Magnetorheological Fluids for Extreme Environments: Stronger, Lighter, Hotter

by

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B.S., Mechanical Engineering, Middle East Technical University (1999)

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ABSTRACT

The controllable properties of magnetorheological (MR) fluids offer reliable and efficient actuation means to a number of far-ranging engineering applications. In this thesis we are motivated by the applications of MR fluids in oil & gas exploration and production. These applications also bring about a number of operational requirements for the fluid such as generating large magnetically induced shift in rheological properties with tolerance to elevated temperatures and low fluid density in order to maintain manageable hydrostatic downhole pressures.

In this thesis we investigate a number of these fluid design constraints. Firstly, the evolution of the rheological properties of MR fluids over a wide range of magnetic field and temperature was investigated. A magnetorheometry fixture with a unique combination of high-field and high-temperature capability was manufactured. With the experimental measurements and the results from a numerical model of interparticle magnetic interaction, a scaling law was identified between the applied magnetic field and the resulting MR yield stress.

The aggregation phenomena and the evolution of fluid microstructure were also investigated in microfluidic geometries with strong particle-wall interactions. The results of this study highlighted design features and operational techniques that can improve the performance of MR fluid valves. Investigation of fluid flow in non-uniform magnetic fields showed that in these regions the motion of the particle phase is governed by a balance between hydrodynamic and magnetophoretic forces.

Finally, the flow of MR fluids in spatially-inhomogeneous magnetic and deformation fields was studied. A slit-flow magnetorheometer was manufactured to measure the bulk MR response of the fluid under non-uniform fields. In order to understand the parameters governing these flows and to develop a predictive tool for further investigations, a two-fluid suspension-balance constitutive model was developed which captures the key features of multi-phase flow and fluid
anisotropy. The model was numerically implemented using the finite element method and was used to study the transport of MR fluids in spatially-inhomogeneous flows such as those encountered in contraction and expansion channels. This model provides insight into the design and optimization of MR fluid devices that can enhance the magnetically-controlled gain in flow resistance under downhole conditions.

Thesis Supervisor: Gareth H. McKinley
Title: Professor of Mechanical Engineering
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Chapter 1

Introduction and Background

Magnetorheological (MR) fluids are suspensions of micron-sized magnetizable particles that undergo a large and reversible change in the rheological properties under an applied magnetic field (Rabinow, 1948; Rankin et al., 1998; Klingenberg, 2001; de Vicente et al., 2011). The magnetically controllable properties have enabled a number of far-ranging engineering applications utilizing these fluids including automotive suspension (Grad, 2006), prosthetic leg (Herr and Wilkenfeld, 2003), high-precision finishing (Kordonski and Shorey, 2007), automotive clutch (Kieburg et al., 2006), and actively-controlled damper for bridge stay cables (Duan et al., 2006), helicopter rotors (Kamath et al., 1999), automotive seats (Jolly et al., 1998), and oil & gas drill bits (Cobern et al., 2007).

The rheology and microstructure of MR fluids show many similarities to those of electrorheological (ER) fluids (Bossis et al., 1997; Shkel and Klingenberg, 2001). In both MR and ER fluids an applied field causes electromagnetic interparticle interactions and leads to chaining and formation of column-like microstructures (Halsey and Toor, 1990; Fermigier and Gast, 1992; Promislow et al., 1995; Bossis et al., 1997; Tao, 2001; Climent et al., 2004; Deshmukh, 2007). In the absence of hydrodynamic forces caused by bulk fluid flow, the fluid microstructure is mainly under the influence of magnetostatic and contact forces. The flow of the suspension is possible only when the percolated network of particle chains is broken.
Figure 1.1: Images of MR fluids in the (a) macro- and (b) micro-scale. The images on the right are captured in a pressure-driven flow of an MR fluid through a microfluidic channel, as described in detail in Chapter 5. In the presence of an applied magnetic field, the particles in the fluid form long column-like structures and the fluid becomes solid-like. With the removal of the field, much of this change in the structure and rheology is reversed.

Observed from the macroscopic point-of-view, the resistance of the particle network against mechanical stress can be described in terms of a field-dependent yield stress, $\tau_y(H)$, where $H$ is the local magnitude of the magnetic field strength. If the stress in the material is below this critical value, the material response is predominantly elastic (Weiss et al., 1994; Li et al., 1999). When this stress is exceeded, the percolated structures are disrupted and the sample flows irreversibly as a liquid.
The relation between the extra stress \((\tau - \tau_y(H))\) and the shear rate, \(\dot{\gamma}\), can be described in the framework of a generalized Newtonian fluid model (where the fluid viscosity is a function of shear rate (Bird et al., 1987, Chapter 4)). The model most commonly used in MR fluid literature is the Bingham model (Klingenberg, 2001),

\[
\tau = \tau_y(H) + \eta_p \dot{\gamma},
\]

where \(\eta_p\) is the plastic viscosity, which is often found to be independent of the applied field (Klingenberg et al., 1991a). Studies probing wider ranges of shear rate have found better fits to experimental data with both the Herschel-Bulkley model (Marksmeier et al., 1998):

\[
\tau = \tau_y + k \dot{\gamma}^n,
\]

and the Casson model (Gabriel and Laun, 2009),

\[
\sqrt{\tau} = \sqrt{\tau_y} + \sqrt{\eta_c \dot{\gamma}}. \tag{1.3}
\]

Here, \(n\) is the power index, \(k\) is the consistence parameter, and \(\eta_c\) is the Casson viscosity.

To understand the mechanisms that lead to the field-dependent yield stress, and to be able to describe this dependence in a quantitative fashion, we review the equations that govern the electromagnetic fields and their coupling to the mechanical domain through body forces. In Section 1.1, we review electromechanical phenomena from a continuum perspective. The governing equations identified in this review are applied to the micromechanics of particles in Section 1.2 to determine the dimensionless parameters important in flows of MR fluids. Finally, in Section 1.3, the scope of this thesis is outlined.
1.1 The electromagnetic domain – fields and forces

The quantitative descriptions of electromagnetism in this thesis will use the classical interpretation as governed by the phenomenological Maxwell’s Equations in their magnetoquasistatic limit (Melcher, 1981, Sect. 2.3). The governing equations are Ampere’s Law \( \nabla \times \mathbf{H} = J_f \) and magnetic Gauss’s Law \( \nabla \cdot \mathbf{B} = 0 \) where \( \mathbf{H} \) is the magnetic field strength and \( \mathbf{B} \) is the magnetic flux density. The free current density, \( J_f \), is generally not present within MR fluids and therefore will not be considered for the rest of this discussion. The bound current is captured in terms of the magnetization, \( \mathbf{M} \), which is used to relate the two magnetic field quantities as \( \mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}) \). The constitutive relation of the media can be described by \( \mathbf{B} = \mu \mathbf{H} \), where \( \mu \) is the magnetic permeability, or \( \mathbf{M} = \chi_m \mathbf{H} \), where \( \chi_m \) is the magnetic susceptibility of the material.

Since we are interested in the forces on magnetizable media, it is instructive to review how they are related to the electromagnetic fields. The fundamental electromagnetic force is described by the empirically obtained Lorentz force, \( f_L = q (\mathbf{E} + \mu_0 \mathbf{v} \times \mathbf{H}) \), where \( q \) is the charge and \( \mathbf{v} \) is the velocity of a test particle (Melcher, 1981). This force can be related to the force on a magnetic dipole, \( \mathbf{m} \) by the relation (see (Zahn, 1987, Sect. 5-8-1) for a detailed derivation)

\[
\mathbf{f}_{mag} = \mu_0 \mathbf{m} \cdot \nabla \mathbf{H}.
\]  

If one makes the approximation of non-interacting dipoles, a magnetic force density can be found by using the relation \( \mathbf{M} = n \mathbf{m} \) where \( n \) is the number density of dipoles (Melcher, 1981, Sect. 3.6):
which is the Kelvin magnetization force density. This definition is used extensively in the ferrofluid literature (see for example, (Rosensweig, 1985)).

An alternate approach for deriving a macroscopic force density is to use thermodynamic techniques. The detailed derivations using this technique are provided by Stratton (1941), Landau and Lifshitz (1960) and Melcher (1981). In a general form for magnetically linear, compressible media,

$$ F^M_{K-H} = -\frac{1}{2} H^2 \nabla \mu(\rho) + \nabla \left( \frac{1}{2} \rho \frac{d\mu}{d\rho} H^2 \right), $$

(1.6)

which is often referred to as the magnetic Korteweg-Helmholtz force density. $\rho$ is the mass density of the media. This form captures both the magnetization component (first term) and the magnetostriction component (second term) of the force. The latter component is caused by the compressibility of the media.

For the materials considered in MR fluids (both the base fluid and the particles) the incompressibility approximation is often well-founded; however, if we consider the suspension from a continuum perspective, the magnetostriction term can be important. This is caused by the variable particle volume fraction, $\phi$, causing localized concentrations of magnetic dipoles, much like density in a compressible material. The force density of Zahn and Rhee (1984) can be used in this case to find

$$ F^M_{MR} = -\frac{1}{2} H^2 \nabla \mu(\phi) + \nabla \left( \frac{1}{2} \phi \frac{d\mu}{d\phi} H^2 \right). $$

(1.7)
In the solution of ferrohydrodynamic flow problems, it is often convenient to describe the magnetic body force as the divergence of the Maxwell stress tensor \( \mathbf{F}^M = \nabla \cdot \mathbf{\tau}^M \). The Maxwell stress tensor for Equation (1.7) is:

\[
\mathbf{\tau}^M_{MR} = \mu(\phi) \mathbf{H} \mathbf{H} - \frac{1}{2} \delta H^2 \left( \mu(\phi) - \phi \frac{\partial \mu}{\partial \phi} \right).
\]  

(1.8)

where \( \delta \) is the unit tensor.

The magnetic force on a body can be determined by calculating the integral, \( \oint \mathbf{n} \cdot \mathbf{\tau}^M dS \) on a closed surface enclosing the body. We should also note that the integrals taken on open surfaces do not lead to physically significant quantities (Melcher, 1981; Rinaldi and Brenner, 2002).

It is instructive to review the governing equation for ER fluids, in order to understand some of its analogies to MR fluids. In an ER fluid that is absent of free charge the Maxwell’s equations in the electroquasistatic limit are \( \nabla \times \mathbf{E} = 0 \) and \( \nabla \cdot \mathbf{D} = 0 \), where \( \mathbf{E} \) is the electric field and \( \mathbf{D} \) is the displacement field. The magnetoquasistatic equations in the absence of free current density, \( \mathbf{J}_f \), reduce to \( \nabla \times \mathbf{H} = 0 \) and \( \nabla \cdot \mathbf{B} = 0 \). These governing equations are the same if we replace \( \mathbf{B} \) with \( \mathbf{D} \), and \( \mathbf{H} \) with \( \mathbf{E} \).
Figure 1.2: Constitutive models of volume fraction dependent relative magnetic permeability \((\mu_r = \mu / \mu_0)\). These models for the equivalent electrostatic problem were derived by Zahn and Rhee (1984). In all the models, the ratio of the dimensions of the particle phase (dark red) and that of the continuous phase (modeled as free space and shown in white) is adjusted such that the model satisfies the volume fraction of the suspension, \(\phi\). Only the ratio of these dimensions affects the model results, their absolute scale has no effect.

The analogy between MR and ER fluids also exists in how the fields relate to the electromagnetic body force. The electroquasistatic Korteweg-Helmholtz force density is
\[ \tau_{k-H}^E = \varepsilon(\rho) \mathbf{E} \mathbf{E} \mathbf{E} - \frac{1}{2} \delta E^2 \left( \varepsilon(\rho) - \rho \frac{\partial \varepsilon}{\partial \rho} \right), \]  

(1.9)

which can be obtained by making the same substitutions to Equation (1.6). We should also note that we have limited our attention to ER fluids without the presence of free charge, for which this analogy strictly applies.

![Diagram](image)

Figure 1.3: The ratio of magnetic susceptibility of the suspension to that of the particle phase as a function of the particle volume fraction, \( \phi \). This quantity needs to be between 0 and 1 for all constitutive models. Furthermore, the parallel and the series models serve as upper and lower bounds to physically meaningful constitutive relations, respectively. For the parallel model, the susceptibility ratio does not depend on the particle permeability, and therefore the two line plots overlap.

The constitutive models for volume fraction dependence of electrical permittivity for electrofluidized beds were reviewed by Zahn and Rhee (1984). These models are converted to the analogous problem in magnetic suspensions of particles and presented in Figure 1.2. It is
illustrative to compare these models by plotting the ratio of magnetic susceptibilities of the suspension and the particle phase (Figure 1.3). This quantity needs to be between 0 and 1 for all constitutive models.

1.2 Microscopic forces in MR suspensions and dimensionless parameters

For design purposes, it is helpful to collapse rheological measurements over a range of field strength, temperature, shear rate, and volume fraction into dimensionless master curves. Suitable non-dimensional parameters specific to the flow of MR fluids can be determined by evaluating the different microscopic forces acting on the particles.

![Diagram](image)

Figure 1.4: (a) The interparticle magnetic force can be determined by making the uniform magnetization approximation. Within this approximation the fields and the magnetic force can be calculated in the equivalent problem of two dipoles located as shown in (b). The uniform magnetization within a spherical particle occurs in two limiting cases: 1) in a magnetically saturated particle or 2) in a particle isolated in a uniform magnetic field as illustrated in (c)
To determine a relation that accurately describes the scale of the interparticle magnetic force, the many-body problem existing in a real fluid is simplified to one that involves two spherical particles. Even with this simplification the exact solution to the magnetic field boundary value problem does not lend itself to a closed form solution that can be used in a dimensionless parameter (Dassios et al., 2006). For this reason, the problem is often reduced to one of two limits in which magnetization within the particle is uniform (Klingenberg et al., 2007). The first limit is for a particle which is magnetically saturated. This approximation becomes more accurate in high magnetic fields. The second limit, which gives uniform magnetization is for particles that only experience the externally applied (uniform) field, as illustrated in Figure 1.4c. In this case the solution to the magnetic field problem of a sphere of susceptibility, $\chi_m$, in a uniform field in free space is (Zahn, 1987, Sect. 5-7-2)

\[
H = \begin{cases} 
\frac{3}{\chi_m + 3} H_0 \hat{z} & r \leq a \\
H_0 \hat{z} + \left[ \frac{\chi_m}{\chi_m + 3} \left( \frac{a}{r} \right)^3 H_0 \left( 2 \cos \theta \hat{r} + \sin \theta \hat{\theta} \right) \right] & r > a
\end{cases}
\]  

(1.10)

We can see that the magnetization of the particle, $M_p = \chi_m H|_{r \leq a}$, is uniform. The field outside the particle is equivalent to the combination of the uniform external field and the field around a dipole (the term in the brackets) of strength $m = VM_p$, where $V$ is the volume of the particle (Zahn, 1987, Sect. 5-5-1):

\[
H|_{r > a} = H_0 \hat{z} + \frac{M_p}{3} \left( \frac{a}{r} \right)^3 \left( 2 \cos \theta \hat{r} + \sin \theta \hat{\theta} \right).
\]  

(1.11)
This result demonstrates why it is convenient to make the uniform magnetization approximation, since the problem can be reduced from one of continuous bodies to discrete dipoles.

It is also possible to determine the force on the particle by calculating the force on the equivalent dipole. We can see this by considering an integration of the Maxwell stress tensor on a closed surface containing only one of the particles. Since the field generated by the particle is the same as the field generated by the dipole, the integrals and therefore the forces are equivalent. The force on a dipole of strength, $m = VM_p$, in the field of a dipole of same strength located $2a$ apart (as illustrated in Figure 1.4a) can be calculated using Equation (1.4):

$$f_{mag} = \mu_0 m \cdot \nabla |H| = \frac{\pi}{6} \mu_0 a^2 M_p^2. \quad (1.12)$$

We should also note that the average particle magnetization $\langle M_p \rangle$ can be related to the suspension magnetization $M_{fluid}$ by the relation:

$$\langle M_p \rangle = \frac{M_{fluid}}{\phi}. \quad (1.13)$$

This relation is very useful in the experimental studies of MR fluid because the fluid magnetization is a macroscopically measurable quantity.

The relative importance of the MR effect in a shear flow can be estimated by comparing the magnetic force to the viscous force exerted on the particle. The Reynolds number at the particle length scale is normally low; therefore, a good estimate of the viscous force acting on a particle can be obtained by Stokes drag on a sphere, $f_d = 6\pi \eta_0 a^2 \dot{\gamma}$. Here $\eta_0$ is the suspension
viscosity at the high shear rate limit and $\dot{\gamma}$ is the shear rate. The ratio of these two particle forces results in the Mason number

$$Mn = \frac{\text{viscous force}}{\text{magnetic force}} = \frac{36\eta_0 \dot{\gamma}}{\mu_0 M_p^2}.$$  

(1.14)

The suspension viscosity, $\eta_0(\phi)$, is typically used in the definition of $Mn$ to account for (in a mean field sense) the increase in viscous drag caused by neighboring particles. At high volume fractions, MR suspensions may exhibit a small yield stress even in the off-state due to jamming effects, and therefore the viscosity of the suspension is also a function of shear rate. However, the viscosity is often found to plateau at high shear rates (Stickel and Powell, 2005) and this high-shear plateau value of the viscosity is commonly utilized to estimate viscous stress.

If we assign representative numerical values to the quantities ($\dot{\gamma}_0 = 10 s^{-1}$, $M_p = 10^6 A/m$, $a = 2 \times 10^{-6} m$, $\eta_0 = 0.2 Pa.s$) typical for a flow condition of interest in a typical oilfield application for MR fluids, we find $Mn = O(10^{-4})$. Therefore, in terms of bulk rheology we can expect to find the field-dependent yield stress much more pronounced than the viscous stress resulting from shearing flow.

Comparing the magnetic force to Brownian force ($F_B \sim k_B T / a$) gives another dimensionless parameter

$$\lambda = \frac{\text{magnetic force}}{\text{Brownian force}} = \frac{\pi \mu_0 a^3 M_p^2}{6 k_B T} = O(10^9),$$  

(1.15)
where \( k_B \) is the Boltzmann constant and \( T = 450K \) is the temperature. It is clear that the magnetic force in common flow situations of MR fluids dominates over Brownian forces even at the elevated temperatures of downhole conditions.

It is further possible to estimate the order of magnitude of MR stresses by recognizing that the magnetic force acts on the area including the particle and the surrounding base fluid, which on average is \( A_p = \pi a^2 / \phi \). Therefore, the magnetic shear stress scaling is

\[
\tau_{mag} = \mu_0 \phi M_p^2 / 6. (1.16)
\]

This equation suggests a linear dependence of MR stress on volume fraction. This dependence has been reported for MR fluids using approximate analytical methods (Ginder et al., 1996) and for ER fluids using experimental (Marshall et al., 1989) and numerical methods (Klingenberg et al., 1991a) (in the volume fraction range \( 0 < \phi < 0.35 \)). However, the latter ER study has also shown that the relation is more complex above the mentioned limits on \( \phi \).

In general, the dimensionless yield stress exhibited by an MR fluid is expected to be a function of \( Mn, \lambda \), and \( \phi \). Because \( \lambda \gg 1 \) for the suspensions of interest here, the measured yield stress can be described as a function of \( Mn \) and \( \phi \) only, and may be expressed in dimensionless form

\[
\frac{\tau_y}{\tau_{mag}} = \Phi(Mn, \phi). (1.17)
\]

With the analysis of the important forces on MR suspension particles, we were able to obtain two dimensionless parameters, and a shear stress scaling. These parameters can help us
understand the rheological measurements of MR fluids, and will be discussed further in Chapter 4.

1.3 Scope of thesis

With the review of literature in MR fluids we can see that the controllable properties of MR fluids are used in a number of engineering applications. The desirable attributes of MR fluid devices such as their low power consumption, simple electromechanical design, and “continuous variability” of fluid resistance provide a strong motivation for their use in applications for oil & gas exploration and production (Cobern et al., 2007; Bhavsar et al., 2008). These applications also bring about a number of operational requirements for the MR fluid such as generating a large field-dependent yield stress with tolerance to elevated temperatures (e.g. 150°C) and low fluid density to limit the hydrostatic downhole pressures. Similar motivations for research can be identified for other applications in the aerospace and robotics industries.

In this thesis we investigate a number of fluid design constraints that are important in the successful commercial developments of MR fluid technology for the oil & gas applications. In Chapter 2, we present the design, manufacturing, and the functional experimentation on two instruments to measure different aspects of the field-dependent rheological properties of MR fluids. The first device is a magnetorheology accessory to a commercial torsional shear rheometer which operates in a regime of high-field (1T) and high-temperature (150°C). The main purpose of the second device, an annular-slit flow magnetorheometer, is to investigate the rheological properties of MR fluids in spatially inhomogeneous magnetic and deformation fields.
In Chapter 3, we evaluate the evolution of the rheological properties of MR fluids with temperature. The bulk rheology of the MR fluid is characterized as a function of magnetic field and temperature. The results are analyzed and compared to the effects of physical properties of the fluid that change with temperature.

The effects of the magnetic field on the MR yield stress are evaluated in Chapter 4. A computational model of particle interaction is developed to quantify the evolution of microscopic magnetic forces with applied field. Motivated by the results of this model, we evaluate the existing scaling laws of MR yield stress with applied field and develop a new power law applicable in a wide range of magnetic fields.

The aggregation phenomena and the evolution of fluid microstructure are investigated in Chapter 5, in MR fluid flows with strong particle-wall interactions. A new microfluidic device construction, closely replicating the magnetic flow conditions in actual devices, is developed. The flow and the migration phenomena in MR fluid particles are investigated in both homogeneous and inhomogeneous magnetic fields.

In Chapter 6, the transport of MR fluids in spatially inhomogeneous magnetic and deformation fields is studied. The annular slit-flow magnetorheometer measurements are used to investigate the parameters governing these flows. A continuum-level model that captures the key features of multi-phase flow and fluid anisotropy is developed. The model is numerically implemented using the finite element method and used to study the transport of MR fluids through contraction and expansion channels.

Finally, in Chapter 7, the conclusions of this thesis are presented and opportunities for future research in the field are identified.
Chapter 2

Magnetorheometry

All the investigations conducted in this thesis into the evolution of rheological properties with temperature (Chapter 3) and magnetic field (Chapter 4), as well as the study of magnetorheology under inhomogeneous fields (Chapter 6), include components of experimental measurements of bulk rheology of MR fluids. In this chapter, two devices developed to explore different aspects of MR fluid rheology are described and their design and operation are outlined. In Section 2.1 a literature survey of custom-built magnetorheology instruments with intermediate and high-field capabilities is provided. In Section 2.2, the design and functional experimentation of the magnetorheometry accessory to a commercial torsional shear rheometer are presented. Finally, in Section 2.3, the design of the annular slit-flow magnetorheometer is described.

2.1 A survey of devices for macroscopic measurements of magnetorheology

Bulk rheological characterization of MR fluids is conducted by the simultaneous application of magnetic fields and shearing strain fields to the fluid samples in the rheometer. The magnitude of the MR response observed depends on the relative orientations of these two vector fields (Kuzhir et al., 2003a; Kuzhir et al., 2003b; Kuzhir et al., 2009). The vast majority of experimental instrumentation generates magnetic field that are aligned in the perpendicular direction to both...
the flow streamlines and the vorticity (de Vicente et al., 2011), presumably because this field orientation occurs in most MR fluid devices (e.g., dampers, clutches).

Table 2-1: A survey of custom-designed high-flux magnetorheology accessories, associated rheometers, and the maximum accessible magnetic flux density, $B_{max}$. CTS: commercial torsional shear rheometer.

<table>
<thead>
<tr>
<th>Reference</th>
<th>Geometry and rheometer</th>
<th>$B_{max}$ ($T$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Laun et al. (1996)</td>
<td>Double concentric cylinder, CTS</td>
<td>0.4</td>
</tr>
<tr>
<td></td>
<td>Custom designed capillary rheometer</td>
<td>0.4</td>
</tr>
<tr>
<td>Rankin et al. (1999)</td>
<td>Parallel plate, CTS</td>
<td>0.4</td>
</tr>
<tr>
<td>Park et al. (2001)</td>
<td>Parallel plate, CTS</td>
<td>0.33</td>
</tr>
<tr>
<td>Horvath et al. (2002)</td>
<td>Parallel plate, CTS</td>
<td>2.0</td>
</tr>
<tr>
<td>Genc and Phule (2002)</td>
<td>Double concentric cylinder, custom designed</td>
<td>0.78</td>
</tr>
<tr>
<td>Deshmukh and McKinley (2004)</td>
<td>Parallel plate, CTS</td>
<td>0.64</td>
</tr>
<tr>
<td>Laeuguer et al. (2005)</td>
<td>Anton Paar® MRD-1T device, CTS</td>
<td>1.0</td>
</tr>
<tr>
<td>Ulicny et al. (2005)</td>
<td>Concentric cylinder, custom design</td>
<td>~1T</td>
</tr>
<tr>
<td>Wang and Gordaninejad (2006)</td>
<td>Custom designed capillary</td>
<td>0.35</td>
</tr>
<tr>
<td>Gabriel and Laun (2009)</td>
<td>Custom designed capillary</td>
<td>0.3</td>
</tr>
<tr>
<td>Laun et al. (2010)</td>
<td>Twin gap parallel plate, CTS</td>
<td>1.5</td>
</tr>
</tbody>
</table>

In Table 2-1, we survey published reports of rheometers and rheometer accessories that have been built to investigate MR fluid rheology in the intermediate and high applied field ranges. Torsional shearing rheometers have been utilized most often in these studies because of their ease of loading the MR fluid sample and the ability to construct a custom magnetorheology accessory that can be fitted to a commercial instrument utilizing instrument grade sensors and control systems. The development of capillary magnetorheometers has mainly been motivated by the need for rheological investigations at high shear rates. Because of the high rates of viscous
dissipation that occur in such devices (at high stresses and high deformation rates), exploring this regime using a small sample of fluid poses challenges in heat removal. The continuous convective removal of heat from a capillary rheometer has been shown to be an effective solution to this problem. It should also be noted that, none of the devices with high flux capability \( (B_{\text{max}} \sim O(1T)) \) have been used to investigate magnetorheology under high temperatures \( (T_{\text{max}} \geq 100^\circ C) \).

### 2.2 Magnetorheometry accessory to torsional shear rheometer

The magnetorheometry fixture is designed to work with the TA Instruments® AR-G2™ stress-controlled rheometer and is centralized to the rheometer spindle axis by means of complementary mating mechanical features that align with the rheometer base casting. The sample space within this fixture is compatible with a 20\( mm \) parallel plate geometry with a maximum sample thickness, \( h = 0.5\, mm \). General construction features of the new MR fixture are illustrated with a schematic diagram and a photograph in Figure 2.1.

#### 2.2.1 Mechanical design and material selection

The fixture generates a magnetic field in the sample by the use of a toroidal electromagnetic coil. The magnetic field is channeled primarily through a series of magnetically-permeable components. Often referred to as a magnetic circuit, the series of magnetic components are used to minimize the required electrical power input and to focus the field as uniformly as possible through the MR fluid sample. The MR fluid sample is located in a narrow gap between two permeable components near the center of the fixture. Magnetically permeable components are manufactured out of 1018 carbon steel for its high relative permeability \( (\mu_r) \) and low coercivity

45
(He) properties. These magnetic properties are highly dependent on processing, heat treating, and machining and were not measured for the specific material used. For reference, they have been reported for 1010 carbon steel, a similar grade of material, to be $\mu_{r,1010CS} = 3800$ and $H_{c,1010CS} = 80-160A/m$ (Stedfeld and Zorc, 1995). Minimizing coercivity is important to minimize the remnant field that remains within the fixture upon removal of the magnetic field.

Figure 2.1:(a) Schematic diagram and (b) photograph of magnetorheometry fixture. Dimensions are given in millimeters.

Components that are not a part of the magnetic circuit often need to have a low magnetic permeability so that the leakage field is minimized. Contrary to conduction or heat transfer problems where highly insulating materials can be selected to isolate the flux of electric current and heat respectively, the range of available material properties does not allow the designer to vary magnetic permeability within power magnetic circuits much beyond a factor of $\sim 10^2$ at high fields. Although diamagnetic materials are less permeable than free space (e.g. relative
permeability of bismuth, $\mu_{r,Bi} = 0.9999834$ (Serway, 1990), this effect is often negligible when compared to ferromagnetism (e.g., steel $\mu_{r,\text{steel}} = 1000$). The same can be said about common paramagnetic materials such as aluminum ($\mu_{r,\text{Al}} = 1.000023$ (Serway, 1990)). Therefore, in analysis and design of static magnetic field generators and actuators non-ferromagnetic materials can be freely used near a magnetic circuit with little regard to their specific magnetic characteristics. Consequently, we selected aluminum for the non-magnetic spacer, the bobbin for the coil winding, and the channel plate that houses the upper spiral flow channels. The upper plate geometry was manufactured of a titanium alloy. Using a magnetic material in the upper plate geometry is advantageous in better focusing of the field and reducing particle slip (Ocalan and McKinley, 2010; Laun et al., 2011). The latter benefit is due to the attractive magnetic forces between the plates and particles. However, a magnetic upper plate placed within the applied field will experience a magnetic body force in both the axial and transverse directions. Design estimates suggest that at large fields ($B \sim 1T$) these forces are significantly larger than the normal load capabilities of the thrust bearings of commercial rheometers. Because of these concerns, a non-magnetic material is selected for the upper plate.

Electrical conductivity is also an important property that affects material selection in electromagnetic design. However, the operation of the magnetorheology fixture is often in a regime where the effects of conductivity are negligible. Under transient magnetic fields, electric field is generated within a material element according to Faraday’s Law, which in turn induces currents depending on the electrical conductivity of the material. If large enough, these induced currents may affect the magnetic field and the magnetorheological measurements. This can be quantified by the magnetic diffusion time constant, $\tau_m = \mu_0 \sigma d^2/\pi^2$ (Woodson and Melcher,
where $\sigma$ is the electrical conductivity and $d$ is the length scale of interest. In the magnetorheology fixture this time scale is in the order of 10ms which is much smaller than the stabilization periods allowed for the rheological measurements in this study.

It is also possible to generate a time-varying magnetic field on a material element by advecting it through a spatially-varying magnetic field. Although the titanium upper plate of the fixture is rotating within a magnetic field, in a perfectly aligned system, there would be no magnetic field transients caused by this rotation because of the axisymmetry of the field. Therefore, the conductivity of the upper plate is not an important quantity to control in the design of the instrument. We should also note that the time scale analysis of the magnetic diffusion and the conclusions on the effects of time-varying magnetic fields are in agreement with the experimental findings of Laun and Gabriel (2007). Furthermore, all experimental work presented herein, have been conducted in conjunction with measurements of the steady magnetic field to ensure that transient fields have no measurable effect on the results.

In a solenoidal electromagnetic coil, such as the one used in the present magnetorheology fixture, the best alignment of the field lines is commonly achieved near the axial midpoint of the coil. Therefore, it is desirable to design the sample location near this point. However, the presence of a coil around the sample geometry would cause difficulties in loading and trimming the sample. For this reason, the physical location of the windings is designed to be completely below the sample plane whereas the magnetic core is continued above the sample through the customs designs of the top covers which are machined from carbon steel to help close the magnetic circuit.
The large currents present in the windings during application of the magnetic field mean that there is considerable resistive heating (\(\sim 200W\)) that takes place within the fixture. If it is not dissipated effectively, this heating may raise the temperature of the fluid sample and possibly cause the components of the fixture to overheat. For these reasons, we have included fluid circulation channels within the fixture. The flow path consists of two spirally cut channels (Figure 2.2) on each side of the electromagnetic coil that are linked with an annular flow section past the winding. On each end of the channels a quick-connect pressure fitting is provided. All component interfaces have been sealed with o-rings. During operation at elevated temperatures the circulation fluid temperature can reach 150°C. Furthermore, due to resistive heating and thermal diffusion, other locations in the fixture may reach locally higher temperatures. To enable safe and stable experiments at this operating point, fluorocarbon o-ring seals, polyimide magnet wire and fluorinated ethylene propylene lead wires were used. Details of the thermal design are presented in a Section 2.3.

Figure 2.2: Model of the central plate. A spiral flow channel is machined onto the plate. A similar flow channel is machined on the lower magnetic plate. These two channels are linked with the annular flow path between the magnetic core and the bobbin.
To accurately measure the rheological properties of the MR fluid, wall slip (Barnes, 1995; Bertola et al., 2003) in the rheometer geometry needs to be avoided. This is especially important for the magnetorheology fixture as the strong particle-particle magnetic interactions, which lead to the bulk MR response, are not present between the non-magnetic plates and the first layer of adjacent particles. In the experiments we use roughened fixture surfaces to enhance interaction between the walls of the geometry and the MR fluid particles. The non-magnetic spacer forming the lower plate was roughened by a sandblasting operation to a surface roughness of 3.83 microns (rms) and the upper plate was used with 600-grit adhesive-backed sandpaper (average particle size 20μm) adhered to its surface. This method eliminated slip in static yield stress measurements, and was verified by independent creep tests.

As a result of the discontinuity of magnetic permeability at the perimeter of the sample, an outward radial magnetic body force on the MR fluid is generated. This radial body force is resisted only by the surface tension of the fluid sample. When the magnetic field is large enough the magnetized material can be ejected out of the geometry. Similar effects have been observed previously (Deshmukh, 2007; Laun et al., 2008a). In our experiments sample integrity was ensured by using an elastomeric ring with a rectangular cross-section stretched around the top geometry, as illustrated in Figure 2.1.

2.2.2 Magnetic design

The magnetorheology fixture can generate a magnetic flux density through the sample of up to $B = 1T$. The main goal in the magnetic design of the instrument was to generate this level of field within the size restrictions placed by the geometry of the rheometer with minimal amount of variation in flux density. Design analysis was mainly conducted using finite element
methods. In preliminary sizing of components, approximate analytical relations, which are obtained through application of integral forms of Ampere’s Law and Gauss’ Law, were used. These relations can be found in many electromechanical references and literature (Avallone and Baumeister III, 1996; Laun et al., 1996). Because of the complex geometry of the magnetorheology fixture and the non-linear response of the material (which are not incorporated in the approximate analytic relations) employing numerical simulation is essential to accurately predict the actual magnetic field that is realized in the new MR fixture.

Since the magnetic flux density is divergenceless \((\nabla \cdot \mathbf{B} = 0)\) a magnetic vector potential can be defined such that, \(\mathbf{B} = \nabla \times \mathbf{A}\) and \(\nabla \cdot \mathbf{A} = 0\). The second equation, which is called the Coulomb gage, is required since defining only the curl does not uniquely identify the vector field \(\mathbf{A}\). Substituting these equations into differential form of Ampere’s Law results in a vector Poisson’s equation in three space coordinates

\[
\nabla^2 \mathbf{A} = -\mu \mathbf{J}.
\]

By invoking the axisymmetry condition, the problem is reduced to a one dimension

\[
\left[\nabla^2 \mathbf{A}\right]_\theta = -\mu J_\theta
\]

(2.2)

for the magnetorheometry fixture where,

\[
\mathbf{B} = -\frac{\partial A_\theta}{\partial z} \mathbf{i}_r + \frac{1}{r} \frac{\partial}{\partial r} (r A_\theta) \mathbf{i}_z
\]

(2.3)

The general formulation for analytical boundary value problems is described in many electromagnetic texts (e.g., Zahn (1987)). The specifics of numerical formulation and solution methods are discussed in more dedicated texts (see for example Ida and Bastos (1997)).
The numerical analysis was conducted with Maxwell® 2D developed by Ansoft Corporation. The magnetorheometry fixture was modeled using an axisymmetric model formulation. Design iterations were made to size components and to minimize leakage and saturation of the magnetic circuit components. After determining the feasibility of generating 1T flux density through the sample within the space constraints posed by the rheometer, the variation of magnetic field in this region was minimized using heuristic iterative methods. Because of the specific geometrical features in the vicinity of the sample axis and the perimeter, the magnetic field uniformity deteriorates near these two regions.

