Length}[\text{Extract}[\text{Layup}, \{1, B\text{Filli} + 1\}]]);\)
\(\sigma\text{A} = \text{Extract}[\text{CLPTSystem}, \{1, B\text{Filli}\}];\)
\(\sigma\text{B} = \text{Join}[\text{Table}[0, \{\text{Bi}, 1, \text{LDiff}\}, \{\text{Bj}, 1, 3\}], \text{Extract}[\text{CLPTSystem}, \{1, B\text{Filli} + 1\}]];\)
\(\sigma11\text{Diff} = \text{Drop}[\sigma\text{B} - \sigma\text{A}, \{\text{Length}[\sigma\text{B}], \{2, 3\}];\)
\(\sigma12\text{Diff} = \text{Drop}[\sigma\text{B} - \sigma\text{A}, \{\text{Length}[\sigma\text{B}], \{1, 2\}];\)
\(\sigma13\text{Fill} = \text{Table}[0, \{i, 1, \text{Length}[\sigma11\text{Diff}]\}, \{j, 1, 1\}];\)
\(\text{BB} = \text{Join}[	ext{BB}, \sigma11\text{Diff}, \sigma12\text{Diff}, \sigma13\text{Fill}];\)
\)
\(\text{Return}[[\text{BB}, \text{CLPTSystem}]];\)
\)
\(* \text{BEGIN MAIN ROUTINE} \)
\(* \text{END MAIN ROUTINE} \)
\)
(* All subroutines must be read before this script can be executed. This script denotes the "main" routine of the implementation of the model. *)
\)
(* Initializes storage for eigenfunction and final linear system matrices. *)
\(\text{RegESystem} = 0 \ast \text{Transpose}[\{\text{Range}[\text{NRegions}]\}];\)
\(\text{RegESystem} = \text{PadRight}[\text{RegESystem}, \{1, \text{NRegions}\}];\)
\(\text{OmSystem} = 0 \ast \text{Transpose}[\{\text{Range}[2 \ast (\text{NRegions} - 1)]\}];\)
\(\text{OmSystem} = \text{PadRight}[\text{OmSystem}, \{1, \text{NRegions}\}];\)
\(\text{OmCSystem} = 0 \ast \text{Transpose}[\{\text{Range}[\text{NRegions} - 2]\}];\)
OmCSystem = PadRight[OmSystem, {1, NRegions - 2}];
GammaSystem = 0 * Transpose[{Range[2 * (NRegions - 1)]}];
GammaSystem = PadRight[GammaSystem, {2, 2 * (2 * NRegions - 3)}];
ThetaSystem = 0 * Transpose[{Range[2 * (NRegions - 1)]}];
ThetaSystem = PadRight[ThetaSystem, {1, 2 * (2 * NRegions - 3)}];

(* For all Regions *)
For[w = 1, w ≤ NRegions, w++,
(* Solves and stores eigenfunction problem solution. *)
CurReg = Extract[Layup, {1, w}];
EFunc = Eigenfunction[CurReg];
RegESystem = ReplacePart[RegESystem, EFunc, {1, w}];
If[w == 1, (* For Region A *)
(* Calculates and stores Omega, Gamma, and Theta matrices. *)
OmTemp = Omega[Extract[RegESystem, {1, w}], CurReg, {-∞, 0}];
{GFTemp, GGTemp, ThTemp} = GammaThetaMat[Extract[RegESystem, {1, w}],
CurReg, {0, -∞}];
OmSystem = ReplacePart[OmSystem, OmTemp, {1, w}];
GammaSystem = ReplacePart[GammaSystem, GFTemp, {1, w}];
GammaSystem = ReplacePart[GammaSystem, GGTemp, {2, w}];
ThetaSystem = ReplacePart[ThetaSystem, ThTemp, {1, w}];
]
(* For final Region in problem *)
(* Calculates and stores Omega, Gamma, and Theta matrices. *)
OmTemp = Omega[Extract[RegESystem, {1, w}], CurReg, {Extract[Len, w - 1],
Extract[Len, w]}];
{GFTemp, GGTemp, ThTemp} = GammaThetaMatCut[Extract[RegESystem, {1, w}],
CurReg, {Extract[Len, w - 1], Extract[Len, w]}, Extract[Layup, {1, w - 1}]];
OmSystem = ReplacePart[OmSystem, OmTemp, {1, w}];
GammaSystem = ReplacePart[GammaSystem, GFTemp, {1, 2 * (2 * w - 3)}];
GammaSystem = ReplacePart[GammaSystem, GGTemp, {2, 2 \* (2 \* w - 3)}];
ThetaSystem = ReplacePart[ThetaSystem, ThTemp, {1, 2 \* (2 \* w - 3)}];

(* Assemble Matrix in Final linear system. *)

AMatrix = MainAssembly[OmSystem, GammaSystem, ThetaSystem, Layup];

(* Assemble right-hand Vector *)

{BVec, CLPTSol} = BVector[OmSystem, Layup];

(* Solves linear system for unknown stress coefficients. *)

Solution = LinearSolve[AMatrix, BVec];

BEGIN SOLUTION ASSEMBLY AND POSTPROCESS SUBROUTINES

(* This subroutine calculates the specific forms of the stresses via substitution of eigenvalues,
eigenvectors, and unknown stress coefficients in the assumed stress shapes. *)

(* Matrix of unknown stress coefficients *)

SolutionBreak = Table[0, {i, 1, 1}, {j, 1, NRegions}, {k, 1, 2}];

(* Matrix of x_1-variation of ply stresses *)

XAssembly = Table[0, {i, 1, 3}, {j, 1, NRegions}, {k, 1, 2}];

(* Matrix of x_3-variation of ply stresses *)

ZAssembly = Table[0, {i, 1, 5}, {j, 1, NRegions}];

(* Output matrix of ply stresses for each ply in each region *)

PlyStresses = Table[0, {i, 1, 5}, {k, 1, NRegions}];

(* For all Regions *)

For[Si = 1, Si \leq NRegions, Si++,

(* Extracts Region Eigenvalues and Eigenvectors. *)

RegionEigenvalues = ValExtract[Extract[RegESystem, {1, Si}]];
RegionEigenvectors = VecExtract[Extract[RegESystem, {1, Si}]];

(* initializes x_1-variation derivative matrices. * )
RegAssembly = Table[0, {p, 1, 2 * Length[RegionEigenvalues]/3}, {q, 1, Length[RegionEigenvalues]}];
DRegAssembly = Table[0, {p, 1, 2 * Length[RegionEigenvalues]/3}, {q, 1, Length[RegionEigenvalues]}];
DDRegAssembly = Table[0, {p, 1, 2 * Length[RegionEigenvalues]/3}, {q, 1, Length[RegionEigenvalues]}];

RegAssembly2 = RegAssembly;
DRegAssembly2 = DRegAssembly;
DDRegAssembly2 = DDRegAssembly;

SolutionBreakT = Table[0, {i, 1, Length[Extract[OmSystem, {1, Si}]]}];

(* For all Regions except the final Region, calculates x_1-variation terms and relevant unknown stress coefficients. *)

If[Si != NRegions,

RegAssembly = Table[Extract[RegionEigenvectors, {q, p}] * Extract[RegionEigenvalues, q] * (z - Extract[Len, Si]), {p, 1, 2 * Length[RegionEigenvalues]/3}, {q, 1, Length[RegionEigenvalues]}];

DRegAssembly = Table[Extract[RegionEigenvectors, {q, p}] * Extract[RegionEigenvalues, q] * (z - Extract[Len, Si]) * Extract[RegionEigenvalues, q], {p, 1, 2 * Length[RegionEigenvalues]/3}, {q, 1, Length[RegionEigenvalues]}];

DDRegAssembly = Table[Extract[RegionEigenvectors, {q, p}] * Extract[RegionEigenvalues, q] * (z - Extract[Len, Si]) * (Extract[RegionEigenvalues, q])^2, {p, 1, 2 * Length[RegionEigenvalues]/3}, {q, 1, Length[RegionEigenvalues]}];

SolutionBreakT = Take[Solution, {Sum[(2 - KroneckerDelta[1, j]) * Length[Extract[OmSystem, {1, j}]]], {j, 1, Si}] - Length[Extract[OmSystem, {1, Si}]] + 1, Sum[(2 - KroneckerDelta[1, j]) * Length[Extract[OmSystem, {1, j}]], {j, 1, Si}]]];

(* For all Regions except Region A, calculates x_1-variation terms and relevant unknown stress coefficients. *)
If $\text{Si} \neq 1,$

$$\text{RegAssembly2} = \text{Table}[\text{Extract}[	ext{RegionEigenvectors}, \{q, p\}] e^{-\text{Extract}[	ext{RegionEigenvalues}, q] (x - \text{Extract}[\text{Len}, \text{Si} - 1])}, \{q, 1, \text{Length}[	ext{RegionEigenvalues}]\}] / 3, \{q, 1, \text{Length}[	ext{RegionEigenvalues}]\}]$$

$$\text{DRegAssembly2} = \text{Table}[\text{Extract}[	ext{RegionEigenvectors}, \{q, p\}] e^{-\text{Extract}[	ext{RegionEigenvalues}, q] (x - \text{Extract}[\text{Len}, \text{Si} - 1])}, \{p, 1, 2 \times \text{Length}[	ext{RegionEigenvalues}] / 3\}, \{q, 1, \text{Length}[	ext{RegionEigenvalues}]\}]$$

$$\text{DDRegAssembly2} = \text{Table}[\text{Extract}[	ext{RegionEigenvectors}, \{q, p\}] e^{-\text{Extract}[	ext{RegionEigenvalues}, q] (x - \text{Extract}[\text{Len}, \text{Si} - 1])}, \{p, 1, 2 \times \text{Length}[	ext{RegionEigenvalues}] / 3\}, \{q, 1, \text{Length}[	ext{RegionEigenvalues}]\}] (\text{Extract}[	ext{RegionEigenvalues}, q])^2$$

$$\text{SolutionBreakT2} = \text{Take}[	ext{Solution}, \{\text{Sum}[(2 - \text{KroneckerDelta}[1, j]) \times \text{Length}[	ext{Extract}[	ext{OmSystem}, \{1, j\}], \{j, 1, \text{Si} - 1\}] + 1, \text{Sum}[(2 - \text{KroneckerDelta}[1, j]) \times \text{Length}[	ext{Extract}[	ext{OmSystem}, \{1, j\}], \{j, 1, \text{Si}\}] - \text{Length}[	ext{Extract}[	ext{OmSystem}, \{1, \text{Si}\}])]\}]$$

(* Assembles and stores x..1-variation matrix. *)

$$\text{XAssembly} = \text{ReplacePart}[	ext{XAssembly}, \text{RegAssembly}, \{1, \text{Si}, 1\}]$$

$$\text{XAssembly} = \text{ReplacePart}[	ext{XAssembly}, \text{DRegAssembly}, \{2, \text{Si}, 1\}]$$

$$\text{XAssembly} = \text{ReplacePart}[	ext{XAssembly}, \text{DDRegAssembly}, \{3, \text{Si}, 1\}]$$

$$\text{XAssembly} = \text{ReplacePart}[	ext{XAssembly}, \text{RegAssembly2}, \{1, \text{Si}, 2\}]$$

$$\text{XAssembly} = \text{ReplacePart}[	ext{XAssembly}, \text{DRegAssembly2}, \{2, \text{Si}, 2\}]$$

$$\text{XAssembly} = \text{ReplacePart}[	ext{XAssembly}, \text{DDRegAssembly2}, \{3, \text{Si}, 2\}]$$

(* Stores unknown stress coefficients. *)

$$\text{SolutionBreak} = \text{ReplacePart}[	ext{SolutionBreak}, \text{SolutionBreakT}, \{1, \text{Si}, 1\}]$$

$$\text{SolutionBreak} = \text{ReplacePart}[	ext{SolutionBreak}, \text{SolutionBreakT2}, \{1, \text{Si}, 2\}]$$

(* Initializes x..3-variation matrix. *)

$$\text{NN} = \text{Length}[	ext{Extract}[	ext{Layup}, \{1, \text{Si}\}]]$$

$$\text{Temp11} = \{\text{Null}\}$$

$$\text{Temp12} = \{\text{Null}\}$$
Temp13 = {Null};
Temp23 = {Null};
Temp33 = {Null};

(* For each ply, calculates x3-variations. *)
For[Pliesi = 1, Pliesi \leq NN, Pliesi+++,
  StressFill = Pliesi - 2;
  CurrG11 = S11F[NN, Pliesi, Extract[Layup, {1, Si}], z];
  CurrG12 = S12F[NN, Pliesi, Extract[Layup, {1, Si}], z];
  CurrG13 = S13F[NN, Pliesi, Extract[Layup, {1, Si}], z];
  CurrG23 = S23F[NN, Pliesi, Extract[Layup, {1, Si}], z];
  CurrG33 = S33F[NN, Pliesi, Extract[Layup, {1, Si}], z];
  G11j = Join[0 * Range[StressFill], CurrG11, 0 * Range[NN - 1 - Length[CurrG11]] - If[StressFill > 0, StressFill, 0], 0 * Range[NN - 1]];
  G12j = Join[0 * Range[NN - 1], 0 * Range[StressFill], CurrG12, 0 * Range[NN - 1 - Length[CurrG12] - If[StressFill > 0, StressFill, 0]]];
  G13j = Join[0 * Range[StressFill], CurrG13, 0 * Range[NN - 1 - Length[CurrG13] - If[StressFill > 0, StressFill, 0]], 0 * Range[NN - 1]];
  G23j = Join[0 * Range[NN - 1], 0 * Range[StressFill], CurrG23, 0 * Range[NN - 1 - Length[CurrG23] - If[StressFill > 0, StressFill, 0]]];
  G33j = Join[CurrG33, 0 * Range[NN - 1]];}
Temp11 = Join[Temp11, {G11j}];
Temp12 = Join[Temp12, {G12j}];
Temp13 = Join[Temp13, {G13j}];
Temp23 = Join[Temp23, {G23j}];
Temp33 = Join[Temp33, {G33j}];
}
Temp11 = Drop[Temp11, 1];
Temp12 = Drop[Temp12, 1];
Temp13 = Drop[Temp13, 1];
Temp23 = Drop[Temp23, 1];
Temp33 = Drop[Temp33, 1];

(* Stores x_3-variations in to matrix. *)

ZAssembly = ReplacePart[ZAssembly, Temp11, {1, Si}];
ZAssembly = ReplacePart[ZAssembly, Temp12, {2, Si}];
ZAssembly = ReplacePart[ZAssembly, Temp13, {3, Si}];
ZAssembly = ReplacePart[ZAssembly, Temp23, {4, Si}];
ZAssembly = ReplacePart[ZAssembly, Temp33, {5, Si}];

(* Final Assembly of Ply Stresses *)

PlyStresses = ReplacePart[PlyStresses, Extract[ZAssembly, {1, Si}]
.Sum[Extract[XAssembly, {1, Si, Sk}].Extract[SolutionBreak, {1, Si, Sk}], {Sk, 1, 2}], {1, Si}];
PlyStresses = ReplacePart[PlyStresses, Extract[ZAssembly, {2, Si}]
.Sum[Extract[XAssembly, {1, Si, Sk}].Extract[SolutionBreak, {1, Si, Sk}], {Sk, 1, 2}], {2, Si}];
PlyStresses = ReplacePart[PlyStresses, Extract[ZAssembly, {3, Si}]
.Sum[Extract[XAssembly, {2, Si, Sk}].Extract[SolutionBreak, {1, Si, Sk}], {Sk, 1, 2}], {3, Si}];
PlyStresses = ReplacePart[PlyStresses, Extract[ZAssembly, {4, Si}]
.Sum[Extract[XAssembly, {2, Si, Sk}].Extract[SolutionBreak, {1, Si, Sk}], {Sk, 1, 2}], {4, Si}];
PlyStresses = ReplacePart[PlyStresses, Extract[ZAssembly, {5, Si}]
.Sum[Extract[XAssembly, {3, Si, Sk}].Extract[SolutionBreak, {1, Si, Sk}], {Sk, 1, 2}], {5, Si}];

BEGIN PLOTTING SUBROUTINES

(* Note: Plotting subroutine is hard-coded to handle plotting two regions. Hard-coding of the subroutine was chosen as a general plotting subroutine was found to be more inefficient than hard-coding for cases with less than 6-7 dropoffs. *)

(* Function Call: [Ply # in Region A, Ply # in Region B, CLPTSolution Matrix, PlyStress-Solution Matrix]. *)
StressPlotGeneration[PlotPlyA_, PlotPlyB_, CLPTSol_, PlyStresses_]:= 
If[True,
RXXA = Re[Extract[ComplexExpand[Extract[PlyStresses, {1, 1}]], PlotPlyA]] + 1 * Extract[Extract[CLPTSol, {1, 1}], {PlotPlyA, 1}];
RXXB = Re[Extract[ComplexExpand[Extract[PlyStresses, {1, 2}]], PlotPlyB]] + 1 * Extract[Extract[CLPTSol, {1, 2}], {PlotPlyB, 1}];
RXYA = Re[Extract[ComplexExpand[Extract[PlyStresses, {2, 1}]], PlotPlyA]] + 1 * Extract[Extract[CLPTSol, {1, 1}], {PlotPlyA, 3}];
RXYB = Re[Extract[ComplexExpand[Extract[PlyStresses, {2, 2}]], PlotPlyB]] + 1 * Extract[Extract[CLPTSol, {1, 2}], {PlotPlyB, 3}];
RXZA = Re[Extract[ComplexExpand[Extract[PlyStresses, {3, 1}]], PlotPlyA]]; 
RXZB = Re[Extract[ComplexExpand[Extract[PlyStresses, {3, 2}]], PlotPlyB]]; 
RYZA = Re[Extract[ComplexExpand[Extract[PlyStresses, {4, 1}]], PlotPlyA]]; 
RYZB = Re[Extract[ComplexExpand[Extract[PlyStresses, {4, 2}]], PlotPlyB]]; 
RZZA = Re[Extract[ComplexExpand[Extract[PlyStresses, {5, 1}]], PlotPlyA]]; 
RZZB = Re[Extract[ComplexExpand[Extract[PlyStresses, {5, 2}]], PlotPlyB]]; 
Return[{RXXA, RXYA, RXZA, RYZA, RZZA, RXXB, RXYB, RXZB, RYZB, RZZB}];
];
{RXXA, RXYA, RXZA, RYZA, RZZA, RXXB, RXYB, RXZB, RYZB, RZZB} = StressPlotGeneration[3, 1, CLPTSol, PlyStresses];
(* Plot parameters, set to liking/format requirements. *)
Col = Black;
DashNumber = Thick;
L1 = Extract[RLengths, 1];
LP = If[L1 > 1, 1, L1];
zloc = -.125/2;(* Desired x.3-location for lengthwise plots. *)
xloc = .01;(* Desired x.1 - location for through - thickness plots. 
(* Plots and Shows Figures *)
Outplot = Show[Plot[(RXXA)//.z \rightarrow zloc, \{z, -1, 0\}, PlotRange \rightarrow \{-1, 1\}, \{0, 2.5\}], AxesOrigin \rightarrow \{0, 0\}, PlotStyle \rightarrow \{Col, Thick, DashNumber\}, AspectRatio \rightarrow .7, WorkingPrecision\rightarrow20, Frame \rightarrow True, FrameTicks\rightarrow\{Automatic, None\}, \{Automatic, None\}], TicksStyle \rightarrow Directive[18], LabelStyle \rightarrow Directive[24]], Plot[(RXXB)//.z \rightarrow zloc, \{x, 0, LP\}, PlotStyle \rightarrow \{Col, Thick, DashNumber\}], PlotRange \rightarrow All]]

Outplot2 = Show[Plot[(RXZA)//.z \rightarrow zloc, \{z, -1, 0\}, PlotRange \rightarrow \{-1, 1\}, \{0, 0.70\}], AxesOrigin \rightarrow \{0, 0\}, PlotStyle \rightarrow \{Black, Thick, DashNumber\}, AspectRatio \rightarrow .7, WorkingPrecision\rightarrow20, Frame \rightarrow True, FrameTicks\rightarrow\{Automatic, None\}, \{Automatic, None\}], TicksStyle \rightarrow Directive[18], LabelStyle \rightarrow Directive[24]], Plot[(RXZB)//.z \rightarrow zloc, \{x, 0, LP\}, PlotStyle \rightarrow \{Black, Thick, DashNumber\}], PlotRange \rightarrow All]]

Outplot3 = Show[Plot[(RZZA)//.z \rightarrow zloc, \{z, -1, 0\}, PlotRange \rightarrow \{-1, 1\}, \{-1, 0.6\}], AxesOrigin \rightarrow \{0, 0\}, PlotStyle \rightarrow \{Black, Thick, DashNumber\}, AspectRatio \rightarrow .7, WorkingPrecision\rightarrow20, Frame \rightarrow True, FrameTicks\rightarrow\{Automatic, None\}, \{Automatic, None\}], TicksStyle \rightarrow Directive[18], LabelStyle \rightarrow Directive[24]], Plot[(RZZB)//.z \rightarrow zloc, \{x, 0, LP\}, PlotStyle \rightarrow \{Black, DashNumber\}], PlotRange \rightarrow All]]

Outplot4 = Show[Plot[(RXYA)//.z \rightarrow zloc, \{z, -1, 0\}, PlotRange \rightarrow \{-1, 1\}, \{0, 1.4\}], AxesOrigin \rightarrow \{0, 0\}, PlotStyle \rightarrow \{Col, Thick, DashNumber\}, AspectRatio \rightarrow .7, WorkingPrecision\rightarrow20, Frame \rightarrow True, FrameTicks\rightarrow\{Automatic, None\}, \{Automatic, None\}], TicksStyle \rightarrow Directive[18], LabelStyle \rightarrow Directive[24]], Plot[(RXYB)//.z \rightarrow zloc, \{x, 0, LP\}, PlotStyle \rightarrow \{Col, Thick, DashNumber\}], PlotRange \rightarrow All]]

Outplot5 = Show[Plot[(RYZA)//.z \rightarrow zloc, \{z, -1, 0\}, PlotRange \rightarrow \{-1, 1\}, \{-2, 1\}], AxesOrigin \rightarrow \{0, 0\}, PlotStyle \rightarrow \{Black, Thick, DashNumber\}, AspectRatio \rightarrow .7, WorkingPrecision\rightarrow20, Frame \rightarrow True, FrameTicks\rightarrow\{Automatic, None\}, \{Automatic, None\}], TicksStyle \rightarrow Directive[18], LabelStyle \rightarrow Directive[24]],
Plot[(RYZB) /. z -> zloc, {x, 0, LP}, PlotStyle -> {Black, Thick, DashNumber}, PlotRange -> All]

(* Outputs plots to .jpeg file format. *)
Export["jSt1xxVQ.jpeg", Outplot, ImageSize->550];
Export["jSt1xzVQ.jpeg", Outplot2, ImageSize->550];
Export["jSt1zzVQ.jpeg", Outplot3, ImageSize->550];
Export["jSt1xyVQ.jpeg", Outplot4, ImageSize->550];
Export["jSt1yzVQ.jpeg", Outplot5, ImageSize->550];

BEGIN THROUGH-THICKNESS PLOTTING SUBROUTINE

(* Again, this is hard-coded for savings in runtime. The following is commented out, as it assumes 16 plies or sublayers total in the laminate. It is provided for reference and future use and development. *)

(*
{RXXA1, RXYA1, RXZA1, RYZA1, RZZA1, RXXB1, RXYB1, RXZB1, RYZB1, RZZB1}
= StressPlotGeneration[1, 1, CLPTSol, PlyStresses];
{RXXA2, RXYA2, RXZA2, RYZA2, RZZA2, RXXB2, RXYB2, RXZB2, RYZB2, RZZB2}
= StressPlotGeneration[2, 1, CLPTSol, PlyStresses];
{RXXA3, RXYA3, RXZA3, RYZA3, RZZA3, RXXB3, RXYB3, RXZB3, RYZB3, RZZB3}
= StressPlotGeneration[3, 1, CLPTSol, PlyStresses];
{RXXA4, RXYA4, RXZA4, RYZA4, RZZA4, RXXB4, RXYB4, RXZB4, RYZB4, RZZB4}
= StressPlotGeneration[4, 2, CLPTSol, PlyStresses];
{RXXA5, RXYA5, RXZA5, RYZA5, RZZA5, RXXB5, RXYB5, RXZB5, RYZB5, RZZB5}
= StressPlotGeneration[5, 1, CLPTSol, PlyStresses];
{RXXA6, RXYA6, RXZA6, RYZA6, RZZA6, RXXB6, RXYB6, RXZB6, RYZB6, RZZB6}
= StressPlotGeneration[6, 2, CLPTSol, PlyStresses];
{RXXA7, RXYA7, RXZA7, RYZA7, RZZA7, RXXB7, RXYB7, RXZB7, RYZB7, RZZB7}
= StressPlotGeneration[7, 3, CLPTSol, PlyStresses];

*)
\{RXXA8, RXYA8, RXZA8, RYZA8, RZZA8, RXXB8, RXYB8, RXZB8, RYZB8, RZZB8\} = StressPlotGeneration[8, 4, CLPTSol, PlyStresses];

\{RXXA1T, RXZA1T, RZZA1T\} = \{RXXA1, RXZA1, RZZA1\} /. z \rightarrow -Z + 15 \times .125/4;
\{RXXA2T, RXZA2T, RZZA2T\} = \{RXXA2, RXZA2, RZZA2\} /. z \rightarrow -Z + 13 \times .125/4;
\{RXXA3T, RXZA3T, RZZA3T\} = \{RXXA3, RXZA3, RZZA3\} /. z \rightarrow -Z + 11 \times .125/4;
\{RXXA4T, RXZA4T, RZZA4T\} = \{RXXA4, RXZA4, RZZA4\} /. z \rightarrow -Z + 9 \times .125/4;
\{RXXA5T, RXZA5T, RZZA5T\} = \{RXXA5, RXZA5, RZZA5\} /. z \rightarrow -Z + 7 \times .125/4;
\{RXXA6T, RXZA6T, RZZA6T\} = \{RXXA6, RXZA6, RZZA6\} /. z \rightarrow -Z + 5 \times .125/4;
\{RXXA7T, RXZA7T, RZZA7T\} = \{RXXA7, RXZA7, RZZA7\} /. z \rightarrow -Z + 3 \times .125/4;
\{RXXA8T, RXZA8T, RZZA8T\} = \{RXXA8, RXZA8, RZZA8\} /. z \rightarrow -Z + 1 \times .125/4;
\{RXXB5T, RXZB5T, RZZB5T\} = \{RXXB5, RXZB5, RZZB5\} /. z \rightarrow -Z + 7 \times .125/4;
\{RXXB6T, RXZB6T, RZZB6T\} = \{RXXB6, RXZB6, RZZB6\} /. z \rightarrow -Z + 5 \times .125/4;
\{RXXB7T, RXZB7T, RZZB7T\} = \{RXXB7, RXZB7, RZZB7\} /. z \rightarrow -Z + 3 \times .125/4;
\{RXXB8T, RXZB8T, RZZB8T\} = \{RXXB8, RXZB8, RZZB8\} /. z \rightarrow -Z + 1 \times .125/4;

(*
Outplot6 = Show[ParametricPlot[{Extract[RXXA8T // .x -> 0, 1], Z}, {Z, 0, .125/2},
PlotRange -> {{0, 3}, {0, .125 * 2}}, PlotStyle -> {Black, Thick, DashNumber},
AspectRatio -> .7, Frame -> True, FrameTicks -> {{Automatic, None}, {Automatic, None}},
TicksStyle -> Directive[18], LabelStyle -> Directive[24]],
ParametricPlot[{Extract[RXXA7T // .x -> 0, 1], Z}, {Z, .125/2, .125},
PlotStyle -> {Black, Thick, DashNumber}, PlotRange -> All],
ParametricPlot[{Extract[RXXA6T // .x -> 0, 1], Z}, {Z, .125, .125 * 3/2},
PlotStyle -> {Black, Thick, DashNumber}, PlotRange -> All],
ParametricPlot[{Extract[RXXA5T // .x -> 0, 1], Z}, {Z, .125 * 3/2, .125 * 2},
PlotStyle -> {Black, Thick, DashNumber}, PlotRange -> All]]
(*
Outplot7 = Show[ParametricPlot[{Extract[RXZB8T // .x -> xloc, 1], Z}, {Z, 0, .125/2},
PlotStyle -> {Black, Thick, DashNumber}, PlotRange -> All}]
*)