The source for field non-uniformity near the sample axis is the rheometer spindle access hole in the portion of the magnetic circuit that is above the sample. This undesirable variation in magnetic field causes inaccuracies in results and is difficult to correct for because of the non-linear response of MR fluids to applied field. The variation can be minimized by limiting the diameter of the shaft and the size of the access hole, however this comes at the cost of introducing additional torsional compliance to the rotating spindle. To minimize this added torsional compliance, a hole geometry with varying diameter was designed, as shown in Figure 2.1. To avoid interference, a small notch on the rheometer spindle was designed right above the plate with edges rounded for minimizing stress concentration. The access hole diameter is 7.3 mm. The improvement to the magnetic field uniformity is presented in contour and line plots in Figure 2.3.
Figure 2.3: Design of the upper plate geometry and the magnetic covers. Reducing the access hole diameter for the rheometer spindle near the sample increases field uniformity. a) Schematic diagram of the gap region. b) Contour plot of magnetic flux density and flux lines in the air gap for notched and uniform diameter spindle. The coil current in the simulation is 5A. The line plots are drawn at the axial location of $z = 250\mu m$ (denoted by line AA) above bottom magnetic core (mid-span of typical sample height, $h = 500\mu m$).

The compliance of the rheometer spindle may have no impact on the measurement (e.g., after steady-state is achieved in steady shear experiments), but any impact that does occur can be easily corrected for by a linear constitutive relation. In commercial rheometers, including the AR-G2™, the compliance in the spindle is corrected for by built-in calibrations and software
functions. One limitation to this correction is in the case of torsional compliance that is so high that additional spindle dynamics are introduced in the frequency range of oscillatory experiments. This behavior can be estimated by a lumped parameter simple oscillator model in which the stiffness element is equal to the reciprocal of the torsional compliance and the mass element equal to the mass moment of inertia of the test geometry below the notch. In this analysis, additional viscous damping provided by the sample can be ignored to achieve a conservative estimate, as any additional damping would be favorable to the analysis. The undamped resonant frequency is estimated to be 6.4 kHz, which is well above the frequency range of interest. An experimental validation of this analysis was conducted in which a sinusoidal torque was applied to the geometry with constant amplitude and frequency sweep up to 100Hz (maximum operating frequency for the AR-G2™ rheometer). No effects of added dynamics were observed.

It can be seen from the plots of \( B(r, z) \) in Figure 2.3 that, there is a significant variation of the magnetic field near the perimeter of the magnetic core; this is caused by the discontinuity of magnetic permeability in this region. Using a sample geometry that is smaller in diameter than the magnetic core is not a valid approach because the magnetic permeability of the sample itself leads to a significant variation near its own perimeter. Contour and line plots of magnetic flux density of a 12mm diameter sample of Lord® MRF-132DG fluid in the magnetorheology fixture are presented in Figure 2.4a to illustrate this effect. The fluid properties in this model includes the full non-linear magnetization curve of the MR fluid (Lord, 2008). It can be seen from Figure 2.4 that, there is a large increase in the magnetic field near the sample perimeter. The flux lines near the perimeter indicate that the cause of the non-uniformity is the discontinuity of
permeability, focusing more magnetic field through the sample. Matching the sample diameter with the magnetic core amplifies this non-uniformity even more, as illustrated in Figure 2.4b.

A method of reducing the magnetic field non-uniformity near the sample perimeter is to place the sample axially near the center of the air gap by the use of a non-magnetic spacer, as illustrated in Figure 2.4c. In this case, the magnetic field is still focused around the perimeter; however, spatially this also corresponds to an area where the magnetic field is de-focusing and spreading radially outward. These two effects oppose each other and the non-uniformity near the sample perimeter is reduced. A similar conclusion had been previously reached in the study by Laun and coworkers (2008b); by analyzing the field variations in a commercial magnetorheology accessory these researchers recommended a straightforward modification to the design features of the magnetic circuit.

Spatially-resolved measurements of the magnetic field were taken to evaluate the accuracy of finite element models. An F.W. Bell® 5180 gaussmeter (with a circular sensing region of diameter 0.381mm) was used in a groove designed within the non-magnetic spacer. The magnetic flux density obtained by finite element models and the measurements show good agreement (Figure 2.5). In contrast with the previous plots shown in Figure 2.3 & Figure 2.4, where magnetic flux density within the MR fluid sample is presented, the curves shown in Figure 2.5 are for the magnetic flux density \( B(r) \) at the sensor location. Another important feature to note in the radial variation of magnetic field is that, near the mid-span radius of the sample geometry, the field is nearly uniform in radial position and maintains a constant plateau value in the neighborhood of this radius.
Figure 2.4: Magnetic flux density and flux line contours for a) 12mm diameter sample, b) 20mm diameter sample placed directly on magnetic core and c) 20mm diameter sample on 1.6mm thick non-magnetic spacer; d) Line plots of magnetic flux density along radius at the axial mid-plane of $h=500\mu m$. 
Figure 2.5: Finite element results compared with experimental measurements for sample gap filled with a) air and b) MR fluid (Lord MRF-132DG). Finite element results plotted along the radius of the MR fixture on the plane of the Hall Effect sensor ($z = 572\mu m$) to provide the most representative comparison.

The Hall effect probe of the gaussmeter cannot be operated at the elevated temperatures ($T \sim 150^\circ C$) targeted in this study. To provide a measurement of magnetic field at temperatures above $70^\circ C$, a custom secondary coil, pictured in Figure 2.6, was manufactured. The total
number of windings was \( N = 100 \), with 50 turns in each direction. This type of winding configuration provides maximum sensitivity to the magnetic field at the midspan radius, \( r_m \); but no sensitivity to magnetic field in radial positions below minimum radius, \( r_{\min} = 6.4\text{mm} \), or above maximum radius, \( r_{\max} = 15.9\text{mm} \). Therefore, the high-sensitivity region of the secondary coil corresponds with the magnetic flux plateau in the sample chamber. Time integration of Faraday's Law gives the relation between the voltage across the coil and a series of spatial integrals of magnetic field:

\[
\int V dt = \int_{CW} B_z dS - \int_{CCW} B_z dS = K_w B_z
\]  

(2.4)

Distinction between the clockwise (CW) and counterclockwise (CCW) directions (as viewed in \(+z\) direction) is made since their contribution to the measured voltage is in opposite polarity. With the knowledge of the spatial distribution of magnetic field and windings, or by making simplifying approximations the coefficient of proportionality, \( K_w \), between the axial magnetic field and the time integral of voltage may be found. In the magnetorheometer setup the secondary coil was connected to an integrating fluxmeter (Lakeshore® Model 480\textsuperscript{TM}). The measurement output of the fluxmeter, i.e. the time integral of voltage, was calibrated with the Hall effect probe measurements to experimentally determine \( K_w = 2.16 \times 10^{-3} \text{VsT}^{-1} \).

### 2.2.3 Thermal design

The magnetorheometry fixture is designed for sample temperatures up to 150°\(\text{C} \). At this temperature the electromagnetic coil can dissipate up to 200\(\text{W} \) of resistive heat. The two main goals in the thermal design of the instrument are: 1) to control sample temperature and 2) to
remove heat from the electromagnetic coil efficiently such that the change in sample temperature due to resistive heating is minimal. This was accomplished by fluid circulation within the fixture. The fluid circulation path, illustrated in Figure 2.2, is made up of two spiral grooves on the plates mating with each of the coil bobbin walls, which are joined with an annular flow channel between the magnetic core and the bobbin.

![Photograph of secondary coil](image)

Figure 2.6: Photograph of secondary coil. This coil was used in elevated temperature experiments to measure field strength \( r_{\text{min}} = 6.4\text{mm}, r_{\text{max}} = 15.9\text{mm} \).

The majority of the resistive heat generated in the electromagnetic windings is carried away from the fixture by conduction across the coil windings and the bobbin and then, in turn, by convection to the circulation fluid. For use in the numerical calculations of this heat transfer problem, the effective thermal conductivity of windings was estimated using a correlation (Agarwal et al., 2006) employed for long-fiber unidirectional composite laminates of high-conductivity fibers embedded in low-conductivity matrix:

\[
k_{\text{eff}} = k_w \frac{1 + \phi_c}{1 - \phi_c}
\]  

(2.5)
where, $k_{ins}$ is the thermal conductivity of the electrical insulation, and $\phi_c$ is the conductor volume fraction in the electromagnetic coil. This correlation is only valid in the directions orthogonal to the fibers (i.e. in the radial and axial directions in our geometry); however, as a result of the axisymmetry, these are the only relevant directions in the heat transfer problem. Convective heat transfer coefficients in each flow channel were calculated by utilizing well-known relations for pressure-driven flow, provided in many heat transfer texts such as that of Mills (1999).

To determine the rate-limiting heat transfer process, the Biot number, $Bt = h_c (H/2)/k_{eff}$, can be used. Here, $h_c = 3000 \text{Wm}^{-2}\text{K}^{-1}$ is the convective heat transfer coefficient, which is found as a function of the Reynolds number and Prandtl number in the circulating fluid, and $H$ is the height of the coil. The convective heat transfer calculations are presented in detail in Appendix A. This dimensionless parameter is found to be approximately 50 for this device, which indicates that conduction is the slower heat transfer mechanism. To estimate the maximum coil temperature resulting from this process a numerical model of heat transfer was generated using Quickfield™ developed by Tera Analysis Ltd. This problem is governed by the Poisson's equation $\nabla^2 T = -q''$, where $q''$ is the volumetric heat generation. The top, bottom and inner surface boundary conditions correspond to convective boundaries and the outside surface is a no flux condition. Due to the thermal insulation this last condition is a conservative approximation to the natural convection present on this vertical cylindrical surface ($h_c \approx 5 \text{Wm}^{-2}\text{K}^{-1}$).

For comparison with numerical results, an approximate analytical solution was also calculated. Neglecting convective heat transfer in the annular flow channel and the conductive resistance posed by the bobbin, the temperature of the coil is found to be:
\[ T(z) = T_f + \frac{QH}{\pi k_e f (r_o^2 - r_i^2)} \left( \frac{1}{Bt} + \frac{z}{2H^2} - \frac{z^2}{2H} \right) \] (2.6)

where, \( T_f \) is the circulation fluid temperature, \( Q \) is the total resistive heat generated, and \( r_i \) and \( r_o \) are the inner and outer radii of the windings.

Figure 2.7: Axial distribution of temperature rise for (i) 1D analytical model in red and (ii) insulated outer wall of 2D numerical model (blue). Background is the contour plot of temperature rise in the numerical model. The contour bar shows temperature rise \( \Delta T \) in °C.

The results of the analysis are presented in Figure 2.7 as a contour plot of the numerical analysis and a line plot comparing the maximum axial variations of temperature obtained by the analytical (1D) and numerical (2D) methods. There is good agreement between the models however the 1D solution slightly overestimates the maximum temperature in the center of the winding and the bobbin because heat rejection to the inner surface in this model is neglected.
Figure 2.8: Evolution of sample chamber temperature and average coil temperature after a step increase in the current input of 5A at $t = 0$. Data taken until steady-state is reached. The coil temperature climbs $\Delta T = 55^\circ C$; however this results in a change of only $4^\circ C$ in sample temperature. The average coil temperature is calculated from the change in the electrical resistance of the winding.

To evaluate the capability of the instrument to isolate the resistive heat that is generated, an experiment was conducted in which cooling fluid was continuously circulated and a step in current of $I = 5A$ was applied to the instrument while monitoring the sample chamber temperature and coil resistance. The change in the coil resistance $\Delta R$ can be related to the average temperature of the coil

$$T_{av} = T_0 + \frac{1}{\alpha_{Cu}} \frac{\Delta R}{R_0}$$  \hspace{1cm} (2.7)

where $\alpha_{Cu} = 0.00393 \, K^{-1}$ is the temperature coefficient of resistivity, and $T_0$ and $R_0$ are the initial temperature and coil resistance, respectively (Avallone and Baumeister III, 1996, Sect.15-
This measurement is a desirable alternative to inserting temperature sensors in the coil and risking breaking the insulation integrity during the installation and removal. Evolutions of average coil temperature and sample chamber temperature are presented in Figure 2.8. It can be seen from the plots that the temperature rise of the bottom plate of the geometry is only 4°C even at $I = 5A$ corresponding to a magnetic flux density $B = 1.0T$.

### 2.3 Annular slit-flow rheometer

While most MR fluid devices contain regions where the fields are uniform, significant deviations from this flow regime are present in virtually all of them. For example, let us consider the active flow channel of an MR fluid damper, in which the flow resistance is controlled by modulating a magnetic field. While the flow and the magnetic fields in the majority of the channel can be considered uniform, the contraction flow into and the expansion flow out of this channel impose significant field non-uniformities.

To probe the MR fluid rheology in regions containing inhomogeneous deformation and magnetic fields, the annular slit-flow magnetorheometer was constructed. In contrast with the magnetorheometry accessory, described in the previous section, the flow and magnetic fields in this device are non-uniform and in many aspects similar to an MR fluid damper. To improve the sensitivity of the instrument to the inhomogeneous flows, a departure was taken from a classical damper design and the uniform flow section was designed to be either short relative to the rest of the flow path or altogether absent.
2.3.1 Design and mechanical construction

The annular slit-flow rheometer consists of a magnetic housing, a piston containing an electromagnetic coil, and a linear actuator-sensor instrument (TA.XTplus™ Texture Analyzer by Stable Micro Systems®). The piston is lowered into the housing containing the MR fluid as illustrated in Figure 2.9. The fluid displaced by the piston is forced up through the annular flow channel and into a reservoir area above the channel. The pressure drop caused by the flow of the fluid from the housing into the reservoir causes an upward force on the piston, which is measured by the linear actuator-sensor.

![Diagram of annular slit-flow magnetorheometer](image)

Figure 2.9: Annular slit-flow magnetorheometer. (a) Schematic diagram of device cross-section. The piston, driven downward with a linear actuator, displaces the MR fluid contained in the housing through the annular flow channel. The fluid resistance to this motion is measured through a force transducer. (b) Photograph of the components. The piston is inverted in the picture.
To minimize the noise on the force measurements, the instrument was designed without any dynamic seals reciprocating with the relative motion of the piston and housing, and the contact surfaces were maintained to be free of MR fluid particles. This was achieved by maintaining a fluid reservoir directly above the annular flow channel such that the fluid used in the experiment does not come into contact with surfaces where the relative motion occurs. The contact between the piston and the housing is made through the sliding of the plastic (virgin polytetrafluoroethylene) bearing. The bearing maintains the concentricity of the annular flow channel while supporting loads generated by magnetic attraction and misalignment between the piston and the housing.

![Figure 2.10: Channel dimensions. The contraction and expansion of the throat are hyperbolic following Equation (2.8).](image)

The non-uniformity of magnetic and flow fields was controlled by the geometry of the throat design located on the piston. Two alternate throat designs were used in the experiments, namely HYP1 and HYP2. Both of the geometries are machined with hyperbolic contraction and expansion sections following

\[ y = \frac{C}{a + x}, \]  

(2.8)
where \( a = L_c h_c / (h_u - h_c) \) and \( C = L_u h_c / (h_u - h_c) \). The dimensions and the coordinate system where Equation (2.8) is valid for the contraction geometry are illustrated in Figure 2.10. The physical dimensions of the throat designs are presented in Table 2.2.

The magnetic field in the instrument is generated by an electromagnetic coil. The coil consists of 1916 turns of 32AWG polyimide insulated magnet wire with 29AWG FEP lead wires. The measured DC resistance of the coil is 105.5Ω.

Table 2.2: Physical dimensions of piston geometries. Dimensions are given in mm.

<table>
<thead>
<tr>
<th>Geometry</th>
<th>( L_c )</th>
<th>( L_u )</th>
<th>( h_c )</th>
<th>( h_u )</th>
</tr>
</thead>
<tbody>
<tr>
<td>HYP1</td>
<td>5.1</td>
<td>0</td>
<td>1.3</td>
<td>5.1</td>
</tr>
<tr>
<td>HYP2</td>
<td>1.27</td>
<td>7.6</td>
<td>1.3</td>
<td>5.1</td>
</tr>
</tbody>
</table>

The technical drawings used to manufacture the components of the device are presented in Appendix B.

### 2.3.2 Simulation and measurements of magnetic field

A magnetostatic finite element model was developed to size components and to quantify the magnitude and distribution of the magnetic field in the flow channel. Similar to the analysis in Subsection 2.2.2, the problem is governed by the scalar Poisson’s equation of the vector potential in the azimuthal direction (Equation (2.2)). The problem was discretized and simulated with Comsol Multiphysics®. The axisymmetric model is illustrated in Figure 2.11(a). The magnetic permeability of the piston and housing was \( \mu_{\text{steel}} = 1200 \). The permeability of the
modeled MR fluid was $\mu^{\text{MRF}} = 9$. Since the device was designed to be in the linear regime, these values were taken to be field-independent.

![Contour plots of the magnetostatic finite element model](image)

**Figure 2.11:** (a) 2D axisymmetric magnetostatic finite element model of annular slit-flow rheometer with HYP1 geometry. Contour plots of the norm of magnetic field strength, $|\mathbf{H}|$ in an air-filled (b) and MR fluid-filled (c) housing with coil current $I_{\text{coil}} = 181\text{Amp-turns}$. Although the field strength in the MR fluid filled case is less, the magnetic flux density, $|\mathbf{B}|$ is greater.

Following the manufacturing of the device, the magnetic field generated in the flow channel was characterized by measuring the field in the radial direction. The peak value of this field component, plotted in Figure 2.12, occurs near the axial center point of the channel. The field measurements from both throat geometries show a linear relationship, $B_{\text{max}} = K_f V_c$, where $V_c$ is the coil voltage. The constant of proportionality, $K_f$ was found to be $8.70 \times 10^{-3} T/V$ and $7.97 \times 10^{-3} T/V$ for HYP1 and HYP2 geometries, respectively. We should also note that the
linear relationship between applied voltage and magnetic field is a validation of the field-independent regime for magnetic properties.

![Graph showing linear relationship between applied voltage and magnetic field.](image)

Figure 2.12: Measurements of the peak value of the magnetic flux density in the annular slit-flow magnetorheometer flow channel as a function of the coil voltage. Measurements were taken while varying the spatial position of the probe tip and the peak value of flux density was recorded. The solid lines are linear curve-fits to the data points. The linearity of the results demonstrates that the device is in the low-field regime where the magnetic permeability of the components is field-independent.

Although the exact distributions of the magnetic and flow fields dictate the flow resistance experienced by the piston, the peak magnetic field is an important quantity to report in experimental measurements because it indicates the scale of the magnetic field in the channel. It is not possible to directly measure this quantity when the fluid is loaded in the instrument;
however, with the results of the finite element model, it can be correlated to the measurements made in an air-filled channel:

\[ B_{\text{max}}^{\text{MRF}}(V_C, \mu_r^{\text{MRF}}) = K_\mu^{\text{MRF}} B_{\text{max}}^{\text{air}}(V_C), \tag{2.9} \]

where \( K_\mu \) is the reluctive correction factor. We can understand the need for this correction by recognizing that the reluctance of the magnetic circuit reduces when a more permeable fluid (MR fluid) replaces air. Therefore, the device is capable of generating a larger amount of magnetic flux for a given current. The correction factor is calculated by comparing the peak flux density at the radial center of the channel, obtained from the finite element model, with and without the presence of the MR fluid. Using this method, the reluctive correction factor for a range of fluid permeability values is plotted in Figure 2.13.

![Figure 2.13: Reluctive correction factor as a function of the magnetic permeability of the fluid. This factor accounts for the reduction of the reluctance of the device magnetic circuit by the presence of the MR fluid. Using Equation (2.9), we can relate the magnetic flux density measurement made in air to that in an MR fluid, which cannot be directly measured.](image_url)
Chapter 3

Magnetorheology at elevated temperatures

MR fluid devices and actuators have been investigated for a number of applications, such as automotive clutch, shock damper, prosthetic knee and specialty polishing applications (Jolly et al., 1998; Klingenberg, 2001; Kavlicoglu et al., 2006; Kordonski and Shorey, 2007). Recently, applications of MR fluids have also been investigated for oil and gas exploration and production (Cobern et al., 2007; Bhavsar et al., 2008). The subterranean operating temperatures for oilfield applications can be relatively high (~150°C) as a result of the geothermal gradient. Furthermore, actuators are often operated at their high-force limit. This is due to constricted physical space within the borehole causes limitations in scaling devices. From the MR fluid point of view this means high shear stresses and thus high magnetic fluxes. These stringent requirements posed by potential oilfield applications are the main motivators for the present study of the high-flux response of MR fluids at elevated temperatures.

3.1 Introduction and background

3.1.1 Experimental measurements of the thermal dependence in magnetorheological effects

A number of previous experimental studies have explored the effects of temperature on MR fluids and devices for conditions close to ambient temperature and intermediate fields. These
studies are summarized in Table 3.1. To provide a quantitative comparison between measured reductions in magnetorheological yield stress we define an average normalized sensitivity of the measured MR yield stress by:

\[
\langle S_\tau \rangle = \frac{1}{\tau_0} \frac{\Delta \tau}{\Delta T}
\] (3.1)

where \(\tau_0\) is the yield stress at the reference temperature, \(T_0\), and \(\Delta \tau\) is the measured change in yield stress corresponding to a change in temperature, \(\Delta T\). Li and coworkers (2002) reported a reduction of magnetorheological yield stress in a commercial fluid (Lord® MRF-132LD) corresponding to \(\langle S_\tau \rangle = -2.2 \times 10^{-3} / ^\circ C\). Zschunke and coworkers (2005) studied the thermal sensitivity of another fluid (Lord® MRF-132AD), which is similar in composition to the previously mentioned fluid (Ponticel, 2002). When their data are fitted with a Bingham plastic model, the sensitivity of the yield stress is nearly threefold higher than that reported by Li and coworkers. Sahin and coworkers (2009) conducted experiments at elevated temperature on a magnetorheological grease formulated by suspending carbonyl iron particles in a commercially available grease and report a very similar result of \(\langle S_\tau \rangle = -3.7 \times 10^{-3} / ^\circ C\). Finally, Batterbee and Sims (2008) studied the effects of temperature on a commercial magnetorheological damper. The force response of the damper was characterized by a constitutive model including a yielding force that arises from the yield stress in the MR fluid. They observed a decreasing trend in this yield force with a temperature sensitivity in the same range as the mentioned rheological studies.
Table 3.1: Experimental studies of the thermal sensitivity of MR fluids. Here, $B_{\text{max}}$ is the highest magnetic flux density at which the yield stress was evaluated. All studies reported significant reduction in the magnetorheological yield stress at elevated temperatures. To provide a quantitative comparison, the average normalized sensitivity $\langle S_r \rangle$ (Equation (3.1)) is tabulated at the highest magnetic field studied in the respective study.

<table>
<thead>
<tr>
<th>Study of thermal sensitivity</th>
<th>Fluid / Device</th>
<th>$B_{\text{max}}$ (T)</th>
<th>$T_{\text{min}} / T_{\text{max}}$ (°C)</th>
<th>$\langle S_r \rangle$ ($10^4/°C$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Li et al. (2002)</td>
<td>Lord® MRF-132LD</td>
<td>0.40</td>
<td>10 / 60</td>
<td>-22</td>
</tr>
<tr>
<td>Zschunke et al. (2005)</td>
<td>Lord® MRF-132AD</td>
<td>0.58</td>
<td>20 / 90</td>
<td>-73</td>
</tr>
<tr>
<td>Sahin et al. (2009)</td>
<td>Carbonyl iron in grease</td>
<td>0.53</td>
<td>10 / 70</td>
<td>-37</td>
</tr>
</tbody>
</table>

All of the discussed studies have identified significant and varying thermal sensitivity of magnetorheological yield stress even within the moderate levels of temperature increase explored ($T_{\text{max}} \leq 100°C$). In applications such as in oil and gas production that demand higher temperature operations than previously studied, it is critical to be able to predict the extent of thermal reductions in the magnetorheological stress, so that appropriate design tolerances can be incorporated into devices.

### 3.1.2 Theoretical predictions to changes in magnetorheology with temperature

All the microscopic forces considered in the functional relationship $\Phi(Mn, \phi)$ given in Equation (1.17) depend on physical quantities that evolve with the ambient operating temperature, $T$. The dependence of magnetic force on temperature arises from the effects of thermal fluctuations on ferromagnetic ordering within the particles. In the absence of an applied field, the ferromagnetic ordering is completely lost at the Curie temperature and the material behavior becomes
paramagnetic (Blundell, 2001). Below the Curie temperature, the saturation magnetization of the ferromagnetic material is a monotonically decreasing function of temperature (Jiles, 1991) and has been well characterized by experimental measurements. Measurements made on polycrystalline iron by Crangle and Goodman (1971) are shown in Figure 3.1 as an example. A functional form of the dependence of saturation magnetization on temperature, \( M_s(T) \), can be obtained using a mean field approximation also known as the Weiss model of ferromagnetism, as illustrated in the figure. This model approximates the exchange interaction of neighboring atoms in a crystal by an effective field to solve for the ferromagnetic behavior of the material. This is not an accurate description of the interaction in metallic crystals because the ferromagnetism in these materials is largely caused by the conduction electrons; however, as shown in the Figure 3.1, the Weiss model captures the general thermal response well across the complete range of temperatures. A detailed derivation of this and other models of magnetically-ordered condensed matter can be found in Blundell (2001).

Shul'man and coworkers (1980) studied the effects of temperature on a colloidal ferrofluid system near the Curie temperature. The strongly decreasing trend in the field dependent response near this critical temperature in their system \( T_c^{FF} = 145^\circ C \) was demonstrated. In contrast with the ferrofluid used this earlier study, magnetorheological fluid particles are often composed of materials with very high Curie points (e.g. for iron, \( T_c^{iron} = 770^\circ C \)). Thermal sensitivity is thus expected to be small. The slope of the saturation magnetization versus temperature curve that characterizes the thermal sensitivity becomes increasingly significant as the operating temperature approaches the Curie temperature. Using the measurements of Crangle and Goodman (1971), we can evaluate the sensitivity of the saturation magnetization of iron at 20°C:
where $M_s^0$ is the saturation magnetization at the reference temperature. The magnitude increases only slightly at higher temperatures (e.g. $-1.7 \times 10^{-4} / ^\circ C$ at 130°C). Equation (1.15) suggests a quadratic dependence of the yield stress on saturation magnetization. Therefore, the sensitivity of the yield stress solely due to this effect is expected to be:

$$S_{\tau M} = 2S_M = -2.2 \times 10^{-4} / ^\circ C$$

(3.3)

Compared to the previous measurements of yield stress at elevated temperatures (Table 3.1), the change in magnetization contribution is at least an order of magnitude less than the actual measured thermal reduction in bulk yield stress.

Figure 3.1: Evolution of saturation magnetization of polycrystalline iron with temperature. The locations corresponding to $20^\circ C (T/T_c = 0.28)$ and $150^\circ C (T/T_c = 0.41)$ are marked to provide a quick reference.
The Brownian force acting on the suspension particles is also higher as the background temperature is increased (Li et al., 2002); however, as seen by the large numerical value of $\lambda$ (Equation (1.15)), the magnetic force completely dominates over Brownian forces at all temperatures. For example, the increase in the Brownian force resulting from a temperature rise from $25^\circ$C to $770^\circ$C (i.e. Curie temperature of iron) only corresponds to half an order of magnitude change in $\lambda$. Therefore, dimensional analysis suggests that the effects of thermal fluctuations on the particle structures in an MR fluid should have a negligible effect on the tendency for particle chaining and a resulting magnetorheological yield stress.

The suspension viscosity in the absence of an external magnetic field is also a strong function of temperature that is largely due to the changes in the matrix fluid viscosity, which typically varies according to the Arrhenius relationship (Bird et al., 1987):

$$
\eta_0(T) = \eta_0(T_0) e^{\frac{\Delta H}{RT_0}}
$$

where $T_0$ is the reference temperature, $\Delta H$ is the activation energy for flow and $R$ is the universal gas constant. From dimensional considerations we have argued that the suspension viscosity appears only in the expression for viscous drag on particle chains and is therefore only present under flowing conditions. Therefore, the static yield stress that dominates the MR response should not be affected by the changes in the matrix fluid viscosity.

It thus remains unclear as to what particular physical mechanism leads to the previously reported thermally induced changes of MR yield stress. For accurate design of downhole devices we need to document these changes and therefore pursue direct experimentation under appropriate thermal and magnetic conditions.
3.1.3 Scope and motivation

In the survey presented in this section, we have identified two outstanding open issues in experimental magnetorheology. First, we have shown that the theoretical predictions for reduction of yield stress are an order of magnitude smaller than the measured values reported in literature. Second, high-flux magnetorheology has not been investigated under elevated temperatures. In the present work, we study a higher range of temperatures and magnetic fields than those investigated to date. To explore high flux and high temperature simultaneously, a new magnetorheology fixture was designed and built to work in conjunction with a commercial torsional shear rheometer, as described in detail in Chapter 2. In Section 3.2, experimental results obtained with the new experimental setup are presented. In Section 3.3, the findings are summarized and the conclusions are presented.

3.2 Rheological measurements at elevated temperatures using the torsional shear rheometer accessory

To evaluate the MR fluid response to the imposed magnetic field, a series of shear-rate-controlled experiments were conducted on MRF-132DG fluid manufactured by Lord Corporation®. To ensure repeatability in the measurements a test protocol developed by Deshmukh (2007) was followed. Prior to the experiments the fluid samples were mixed for more than 8 hours on a benchtop roller mixer (Wheaton®, Model 348921UL) at approximately 10rpm. After the fluid sample was loaded in the rheometer, a field in excess of 0.5T was applied for more than 30 seconds. This process was done to allow the fluid to form stable structures within the sample chamber. The field was removed and the sample then was pre-sheared at \( \dot{\gamma}_p = 300s^{-1} \). At this state, the fluid was assumed to be independent of history. The repeatability
of experimental measurements in previous work (Deshmukh, 2007) and the present study strongly supports this assertion.

3.2.1 Measurements at ambient conditions

The steady shear measurements are presented in Figure 3.2. The magnetic flux density reported is the plateau value measured at a mid-span radius of \( r_{mid} = 5\, \text{mm} \) by using the gaussmeter probe (for ambient temperature experiments) and the calibrated secondary coil (for experiments at elevated temperatures). The local shear rate in the parallel plate geometry is zero on the axis of the geometry and linearly increases to its maximum value, \( \dot{\gamma}_R \), at the geometry rim assuming there is no slip on the roughened fixtures. The reported shear stress values account for the non-uniform shear rate in the sample with a Rabinowitsch type correction (Macosko, 1994):

\[
\tau = \frac{\mathcal{J}}{2\pi R^3} \left[ 3 + \frac{d \ln \mathcal{J}}{d \ln \dot{\gamma}_R} \right],
\]

where \( \mathcal{J} \) is the torque on rheometer spindle and \( R \) is plate radius. For Newtonian fluids, the second term in the brackets is unity. The stress obtained by making this approximation is called the Newtonian or apparent stress, \( \tau_a = 2\mathcal{J}/\pi R^3 \). The rate-dependent viscosity is reported at rim shear rate and calculated using the expression

\[
\eta_R = \frac{\tau}{\dot{\gamma}_R},
\]
Figure 3.2: Experimental results for rate-controlled steady shear viscosity of a magnetorheological fluid (MRF-132DG). (a) The dynamic yield stress values for each magnetic field point can be easily seen in the vertical asymptotes of the viscosity versus shear stress plot. There is a small yield stress also present in the field-off data; however, this is very weak compared to field-on yield stress ($\tau_{y}^{ff} < 5\text{Pa}$). In the high shear limit the field-on plots tend toward the field-off viscosity as the viscous stress on particles becomes the same order of
magnitude as the magnetic stress ($Mn - 1$). (b) The measured shear stress is essentially flat over four orders of magnitude increase in the shear rate. The fluid rheology is dominated by the field dependent yield stress $\tau_y(B)$.

Plots of the viscosity variation with decreasing shear stress (Figure 3.2a) show that at all field levels the viscosity tends to infinity while the stress approaches an asymptotic value. We measure dynamic yield stress as this asymptotic value. In the shear stress-shear rate plots (Figure 3.2b), the dominance of yield stress in the rheology of MR fluids is evident. These plots are essentially flat over four orders of magnitude in shear rate. In the field-off state the fluid has a yield stress as expected from a high volume fraction suspension (the fluid is 32% v/v). However, as compared to the field-on yield stress, this value is exceedingly small.

It is important to understand the evolution of the MR yield stress with applied field for design purposes. Plots of the yield stress with varying magnetic field strength, $H$ and magnetization, $M$ is presented in Figure 3.3. In the lower field levels both of these variations follow previously described scaling laws by Ginder and coworkers (1996) in the case of field strength and Klingenberg and coworkers (2007) for magnetization. However, in high field regime the variation significantly deviates from these scaling laws.

The measurements of MR fluid rheology were also fitted with Bingham (Equation (1.1)), Herschel-Bulkley (Equation (1.2)) and Casson (Equation (1.3)) models. The fitted parameters and presented in Table 3.2.
Figure 3.3: Evolution in magnetorheological shear stress with magnetic field strength $H$, and magnetization, $M$. In the lower fields the scaling laws identified by Ginder and coworkers, and Klingenberg and coworkers accurately describe the evolution of MR fluid rheology. In higher fields, however, there is significant deviation from both of the scaling laws.

3.2.2 Magnetorheological stress at elevated temperatures

The effect of temperature on magnetorheological response is evaluated for three levels of magnetic flux density and presented in Figure 3.4. Measurements on five samples were taken at three temperatures each: 20, 75 and 130°C. At the highest two temperatures, the flux density was measured with the secondary coil and integrating fluxmeter. At 0.3T, the current necessary to generate this field was the same at all temperature levels tested. However, at higher flux densities, the current requirement was found to be higher at elevated temperatures (leading to a magnetic field correction up to approximately 5%). This is attributed to the high temperature properties of the magnetic material used in the magnetorheometer. Therefore, making measurements at elevated temperature without a calibrated magnetic field sensor in place may induce an undesirable device-specific systemic error into the measurements.
Table 3.2: Rheological model fit for steady shear data. Casson and Herschel-Bulkley model fits are superior as compared to the Bingham model.