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Outplot8 = Show[ParametricPlot[{Extract[RZZA8T//.x -> 0, 1], Z}, {Z, 0, .125/2},
PlotStyle -> {Black, Thick, DashNumber}, PlotRange -> All],
ParametricPlot[{Extract[RZZA7T//.x -> 0, 1], Z}, {Z, .125/.4, .125},
PlotStyle -> {Black, Thick, DashNumber}, PlotRange -> All],
ParametricPlot[{Extract[RZZA6T//.x -> 0, 1], Z}, {Z, .125/.3/2, .125*2},
PlotStyle -> {Black, Thick, DashNumber}, PlotRange -> All],
ParametricPlot[{Extract[RZZA5T//.x -> 0, 1], Z}, {Z, .125*3/2, .125*2},
PlotStyle -> {Black, Thick, DashNumber}, PlotRange -> All],
ParametricPlot[{Extract[RZZA4T//.x -> 0, 1], Z}, {Z, .125*4/2, .125*5/2},
PlotStyle -> {Black, Thick, DashNumber}, PlotRange -> All],
ParametricPlot[{Extract[RZZA3T//.x -> 0, 1], Z}, {Z, .125*5/2, .125*6/2},
PlotStyle -> {Black, Thick, DashNumber}, PlotRange -> All],
ParametricPlot[{Extract[RZZA2T//.x -> 0, 1], Z}, {Z, .125*6/2, .125*7/2},
PlotStyle -> {Black, Thick, DashNumber}, PlotRange -> All],
ParametricPlot[{Extract[RZZA1T//.x -> 0, 1], Z}, {Z, .125*7/2, .125*8/2},
PlotStyle -> {Black, Thick, DashNumber}, PlotRange -> All}]*)
Appendix B

Source Code Listing for Analysis of Step 3

The source code of the program for analysis of the Step 3 problem of a laminate with multiple finite length dropoffs along its length is listed in this appendix. The general structure for running this script follows that of the code in Appendix A and as described in the introduction to that Appendix. All significant differences between the code in Appendix A for Steps 1 and 2, and that in Appendix B for Step 3 are noted in the comments of the code. All code is written for Mathematica Release 7.x, although accurate results for the code have been obtained on distributions 8.0 and greater.
BEGIN LAMINATE CONFIGURATION INITIALIZATION

(* The following is a list of the material parameters of plies used in the current work. Each line acts as its own set of material properties. The format of the variable is \{E_l, E_t, \nu_{lt}, \nu_{tz}, G_{lt}, G_{tz}, PlyAngle\}. *)

AS90 = \{130, 9.0, 0.28, 0.28, 4.8, 3.51, 90\};
ASp45 = \{130, 9.0, 0.28, 0.28, 4.8, 3.51, 45\};
ASm45 = \{130, 9.0, 0.28, 0.28, 4.8, 3.51, -45\};

(*Measure of Ply/Sublayer Thickness. Entered as a variable, as all problems considered for the work have had equal ply or sublayer thickness. However, the current code is robust enough to support ply/sublayers with differing thickness.*)
\[ \text{Thk} = .125; \]

(* The Layup Variable Denotes the laminate layups in all regions. Each entry in the matrix below denotes an Independent region. Layup is a matrix of submatrices, where each submatrix describes the layup in a single region. The format for each submatrix is a number of rows equaling the number of plies/sublayers in a region, and each row is denoted as \{Ply/SublayerThickness, Ply/SublayerMaterialParameters\}. Externally dropped plies are indicated via reduction of the number of plies from Region to Region. *)

\[
\text{Layup} = \begin{pmatrix}
\text{Thk} & \text{AS4} \\
\text{Thk} & \text{AS4} \\
\text{Thk} & \text{AS4} \\
\text{Thk} & \text{AS4}
\end{pmatrix}
\begin{pmatrix}
\text{Thk} & \text{AS4} \\
\text{Thk} & \text{AS4} \\
\text{Thk} & \text{AS4}
\end{pmatrix}
\begin{pmatrix}
\text{Thk} & \text{AS4} \\
\text{Thk} & \text{AS4}
\end{pmatrix};
\]

(* Internal variable for number of regions *)

\[ \text{NRegions} = \text{Length}[\text{Transpose}[\text{Layup}]]; \]

(* Internal variable for total laminate thickness; used mainly for normalization of results. *)

\[ \text{NDrops} = \text{NRegions} - 1; \]

(* Internal variable for total laminate thickness; used mainly for normalization of results. *)

\[ \text{LT} = 2 \times \text{Extract}[\text{Total}[\text{Extract}[\text{Layup}, \{1, 1\}], 1], 1]; \]

(* Vector of absolute region lengths. It is assumed Region A is infinite in Length. The number of entries in RLengths must equal NRegions - 1. *)

\[ \text{RLengths} = \{1, 1\}; \]

(* Vector that stores region boundaries in the global x_1-system. This initializes the vector. *)

\[ \text{Len} = \{0\}; \]

(* Operates on Len and RLengths to obtain and store Region Boundary information. *)

\[ \text{For}\[i = 1, i \leq \text{NDrops}, i++, \]
\[ \text{If}\[i==1, \]
\[ \text{Len} = \text{Append}[\text{Len}, \text{Extract}[\text{RLengths}, i]]; \]
\[ , \]
\[ \text{Len} = \text{Append}[\text{Len}, \text{Extract}[\text{RLengths}, i] + \text{Extract}[\text{Len}, i]]; \]
\[ ]; \]
BEGIN SUBROUTINE DEFINITION AND COMPILATION

(* Rotates ply material properties in planar form for use in calculating CLPT solution. For _ is a switch that denotes either forward rotation to calculate CLPT A-matrices or to rotate back into the ply coordinate frame to calculate ply stresses. *)

TwoDElas[{El_, Et_, vlt_, vtz_, Glt_, Gtz_, th_}, For_]:=
If[True,
theta = (-1 + 2 * KroneckerDelta[For, 1]) * th/180 * pi;
c = Cos[theta];
s = Sin[theta];
vtl = vlt * Et/El;
Div = (1 - vlt * vt);        
e1111 = El/Div;
e2222 = Et/Div;
e1122 = vlt * Et/Div;
e1212 = Glt;

\[
\begin{pmatrix}
Q1111 \\
Q2222 \\
Q1122 \\
Q1212 \\
Q1112 \\
Q1222
\end{pmatrix}
= 
\begin{pmatrix}
c^4 & s^4 & 2*c^2*s^2 & 4*c^2*s^2 \\
s^4 & c^4 & 2*c^2*s^2 & 4*c^2*s^2 \\
c^2*s^2 & c^2*s^2 & c^4+s^4 & -4*c^2*s^2 \\
c^2*s^2 & c^2*s^2 & -2*c^2*s^2 & (c^2-s^2)^2 \\
s*c^3 & -s^3*c & (c*s^3-c^3*s) & 2*(c*s^3-c^3*s) \\
s^3*c & -s*c^3 & c^3*s-c*s^3 & 2*(c^3*s-c*s^3)
\end{pmatrix}
\begin{pmatrix}
e1111 \\
e2222 \\
e1122 \\
e1212
\end{pmatrix}
\]

Return \[
\begin{pmatrix}
Q1111 & Q1122 & 2*Q1112 \\
Q1122 & Q2222 & 2*Q1212 \\
Q1112 & Q2222 & 2*Q1212
\end{pmatrix}
\]
];
(* Calculates full Material Compliance Matrix rotated by angle th. *)

Compliance[{El_, Et_, \nu lt_, \nu tz_, Glt_, Gtz_, th_}]:= 
If[True, 
theta = -th/180 * \pi;

c = Cos[theta];

s = Sin[theta];

\[K = \begin{pmatrix}
c^2 & s^2 & 0 & 0 & 0 & 2 * c * s \\
s^2 & c^2 & 0 & 0 & 0 & -2 * c * s \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & c & s & 0 \\
0 & 0 & 0 & -1 * s & c & 0 \\
-c * s & c * s & 0 & 0 & 0 & c^2 - s^2 \\
\end{pmatrix};

IK = Inverse[K];

Return Transpose[IK]. 

(* Calculates material matrices relevant to CLPT analyses. Let_ denotes what material matrix (A,B,or D) to return. *)

LPTM[PlyA_, Let_] := 
If[True, 
Matrix = {{}, {}};

Matrix = 0 * Transpose[Range[3]]; 

Matrix = PadRight[Matrix, {3, 3}];
TotalT = Total[Extract[Transpose[PlyA], 1]];
zupper = TotalT;
ErrorFlag = False;
For[i = 1, i ≤ Length[PlyA], i++,
zlower = zupper - Extract[PlyA, {i, 1}];
If[Let === A, 
Multi = zupper - zlower;
If[Let === D, 
Multi = 1/3 * (zupper³ - zlower³) ;
If[Let === B, 
Multi = 1/2 * (zupper² - zlower²) ;
If[ErrorFlag == False, 
Print[“Error - Unspecified Matrix for CLPT. Enter in letter A, B, or D.”];
ErrorFlag = True; ];
Multi = 0; ]; ]; ];
Adde = Multi * TwoDElas[Extract[PlyA, {i, 2}], 0];
Matrix = Matrix + Adde;
zupper = zlower;
];
Return[Matrix]; ];

(* Calculates h11 matrices as denoted in Equations (4.10a-i) using the expressions in Equations (7.3a) and (7.3b) for ply CPly. at local x.3-coordinate GPoint. *)
S11F[Nn_, CPly_, Ply_, GPoint_] :=
If[True,
TopTerm = {1/Extract[Ply, {CPly, 1}]};
BottomTerm = {-1/Extract[Ply, {CPly, 1}]};
If[CPly == 1,
Return[BottomTerm];
,
If[CPly == Nn,
Return[TopTerm];
,
Return[Join[TopTerm, BottomTerm]];]};
];
];
(* Calculates h12 matrices as denoted in Equations (4.10a-i) using the expressions in Equa-
tions (7.3a) and (7.3b) for ply CPly_. at local x.3-coordinate GPoint_. *)

S12F[Nn_, CPly_, Ply_, GPoint_] :=
If[True,
TopTerm = {1/Extract[Ply, {CPly, 1}]};
BottomTerm = {-1/Extract[Ply, {CPly, 1}]};
If[CPly == 1,
Return[BottomTerm];
,
If[CPly == Nn,
Return[TopTerm];
,
Return[Join[TopTerm, BottomTerm]];]};
];
(* Calculates h13 matrices as denoted in Equations (4.10a-i) using the expressions in Equations (7.3a) and (7.3b) for ply CPly at local x3-coordinate GPoint. *)

S13F[NNn, CPly, Ply, GPoint] :=
If[True, 
TopTerm = {-(GPoint/Extract[Ply, {CPly, 1}] - 1/2)};
BottomTerm = {(GPoint/Extract[Ply, {CPly, 1}] + 1/2)};
If[CPly == 1, Return[BottomTerm]; , If[CPly == NNn, Return[TopTerm]; , Return[Join[TopTerm, BottomTerm]]; ]; ]; ];

(* Calculates h23 matrices as denoted in Equations (4.10a-i) using the expressions in Equations (7.3a) and (7.3b) for ply CPly at local x3-coordinate GPoint. *)

S23F[NNn, CPly, Ply, GPoint] :=
If[True, 
TopTerm = {-(GPoint/Extract[Ply, {CPly, 1}] - 1/2)};
BottomTerm = {(GPoint/Extract[Ply, {CPly, 1}] + 1/2)};
If[C Ply == 1,
Return[BottomTerm];
,
If[C Ply == N n,
Return[TopTerm];
,
Return[Join[TopTerm, BottomTerm]];
];
];
];
];

(* Calculates h33 matrices as denoted in Equations (4.10a-i) using the expressions in Equations (7.3a) and (7.3b) for ply C Ply at local x_3-coordinate G Point. *)

S33F[N n_, C Ply_, P ly_, G Point_] :=
If[True,
Outvec = 0 * Range[N n - 1];
For[k = 1, k <= 1 * (C Ply - 1), k++,
Outvec = ReplacePart[Outvec, -(Extract[P ly, {k, 1}] + Extract[P ly, {k + 1, 1}])/2, k];
];
If[C Ply != 1,
Outvec = ReplacePart[Outvec, Extract[Outvec, C Ply - 1] +
Extract[P ly, {C Ply, 1}]/2 * (G Point/Extract[P ly, {C Ply, 1}] - 1/2)^2, C Ply - 1];
];
If[C Ply != N n,
*(G Point/Extract[P ly, {C Ply, 1}] + 1/2)^2, C Ply];
];
Return[Outvec];
(* Sets up the eigenfunction problem in Equation 4.22 and solves for the eigenvalues and eigenvectors. *)

Eigenfunction[PlyA.]:= If[True, Nn = Length[PlyA];
(* Implements Gaussian Quadrature tables for integration of h-matrices in x_3. *)
GP = \{-\sqrt{(3 + 2\sqrt{6/5})/7}, -\sqrt{(3 - 2\sqrt{6/5})/7}, \sqrt{(3 + 2\sqrt{6/5})/7}\};
GW = \{(18 - \sqrt{30})/36, (18 + \sqrt{30})/36, (18 + \sqrt{30})/36, (18 - \sqrt{30})/36\};
WGP = {};
WGW = {};
(* Components of Matrices in the Eigenfunction problem *)
E0 = Table[0, \{2 Nn - 2\}, \{2 Nn - 2\}];
E1 = E0;
E2 = E0;
E3 = E0;
E4 = E0;
E5 = E0;
E6 = E0;
E7 = E0;
E8 = E0;
(* Further modifies Gaussian Quadrature tables for integration of h-matrices in x_3. This code handles plies or sublayers of differing thicknesses. *)
For[i = 1, i <= Nn, i++,
WGP = Append[WGP, Extract[PlyA, \{i, 1\}]/2 * GP];
WGW = Append[WGW, Extract[PlyA, \{i, 1\}]/2 * GW];
]
(* Calculates reduced compliance matrix terms. *)