<table>
<thead>
<tr>
<th>$B(T)$</th>
<th>Bingham</th>
<th>Herschel-Bulkley</th>
<th>Casson</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\tau_y^B$</td>
<td>$\eta_p^B$</td>
<td>$R^2$</td>
</tr>
<tr>
<td></td>
<td>(Pa)</td>
<td>(Pa.s)</td>
<td></td>
</tr>
<tr>
<td>0.025</td>
<td>107</td>
<td>0.32</td>
<td>0.956</td>
</tr>
<tr>
<td>0.050</td>
<td>413</td>
<td>0.48</td>
<td>0.927</td>
</tr>
<tr>
<td>0.075</td>
<td>903</td>
<td>0.66</td>
<td>0.913</td>
</tr>
<tr>
<td>0.100</td>
<td>1404</td>
<td>0.92</td>
<td>0.933</td>
</tr>
<tr>
<td>0.200</td>
<td>5372</td>
<td>9.09</td>
<td>0.803</td>
</tr>
<tr>
<td>0.300</td>
<td>10612</td>
<td>11.84</td>
<td>0.824</td>
</tr>
<tr>
<td>0.400</td>
<td>16829</td>
<td>11.11</td>
<td>0.875</td>
</tr>
<tr>
<td>0.500</td>
<td>22517</td>
<td>12.92</td>
<td>0.829</td>
</tr>
<tr>
<td>0.600</td>
<td>27976</td>
<td>15.04</td>
<td>0.898</td>
</tr>
<tr>
<td>0.700</td>
<td>32783</td>
<td>16.19</td>
<td>0.787</td>
</tr>
<tr>
<td>0.800</td>
<td>36360</td>
<td>19.80</td>
<td>0.762</td>
</tr>
<tr>
<td>0.900</td>
<td>38683</td>
<td>22.97</td>
<td>0.731</td>
</tr>
<tr>
<td>1.000</td>
<td>39445</td>
<td>3.34</td>
<td>0.137</td>
</tr>
</tbody>
</table>

The high temperature results show a systematic and statistically significant decline in magnetorheological stress with temperature. In Section 1, an average sensitivity parameter, $\langle S_e \rangle$, was defined to quantitatively compare the changes observed by different studies of magnetorheology in the literature and the changes predicted by theory. This parameter and the dynamic yield stress are tabulated for the elevated temperature measurements in Table 3.3.
Figure 3.4: Evolution in magnetorheological shear stress with increasing temperature. Black squares, blue circles and red triangles are data at $T = 20^\circ C$, $T = 75^\circ C$ and $T = 130^\circ C$ respectively. Five samples were measured at each temperature and magnetic field level. Bars designate standard deviation $\pm \sigma$. There is a systematic and statistically significant decrease in stress with temperature, which becomes larger at high temperatures and field strengths.

3.3 Discussion and conclusions

A new magnetorheometry fixture was designed to be used with a commercial torsional shear flow rheometer. Using finite element methods, the magnetic field uniformity in the sample chamber was evaluated and improved by implementing changes to design features of the fixture. The numerical results were verified with spatially resolute measurements of magnetic field. The design of the fixture also allows for magnetic field measurements while the sample is under test. During rheological measurements under ambient temperature conditions, a commercial Hall effect probe was used. For high-temperature measurement, a custom-wound secondary coil was used in conjunction with an integrating fluxmeter. The fixture design includes oil circulation flow channels for sample temperature control. This heating-cooling system was shown to be
effective in regulating the sample temperature even under maximum resistive heat generated in the electromagnetic coils.

Table 3.3: Evolution of magnetorheological yield stress with temperature. The dynamic yield stress, $\tau^0$, is measured as the asymptotic value in the fluid stress as the viscosity tends to infinity. The average normalized sensitivity parameter, $\langle S_r \rangle$, is reported for comparison with the values in Section 1 in terms of an average value between 20°C to 75°C and 20°C to 130°C.

<table>
<thead>
<tr>
<th>$B(T)$</th>
<th>$\tau^0$ (kPa)</th>
<th>$\langle S_r \rangle$ $(10^4/°C)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>20°C</td>
<td>75°C</td>
</tr>
<tr>
<td>0.3</td>
<td>10.8</td>
<td>10.2</td>
</tr>
<tr>
<td>0.6</td>
<td>28.5</td>
<td>27.0</td>
</tr>
<tr>
<td>0.9</td>
<td>41.6</td>
<td>39.3</td>
</tr>
</tbody>
</table>

The fixture was used to evaluate magnetorheological stress at ambient and elevated temperatures. Comparison of these experiments shows a clear trend of reduction in magnetorheological yield stress with temperature. A sensitivity parameter was defined to provide a quantitative comparison of yield stress reductions measured in the literature. The change in magnetorheological yield stress measured in this work is less than what is found in previous literature. However, there is still a disparity between theory (Equation (3.27)) and experiments (Table 3.3). One possible contributor to this may be the effect of temperature on volume fraction.

In most applications of MR fluids, the volume fraction of magnetic particles is also a function of temperature. As the hydrocarbon oils used commonly as the suspending medium have higher coefficients of expansion than the particles, during an isobaric increase in temperature, the volume fraction of the fluid decreases. This change in volume fraction may also lead to a change in MR yield stress. The volume fraction at elevated temperature is:
\[
\phi_f = \frac{\phi_0 (1 + \alpha_p \Delta T)}{1 + \phi_0 \alpha_f \Delta T + (1 - \phi_0) \alpha_b \Delta T}
\] (3.7)

where \(\phi_0\) is the initial particle volume fraction, and \(\alpha_b\) and \(\alpha_p\) are volumetric thermal expansion coefficients for the base fluid and particles respectively. Using this relation, we can find the sensitivity of volume fraction to temperature by

\[
S_\phi = \lim_{\Delta T \to 0} \frac{1}{\phi_0} \frac{\phi_f - \phi_0}{\Delta T} = \alpha_p - \alpha_f + \phi_0 (\alpha_f - \alpha_p)
\] (3.8)

Typical coefficients of volumetric expansion are \(6.3 \times 10^{-4} / ^\circ C\) for mineral oils (Khonsari and Booser, 2001) and \(3.54 \times 10^{-5} / ^\circ C\) for iron (calculated from coefficient of linear expansion given by Lide (2009)). For a fluid with 32% volume fraction, the normalized sensitivity is \(-4.0 \times 10^{-4} / ^\circ C\).

This sensitivity is significant as compared to those calculated in Section 3.1; however, to calculate the change in yield stress due to the change in volume fraction, the relation between these terms needs to be defined. At low volume fractions this relation is approximately linear as found by the non-interacting chain model of Ginder and coworkers (1996). The linear relationship was also found by Klingenberg and coworkers (1991a) for the electrical counterparts of MR fluids, electrorheological fluids at low volume fractions. Using a point dipole model and experimental methods they showed a region where the yield stress is linearly related to volume fraction \((0 < \phi < 0.35)\). However above this range the yield stress reached a maximum and the relation was more complex. Further studies of the functional dependence of field-dependent yield stress on volume fraction may assist in better quantifying this dependence; however, it is
clear that the volume fraction change in MR fluids can have an important effect on the evolution of its rheological properties with temperature.
Chapter 4

Quantitative scaling of magnetorheology with applied magnetic field

4.1 Introduction

It is important to understand the evolution of magnetorheological stress with applied field for fluid and device design purposes. Power law relations between MR fluid (and equivalently for its electrical counterpart, electrorheological fluid) shear stress and various electromagnetic quantities have been identified in numerous experimental, analytical and numerical studies. These studies, the scaling laws they identified, and the regime of applicability of these laws are reviewed in this section.

In the linear (low-field) regime of ER fluids, the polarization of each particle is proportional to the applied field, resulting in the interparticle force being proportional to the square of the applied field. This occurs as the polarization and the electric field caused by the particles each linearly depend on the applied field resulting in the quadratic field dependence of interparticle force. This effect was confirmed by the electrorheological measurements of Marshall and coworkers (1989) and simulations of Klingenberg and Zukoski (1991a), showing that the yield stress of an ER fluid is quadratically dependent on electric field ($\tau \propto E^2$).
Similar arguments also apply for the linear regime response of MR fluids and therefore quadratic dependence on the applied field is predicted. However, the non-linear nature of ferromagnetic phenomena in the range of practical interest renders the MR fluid response more complex. The cause of this non-linearity is the saturation of magnetization; at high applied fields the material contribution to the magnetic field is independent of applied field. The evolution of magnetization up to this point is non-linear in nature and is often well described by the Frohlich-Kennelly relation (Jiles, 1991):

\[ M = \frac{H}{\sqrt{\chi^0 + H/M_s}} \]  

At the extreme limits of applied field, this relation approaches expected limits: 
\[ M \approx \chi^0 H \] (for \( H \ll M_s / \chi^0 \)) and 
\[ M = M_s \] (for \( H \gg M_s / \chi^0 \)). Although the Frohlich-Kennelly relation is empirical, it often describes magnetic response with good accuracy with only two fitting parameters: initial magnetic susceptibility, \( \chi^0 \), and saturation magnetization, \( M_s \). An example of this model fitted to a commercially available MR fluid magnetic response is presented in Figure 4.1.

Finite element methods have been shown to provide valuable insight into microscopic forces in ER and MR fluids, which also provides clues on how these interactions affect the bulk fluid properties. Davis utilized these models for ER fluids by making microstructural assumptions in terms of particle chains (Davis, 1992a), and particles in a periodic lattice (Davis, 1992b). Ginder and coworkers (1996) utilized finite element analysis for MR fluids by analyzing the forces on a particle in an infinite chain by making uniform and affine deformation of the chain. As the total force in each particle is identically zero, they proposed to integrate an
electromagnetic stress tensor (derived from the Maxwell stress tensor) in the plane between the particles. Klingenberg and coworkers (2007) analyzed the attractive force between two MR fluid particles aligned along an applied field using an axisymmetric finite element model. By using this approach they could evaluate the electromagnetic stress integral over a closed surface as the total force on the particle is non-zero. However, their analysis did not study the separation between the particles and the effects of this parameter on magnetic force. We demonstrate in the following sections that this is a critical parameter that affects how the magnetic force scales with the applied field.

Figure 4.1: Frohlich-Kennelly fit to magnetic measurements of an MR fluid (Lord® MRF-132DG). The measurements are digitized from the data sheet of the fluid (Lord, 2008). Fitted parameters are the saturation magnetization, $M_s = 620 \text{kA/m}$, and initial susceptibility, $\chi^0 = 8.06$.

There have been numerous studies of MR fluid rheological response to applied field, both experimental and theoretical, and this response has been described by a magnetic quantity. As illustrated in Figure 4.2, many studies have found power law relations:

$$\tau_y \propto H^h \propto B^b \propto M^m$$  \hspace{1cm} (4.2)
In the linear regime, \( h = b = m = 2 \) as a result of quadratic dependence of magnetic force on applied field and the field independent magnetic susceptibility. This regime \( (b = 2) \) was identified by Laun and coworkers (1996). However, Deshmukh and McKinley (2004) found a subquadratic dependence \( (b = 3/2) \) within a comparable range of magnetic flux densities. Similar scaling laws were found by Parziale and Tilton (1950), Park et al. (2001), Genc and Phule (2002) and See (2003). All studies referenced in this paragraph used micron-sized carbonyl iron suspensions to evaluate scaling laws.

Magnetorheometry instruments with higher field capabilities were used to study the fluid response at the high-flux regime by Burguera et al. (2008) on an MR fluid with bimodal distribution of iron particles and Laun et al. (2008a) on a suspension of micron-sized carbonyl iron particles. At high fields \(~ IT\), effects of magnetic saturation of particles were observable and the sensitivity of yield strength on applied field was found to be diminishing \( (b \to 0) \).

Ginder and coworkers (1996) offered a theoretical explanation to the subquadratic response by utilizing the finite element method and approximate analytical methods on infinite particle chains. The resulting power law relation was between yield stress and magnetic field strength \( (h = 3/2) \). This law was introduced for intermediate field strength which corresponds to the regime bound by magnetic saturation of the polar points of contact between neighboring particles on a chain as the lower limit, and complete saturation of the particles as the upper limit. However, the transition between the subquadratic regime and the saturated regime was not described.
Figure 4.2: Power law scaling of yield strength in the literature. The horizontal extents of boxes indicate the magnetic flux densities where the scaling laws were identified. See Equation (4.2) for the definition of the power law exponents $b$, $h$ and $m$. There are overlapping regions in magnetic flux density where quadratic ($b = 2$) and sub-quadratic ($b = 3/2$) dependences were identified experimentally. In more recent studies of higher magnetic fields, saturation of rheological properties was also identified. The theoretical scaling laws found in literature do not provide a comprehensive explanation of the experimental findings.

In an alternate view, the non-linear magnetic response of the material was proposed to be the only contribution to deviation from quadratic field dependence. This non-linearity was proposed to be accounted for by scaling the yield strength with magnetization ($m = 2$) rather than the applied field. This scaling law was initially proposed for inverse ferrofluids (de Gans et al., 2000), and later shown to be applicable to suspended magnetic particles (Domínguez-Garcia et al., 2005) and magnetorheological fluids (Klingenberg et al., 2007).
As discussed in Section 1.1, the magnetic field can be separated into two parts: the free-current and bound-current generated field, where the latter field is the material contribution. In magnetorheological experiments, often the magnetic flux density, \( B \), is measured in a location within the magnetorheometer and boundary condition equations are invoked to calculate the field inside the MR fluid. In the case of a boundary perpendicular to the field, the calculation is trivial as the field in this direction, \( B_\perp \), is equal on both sides of the boundary. Clearly this measurement captures a single aspect of magnetism within the MR fluid element; in this case the magnetic flux density that includes both the bound and free current contributions. However individual measurements of each of these contributions are not made. For this measurement, the experimentalists use a magnetometer, which measures the material contribution to magnetic field in response to an applied field. These measurements can be correlated to the rheological measurements of yield stress, because the magnetic flux density is measured in the rheological experiments and the same quantity can be measured for the magnetization experiments. The results of such a set of experiments is illustrated in Figure 4.3 by a three dimensional curve in the \( \tau_y - B - M \) space. It can be seen that the rheological measurements, or the projection of the curve on the \( \tau_y - B \) plane, may lead to a different scaling law than the projection of the plot on the \( \tau_y - M \) plane as a result of the non-linear relation between the magnetic quantities. By making the magnetization measurements on the MR fluid it is possible to identify scaling laws, which may not be possible by using solely rheological measurements.

Klingenberg and coworkers (2007), analyzing experimental data from both classes of fluids, showed that the scaling laws for ER fluids do not apply well to MR fluids. Interparticle attraction between magnetic particles with non-linear material properties was studied using axisymmetric finite element models. By using the results from numerical and experimental
studies they attributed the difference between the scaling laws for ER and MR fluids in part to the non-linearity in magnetic properties in MR fluid particles. In the following sections, we illustrate the validity of their approach in both numerical and experimental aspects. However, although the quadratic scaling of yield stress with magnetization \( m = 2 \) is valid in a portion of the applied field, the geometric features within the fluid (mainly in the contact areas between the suspension particles) lead to a different scaling law at higher fields. This regime of fluid response was not identified by this study because of the selections in parameters used in the numerical models. Furthermore, the rheological data used to evaluate the scaling law were analyzed on plots that span many orders of magnitudes in viscosity and shear rate, which do not present the higher level of accuracy that is desirable.

Figure 4.3: Evolution of magnetic and rheological quantities in \( \tau_y - B - M \) space. Experimental studies have mostly characterized the yield stress \( \tau_y \) as a function of magnetic flux density \( B \). This dependency can be viewed as a projection of a three dimensional curve in the illustrated space onto the \( \tau_y - B \) plane. However, by incorporating magnetic measurements of the fluid, different dependencies such as \( \tau_y(M) \) may be analyzed.

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In the present study, all the parameters in a finite element model of MR fluid particles are studied. The results are shown to be in agreement with previous studies and furthermore, show that there are two regimes of interparticle force response. In the experimental aspect, instead of presenting the rheological results in plots that span many orders of magnitudes in both axes, shear stress is plotted in both dimensional and non-dimensional forms. This choice of axes leads to a clear identification of different regimes of fluid response. In Section 4.2, the axisymmetric finite element model developed to analyze interparticular attraction of particles in a uniform applied magnetic field is described. In Section 4.4 the results from the numerical model and the experimental measurements were utilized to describe a new scaling law. This scaling law is compared to previously proposed laws to demonstrate its accuracy in obtaining a master curve of magnetorheological response. Finally, in Section 4.4, results are summarized and conclusions from the work are presented.

### 4.2 Numerical model of interparticular force

A finite element model was developed to analyze the magnetostatic interaction between MR fluid particles. As the dominant force in low Mason number MR fluid flow, scaling of magnetic force with applied field has close correlation with scaling of magnetorheological shear strength. The model was developed for a pair of particles while neglecting the effects of other particles in the vicinity. This simplification allows for a greater computational effort to be spent on refining the spatial resolution of the model and thus increasing the accuracy of force prediction.

Maxwell’s equations in the current free, magnetostatic limit \((\nabla \cdot \mathbf{B} = 0, \nabla \times \mathbf{H} = 0)\) and the magnetic constitutive relation \((\mathbf{B} = \mu \mathbf{H})\) are the governing equations where \(\mu = \mu_0 (1 + \chi_m)\).
Since the magnetic field strength is curl free, it can be described as the gradient of a scalar potential \( \mathbf{H} = -\nabla \Phi \). Therefore, the governing equations can be combined in the scalar equation: \( \nabla \cdot (\mu \nabla \Phi) = 0 \). For field independent material properties, this equation reduces to the Laplace’s equation. The governing equation was discretized using the variational method (Ida and Bastos, 1997) and implemented in Matlab®. The code generated as a part of this implementation is presented in Appendix C. Due to the symmetries in the geometry of the two particle interaction, an axisymmetric finite element model (Figure 4.4) consisting of a magnetizable spherical particle modeled in cylindrical surroundings was used. The outer wall boundary condition is zero normal flux. The boundary conditions on the top and the bottom planes are assigned magnetic scalar potential. This boundary condition also imposes the magnetic field to be normal to these surfaces, constituting a symmetry boundary condition. In the inset of Figure 4.4 the symmetries are opened to illustrate the modeled system. The resulting solution is for an infinite chain of particles where the gaps between the neighboring particles are alternatingly small and large. The large gap \((8a)\) is much larger than the small gap \((0.01a, \text{ except when this parameter is studied})\); therefore, the effect of magnetic attraction from the other particles in the infinite chain is negligibly small. This was verified by numerically evaluating the sensitivity of calculated force on the large gap. Magnetization of the particle material is modeled with the Frohlich-Kennelly relation with saturation magnetization \( M_s = 2.1T/\mu_0 \) and initial relative permeability \( \mu_r^0 = 1000 \) for iron. The non-linear set of equations is solved using the method of successive approximations. Force on the particle is found by integrating the Maxwell stress tensor (Melcher, 1981), \( T_{ij} = H_i B_j - (1/2) \delta_{ij} \mu_0 H_s H_s \), on the outer surfaces of the model, enclosing a particle.
\[
F_{part} = \oint n_i T_{iz} \, dS
\] (4.3)

\( \delta_y \) is the Kronecker delta, \( n_i \) is the unit normal to surface and summation on repeated indices is implied.

Figure 4.4: 2-D axisymmetric finite element model geometry (not to scale). Outer radial boundary condition is zero flux. The upper and lower boundaries are applied scalar potential which effectively imposes normal flux lines and a symmetry condition. The inset illustrates the opened symmetries.

Solutions obtained for a range of applied magnetic field are presented in Figure 4.5. The line plots of magnetization are taken along the axis joining the particles. The magnetic force is non-dimensionalized by saturation force, which was obtained by evaluating Equation (1.12) at saturation magnetization. The force is plotted against average magnetization of the particle (Figure 4.5b). It can be observed from plots that the field strengths that lead to quadratic
dependence on magnetization, do not induce saturation within the particle. At field strengths which the contact point saturates, the force scaling shifts to a subquadratic dependence on magnetization. In a large range of this region the power law exponent is 3/2. We define the average magnetization at which this shift occurs as the critical magnetization, $M_c$. It should be noted that this is an average quantity relating to the particle only. However, with the aid of Equation (1.13) this quantity can be related to a macroscopic fluid measurement. The saturation is induced in the contact point due to the field focusing effect of two spherical bodies near contact. At higher fields, the saturation zone expands towards and past the center of the particle until the whole particle is saturated. It should also be noted that at complete saturation, the force prediction of the model tends to the exact relation of Equation (1.12).

The critical magnetization can be determined from the model; however, its value is highly sensitive to how the contact point is modeled. This is demonstrated by two examples. First, the gap between the contacting particles is varied (Figure 4.7). It can be seen from the results that an increase in the gap also results in an increase in critical magnetization. This is due to the change in field focusing between the particles at different values of the gap. The intensification of the field near the contact becomes more diffuse as the gap increases and therefore the shift in force scaling law occurs at larger fields. In the second example, the focusing caused by the contact point is completely removed by modeling cylindrical particles (Figure 4.7). The cylindrical geometry has the advantage of the uniform cross-section in the axial direction while still maintaining the axisymmetry. In this model, the contact point saturation occurs along with the rest of the particle; therefore, a quadratic scaling law is maintained throughout. A variety of aspect ratios of the cylinder also showed the same results.
Figure 4.5: (a) Line plots of magnetization along particle axis. Dashed and solid lines are of data points in quadratic and subquadratic ranges, respectively, as seen in plot of magnetostatic force scaling with particle average magnetization (b). In this plot, the dashed line is a curve fit of the quadratic power law relation to the lowest field data point. Black circles in the quadratic regime correspond to the line plots where contact point saturation has not occurred (black dashed lines) the red squares correspond to red solid lines where the saturation zone in the vicinity of the contact point is progressively expanding. The applied magnetic scalar potential for the point-n in this plot is $\chi_s = 2.23 \times 10^{3+0.114}$. (c) and (d) show contour plots of non-dimensional magnetization (zoomed into particle to show feature) for average magnetization of $0.031M_s$ and $0.449M_s$ respectively.
Figure 4.6: Interparticular force calculations with varying particle distance. For low field strength, the relation between average magnetization and magnetic force is quadratic as predicted by linear theory. Dashed lines are curve fits of the quadratic power law relation to lowest field data point. The relation becomes subquadratic when magnetic saturation occurs at the contact point or at the critical average magnetization ($M_c$). The plots show this relation for different gaps between the particles. The critical magnetization is highly dependent on the gap, or in general the geometry of contact between the particles.

From these models we conclude that the geometry of the MR fluid particles contribute to the non-linear response of interparticular force. This result is in agreement with the particle chain model of Ginder and coworkers (1996). Furthermore, two regimes of scaling laws are identified between the interparticular force and average magnetization. The shift occurs at a critical magnetization where the contact point saturates. The model dependent nature of this critical magnetization poses challenges in calculation of its value because of the sensitivity of this parameter to interparticle spacing and the particle shape. It is clear that the sphericity of the particle will affect the level of focusing that can occur at the contact points. Furthermore, the
stabilizing agents present in many MR fluids may cause the contact geometry to be affected by steric repulsion

![Contour plot of non-dimensionalized magnetization](image)

![Graph showing quadratic dependence](image)

Figure 4.7: Interparticular force calculations for cylindrical particles. (a) Contour plot of non-dimensionalized magnetization ($M / M_s$) at $\langle M_p \rangle = M_s / 10$. (b) The interparticular force shows quadratic dependence on average magnetization independent of the applied field. Dashed line is the curve fit of the quadratic power law relation to lowest field data point.

In this model, the interparticular force was accurately calculated for two spherical particles in an applied uniform field and the scaling laws were identified between this force and average magnetization. In the low Mason number regime of MR fluid flow, the magnetic force is the dominant force acting on a particle, and therefore the changes in the scaling laws of this force affects how MR fluid stresses scale. The model neglects the effects of other particles in the vicinity of the contact modeled, and furthermore, the principle of superposition cannot be invoked because of the non-linear material model. However, the critical effect identified by the model, the contact point saturation, occurs when this material point is in the non-linear magnetic property regime; the rest of the particle is in the linear regime with a relatively low field. This
suggests that other contact points on the particles do not significantly contribute to the effects quantified. As a result, the conclusions reached on the interparticular magnetic force are expected to be applicable to MR fluid stress which is discussed in detail in the next section.

4.3 Scaling of magnetorheological stress with applied field

To evaluate scaling of magnetorheological shear stress with applied field, the experimental measurements were converted to dimensionless form. As mentioned in Chapter 1, the Brownian disturbance to suspended particles is normally negligible ($\lambda \gg 1$). Therefore, in the following analysis, we neglect the dependence on thermal disturbances and present the dimensionless rheological data in the form:

$$\frac{\tau_y}{\tau_{mag}} = \Phi(Mn) \quad (4.4)$$

The magnetic stress scaling is

$$\tau_{mag} = \frac{F_{mag}}{A_p} \quad (4.5)$$

where $A_p$ is the effective area in the fluid the magnetic force is distributed on. On average this area is $A_p = \pi a^2 / \phi$.

In Chapter 1, interparticular magnetic force was estimated (Equation (1.13)) by making the uniform magnetization approximation. It was also shown that this leads to a quadratic dependence in magnetization. In this section, the goal is to obtain an expression that leads to an accurate description of the dependence without relying on approximations. First, two previously
described MR stress scalings are used to non-dimensionalize the experimental results and their effectiveness in obtaining master curves are evaluated. Then the new scaling law based on numerical simulations is described and applied to the experimental data.

By using numerical and approximate analytical methods Ginder and coworkers (1996) obtained an MR yield stress scaling in which \( h = \frac{3}{2} \) (according to Equation (4.2)). By using analytical methods, they obtained a closed form expression for the MR yield stress. Here, this expression is used to calculate \( F_{\text{mag}} \) using Equation (4.5) and substituting this into Equation (1.14) to obtain \( M_n \). The resulting definitions are:

\[
\tau_{\text{mag}}^G = \sqrt{6\phi\mu_0 M_s^{1/2}H^{3/2}}
\]

\[
M_n^G = \frac{\sqrt{6\eta_0 \dot{\gamma}}}{\mu_0 M_s^{3/2}H^{3/2}}
\]

The superscript \( G \) is used to distinguish this definition from the others presented herein.

The experimental measurements obtained under a steady-shear deformation (Figure 3.2) are non-dimensionalized using these definitions (Figure 4.8(a)). At low field strengths (corresponding to \( B < 0.3T \)) the data collapse fairly well; however, at higher levels of applied field a systemic error is apparent in the scaling. This is an expected result because this theory is valid in intermediate fields. For the MR fluid studied here, this regime is found to be less than 0.3T. To obtain a quantitative measurement of the collapse, data taken for the lowest three orders of magnitude in \( M_n \) is statistically analyzed for its mean and normalized standard deviation. The statistical analysis is restricted to this subset of data points because they correspond to a flat portion of the rheological response. The normalized standard deviation was found to be 50.1%.
Figure 4.8: Rheological data non-dimensionalized by (a) sub-quadratic applied field, $h = 3/2$ in Equation (4.2) (Ginder et al., 1996) (b) quadratic magnetization, $m = 2$ (Klingenberg et al., 2007) definition of Mason number. At high fields ($B > 0.3T$) both scalings deviate from measurements; however, the latter definition is better over the whole range of fields. The quality of the collapse of the data is measured by computing normalized standard deviation in the flat
regions of the curves. The analytical relation for yield stress given by Ginder and coworkers (1996) give a significantly better estimate of this property as \( \frac{\tau}{\tau_{mag}} \approx 1 \) for \( Mn \ll 1 \).

In Figure 4.8(b), the experimental data is non-dimensionalized by using the scaling proposed by Klingenberg and coworkers (2007). In their study a definition of \( Mn \) was provided. Similar to the procedure above Equations (1.14) and (4.5) are used to calculate the corresponding \( \tau_{mag} \) for this scaling. These two parameters are:

\[
\tau_{mag}^{*} = \frac{4}{3} \phi \mu_0 \left( M_r \right)^2
\]

\[
Mn^{*} = \frac{9 \eta_0 \dot{y}}{2 \mu_0 \left( M_r \right)^2}
\]

\( \left( M_r \right) \) is the average magnetization of particulate phase in MR fluid and can be related to fluid magnetization by Equation (1.13). The magnetization data for the fluid used in this study was available from the manufacturer (Lord, 2008). Similar to the previous scaling this definition of \( Mn \) collapses the data in low fields but the collapse is poor in high fields. However for the range of fields studied the collapse is better than the previously discussed scaling law as seen from the normalized standard deviation of 23.5%, as compared to 50.1%.
Figure 4.9: Graphical representation of the extensive definition of Mason number. Reciprocal Mason number is plotted to illustrate the similarity of the curve to the numerical results presented in Figure 4.5. The dependence in magnetization shifts between quadratic and subquadratic at a critical magnetization $M_c$. The equations describing this curve are defined such that the transition at $M_c$ is continuous and the $Mn(\langle M_p \rangle = M_s)$ corresponds to the value calculated by using the exact relation of magnetic force at saturation defined in Equation (1.12).

As determined in Section 3, the magnetic attractive force between MR fluid particles scales quadratically with average magnetization at low fields. This result agrees with that determined by non-dimensionalizing experimental data by using the scaling provided by Klingenberg and coworkers. It is further evident that above a critical magnetization this scaling no longer applies. From the numerical analysis, we see that this scaling is subquadratic and approximately results in $m = 3/2$ (according to Equation (4.2)) above the critical magnetization. Based on these observations, we propose the following definition of $Mn$ applicable to an extensive magnetic field range:
\[
Mn^{ex} = \begin{cases} 
\frac{36\eta_0\dot{\gamma}}{\mu_0\langle M_p \rangle^2} \left( \frac{M_C}{M_s} \right)^{1/2} & \langle M_p \rangle < M_C \\
\frac{36\eta_0\dot{\gamma}}{\mu_0\langle M_p \rangle^{3/2}} \left( \frac{1}{M_s} \right)^{1/2} & \langle M_p \rangle > M_C 
\end{cases}
(4.10)
\]

which is also illustrated by a plot of \(Mn^{-1}\) in Figure 4.9. As seen from the definition, there are two regions of different power law dependence on magnetization separated by a critical magnetization, \(M_C\). As discussed in Section 2, this critical magnetization is dependent on the geometry of interaction between the MR fluid particles. Therefore, with current methods, the only reliable way to determine this parameter is by rheological measurements. The constants in parentheses in the equations are included in the definition in order to: (1) have a continuous transition between the two regions and (2) provide a good order of magnitude estimate of the whole curve. As discussed in Section 1.1, an analytical expression of the magnetic force can be attained for the case of uniform magnetization. This condition is satisfied for a magnetically saturated particle. Therefore, the best estimate for \(Mn\) corresponds to this magnetization where \(F_{mag} = \pi\mu_0 a^2 M_s^2 / 6\) and as a result \(Mn = 36\eta_0\dot{\gamma} / \mu_0 M_s^2\). This corresponds to the \(Mn\left(\langle M_p \rangle = M_s\right)\) for Equation (4.10).
Magnetorheological suspension

\{B(H)\}_{pos} \rightarrow \text{Magnetic constitutive relation – Equation (2)} \rightarrow \text{Curve fitting to F-K Relation – Equation (7)} \rightarrow \{M_s\}_{pos} \rightarrow \text{Particle phase} \rightarrow M_s

\{M(B)\}_{pos} \rightarrow \text{Equation (2)}

\text{Figure 4.10: Flow chart for obtaining the required magnetic parameters for scaling from experimental measurements on MR fluids. Magnetometer measurements can be defined as a relation between any two of the magnetic parameters } B, H, \text{ and } M. \text{ The parameters can be changed using the magnetic constitutive relation. Saturation magnetization of the fluid can be determined by fitting the Frohlich-Kennelly relation to these measurements. The particle phase parameters used in the equations can be attained by using the measurements made on the bulk fluid and using Equation (1.13).}

The corresponding magnetic stress scaling is:

\[ \tau_{mag}^{ex} \begin{cases} \frac{\mu_0 \phi (M_p)^2}{6} \left( \frac{M_s}{M_c} \right)^{1/2} & \langle M_p \rangle < M_C \\ \frac{\mu_0 \phi (M_p)^{3/2}}{6} M_s^{1/2} & \langle M_p \rangle > M_C \end{cases} \] (4.11)

It is illustrative to go through the steps needed to non-dimensionalize the experimental data points according to this scaling. In the original form the data is available in shear stress versus shear rate for each level of magnetic flux density (B). The parameters needed for non-dimensionalization are \( \tau_{mag}^{ex} \) and \( Mn^{ex} \) for each data point. To calculate these \( \langle M_p (B) \rangle \), \( M_s \) and \( M_C \) need to be determined from the experimental data of magnetization of the bulk fluid. A flow chart of the parameter manipulations required is illustrated in Figure 4.10. The magnetization measurements on the bulk fluid can be converted to that of the particle phase by the use of
Equation (1.13). These measurements of magnetization $M$ are often made as a function of applied field, $H$; however, it is straightforward to convert either one of these parameters using the magnetic constitutive relation $B = \mu_0 (H + M)$. By the use of these two relations, $\langle M_p (B) \rangle$ can be obtained. Saturation magnetization, $M_s$, can be determined by curve fitting the Frohlich-Kennelly relation given in equation (4.1) to the magnetization data. If sufficiently high fields are employed in these measurements, the saturation magnetization can also be taken as the highest value measured. In this case these two methods should result in a very similar value. The final parameter needed is critical magnetization, $M_c$. This parameter, in essence, determines at what field level the rheological measurements shift to a sub-quadratic dependence on magnetization from quadratic dependence. Although it can be chosen as one of the discrete particle magnetizations corresponding to a magnetic field, a better collapse of data can be obtained if it is left as a free fitting parameter and curve fitted to all the measurements. For the measurements presented here, it was found to be $5.0 \times 10^5 \text{ A/m}$, which corresponds to a particle magnetization corresponding to a value of magnetic flux density between $0.2T$ and $0.3T$.

Table 4.1: Rheological model fit for non-dimensionalized steady shear data. As it was found for the dimensional case, Casson and Herschel-Bulkley models provide better fits as compared to the Bingham model.

<table>
<thead>
<tr>
<th></th>
<th>$\tilde{\tau}_y$</th>
<th>$k$</th>
<th>$n$</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bingham</td>
<td>0.2462</td>
<td>0.7089</td>
<td></td>
<td>0.8825</td>
</tr>
<tr>
<td>Herschel-Bulkley</td>
<td>0.2374</td>
<td>0.6444</td>
<td>0.605</td>
<td>0.9727</td>
</tr>
<tr>
<td>Casson</td>
<td>0.2358</td>
<td>0.2075</td>
<td></td>
<td>0.9755</td>
</tr>
</tbody>
</table>

Experimental data non-dimensionalized by the new definition is presented in Figure 3.4. The collapse of the data onto a master curve is significantly better than the previously described
scaling laws (5.4% normalized standard deviation in the flat portion of the curve as compared to 50.1% and 23.5%). All the data points in the non-dimensional form were fitted to Bingham, Herschel-Bulkley and Casson models in the form

\[
\left( \frac{\tau}{\tau_{mag}} \right)^m = \left( \tilde{\tau}_y \right)^m + \left( \tilde{k} \cdot Mn^n \right)^m
\] (4.12)

where, \( \tilde{\tau}_y \) is the dimensionless yield stress. \( \tilde{k} \) is a model parameter; for the Bingham model it is also a dimensionless plastic viscosity. The power index \( m \) is equal to unity for Bingham and Herschel-Bulkley models and to 1/2 for the Casson model. The power index \( n \) is a model parameter for Herschel-Bulkley model and equal to unity for Bingham and Casson models. The results and the quality of the curve fits are presented in Table 4.1 and also illustrated in Figure 4.11.