For[i = 1, i ≤ Nn, i++,
CompI = Compliance[Extract[PlyA, {i, 2}]];
RComp =
{Extract[CompI, 1, 1], Extract[CompI, 1, 3], 0, 0, Extract[CompI, 1, 6]},
{Extract[CompI, 1, 3], Extract[CompI, 3, 3], 0, 0, Extract[CompI, 3, 6]},
{0, 0, Extract[CompI, 4, 4], Extract[CompI, 4, 5], 0},
{0, 0, Extract[CompI, 4, 5], Extract[CompI, 5, 5], 0},
{Extract[CompI, 1, 6], Extract[CompI, 2, 6], 0, 0, Extract[CompI, 6, 6]} -
1/Extract[CompI, {2, 2}]*
{Extract[CompI, 1, 2]^2, Extract[CompI, 1, 2] Extract[CompI, 3, 2], 0, 0,
Extract[CompI, 1, 2] Extract[CompI, 6, 2]},
{Extract[CompI, 1, 2] Extract[CompI, 3, 2], Extract[CompI, 3, 2]^2, 0, 0,
Extract[CompI, 3, 2] Extract[CompI, 6, 2]},
{0, 0, Extract[CompI, 4, 2]^2, Extract[CompI, 4, 2] Extract[CompI, 5, 2], 0},
{0, 0, Extract[CompI, 4, 2] Extract[CompI, 5, 2], Extract[CompI, 5, 2]^2, 0},
{Extract[CompI, 1, 2] Extract[CompI, 6, 2], Extract[CompI, 3, 2] Extract[CompI, 6, 2],
0, 0, Extract[CompI, 6, 2]^2};
(* Assembles of matrix components for the eigenfunction problem. *)

Fill = i - 2;

For[j = 1, j ≤ 4, j++,
CurrS11 = S11F[Nn, i, PlyA, Extract[WGP, {i, j}]];
CurrS12 = S12F[Nn, i, PlyA, Extract[WGP, {i, j}]];
CurrS13 = S13F[Nn, i, PlyA, Extract[WGP, {i, j}]];
CurrS23 = S23F[Nn, i, PlyA, Extract[WGP, {i, j}]];
CurrS33 = S33F[Nn, i, PlyA, Extract[WGP, {i, j}]];
S11j = Join[0 * Range[Fill], CurrS11, 0 * Range[Nn - 1 - Length[CurrS11]
-If[Fill > 0, Fill, 0], 0 * Range[Nn - 1]];}
S12j = Join[0 * Range[Nn - 1], 0 * Range[Fill], CurrS12, 0 * Range[Nn - 1 - Length[CurrS12] - If[Fill > 0, Fill, 0]]];
S13j = Join[0 * Range[Fill], CurrS13, 0 * Range[Nn - 1 - Length[CurrS13] - If[Fill > 0, Fill, 0]], 0 * Range[Nn - 1] - Length[CurrS13] - If[Fill > 0, Fill, 0]]];
S23j = Join[0 * Range[Nn - 1], 0 * Range[Fill], CurrS23, 0 * Range[Nn - 1 - Length[CurrS23] - If[Fill > 0, Fill, 0]], 0 * Range[Nn - 1] - Length[CurrS23] - If[Fill > 0, Fill, 0]]];
S33j = Join[CurrS33, 0 * Range[Nn - 1] - Length[CurrS33], 0 * Range[Fill], CurrS33, 0 * Range[Nn - 1] - Length[CurrS33], 0 * Range[Fill], CurrS33, 0 * Range[Nn - 1] - Length[CurrS33], If[Fill > 0, Fill, 0], 0 * Range[Nn - 1] - Length[CurrS33], If[Fill > 0, Fill, 0]]];

E0 = E0 + 2 * Extract[WGW, {i, j}] * Transpose[{S33j}].{S33j} * Extract[RComp, {2, 2}];
E1 = E1 - 2 * Extract[WGW, {i, j}] * Transpose[{S13j}].{S13j} * Extract[RComp, {3, 3}];
E2 = E2 - 2 * Extract[WGW, {i, j}] * Transpose[{S23j}].{S23j} * Extract[RComp, {4, 4}];
E3 = E3 + 2 * Extract[WGW, {i, j}] * Transpose[{S33j}].{S11j} + Transpose[{S11j}].{S33j} * Extract[RComp, {1, 2}];
E4 = E4 - 2 * Extract[WGW, {i, j}] * Transpose[{S13j}].{S23j} + Transpose[{S23j}].{S13j} * Extract[RComp, {3, 4}];
E5 = E5 + 2 * Extract[WGW, {i, j}] * Transpose[{S12j}].{S33j} + Transpose[{S33j}].{S12j} * Extract[RComp, {2, 5}];
E6 = E6 + 2 * Extract[WGW, {i, j}] * Transpose[{S11j}].{S11j} * Extract[RComp, {1, 1}];
E7 = E7 + 2 * Extract[WGW, {i, j}] * Transpose[{S12j}].{S12j} * Extract[RComp, {5, 5}];
E8 = E8 + 2 * Extract[WGW, {i, j}] * Transpose[{S11j}].{S12j} + Transpose[{S12j}].{S11j} * Extract[RComp, {1, 5}];
];
];

(* Assembles matrices in the eigenfunction problem. *)
AA = E0;
BB = E1 + E2 + E3 + E4 + E5;
CC = E6 + E7 + E8;

(* Calculates eigenvalues and eigenvectors of the eigenfunction problem. *)
EMat1 = Transpose[Join[Transpose[Join[−CC, Table[0, {2 * Nn + 2}, {2 * Nn + 2}]]], Transpose[Join[Table[0, {2 * Nn + 2}, {2 * Nn + 2}], AA]]];
EMat2 = Transpose[Join[Transpose[Join[BB, AA]], Transpose[Join[AA, Table[0, {2 * Nn - 2}, {2 * Nn - 2}]]]]];
EValues2 = Eigenvalues[{EMat1, EMat2}];
EVectors = Eigenvectors[{EMat1, EMat2}];
EValues = 0 * EValues2;
(* Formats eigenvalue and eigenvector data for use later in the code. *)
For[i = 1, i <= Length[EValues2], i++,
  EValues = ReplacePart[EValues, (Extract[EValues2, i])^1/2, i];
Outmat = {{{}, {}}};
Outmat = 0 * Transpose[{Range[Length[EValues] + 1]}];
Outmat = PadRight[Outmat, {Length[EValues] + 1, Length[EValues] + 1}];
Outmat = ReplacePart[Outmat, EValues, 1];
For[i = 1, i <= Length[EValues2], i++,
  Outmat = ReplacePart[Outmat, Extract[EVectors, i], i + 1];
];
Return[Outmat];
(* ******************************************************************************
  (* Extracts eigenvalue data from matrices generated in Eigenfunction. *)
ValExtract[Region_] :=
  Return[
    Extract[Transpose[Drop[Transpose[Drop[Region, {2, Length[Region]}]],
{3 * (Length[Region] - 1)/4 + 1, Length[Region] - 1}]], 1];
(* ******************************************************************************
  (* Extracts eigenvector data from matrices generated in Eigenfunction. *)
*)
VecExtract[Region_] :=
If[True,
UnNorm = Transpose[Drop[Transpose[Drop[Drop[Region, {1, 1}],
{3* (Length[Region] - 1)/4 + 1, Length[Region] - 1}],
(Length[Region] - 1)/2 + 1, Length[Region] - 1]]];
NormEigen = Table[Extract[UnNorm, I]/Norm[Extract[UnNorm, I], 2],
{I, 1, Length[UnNorm]}];
Return[NormEigen];
];
(* Calculates derivative of exponential terms in Equations (7.3a) and (7.3b) using eigenvalue
and eigenvector data. *)
Deriv[EValue_, EVec_] :=
If[True,
DerivMatrix = Table[Extract[EVec, {p, q}] * Extract[EValue, q],
{p, 1, Length[EValue]/3}, {q, 1, Length[EValue]}];
Return[DerivMatrix];
];
(* Assembles Omega and Omega-Cross matrices for region Region_. The Omega-Cross matrices are used only in configurations with more than one dropoff for Step 3. The switch CrossTerm_ indicates if the subroutine should calculate the Omega or Omega-Cross matrices for a given region. CrossTerm_ takes on the values of "Crossed" and "False." "Crossed" denotes an Omega-Cross matrix; "False" indicates an Omega matrix. This switch is always set to "False" for Steps 1 and 2. *)
Omega[Region_, PlyA_, LVec_, CrossTerm_] :=
If[True,
Nn = Length[PlyA];
(* Implements Gaussian Quadrature tables for integration of h-matrices in x_3. *)

$$GP = \left\{-\sqrt{\frac{3 + 2\sqrt{6/5}}{7}}, \sqrt{\frac{3 - 2\sqrt{6/5}}{7}}, \sqrt{\frac{3 - 2\sqrt{6/5}}{7}}, \sqrt{\frac{3 + 2\sqrt{6/5}}{7}}\right\};$$

$$GW = \left\{\frac{(18 - \sqrt{30})}{36}, \frac{(18 + \sqrt{30})}{36}, \frac{(18 + \sqrt{30})}{36}, \frac{(18 - \sqrt{30})}{36}\right\};$$

WGP = {};

WGW = {};

(* Extracts eigenvalue and eigenvector entries for current region. *)

$$OEValue = ValExtract[Region];$$

$$OEVector = VecExtract[Region];$$

(* Separates eigenvector terms to terms acting on interfacial stress functions F(x_1) and those acting on G (x_1). *)

$$EVecF = Drop[Transpose[OEVector], \{Length[Transpose[OEVector]]/2 + 1, Length[Transpose[OEVector]]\}];$$

$$EVecG = Drop[Transpose[OEVector], \{1, Length[Transpose[OEVector]]/2\}];$$

(* Calculates derivatives of F(x_1) and G(x_1) with respect to x_1. *)

$$EVecFPrime = Deriv[OEValue, EVecF];$$

$$EVecFDPrime = Deriv[OEValue, EVecFPrime];$$

$$EVecGPrime = Deriv[OEValue, EVecG];$$

(* Initializes components of Omega or Omega-Cross matrix. *)

$$E0 = Table[0, \{3 \times Nn - 3\}, \{3 \times Nn - 3\}];$$

$$E1 = E0;$$

$$E2 = E0;$$

$$E3 = E0;$$

$$E4 = E0;$$

$$E5 = E0;$$

$$E6 = E0;$$

$$E7 = E0;$$

$$E8 = E0;$$

(* Initializes output matrix. *)
OmegaOut = Table[0, {3 * Nn - 3}, {3 * Nn - 3}];
(* Modifies Gaussian Quadrature Terms for varying ply/sublayer thicknesses. *)
For[i = 1, i <= Nn, i++,
WGP = Append[WGP, Extract[PlyA, {i, 1}]/2 * GP];
WGW = Append[WGW, Extract[PlyA, {i, 1}]/2 * GW];
]
(* Calculates reduced compliance matrix *)
For[i = 1, i <= Nn, i++,
CompI = Compliance[Extract[PlyA, {i, 2}]];
RComp =
{Extract[CompI, 1, 1], Extract[CompI, 1, 3], 0, 0, Extract[CompI, 1, 6]},
{Extract[CompI, 1, 3], Extract[CompI, 3, 3], 0, 0, Extract[CompI, 3, 6]},
{0, 0, Extract[CompI, 4, 4], Extract[CompI, 4, 5], 0},
{0, 0, Extract[CompI, 4, 5], Extract[CompI, 5, 5], 0},
{Extract[CompI, 1, 6], Extract[CompI, 2, 6], 0, 0, Extract[CompI, 6, 6]} -
1/Extract[CompI, {2, 2}]* 
{{Extract[CompI, 1, 2]^2, Extract[CompI, 1, 2] Extract[CompI, 3, 2], 0, 0, Extract[CompI, 1, 2] Extract[CompI, 6, 2]},
{Extract[CompI, 1, 2] Extract[CompI, 3, 2], Extract[CompI, 3, 2]^2, 0, 0, Extract[CompI, 3, 2] Extract[CompI, 6, 2]},
{0, 0, Extract[CompI, 4, 2]^2, Extract[CompI, 4, 2] Extract[CompI, 5, 2], 0},
{0, 0, Extract[CompI, 4, 2] Extract[CompI, 5, 2], Extract[CompI, 5, 2]^2, 0},
{Extract[CompI, 1, 2] Extract[CompI, 6, 2], Extract[CompI, 3, 2] Extract[CompI, 6, 2], 0, 0, Extract[CompI, 6, 2]^2}};
(* Calculates output matrix components *)
Fill = i - 2;
For[j = 1, j <= 4, j++,
CurrS11 = S11F[Nn, i, PlyA, Extract[WGP, {i, j}]];
CurrS12 = S12F[Nn, i, PlyA, Extract[WGP, {i, j}]];

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CurrS13 = S13F[Nn, i, PlyA, Extract[WGP, {i, j}]];  
CurrS23 = S23F[Nn, i, PlyA, Extract[WGP, {i, j}]];  
CurrS33 = S33F[Nn, i, PlyA, Extract[WGP, {i, j}]];  
s11j = {Join[0 * Range[Fill], CurrS11, 0 * Range[Nn - 1 - Length[CurrS11]  
  -If[Fill > 0, Fill, 0]]};  
s12j = {Join[0 * Range[Fill], CurrS12, 0 * Range[Nn - 1 - Length[CurrS12]  
  -If[Fill > 0, Fill, 0]]};  
s13j = {Join[0 * Range[Fill], CurrS13, 0 * Range[Nn - 1 - Length[CurrS13]  
  -If[Fill > 0, Fill, 0]]};  
s23j = {Join[0 * Range[Fill], CurrS23, 0 * Range[Nn - 1 - Length[CurrS23]  
  -If[Fill > 0, Fill, 0]]};  
s33j = {CurrS33};  
E0 = E0 + 2 * Extract[WGW, {i, j}] * Transpose[s33j.EVecFDPrime].s33j.EVecFDPrime  
  *Extract[RComp, {2, 2}];  
E1 = E1 + 2 * Extract[WGW, {i, j}] * Transpose[s13j.EVecFPrime].s13j.EVecFPrime  
  *Extract[RComp, {3, 3}];  
E2 = E2 + 2 * Extract[WGW, {i, j}] * Transpose[s23j.EVecGPrime].s23j.EVecGPrime  
  *Extract[RComp, {4, 4}];  
E3 = E3 + 2 * Extract[WGW, {i, j}] * (Transpose[s33j.EVecFDPrime].s11j.EVecF  
  + Transpose[s11j.EVecF].s33j.EVecFDPrime) * Extract[RComp, {1, 2}];  
E4 = E4 + 2 * Extract[WGW, {i, j}] * (Transpose[s13j.EVecFPrime].s23j.EVecGPrime  
  S + Transpose[s23j.EVecGPrime].s13j.EVecFPrime) * Extract[RComp, {3, 4}];  
E5 = E5 + 2 * Extract[WGW, {i, j}] * (Transpose[s33j.EVecFDPrime].s12j.EVecG  
  + Transpose[s12j.EVecG].s33j.EVecFDPrime) * Extract[RComp, {2, 5}];  
E6 = E6 + 2 * Extract[WGW, {i, j}] * Transpose[s11j.EVecF].s11j.EVecF  
  *Extract[RComp, {1, 1}];  
E7 = E7 + 2 * Extract[WGW, {i, j}] * Transpose[s12j.EVecG].s12j.EVecG  
  *Extract[RComp, {5, 5}];  
E8 = E8 + 2 * Extract[WGW, {i, j}] * (Transpose[s11j.EVecF].s12j.EVecG  
  + Transpose[s12j.EVecG].s33j.EVecFDPrime) * Extract[RComp, {2, 5}];  
E9 = E9 + 2 * Extract[WGW, {i, j}] * Transpose[s33j.EVecFDPrime].s33j.EVecFDPrime  
  *Extract[RComp, {1, 1}];  
E10 = E10 + 2 * Extract[WGW, {i, j}] * Transpose[s11j.EVecF].s11j.EVecF  
  + Transpose[s11j.EVecF].s33j.EVecFDPrime) * Extract[RComp, {2, 5}];  
E11 = E11 + 2 * Extract[WGW, {i, j}] * Transpose[s23j.EVecG].s23j.EVecGPrime  
  *Extract[RComp, {3, 3}];  
E12 = E12 + 2 * Extract[WGW, {i, j}] * Transpose[s33j.EVecFDPrime].s33j.EVecFDPrime  
  *Extract[RComp, {2, 5}];  
E13 = E13 + 2 * Extract[WGW, {i, j}] * Transpose[s11j.EVecF].s11j.EVecF  
  + Transpose[s11j.EVecF].s33j.EVecFDPrime) * Extract[RComp, {2, 5}];  
E14 = E14 + 2 * Extract[WGW, {i, j}] * Transpose[s23j.EVecG].s23j.EVecGPrime  
  *Extract[RComp, {3, 3}];  
E15 = E15 + 2 * Extract[WGW, {i, j}] * Transpose[s33j.EVecFDPrime].s33j.EVecFDPrime  
  *Extract[RComp, {2, 5}];  
E16 = E16 + 2 * Extract[WGW, {i, j}] * Transpose[s11j.EVecF].s11j.EVecF  
  + Transpose[s11j.EVecF].s33j.EVecFDPrime) * Extract[RComp, {2, 5}];  
E17 = E17 + 2 * Extract[WGW, {i, j}] * Transpose[s23j.EVecG].s23j.EVecGPrime  
  *Extract[RComp, {3, 3}];  
E18 = E18 + 2 * Extract[WGW, {i, j}] * Transpose[s33j.EVecFDPrime].s33j.EVecFDPrime  
  *Extract[RComp, {2, 5}];  
E19 = E19 + 2 * Extract[WGW, {i, j}] * Transpose[s11j.EVecF].s11j.EVecF  
  + Transpose[s11j.EVecF].s33j.EVecFDPrime) * Extract[RComp, {2, 5}];  
E20 = E20 + 2 * Extract[WGW, {i, j}] * Transpose[s23j.EVecG].s23j.EVecGPrime  
  *Extract[RComp, {3, 3}];  
E21 = E21 + 2 * Extract[WGW, {i, j}] * Transpose[s33j.EVecFDPrime].s33j.EVecFDPrime  
  *Extract[RComp, {2, 5}];
OmegaOut = OmegaOut + E0 + E1 + E2 + E3 + E4 + E5 + E6 + E7 + E8;

(* Omega and Omega-Cross Matricies have similar form. The following lines of code modify the output matrix to be either the Omega Matrix or Omega-Cross matrix of a region. *)

If[CrossTerm === Crossed, (* FOR STEP 3 ONLY: OMEGA-CROSS MATRIX *)
TempTable = Table[If[I != J, (e(Extract[LVec,1] - Extract[LVec,2]) * Extract[OEValue,I] - e(Extract[LVec,1] - Extract[LVec,2]) * Extract[OEValue,J]) / (Extract[OEValue, I] - Extract[OEValue, J]) - Extract[OEValue, I]) * (Extract[LVec,2] - Extract[LVec,1]), {I, 1, Length[OmegaOut]}, {J, 1, Length[OmegaOut]}];
FinalOmegaOut = Table[Extract[OmegaOut, {I, J}]* Extract[TempTable, {I, J}], {I, 1, Length[OmegaOut]}, {J, 1, Length[OmegaOut]}];
,
(* Else return Omega matrix *)
];
Return[FinalOmegaOut];
];

(* Calculates values used for enforcing free-surface and stress-continuity across dropoffs in Sigma_{11} for Ply CPly_ at local x_3-coordinate zval_. Terms here are assembled into matrix GammaA. Cross_ dentoes a switch used to calculate terms for intermediate regions in Step *)
3. Switch values are "True" and "False." *

\[ S11BC\{EVecF_, EValue_, Ply_, LVec_, zval_, CPly_, Cross_\} := \]
If[True,
\[ t = Extract[Ply, \{CPly, 1\}] ; \]
If[CPly == 1,
\[ Vec11 = -1/t \times (Extract[EVecF, \{CPly\}]); \]
, If[CPly == Length[Ply],
\[ Vec11 = 1/t \times (Extract[EVecF, \{CPly - 1\}]); \]
, \[ Vec11 = 1/t \times (Extract[EVecF, \{CPly - 1\}]) - 1/t \times (Extract[EVecF, \{CPly\}]); \]
];
];
If[Cross == True, (* FOR STEP 3 ONLY *)
\[ OutVec11 = Table[Extract[Vec11, j] \times e^{-Extract[EValue, j]} \times (Extract[LVec, 2] - Extract[LVec, 1]), \{i, 1, 1\}, \{j, 1, 3 \times Length[Ply] - 3\}]; \]
, \[ OutVec11 = Table[Extract[Vec11, j] \times e^{Extract[EValue, j]} \times 0, \{i, 1, 1\}, \{j, 1, 3 \times Length[Ply] - 3\}]; \]
];
Return[OutVec11];
];

(* Calculates values used for enforcing free-surface and stress-continuity across dropoffs in Sigma_12 for Ply CPly_ at local x_3-coordinate zval_. Terms here are assembled into matrix GammaA. Cross_ denotes a switch used to calculate terms for intermediate regions in Step 3. Switch values are "True" and "False." *)
S12BC[EVecG_, EValue_, Ply_, LVec_, zval_, CPly_, Cross_] :=
  If[True,
    t = Extract[Ply, {CPly, 1}];
    If[CPly == 1,
      Vec12 = -1/t * (Extract[EVecG, {CPly}]);
    ,
      If[CPly == Length[Ply],
        Vec12 = 1/t * (Extract[EVecG, {CPly - 1}]);
      ,
        Vec12 = 1/t * (Extract[EVecG, {CPly - 1}]) - 1/t * (Extract[EVecG, {CPly}]);
      ];
    ];
    ];
    If[Cross == True, (* FOR STEP 3 ONLY *)
      OutVec12 = Table [Extract[Vec12, j] * e^-Extract[EValue, j] * (Extract[LVec, 2] - Extract[LVec, 1]),
        {i, 1, 1}, {j, 1, 3 * Length[Ply] - 3}];
      ,
      OutVec12 = Table [Extract[Vec12, j] * e^Extract[EValue, j] * 0, {i, 1, 1},
        {j, 1, 3 * Length[Ply] - 3}];
      ];
    Return[OutVec12];
  ];

(* Calculates values used for enforcing free-surface and stress-continuity across dropoffs in Sigma_13 for Ply CPly_ at local x.3-coordinate zval_. Terms here are assembled into matrix GammaA. Cross_ denotes a switch used to calculate terms for intermediate regions in Step 3. Switch values are "True" and "False." *)

S13BC[EVecF_, EValue_, Ply_, LVec_, zval_, CPly_, Cross_] :=
  If[True,
\[ t = \text{Extract}[\text{Ply}, \{\text{CPly}, 1\}] \];
If[\text{CPly} == 1, 
Vec13 = (\text{Extract}[\text{EVecF}, \{\text{CPly}\}]) \ast (zval/t + 1/2);
, 
If[\text{CPly} == \text{Length}[\text{Ply}], 
Vec13 = -(\text{Extract}[\text{EVecF}, \{\text{CPly} - 1\}]) \ast (zval/t - 1/2);
, 
Vec13 = (\text{Extract}[\text{EVecF}, \{\text{CPly}\}]) \ast (zval/t + 1/2) - (\text{Extract}[\text{EVecF}, \{\text{CPly} - 1\}]) \ast (zval/t - 1/2);
];
];
]

If[\text{Cross}==True, (* FOR STEP 3 ONLY *)
OutVec13 = Table [(\text{Extract}[\text{Vec13}, j]) \ast e^{-\text{Extract}[\text{EValue}, j] \ast (\text{Extract}[\text{LVec}, 2] - \text{Extract}[\text{LVec}, 1])} \ast \text{Extract}[\text{EValue}, j], \{i, 1, 1\}, \{j, 1, 3 \ast \text{Length}[\text{Ply}] - 3\}];
, 
OutVec13 = Table [(\text{Extract}[\text{Vec13}, j]) \ast e^{\text{Extract}[\text{EValue}, j] \ast 0} \ast \text{Extract}[\text{EValue}, j], \{i, 1, 1\}, \{j, 1, 3 \ast \text{Length}[\text{Ply}] - 3\}];
];
Return[OutVec13];
];

(* Assembles Gamma and Theta Matricies for region Region. in the negative x_1-direction of a dropoff. Cross. is a switch used in Step 3 for intermediate region calculations. Switch values are "True" and "False." *)

\text{GammaThetaMat}[\text{Region.}, \text{PlyA.}, \text{LVec.}, \text{Cross.}] :=
If[\text{True}, (* Initialize Output matrices. *)
\text{GaAF} = \{\text{Null}\};
GaAG = {Null};
ThA = {Null};

(* Extract Region Eigenvalues/Eigenvectors. *)
GTEValue = ValExtract[Region];
GTEVector = VecExtract[Region];

(* Separate Eigenvector terms to those acting on F(x_1) and G(x_1). *)
EVecF = Drop[Transpose[GTEVector], {Length[Transpose[GTEVector]]}/2 + 1];
EVecG = Drop[Transpose[GTEVector], {1, Length[Transpose[GTEVector]]}/2];

(* Assembles Gamma and Theta Matrices. *)
For[i = 1, i < Length[PlyA], i++,
    GaAF = Join[GaAF, S11BC[EVecF, GTEValue, PlyA, LVec, 1/2 * Extract[PlyA, {i, 1}], i, Cross]];
    GaAG = Join[GaAG, S12BC[EVecG, GTEValue, PlyA, LVec, 1/2 * Extract[PlyA, {i, 1}], i, Cross]];
    ThCo = If[Cross == True, -1, 1];
    ThA = Join[ThA, ThCo * S13BC[EVecF, GTEValue, PlyA, LVec, 0, i, Cross]];]

(* "Cleans up" initialiation of output matrices. *)
GaAF = Drop[GaAF, 1];
GaAG = Drop[GaAG, 1];
ThA = Drop[ThA, 1];
GaAF = Take[GaAF, {1, Length[GaAF] - 1}];
GaAG = Take[GaAG, {1, Length[GaAG] - 1}];
ThA = Take[ThA, {1, Length[ThA] - 1}];
Return[{GaAF, GaAG, ThA}];
(* Assembles Gamma and Theta Matricies for region Region... in the positive x..1-direction
of a dropoff. Cross... is a switch used in Step 3 for intermediate region calculations. Switch
values are "True" and "False." *)

\[
\text{GammaThetaMatCut}[\text{Region}..., \text{PlyB}..., \text{LVec}_..., \text{PlyA}..., \text{Cross}..] :=
\]

\[
\text{If}[\text{True},
\]

(* Initalize Output matrices. *)

\[
\text{GaAF} = \{\text{Null}\}; \\
\text{GaAG} = \{\text{Null}\}; \\
\text{ThA} = \{\text{Null}\};
\]

(* Extract Region Eigenvalues/Eigenvectors. *)

\[
\text{GTEValue} = \text{ValExtract}[\text{Region}]; \\
\text{GTEVector} = \text{VecExtract}[\text{Region}];
\]

(* Seperate Eigenvector terms to those acting on F(x..1) and G (x..1). *)

\[
\text{EVecF} = \text{Drop}[\text{Transpose}[\text{GTEVector}], \{\text{Length}[\text{Transpose}[\text{GTEVector}]]/2 \\
\text{+}1, \text{Length}[\text{Transpose}[\text{GTEVector}]]\}]; \\
\text{EVecG} = \text{Drop}[\text{Transpose}[\text{GTEVector}], \{1, \text{Length}[\text{Transpose}[\text{GTEVector}]]/2\}];
\]