The new scaling law was also evaluated for MR fluid measurements made at elevated temperatures (130°C) as presented in Figure 4.12. \( Mn^{ex} \) and \( \tau_{mag} \) were calculated using the magnetization data of the fluid for ambient conditions. The non-dimensionalized data (Figure 4.12b) are collapsed very well onto a master curve. To quantify the collapse, standard deviation of the data in the flat region of the curve is calculated to be 1.1%.
Figure 4.11: Rheological data non-dimensionalized by extensive definition of Mason number. The collapse of the data is superior compared to the definitions based on previously described scaling laws. These data points were fitted with non-dimensional Bingham, Herschel-Bulkley and Casson models. The plots for the latter two fits are virtually overlapping in this range and are superior to the fit by the Bingham model.
Figure 4.12: Steady shear measurements of MRF-132DG at 130°C. (a) Bulk magnetorheological stress data plotted in dimensional form. As observed in the ambient temperature data, the curves are flat and dominated by yield stress. (b) Data non-dimensionalized with the new scaling law. The quality of collapse of the data to a master curve is quantified by calculating a standard deviation from a mean constant in Mason number. The standard deviation is found to be 3.7% for $Mn^{ex} \leq 3 \times 10^{-6}$.
4.4 Conclusions

A computational model for magnetizable particle interaction in an applied magnetic field was studied. The results of the study showed that the interaction between the particles is linear only in the low magnetic field regime which results in a quadratic dependence particle force on average particle magnetization. The first appreciable deviation from this linear interaction occurs at field levels that cause the contact point between the particles to magnetically saturate. These observations agree with the particle chain model of Ginder and coworkers (1996). With the present model it was further identified that if the magnetic force between the particles is quantified with respect to average particle magnetization, two regions of force scaling are observed for all applied magnetic field levels. This result was used to propose a new scaling law and a corresponding definition for a Mason number that includes two regions of power law scaling of MR stress with fluid magnetization. Using numerical models, it was demonstrated that the critical point at which the shift in force scaling occurs is dependent on the geometry of interaction between the particles, and therefore it is a particle shape dependent parameter.

Measurements made using a custom-built magnetorheology rheometer accessory was used to evaluate MR fluid rheological properties with respect to applied field. These measurements were non-dimensionalized using power law scalings that were described in the literature and the new scaling that was proposed here. The results suggest that the new scaling can be used to quantify the evolution of magnetorheological properties with a high level of accuracy.
Further work may make it possible to analytically predict the critical magnetization point at which the magnetorheological scaling shifts. As this is identified to be a geometry dependent property, it may be possible to quantify its dependence on particle shape and size distribution.
Chapter 5

Rheology and microstructural evolution in pressure-driven flow of a magnetorheological fluid with strong particle-wall interactions

5.1 Introduction

Microscopy-based studies into the microstructural evolution of field responsive fluids such as electrorheological (ER) and magnetorheological (MR) fluids have provided significant insight into the processes that result in their controllable bulk properties. The focus of these studies has been in the areas of chain deformation and breakup (Klingenberg and Zukoski, 1990) and in particle aggregation & disaggregation (Shulman et al., 1986; Fermigier and Gast, 1992; Melle et al., 2003; Dominguez-Garcia et al., 2005; Deshmukh, 2007). Quantitative understanding of the bulk rheological properties of these so-called 'smart' fluids, such as the field-dependent yield stress, viscoelastic modulus and material response time, is intimately linked to these microstructural processes.

The dynamics and extent of local aggregation phenomena in MR fluids is governed by the dimensionless ratios of the magnetic forces to the other forces acting on particles including Brownian forces (Fermigier and Gast, 1992) and hydrodynamic forces (Melle et al., 2003; Deshmukh, 2007). The relative importance of these dimensionless parameters depends on the
specific flow regime of interest. It is important to distinguish the origins of the magnetic force so that we can understand the physical mechanisms governing the dynamics of aggregation. By making the uniform magnetization approximation, the magnetic force on a spherical particle can be treated as the force on a magnetic dipole of strength \( m = VM_p \), where \( V \) is the volume of the particle and \( M_p \) is the magnetization (Zahn, 1987, Sect. 5-5-1). This approximation is exact in two limits (Klingenberg et al., 2007): (1) When the particle has reached saturation magnetization, \( M_p = M_s \), and (2) When the particle is isolated in a uniform magnetic field \( H \), for which

\[
m = VM_p = 4\pi a^3 \frac{\chi_m}{\chi_m + 3} H, \tag{5.1}
\]

where \( \chi_m \) is the magnetic susceptibility of the particle (Zahn, 1987, Sect. 5-7-2).

A magnetic force on the dipole only exists in an inhomogeneous field (for which \( \nabla H \neq 0 \)) and is determined by:

\[
f_{\text{mag}} = \mu_0 m \cdot \nabla H, \tag{5.2}
\]

where \( \mu_0 \) is the magnetic permeability of free space. The magnetic force generated between two particles placed in a homogeneous field can be viewed as arising from the distorted field lines of each magnetically-susceptible particle. This interparticle force for the case of two spherical particles in contact is (Klingenberg et al., 2007):

\[
f_{\text{int}} = \frac{\pi}{6} \mu_0 a^2 M_p^2, \tag{5.3}
\]

and varies quadratically with magnetization. The challenge is to understand how such pairwise interactions between two particles are related to the overall properties of a dense suspension of
MR-active particles which form chained structures along the field lines. Time-resolved microscopic observations under both equilibrium and flowing conditions are essential to make progress in understanding.

The evolution in microstructure of a quiescent MR suspension was studied by Fermigier and Gast (1992). In the absence of background flow, the aggregation and disaggregation phenomena was shown to be governed solely by the dimensionless ratio of the interparticle magnetic force to the Brownian force

\[ \lambda = \frac{f_{int}}{f_B} = \frac{\mu_0 a^3 M_p^2}{6k_B T}. \]  

Here, we have used the Brownian force scale, \( f_B \sim k_B T / a \) (Sharma et al., 2009) where \( k_B \) is the Boltzmann constant and \( T \) is the absolute temperature.

In the presence of hydrodynamic forces, additional dimensionless parameters are important. At the length scale of the individual particle, the Reynolds number is often small and the hydrodynamic force on the particle can be evaluated using the Stokes drag formula

\[ f_d = 6\pi \eta_0 a V, \]

where \( \eta_0 \) is the matrix viscosity. In a simple shear flow, the local velocity in this relation is \( V = \dot{\gamma} a \) where \( \dot{\gamma} \) is the shear rate. The ratio of the magnetic force to the hydrodynamic force results in a dimensionless group commonly referred to as the Mason number (Marshall et al., 1989 (for equivalent problem in ER fluids); Klingenberg et al., 2007)

\[ Mn = \frac{f_d}{f_{int}} = \frac{36\eta_0 \dot{\gamma}}{\mu_0 M_p^2}. \]  

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Studies conducted using rotating magnetic fields (Melle et al., 2003) and pressure-driven flow in microfluidic channels (Haghgooie, 2006; Deshmukh, 2007) showed that in homogeneous magnetic fields, the aggregation dynamics of magnetically-susceptible particles in the presence of hydrodynamic forces are governed by the Mason number.

If the applied field is spatially inhomogeneous, the magnetic force on the particles is caused not only by the perturbation resulting from other neighboring particles but also by the background field gradients. We can estimate this magnetophoretic force by approximating the field to be locally uniform at the length scale of the particle, and therefore inserting Equation (5.1) into Equation (5.2):

\[ f_{\text{mag}} = \mu_0 \left( \frac{4}{3} \pi \alpha^3 \right) \mathbf{M}_p \cdot \nabla \mathbf{H} = 4\pi \mu_0 \alpha^3 \beta H (\nabla H) \]  

(5.6)

where the effective susceptibility, \( \beta = \chi_m / (\chi_m + 3) \). To obtain the second equality, we made use of the vector identity \( \mathbf{H} \cdot \nabla \mathbf{H} = H \nabla H - \mathbf{H} \times (\nabla \times \mathbf{H}) \) and the magnetostatic Ampere’s Law in the absence of free current \( (\nabla \times \mathbf{H} = 0) \) (Rosensweig, 1979). Comparing this force to Stokes drag, we find another dimensionless parameter:

\[ \Psi' = \frac{f_{\text{mag}}}{f_d} = \frac{2\mu_0 \alpha^3 \beta H}{3\eta_0 V_b} \frac{\partial H}{\partial x}, \]  

(5.7)

where the background flow velocity, \( V_b \), is in the x-direction. Since we are considering the balance between the magnetophoretic force and the hydrodynamic force acting on a particle, the velocity scale \( V_b \) in this case is left in terms of the local velocity of the background flow as opposed to the shear rate that is incorporated in the definition of the Mason number. In other
words, magnetostatic forces can affect particle motion even in uniform translational flows of an MR suspension for which $\mathbf{V} = V_e \mathbf{e}_z$ if the magnetic field is inhomogeneous.

Kuzhir and coworkers (2005) studied the flow of non-magnetic particles suspended in a ferrofluid (a magnetic colloidal system) in non-uniform magnetic fields. An important dimensionless parameter identified in their two-phase flow model was the quantity $A = \left( 9 \eta_0 x_{max} \nu \right) / \left( 2 \alpha_0^2 \mu_0 M_s H_{max} x_{max} \right)$ where $H_{max}$ and $x_{max}$ are the relevant scales for magnetic field intensity and distance of field decay. This quantity is related to Equation (5.7) and in fact we can see that $A = \Psi^{-1}$ by describing the saturation magnetization in terms of the local field and effective susceptibility, $M_s = 3 \beta H$ and recognizing that the inhomogeneous gradient $(\partial H / \partial x) = (H_{max} / x_{max})$. Microstructure and aggregation phenomena in non-uniform magnetic fields from a microscopic perspective, however, have not been investigated for either suspension of non-magnetic particles in ferrofluids or for MR fluids.

In order to quantify the relative magnitude of the forces in an MR suspension, it is necessary to design and construct flow channels that allow simultaneous observation of microstructural evolution as well as macroscopic quantities such as the field-dependent yield stress or pressure drop. In the present study we describe a fabrication process that enables us to construct microfluidic channels in which we can simultaneously observe the formation in fluid flow as well as the resulting changes in the hydrodynamic resistance exerted by the MR fluid as an external magnetic field is applied.

The interaction between the individual MR fluid particles and the surfaces that retain the fluid is extremely important because it can limit the extent of momentum transfer to/from the fluid through the liquid/solid interface. Deshmukh (2007) used high-speed video microscopy to
investigate the aggregation phenomena in pressure-driven flow of MR fluids in smooth glass and poly-dimethylsiloxane (PDMS) microfluidic channels. It was found that, in the absence of a strong particle-wall interaction there is significant slip of the particles and field-induced aggregates along channel walls. The consequences of this on the macroscopic pressure drop were not measured, but any wall-slip is expected to reduce the resulting shear stress exerted by the MR fluid on the wall. Particle-wall interaction is commonly enhanced in applications and in rheometric studies by controlling the surface roughness and selecting ferromagnetic channel materials. These enhancement methods were demonstrated to increase the field-dependent shear stress exerted by MR fluids in the bulk rheological study conducted by Laun and coworkers (2011). In MR fluid applications, ferromagnetic channel walls are often utilized (Jolly et al., 1999; Herr and Wilkenfeld, 2003) and studies of chain formation and flow adjacent to ferromagnetic walls therefore has more engineering significance.

In the present study, we investigate the microstructural evolution of MR fluids in microchannels with ferromagnetic side walls. The strong interaction between the MR fluid particles and the channel walls provides a means to study aggregation and flow phenomena in these field-responsive fluids with boundary conditions replicating those realized in actuators. We also explore the flow of MR fluid through more complex channels such as contraction and expansion geometries that lead to inhomogeneous magnetic fields induced by the ferromagnetic channel walls.

5.2 Experimental methods

The rectilinear channels used in this study were manufactured by adhering two ferromagnetic sheets with a separation of $h = 1.2 \text{mm}$ between two glass microscope cover slips.
as illustrated in Figure 5.1b. In order to generate a uniform field in the flow path, additional external ferromagnetic plates were assembled on each side of the microfluidic channel. The metallic components were manufactured from sheets of low carbon steel (1008/1010 carbon steel for 0.3\textit{mm} thick pieces and CS Type B for 1\textit{mm} pieces) using hydraulic shears and deburred using sandpaper.

The benefits provided by the external plates can be understood from the two dimensional finite element model, which was implemented in Comsol Multiphysics®. The model is solved for the magnetic vector potential in the out-of plane direction, $A_z$. In the absence of free current, the governing equation reduces to a scalar Laplace’s equation ($\nabla^2 A_z = 0$). The relative magnetic permeability in the model is unity except for the ferromagnetic sheets where it is $\mu_r = 10^3$. The boundary conditions, which are imposed far from the geometrical features, are defined as constant vector potential conditions on the top and the bottom surfaces, and perpendicular field conditions on the side walls. These conditions generate a uniform applied field far from the channel features. The results of the first simulation, presented in Figure 5.2a, show that in the absence of the external plates the field uniformity is very poor with a strong gradient towards the sharp corners of the ferromagnetic channel walls. MR fluid flowing in this type of channel would experience an undesirable magnetophoretic force towards the channel walls. In the second simulation (Figure 5.2b) the external plates located above the glass cover slips are also included. In this case the field uniformity is significantly improved with the fringing fields shifted away from the microfluidic flow channel.
Figure 5.1: Experimental setup and microchannel channel construction. (a) The applied magnetic field is generated using a custom-built electromagnet. (b) Cross-sectional view of the microchannel in the flow direction. The channels used in this study are manufactured using a multilayered sandwich construction to provide a thin plane of fluid flow while maintaining the magnetic field uniformity. (c) Top view of straight flow channel. Two adapters are adhered on either end of the flow channel creating a pressure-retaining joint. The external plates are shorter in length in order to promote the seal between the microscope slides and the adapter. (d) Top view of the expansion and contraction flow channels. The side walls in the larger width section are aluminum. The channel dimensions (in mm) shown in the figure are utilized in all flow channels unless otherwise noted.

The entire assembly is designed to be placed on the stage of an inverted microscope (Nikon TE-2000S) and viewed with microscope objectives (2X 0.06NA objective) and a monochromatic CCD camera (Matrix Vision BlueFOX). The depth of field of this imaging system is 240μm and the image resolution is 3.4μm/pixel.
Figure 5.2: 2D magnetostatic finite element model for evaluation of field uniformity in the microchannels. Contours plotted for a zoomed section of the model to show the important features. (a) Without the external plates there is focusing of the field at the channel walls that would produce an undesirable magnetophoretic force on the particles. (b) The field uniformity in the microfluidic channel (shown by dashed lines) can be significantly improved with the addition of external plates.

During the flow visualization experiments, the channel is placed between the poles of a custom built electromagnet as illustrated in Figure 5.1a. The field applied by the magnet was calibrated using a Gauss probe placed in the mid-point between the poles of the magnet. The flux density at this point was found to be $B_M = 22mT$ at a DC coil potential drop $V_c = 25V$. Because of the dimensional constraints of the probe, it was not possible to conduct this measurement within the microchannels after assembly. The amplification of the field in the narrow gap, caused by the channel walls was estimated using the results of the finite element model, as presented in Figure 5.2b. The magnetic flux density in the channel is $B_c = 6B_M$. 
Figure 5.3: Image processing conducted to identify void clusters. (a) Original image captured at $Mn = 1.6 \times 10^{-3}$ in a rectilinear flow channel with the dimensions $w = 1.2mm$, $h = 0.3mm$. Flow is from bottom to top. (b) The image is binarized using thresholding and processed with a cluster identification algorithm. The clusters are illustrated with a randomized coloring scheme. (c) The image consists of a bimodal distribution of pixels as seen in the histogram. Thresholding, applied at the local minimum of the distribution as indicated, effectively distinguishes between particles and voids.

Prior to each experiment the ferromagnetic walls of the flow channel was demagnetized using a time-varying magnetic field in the form

$$B_c(t) = B_0 \cos(2\pi ft)e^{-\alpha t}. \quad (5.8)$$

Here $B_0$ is the initial amplitude of the signal which is normally set as the largest field applied since the preceding demagnetization cycle. The constants $f$ and $\alpha$ are normally selected in the ranges $1-10Hz$ and $0.3-1s^{-1}$ respectively.
The MR fluid used in the experiments was prepared by suspending 1% v/v carbonyl iron particles (BASF® CR-grade) with a mean diameter of $7\mu m$ in a mixture of PDMS fluid and 10% w/w surfactant (Gelest™ DMS-T31 and DMS-S31 respectively, both fluids with viscosity $\eta_0 = 1Pa.s$). The surfactant was incorporated in the fluid to promote the dispersion of the particles in the matrix fluid. The preparation procedure of the fluid was similar to that of Deshmukh (2007). The fluid was first mixed in an ultrasonic bath for 30 minutes followed by a cycle in a conditioning mixer (Thinky® AR-310). The cycle consisted of 2 minutes of mixing, 1 minute of degassing and 1 minute of mixing. These steps were found to repeatably prepare a well dispersed sample without the presence of air inclusions. The fluid was pumped through the channel using a syringe pump (Harvard Apparatus® PHD 4400) with a constant flow rate, $Q$. The pressure drop across the channel was measured with a differential pressure transducer (Honeywell 26PCBFA6D) with a measurement range of $34kPa$.

The resulting 2D video images were analyzed with a cluster identification algorithm in order to quantify the observations made on the aggregation phenomena. These images are a 2D projection of the 3D structure across the 0.3$mm$ height of the channel. Because of the strong particle-wall interactions, under applied fields, the particle density near the walls is relatively high, causing the particle phase to appear as the continuous phase in most of these images (see Figure 5.3). The cluster identification algorithm was used to seek clusters of voids instead of particle aggregates. Following the methodology of Deshmukh (2007), grayscale images captured from the video camera were binarized using a thresholding algorithm. The level of thresholding was selected using the minimum histogram method (Russ, 2002). We can understand the utility of this method by realizing that the original image is made up of a bimodal distribution of dark and bright pixels, as shown in Figure 5.3c. The appropriate value for thresholding is located in
between the peaks in the histogram. Exploratory calculations showed that the exact value chosen within this valley does not materially affect the results of this analysis because the number of clusters is so much smaller than the total number of pixels. The black-and-white image obtained with this method was processed with a cluster identification algorithm using the Matlab® Image Analysis Toolbox. The output of this algorithm is illustrated in Figure 5.3b.

5.3 Results

Images recorded during a typical straight channel experiment are presented in Figure 5.4. In a demagnetized channel, the flow of the MR fluid occurs without aggregation of the particles. When the field is applied, the aggregates that form can be categorized into three groups by the number of channel walls the aggregate is in contact with: 0 (free-flowing clusters), 1 or 2 (channel-spanning). The aggregates that belong to the latter group come to a complete stop within the flow channel under the influence of strong particle-wall interactions. The deformed shape of these chains arises from the local balance of hydrodynamic and magnetic dipolar force acting on the particle. The aggregates contacting only one of the channels also come to a stop at the channel wall; however, they continue to rotate under the influence of the flow until they contact another aggregate or until the magnetic torque on the chained aggregate becomes equal to the hydrodynamic torque. Finally, the aggregates that do not have any contact with the channel walls flow continuously under the influence of the hydrodynamic forces; however, many eventually merge with a wall-contacting or channel-spanning aggregate and stop flowing. The matrix phase continues to flow continuously through the channel and there is thus a relative “slip” between the particle phase and the suspending solvent. Upon removal of the magnetic field, most of the particles in the channel freely flow once more. However, the particles close to
the walls remain adhered to the interface (Figure 5.4c), forming a particle-enriched layer on both sides of the channel. These layers contribute to the residual extra pressure drop $\Delta P_{rem}$ across the channel as seen in Figure 5.5.

Figure 5.4: Aggregation and disaggregation of MR fluid flow in a ferromagnetic microchannel ($w=1.2\,mm$, $h=0.3\,mm$) during a typical experiment cycle. Flow is in the upward direction and the applied field is in the horizontal orientation. (a) Flow in a demagnetized channel without applied field. $Q=10\,\mu l/min$ corresponding to a bulk velocity of $V_b=4.6\times10^{-4}\,m/s$ and a Reynolds number $Re_w=1.4\times10^{-4}$. (b) Aggregated chain structures with $Mn=6.3\times10^{-4}$. (c) Evolution of microstructure $t=7s$ after field turned off. There is a layer of particles that are magnetically adhered to each channel wall. The dashed lines indicate the effective channel width $w_{eff}$ in this flow regime. (d) Flow after demagnetizing field has been applied.

We can think of the extra pressure drop in the channel (in addition to the original pressure drop $\Delta P_0$, prior to the application of the magnetic field) as a geometric factor, since parts of the
channel adjacent the ferromagnetic walls are effectively blocked. In a 2D rectilinear flow of an MR fluid with the channel width $w$ defined in the direction of the magnetic field and the channel height, $h$ defined in the orthogonal direction (as shown in Figure 5.1) with $w \gg h$, the additional pressure drop can be described in terms of an effective channel width

$$w_{eff}^{w \gg h} = w \left(1 + \frac{\Delta P_{rem}}{\Delta P_0} \right)^{-1}$$

(5.9)

For common MR fluid valve applications, however, the aspect ratio of the channel width and height are reversed with $w \ll h$. In this case, the flow can again be approximated as 2D, however now the effective width is

$$w_{eff}^{w \ll h} = w \left(1 + \frac{\Delta P_{rem}}{\Delta P_0} \right)^{-\frac{1}{3}}$$

(5.10)

For the measurements presented in Figure 5, with the defined pressure drops $\Delta P_0 = 11.3kPa$ and $\Delta P_{rem} = 3.1kPa$, the resulting effective width is $w_{eff}^{w \ll h} = 0.92$. This is equivalent to two particle diameters on each side of the channel for this narrow width ($w \ll h$) rectilinear flow.

When a demagnetizing field protocol is applied (Equation (5.8)), the remnant field in the channel walls is effectively eliminated and all particles and aggregates flow out of the channel under the action of hydrodynamic stresses. The pressure drop across the channel also returns to its original value at the end of the demagnetizing step.
Figure 5.5: Differential pressure measured across the flow channel during a typical experiment in a rectilinear flow channel with the dimensions $w = 0.35\, \text{mm}$, $h = 1.2\, \text{mm}$. The dashed line is an exponential decay curve fitted to the field-off region of the curve. After the removal of the magnetic field, the pressure drop does not reduce to its original value, $\Delta P_0$. This is caused by the particle-rich layer shown in Figure 5.4c. Using Equation (5.10), we can represent the new flow resistance in terms of an effective channel width $w_{\text{eff}}^* = 0.92$.

The aggregate structures formed over a wide range of Mason numbers are presented in Figure 5.6. The magnetic field is applied rapidly on a demagnetized flow channel and the images shown are recorded when the field-induced structure formation in the flow has reached a steady state. There is a general trend of increase in the scale of the aggregated structures with decreasing $Mn$, which can be quantitatively identified by the cluster detection algorithm, as presented in the plots of the void extents shown in Figure 5.7. For this concentration of MR particles we observe a maximum in the void spacing at $Mn = 1.6 \times 10^{-3}$. This local maximum arises because at this Mason number, large clusters develop from discrete chains in the fluid colliding and merging with clusters already attached to the wall. As the field strength increases
further corresponding to a lower $Mn$, the majority of the particles form channel-spanning chains and cease flowing entirely before aggregation with other chains can occur.

Figure 5.6: Evolution in aggregate microstructure of MR fluid flow. The Mason numbers in the images are (a) $\infty$ (field-off), (b) 0.16, (c) 0.018, (d) 0.0016 and (e) 0.00034. The channel dimensions are $w=1.2\,mm$, $h=0.3\,mm$.

The aggregation and flow phenomena in the contraction and expansion geometries were also studied under a low-frequency (0.1Hz) periodic applied field. The field was constantly on for the first half of the cycle followed by a demagnetizing field ($f=10\,Hz, \alpha = 0.44$ in Eq. (5.8)) for the second half. With this selection of parameters, the decay in the demagnetizing field is rapid enough that the field is effectively removed entirely for the last 40% of the cycle. A sequence of images obtained in these flow conditions are illustrated in Figure 5.8. The aggregation in the inhomogeneous field sections outside the rectilinear channel shows a preferred direction aligned with the local field. Furthermore, an increase or a decrease in the convective velocity of the particle clusters is also observed because of the magnetic field gradients in these converging/diverging flows. The magnitude of this magnetophoretic response can be evaluated
using the dimensionless parameter $\Psi'$ defined in Equation (5.7). For these flow geometry the field gradient can be approximated by $|\partial H/\partial x| = B_c/(\mu_0 w)$. The effective susceptibility, $\beta$ was taken to be unity, in accordance with the infinitely permeable particle approximation. A local reversal in flow direction of the particulate phase was also observed in the expansion channel under a high magnetic field $B_c = 0.1 mT$ and low flow rate $Q = 10 \mu l/min$ corresponding to $\Psi' = -0.12$. This reversal can be observed by following the particles located in the circled region in Figure 5.8b. When the field is turned on these particles not only form an aggregate but also flow back into the narrow channel under the influence of magnetophoretic forces.

Figure 5.7: Horizontal and vertical void extents versus $Mn^{-1}$ as identified by the cluster detection algorithm. The extents of a void in the horizontal, $x_v$ and vertical, $y_v$, directions are equal to the respective side of the smallest upright rectangle that can completely cover the void. In the plots the mean (diamond symbols), standard deviation (red horizontal bars closer to the mean) and the maximum & minimum (blue bars) of the extents are plotted. The maximum void size is achieved for $Mn = 1.6 \times 10^{-3}$.
5.4 Conclusions and discussion

We have described a unique microfluidic channel setup with ferromagnetic walls that enables direct microscopic imaging of the flow of an MR fluid with strong particle-wall interaction. The forces caused by this strong particle-wall interaction were demonstrated to be large enough to effectively eliminate the local slip of the particulate phase against the channel walls. Under these boundary conditions the aggregation phenomena in rectilinear flow channels were studied by varying the Mason number.

Under the influence of an externally-applied magnetic field, the particles in the fluid form anisotropic aggregates with an average cluster size that varies as a function of the Mason number. At low Mason numbers the aggregate size increases with this parameter; however the aggregate size reaches a maximum at a critical Mason number then shows a decreasing trend. This was linked to a longer time-scale convective aggregation phenomena, which occurs as free-flowing aggregates collide and merge with aggregates that are already attached to a wall. At higher fields (lower $Mn$), this aggregation process loses its importance, as more of the particles immediately form channel spanning aggregates with the application of the magnetic field, stop flowing, and therefore stop aggregating.
Figure 5.8: Contraction and expansion flow of MR fluid (width ratio, $w_1/w_2 = 3$). Square wave magnetic field with a rapid demagnetization cycle field-off state. The rectilinear section dimensions are $w_2 = 1.2\, mm$, $h = 0.3\, mm$. The field is turned on at time $t = 0$. (a) Expansion flow at $Mn = 3.8 \times 10^{-3}$, $\Psi' = -0.011$. (b) Expansion flow at $Mn = 3.4 \times 10^{-4}$, $\Psi' = -0.12$. (c) Contraction flow at $Mn = 3.2 \times 10^{-3}$, $\Psi' = 0.012$. The magnetophoretic force caused by the field gradient in the expansion flow channel at lower levels of $\Psi' \lesssim -0.1$, as observed in (b), is large enough to reverse the flow of particles clusters back into the narrow channel. This can be seen by following the cluster of particles indicated with the red ellipse.
The magnetic field gradient in the entry and exit regions of the converging/diverging flow channels generate additional magnetophoretic forces on the particles that can be significant when compared with the hydrodynamic force. The relative importance of the magnetophoretic force in relation to the hydrodynamic force can be determined by the dimensionless parameter $\Psi'$ given in Equation (5.7). For flows with $\Psi' \leq -0.1$, the magnetophoretic force can drive secondary flows. With the combination of the magnetophoretic force in the entry & exit regions and the no-slip condition on the particulate phase, a slow increase in particle volume fraction in the converging/diverging channels was observed in experiments with large values of this dimensionless magnetophoretic parameter.

By directly imaging the steady flow of an MR suspension in an inhomogeneous magnetic field it is clear that a simple isotropic continuum model such as a Bingham or Casson model with a field-dependent yield stress $\tau_y(B)$ will be insufficient for describing the dynamics of MR suspensions in real devices. An alternate approach that we are currently exploring is to self-consistently incorporate the anisotropic and field-dependent particle level stresses in a two-fluid suspension balance model of the type considered by Nott and Brady (1994), and Fan and coworkers (2000).

An additional benefit in utilizing these types of models is their treatment of the multi-body effects in the evaluation of the hydrodynamic and magnetic forces in a suspension. For defining the magnetophoretic parameter $\Psi'$ we analyzed the micromechanics of isolated particles in order to obtain analytical expressions for the forces acting on an individual magnetizable particle in an MR suspension. In doing so we neglected multi-body effects, which is consistent with the approach used to develop other dimensionless parameters such as $Mn$ and $\lambda$. Using a continuum description of the many-body interactions and fitting the resulting predictions to
experimental observations, it may be possible to describe the forces in the MR suspension with a higher degree of accuracy which can allow quantitative analytical and computational studies of the complex two-phase flows in inhomogeneous fields.
Chapter 6

Transport of MR fluids under spatially-inhomogeneous magnetic and flow fields – phenomenology and continuum modeling

An important goal of studying the rheology of MR fluids is to be able to develop constitutive models that can be used in computational analysis of fluid flow, mainly by engineers in the development of devices. In many of these devices considered for mechanical engineering applications (e.g. dampers, clutches), a large part of the magnetically-controlled “active” flow channel presents a parallel flow problem in which the magnetic field is orthogonal to the flow streamlines. The great majority of the experimental literature in MR fluids also restrict their attention to this limited flow problem (see the recent review by de Vincente et al. (2011) and the survey of magnetorheology devices in Table 1.1), presumably motivated by the engineering relevance.

Although a large portion of the active flow channel is in fully-developed parallel flow with orthogonal shearing and magnetic fields, there are parts of the flow that deviate from this in virtually all devices (e.g. contraction and expansion geometries in pressure-driven flow). Furthermore, the fluid dynamics of the two-phase flow in these regimes, may also impact the response of the fluid in the straight channels. Motivated primarily by the potential utility of
gaining further understanding into these fluid mechanics problems, in this chapter, we will investigate the flow of MR fluids under spatially non-uniform magnetic and deformation fields.

The main goals of this chapter are to: 1) address the complexities associated with the transport of MR fluids under non-uniform magnetic and flow fields, 2) develop a rheological model capturing the measured and observed behavior in these flow conditions and 3) study the prototypical flow problem of contraction and expansion flow by performing numerical simulations that incorporate the new constitutive equation.

In Section 6.1, existing rheological models for magnetorheological response are surveyed. In Section 6.2, bulk rheological measurements conducted in non-uniform flow and magnetic fields are presented. These measurements were obtained with the newly developed annular slit-flow magnetorheometer, which was described in Chapter 2. The bulk measurements are also contrasted with the uniform field conditions obtained with the torsional shear rheometer accessory which were presented in Chapters 3 and 4. In Section 6.3, the governing equations of the continuum model are derived. In Section 6.4, the model is implemented numerically using the finite element technique for the limit of zero particle phase slip. This restriction is relaxed in Section 6.5 and a model qualitatively describing the particle volume fraction enhancements in a spatially resolved manner is presented. Finally, in Section 6.6, conclusions from the study are presented.

6.1 Introduction and background

With the focus of the chapter on flow of MR fluids in non-uniform and non-orthogonal fields, in this section we will review the experimental studies and rheological models in the literature as they relate to these flow conditions. The existing constitutive models for predicting the
magnetorheological response can be divided into three categories: phenomenological models of bulk rheology, microscopic models, and macroscopic continuum models. In the first three subsections, these constitutive models are summarized, and their relative strengths and weaknesses are evaluated. In the final subsection, experimental studies in non-uniform and non-orthogonal fields are discussed.

6.1.1 Phenomenological bulk rheology models

Phenomenological models are commonly used in rheological studies of MR fluids because of their well-established mathematical framework and the straightforward methodology of incorporating the measurements into solutions of fluid dynamics problems. The MR fluids are experimentally characterized by the simultaneous application of magnetic field and either a constant shear rate or constant shear stress. The data from these experiments are then fitted to predictions of a rheological equation of state or “constitutive model”. Although the Bingham plastic model is commonly used (Rankin et al., 1998), recent studies have shown superior rheological fits with Herschel-Bulkley (Lee and Wereley, 1999) and Casson (Gabriel and Laun, 2009) models.

An important limitation of the bulk rheological models proposed thus far is the requirements placed on flow geometry and field directionality. The general tensorial form of the models discussed is exclusively for isotropic fluids. By contrast, the microstructure of MR fluids is anisotropic when subjected to an external magnetic field (as seen in the microfluidic observations presented in Chapter 5), as the magnetizable particles tend to align in the direction of the field and form structures that have a preferred directionality. The experimental measurements are often made by imposing a fluid flow field in which the velocity and vorticity
are orthogonal to each other and both are orthogonal to the magnetic field. As a result, MR fluid flow simulations, which make use of these rheological models, are restricted to flow situations which satisfy these orthogonality requirements.

Another feature ignored in the phenomenological models is the two-phase nature of MR fluids. Depending on the loading conditions of the suspension, the flow of the base fluid may differ from the flow of the particles, which can lead to an inhomogeneity in fluid properties. An example of this type of flow with rheologically-observable consequences occurs in torsional magnetorheometers when there is an undesirable radial gradient in the magnetic field imposed on the fluid. The magnetophoretic force generated by the field gradient leads to a radial migration of particles that can be observed through time-resolved measurements of the torque required to maintain constant rotational speed (Laun et al., 2008b).

For other classes of materials that show both anisotropy and yield stress, more complex models have been proposed. In a classic paper by Hill (1948) a six parameter yield criterion is described that was motivated by addressing the anisotropy observed in cold-worked polycrystalline metals. In an alternate approach, Spencer (1992) proposed an anisotropic yield criterion for fiber reinforced composites using tensor invariants. Robinson et al. (2002) used the yield criterion of Spencer and formulated a constitutive model for a transversely isotropic Bingham fluid. Their model was motivated by fibrous or disk-like particle-filled polymer melts in which the material has a preferred direction and axial symmetry around this direction. These advancements in anisotropic material models have not been paralleled for MR fluids which also show a preferred axis aligned in the direction of the magnetic field.
6.1.2 Microscopic models

Microscopic models of MR fluids forego the continuum description of the fluid at macroscopic length scales, and treat the suspended particles as discrete rigid bodies. The hydrodynamic and magnetic forces on an individual particle resulting from the bulk field and interactions with other particles are often captured with models of these forces that are functions of spatial and physical parameters. As the dimensions of the particles can be many orders of magnitude smaller than a characteristic flow dimensions, a representative subsection of a fluid flow and/or magnetostatic problem is often modeled to maintain manageable computational requirements. Even with restricted domain solutions, the coupled fluid flow and magnetostatic problem may require an extensive computational effort because of the large number of model variables and fine temporal step size required for a converged solution.

The point dipole approximation is often used in modeling magnetic interactions between two particles (Klingenberg et al., 1991a; Furst and Gast, 2000; Tao, 2001). These far-field models are developed by approximating the magnetic interactions of particles with those of dipoles located in the center of the particle. Although this method leads to an efficient solution to the complex multi-body problem, its accuracy in the prediction of interparticle magnetic force limited (see Jones et al. (1989) for the electrostatic equivalent problem) since the particle interactions in an MR fluid is inherently a near-field problem. The classic magnetostatics problem of a linearly magnetizable sphere in uniform applied field (with corresponding analogs in electrostatics and potential fluid flow) has a solution that can be described as the superposition of a uniform applied field and a dipolar field induced by the sphere. It is important to note that this solution is exact even in the near-field. However, at the microscopic scale, the magnetic field is typically not spatially-uniform generally because of the neighboring particles. This results in a
non-uniform field distribution within the particle and the external field generated by this particle is no longer dipolar in character and also includes contributions from higher order poles.

Several studies have included the effects of higher order poles to address shortcomings of the point dipole approximation for both the magnetorheological (Bossis et al., 1997; Ly et al., 1999; Wang, 2007) and the electrorheological (Klingenberg et al., 1991b; Bonnecaze and Brady, 1992; Clercx and Bossis, 1993) fluid modeling problems. Multipolar expansion presupposes the principle of linear superposition, and thus the linearity of the differential equations governing the problem. The wide linear extents of electrical properties of ER fluids allow this to be a powerful method. For magnetorheological fluids, however, the nonlinear nature of the ferromagnetic phenomenon poses serious limitations to the applicability of multipole expansion methods. The magnetic susceptibility of a material is a non-linear, field-dependent function at commonly observed field levels. Furthermore, the geometry of the interface between spheres focuses the field lines towards contact points as demonstrated in the finite element model in Chapter 4 (see Figure 4.5d). The resulting high magnetization levels that arise from this focusing cause non-linear effects to become observable in bulk rheological properties at lower field levels.

The limitations of obtaining an accurate model of interparticular force with point dipole and multipole approximations have motivated many researchers to follow a finite element modeling approach (Davis, 1992b; Ginder et al., 1996; Klingenberg et al., 2007) and motivated the work presented in Chapter 4. Although very powerful in evaluating exact solution of the non-linear magnetostatic problems faced in MR fluids, this method has significant computational demands. The particles within the fluid and the base fluid need to be meshed for every geometrical situation and a set of non-linear equations for the flow and magnetic field variables need to be solved simultaneously. Because of these computational challenges, application of the
finite element method in a large-scale fluid flow problem does not appear to be a realistic goal within the constraints of current computational capability.

6.1.3 Macroscopic models

In macroscopic models of magnetorheological fluids, a continuum approximation for the fluid is made in terms of both the magnetic and the rheological properties (Rosensweig, 1995; Bossis et al., 1997; Shkel and Klingenberg, 1999; von Pfeil et al., 2002; Kuzhir et al., 2003a; Lopez-Lopez et al., 2010). In contrast to phenomenological models discussed in Section 6.1.1, however, the coupled nature of the problem is recognized and used to self-consistently obtain rheological properties from magnetostatic ones. Although the material model is defined macroscopically, it is important to note that the field-induced microscopic fluid structure is ultimately responsible for the key changes in the MR fluid rheology with the applied field. Therefore, macroscopic models are not complete until a microstructural description is included which incorporates microscopic or phenomenological aspects.

Macroscopic models of magnetizable and polarizable particles have also been used in systems other than MR fluids. Some examples include fluidized magnetizable beds (Rosensweig et al., 1982), electrofluidized beds (Zahn and Rhee, 1984) and ferrofluids (Rosensweig, 1985; Zahn, 1989). Although these studies relate to material systems with different rheological properties, they are governed by underlying physical mechanisms similar to those that control the rheology of MR fluids. Rosensweig (1995) developed a macroscopic model for MR and ER fluids in which, the unsymmetric stress tensor caused by the misaligned applied field and magnetization vectors were used to relate these electromagnetic quantities to the macroscopic fluid yield stress. The study showed that by making approximations about the fluid
microstructure (such as laminar or cylindrically arranged particles phase) it is possible to obtain closed-form analytical relations for the yield stress in terms of magnetic properties of the fluid.

Shkel and Klingenberg (1999) proposed a model that captures electrorheological and electrostrictive behavior in ER fluids. They start the derivation of the model from the expression for stress field in anisotropic media derived by Landau and Lifshitz (1960). The model relates material parameters in the electrical domain to an electrical stress for linearly polarizable particles. The electrical domain parameters were calculated for an isolated chain of polarizable particles. In a later study (Shkel and Klingenberg, 2001), the authors described similar properties for magnetizable chains including non-linear material parameters (e.g. magnetic field-dependent susceptibility) in which the non-uniform spatial distribution of magnetization within the particles was neglected.

Von Pfeil and coworkers (2002; 2003; 2004) developed a macroscopic fluid model that captures the pattern formation observed in ER fluids. In these studies, the researchers have developed a two-phase continuum model for ER fluids based on previous models of dense suspensions (Nott and Brady, 1994; Morris and Brady, 1998; Morris and Boulay, 1999). The model was demonstrated to capture many of the key features observed in ER fluid microstructure, such as the formation of columns in quiescent suspensions, and anisotropic structures such as planes of aligned particles in sheared suspensions; however, predictions of the bulk rheological behavior resulting from the structures were not made. As stated by the authors, the model in its current form lead to a prediction of zero yield stress (von Pfeil et al., 2003). In Section 6.3, the reasoning behind this limitation is discussed and a similar model is extended to capture the yield stress behavior of MR fluids. In the newly-developed model the anisotropic yield stress observed in MR fluids is linked to the magnetic body couple, which is generated on
the particle phase because of its microstructure. The new model predicts the formation of chains in a quiescent suspension, similar to the model of von Pfeil and coworkers (2003), and extends this model to capture bulk rheological behavior of MR fluids.

A two-phase model was also developed for a suspension of non-magnetic particles in a ferrofluid (Kuzhir et al., 2005). The model was applied to channel flow of this suspension through a non-uniform magnetic field for an application in which a magnetically-induced channel blockage is desirable.

### 6.1.4 Experimental measurements of MR fluid rheology in non-uniform and non-orthogonal flow and magnetic fields

Flow of MR fluids in uniform channels with non-orthogonal fields was studied by Kuzhir and coworkers (2003a; 2003b). In these studies, the researchers performed experimental measurements and developed a macroscopic model for this flow configuration and applied it to pressure-driven rectilinear flow in planar and cylindrical geometries. In their experimental work, the magnetically-induced resistance to flow in the cylindrical geometry was measured as a function of the angle between the flow streamlines and the magnetic field. These measurements are utilized in Section 6.3 to compare to those of the newly developed rheological model.

The axisymmetric contraction flow of MR fluids was studied by Kuzhir and coworkers (Kuzhir et al., 2009). In this study, the flow resistance of an MR fluid was measured for magnetic fields applied along the axis of the channel and also in a transverse direction. The former orientation was shown to exhibit a greater flow resistance. A macroscopic model was also developed to capture this dependence by analyzing the strength of chains aligned with the axis of
the flow channel and linking the fracture strength of the chains to the magnetic and flow field variables.

Finally, in a recent conference presentation, Goncalves and coworkers (2010) demonstrated an increase in apparent yield stress of MR fluids in valves. In an MR fluid valve, a material element of the fluid always experiences an inhomogeneity in the magnetic field as it flows into and out of the valve. Goncalves and coworkers show that in certain conditions, the resistance to flow of an MR fluid in a non-uniform field may differ from the one determined from uniform field measurements. The researchers also found that this difference was greater for lower volume fraction fluids.

6.2 Bulk measurements of magnetorheology under spatially inhomogeneous magnetic and flow fields

The microfluidic study described in Chapter 5 provided an insight into the aggregation and microstructure evolution of an MR fluid in a pressure-driven flow through a non-uniform magnetic field and an inhomogeneous deformation. The results of this study showed that, the flow of the particle phase is governed by the balance between the hydrodynamic forces and the magnetophoretic forces. It was further observed that certain orientations of the field gradients in these flow channels lead to a local increase in volume fraction which may be a positive attribute for an MR fluid device in which the largest possible change between the field-off and field-on conditions is desired.

To evaluate these observations on the macro-scale and to study MR fluid rheology in non-uniform fields, an experimental investigation was undertaken using the annular slit-flow
magnetorheometer. This device embodies a flow path consisting of straight annular channel plus contraction and expansion sections. The detailed description and the design of this device were presented in Section 2.2.

For the experiments, the housing is filled with 45mL of MR fluid, which allows for a maximum stroke up to 13mm to ensure the displaced fluid remains below the level of the plastic bearing when the piston is stroked down. The latter limit is important for maintaining a low friction sliding surface between the bearing and the housing.

The magnetic field is reported in terms of the maximum field in the throat of the geometry,

$$H_{\text{max}} = \frac{B_{\text{max}}^{\text{MRF}}(V_c, \mu_r^{\text{MRF}})}{\mu_0 \mu_r^{\text{MRF}}}$$

where $\mu_0$ is the permeability of free space and $\mu_r^{\text{MRF}}$ is the relative permeability of the MR fluid. The maximum flux density, $B_{\text{max}}^{\text{MRF}}$, which is a function of the coil voltage, $V_c$, and the permeability of the MR fluid, is calculated using Equation (2.9), and can be varied up to 120mT for this device.

The device was demagnetized every time the desired value of the magnetic field was lower than a previously-applied magnetic field. This demagnetization procedure is imposed by first lowering the piston to the bottommost position and applying an AC magnetic field at 10Hz frequency and a peak-to-peak amplitude that is more than twice the largest DC field used since the previous demagnetization. With the AC field on, the piston is moved upward to the topmost position used in the experiment (typically 11mm above bottommost position). Keeping the piston
stationary, the amplitude of the AC signal is slowly reduced to zero, over an approximately 30-second period. Following this procedure ensures that all the material points in the device experience a demagnetizing field of sufficient amplitude.

Figure 6.1: Constant magnetic field, constant velocity experiments with the annular slit-flow magnetorheometer. The measurements are taken with (a) HYP1 geometry at $H_{\text{max}} = 6.1\, kA/m$, (b) HYP2 geometry at $H_{\text{max}} = 3.4\, kA/m$. For the field-off plots the piston velocity, $v_p = 0.1\, mm/s$ whereas in the field-on plots, $0.1\, mm/s \leq v_p \leq 10\, mm/s$. For other results presented in this section, the background frictional component is corrected for by subtracting the field-off data from each experiment. The experiments in lower fields (and field-off) often start with a negative value since the friction force at this point still points upward caused by the return stroke.

All the experiments described in this subsection were conducted with a commercial MR fluid (MRF-132DG manufactured by Lord Corporation®). Prior to the experiments the fluid
samples were mixed for more than 8 hours on a benchtop roller mixer (Wheaton®, Model 348921UL) at approximately 10rpm.

Before each experiment, the coil voltage (thus indirectly the magnetic field, which was demonstrated to have a linear relationship with the voltage in Figure 2.12) was adjusted and fixed at a particular value. A constant-velocity stroke was imposed by the linear actuator and the resisting force was recorded. Data obtained from two sets of experiments are presented in Figure 6.1. We should also note that the return stroke of the piston is always made with the field turned off to avoid cavitation.

Figure 6.2: Measurements of resistance force as a function of piston velocity for (a) HYP1 and (b) HYP2 geometries. The magnetic field is reported in terms of $H$ (Equation (6.1)). The resistance force shows a finite and field-dependent component of the resistance force as the deformation rate goes to zero. We refer to this component as the yield force.

It can be seen that, although the velocity is constant, the force measured varies with displacement. This can be attributed to contact friction, squeeze-flow between the piston and the housing bottom surface and hydrostatic pressure caused by the rising fluid level in the housing.
The hydrostatic component is estimated to be $0.07N/mm$ and thus corresponds to a maximum value of $0.7N$ for a stroke of $10mm$. By using the lubrication approximation, the squeeze flow contribution at the bottom of the tube is found to be $0.7N$ at the bottommost position of the piston ($1mm$ above contact) at the highest velocity of $10mm/s$, taking the Newtonian (field-off) viscosity of the MR fluid. Since these values are small compared to the observed deviation from the mean value, the systematic contribution to the force is attributed to the contact friction. In fact, we can see that this component of friction can largely be corrected for by subtracting the field-off data at the lowest piston velocity, $v_p = 0.1mm/s$ (red curves) from all of the other measurements. Following this procedure, an average force for each magnetic field and piston velocity is calculated by correcting for contact friction and averaging the rest of the data points in a range $1.5mm < x_p < 7mm$ where the data shows no signs of end effects. These averaged forces are presented in Figure 6.2 as a function of the imposed piston speed.

The fluid response measured with this instrument is qualitatively similar to the measurements made under uniform magnetic field with the torsional shear magnetorheometry accessory (Figure 3.2b). Here we can see that as the deformation rate, measured through the piston velocity, goes to zero, we recover a field-dependent yield term. We refer to this resistance force as the yield force, $F_y$. This term is larger for the HYP2 geometry because of its narrow straight flow section.
Figure 6.3: Yield component of the fluid resistance, as measured by yield force or yield stress, versus magnetic field strength. Using the torsional shear rheometer, we obtain a subquadratic scaling law ($\tau_y \sim H^{1.5}$) in its viscometric flow (parallel-plate) and uniform magnetic field. By contrast, yield force measurements in the annular slit-flow magnetorheometer exhibit a superquadratic scaling with the field ($F_y \sim H^{2.3}$) for both HYP1 and HYP2 geometries. The magnetic field for these geometries is reported in terms of the maximum field strength, $H_{\text{max}}$ (Equation (6.1)).

The measurements of fluid resistance may be qualitatively similar to the uniform field case; however, when we quantify how the yield component of the resistance scales with the magnetic field, the results differ significantly. In Figure 6.3, the yield stress measurements in the torsional shear rheometer and yield force measurements from the annular slit-flow magnetorheometer are plotted as a function of the applied field. The yield force data is an average of two experiments for each geometry, conducted with different fluid samples. The torsional rheometer measurements are conducted by subjecting the fluid to a viscometric flow (in
a parallel-plate geometry) and a uniform magnetic field. In this case, we obtain a subquadratic scaling law between the yield stress and the applied field ($\tau \sim H^{1.5}$). This scaling law is appropriate to the field range of interest in this study. A detailed discussion on this subject is provided in Chapter 4. The results from the annular slit-flow magnetorheometer, on the other hand, obey a superquadratic scaling law ($F \sim H^{2.3}$) for both HYP1 and HYP2 geometries. This suggests that the magnetic field has a bigger contribution to the fluid resistance than just changing the yield stress.

A further evidence of this behavior can be seen in the experiments presented in Figure 6.4. In this figure, results from a set of consecutive experiments are plotted, in which the applied coil voltage ($V_c = 1.19V$ corresponding to $H_{\text{max}} = 1.7kA/m$) and the piston speed ($v_p = 0.1mm/s$) were kept constant during the measurements. We recall that during the return stroke the applied voltage was turned off to prevent cavitation of the fluid. Prior to the second experiment a coil voltage of $V_c = 6.25V$ ($H_{\text{max}} = 8.6kA/m$) was applied for approximately 30s and then reduced to $V_c = 1.19V$ before commencing the experiment. Experiments #3-#6 were repetitions of the first experiment, but this time without applying a higher field beforehand.
Figure 6.4: (a) Resistance force as a function of piston displacement for six consecutive experiments with a constant coil voltage, $V_c = 1.19V$, and piston velocity, $v_p = 0.1mm/s$. The background friction is corrected for on all plots by subtracting the measurements taken at $V_c = 0$, $v_p = 0.1mm/s$. Prior to the second experiment a coil voltage of $V_c = 6.25V$ is applied for approximately 30s and reduced to the experiment value. (b) Time-resolved force on the piston over the complete stroke averaged for each test. The experiments show a permanent shift in force, $\Delta F_1$, and a shift which is lost after mixing that occurs during the field-off return stroke, $\Delta F_2$. These shifts are attributed to remnant magnetization and increasing particle volume fraction at the throat, respectively.

We can see that the application of the high magnetic field prior to the second experiment generates a very significant shift in the measured resistance force ($\Delta F_1 + \Delta F_2$ as illustrated in Figure 6.4). We can attribute this shift to two factors: 1) a change in particle volume fraction within the narrow gap and 2) remnant (permanent) magnetization of the device. We can see that a part of this shift ($\Delta F_2$) is lost during the return stroke of Experiment #2, while the remaining shift is retained ($\Delta F_1$) for all the following experiments. Therefore we can attribute the
component invariant to fluid mixing, $\Delta F_1$, to remnant magnetization and $\Delta F_2$, to a shift in particle volume fraction near the throat of the geometry.

Through the measurements presented in Figure 6.3 and Figure 6.4, we see a strong indication that non-uniform magnetic fields generate a measurable and significant change to MR fluid rheology. As observed in the microfluidic study (Chapter 5) this change appears to be caused by the non-uniform field modifying the local volume fraction of magnetizable particles in the active region of the MR fluid device. With these phenomenological observations, we now turn our attention to developing a model for MR fluids that is capable of capturing the effective rheological properties of the suspension and the relative motion between the two constituting phases by coupling the effects of field gradients on the average stress field resulting from the anisotropic chaining of the particles.

### 6.3 Suspension balance model

In this section, a two-phase continuum model for MR fluid flow is derived. The model follows the multiphase flow averaging method of Drew and Lahey (1992). Following this method, similar models were developed to capture particle migration in dense suspensions (Nott and Brady, 1994; Morris and Brady, 1998; Morris and Boulay, 1999). More recently, a continuum model capturing phase separation in ER and MR fluids for quiescent and shear flow conditions was developed (von Pfeil et al., 2002; von Pfeil et al., 2003; von Pfeil and Klingenberg, 2004). This model, however, does not capture the field-dependent yield stress behavior and thus cannot be used to make predictions of the rheological response of the MR fluids presented in Figures 6.3 and 6.4.
In Subsection 6.3.1, we outline the steps required to obtain an averaged continuum two-phase flow model. In Subsection 6.3.2, the particle phase stress model for capturing the anisotropic yield stress of MR fluids is described. In Subsection 6.3.3, the model is linearized around the uniform flow and magnetic field condition to gain further insight into the local stability of the two-phase model. In Subsection 6.3.4, a dimensionless parameter that quantifies the importance of particle phase slip is derived using the components of stress acting on the particle phase. For flow conditions that generate negligible slip (as determined by evaluating the magnitude of this newly defined dimensionless parameter) the two-phase model is simplified to a single-phase model in Subsection 6.3.5. And finally, analytical solutions to two parallel flow problems under uniform but arbitrary magnetic fields are presented in Subsection 6.3.6; these are used to compare the predictions of the newly developed model to measured values of yield stress from the literature.

### 6.3.1 Governing equations

At the length scale of MR fluid particles, the conservation equations take the form of a continuity equation:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}_m) = 0, \quad (6.2)
\]

and the Cauchy momentum equation:

\[
\rho \frac{D\mathbf{v}_m}{Dt} = \nabla \cdot \mathbf{\tau} + \mathbf{f}_b. \quad (6.3)
\]
Here, \( \rho \) is the local density (the particle phase or the matrix fluid, whichever occupies the space at a given instant of time), \( v \) is the local microscopic velocity, \( \tau \) is the stress and \( f_b \) is the body force.

For obtaining a continuum model, these conservation equations described in terms of the microscopic quantities need to be averaged over the scale of a representative volume element. The averaging procedure that must be conducted to obtain the governing equations of the two-phase flow were derived by Drew and Lahey (1992) and Nott and Brady (1994). Here an outline of the procedure, as it applies to MR fluid flow, is provided as a reference. To obtain the conservation equations for the particle phase, the microscopic conservation equations are multiplied with the phase indicator function

\[
X_p(x) = \begin{cases} 
1 & \text{x in the particle phase} \\
0 & \text{x in the fluid phase}
\end{cases},
\]

and ensemble averaged:

\[
\left\langle f(x,t) \right\rangle = \int f(x,t;l)dm(l),
\]

where \( f \) is the averaged quantity, \( \xi \) is the set of all processes of interest and \( dm(l) \) is the measure of observing process \( l \). It should also be noted that the ensemble average is equivalent to volume averaging for homogeneous flow and to time averaging for steady flow. This procedure results in the particle phase conservation of mass equation:

\[
\frac{\partial \phi}{\partial t} + \nabla \cdot (\phi V) = 0,
\]
and the corresponding momentum conservation equation:

\[ \rho_p \phi \frac{D_p V}{Dt} = f_p + F_p + \nabla \cdot \tau_p , \tag{6.7} \]

where, \( \rho_p \) is the particle density, \( V \) is the averaged particle phase velocity, \( f_p \) is the body force on the particle phase, \( F_p \) is the drag force on the particle phase, \( \tau_p \) is the particle phase stress and \( D_p / Dt \) is the material derivative following the average particle motion. \( \phi \) is the average fraction that can be determined by Equation (6.5), setting \( f = X_p \). The parameter \( \phi \) is often interpreted as the volume fraction, as done in this study; however, we should note that this is strictly correct for suspensions, which are spatially-homogeneous at the averaging length scale.

The averaged conservation equations for the entire suspension are obtained by averaging Equations (6.2) and (6.3); however, this time without multiplying by the phase indicator function. This procedure results in the suspension mass conservation:

\[ \nabla \cdot v = 0 , \tag{6.8} \]

and momentum conservation:

\[ \frac{D(\rho v_m)}{Dt} = \nabla \cdot \tau_s + f_s , \tag{6.9} \]

where \( v \) is the velocity, \( \tau_s \) is the stress, and \( f_s \) is the body force for the averaged suspension.

Equations (6.6)-(6.9) constitute the conservation equations related with the fluid flow problem. We limit our attention to inertia-free flows where the Reynolds number is small at the
length scale of the particle as well as the length scale representative of the flow geometries; we thus neglect the left-hand side of Equations (6.7) and (6.9).

The electrodynamics problem, which needs to be solved simultaneously with the equations for fluid flow, is governed by Maxwell’s equations (Zahn, 1987). For common problems involving MR fluids, these equations can be used in the magnetoquasistatic limit (Melcher, 1981). By noting that fluid is commonly absent of free current, the governing equations can be expressed as the divergence-free magnetic flux density and curl-free magnetic field strength \((\nabla \cdot \mathbf{B} = 0, \nabla \times \mathbf{H} = 0)\). These equations can also be shown to be valid for averaged field variables (Rosensweig, 1985). The second relation allows the definition of the magnetic field strength as a gradient of the magnetic scalar potential:

\[
\mathbf{H} = -\nabla \chi. \tag{6.10}
\]

The constitutive relationship between the applied field and resulting magnetic flux is:

\[
\mathbf{B} = \mu \mathbf{H}, \tag{6.11}
\]

where \(\mu\) is the magnetic permeability, which is a second-order tensor for anisotropic materials.

In a quiescent MR fluid in an arbitrary applied field, the particles align in the direction of the field. Although this alignment is the cause of the anisotropy in the material magnetic behavior, it can be seen that the only component that is relevant in relation (6.11) is the \(\mu_{11}\) component of the tensor where the 1-direction is aligned with the local magnetic field. Combining (6.10) and (6.11) results in an expression analogous to the Fourier law of heat conduction with \(\mathbf{B} = -\mu \nabla \chi\).

In a mechanically deformed suspension, the picture is more complex. Through the bulk rheological measurements of Chapter 3 and microfluidic observations of Chapter 5, we know
that as the Mason number $Mn \to 0$ the MR suspension behavior approaches that of a quiescent suspension. In bulk rheological terms, the shear stress in the fluid approaches the yield stress as the deformation rate goes to zero; whereas in microstructural terms, the particle chains align with the applied field as the magnetic torque becomes dominant as compared to the hydrodynamic torque. Following these phenomenological observations, we will approximate magnetic behavior of MR fluids for $Mn \ll 1$ as a quiescent suspension. This approximation allows us to reduce the magnetic permeability tensor to a scalar.

We should also note that in a flow regime where the hydrodynamic forces dominate magnetic forces ($Mn \gg 1$), both the fluid rheology and the microscopic aggregation behavior are very similar to those in an unmagnetized suspension. In this case, we can approximate the suspension as magnetically isotropic (again, the magnetic permeability $\mu$ is a scalar). Although this treatment is straightforward, the flow phenomena of interest in this modeling effort do not fall into this regime and therefore, are not further developed.

The magnetic permeability is strongly dependent on the particle volume fraction, $\phi$. In the limiting cases where $\phi = 0$ or $\phi = 1$, the magnetic permeability of the suspension is equal to that of the continuous and particulate phase respectively. The commonly used constitutive relations for the permeability as a function of volume fraction were reviewed in Section 1.1.

By inserting the magnetic constitutive relation into the condition of divergence free magnetic flux density, we obtain the final governing equation of the continuum model:

$$\nabla \cdot (\mu(\phi) \nabla \mathbf{H}) = 0. \quad (6.12)$$
The base fluid suspending the MR fluid particles is modeled as an isotropic incompressible Newtonian fluid. This results in an averaged suspension stress of

$$\tau_s = -p\delta + \eta (\nabla v + \nabla v^T) + \tau_p,$$

(6.13)

where $p$ is the average suspension pressure, $\delta$ is the unit tensor, $\eta$ is the viscosity of the base fluid and $\tau_p$ is the particle phase stress. With this constitutive relation the drag force on the particle phase can be approximated in a mean field approach as (Nott and Brady, 1994):

$$F_p = \frac{9\eta \phi}{2a^2 f(\phi)}(v - V),$$

(6.14)

where $a$ is the particle radius and $f(\phi)$ is the hindered mobility function. This function is a monotonously decreasing function with $f(0) = 1$. The nature of this function has been investigated by numerous experimental and theoretical studies of sedimentation, which were reviewed by Davis and Acrivos (1985).

The particle stress constitutes of three components:

$$\tau_p = \tau^H + \tau^M + \tau^Y,$$

(6.15)

which are the hydrodynamic, magnetic, and yield stress, respectively. In this model, only the shear component of the hydrodynamic stress component is considered:

$$\tau^H = \eta_p(\phi)(\nabla v + \nabla v^T),$$

(6.16)

where $\eta_p(\phi)$ is the particle phase viscosity and can be related to the volume fraction-dependent suspension viscosity.
\[ \eta_s(\phi) = \eta + \eta_p(\phi). \quad (6.17) \]

The magnetic (Maxwell) stress on the particle phase is given by the relation:

\[ \tau^M = \mu(\phi)HH - \delta \frac{1}{2} H^2 \left[ \mu(\phi) - \phi \frac{d\mu}{d\phi} \right], \quad (6.18) \]

where \( \delta \) is the unit tensor. The derivation of this equation and the conditions that are required for its use in describing the magnetic stress in continuous media are discussed in Section 1.1. This equation captures both the magnetization and the magnetostriction (final term) force in the particulate phase. The latter component is a part the electromagnetic stress tensor because of the compressible nature of the particle phase (or in other words, the variable volume fraction of magnetizable media within a unit volume of the MR suspension).

Although the magnetic body force on the particle phase is captured by the Maxwell stress in Equation (6.18), this definition results in a zero yield stress prediction (von Pfeil et al., 2003). The origin of the yield stress observed in experiments can be understood by observing the microstructure of particle aggregation and deformation. In the following subsection, these aspects are reviewed and constitutive relations that can be used in the suspension balance model are developed.

### 6.3.2 Magnetic body couple and anisotropic yield stress

The observations made of the MR and ER fluid microstructures in the literature (Klingenberg and Zukoski, 1990; Fermigier and Gast, 1992; Deshmukh, 2007) as well as those presented in Chapter 5 reveal a complex network of particles with a preferred alignment in the direction of the local field. These structures are chain-like when the volume fraction of particles
is low \(O(1\%)\); however, significantly more complex structures are observed at higher volume fractions, forming percolated networks and interchain links (Fermigier and Gast, 1992). These three-dimensional networks supporting the chains and columns that form in the preferred (magnetic field) direction have been linked to the very large values of elastic moduli measured in these fluids and the low strains to failure (Pan and McKinley, 1997).

![Diagram](image)

Figure 6.5: Schematic diagram illustrating the evolution of MR fluid microstructure under deformation. In the undeformed state under an applied field, the fluid is made up of columnar structures that are normally cross-linked with shorter randomly-oriented bridging chains. Noting that the 1-direction is aligned with the applied field, a deformation in the form \(\frac{\partial u_2}{\partial x_1}\) results in a misalignment of the chains and the field. In this state a magnetic body couple acts on the particle phase to restore the chain orientation to alignment with the field. Deformation in the \(\frac{\partial u_1}{\partial x_2}\) direction does not cause this misalignment; however, the randomly oriented short chains can still be broken as a result of this component of strain as well.

The microstructural picture for a sheared MR fluid under an applied magnetic field is illustrated in the schematic diagram of Figure 6.5. Under the influence of the magnetic field, the
particles form chains and columnar structures that are interlinked with randomly-oriented shorter chains. We shall recall that the 1-direction is used as the direction of the field as a convention. If we consider a deformation field that comprises of only a non-zero deformation gradient in the \( \partial u_2/\partial x_1 \) component, the directional structures are deformed under this strain field. This microstructural shift is not captured in the Maxwell stress of Equation (6.18). Furthermore, the average magnetization in the deformed chains is misaligned from the magnetic field, which is not captured in an isotropic magnetic constitutive relation (Equation (6.11)). Here, we can make the observation that the misaligned magnetization and the field vectors result in a body couple density on the particle phase,

\[
g = \mu_0 \mathbf{M} \times \mathbf{H},
\]  

(6.19)

which is not captured in the Maxwell stress (Rosensweig, 1995). This is clearly not a deficiency in the theory behind this relation, but an effect caused by the anisotropy of MR fluids which violates the isotropic material assumption used in the development of the body force density. We take the approach of using the body couple to describe the discrepancy between the isotropic magnetic force density and the actual magnetic interactions in the fluid.

The complementary component in the deformation gradient tensor \( \partial u_1/\partial x_2 \) does not result in a misalignment between the magnetization and field vectors, as illustrated in Figure 6.5; however, this deformation can break the randomly-oriented short chains that contribute to the strength of MR fluids resulting in yielding of the fluid. The constitutive model for the body couple density therefore needs to capture these aspects correctly in order to represent the anisotropic response of MR fluid to deformation.
The body couple density transfers angular momentum to the particle phase. It is approximated that the processes leading to angular momentum transfer between the particle phase and the fluid phase is perfectly efficient with zero dissipative losses. If described under a framework of fluids with internal rotation (Dahler and Scriven, 1961; Dahler and Scriven, 1963; Rosensweig, 1985) the conservation of internal angular momentum equation is:

$$\rho \frac{Ds}{Dt} = g + \nabla \cdot C - \varepsilon : \sigma ,$$  \hspace{1cm} (6.20)

where $s$ is the internal angular momentum, $C$ is the couple stress tensor, $\varepsilon$ is the third-order alternating tensor and $\sigma$ is the total stress. In this formalism, if we set the internal angular momentum and the couple stress tensor to zero, it is clear that the body couple can be directly linked to the fluid stress. Common constitutive relations used to describe these quantities are (Rosensweig, 1985):

$$C = \lambda' (\nabla \cdot \omega) \delta + \eta' (\nabla \omega + \nabla \omega^T),$$  \hspace{1cm} (6.21)

$$s = I \omega ,$$  \hspace{1cm} (6.22)

where $I$ is the average moment of inertia per unit mass, $\omega$ is the internal spin rate, and $\lambda'$ and $\eta'$ are the bulk and shear coefficients of the spin viscosity, respectively. It can be seen that both of these constitutive relations result is zero couple stresses or angular momentum when $\omega$ vanishes. In this limit the yield component of the particle phase stress (as defined in Equation (6.15)) is

$$\tau_y^\gamma = - \frac{1}{2} \epsilon_{yk} g_k ,$$  \hspace{1cm} (6.23)
which is a statement of the conservation of angular momentum of the continuum.

We can understand the nature of the body couple by reviewing bulk rheological measurements on MR fluids subjected to an oscillatory shear deformation (Figure 6.6). The MR fluid stress is largely linear in the pre-yielding regime; however, when the fluid is deformed beyond a critical strain, \( \gamma_c \), the linear increase of stress abruptly stops and reaches a saturated value. The stress corresponding to this shift is the yield stress, \( \tau_y \).

![Figure 6.6: Stress-controlled oscillatory shear experiment of MR fluid (Lord® MRF-132DG) conducted in the magnetorheology accessory (described in Chapter 2). The measurements are presented in terms of the apparent yield stress, \( \tau_a \), versus the shear strain at the rim of the parallel-plate geometry. The measurements of stress are linear with the strain below the yield stress (and the corresponding critical strain, \( \gamma_c \)). Above this stress, the magnetically-induced contribution saturates.](image)

In the small strain limit, these observations on the fluid response can be captured by the constitutive relation:
\[ g_3 = -2G' (\phi, H) \frac{\partial u_2}{\partial x_1}, \]  

(6.24)

in a 2-D flow in the 1-2 plane, where \( G' \) is the elastic modulus of the MR fluid and \( \mathbf{u} \) is the suspension displacement vector. It can be seen that inserting this relation into Equation (6.23) results in an elastic stress contribution with individual terms in a familiar form

\[
\tau^\gamma = \begin{bmatrix}
0 & G'(\phi, H) \frac{\partial u_2}{\partial x_1} \\
-G'(\phi, H) \frac{\partial u_2}{\partial x_1} & 0
\end{bmatrix}. \tag{6.25}
\]

To determine the correct constitutive relation for large deformations (post-yield), we need to consider the fluid microstructure. At the microscopic level, the saturation of the magnetically-induced stress can be linked to the plastic failure of the magnetically formed structures at the critical strain, \( \gamma_c \) (Klingenberg and Zukoski, 1990). It is important to note that the microstructural failure may occur in both the main (directional) chains, as well as the cross-linking chains as shown in Figure 6.5 (Pan and McKinley, 1997). Therefore, the critical strain in a multi-dimensional model needs to be an isotropic measure, which can be captured by magnitude of the strain tensor at point of yield.

In constitutive form we can incorporate these observations by expressing the body couple density as

\[
g_3 = -2G'(\phi, H) \left. \frac{\partial u_2}{\partial x_1} \right|_{\gamma = \gamma_c} = -2\tau_\gamma (\phi, H) \left[ \frac{\partial u_2 / \partial x_1}{\gamma} \right]_{\gamma = \gamma_c}, \tag{6.26}
\]
where, the scalar yield stress is taken to be \( \tau_y = G' \gamma_c \). The magnitude of the strain tensor is indicated \( \gamma = \sqrt{\gamma_0 \gamma_1 / 2} \) where \( \gamma_0 = \partial u_1 / \partial x_j + \partial u_j / \partial x_1 \). This constitutive relation captures the aspects of the described phenomenology; however, while solving fluid mechanics problems it may not be possible to calculate the quantity in the brackets since this term requires the elastic part of the strain tensor to be determined form the total strain (which may include a plastic term). This quantity is represents the ratio of the strain in the preferred direction to the magnitude of strain at the point of yielding in a material element. We can approximate the term in the brackets by the same ratio at total strain, which is effectively a statement that the direction of plastic deformation in the material coincides with that of elastic deformation. Therefore, generalizing for the three dimensional case, the constitutive relation is

\[
g = \begin{cases} 
2G'(\phi, H) \left( \frac{\partial u_1}{\partial x_1} i_2 - \frac{\partial u_2}{\partial x_1} i_3 \right) & \gamma < \gamma_c \\
2\tau_y (\phi, H) \left( \frac{\partial u_1}{\partial x_1} i_2 - \frac{\partial u_2}{\partial x_1} i_3 \right) & \gamma \geq \gamma_c 
\end{cases}
\]

(6.27)

In solutions of the flow problems in which the pre-yielding behavior of the fluid does not play an important role, it is convenient to formulate the constitutive model in terms of velocity gradients. By letting \( \gamma_c \to 0 \) while maintaining \( \tau_y \) constant (which requires \( G' \to \infty \)) we obtain

\[
g = 2\tau_y (\phi, H) \left( \frac{\partial v_1}{\partial x_1} i_2 - \frac{\partial v_2}{\partial x_1} i_3 \right).
\]

(6.28)

Here the magnitude of the strain rate is \( \dot{\gamma} = \sqrt{\dot{\gamma}_0 \dot{\gamma}_1 / 2} \), where \( \dot{\gamma}_0 = \partial v_1 / \partial x_j + \partial v_j / \partial x_1 \). This limit can be intuitively understood by noticing that when the modulus goes to infinity, the elastic strain in the material goes to zero. Since the material does not have any elastic deformation, the
deformation of the magnetically formed chains therefore follows the deformation rate at all material elements in the continuum.