(* Assembles Gamma and Theta Matricies. *)

\[
\text{For}[i = 1, i < \text{Length}[\text{PlyA}], i++]
\]

\[
\text{If}[i > \text{Length}[\text{PlyA}] - \text{Length}[\text{PlyB}],
\]

\[
k = i - (\text{Length}[\text{PlyA}] - \text{Length}[\text{PlyB}]);
\]

\[
\text{GaAF} = \text{Join}[\text{GaAF}, \text{S11BC}[\text{EVecF}, \text{GTEValue}, \text{PlyB}, \text{LVec}, 1/2
\text{*Extract}[\text{PlyB}, \{k, 1\}], k, \text{Cross}]]; \\
\text{GaAG} = \text{Join}[\text{GaAG}, \text{S12BC}[\text{EVecG}, \text{GTEValue}, \text{PlyB}, \text{LVec}, 1/2
\text{*Extract}[\text{PlyB}, \{k, 1\}], k, \text{Cross}]]; \\
\text{ThCoC} = \text{If}[\text{Cross}==\text{True}, -1, 1]; \\
\text{ThA} = \text{Join}[\text{ThA}, \text{ThCoC} \ast \text{S13BC}[\text{EVecF}, \text{GTEValue}, \text{PlyB}, \text{LVec}, 0, k, \text{Cross}]]; \\
\]

\[
\text{GTFill} = \text{Table}[0\{\text{FillI}, 1, 1\}, \{\text{FillJ}, 1, 3 \ast \text{Length}[\text{PlyB}] - 3\}]; \\
\text{GaAF} = \text{Join}[\text{GaAF}, \text{GTFill}];
\]
GaAG = Join[GaAG, GTFill];
ThA = Join[ThA, GTFill];

(* "Cleans up" initialization of output matrices. *)
GaAF = Drop[GaAF, 1];
GaAG = Drop[GaAG, 1];
ThA = Drop[ThA, 1];
GaAF = Take[GaAF, {1, Length[GaAF] - 1}];
GaAG = Take[GaAG, {1, Length[GaAG] - 1}];
ThA = Take[ThA, {1, Length[ThA] - 1}];
Return[-1 * {GaAF, GaAG, ThA}];

(* Assembles the matrix used in the final linear system in Equation 7.9a. The manner in
which the matrix is assembled is by assembling all independent columns in the matrix and
filling the rest of the matrix, as the matrix is always symmetric. OmCSystem_ contains the
Omega-Cross matrices used in Step 3. *)
MainAssembly[OmSystem_, OmCSystem_, GammaSystem_, ThetaSystem_, Layup_] :=
If[True,
(* Defines local variables for number of regions and number of dropoffs present in
laminate. *)
NRegions = Length[Transpose[Layup]]; 
NDrops = NRegions - 1;
(* Initializes matrix of independent columns in output matrix. *)
ANumber = 2 * NRegions - 2;
ABCRow = 3 * NDrops;
AComp = 0 * Transpose[{Range[ANumber]}];
AComp = PadRight[AComp, {1, ANumber}];
(* Counter used for Boundary-condition matrix assembly. *)
BCCounter = 1;
(* Assembles Omega and Omega-Cross matrices into output matrix. *)
For[i = 1, i ≤ ANumber, i++,
ATemp = Table[0, {U, 1, 1}, {V, 1, 3 * Length[Extract[Layup, {1, Ceiling[(i + 1)/2]]] - 3]}];
For[j = 1, j ≤ ANumber, j++,
If[i == j,
AMult = If[j ≠ 1 && Mod[j, 2] == 1, -1, 1];
ATemp = Join[ATemp, AMult * Extract[OmSystem, {1, Ceiling[(j + 1)/2]]]];,
If[i ≠ 1 && i ≠ ANumber && (Mod[i, 2] == 1 && i - 1 == j || Mod[i, 2] == 0 && j - 1 == i),
ATemp = Join[ATemp, Extract[OmCSystem, {1, Ceiling[(i - 1)/2]]]];,
OFill = Table[0, {m, 1, 3 * Length[Extract[Layup, {1, Ceiling[(j + 1)/2]]] - 3},
{n, 1, (3 * Length[Extract[Layup, {1, Ceiling[(i + 1)/2]]] - 3)}];
ATemp = Join[ATemp, OFill];];
];
(* Assembles Gamma and Theta matrices into output matrix. *)
BCMultTable = Table[0, {J, 1, NDrops}];
If[i == 1,
BCMultTable = ReplacePart[BCMultTable, 1, i];,
If[i == ANumber,
BCMultTable = ReplacePart[BCMultTable, 1, NDrops];,
BCMultTable = ReplacePart[BCMultTable, 1, Ceiling[(i - 1)/2]];
BCMultTable = ReplacePart[BCMultTable, 1, Ceiling[(i + 1)/2]];
]
]
BCTemp = {Null};
For[j = 1, j <= NDrops, j++,
  If[Extract[BCMultTable, j] == 1,
    BCTemp = Join[BCTemp, Extract[GammaSystem, {1, BCCounter}]];
    BCTemp = Join[BCTemp, Extract[GammaSystem, {2, BCCounter}]];
    BCTemp = Join[BCTemp, Extract[ThetaSystem, {1, BCCounter}]];
    BCCounter++;
  ,
    BCFill = Table[0, {I, 1, (3 * Length[Extract[Layup, {1, j}]] - 3)},
                {J, 1, (3 * Length[Extract[Layup, {1, Ceiling[(i + 1)/2]}]] - 3)}];
    BCTemp = Join[BCTemp, BCFill];
  ];
]
BCTemp = Drop[BCTemp, 1];
(* Cleans up initialization of independent-column matrix. *)
ATemp = Join[ATemp, BCTemp];
ATemp = Drop[ATemp, 1];
AComp = ReplacePart[AComp, ATemp, {1, i}];
]
(* Assembles symmetric half of output matrix from previously-calculated terms.*)
ANonSym = Transpose[Extract[AComp, {1, 1}]];
For[Ai = 2, Ai <= ANumber, Ai++,
  ANonSym = Join[ANonSym, Transpose[Extract[AComp, {1, Ai}]]];
];
OmTotalLength = 0;
For[OmLi = 1, OmLi ≤ NRegions, OmLi++,

OmTotalLength = OmTotalLength + (2 − KroneckerDelta[1, OmLi]
−KroneckerDelta[NRegions, OmLi]) * Length[Extract[OmSystem, {1, OmLi}]];
];

AMatrix = Table[
If[I ≤ OmTotalLength, Extract[ANonSym, {I, J}],
If[J ≤ OmTotalLength, Extract[ANonSym, {J, I}],
0]], {I, 1, Length[Transpose[ANonSym]]}, {J, 1, Length[Transpose[ANonSym]]}];
Return[AMatrix];
]

(* Calculates and assembles Right-Hand Vector in Equation 7.9a, as well as far-field stresses from CLPT. *)

BVector[OmSystem_, Layup_] :=
If[True,

(* Defines local variables for number of regions and number of dropoffs present in laminate. *)

NRegions = Length[Transpose[Layup]];
NDrops = NRegions − 1;

(* Initializes output vector. *)

Nn = Sum[(2 − KroneckerDelta[1, i] − KroneckerDelta[NRegions, i])
*Length[Extract[OmSystem, {1, i}]], {i, NRegions}];

BB = Table[0, {i, 1, Nn}, {j, 1, 1}];

(* Far-field load per unit length on the laminate. This follows CLPT conventions in the form P_11, P_22, P_12. *)

Load = (1
0
0
);

(* Calculates CLPT solution. *)
CLPTSystem = 0 * Transpose[{Range[NRegions]}];
CLPTSystem = PadRight[RegESystem, {1, NRegions}];
For[CLPTi = 1, CLPTi ≤ NRegions, CLPTi++,
(* Calculates Ply Strains. *)
AMat = 2 * LPTM[Extract[Layup, {1, CLPTi}], A];
eStrain = LinearSolve[AMat, Load];
(* Calculates ply/sublayer stresses. *)
σTemp = Table[0, {Q, 1, Length[Extract[Layup, {1, CLPTi}]]}, {X, 1, 3}];
For[CLPTj = 1, CLPTj ≤ Length[Extract[Layup, {1, CLPTi}]], CLPTj++,
CLPTSol = TwoDElas[Extract[Extract[Layup, {1, CLPTi}], {CLPTj, 2}], 1].eStrain;
σTemp = ReplacePart[σTemp, Extract[CLPTSol, {1, 1}], {CLPTj, 1}];
σTemp = ReplacePart[σTemp, Extract[CLPTSol, {2, 1}], {CLPTj, 2}];
σTemp = ReplacePart[σTemp, Extract[CLPTSol, {3, 1}], {CLPTj, 3}];
];
CLPTSystem = ReplacePart[CLPTSystem, σTemp, {1, CLPTi}];
]
(* Assembles output vector. *)
For[BFilli = 1, BFilli ≤ NDrops, BFilli++,
LDiff = (Length[Extract[Layup, {1, BFilli}]] - Length[Extract[Layup, {1, BFilli + 1}]]);
σA = Extract[CLPTSystem, {1, BFilli}];
σB = Join[Table[0, {Bi, 1, LDiff}, {Bj, 1, 3}], Extract[CLPTSystem, {1, BFilli + 1}]];
σ11Diff = Drop[σB - σA, {Length[σB]}, {2, 3}];
σ12Diff = Drop[σB - σA, {Length[σB]}, {1, 2}];
σ13Fill = Table[0, {i, 1, Length[σ11Diff]}, {j, 1, 1}];
BB = Join[BB, σ11Diff, σ12Diff, σ13Fill];
];
Return[{BB, CLPTSystem}];
}
BEGIN MAIN ROUTINE

(* All subroutines must be read before this script can be executed. This script denotes the
"main" routine of the implementation of the model. *)

(* Initializes storage for eigenfunction and final linear system matrices. *)
RegESystem = 0 * Transpose[{Range[NRegions]}];
RegESystem = PadRight[RegESystem, {1, NRegions}];
OmSystem = 0 * Transpose[{Range[2 * (NRegions - 1)]}];
OmSystem = PadRight[OmSystem, {1, NRegions}];
OmCSystem = 0 * Transpose[{Range[NRegions - 2]}];
OmCSystem = PadRight[OmCSystem, {1, NRegions - 2}];
GammaSystem = 0 * Transpose[{Range[2 * (NRegions - 1)]}];
GammaSystem = PadRight[GammaSystem, {2, 2 * (2 * NRegions - 3)}];
ThetaSystem = 0 * Transpose[{Range[2 * (NRegions - 1)]}];
ThetaSystem = PadRight[ThetaSystem, {1, 2 * (2 * NRegions - 3)}];

(* For all Regions *)
For[w = 1, w < NRegions, w++,

(* Solves and stores eigenfunction problem solution. *)
CurReg = Extract[Layup, {1, w}];
EFunc = Eigenfunction[CurReg];
RegESystem = ReplacePart[RegESystem, EFunc, {1, w}];

(* For Region A *)
If[w == 1,

(* Calculates and stores Omega, Gamma, and Theta matrices. *)
OmTemp = Omega[Extract[RegESystem, {1, w}], CurReg, {-\infty, 0}, Null];
{GFTemp, GGTemp, ThTemp} = GammaThetaMat[Extract[RegESystem, {1, w}],
CurReg, {0, -\infty}, False];
OmSystem = ReplacePart[OmSystem, OmTemp, {1, w}];
GammaSystem = ReplacePart[GammaSystem, GFTemp, {1, w}];
GammaSystem = ReplacePart[GammaSystem, GGTemp, {2, w}];
ThetaSystem = ReplacePart[ThetaSystem, ThTemp, {1, w}];

(* For final Region in problem *)
If[w == NRegions,
(* Calculates and stores Omega, Gamma, and Theta matrices. *)
OmTemp = Omega[Extract[RegESystem, {1, w}], CurReg, {Extract[Len, w - 1],
Extract[Len, w]}, Null];
{GFTemp, GGTemp, ThTemp} = GammaThetaMatCut[Extract[RegESystem, {1, w}],
CurReg, {Extract[Len, w - 1], Extract[Len, w]}, Extract[Layup, {1, w - 1}], False];
OmSystem = ReplacePart[OmSystem, OmTemp, {1, w}];
GammaSystem = ReplacePart[GammaSystem, GFTemp, {1, 2 * (2 * w - 3)}];
GammaSystem = ReplacePart[GammaSystem, GGTemp, {2, 2 * (2 * w - 3)}];
ThetaSystem = ReplacePart[ThetaSystem, ThTemp, {1, 2 * (2 * w - 3)}];
,
(* FOR INTERMEDIATE REGIONS - STEP 3 ONLY *)
(* Calculates and stores Omega, Omega-Cross, Gamma, and Theta matrices. *)
OmTemp = Omega[Extract[RegESystem, {1, w}], CurReg, {Extract[Len, w - 1],
Extract[Len, w]}, Null];
OmCrossTemp = Omega[Extract[RegESystem, {1, w}], CurReg, {Extract[Len, w - 1],
Extract[Len, w]}, Crossed];
{GFTempB1, GGTempB1, ThTempB1} = GammaThetaMatCut[Extract[RegESystem,
{1, w}], CurReg, {Extract[Len, w - 1], Extract[Len, w]}, Extract[Layup,
{1, w - 1}], False];
{GFTempF1, GGTempF1, ThTempF1} = GammaThetaMatCut[Extract[RegESystem,
{1, w}], CurReg, {Extract[Len, w - 1], Extract[Len, w]}, Extract[Layup,
{1, w - 1}], True];
{GFTempB2, GGTempB2, ThTempB2} = GammaThetaMat[Extract[RegESystem,
\{1, w\}, CurReg, \{Extract[Len, w], Extract[Len, w + 1]\}, True];
{GFTempF2, GGTempF2, ThTempF2} = GammaThetaMat[Extract[RegESystem,
{1, w}], CurReg, \{Extract[Len, w], Extract[Len, w + 1]\}, False];
OmSystem = ReplacePart[OmSystem, OmTemp, \{1, w\}];
OmCSystem = ReplacePart[OmCSystem, OmCrossTemp, \{1, w - 1\}];
GammaSystem = ReplacePart[GammaSystem, GFTempB1, \{1, 4 * w - 6\}];
GammaSystem = ReplacePart[GammaSystem, GGTempB1, \{2, 4 * w - 6\}];
ThetaSystem = ReplacePart[ThetaSystem, ThTempB1, \{1, 4 * w - 6\}];
GammaSystem = ReplacePart[GammaSystem, GFTempB2, \{1, 4 * w - 5\}];
GammaSystem = ReplacePart[GammaSystem, GGTempB2, \{2, 4 * w - 5\}];
ThetaSystem = ReplacePart[ThetaSystem, ThTempB2, \{1, 4 * w - 5\}];
GammaSystem = ReplacePart[GammaSystem, GFTempF1, \{1, 4 * w - 4\}];
GammaSystem = ReplacePart[GammaSystem, GGTempF1, \{2, 4 * w - 4\}];
ThetaSystem = ReplacePart[ThetaSystem, ThTempF1, \{1, 4 * w - 4\}];
GammaSystem = ReplacePart[GammaSystem, GFTempF2, \{1, 4 * w - 3\}];
GammaSystem = ReplacePart[GammaSystem, GGTempF2, \{2, 4 * w - 3\}];
ThetaSystem = ReplacePart[ThetaSystem, ThTempF2, \{1, 4 * w - 3\}];
);
);
);
(* Assemble Matrix in Final linear system. *)
AMatrix = MainAssembly[OmSystem, OmCSystem, GammaSystem, ThetaSystem, Layup];
(* Assemble right-hand Vector *)
\{BVec, CLPTSo1\} = BVector[OmSystem, Layup];
(* Solves linear system for unknown stress coefficients. *)
Solution = LinearSolve[AMatrix, BVec];
(* BEGIN SOLUTION ASSEMBLY AND POSTPROCESS SUBROUTINES *)
(* This subroutine calculates the specific forms of the stresses via substitution of eigenvalues, eigenvectors, and unknown stress coefficients in the assumed stress shapes. *)

(* Matrix of unknown stress coefficients *)
SolutionBreak = Table[0, {i, 1, 1}, {j, 1, NRegions}, {k, 1, 2}];

(* Matrix of x_1-variation of ply stresses *)
XAssembly = Table[0, {i, 1, 3}, {j, 1, NRegions}, {k, 1, 2}];

(* Matrix of x_3-variation of ply stresses *)
ZAssembly = Table[0, {i, 1, 5}, {j, 1, NRegions}];

(* Output matrix of ply stresses for each ply in each region *)
PlyStresses = Table[0, {i, 1, 5}, {k, 1, NRegions}];

(* For all Regions *)
For[Si = 1, Si <= NRegions, Si++,

(* Extracts Region Eigenvalues and Eigenvectors. *)
RegionEigenvalues = ValExtract[Extract[RegESystem, {1, Si}]];
RegionEigenvectors = VecExtract[Extract[RegESystem, {1, Si}]];

(* Initializes x_1-variation derivative matrices. *)
RegAssembly = Table[0, {p, 1, 2*Length[RegionEigenvalues]/3}, {q, 1, Length[RegionEigenvalues]}];
DRegAssembly = Table[0, {p, 1, 2*Length[RegionEigenvalues]/3}, {q, 1, Length[RegionEigenvalues]}];
DDRegAssembly = Table[0, {p, 1, 2*Length[RegionEigenvalues]/3}, {q, 1, Length[RegionEigenvalues]}];
RegAssembly2 = RegAssembly;
DRegAssembly2 = DRegAssembly;
DDRegAssembly2 = DDRegAssembly;

SolutionBreakT = Table[0, {i, 1, Length[Extract[OmSystem, {1, Si}]]}];
SolutionBreakT2 = SolutionBreakT;

(* For all Regions except the final Region, calculates x_1-variation terms and relevant unknown stress coefficients. *)
If Si ≠ NRegions,

\[
\text{RegAssembly} = \text{Table}[\text{Extract}[\text{RegionEigenvectors}, \{q, p\}] * e^{\text{Extract}[\text{RegionEigenvalues}, q] * (z - \text{Extract}[\text{Len}, Si])}, \{p, 1, 2 \times \text{Length}[\text{RegionEigenvalues}] / 3\}, \{q, 1, \text{Length}[\text{RegionEigenvalues}]\}];
\]

\[
\text{DRegAssembly} = \text{Table}[\text{Extract}[\text{RegionEigenvectors}, \{q, p\}] * e^{\text{Extract}[\text{RegionEigenvalues}, q] * (z - \text{Extract}[\text{Len}, Si])} * \text{Extract}[\text{RegionEigenvalues}, q], \{p, 1, 2 \times \text{Length}[\text{RegionEigenvalues}] / 3\}, \{q, 1, \text{Length}[\text{RegionEigenvalues}]\}];
\]

\[
\text{DDRegAssembly} = \text{Table}[\text{Extract}[\text{RegionEigenvectors}, \{q, p\}] * e^{\text{Extract}[\text{RegionEigenvalues}, q] * (z - \text{Extract}[\text{Len}, Si])} * (\text{Extract}[\text{RegionEigenvalues}, q])^2, \{p, 1, 2 \times \text{Length}[\text{RegionEigenvalues}] / 3\}, \{q, 1, \text{Length}[\text{RegionEigenvalues}]\}];
\]

\[
\text{SolutionBreakT} = \text{Take}[\text{Solution}, \{\text{Sum}[(2 - \text{KroneckerDelta}[1, j]) \times \text{Length}[\text{Extract}[\text{OmSystem}, \{1, j\}]], \{j, 1, Si\}] - \text{Length}[\text{Extract}[\text{OmSystem}, \{1, Si\}]] + \text{Sum}[(2 - \text{KroneckerDelta}[1, j]) \times \text{Length}[\text{Extract}[\text{OmSystem}, \{1, j\}]], \{j, 1, Si\}]] ];
\]

(* For all Regions except Region A, calculates x_1-variation terms and relevant unknown stress coefficients *)

If Si ≠ 1,

\[
\text{RegAssembly2} = \text{Table}[\text{Extract}[\text{RegionEigenvectors}, \{q, p\}] * e^{\text{Extract}[\text{RegionEigenvalues}, q] * (z - \text{Extract}[\text{Len}, Si - 1])}, \{p, 1, 2 \times \text{Length}[\text{RegionEigenvalues}] / 3\}, \{q, 1, \text{Length}[\text{RegionEigenvalues}]\}];
\]

\[
\text{DRegAssembly2} = \text{Table}[\text{Extract}[\text{RegionEigenvectors}, \{q, p\}] * e^{\text{Extract}[\text{RegionEigenvalues}, q] * (z - \text{Extract}[\text{Len}, Si - 1])} * -\text{Extract}[\text{RegionEigenvalues}, q], \{p, 1, 2 \times \text{Length}[\text{RegionEigenvalues}] / 3\}, \{q, 1, \text{Length}[\text{RegionEigenvalues}]\}];
\]

\[
\text{DDRegAssembly2} = \text{Table}[\text{Extract}[\text{RegionEigenvectors}, \{q, p\}] * e^{\text{Extract}[\text{RegionEigenvalues}, q] * (z - \text{Extract}[\text{Len}, Si - 1])} * (\text{Extract}[\text{RegionEigenvalues}, q])^2, \{p, 1, 2 \times \text{Length}[\text{RegionEigenvalues}] / 3\}, \{q, 1, \text{Length}[\text{RegionEigenvalues}]\}];
\]

\[
\text{SolutionBreakT2} = \text{Take}[\text{Solution}, \{\text{Sum}[(2 - \text{KroneckerDelta}[1, j]) \times \text{Length}[\text{Extract}[\text{OmSystem}, \{1, j\}]], \{j, 1, Si - 1\}] + 1,
\]

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\[ \text{Sum}[(2 - \text{KroneckerDelta}[1, j]) \times \text{Length}[\text{Extract}[\text{OmSystem}, \{1, j\}], \{j, 1, \text{Si}\}]
- \text{Length}[\text{Extract}[\text{OmSystem}, \{1, \text{Si}\}])]; \]

\[
\]

(* Assembles and stores x_1-variation matrix. *)
\[
\text{XAssembly} = \text{ReplacePart}[\text{XAssembly}, \text{RegAssembly}, \{1, \text{Si}, 1\}];
\]
\[
\text{XAssembly} = \text{ReplacePart}[\text{XAssembly}, \text{DRegAssembly}, \{2, \text{Si}, 1\}];
\]
\[
\text{XAssembly} = \text{ReplacePart}[\text{XAssembly}, \text{DDRegAssembly}, \{3, \text{Si}, 1\}];
\]
\[
\text{XAssembly} = \text{ReplacePart}[\text{XAssembly}, \text{RegAssembly2}, \{1, \text{Si}, 2\}];
\]
\[
\text{XAssembly} = \text{ReplacePart}[\text{XAssembly}, \text{DRegAssembly2}, \{2, \text{Si}, 2\}];
\]
\[
\text{XAssembly} = \text{ReplacePart}[\text{XAssembly}, \text{DDRegAssembly2}, \{3, \text{Si}, 2\}];
\]

(* Stores unknown stress coefficients. *)
\[
\text{SolutionBreak} = \text{ReplacePart}[\text{SolutionBreak}, \text{SolutionBreakT}, \{1, \text{Si}, 1\}];
\]
\[
\text{SolutionBreak} = \text{ReplacePart}[\text{SolutionBreak}, \text{SolutionBreakT2}, \{1, \text{Si}, 2\}];
\]

(* Initializes x_3-variation matrix. *)
\[
\text{NN} = \text{Length}[\text{Extract}[\text{Layup}, \{1, \text{Si}\}]);
\]
\[
\text{Temp11} = \{\text{Null}\};
\]
\[
\text{Temp12} = \{\text{Null}\};
\]
\[
\text{Temp13} = \{\text{Null}\};
\]
\[
\text{Temp23} = \{\text{Null}\};
\]
\[
\text{Temp33} = \{\text{Null}\};
\]

(* For each ply, calculates x_3-variations. *)
\[
\text{For}[\text{Pliesi} = 1, \text{Pliesi} \leq \text{NN}, \text{Pliesi}++;,
\]
\[
\text{StressFill} = \text{Pliesi} - 2;
\]
\[
\text{CurrG11} = \text{S11F}[\text{NN}, \text{Pliesi}, \text{Extract}[\text{Layup}, \{1, \text{Si}\}, z];
\]
\[
\text{CurrG12} = \text{S12F}[\text{NN}, \text{Pliesi}, \text{Extract}[\text{Layup}, \{1, \text{Si}\}, z];
\]
\[
\text{CurrG13} = \text{S13F}[\text{NN}, \text{Pliesi}, \text{Extract}[\text{Layup}, \{1, \text{Si}\}, z];
\]
\[
\text{CurrG23} = \text{S23F}[\text{NN}, \text{Pliesi}, \text{Extract}[\text{Layup}, \{1, \text{Si}\}, z];
\]
\[
\text{CurrG33} = \text{S33F}[\text{NN}, \text{Pliesi}, \text{Extract}[\text{Layup}, \{1, \text{Si}\}, z];
\]
\[
\text{G11j} = \text{Join}[0 \times \text{Range}[\text{StressFill}], \text{CurrG11}, 0 \times \text{Range}[\text{NN} - 1
\]
-Length[CurrG11] = If[StressFill > 0, StressFill, 0] * Range[NN - 1];
G12j = Join[0 * Range[NN - 1], 0 * Range[StressFill], CurrG12,
0 * Range[NN - 1 - Length[CurrG12] - If[StressFill > 0, StressFill, 0]]];
G13j = Join[0 * Range[StressFill], CurrG13, 0 * Range[NN - 1
- Length[CurrG13] - If[StressFill > 0, StressFill, 0]]], 0 * Range[NN - 1];
G23j = Join[0 * Range[NN - 1], 0 * Range[StressFill], CurrG23,
0 * Range[NN - 1 - Length[CurrG23] - If[StressFill > 0, StressFill, 0]]];
G33j = Join[CurrG33, 0 * Range[NN - 1]];
Temp11 = Join[Temp11, {G11j}];
Temp12 = Join[Temp12, {G12j}];
Temp13 = Join[Temp13, {G13j}];
Temp23 = Join[Temp23, {G23j}];
Temp33 = Join[Temp33, {G33j}];
];
Temp11 = Drop[Temp11, 1];
Temp12 = Drop[Temp12, 1];
Temp13 = Drop[Temp13, 1];
Temp23 = Drop[Temp23, 1];
Temp33 = Drop[Temp33, 1];
(* Stores x.3-variations in to matrix. *)
ZAssembly = ReplacePart[ZAssembly, Temp11, {1, Si}];
ZAssembly = ReplacePart[ZAssembly, Temp12, {2, Si}];
ZAssembly = ReplacePart[ZAssembly, Temp13, {3, Si}];
ZAssembly = ReplacePart[ZAssembly, Temp23, {4, Si}];
ZAssembly = ReplacePart[ZAssembly, Temp33, {5, Si}];
(* Final Assembly of Ply Stresses *)
PlyStresses = ReplacePart[PlyStresses, Extract[ZAssembly, {1, Si}]
.Sum[Extract[XAssembly, {1, Si, Sk}].Extract[SolutionBreak, {1, Si, Sk}],
{Sk, 1, 2}], {1, Si}];
PlyStresses = ReplacePart[PlyStresses, Extract[ZAssembly, {2, Si}]
.Sum[Extract[XAssembly, {1, Si, Sk}].Extract[SolutionBreak, 
{1, Si, Sk}], {Sk, 1, 2}], {2, Si}];
PlyStresses = ReplacePart[PlyStresses, Extract[ZAssembly, {3, Si}]
.Sum[Extract[XAssembly, {2, Si, Sk}].Extract[SolutionBreak, {1, Si, Sk}], 
{Sk, 1, 2}], {3, Si}];
PlyStresses = ReplacePart[PlyStresses, Extract[ZAssembly, {4, Si}]
.Sum[Extract[XAssembly, {2, Si, Sk}].Extract[SolutionBreak, {1, Si, Sk}], 
{Sk, 1, 2}], {4, Si}];
PlyStresses = ReplacePart[PlyStresses, Extract[ZAssembly, {5, Si}]
.Sum[Extract[XAssembly, {3, Si, Sk}].Extract[SolutionBreak, {1, Si, Sk}], 
{Sk, 1, 2}], {5, Si}];
];