6.3.3 Linearized suspension balance model

The non-linear nature of the two-phase suspension balance model poses significant challenges for analytically studying MR fluid flow in all but very simple flow problems. This is the main driver for the numerical methods used in the following sections of this chapter. In this subsection however, the model is linearized at an equilibrium point in order to study the flow and aggregation dynamics and understand the response of the model to small variations. The methodology of Zahn and Rhee (1984) is used in the linearization and the following complex analysis.

The governing Equations (6.6)-(6.9),(6.12) are linearized around conditions of uniform flow velocity, \( \mathbf{V}_0 \), and magnetic field, \( H_0 \mathbf{i}_1 \), where \( \mathbf{i}_1 \) is the unit vector in the 1-direction. The convention of using this direction in alignment with the magnetic field is retained and the flow is allowed to be in an arbitrary direction. In this equilibrium base flow the volume fraction \( \phi_0 \) and pressure \( p_0 \) are uniform. As the averaged suspension is assumed to be incompressible, the selection of the equilibrium pressure is arbitrary and does not affect the results. As a summary, the variables in this homogeneous equilibrium base flow are

\[
\mathbf{v} = \mathbf{V} = \mathbf{V}_0 \mathbf{i}_2, \phi = \phi_0, \chi = -H_0 x_1, p = p_0
\]  

(6.29)

where, \( x_1 \) is the component of the position vector in the 1-direction. Small perturbations from this equilibrium solutions are allowed for all variables.
\[ v = V_0 + v', \quad (6.30) \]

\[ V = V_0 + V', \quad (6.31) \]

\[ \phi = \phi_0 + \phi', \quad (6.32) \]

\[ \chi = -H_0 x_i + \chi', \quad (6.33) \]

\[ p = p_0 + p', \quad (6.34) \]

where, the primed variables indicate the perturbations. Substituting these variables into the governing equations ((6.6)-(6.9),(6.12)) and retaining the terms that are at most first order in terms of the perturbations, we obtain

\[ \mu(\phi_0) \frac{\partial^2 \chi'}{\partial x_i \partial x_i} - \frac{d\mu}{d\phi_{\phi_0}} H_0 \frac{\partial \phi'}{\partial x_i} = 0, \quad (6.35) \]

\[ \frac{\partial v_i'}{\partial x_i} = 0, \quad (6.36) \]

\[ \frac{\partial}{\partial x_j} \left[ -p' \delta_j + \eta \left( \frac{\partial v_j'}{\partial x_j} + \frac{\partial v_i'}{\partial x_j} \right) \right] + \frac{9\eta \phi_0}{2a^2 f(\phi_0)} (V_i' - v_i') = 0, \quad (6.37) \]

\[ \frac{\phi_0}{2} H_0^2 \frac{d^2 \mu}{d\phi^2} \frac{\partial \phi'}{\partial x_i} - \frac{\phi_0}{2} H_0 \frac{d\mu}{d\phi} \frac{\partial^2 \chi'}{\partial x_i \partial x_i} + \eta_p (\phi_0) \frac{\partial}{\partial x_i} \left( \frac{\partial v_j'}{\partial x_j} + \frac{\partial v_i'}{\partial x_j} \right) + \frac{\partial r'}{\partial x_j}, \quad (6.38) \]

\[ = \frac{9\eta \phi_0}{2a^2 f(\phi_0)} (V_i' - v_i') \]

\[ \frac{\partial \phi'}{\partial t} + V_0 \frac{\partial \phi'}{\partial x_i} + \phi_0 \frac{\partial V_i'}{\partial x_i} = 0, \quad (6.39) \]
where summation is implied for all repeated indices. Taking the divergence of (6.38) and substituting (6.36) and (6.39) we obtain the following governing equation

\[
\frac{\phi_0}{2} H_0 \left. \frac{d^2 \mu}{d \phi^2} \right|_{\phi_0} \frac{\partial^2 \phi'}{\partial x_i \partial x_j} - \phi_0 H_0 \left. \frac{d \mu}{d \phi} \right|_{\phi_0} \frac{\partial^3 \chi'}{\partial x_i \partial x_j \partial x_k} + \frac{9 \eta}{2 \alpha^2 f(\phi_0)} \left( \frac{\partial \phi'}{\partial t} + V_0 \frac{\partial \phi'}{\partial x_j} \right) = 0. \tag{6.40}
\]

It should be noted that the body force term associated with the yield stress is divergence free and therefore does not appear in this equation. We can see this by recognizing that this body force density is

\[
f' = \frac{1}{2} \nabla \times g, \tag{6.41}
\]

and that divergence of a curl is identically zero.

Because the equations are linear and have constant-coefficients, we can express the perturbation variables in the form

\[
\phi' = \text{Re} \{ \hat{\phi} \exp(st - jk \cdot x) \}, \tag{6.42}
\]

\[
\chi' = \text{Re} \{ \hat{\chi} \exp(st - jk \cdot x) \}, \tag{6.43}
\]

where \( s \) is the (complex) frequency, \( k \) is the wave vector, \( x \) is the position vector, and \( \hat{\phi} \) and \( \hat{\chi} \) are the (complex) amplitudes of the perturbation. Substituting these forms in (6.40) we obtain the following expression for the complex frequency

\[
s = \frac{2\phi_0 H_0^2 \alpha^2 f(\phi_0)}{9 \eta} \left[ \frac{1}{2} \left. \frac{d^2 \mu}{d \phi^2} \right|_{\phi_0} k^2 - \frac{1}{\mu(\phi_0)} \left( \left. \frac{d \mu}{d \phi} \right|_{\phi_0} \right)^2 \right] + jk \cdot V_0. \tag{6.44}
\]
Using the Lorentz sphere model (described in Figure 1.2) for the constitutive model of magnetic permeability, we obtain

\[ s = \frac{2\phi_0^2 \mu_0 H_0^2 a^2 f(\phi_0)}{3\eta(1-\beta\phi_0)^3} \left[ \frac{2(1-\beta\phi)}{1+2\beta\phi_0} k^2 + k_2^2 + k_3^2 \right] + jk \cdot V_0, \]  

(6.45)

where the effective susceptibility, \( \beta = (\mu_r - 1)/(\mu_r + 2) \). The real part of the complex frequency determines the growth rate of a given perturbation, and hence a stable base flow with a negative value indicating a decaying perturbation. This quantity is plotted in Figure 6.7 for perturbations where \( k_3 = 0 \); however, no generality is lost since \( k_2 \) and \( k_3 \) affect the complex frequency in the same manner. It can be seen from this plot that the perturbations with variation only in the applied field direction are stable whereas the perturbations with variations only in an orthogonal direction to the applied field are unstable. The most unstable, and therefore the fastest growing perturbations have \( |k_2| \to \infty \) and \( |k_3| \to \infty \). Therefore the model predicts the formation of thin chains aligned in the direction of the field. Since the yield component of stress does not affect the complex frequency, Equation (6.45) is in quantitative agreement with the linearized model of von Pfeil et al. (2002) even though this earlier model does not include a yield term.

To summarize, we can see that the spatially-homogeneous solutions to the two-phase model with a magnetic body force are always unstable with respect to the chain-forming perturbations and the yield component of stress (as described by Equation (6.23)) does not provide a stabilizing effect. This instability poses a significant challenge for the effective numerical implementation of this model. For this reason, we next turn our attention to what can be learned analytically from the model so that we can make a quantitative estimate on the two-phase nature of a given flow problem.
Figure 6.7: The real part of the complex frequency as a function of wavevectors in the field direction, $k_1$ and an orthogonal direction $k_2$. A perturbation (with a given wavevector) is stable if the real part of the frequency is negative. Perturbations with variation in the field direction are stable, whereas perturbations in the orthogonal direction are unstable. The most unstable perturbation corresponds to $k_1 = 0, |k_2| \to \infty, |k_3| \to \infty$.

6.3.4 Dimensionless magnetic slip parameter

The complex frequency calculated in Equation (6.45) can be thought of as a balance between a growth rate of a given perturbation, which appears in the real part, to its advection rate, which is the imaginary part. In fact, the problem we would like to address of particle phase slip is a balance between a growth rate of the local volume fraction in non-uniform fields to hydrodynamic (advective) forces that carry particles (and matrix) away from these locally enriched regions. It is indeed possible to quantify this balance by considering the relevant particle phase forces in the full non-linear model.

The magnetic migration of the particle phase occurs because of the magnetophoretic forces in the direction orthogonal to the field. In the field direction (1-direction) the particles migrate into columnar structures with a very small time scale, often in the order of milliseconds.
(Deshmukh, 2007) and form stable structures. The migration phenomenon of interest occurs over much larger time scales as identified in microfluidic observations (as illustrated in Figure 5.8). By considering a balance between the magnetic (Equation (1.7)) and the hydrodynamic forces (Equation (6.14)) on the particle phase \( \left[ F^M = -F_p \right] \) in 2D flow, we can obtain the ratio

\[
\Psi = \frac{V_{\text{slip}}}{v_2} = \frac{2a^2 f(\phi)}{9\eta v_2} \left[ \frac{d\mu}{d\phi} H \frac{\partial H}{\partial x_2} + \frac{1}{2} \frac{d^2\mu}{d\phi^2} H^2 \frac{\partial \phi}{\partial x_2} \right],
\]

where the magnetically induced slip velocity, \( V_{\text{slip}} = V_2 - v_2 \).

In Chapter 5 we derived a similar parameter analyzing the micromechanics of a single isolated particle \( \Psi' \) (Equation (5.7)). If we compare this equation to the newly obtained definition, we can see that the micromechanical approach finds the slip parameter to be similar to the first term in the brackets in Equation (6.46) but does not predict the second term. Essentially, the earlier definition does not capture the multi-body effects, such as magnetostriction (second term in the brackets) and hindered mobility (through the hindered mobility function \( f(\phi) \)) of the particle phase. It is important to include these terms in analyzing flow problems in which the interaction between the particles are significant. Both hydrodynamic and magnetic interactions increase as the separation distance between the particles reduces (as discussed in Section 1.1), therefore it is important to capture interparticle interactions for MR suspensions with higher particle volume fractions \( \phi \). The new form, therefore, enables us to make a better estimate of the particle phase slip phenomenon in concentrated suspensions.
6.3.5 Single-phase model for no-slip flow conditions

For flow problems in which $|\Psi| \ll 1$ the slip in the particle phase is negligible and thus the suspension can be treated as a single phase continuum. This treatment results in a significant reduction in the complexity of the model as the governing equations constitute of only the conservation equations of the average suspension:

$$\frac{\partial v_i}{\partial x_i} = 0, \quad (6.47)$$

$$0 = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j} \left[ \eta_s(\phi) \left( \frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} \right) \right] + \frac{\partial}{\partial x_j} \tau^M_{ji} + \frac{\partial}{\partial x_j} \tau^\gamma_{ji}, \quad (6.48)$$

$$\frac{\partial}{\partial x_i} \left( \mu(\phi) \frac{\partial \chi}{\partial x_i} \right) = 0, \quad (6.49)$$

where, the magnetic stress (from Equation (6.18)),

$$\tau^M_{ji} = \mu(\phi) H_i H_j - \delta_{ij} \frac{H^2}{2} \left[ \mu(\phi) - \phi \frac{d\mu}{d\phi} \right], \quad (6.50)$$

and the yield component of stress (from Equation (6.23) and (6.28)),

$$\tau^\gamma_{ji} = \frac{\tau_y(\phi, H)}{\dot{\gamma}} \left( \epsilon_{ij}^2 \frac{\partial v_3}{\partial x_i} - \epsilon_{ij}^3 \frac{\partial v_2}{\partial x_i} \right). \quad (6.51)$$

This stress component is calculated in the limit where $G' \rightarrow \infty$ as described in Equation (6.28). Summation is implied on all repeated indices.

For both numerical and analytical methods, it is beneficial to non-dimensionalize the governing equations. To be able to do this however, we need to understand how each of the
quantities scale with other variables in terms of constitutive models. For the volume fraction and field dependence of yield stress we use the form

\[ \tau_y(\phi, H) = K_r \phi H^{3/2}, \]  

(6.52)

where, \( K_r \) is a fluid-dependent constant of proportionality, which can be identified through torsional shear measurements. In the SI system it has the units of \( HA^{1/2} m^{-3/2} \). A subquadratic scaling law of yield stress was experimentally identified by Parziale and Tilton (1950) and later linked to magnetic saturation at the contact point by Ginder and coworkers (1996). This scaling law was also evaluated for our measurements of yield stress in Chapter 4, in which it showed a good collapse of data (Figure 4.8a) in the low field regime \( B < 0.3T \). A linear relation between the volume fraction and the yield stress for ER fluids was identified experimentally by Marshall and coworkers (1989) and numerically by Klingenberg and coworkers (1991a) for \( \phi < 0.35 \). Felt and coworkers (1996) found a linear relationship in the range \( 0.014 < \phi < 0.12 \) for MR fluid. The more rapid than linear relationship at higher volume fractions is attributed to the complex structures formed under the influence of magnetic field in these suspensions (de Vicente et al., 2011). For the remainder of this study, the linear relationship is used, as defined in Equation (6.52).

The constitutive relation for the suspension viscosity is taken to be

\[ \eta_s(\phi) = \eta \left( 1 - \frac{\phi}{\phi_m} \right)^{-[\phi]d_m}, \]  

(6.53)
as described by Krieger and Dougherty (1959) where $\phi_m$ is the maximum packing fraction. Their equation collapses to Einstein’s theoretical prediction in the low volume fraction limit when the intrinsic viscosity, $[\eta] = 2.5$.

The volume fraction dependent magnetic permeability of the suspension is described using the parallel (column) model (this and other constitutive relations were reviewed in Figure 1.2). In the dimensionless form this model defines the relative permeability as

$$\mu_r(\phi) = 1 + \phi (\mu_r^p - 1), \quad (6.54)$$

where $\mu_r^p$ is the relative magnetic permeability of the particle phase.

For pressure-driven flows, which are of primary interest to this study, the governing equations are non-dimensionalized substituting $\tilde{\chi} = x / H_0 L$, $\tilde{\rho} = p / \Delta p$, $\tilde{\tau} = \tau / K_i H_0^{3/2}$, $\tilde{x}_i = x_i / L$, $\tilde{v}_i = v_i / \eta / \Delta p L$, $\mu_r(\phi) = \mu(\phi) / \mu_0$ and $\tilde{\eta}_s(\phi) = \eta_s(\phi) / \eta$, where $L$, $H_0$ and $\Delta p$ are the length, magnetic field strength, and applied pressure differential of the MR fluid flow. It is important to note that both the horizontal and the vertical dimensions in the model are scaled with the same length scale, $L$. Dropping tildes we obtain the following dimensionless expressions:

$$\frac{\partial v_i}{\partial x_i} = 0, \quad (6.55)$$

$$0 = -\frac{\partial p}{\partial x_j} + \frac{\partial}{\partial x_j} \left[ \eta_s(\phi) \left( \frac{\partial v_j}{\partial x_j} + \frac{\partial v_i}{\partial x_i} \right) \right] + Mn \frac{\partial}{\partial x_j} \tau^{\mu\nu} + Bi \frac{\partial}{\partial x_j} \tau^{\nu}, \quad (6.56)$$
\[
\frac{\partial}{\partial x_i} \left( \mu_r(\phi) \frac{\partial \gamma}{\partial x_i} \right) = 0. \tag{6.57}
\]

where the non-dimensional magnetic component of the stress is

\[
\tau^M_{ji} = \mu_r(\phi) H_i H_j - \delta_{ij} \frac{H^2}{2} \left[ \mu_r(\phi) - \phi \frac{d\mu_r}{d\phi} \right], \tag{6.58}
\]

and the yield component of the stress,

\[
\tau^\gamma_{ji} = \frac{\phi H^2}{\gamma} \left( \varepsilon_{i2} \frac{\partial v_2}{\partial x_i} - \varepsilon_{i3} \frac{\partial v_3}{\partial x_i} \right). \tag{6.59}
\]

For the remainder of this chapter, governing equations are used only in their dimensionless form. The two dimensionless parameters that appear in the momentum equation are the Mason number,

\[
Mn_p = \frac{\Delta p}{\mu_0 H_0^2}, \tag{6.60}
\]

which quantifies the relative magnitude of viscous stress to the magnetic stress, and the magnetic Bingham number,

\[
Bi_p = \frac{K_r H_0^{3/2}}{\Delta p}, \tag{6.61}
\]

which quantifies the relative magnitude of the yield stress to the viscous stress. The subscript p is used to denote that these definitions are applicable in pressure-driven flow problems of MR fluids. The particle phase slip parameter can be written in terms of the non-dimensionalized quantities as:
A particular flow problem can be studied by varying the two dimensionless parameters in any manner; however, it is beneficial to follow particular paths in the $Mn_p - Bi_p$ space to compare with experimental studies effectively. A difficulty arises if the path followed in the computational study requires a varying $K_r$, which means experimentally every data point on the curve needs to be acquired with a different MR fluid. The paths where $K_r$ is constant are illustrated in Figure 6.8. The path $Mn_pBi_p = const.$ represents the experiments in which the applied field on the fluid is maintained constant and the applied pressure difference is varied. For $Mn_pBi_p^{4/3} = const.$ the applied magnetic field is varied while the differential pressure is kept constant.

![Figure 6.8: Paths of constant $K_r$ in $Mn_p - Bi_p$ space. If the dimensionless parameters are studied following these paths, the results, when re-dimensionalized, are obtained for a particular MR fluid.](image-url)
We now consider the example of the dimensional parameters for a commercial MR fluid (Lord MRF-132DG) for which \( K_c = 1.68 \times 10^{-3} \) and \( \phi = 0.32 \). The curve \( M_{p}B_{i} = 10 \) corresponds to a varying pressure difference in which the applied field is \( H = 1.79 \times 10^4 \), which is in the range of the field dependence of yield stress as defined by Equation (6.52).

### 6.3.6 Analytical solutions to parallel flow of MR fluid under a uniform magnetic field of arbitrary direction

Analytical solutions of MR fluid flow problems using the newly developed constitutive relation given in Equation (6.28) provide a useful tool for understanding the predictions of the model. In this subsection two parallel flow problems, which are illustrated in Figure 6.9, are analytically solved. The applied field is spatially-uniform but is aligned with an arbitrary direction, denoted by \( \theta \). The bounding channel walls are magnetic with magnetic permeability matching that of the fluid. Under these conditions, the magnetic body force is zero in the fluid and there is no magnetic stress jump at the boundaries.

In the first flow problem a wall driven flow of MR fluid between two infinite planar surfaces is generated by the relative displacement of the surfaces with a steady velocity of \( V_{wall} = \dot{\gamma}_0 h \), generating a parallel flow \( v = \dot{\gamma}_0 x_i \). The magnitude of the shear rate tensor \( \dot{\gamma} = \dot{\gamma}_0 \) is uniform. The velocity gradient in the field direction is found by applying a rotational transformation to this tensor to find its component in the 1-2 coordinate system as

\[
\frac{\partial v_2}{\partial x_1} = \dot{\gamma}_0 \cos^2 \theta .
\]  

(6.63)

Using the constitutive relation of Equation (6.28) the magnetic body couple is
\[ g_3 = -2\tau_y \frac{\partial v_y}{\partial x_1} \frac{\dot{\gamma}}{} = -2\tau_y \cos^2 \theta, \]  

(6.64)

and the yield component of particle stress is

\[ \tau^y = \begin{bmatrix} 0 & \tau_y \cos^2 \theta \\ -\tau_y \cos^2 \theta & 0 \end{bmatrix}. \]  

(6.65)

Figure 6.9: Two parallel shear flow problems that can be solved analytically: (a) wall-driven flow, (b) pressure-driven flow. The applied magnetic field is uniform; however, it is aligned in an arbitrary direction with respect to the flow.

Noting that this stress is invariant to rotational transformation, we obtain the shear stress on the left wall caused by the fluid flow as:

\[ \tau_{xy} = \tau_y \cos^2 \theta + \eta_s(\phi) \dot{\gamma} \]  

(6.66)
This is equivalent to the constitutive relation of a Bingham fluid (Equation (1.1)) with an effective yield stress

$$
\tau_{eff}^y = \tau_y \cos^2 \theta .
$$

(6.67)

We now turn our attention to the pressure-driven flow problem illustrated in Figure 6.9b. We seek a parallel flow solution with a yielded zone of thickness $h_y$ near the walls, where the shear rates are high, and an unyielded plug is in the center of the channel $\left( v = v_\gamma(x)i_\gamma \right)$. The velocity gradient, $\partial v_2 / \partial x_1 = (\partial v_\gamma / \partial x) \cos^2 \theta$ and $\dot{\gamma} = |\partial v_\gamma / \partial x|$. In the yielded region near the left wall $\tau_{ij} = \epsilon_{ij} \tau_y \cos^2 \theta$. With the divergence-free magnetic stress component the momentum conservation equation in the flow direction reduces to:

$$
0 = -\frac{\partial p}{\partial y} + \eta_\epsilon(\phi) \frac{\partial^2 v_\gamma}{\partial x^2} + \frac{\partial \tau_{i\gamma}^y}{\partial x} .
$$

(6.68)

Satisfying the no slip boundary condition $\left( v_\gamma(x = 0) = 0 \right)$ and continuity of shear stress at yield surface $\left( dv_\gamma / dx \right)_{x = 0} = 0$ the thickness of the yielded zone is found to be

$$
h_y = \frac{h}{2} + \frac{\tau_y \cos^2 \theta}{\partial p / \partial y} .
$$

(6.69)

The velocity profile is
and the resulting flow rate found by integrating the velocity field is

\[
Q' = \frac{\tau_B h^2}{6\eta_s(\phi)} \left[ 1 - \frac{3}{2} \left( \frac{\tau_y \cos^2 \theta}{\tau_B} \right) + \frac{1}{2} \left( \frac{\tau_y \cos^2 \theta}{\tau_B} \right)^{3/2} \right],
\]

where the wall shear stress \( \tau_B = (h/2)(-\partial p/\partial y) \). This result is also the equivalent to the Bingham fluid slit-flow solution (Bird et al., 1987, Sect. 4.5) with the effective yield stress satisfying (6.67).

With the solution to these two flow problems we are able to understand the model prediction in parallel flow of MR fluid under an arbitrary uniform magnetic field as a Bingham fluid with a yield stress given by Equation (6.67). Using this relation, we can compare the model prediction in these cases with the measurements of Kuzhir and coworkers (2003b) and other models. These researchers used measurements of pressure-driven flow of MR fluid through a cylindrical tube, and varied the orientation of the tube with respect to the magnetic field to obtain angular dependent measurements of the MR yield stress. In the Part I of their work (Kuzhir et al., 2003a) they developed a model for flow of MR fluids in which the magnetic field is non-orthogonal to the flow using a stability analysis of chain-like aggregated of MR fluid particles. As a final comparison, the isotropic Bingham model is also considered. In this continuum model the isotropic yield stress prediction is not dependent on the relative alignment of the flow and
field directions. The comparison of these predictions with the new model is illustrated in Figure 6.10 for the ratio \( \tau_y / \tau_{y,\theta=0} \). By inserting Equation (6.67) into this ratio, the angular dependence is found to be in the form:

\[
\frac{\tau_y}{\tau_{y,\theta=0}} = \cos^2 \theta.
\] (6.72)

The predicted behavior of the new model is in good agreement with the measurements. In fact, this model better matches the measurements from Kuzhir and coworkers (2003a) than their model.

Figure 6.10: Comparison of model prediction (Equation (6.72)) in parallel flow of MR fluid under a uniform magnetic field of arbitrary direction with other models and experimental data of Kuzhir and coworkers (2003b). These experimental measurements were obtained in a pressure-driven flow of MR fluid through cylindrical tubes with varying orientation with respect to the magnetic field. \( \theta = 0 \) corresponds to the case which the magnetic field is orthogonal to the flow streamlines. The isotropic Bingham model prediction is not dependent on \( \theta \). The model of Kuzhir and coworkers is based on an analysis of chain-like aggregates of MR fluid.
6.4 Finite element methods

A computational model of the single-phase continuum model was constructed using the finite element method to address flow problems that cannot be solved analytically. The governing system of equations on the element level was derived using the Galerkin method. The numerical methods utilized in this work are discussed in detail by Bathe (1982), Cuvelier, Segal & van Steenhoven (1986) and Reddy & Gartling (2001).

6.4.1 Finite element model

In this subsection, the non-linear partial differential equations governing the single-phase flow of MR fluids (Equations (6.55)-(6.57)) are spatially discretized using the Galerkin finite element method. This method converts the PDEs into a set of non-linear algebraic equations on nodal variables. We restrict our attention to 2D flows without any variation in the third direction; however, with straightforward modifications to the numerical methods, it is possible to generalize the discussion to 2D axisymmetric and 3D flows.

In the current model, nine-node quadrilateral elements are used. For velocity components and the magnetic scalar potential a biquadratic interpolation scheme is used to relate these field variables to the nine nodal variables. For pressure however, a bilinear scheme is used with the pressure nodal variable being present only on the corner nodes of the element. The different order interpolation schemes is needed for the compatibility of the numerical scheme with Ladyzhenskaya-Babuska-Brezzi condition (Reddy and Gartling, 2001, Sect. 4.5.2).
Figure 6.11: Nine-node finite element. Velocity components and the magnetic scalar potential nodal variables are defined in all nine nodes. The pressure nodal variable is defined only on the four corner nodes. The integration conducted to obtain the element equations are evaluated numerically on the 3×3 Gauss integration points.

The interpolation (shape) functions relate both the position and the field variables to those of the element nodes. The biquadratic interpolation function is

$$\Gamma = \frac{1}{4} \begin{vmatrix} (1+r)(1+s)(r+s-1)+(1-r^2)(1-s^2) \\ (1-r)(1+s)(-r+s-1)+(1-r^2)(1-s^2) \\ (1-r)(1-s)(-r-s-1)+(1-r^2)(1-s^2) \\ (1+r)(1-s)(r-s-1)+(1-r^2)(1-s^2) \\ 2(1-r^2)(1+s)-2(1-r^2)(1-s^2) \\ 2(1-s^2)(1-r)-2(1-r^2)(1-s^2) \\ 2(1-r^3)(1-s)-2(1-r^2)(1-s^2) \\ 2(1-s^3)(1+r)-2(1-r^2)(1-s^2) \\ 4(1-r^2)(1-s^2) \end{vmatrix} = (6.73)$$
where, \( r \) and \( s \) are the normalized coordinates for the element. The bilinear interpolation function,

\[
\Phi = \frac{1}{4} \begin{bmatrix}
1 + r & 1 + s \\
1 - r & 1 + s \\
1 + r & 1 - s \\
1 - r & 1 - s \\
\end{bmatrix},
\]

(6.74)

The field variables used in the governing PDEs are related to the nodal vectors as \( \nu_i = \Gamma^T \hat{\nu}_i, \)

\( \chi = \Gamma^T \hat{\chi} \) and \( p = \Gamma^T \hat{p}. \)

We will first consider the magnetostatic governing equation (6.57) by multiplying the equation with the weighting function \( R \) and integrating over the whole volume of the element, \( \Omega^e \)

\[
\int_{\Omega^e} \frac{\partial}{\partial x_i} \left( \mu_r(\phi) \frac{\partial \chi}{\partial x_i} \right) dA = 0.
\]

(6.75)

where \( dA = dx dy \) is defined in the \((x, y)\) global coordinates. We can integrate by parts in order to reduce the order of differentiation and obtain the weak form

\[
\int_{\Omega^e} \frac{\partial R}{\partial x_i} \mu_r(\phi) \frac{\partial \chi}{\partial x_i} dA = \int_{\partial \Omega^e} R \mu_r(\phi) \frac{\partial \chi}{\partial x_i} n_i dl,
\]

(6.76)

where, \( n_i \) is the component of the unit normal in the \( i \)-direction. The integral on the right side is taken on the closed boundary of the element \( \partial \Omega^e. \) The choice in the weighing function is completely arbitrary; however, in the Galerkin method the function is equated to the
interpolation function, \( \Gamma \). We also recognize that the integrand on the right side includes the magnetic flux density, \( B \). By making these substitutions we obtain

\[
\left( \int_{\alpha} \frac{\partial \Gamma}{\partial x_i} \mu_r \phi \frac{\partial \Gamma^T}{\partial x_i} dA \right) \hat{\nabla} \cdot \hat{\mathbf{x}} = -\int_{\partial \alpha} \Gamma B n_i dl .
\] (6.77)

This equation is in the form of a set of algebraic equations of the nodal variable \( \hat{\mathbf{x}} \). The surface integral on the right side constitutes the natural boundary condition.

To evaluate these integrals numerically the operators \( dA \) and \( \frac{\partial ( \ )}{\partial x_i} \) need to be expressed in terms of the local normalized coordinates \((r, s)\). As discussed earlier, the position in the element is related to the nodal position with the interpolation functions just like the field variables: \( x = \Gamma \hat{x} \) and \( y = \Gamma \hat{y} \), where \( \hat{x} \) and \( \hat{y} \) are nodal positions in the global coordinates. We define the Jacobian matrix

\[
[J] = \begin{bmatrix}
\frac{\partial x}{\partial r} & \frac{\partial y}{\partial s} \\
\frac{\partial x}{\partial s} & \frac{\partial y}{\partial r}
\end{bmatrix},
\] (6.78)

which allows the transformation of the operators

\[
\begin{bmatrix}
\frac{\partial}{\partial x} \\
\frac{\partial}{\partial y}
\end{bmatrix} = [J]^{-1} \begin{bmatrix}
\frac{\partial}{\partial r} \\
\frac{\partial}{\partial s}
\end{bmatrix},
\] (6.79)

and \( dA = \det[J] dr ds \). Using these two relations the numerical integrals can be calculated for each element in the normalized elemental coordinates.
The methodology for calculating the elemental matrices for the momentum conservation is similar; however, the weighting function needs to be modified to accommodate the two components of the velocity field. The weighting function components can be defined as \( w_1 = [\Gamma \ 0]^T \) and \( w_2 = [0 \ \Gamma]^T \). Multiplying with these functions and integrating over the element we obtain the momentum conservation equation in the form:

\[
\begin{align*}
\int_{\Omega'} \left[ \frac{\partial}{\partial x_j} \left( -p \delta_{ij} + \eta_\phi \left( \frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} \right) + M_{ij} \tau_{ij}^m + B_i \tau_{ji}^\gamma \right) \right] dA &= 0. \\
\end{align*}
\] (6.80)

Integrating by parts reduces the highest order of differentiation to one:

\[
\begin{align*}
\int_{\Omega'} \frac{\partial w_i}{\partial x_j} \left[ -p \delta_{ij} + \eta_\phi \left( \frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} \right) + M_{ij} \tau_{ij}^m + B_i \tau_{ji}^\gamma \right] dA &= \int_{\partial \Omega'} \left[ -p \delta_{ij} + \eta_\phi \left( \frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} \right) + M_{ij} \tau_{ij}^m + B_i \tau_{ji}^\gamma \right] n_j dl, \\
\end{align*}
\] (6.81)

where \( \Omega' \) and \( \partial \Omega' \) are the element volume and surface respectively. By noting that the integrand in the right side contains the surface traction,

\[
T_i = \left[ -p \delta_{ij} + \eta_\phi \left( \frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} \right) + M_{ij} \tau_{ij}^m + B_i \tau_{ji}^\gamma \right] n_j,
\] (6.82)

the equation takes the form:

\[
\begin{align*}
\int_{\Omega'} \frac{\partial w_i}{\partial x_j} \left[ -p \delta_{ij} + \eta_\phi \left( \frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} \right) \right] dA + B_i \int_{\Omega'} \frac{\partial w_i}{\partial x_j} \tau_{ji}^\gamma dA &= -M_{ij} \int_{\Omega'} \frac{\partial w_i}{\partial x_j} \tau_{ij}^m dA \\
&\quad + \int_{\Omega'} w_i T_i dl.
\end{align*}
\] (6.83)
The magnetic stress component, $\tau^M$, is placed on the right side of the equation because the magnetic stress is independent of the velocity field and thus it is purely a forcing term in the equation. We will now consider each term of the equation separately before re-assembling them to obtain the set of elemental equations.

Replacing the field variables with the nodal variables and the weighting functions with the interpolation functions for the first (hydrodynamic) term yields

$$
\int_{\omega} \frac{\partial w_i}{\partial x_j} \left[ -p \delta_{ij} + \eta_s(\phi) \left( \frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} \right) \right] dA = -\left[ \int_{\omega} \frac{\partial \Gamma}{\partial x_i} \Phi' dA \right] \hat{p} \\
+ \left[ \int_{\omega} \frac{\partial \eta_s(\phi)}{\partial x_j} \frac{\partial \Gamma}{\partial x_j} dA \right] \hat{\nu}_j + \left[ \int_{\omega} \eta_s(\phi) \frac{\partial \Gamma}{\partial x_j} dA \right] \hat{v}_i.
$$

The yield component of the stress for the 2D case calculated by the constitutive relations of Equations (6.28) and (6.52) is

$$
\tau^y_{ji} = \frac{1}{2} \varepsilon_{\phi k} s_k = -\frac{\varepsilon_{xj} \phi H^{3/2}}{\dot{\gamma}} \left[ \frac{\partial v_2}{\partial x_1} \right].
$$

The term in the brackets, however, can be discontinuous at $\dot{\gamma} = 0$. Two approaches exist to resolve this issue: 1. regularizing the discontinuity by replacing the discontinuous function with a continuous one that sufficiently approximates the original function (Papanastasiou, 1987) and 2. by using a yield criteria (e.g. Von Mises) to determine yield surfaces that separate the model into fluid (yielded) and solid regions (see for example (Roquet and Saramito, 2003)). The former method is utilized in this work by approximating Equation (6.85) by

$$
\tau^y_{ji} = -\varepsilon_{xj} \phi H^{3/2} \left[ \frac{\partial v_2 / \partial x_1}{\dot{\gamma}} \right] [1 - \exp(-m\dot{\gamma})].
$$
As illustrated in Figure 6.12, the approximate function approaches the original one as the regularization parameter, $m$, is progressively increased. The upper limit of $m$ is determined by the mesh spacing and by the convergence of the non-linear equation solver.

![Figure 6.12: Approximate functional form of yield component of stress (Equation (6.86)). As the regularization parameter $m$ increases the function more closely approximates the original discontinuous function (Equation (6.85)).](image)

Using this constitutive relation the yield (second) term in Equation (6.83) can be calculated as

\[
Bi_p \int_{\Omega} \frac{\partial w_{12}}{\partial x_j} \tau_{12} \, dA = Bi_p \int_{\Omega} e_{ji3} \frac{\partial w_{12}}{\partial x_j} \phi H^{3/2} \left[ \frac{1-\exp(-m\dot{\gamma})}{\dot{\gamma}} \right] \frac{\partial v_2}{\partial x_1} \, dA
\]

\[
= \left. Bi_p \int_{\Omega} e_{ji3} \frac{\partial \Gamma}{\partial x_j} \phi H^{3/2} \left[ \frac{1-\exp(-m\dot{\gamma})}{\dot{\gamma}} \right] \frac{\partial \Gamma}{\partial x_1} \, dA \right|_{\dot{\gamma}}. \tag{6.87}
\]

We should note that the 1- and the 2-directions that appear in this equation are not the directions in the global coordinate system. They are determined by the magnetic field with the 1-direction aligned with the local magnetic field. The relevant velocity gradient is

\[
\frac{\partial v_2}{\partial x_1} = \frac{1}{H^2} \left[ -H_x H_y \frac{\partial v_x}{\partial y} - H_y^2 \frac{\partial v_x}{\partial y} + H_x^2 \frac{\partial v_y}{\partial x} + H_x H_y \frac{\partial v_y}{\partial y} \right]. \tag{6.88}
\]
which is obtained by applying a rotational transformation to the $(\nabla \nu)_\phi$ tensor.

Finally, the magnetic stress term on the right side of Equation (6.83) is

$$-M_n p \int \frac{\partial w}{\partial x_j} \tau^{\mu}_{\nu} dA = -M_n p \int \frac{\partial}{\partial x_j} \left[ \mu_r(\phi) H_i H_j - \delta_i^j \frac{H^2}{2} \left[ \mu_r(\phi) - \phi \frac{d \mu_r}{d \phi} \right] \right] dA. \quad (6.89)$$

The continuity equation (Equation (6.55)) is integrated over the element with a weighting function, $q$:

$$\int_{\alpha'} q \frac{\partial w}{\partial x_i} dA = 0. \quad (6.90)$$

It is not possible to reduce the order of differentiation in the continuity equation with the weak form formulation. Instead the interpolation functions are directly substituted in this equation with $q = \Phi$ to give:

$$\left[ \int_{\alpha'} \Phi \frac{\partial \Gamma^T}{\partial x_i} dA \right] \hat{v}_j = 0. \quad (6.91)$$

Assembling Equations (6.83) and (6.91) in matrix form we obtain:

$$\begin{bmatrix} 2K_{xx} + K_{yy} + B_{xx} & K_{xy} + B_{xy} & -Q_1^T \\ K_{yx} + B_{yx} & K_{xx} + 2K_{yy} + B_{xx} & -Q_2^T \\ -Q_1 & -Q_2 & 0 \end{bmatrix} \begin{bmatrix} \hat{v}_x \\ \hat{v}_y \\ \hat{p} \end{bmatrix} = \begin{bmatrix} -S_{xx} - S_{xy} - S_{ox} + \int_{\partial A'} \Gamma \Gamma^T d\ell \\ -S_{yx} - S_{yy} - S_{oy} + \int_{\partial A'} \Gamma \Gamma^T d\ell \end{bmatrix}, \quad (6.92)$$

where,
\[ K_{ij} = \int_{\alpha} \frac{\partial \Gamma}{\partial x_j} \eta_i(\phi) \frac{\partial \Gamma}{\partial x_i} dA, \quad (6.93) \]

\[ Q_i = \int_{\alpha} \frac{\partial \Gamma}{\partial x_i} \Phi^r dA, \quad (6.94) \]

\[ B_{xx} = \int_{\alpha} B_0 \frac{\partial \Gamma}{\partial y} \left( H_x H_y \frac{\partial \Gamma}{\partial x} + H_y^2 \frac{\partial \Gamma}{\partial y} \right) dA, \quad (6.95) \]

\[ B_{yy} = \int_{\alpha} -B_0 \frac{\partial \Gamma}{\partial y} \left( H_x^2 \frac{\partial \Gamma}{\partial x} + H_x H_y \frac{\partial \Gamma}{\partial y} \right) dA, \quad (6.96) \]

\[ B_{yx} = \int_{\alpha} -B_0 \frac{\partial \Gamma}{\partial x} \left( H_y^2 \frac{\partial \Gamma}{\partial y} + H_x H_y \frac{\partial \Gamma}{\partial y} \right) dA, \quad (6.97) \]

\[ B_{xy} = \int_{\alpha} B_0 \frac{\partial \Gamma}{\partial x} \left( H_y^2 \frac{\partial \Gamma}{\partial y} + H_x H_y \frac{\partial \Gamma}{\partial y} \right) dA, \quad (6.98) \]

\[ B_0 = Bi_\phi H^{-1/2} \left[ \frac{1 - \exp(-m\dot{\gamma})}{\dot{\gamma}} \right], \quad (6.99) \]

\[ S_{ij} = \int_{\alpha} \frac{\partial \Gamma}{\partial x_i} Mn^{-1} \mu_i(\phi) H_j dA, \quad (6.100) \]

\[ S_{0i} = -\int_{\alpha} \frac{\partial \Gamma}{\partial x_i} Mn^{-1} H_j \left[ \mu_i(\phi) - \phi \frac{d\mu_i}{d\phi} \right] dA, \quad (6.101) \]

Although the index \( i \) is repeated in Equation (6.100), there is no summation over this index.
The isotropic Bingham model was also implemented numerically for comparative purposes. This can be accomplished by replacing the constitutive model for the yield component of stress with (Papanastasiou, 1987):

$$\tau_{ji}^{(\text{Isotropic})} = \phi H^{3/2} \left[ \frac{1 - \exp(-m\dot{\gamma})}{\dot{\gamma}} \right] \dot{\gamma}_{ji}. \quad (6.102)$$

The derivation of finite element equations for an isotropic Bingham fluid follows the same procedure outlined above and is described in detail by Reddy and Gartling (2001).

The elemental integrals are numerically evaluated using Gauss integration on 3x3 points (illustrated in Figure 6.11) by the formula (Bathe, 1982):

$$\int_{-1}^{1} \int_{-1}^{1} F(r,s)drds = \sum_{i,j} \alpha_i \alpha_j F(r_i, s_j), \quad (6.103)$$

where, $\alpha_1 = \alpha_3 = 5/9$, $\alpha_2 = 8/9$, $r_1 = r_3 = -s_1 = s_3 = \sqrt{3/5}$ and $r_2 = s_2 = 0$.

The complete set of equations is calculated by assembling all the elemental matrices in the model and applying the boundary conditions. The magnetostatic problem is linear and can be directly solved. The flow problem is non-linear because the matrix $B_{ij}$ is velocity-gradient dependent. For this problem, first a set of linear equations is solved by setting $B_{ij} = 0$. This solution is then used as an initial condition for the non-linear problem. These equations are iteratively solved using the Picard method for which the convergence was found to be enhanced by the use of under-relaxation (Reddy and Gartling, 2001).

For a single-phase flow problem with a steady volume fraction distribution, the solution to the magnetostatic problem of Equation (6.77) is invariant of the flow problem. Therefore this
equation can be initially solved without considering the flow field. Obtaining this solution allows determination of the magnetic and yield components of the stress and allows for the flow problem to be solved.

The velocity gradients and magnetic field components (which are the gradients of the magnetic scalar potential) used in the constitutive relation (Equation (6.86)) pose a numerical challenge when solving the finite element equations. The velocity components and the magnetic scalar potential are continuous across element boundaries but their gradients are not. Therefore stress predicted by the constitutive relation includes an artificial jump at these interfaces. To alleviate this problem, a gradient smoothing scheme was utilized (Lee et al., 1979). In this technique, the gradient quantities utilized in the model (the magnitude of the shear rate tensor, $\dot{\gamma}$ and the components of the magnetic field, $H_x$ and $H_y$) are calculated on the 3x3 Gauss points. The middle node is assigned the value calculated for the middle Gauss point since these two points coincide. The values calculated at the remainder of the Gauss points are linearly extrapolated to the corresponding node by using the middle Gauss point value. Since this is done for every element, each node at the boundary of two or four elements receives a corresponding number of gradient values. A final smoothed value is calculated by averaging the elemental values weighted by the area of the element. The finite element model was implemented in Matlab®. The code generated as a part of this implementation is presented in Appendix D.

6.4.2 Hyperbolic contraction and expansion model

Pressure-driven flow through a hyperbolic contraction and expansion channel was modeled to study the important parameters related to MR fluid flow in non-uniform deformation and magnetic fields. The discretized finite element mesh is illustrated in Figure 6.13.
Figure 6.13: Finite element mesh for pressure-driven flow of an MR fluid through hyperbolic contraction and expansion. (a) The dimensions of the model with contraction ratio, $w_u/w_c = 2$. The upper contraction boundary is defined in the primed coordinate system by Equation (6.104). (b) Typical finite element mesh with refinement coefficient, $k_r = 16$. The mesh refinement is defined by the number of node spacings in each section of the geometry (as shown in drawing) and parameterized with the refinement coefficient.

The upper wall of the contraction section follows the hyperbola

$$y' = \frac{C}{a + x'},$$

(6.104)

where, $a = L_c w_c / (w_u - w_c)$ and $C = L_c w_u w_c / 2 (w_u - w_c)$. We can see that this functional form is equivalent to the one in Equation (2.6) by noticing that $w_c = 2h_c$ and $w_u = 2h_u$. Because of the symmetry of the flow in the upper and lower halves, both the magnetic and flow problems can be
solved by modeling only the upper half as shown in the finite element mesh of Figure 6.13b and applying symmetry boundary conditions along the line $y = w_u/2$.

The geometrical parameters describing the channel shape are identified in Figure 6.13a. The parameters determining the mesh refinement are defined in terms of the number of node spacings per a given section. For example, in the vertical direction there are $N_v$ spacings which correspond to $(N_v + 1)$ nodes and $N_v/2$ elements. The ratios between these spacing parameters were kept constant and the refinement was adjusted by the refinement parameter, $k_r$. Typical geometric and mesh parameters are given in Table 6.1.

Table 6.1: Typical geometric and mesh parameters used in the simulations. The values given are common to all simulation results unless otherwise noted. The total number of unknowns corresponding to these parameters is 4112 electromagnetic variables (magnetic scalar potential) and 9098 fluid mechanics variables (pressure and two velocity components)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_{H1}$</td>
<td>2</td>
</tr>
<tr>
<td>$L_c$</td>
<td>1</td>
</tr>
<tr>
<td>$L_{H2}$</td>
<td>1</td>
</tr>
<tr>
<td>$w_u$</td>
<td>2</td>
</tr>
<tr>
<td>$w_c$</td>
<td>1</td>
</tr>
<tr>
<td>$k_r$</td>
<td>16</td>
</tr>
<tr>
<td>$N_{H1}$</td>
<td>$4k_r$</td>
</tr>
<tr>
<td>$N_{H2}$</td>
<td>$4k_r$</td>
</tr>
<tr>
<td>$N_C$</td>
<td>$4k_r$</td>
</tr>
<tr>
<td>$N_v$</td>
<td>$k_r$</td>
</tr>
</tbody>
</table>
The typical physical parameters used in the simulations are presented in Table 6.2. Since the experimental measurements presented in Section 6.2 as well as Chapters 3 and 4 were taken using a commercial MR fluid (MRF-132DG manufactured by Lord Corporation), the volume fraction, $\phi$, and the particle relative permeability, $\mu^p$, were selected to closely match the properties of this fluid. The latter parameter is used in the column model of volume fraction dependent magnetic permeability of Equation (6.54). If we set this parameter to $\mu^p = 9/\phi$, we recover the measured bulk permeability of the fluid as shown in Figure 4.1.

For the results presented in the remainder of this chapter the typical parameters presented in Table 6.1 and Table 6.2 are utilized and any deviation from these values is noted.

Table 6.2: Typical physical and numerical parameters used in the simulations. The values given are common to all simulation results unless noted otherwise.

<table>
<thead>
<tr>
<th>Physical or numerical quantity</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Volume fraction</td>
<td>$\phi$</td>
<td>0.32</td>
</tr>
<tr>
<td>Max. packing fraction</td>
<td>$\phi_m$</td>
<td>0.68</td>
</tr>
<tr>
<td>Intrinsic viscosity</td>
<td>$[\eta]$</td>
<td>2.5</td>
</tr>
<tr>
<td>Particle radius</td>
<td>$a$</td>
<td>0.01</td>
</tr>
<tr>
<td>Regularization parameter</td>
<td>$m$</td>
<td>100</td>
</tr>
<tr>
<td>Relative permeability of the particle phase</td>
<td>$\mu^p$</td>
<td>$9/\phi$</td>
</tr>
</tbody>
</table>

For the flow problem, boundary conditions of the first kind are set on the channel walls ($v_z = v_y = 0$), the symmetry plane ($v_y = 0$) and on the inlet and exit of the channel ($v_x = 0$). The natural boundary condition on the symmetry plane dictates a zero velocity gradient $\partial v_z/\partial y$ on this surface. Using the natural boundary condition at the channel inlet and exit, a pressure
difference, \( \Delta p = p_{\text{inlet}} - p_{\text{exit}} \) is prescribed. To understand how this is done, we can calculate the surface traction in the x-direction on the channel inlet using Equation (6.82):

\[
T_x = p - 2\eta_s(\phi)\frac{\partial v_x}{\partial x} + Mn_\nu \frac{H^2}{2} \left[ \mu_\nu (\phi) - \phi \frac{d\mu_\nu}{d\phi} \right].
\] (6.105)

The yield component of the stress is zero for all components with repeated indices and therefore, \( \tau_{xx}^Y = 0 \) as described by the constitutive relation of Equation (6.86). The second (viscous) term in the right side of the traction equation is also zero because the velocity field is divergence-free \( (\partial v_x/\partial x = -\partial v_y/\partial y) \) and the other boundary condition on this surface sets \( v_y = 0 \). The third (magnetic) term is non-zero; however, since the magnetic field is the same at the inlet and discharge, it cancels out when the pressure difference is calculated.

We should also note that the governing equations of the single-phase model (Equations (6.55)-(6.57)) are non-dimensionalized using \( \Delta p \). The particular pressure differential simulated in the model can therefore be controlled by adjusting the dimensionless parameters, \( Mn_\nu \) and \( Bi_\nu \). For this reason, \( \Delta p \) is always set to unity in the numerical simulations.

The boundary conditions for the magnetostatic problem are set to satisfy the symmetry at the mid-plane of the channel and at the inlet and exit surfaces. The channel walls are taken to be infinitely permeable which requires the magnetic field lines to terminate orthogonally to each. To satisfy these requirements the magnetic scalar potential is set to \( 1/2 \) on the upper wall and to zero at the symmetry boundary, which generates \( H = 1 \) in the narrow section of the geometry (the field becomes exactly unity as \( L_{\nu_2} \rightarrow \infty \)). Since the scalar potential is a constant on both of
these surfaces, there is no gradient and therefore no magnetic field in the tangential direction. On
the inlet and exit surfaces, the natural boundary condition of $\mathbf{n} \cdot \mathbf{B} = 0$ is applied.

We can understand the nature of the flow and magnetic fields generated by the boundary
conditions by plotting the flow streamlines and magnetic field lines in the Newtonian limit
($Bi_p = Mn_p = 0$) which are illustrated in Figure 6.14.

6.4.3 Results
In any numerical simulation, it is important to establish that the numerical parameters do not
affect the solution obtained for the physical system modeled. Therefore, before proceeding
further with studying other parameters, we first validate the numerical approximations made in
the model.

First, the regularization of the constitutive relation of Equation (6.86) is evaluated. The
regularization parameter, $m$, was used to parameterize the degree of approximation of the
discontinuous constitutive relation (Equation (6.85)). As $m$ is increased, the regularized form
approaches the original constitutive relation. Simulations conducted with varying $m$ are
illustrated in Figure 6.15 in terms of the flow rate per unit length in the out-of-plane direction, in
which the results display a convergent behavior around $m = 100$. The flow rate per unit width is
calculated from the solution of the flow problem by

$$Q' = 2 \sum_{\text{inlet or exit elements}} \int v_x dl = 2 \sum_{\text{inlet or exit elements}} \left[ \int \Gamma dl \right] \hat{v}_x. \quad (6.106)$$
The numerical integral is taken only on the face of the element on the inlet or exit boundary, \( \partial \Omega' \). The factor of two appears in the equation to determine the equal contribution to flow from the symmetric upper and lower halves of the model (the latter being not included in the model). The Mason number also varies on each simulation point, satisfying the relation \( Bi_p Mn_p = 10 \). As illustrated in Figure 6.8, this is one of the two types of paths in the \( Bi_p - Mn_p \) space that can be followed with a constant fluid property, \( K_r \). In this case, the re-dimensionalized results would be a simulation of a series of experiments in which the magnetic field is kept constant and the pressure difference is varied.

![Flow streamlines and magnetic field lines](image)

Figure 6.14: (a) Flow streamlines and (b) magnetic field lines for \( Bi_p = Mn_p^{-1} = 0 \). In this limit, the fluid is Newtonian. The flow boundary conditions are no-slip and no-flux conditions on the channel wall, no-flux condition on the horizontal plane-of-symmetry, constrained vertical flow, and an applied pressure difference (natural boundary condition) in the channel inlet and outlet. The boundary conditions for the magnetic field problem are: applied magnetic scalar potential on the wall, and plane-of-symmetry and the natural boundary condition of zero-normal flux at the channel inlet and exit. The applied scalar potential condition satisfies the required symmetry at the mid-plane as the field lines terminate on this boundary in the normal direction.
In this plot, an analytical validation of the model is also presented. The lubrication approximation can be used to get a closed-form solution to the non-dimensional pressure difference across a hyperbolic contraction or expansion channel with profile given by Equation (6.104)

\[ \Delta P_{hyp} = \frac{3\eta_s(\phi)L_c}{w_u^3} \left[ \left( \frac{w_u}{w_c} \right)^2 + 1 \right] \left[ \frac{w_u}{w_c} + 1 \right] Q'. \]  

(6.107)

In obtaining this relation, the flow through the varying cross-section of the channel is locally approximated with a fully-developed flow through a straight channel of the same cross-section (for a detailed discussion on this method, see Bird et al. (1987), Sect. 1.3). This solution can also be used for the straight portions of the geometry by letting \( w_c = w_u \) and using the corresponding values for \( L_c \) and \( w_u \). Using this method we can find the flow rate per unit length, \( Q' = 0.0116\Delta P \), which is indicated by the dashed line in Figure 6.15. All the plots in this figure approach an asymptotic value as the fluid behavior goes to the Newtonian limit (\( Bi_p \to 0 \)) which is close to the approximate analytical value. Since the hyperbolic contraction is strong (which can be quantified by \( (dy'/dx')_{x=0} = 1 \)) it is not possible to draw very strong conclusions from this comparison; however, the match between the analytical and numerical results shows that the finite element solutions are close to an expected value.
Figure 6.15: Plots of volumetric flow rate per unit width versus $Bi_p$ with varying regularization parameter, $m$ (as defined in the constitutive relation of Equation (6.86)). The regularized constitutive relation approaches the original, and discontinuous, constitutive relation (Equation (6.85)) as $m$ is increased. The dashed line shows the analytical result obtained by the lubrication approximation. As $Bi_p \to 0$ (Newtonian fluid) the plots reach an asymptotic value close to the approximate analytical value.

Another numerical parameter to evaluate in the model is the mesh refinement. As a part of the finite element method, we have replaced the continuum variables defining the fluid flow by a discretized set of nodal variables. The solution of the discretized problem should not depend on the level of spatial refinement of the nodes, which is parameterized by $k_r$ (as shown in Table 6.2). In Figure 6.16, the ratio of the flow rate per unit length, $Q'$, to that at the highest level of refinement ($k_r = 16$) is plotted against $Bi_p$ for four different refinement coefficients. It can be
seen that the two highest refinement coefficients used result in a sufficiently close (within 0.4% of each other) numerical solution.

![Image](image_url)

Figure 6.16: Plots of the ratio of volumetric flow rate per unit length to that at the highest level of refinement \( k_r = 16 \) versus \( Bi_p \). Each plot shows simulation results with varying levels of refinement of the finite element mesh. The refinement is quantified by the parameter \( k_r \) as defined in Table 6.1. The Mason number is also varied in these plots, satisfying the relation \( Bi_p \cdot Mn_p = 10 \). At \( k_r = 16 \), the numerical results are convergent and largely invariant of refinement. This value is used for all of the simulations presented.

As an example of the solution for the flow and magnetic field problems, the contour plots for \( Bi_p = 0.562, \ Mn_p = 17.8 \) are presented in Figure 6.17. As shown in Figure 6.15, the flow field is a strong function of \( Bi_p \), since this parameter controls the amount of dissipation resulting from the plastic flow. The magnetic field solution, on the other hand, is invariant of both the...
dimensionless parameters. Therefore the contour and the field-line plots of Figure 6.17b are universal for all of the simulations presented in this subsection.

Figure 6.17: Contours of (a) flow speed and (b) magnetic field intensity for $Bi_p = 0.562$, $Mn_p = 17.8$. The flow streamlines and the magnetic field lines are superimposed on their respective domain plots. The dimensionless volumetric flow rate per unit length, $Q' = 4.66 \times 10^{-3}$ (calculated using Equation (6.106)).

Although $Bi_p$ affects the flow field significantly, flow rate plots obtained by varying the Mason number are virtually identical for this flow problem. Where $Mn_p$ becomes important is in the magnetically-induced particle phase slip, as it directly appears in the definition of $\Psi$ (Equation (6.62)). This can be seen in the contour plots of $\Psi$ presented in Figure 6.18. At high Mason numbers, the flow of the particle phase is dominated by hydrodynamic drag and little to no slip is expected. At lower values of $Mn$, the magnitude of $\Psi$ becomes significant, indicating that the results obtained from this simulation have limited accuracy because the uniform volume fraction assumption can no longer be justified.
Figure 6.18: Magnetic slip parameter, $\Psi$ (defined in Equation (6.62)) for (a) $Mn = 17.8$, (b) $Mn = 5.62$, and (c) $Mn = 1.78$. All $Bi_p$ satisfying $Bi_p Mn_p = 1$. At high $Mn$, the slip is negligible ($|\Psi| \ll 1$) and the single phase approximation is valid. As $Mn_p \to 0$ the magnetophoretic force becomes increasingly important and the flow velocity reduces (because of the increasing yield stress). Both of these effects contribute to increase relative motion of the fluid and particulate phases.

Since the constitutive relation describing the yield component is regularized, we cannot directly determine which portion of the flow is unyielded. It is still instructive, however, to analyze where in the flow we expect yielding to occur. For this we can define a yield criterion (Papanastasiou, 1987) based on the magnitude of the shear rate tensor $\dot{\gamma} = 10^{-3}$ which is chosen as an order of magnitude lower than the inverse of the regularization parameter $m^{-1}$. A deformation
rate higher than this threshold indicates a portion of the flow that is past the yield point. This is illustrated in the contour plots of Figure 6.19.

![Contour plots](image)

Figure 6.19: The magnitude of shear rate, \( \dot{\gamma} \), can be used to evaluate yielding behavior in the flow. (a) Contour plots of \( \dot{\gamma} \) for \( Bi_p = 0.562 \), \( Mn_p = 17.8 \). Since the constitutive behavior of the fluid is regularized for numerical simulations, the deformation rate is non-zero throughout the flow. A threshold can be defined to give an indication for yielding. In (b) a threshold of \( \dot{\gamma} < 10^{-3} \) is used to predict the yield surface. Gray indicates unyielded zone and the black the yielded zone. In (c) the same contour is presented for \( Bi_p = 0.316 \), \( Mn_p = 31.6 \). With a decreasing Bingham number, the thickness of the unyielded zone decreases.

Finally, it is important to compare the new anisotropic yield stress model to an existing multi-dimensional model. The most appropriate comparison for this purpose is the isotropic Bingham model (6.102). The plots of flow rate, \( Q' \) (Figure 6.20) show that at the same Bingham
number, the isotropic model overestimates the flow resistance, as compared to the simulation results of the new model. This is an expected result because the isotropic model treats the resistance provided by the magnetically aligned structures to be the same in all directions of deformation. The anisotropic model on the other hand, allows for some of the deformations (those not rotating the chains) to take place without an increase in the fluid resistance.

![Graph showing volumetric flow rate per unit length as a function of the magnetic Bingham number](image)

Figure 6.20: Volumetric flow rate per unit length as a function of the magnetic Bingham number for simulation using the newly developed anisotropic model and isotropic model. The isotropic model underestimates the flow rate as the dissipation characteristics of the fluid are overestimated. As $Bi_p \to 0$ both models converge to a value that is close to the analytical solution obtained using the lubrication approximation (shown with the dotted line).

### 6.5 A qualitative model for flows with particle phase slip

By identifying and evaluating a dimensionless parameter for quantifying the importance of magnetically-induced particle phase slip, $\Psi$, we have been able to identify flows in which the particle migration can be neglected and the fluid can be treated as a single-phase continuum. In
this section, we focus on the flows where particle slip is not negligible and seek to gain further understanding by developing a model that can lead to insight about the phenomenon.

If we first consider steady parallel flow of MR fluid in the 2-direction (orthogonal to the magnetic field), the continuity equation of the particle phase (Equation (6.6)) reduces to

\[ \frac{\partial (\phi v_z)}{\partial x_2} = 0. \]  

Using the definition of \( \Psi \) (Equation (6.46)) the particle phase velocity, \( v_z = (1 + \Psi) v_2 \). Since the average suspension velocity is constant, inserting this relation in (6.108), we find the quantity \( \phi (1 + \Psi) = \text{const.} \). Considering a flow problem in which there is no slip of the particle phase at the inlet we can relate the local volume fraction \( \phi \) at any position to the local \( \Psi (\phi) \) by the expression:

\[ \phi = \frac{1}{1 + \Psi (\phi)} \phi_0, \]  

where, \( \phi_0 \) is the inlet volume fraction. We should also note that \( \Psi \) is a strong function of the volume fraction. Therefore this relation is a non-linear equation describing the volume fraction distribution as a function of the local \( \Psi \).

Using this result, we can see that the change in the power law scaling for the magnetorheological stress is driven (at least in part) by the change in volume fraction. Considering for the case \( |\Psi| \ll 1 \) and expanding the denominator in Equation (6.109) we find:

\[ \tau_y \sim \phi H^{3/2} \sim (1 - \Psi) H^{3/2} \sim (1 + \alpha H^2) H^{3/2}, \]  

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where, $\alpha$ is a small parameter determined from Equation (6.46). We can see that $\Psi \sim H^2$ in this limit by noticing that both of the terms in Equation (6.62) scale quadratically with the applied field. This relation alone is not sufficient to predict the final scaling law observed when $\Psi$ becomes large; however, we can see that in regions where a magnetic field gradient exists, magnetophoretic effects result in alteration of the volume fraction and the scaling laws determined from studies under uniform magnetic field need not directly apply.

Figure 6.21: Perturbed volume fraction for $Bi_p = 0.562$, $Mn_p = 1.78$ in a hyperbolic contraction and expansion geometry ($L_{wt} = 0$). The distribution is a solution to Equation (6.109), where $\Psi$ is calculated by the finite element solution of the magnetic and flow fields. The contour levels are cropped to show distribution in the channel. The results show a decrease in volume fraction in the contraction section and an increase in the expansion channel. The largest increase in volume fraction is predicted on the channel walls, which is in agreement with the microscopic observations presented in Figure 5.8.

We can also obtain a spatially-resolved picture of the volume fraction distribution in a given MR fluid flow by incorporating the volume fraction relation in the finite element model. Equation (6.109) is solved using the flow and magnetic field solutions of the uniform $\phi$ problem. No further iterations on $\phi$ were made since, the goal of this analysis is only to determine its distribution in the flow field qualitatively.
These results are illustrated in Figure 6.21 for a hyperbolic contraction and expansion geometry. The volume fraction shows an enhancement in the expansion section and a reduction in the contraction geometry. Furthermore, the largest enhancement to the volume fraction is found near the wall of the channel, which is in agreement with the microscopic observations presented in Figure 5.8.

6.6 Conclusions

In this chapter, the flow of MR fluids in non-uniform magnetic and deformation fields was investigated through experimental measurements and the development of a two-phase model for the rheological response of an MR suspension.

Bulk rheological measurements were conducted under controlled non-uniform fields using the newly-developed annular slit-flow magnetorheometer. Compared against the measurements taken under uniform fields, the results from this instrument showed a different scaling law for the yielding or plastic component of the fluid resistance with increasing magnetic field. In the uniform field case, the results show a subquadratic scaling \( \tau_y \sim H^{1.5} \) in the intermediate field regime (this is the definition of the extents of this regime), which was discussed in detail in Chapter 4. For the non-uniform field conditions of the annular-slit flow magnetorheometer, the scaling law was found to be superquadratic \( F_y \sim H^{2.3} \), indicating an enhanced response of the fluid under these flows.

To understand the physics driving this enhancement and to develop a predictive tool for further investigations into MR fluid flows in non-uniform fields, an anisotropic viscoplastic model was developed. The model was formulated within the framework of a two-phase
continuum model. Similar models existing in the literature do not capture the yield stress behavior of MR fluids. The new model captures this behavior through a magnetically-generated body couple that acts on the fluid to resist shearing deformations of material elements across field lines.

Using the two-phase suspension balance model, the forces that lead to particle phase slip relative to the flowing matrix fluid were quantified and result in a general expression for a dimensionless slip parameter, \( \Psi \) (see Equation (6.46)). The use of this parameter as a tool to quantify the presence of slip in a given geometry and MR fluid flow was demonstrated.

The model was implemented computationally using the finite element method for flows in which the particle phase slip is negligible. Using this numerical model, hyperbolic expansion and contraction flows of MR fluids were studied. The model predictions were also contrasted with the predictions for an isotropic Bingham model.

Finally, a qualitative model for flows with significant slip was developed. The model was used to describe the changes in the volume fraction distributions for such flows. The results qualitatively agree with the microfluidic observations concerning the directionality of particle slip and indicated a mechanism that can lead to the superquadratic scaling laws identified in bulk rheological measurements of the MR fluid in non-uniform magnetic and flow fields.
Chapter 7

Conclusions and future work

The research described in this thesis was motivated by the potential applications for MR fluids in the oil & gas exploration and production industry. These applications bring about a number of operational requirements for the MR fluid such as generating large and controllable levels of magnetically induced shift in rheological properties, tolerance to elevated temperatures (in the range of 150°C), and relatively low bulk fluid densities to maintain manageable hydrostatic downhole pressures. Although the applications may be very different, the same technological drivers can also be identified for aerospace and robotics industries, in which the actuator weight and tolerance to extreme environments are important.

7.1 Conclusions

In this thesis we investigated a number of the fluid design constraints identified for oilfield applications. First, the evolution of the rheological properties of MR fluids over a wide range of magnetic field and temperature was investigated. A magnetorheology accessory to a commercial torsional shear rheometer, with a unique combination of high-field and high-temperature capability was designed and manufactured, as described in Chapter 2. Using this instrument, the rheological properties of MR fluids were characterized as a function of temperature, as presented in Chapter 3. The results, which showed a modest decrease in the MR effect at temperatures representative of a common downhole environment, were placed in

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context by considering the different thermophysical processes that are affected by the temperature.

In Chapter 4, the relationship between the applied magnetic field and the magnitude of the MR yield stress was investigated. Analyzing the bulk rheological measurements showed that the present scaling laws described in the literature have a systematic deviation from the experimental observations in the high-flux regime. To investigate the sources of this deviation, a finite element model was developed to understand interparticle magnetic interaction. By combining experimental measurements and numerical results, a new scaling law was identified between the MR yield stress and applied field and was shown to capture the field behavior across all field regimes.

Using microfluidic techniques, in Chapter 5, the aggregation phenomena and the evolution of fluid microstructure were investigated in flows with strong particle-wall interactions that are characteristic of many engineering applications of MR fluids. To conduct this study, a new process for constructing microfluidic devices was developed to more closely replicate the magnetic field conditions in actual devices. The results of this study highlighted design features and operational techniques that can improve the performance of MR fluid valves. Investigation of the flow and migration of MR fluid particles in non-uniform magnetic fields demonstrated that in regions of inhomogeneous field the particle phase velocity is governed by a balance between hydrodynamic and magnetophoretic forces, resulting in a ‘slip’ relative to the continuous matrix phase. These observations suggest that with the correct design methodology local magnetophoretic forces can be utilized to achieve higher levels of controllable fluid resistance through constructing such regions in MR valves.
Finally in Chapter 6, the bulk flow of MR fluids in spatially inhomogeneous magnetic and deformation fields was studied. A slit-flow magnetorheometer was manufactured to validate the microfluidic findings on the macro-scale and to measure the bulk MR response of the fluid under non-uniform fields. To understand the parameters governing these flows and to obtain a predictive tool suitable for future design investigations, a continuum-level model was developed, which captures the key features of multi-phase flow and fluid anisotropy. The model was implemented numerically using the finite element method and was used to study the flow of a two-phase MR suspension through contraction and expansion channels. This two-phase model provides insight into the design of MR fluid devices that enhance the magnetically-controlled flow resistance through local magnetophoretically-driven variations in the concentration of the MR particles. This numerical model can be used in the future to study other flow geometries.

To summarize, with these studies we have gained further understanding of the parameters governing the MR fluid response as a function of temperature, magnetic field strength and field gradients. We can review the key findings under the three research themes we identified in the title: stronger, lighter and hotter. Using bulk rheological measurements and finite element modeling of interparticle magnetic interaction in the high-flux regime we have identified a universal scaling law that is applicable at all magnetic field levels. This scaling law can allow the designers of MR fluids and actuators to accurately predict the evolution of the MR effect with applied field in cases where it may not be practical to do the measurement beforehand, and thus have a better understanding of how much stronger the MR fluid will be expected to be at a given (high) field. For example, experimental measurements obtained from a magnetorheometer or an MR fluid device with limited magnetic field capabilities can be extrapolated into the high-field regime with good accuracy as demonstrated with the universal curve plotted in Figure 4.11.
Through our investigations of MR fluid flow in spatially-inhomogeneous magnetic and deformation fields we have identified flow regimes in which the local volume fraction of the fluid can be enhanced by the influence of magnetophoretic forces. This enhancement and the tools and parameters identified which drive magnetophoretic separation, allow engineers to design devices and actuators that promote the enhancement and utilize lower particle volume fraction fluids as a result. Because of the high density of the MR fluid particles \( \rho_{iron} \approx 7600\, \text{kg/m}^3 \) as compared to the matrix fluid \( \rho_{oil} \approx 800\, \text{kg/m}^3 \), the reduction of particle volume fraction commonly results in a lighter MR fluid, which can enable density sensitive applications, such as those in the oilfield, aerospace and robotics industries. For example, the hyperbolic contraction and expansion flow simulations illustrated in Figure 6.18, show that by utilizing a higher Mason number flow, it is possible to induce particle phase slip (Figure 6.18c) which leads to a magnetically enhanced volume fraction and a higher shear stress fluid.

Finally, through our study of the evolution of the rheological properties of MR fluids with temperature, we have characterized the expected changes to the performance of MR fluid devices in hotter environmental conditions. The reduction, which was found to be approximately 10\% at a common bottom-hole temperature of 130\(^{\circ}\)C (see Figure 3.4), was found to be closely related to changes in physical parameters affecting the MR yield stress such as saturation magnetization and volume fraction.