(*----------------------------------------------------------*)

BEGIN PLOTTING SUBROUTINES

(* Note: Plotting subroutine is hard-coded to handle plotting two regions. Modification of the following subroutine can be used to plot 3 or more regions as in Step 3. Hard-coding of the subroutine was chosen as a general plotting subroutine was found to be more inefficient than hard-coding for cases with less than 6-7 dropoffs. *)

(* Function Call: [Ply # in Region A, Ply # in Region B, CLPTSolution Matrix, PlyStress-Solution Matrix]. *)

StressPlotGeneration[PlotPlyA, PlotPlyB, CLPTSol, PlyStresses]:=
If[True,
RXXA = Re[Extract[ComplexExpand[Extract[PlyStresses, {1, 1}]], PlotPlyA]] +
1 * Extract[Extract[CLPTSol, {1, 1}], {PlotPlyA, 1}];
RXXB = Re[Extract[ComplexExpand[Extract[PlyStresses, {1, 2}]], PlotPlyB]] +
1 * Extract[Extract[CLPTSol, {1, 2}], {PlotPlyB, 1}];
RXYA = Re[Extract[ComplexExpand[Extract[PlyStresses, {2, 1}]], PlotPlyA]] +
1 * Extract[Extract[CLPTSol, {1, 1}], {PlotPlyA, 3}];
\[ RXYB = \text{Re}\left[ \text{Extract}\left[ \text{ComplexExpand}\left[ \text{Extract}\left[ \text{PlyStresses}, \{2, 2\}\right], \text{PlotPlyB}\right]\right]\right] + 1 \times \text{Extract}\left[ \text{Extract}\left[ \text{CLPTSol}, \{1, 2\}\right], \{\text{PlotPlyB}, 3\}\right]; \]

\[ RXZA = \text{Re}\left[ \text{Extract}\left[ \text{ComplexExpand}\left[ \text{Extract}\left[ \text{PlyStresses}, \{3, 1\}\right], \text{PlotPlyA}\right]\right]\right]; \]

\[ RXZB = \text{Re}\left[ \text{Extract}\left[ \text{ComplexExpand}\left[ \text{Extract}\left[ \text{PlyStresses}, \{3, 2\}\right], \text{PlotPlyB}\right]\right]\right]; \]

\[ RYZA = \text{Re}\left[ \text{Extract}\left[ \text{ComplexExpand}\left[ \text{Extract}\left[ \text{PlyStresses}, \{4, 1\}\right], \text{PlotPlyA}\right]\right]\right]; \]

\[ RYZB = \text{Re}\left[ \text{Extract}\left[ \text{ComplexExpand}\left[ \text{Extract}\left[ \text{PlyStresses}, \{4, 2\}\right], \text{PlotPlyB}\right]\right]\right]; \]

\[ RZZA = \text{Re}\left[ \text{Extract}\left[ \text{ComplexExpand}\left[ \text{Extract}\left[ \text{PlyStresses}, \{5, 1\}\right], \text{PlotPlyA}\right]\right]\right]; \]

\[ RZZB = \text{Re}\left[ \text{Extract}\left[ \text{ComplexExpand}\left[ \text{Extract}\left[ \text{PlyStresses}, \{5, 2\}\right], \text{PlotPlyB}\right]\right]\right]; \]

\[ \text{Return}\left[\{RXXA, RXYA, RXZA, RYZA, RZZA, RXXB, RXYB, RXZB, RYZB, RZZB\}\right]; \]

\{RXXA, RXYA, RXZA, RYZA, RZZA, RXXB, RXYB, RXZB, RYZB, RZZB\} =

\text{StressPlotGeneration}[3, 2, \text{CLPTSol, PlyStresses}];

\begin{verbatim}
(* Plot parameters, set to liking/format requirements. *)

\text{Col} = \text{Black};
\text{DashNumber} = \text{Thick};
\text{L1} = \text{Extract}[\text{RLengths}, 1];
\text{LP} = \text{If}[\text{L1} \geq 1, 1, \text{L1}];
\text{zloc} = -.125/2; (* Desired x_3-location for lengthwise plots. *)
\text{xloc} = .01; (* Desired x_1-location for through-thickness plots. *)

(* Plots and Shows Plots *)

\text{Outplot} = \text{Show}[\text{Plot}[\text{RXXA}]/._. \rightarrow \text{zloc}, \{x, -1, 0\}, \text{PlotRange} \rightarrow \{-1, 1\}, \{0, 2.5\},
\text{AxesOrigin} \rightarrow \{0, 0\}, \text{PlotStyle} \rightarrow \{\text{Col, Thick, DashNumber}, \text{AspectRatio} \rightarrow .7,
\text{WorkingPrecision} \rightarrow 20, \text{Frame} \rightarrow \text{True}, \text{FrameTicks} \rightarrow \{\{\text{Automatic, None},
\{\text{Automatic, None}\}, \text{TicksStyle} \rightarrow \text{Directive}[18], \text{LabelStyle} \rightarrow \text{Directive}[24],
\text{Plot}[\text{RXXB}]/._. \rightarrow \text{zloc}, \{x, 0, \text{LP}\}, \text{PlotStyle} \rightarrow \{\text{Col, Thick, DashNumber},
\text{PlotRange} \rightarrow \text{All}\}]
\text{Outplot2} = \text{Show}[\text{Plot}[\text{RXZA}]/._. \rightarrow \text{zloc}, \{x, -1, 0\}, \text{PlotRange} \rightarrow \{-1, 1\}, \{0, 0.70\},
\text{AxesOrigin} \rightarrow \{0, 0\}, \text{PlotStyle} \rightarrow \{\text{Black, Thick, DashNumber}, \text{AspectRatio} \rightarrow .7,
\text{WorkingPrecision} \rightarrow 20, \text{Frame} \rightarrow \text{True}, \text{FrameTicks} \rightarrow \{\{\text{Automatic, None},
\end{verbatim}
BEGIN THROUGH-THICKNESS PLOTTING SUBROUTINE
(* Again, this is hard-coded for savings in runtime. The following is commented out, as it assumes 16 plies or sublayers total in the laminate. It is provided for reference and future use and development. *)

(*
{RXXA1, RXYA1, RXZA1, RYZA1, RXXB8, RXYB8, RXZB8, RYZB8, RZZB8}
  = StressPlotGeneration[1, 1, CLPTSol, PlyStresses];
{RXXA2, RXYA2, RXZA2, RYZA2, RXXB8, RXYB8, RXZB8, RYZB8, RZZB8}
  = StressPlotGeneration[2, 1, CLPTSol, PlyStresses];
{RXXA3, RXYA3, RXZA3, RYZA3, RXXB8, RXYB8, RXZB8, RYZB8, RZZB8}
  = StressPlotGeneration[3, 1, CLPTSol, PlyStresses];
{RXXA4, RXYA4, RXZA4, RYZA4, RXXB8, RXYB8, RXZB8, RYZB8, RZZB8}
  = StressPlotGeneration[4, 2, CLPTSol, PlyStresses];
{RXXA5, RXYA5, RXZA5, RYZA5, RXXB5, RXYB5, RXZB5, RYZB5, RZZB5}
  = StressPlotGeneration[5, 1, CLPTSol, PlyStresses];
{RXXA6, RXYA6, RXZA6, RYZA6, RXXB6, RXYB6, RXZB6, RYZB6, RZZB6}
  = StressPlotGeneration[6, 2, CLPTSol, PlyStresses];
{RXXA7, RXYA7, RXZA7, RYZA7, RXXB7, RXYB7, RXZB7, RYZB7, RZZB7}
  = StressPlotGeneration[7, 3, CLPTSol, PlyStresses];
{RXXA8, RXYA8, RXZA8, RYZA8, RXXB8, RXYB8, RXZB8, RYZB8, RZZB8}
  = StressPlotGeneration[8, 4, CLPTSol, PlyStresses];
{RXXA1T, RXZA1T, RZZA1T} = {RXXA1, RXZA1, RZZA1}/.z -> -Z + 15 * .125/4;
{RXXA2T, RXZA2T, RZZA2T} = {RXXA2, RXZA2, RZZA2}/.z -> -Z + 13 * .125/4;
{RXXA3T, RXZA3T, RZZA3T} = {RXXA3, RXZA3, RZZA3}/.z -> -Z + 11 * .125/4;
{RXXA4T, RXZA4T, RZZA4T} = {RXXA4, RXZA4, RZZA4}/.z -> -Z + 9 * .125/4;
{RXXA5T, RXZA5T, RZZA5T} = {RXXA5, RXZA5, RZZA5}/.z -> -Z + 7 * .125/4;
{RXXA6T, RXZA6T, RZZA6T} = {RXXA6, RXZA6, RZZA6}/.z -> -Z + 5 * .125/4;
{RXXA7T, RXZA7T, RZZA7T} = {RXXA7, RXZA7, RZZA7}/.z -> -Z + 3 * .125/4;
{RXXA8T, RXZA8T, RZZA8T} = {RXXA8, RXZA8, RZZA8}/.z -> -Z + 1 * .125/4;*)
\{RXXB5T, RXZB5T, RZZB5T\} = \{RXXB5, RXZB5, RZZB5\}/.z \rightarrow -Z + 7 \cdot .125/4;
\{RXXB6T, RXZB6T, RZZB6T\} = \{RXXB6, RXZB6, RZZB6\}/.z \rightarrow -Z + 5 \cdot .125/4;
\{RXXB7T, RXZB7T, RZZB7T\} = \{RXXB7, RXZB7, RZZB7\}/.z \rightarrow -Z + 3 \cdot .125/4;
\{RXXB8T, RXZB8T, RZZB8T\} = \{RXXB8, RXZB8, RZZB8\}/.z \rightarrow -Z + 1 \cdot .125/4;
*
(*
Outplot6 = Show[ParametricPlot[{Extract[RXXA8T//.x->0,1], Z}, {Z, 0, .125/2},
PlotRange \rightarrow \{\{0, 3\}, \{0, .125 \cdot 2\}\}, PlotStyle \rightarrow \{Black, Thick, DashNumber\},
AspectRatio \rightarrow .7, Frame \rightarrow True, FrameTicks\rightarrow\{\{Automatic, None\},
\{Automatic, None\}\}, TicksStyle \rightarrow Directive[18], LabelStyle \rightarrow Directive[24]],
ParametricPlot[{Extract[RXXA7T//.x->0,1], Z}, \{Z, .125/2, .125\},
PlotStyle \rightarrow \{Black, Thick, DashNumber\}, PlotRange \rightarrow All],
ParametricPlot[{Extract[RXXA6T//.x->0,1], Z}, \{Z, .125, .125 \cdot 3/2\},
PlotStyle \rightarrow \{Black, Thick, DashNumber\}, PlotRange \rightarrow All],
ParametricPlot[{Extract[RXXA5T//.x->0,1], Z}, \{Z, .125 \cdot 3/2, .125 \cdot 2\},
PlotStyle \rightarrow \{Black, Thick, DashNumber\}, PlotRange \rightarrow All]
*)
(*
Outplot7 = Show[ParametricPlot[{Extract[RXZB8T//.x->xloc,1], Z}, \{Z, 0, .125/2\},
PlotRange \rightarrow \{\{-5, 1.0\}, \{0, .125 \cdot 2\}\}, PlotStyle \rightarrow \{Black, Thick, DashNumber\},
AspectRatio \rightarrow .7, Frame \rightarrow True, FrameTicks\rightarrow\{\{Automatic, None\},
\{Automatic, None\}\}, TicksStyle \rightarrow Directive[18], LabelStyle \rightarrow Directive[24]],
ParametricPlot[{Extract[RXZB7T//.x->xloc,1], Z}, \{Z, .125/2, .125\},
PlotStyle \rightarrow \{Black, Thick, DashNumber\}, PlotRange \rightarrow All],
ParametricPlot[{Extract[RXZB6T//.x->xloc,1], Z}, \{Z, .125, .125 \cdot 3/2\},
PlotStyle \rightarrow \{Black, Thick, DashNumber\}, PlotRange \rightarrow All],
ParametricPlot[{Extract[RXZB5T//.x->xloc,1], Z}, \{Z, .125 \cdot 3/2, .125 \cdot 2\},
PlotStyle \rightarrow \{Black, Thick, DashNumber\}, PlotRange \rightarrow All]
*)

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(*
Outplot8 = Show[ParametricPlot[{Extract[RZZA8T // .x -> 0, 1], Z}, {Z, 0, .125/2},
PlotRange -> {-.4, 1}, PlotStyle -> {Black, Thick, DashNumber},
AspectRatio -> .7, Frame -> True, FrameTicks -> {Automatic, None},
{Automatic, None}], TicksStyle -> Directive[18], LabelStyle -> Directive[24]],
ParametricPlot[{Extract[RZZA7T // .x -> 0, 1], Z}, {Z, .125/2, .125},
PlotStyle -> {Black, Thick, DashNumber}, PlotRange -> All],
ParametricPlot[{Extract[RZZA6T // .x -> 0, 1], Z}, {Z, .125, .125 * 3/2},
PlotStyle -> {Black, Thick, DashNumber}, PlotRange -> All],
ParametricPlot[{Extract[RZZA5T // .x -> 0, 1], Z}, {Z, .125 * 3/2, .125 * 2},
PlotStyle -> {Black, Thick, DashNumber}, PlotRange -> All],
ParametricPlot[{Extract[RZZA4T // .x -> 0, 1], Z}, {Z, .125 * 4/2, .125 * 5/2},
PlotStyle -> {Black, Thick, DashNumber}, PlotRange -> All],
ParametricPlot[{Extract[RZZA3T // .x -> 0, 1], Z}, {Z, .125 * 5/2, .125 * 6/2},
PlotStyle -> {Black, Thick, DashNumber}, PlotRange -> All],
ParametricPlot[{Extract[RZZA2T // .x -> 0, 1], Z}, {Z, .125 * 6/2, .125 * 7/2},
PlotStyle -> {Black, Thick, DashNumber}, PlotRange -> All],
ParametricPlot[{Extract[RZZA1T // .x -> 0, 1], Z}, {Z, .125 * 7/2, .125 * 8/2},
PlotStyle -> {Black, Thick, DashNumber}, PlotRange -> All]]
*)