### 7.2 Future work

The scope of this thesis focused on several different aspects of MR fluid flow with the central focus on applications in the downhole environment. The next step towards engineering
implementation of MR fluids in various oilfield applications is to consider design and operational suitability. These studies should involve specific engineering aspects of device design in terms of tolerance to the extreme environments and long-term reliability.

In the process of this investigation, we have also made new advancements on experimental methods and constitutive modeling of MR fluids. These advancements may open a door for further work in the area to advance our understanding. In the area of microfluidics, we have demonstrated a method of manufacturing channels that incorporate magnetizable walls that create a strong particle-wall interaction, while maintaining a uniform magnetic field in the flow channel. Important follow-on work in this area would be to make "online" measurements of the evolution in fluid rheology through inlet and exit pressure measurements on the microfluidic channel. The simultaneous acquisition of fluid microstructure and rheology may uncover important aspects of MR fluid flow, which, until now, cannot be achieved with the 'decoupled' studies of rheology and microstructure. We have seen that the microstructural aspects are important in the development of anisotropic constitutive models for MR fluids. A combined microstructure/rheology study can enable further advancements of these models.

In Chapter 6, we developed continuum models for MR fluids with either a single-phase or a two-phase (particles and matrix) description of the fluid. The results of the chapter demonstrate that differences in the local velocity of the two phases of the fluid can lead to significant shifts in the measured resistance to flow through changes in local volume fraction of particles. The current study used a single-phase model of MR fluid flow and utilized the dimensionless magnetophoretic slip parameter to evaluate the validity of this description in a given magnetic and flow field. Numerically implementing a full non-linear two-phase MR fluid model could enable quantitative computational studies in the slip regime that consistently couple
large evolution in the volume fraction and in MR stress. A key challenge in development of such a model is the unstable nature of the electromagnetic stress. A constitutive relation that captures the magnetic body force in a two-phase model, while retaining suspension stability, may enable a successful implementation.

Through the investigations of different aspects of MR fluids that are important for its commercial introduction in the oilfield, we have seen that the fluid can indeed function well in the elevated temperature environment and has the versatility to tolerate design requirements of high shear stress and low bulk fluid density. Based on the conclusions of this work, we may in fact expect the growing use of MR fluids in the automotive segment to cross-over to the oilfield applications. The tools developed in the present study can be used to understand how temperature, magnetic fields and device design can be exploited to make MR fluids that operate stronger, lighter and hotter.
Appendix A

Heat transfer calculations for torsional shear magnetorheometer accessory

In this appendix, the heat transfer calculations for the fluid circulation system are presented. The correlations used for friction factors and heat transfer coefficients can be found in Mills (1999).

The circulator can deliver a maximum pressure of \( P_{\text{max}} = 4.42 \times 10^4 \text{ Pa} \) and a maximum flow rate of \( Q_{\text{max}} = 2.5 \times 10^{-3} \text{ m}^3/\text{s} \). The properties for a typical silicone fluid used in the circulator are the kinematic viscosity \( \nu = 10^{-6} \text{ m}^2/\text{s} \), the density \( \rho = 720 \text{ kg}/\text{m}^3 \), the thermal conductivity \( k = 0.133 \text{ W/(m.K)} \), the constant-pressure specific heat \( C_p = 2553 \text{ J/(kg.K)} \) and the Prandtl number \( Pr = 13.8 \). An iterative solution method is used to determine the flow rate which resulted in \( Q = 1.05 \times 10^{-4} \text{ m}^3/\text{s} \) for a total pressure drop \( \Delta P_t \) contributed by three sources: the connection hose, spiral channels and the annular channel.

The flow in the connection hose is found to be in turbulent since the Reynolds number is

\[
\text{Re}_{D_h} = \frac{\bar{V} D_h}{\nu} = 1.41 \times 10^4, \quad (A.1)
\]
where, the average flow velocity $V_1 = 1.48 \text{m/s}$ and the hydraulic diameter (which is equal to the inner diameter of the round hose) $D_{h1} = 0.0095 \text{m}$. The friction factor is found to be

$$f_1 = \left(0.790 \ln \text{Re}_{D_{h1}} - 1.64\right)^2 = 0.0287,$$  \hspace{1cm} (A.2)

and the resulting pressure drop is

$$\Delta P_1 = f_1 \left( \frac{L}{D_{h1}} \right) \left( \frac{1}{2} \rho V_1^2 \right) = 8.70 \times 10^3 \text{Pa}.$$  \hspace{1cm} (A.3)

In the spiral flow channel the perimeter $P_2 = 0.0254 \text{m}$ and the cross-sectional area $A_{c2} = 4.03 \times 10^{-5} \text{m}^2$ results in the hydraulic diameter

$$D_{h2} = \frac{4A_{c2}}{P} = 0.00635 \text{m}.$$  \hspace{1cm} (A.4)

The flow in this channel is turbulent as well with

$$\text{Re}_{D_{h2}} = \frac{V_2 D_{h2}}{\nu} = 1.65 \times 10^4,$$  \hspace{1cm} (A.5)

where the average velocity $V_2 = 2.60 \text{m/s}$. The friction factor is

$$f_2 = \left(0.790 \ln \text{Re}_{D_{h2}} - 1.64\right)^2 = 0.0275,$$  \hspace{1cm} (A.6)

which is used in determining the pressure drop $\Delta P_2 = 3.38 \times 10^4 \text{Pa}$. The Nusselt number in this channel is
\[ \text{Nu}_{D2} = \frac{(f_2/8)(\text{Re}_{D2}-1000)\text{Pr}}{1+12.7(f_2/8)^{1/2}(\text{Pr}^{2/3}-1)} = 162. \quad (A.7) \]

The convective heat transfer coefficient is

\[ h_{c2} = \frac{\text{Nu}_{D2}k}{D_{h2}} = 3400 \frac{W}{m^2 K} \quad (A.8) \]

Finally, in the annular channel the flow is laminar with \( V_3 = 1.69 m/s \), \( D_{h3} = 0.0013 m \) and \( \text{Re}_{Dh3} = 2.15 \times 10^3 \). The friction factor is \( f_3 = 63/\text{Re}_{Dh3} \) and the pressure drop is \( \Delta P_3 = 1.69 \times 10^3 Pa \). The Nusselt number is \( \text{Nu}_{D3} = 4.82 \) and the heat transfer coefficient is

\[ h_{c3} = 505 W m^{-2} K^{-1} \]

These result in a total pressure drop \( \Delta P_T = \Delta P_1 + \Delta P_2 + \Delta P_3 = 4.42 \times 10^4 Pa \) which is within the capabilities of the circulator as discussed above.
Appendix B

Annular slit-flow magnetorheometer drawings

In this appendix, the technical drawings used in the manufacturing of the annular slit-flow magnetorheometer are presented. All dimensions are in inches.
5-40 UNC - 2B TAP \( \frac{24}{64} \)  
38 DRILL ( 0.102 ) \( \frac{24}{64} \) - ( 3 ) HOLES

\[ \phi 1.375 \]

\[ \phi 1.9375 \]

\[ \phi 1.746\pm0.001 \]

\[ \phi 19.98\pm0.001 \]

\[ \phi 0.188 \]

\[ 0.500 \]

\[ 0.28 \]

\[ 0.575 \]

\[ 0.125 \]

\[ 0.625 - 18.000 UNF - 2B \]

\[ \phi 0.500 \]

\[ \phi 0.28 \]

\[ \phi 0.188 \]

\[ \phi 0.625 \]

\[ \phi 0.365\pm0.005 \]

\[ \phi 0.125 \]

\[ \phi 0.575 \]

\[ \phi 0.681 \]

\[ \phi 0.750 \]

\[ 1.000 \]

\[ \phi 1.88 \]

SECTION A-A

1018 CARBON STEEL

UPPER PISTON

K.X \( \pm0.1 \)

K.XX \( \pm0.01 \)

K.XXX \( \pm0.005 \)

ANG. \( \pm0.5 \)

M. OCALAN
3X Ø.154
Ø120° AS SHOWN

Ø1.375

Ø1.88

Ø1.00

.25

.125±.010

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<td>ANG.</td>
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ALUMINUM

RETAINER PLATE

M. O. ALAN
SECTION A-A

MATERIAL: PTFE

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<tr>
<td>XXXX</td>
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<tr>
<td>ANG</td>
<td>±0.5</td>
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PLASTIC BUSHING
**DETAIL A**

SCALE 1:1

1/4-20 UNC - 2B TAP ± 0.002

#7 DRILL (0.201) ± 0.003 - 1/8" HOLE

SHAPE OF SPLINE IS PROVIDED IN THE 3D CAD FILE

1018 CARBON STEEL

LOWER PISTON

M. D. ALAN
SEE DETAIL A
SCALE 16:1
TYP. 2X

DETAIL A

COIL TUBE

ALUMINUM

| X.X  | ±0.1 |
| X.XX | ±0.01 |
| X.XXX| ±0.005|
| ANG. | ±0.5 |

M. O'CALAN
SECTION A-A

MATERIAL: PTFE

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<tbody>
<tr>
<td>X.X</td>
<td>±0.1</td>
<td></td>
</tr>
<tr>
<td>X.XX</td>
<td>±0.01</td>
<td></td>
</tr>
<tr>
<td>X.XXX</td>
<td>±0.005</td>
<td></td>
</tr>
<tr>
<td>ANG.</td>
<td>±0.5</td>
<td></td>
</tr>
</tbody>
</table>

BOBBIN

M. O'CAHAN
Appendix C

Source code – Numerical model of interparticular force

In Chapter 4, a finite element model was developed to determine the magnetic force between two spherical magnetizable particles in an applied field. The Matlab® source code for this model is provided here.

```matlab
function simpleparticle_2particleattraction_frameSI(gap)

%%%%% This is the main function for FE modeling of interparticle magnetic force. The sub-functions are contained within this function

close all
clear all
clear global all

% Constants and properties
mu0 = 4e-7*pi;
mur = 1e3;
Rpart = .5e-6;
gap = 30e-9;
W = 3.5*Rpart;
Wy = 5*Rpart;
elemsize = 5e-8;
satmag = 2.1/mu0;
alpha = 0.1;
numtol = 1e-12;
magdom = 2;
airdom = 1;
volfac = 0.32;
fscale = pi/6*mu0*Rpart^2*satmag^2;

% Solver parameters
maxnosubsteps = 10000;
```
Linpot = 1e-6;
SoluTol = 1e-7;

appotval = 1e5*(Rpart+Wy);
Shearstr = [];
Bave = [];
Mave = [];
Partforce = [];
PartforceNI = [];
PartforceNO = [];
Partforcebound = [];

for ForIndex = 1:length(appotval)
    appot = appotval(ForIndex);
    optimgap = gap;
    forceout(optimgap);
    appot = appotval;
    Partforce = [Partforce Fchain];
    PartforceNI = [PartforceNI Fchainsurf];
    PartforceNO = [PartforceNO Fchainsurfout];
    Partforcebound = [Partforcebound Fchainbound];
    Shearstr = [Shearstr Forcex*3*Volfrac/(2*pi*Rpart^2)];
    Strain = sqrt(2/Rpart+optimgap+(optimgap/Rpart).^2);
    sum(pi*r0.*Delem.*B)/(pi*W^2*(Rpart+optimgap));
    Mave = [Mave sum(mz)/sum(Vmag)];
    Bave = [Bave sum(pi*r0.*Delem.*B)/(pi*W^2*(Rpart+optimgap))];

figure(10)
loglog(appotval(1:length(Shearstr))/(Rpart+Wy)/Satmag,...
    Partforcebound/Fscale,'-o');
xlabel('H_a_p / M_s')
ylabel('F_p_a_r_t / F_0')

figure(12)
loglog(appotval(1:length(Shearstr))/(Rpart+Wy)/Satmag,...
    abs(Mave)/Satmag,'-o');
xlabel('H_a_p / M_s')
ylabel('\langle M_p \rangle / M_s')
figure(13)
loglog(abs(Mave)/Satmag,Partforcebound/Fscale,'-og');
xlabel('\langle M_p \rangle / M_s')
ylabel('F_p_a_r_t / F_m_a_g')
figure(1)
end

figure(13)
hold on
FitX = abs(Mave)/Satmag;
FitY = Partforcebound/Fscale;
FitCoef = FitY(1)/(FitX(1)^2);
loglog(FitX,FitCoef.*FitX.^2,'--r')

fclock = clock;
save(cat(2,'OptimizerResults_'),num2str(fclock(1)),'_',num2str(fclock(2))...,'_',num2str(fclock(3)),'_',num2str(fclock(4)),num2str(fclock(5))))
figure(20)
saveas(gcf,cat(2,'LinePlot_','num2str(fclock(1)),'\_','num2str(fclock(2))...\_','num2str(fclock(3)),'\_','num2str(fclock(4)),'num2str(fclock(5)))))
figure(1)
saveas(gcf,cat(2,'ContourPlot_','num2str(fclock(1)),'\_','num2str(fclock(2))...\_','num2str(fclock(3)),'\_','num2str(fclock(4)),'num2str(fclock(5)))))

function Minfunc = forceout(x0)
    Initgap = x0;
    
    % Generate FEA location, connectivity and domain matrices, % setup matrices for force integral
    automaticmeshgen;
    setintegral;
    findaxis;
    R = []; Z = [ ]; C = [ ]; B = [ ]; mz = [ ]; rO = [ ]; Delem = [ ];
    for i = 1:length(conmat)
        R = [ R locmat(conmat(i,:)',1)];
        Z = [ Z locmat(conmat(i,:)',2)];
        C = [ C dommat(i)];
    end
    Kmat0 = sparse(length(locmat),length(locmat));
    Qmat00 = sparse(length(locmat),1);
    for i = 1:length(conmat)
        r0(i) = sum(locmat(conmat(i,:)',1))/3;
        Delem(i) = det([ones(3,1),locmat(conmat(i,:)',1)... locmat(conmat(i,:)',2)]);
        q(:,i) = [0 1 -1;1 0 -1 -1 0]*locmat(conmat(i,:)',2);
        f(:,i) = [0 -1 1;1 -1 1 0]*locmat(conmat(i,:)',1);
        Sraw(:,:,i) = q(:,i)*q(:,i)'+f(:,i)*f(:,i);
        if dommat(i) = Magdom
            S = pi*r0(i)*muO/Delem(i)*Sraw(:,:,i);
            Kmat0(conmat(i,:),conmat(i,:)) = Kmat0(conmat(i,:),... conmat(i,:)) + S;
        else
            Vmag(i) = pi*r0(i)*Delem(i);
        end
    end
    maxz = max(locmat(:,2));
    minz = min(locmat(:,2));
    Boundmult = logical(ones(length(locmat)));
    Boundadd = sparse(zeros(length(locmat)));
    for i = 1:size(locmat)
        if (abs(locmat(i,2)-maxz)<NumTol) || (abs(locmat(i,2)-minz)...<NumTol)
            Boundmult(i,:) = zeros(1,length(locmat));
        end
    end
Boundadd(i,i) = 1;
if (abs(locmat(i,2)-maxz)<NumTol)
    Qmat00(i,1) = appot;
end
end

% Initial guess
Kmatint = KmatO;
for i = 1:length(conmat)
    if dommat(i) == Magdom
        S = pi*r0(i)*mu0*mur/Delem(i)*Sraw(:,i,i);
        Kmatint(conmat(i,:),conmat(i,:)) = Kmatint(conmat(i,:),... conmat(i,:)) + S;
    end
end
Kmatint = Kmatint.*Boundmult+Boundadd;
V0 = (Kmatint\Qmat00)*Linpot;
Lastpot = Linpot;
LastV0 = V0;

for j = 1:1
    Substeppot = 1;
    Refn = (Substeppot-Linpot);
    while Lastpot ~= 1
        Qmat0 = Qmat00*Substeppot;
        V0 = LastV0*Substeppot/Lastpot;
        [Vmat,Exitflag] = msolve(V0);
        while -Exitflag && (Refn>l/Maxnosubsteps)
            Refn = Refn/2;
            Substeppot = Lastpot+Refn;
            Qmat0 = Qmat00*Substeppot;
            V0 = LastV0*Substeppot/Lastpot;
            [Vmat,Exitflag] = msolve(V0);
        end
        mz = zeros(1,length(conmat));
        Hr = zeros(1,length(conmat));
        Hz = zeros(1,length(conmat));
        H = zeros(1,length(conmat));
        for i = 1:length(conmat)
            Hr(i) = -1/Delem(i).*g(:,i)'*Vmat(conmat(i,:));
            Hz(i) = -1/Delem(i).*f(:,i)'*Vmat(conmat(i,:));
            H(i) = sqrt(Hr(i)^2+Hz(i)^2);
            if dommat(i) == Magdom
                B(i) = H(i)*mu0*(1+(mur-1)*Satmag/H(i)... /((mur-1+Satmag/H(i))));
                Bz = Hz(i)*mu0*(1+(mur-1)*Satmag/H(i)... /((mur-1+Satmag/H(i))));
                mz(i) = Vmag(i)*(Bz/muO-Hz(i));
            else
                B(i) = H(i)*muO;
            end
        end
        figure(1)
        fill(R/Rpart,Z/Rpart,(B/muO-H)/Satmag,'linestyle','none')

    end
end
axis equal
axis([0,max(locmat(:,1))/Rpart,min(locmat(:,2))...]
/Rpart,max(locmat(:,2))/Rpart])
colorbar
title(['Substeppot = ',num2str(Substeppot),...
', Lastpot = ',num2str(Lastpot),' Exitflag = ',...
num2str(Exitflag)])
ylabel('z / r_p')
xlabel('r / r_p')
Lastpot = Substeppot;
LastV0 = Vmat;
Refn = 1-Substeppot;
Substeppot = 1;
end

H0 = -appot/(max(locmat(:,2))-min(locmat(:,2)));
Pchain = muO*pi/2*((Hz.*Bound'-H0*Bound').^2).*Elemedge;
Forcex = sqrt(1-(Rpart/(Rpart+Initgap))^2)*Fchain;

Fchainsurf = 0;
Fchainsurfout = 0;
Fchainbound = 0;
for i = 1:size(Edgeelement,1)
    Fchainsurf = Fchainsurf + pi*(locmat(Edgepoint(i),1)...)
     +locmat(Edgepoint(i+1),1))*
     ((muelemrec(Edgeelement(i,2))*Hr(Edgeelement(i,2))...
     *Hz(Edgeelement(i,2)))*
     *locmat(Edgepoint(i),2)-locmat(Edgepoint(i+1),2))...
     +muelemrec(Edgeelement(i,2))*Hz(Edgeelement(i,2))^2...
     -muO/2*(Hz(Edgeelement(i,2))^2+Hr(Edgeelement(i,2))^2))...
     *(locmat(Edgepoint(i+1),1)-locmat(Edgepoint(i),1)))
end
for i = 1:length(Boundelement)
    Fchainbound = Fchainbound + pi*(locmat(Boundpoint(i),1)...)
     +locmat(Boundpoint(i+1),1))*
     ((muelemrec(Boundelement(i,1))*Hr(Boundelement(i,1))...
     *Hz(Boundelement(i,1)))*
     *locmat(Boundpoint(i),2)-locmat(Boundpoint(i+1),2))...
     +muelemrec(Boundelement(i,1))*Hz(Boundelement(i,1))^2...
     -muO/2*(Hz(Boundelement(i,1))^2+Hr(Boundelement(i,1))^2))...
     *(locmat(Boundpoint(i+1),1)-locmat(Boundpoint(i),1)))
end
Forcexsurf = sqrt(1-(Rpart/(Rpart+Initgap))^2)*Fchainsurf;
Forcexsurfout = sqrt(1-(Rpart/(Rpart+Initgap))^2)*Fchainsurfout;
Minfunc = -Forcex;
sum(Vmag);

figure(20)
hold on
plot((B(Axiseelement)/(4e-7*pi)-H(Axiselement))/Satmag,...
Axislocs(1:(length(Axislocs)-1))/Rpart,...
'linewidth',1);
axis([0 1 0.5 1])
xlabel('M / M_s')
ylabel('z / r_p')
hold off

function [Vnew,Exitflag] = msolve(V00)
Maxnoloop = 50;
loopcont = 1;
noloop = 1;
muelemrec = zeros(size(conmat,1),1);
while loopcont
    Kmat = Kmat0;
    Qmat = Qmat0;
    for i = 1:length(conmat)
        if dommat(i) == Magdom
            Hrelem = -1/Delem(i)*(q(:,i)'*V00(conmat(i,:)));
            Hzelem = -1/Delem(i)*(f(:,i)'*V00(conmat(i,:)));
            Helem = sqrt(Hrelem^2+Hzelem^2);
            muelem = mu0*(1+(mur-l)*Satmag/Helem...
                          /(mur-1+Satmag/Helem));
            S = pi*r0(i)*muelem/Delem(i)*Sraw(:,i,i);
            Kmat(conmat(i,:),conmat(i,:)) = Kmat(conmat(i,:),conmat(i,:)) + S;
            muelemrec(i) = muelem;
        else
            muelemrec(i) = mu0;
        end
    end

    Kmat = Kmat.*Boundmult+Boundadd;
    Vnew = Kmat\Qmat;
    if abs(V00'*ones(size(V00))-Vnew'*ones(size(V00)))/...
         (V00'*ones(size(V00))) < SoluTol
        loopcont = 0;
    end

    noloop = noloop+1;
    V00 = Vnew;
    if noloop > Maxnoloop;
        loopcont = 0;
    end
end
if noloop > Maxnoloop
    Exitflag = 0;
else
    Exitflag = 1;
end

end % msolve

function automaticmeshgen
    Refr = .75*Rpart;
    Refz = .75*Rpart;-2*Initgap;0;

    gd = [1, 0, 0, Rpart,0,0,0,0,0,0,0,0,0,0; 3, 4, -Rpart, 0, 0, -Rpart, Rpart, ... Rpart, -Rpart, -Rpart, 0,0,0,0; 3, 4, 0, W, W, 0, -Wy, -Wy, ... Rpart+Initgap, Rpart+Initgap, 0,0,0,0]';
    sf = 'R1-C1+(C1-R2)';
    ns = [ 67 82 82 49 50 49];
    dl=decsg(gd,sf,ns);
    [locmat,e,conmat] = initmesh(dl,'Hmax',Elemsize);

    Tris = zeros(length(conmat),1);
    if Initgap < (2*Elemsize)
        for j = 1:2 %ceil(log2(Elemsize/Initgap))
            for i = 1:length(conmat)
                if ( ([1,0]*(sum(locmat(:,conmat(1:3,i)),2)/3)) ... < Refr) &&( ([0,1]*... (sum(locmat(:,conmat(1:3,i)),2)/3))> Refz)
                    Tris(i,1) = 1;
                end
            end
            [locmat,e,conmat]=refinemesh(dl,locmat,e,conmat,...
            find(Tris));
            Tris = zeros(length(conmat),1);
            Refr = Refr/2;
            Refz = Rpart-(Rpart-Refz)/2;
        end
    end

    pdemesh(locmat,e,conmat)
    axis equal

    locmat = locmat';
    dommat = conmat(4,:);'
    conmat = conmat(1:3,:);'
end % automaticmeshgen

function setintegral
    point1 = intersect(find(locmat(:,1)<(0+NumTol)&locmat(:,1)...>
                         (0-NumTol)),... find(locmat(:,2)<(max(locmat(:,2))+NumTol)&locmat(:,2)...>
                         (max(locmat(:,2))-NumTol)));
Bound = zeros(length(conmat),1);
Elemedge = Bound;
Boundelement = [ ];
Boundpoint = point1;
while locmat(point1,1) < (max(locmat(:,1)) - NumTol)
    [rowall,colall,dummy] = find(conmat==point1);
    for i = 1:length(rowall)
        boundcount = 0;
        for j = 1:3
            if (abs(locmat(conmat(rowall(i),j),1)) - max(locmat(:,1))) < NumTol... 
                && (locmat(conmat(rowall(i),j),1)) >= locmat(point1,1))
                boundcount = boundcount +1;
            end
        end
        if boundcount == 2
            searchelem = i;
        end
    end
    Bound(rowall(searchelem)) = 1;
    Boundelement = [Boundelement rowall(searchelem)];
    for i = 1:3
        if abs(locmat(conmat(rowall(searchelem),i),1)) - max(locmat(:,1)) < NumTol...
            && abs(locmat(conmat(rowall(searchelem),i),1)) >= locmat(point1,1))
            point2 = conmat(rowall(searchelem),i);
        end
    end
    Elemedge(rowall(searchelem)) = locmat(point2,1)^2 ...
                           -locmat(point1,1)^2;
    point1 = point2;
    Boundpoint = [Boundpoint point1];
end %while

% Side cylinder (whole model)
point1 = intersect(find(locmat(:,1)<(max(locmat(:,1))... 
+NumTol) & locmat(:,1)>(max(locmat(:,1)) - NumTol)),... 
find(locmat(:,2)<(max(locmat(:,2))+NumTol) & locmat(:,2)... 
>(max(locmat(:,2))-NumTol)));
while locmat(point1,2) > (min(locmat(:,2)) + NumTol)
    [rowall,colall,dummy] = find(conmat==point1);
    for i = 1:length(rowall)
        boundcount = 0;
        for j = 1:3
            if (abs(locmat(conmat(rowall(i),j),1)) ... 
                -max(locmat(:,1))) < NumTol... 
                && (locmat(conmat(rowall(i),j),1)) <= locmat(point1,2))
                boundcount = boundcount +1;
            end
        end
        if boundcount == 2
            searchelem = i;
        end
    end

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end
Boundelement = Boundelement(rowall(searchelem));
for i = 1:3
    if abs(locmat(conmat(rowall(searchelem),i),1)
        - max(locmat(:,1))) < NumTol
        && abs(locmat(conmat(rowall(searchelem),i),2)
        - locmat(point1,2)) > NumTol
        point2 = conmat(rowall(searchelem),i);
    end
end
point1 = point2;
Boundpoint = [Boundpoint point1];
end %while

% Bottom disc (whole model)
point1 = intersect(find(locmat(:,1)<(max(locmat(:,1))
    + NumTol) & locmat(:,1)> (max(locmat(:,1))-NumTol)),
    find(locmat(:,2)<(min(locmat(:,2))+NumTol) & locmat(:,2)
    > (min(locmat(:,2))-NumTol)));
while locmat(point1,1) > (NumTol)
    [rowall,colall,dummy] = find(conmat==point1);
    for i = 1: length(rowall)
        boundcount = 0;
        for j = 1:3
            if (abs(locmat(conmat(rowall(i),j),2)
                - min(locmat(:,2)))<NumTol)
                && (locmat(conmat(rowall(i),j),1)
                <=locmat(point1,1))
                boundcount = boundcount +1;
            end
        end
        if boundcount == 2
            searchelem = i;
        end
    end
Boundelement = Boundelement(rowall(searchelem));
for i = 1:3
    if abs(locmat(conmat(rowall(searchelem),i),2)
        - min(locmat(:,2)))<NumTol
        && abs(locmat(conmat(rowall(searchelem),i),1)
        - locmat(point1,1)) > NumTol
        point2 = conmat(rowall(searchelem),i);
    end
end
point1 = point2;
Boundpoint = [Boundpoint point1];
end %while

point1 = intersect(find(locmat(:,1)<(0+NumTol) & locmat(:,1)
    >(0-NumTol)), find(locmat(:,2)<(Rpart+NumTol)...
    & locmat(:,2)>(Rpart-NumTol)));
Boundout = zeros(length(conmat),1);
Edgepoint = point1;
Edgeelement = [];
Boundin = Boundout;
while locmat(point1,2) > (-Rpart+NumTol)
    [rowall, colall, dummy] = find(conmat==point1);
    for i = 1: length(rowall)
        boundcount = 0;
        for j = 1:3
            if abs(sqrt(locmat(conmat(rowall(i),j),1)^2 + locmat(conmat(rowall(i),j),2)^2)-Rpart) < NumTol 
                && atan2(locmat(conmat(rowall(i),j),2),locmat(conmat(rowall(i),j),1)) <= atan2(locmat(point1,2),locmat(point1,1))
            end
        end
        if boundcount == 2
            if dommat(rowall(i)) == Magdom
                searchelemin = i;
            else
                searchelemout = i;
            end
        end
    end
    Boundin(rowall(searchelemin)) = 1;
    Boundout(rowall(searchelemout)) = 1;
    Edgeelement = [Edgeelement rowall(searchelemout) rowall(searchelemin)];
    for i = 1:3
        if abs(sqrt(locmat(conmat(rowall(searchelemout),i),1)^2 + locmat(conmat(rowall(searchelemout),i),2)^2)-Rpart) < NumTol 
            && atan2(locmat(conmat(rowall(searchelemout),i),2),locmat(conmat(rowall(searchelemout),i),1)) <= atan2(locmat(point1,2),locmat(point1,1))
        end
    end
    point1 = point2;
    Edgepoint = [Edgepoint point1];
end
end % setintegral

function findaxis

point1 = intersect(find(locmat(:,1)<(0+NumTol)&locmat(:,1)>0-NumTol)),
        find(locmat(:,2)<(Rpart+NumTol)&locmat(:,2)>(Rpart-NumTol)));
Axiselement = [];
Axislocs = locmat(point1,2);
while locmat(point1,2) > (-Rpart+NumTol)
    [rowall, colall, dummy] = find(conmat==point1);
    for i = 1: length(rowall)
for \( j = 1:3 \)
   if locmat(conmat(rowall(i),j),1) < NumTol ... 
      && locmat(conmat(rowall(i),j),2) ... 
      < locmat(point1,2)
      point2 = conmat(rowall(i),j);
      Axislocs = [Axislocs 
                  locmat(point2,2)];
      Axiselement = [Axiselement 
                     rowall(i)];
      end
   end
end
point1 = point2;
end
end % forceout
end % Main function
Appendix D

Source code – Finite element model of MR fluid flow

In Chapter 6, a finite element model was developed to study the flow of MR fluids in spatially inhomogeneous deformation and magnetic fields. The numerical implementation includes developing and solving the coupled magnetic and flow problems. The Matlab® source code for this model is presented here.

D.1. main.m

```matlab
%%%% main.m
%%%%
%%%% Main function which calls out sub-functions for generating FE mesh
%%%% and assembling matrices for the electromagnetic and fluid flow
%%%% problems. The equations are solved and saved for each value of the
%%%% parameterized Bingham and Mason numbers.
%%%%

close all
clear all

%%%% Numerical and model parameters
n_para = 1e2; % Regularization parameter
convcrit = 1e-3;
refinement = 16;
Underrelax = 0.25;
Omegalim = -0.9; % These limits are numerical and should
phillim = 0.01; % not be active in the final solution
phiulim = 0.68;

%%%% Fluid mechanics
yieldmodel = 'Unsymmetric'; % 'Bingham'
```

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Papl = 1; % This should be kept constant
volfr0 = 0.32;
Bi_para = 10^(-1);
a = 1e-2;
Mni_para = Bi_para/10;
phim = phiulim; % Maximum packing density

% Electromagnetic parameters
mumodel = 'Columns';%'Lorentz'
mup = 9/0.32; % Particle permeability, set to match experimental fluid
chip = 1000;
Beta = chip/(chip+3);
assignpot = 0.5; % Magnetic potential difference applied on the channel

% Mesh generation

% Simulation

BiMniFRaFRb = [];

for paramind = 1:max(length(Bi_para),length(Mni_para))
    paramind/max(length(Bi_para),length(Mni_para))
    Bisol = Bi_para(paramind);
    Mni = Mni_para(paramind);
    n = n_para(paramind);
    smooth_precalc

    % Initial volume fraction distribution
    volfr = volfr0*ones(nonode,1);

    % Solution to EM problem
    assymatEM;
    solnEM = Kglob\Fglob;
    smooth_h;

    chisol = zeros(nonode,1);
    for i = 1:max(max(IDEM))
        if sign(find(IDEM==i)) == 1
            chisol(find(IDEM==i))=solnEM(i);
        end
    end

    figure(3)
    stem3(nodepos(:,1),nodepos(:,2),chisol,'d','filled');
% Linear solution
Bi = 0;
soln=zeros(max(max(ID)),1);
if yieldmodel == 'Bingham'
    assymat_Bingham
elseif yieldmodel == 'Unsymmetric'
    assymat_antisym
elseif yieldmodel == 'UnsymCasson'
    assymat_antisym_casson
end
soln=Kglob\Fglob;
smooth_gradv;

Bi = Bisol;
norconv = 1;
while abs(norconv) > convcrit
    if yieldmodel == 'Bingham'
        assymat_Bingham
    elseif yieldmodel == 'Unsymmetric'
        assymat_antisym
    elseif yieldmodel == 'UnsymCasson'
        assymat_antisym_casson
    end
    soln=Kglob\Fglob;
    smooth_gradv;
    if exist('psoln','var')
        norconv = sum((soln-psoln)./psoln)/length(soln);
    else
        psoln = soln;
    end
usol = zeros(nonode,1);
vsol = zeros(nonode,1);
psol = zeros(nonode,1);
for i = 1:max(max(ID))
    if mod(find(ID==i),3) == 1
        usol(ceil(find(ID==i)/3))=soln(i);
    elseif mod(find(ID==i),3) == 2
        vsol(ceil(find(ID==i)/3))=soln(i);
    else
        psol(ceil(find(ID==i)/3))=soln(i);
    end
end

figure(2)
quiver(nodepos(:,1),nodepos(:,2),usol,vsol,'linewidth',2)
if exist('Nv2','var')
    rectangle('position', [Lh1,Lv2,Lh2,Lv1-Lv2],...
             'facecolor','b','linestyle','none');
    rectangle('position', [0,Lv1,Lh1+Lh2,0.125],...
             'facecolor','b','linestyle','none');
    axis([0 (Lh1+Lh2) 0 (Lv1+0.125)])
end
if exist('norconv','var')
    title(abs(norconv))
end
psoln = psoln+Underrelax*(soln-psoln);
end %while
flowratecalc;
update_volf
BiMniFRaFRb = [BiMniFRaFRb; Bi Mni ratein];

% Save step in next file
load filecount
save(cat(2, 'results ', num2str(filecount, '%04d')));
filecount = filecount + 1;
save filecount.mat filecount
end

figure
plot(BiMniFRaFRb(:,1),BiMniFRaFRb(:,3),'linewidth',3,'marker','o', ... 
'markerfacecolor','b','markersize',9)
set(gca,'yscale','lin','xscale','log','fontsize',20)
legend('
phi_0','
phi_1')
ylabel('Q''','fontsize',24)
xlabel('Bi_P','fontsize',24)

D.2. genmesh.m

%%%%
%%%% genmesh.m
%%%%
%%%% Generates FE mesh for a pressure-driven hyperbolic contraction and 
%%%% expansion channel.
%%%%

Nh1 = 4*refinement;
Nh2 = 4*refinement;
Nh3 = 4*refinement;
Nc = 4*refinement;
Nv = refinement;
Lh1 = 2;
Lh2 = 1;
Lc = 1;
Lv1 = 1;
Lv2 = 0.5;

meshinc = 2/Nv/(Nv+1);
ymesh = cumsum(([0 Nv:-1:1])*meshinc);
for i = 2:2:Nv
    ymesh(i) = (ymesh(i+1)+ymesh(i-1))/2;
end

nxnode = 1;
for j = 1:(Nh1+1)
    for i = 1:(Nv+1)
        nodemesh(i,j) = nxnode;
        nodepos(nxnode,:) = [(j-1)/Nh1*Lh1, ymesh(i)*Lv1];
nxnode = nxnode + 1;
end
dend
for j = 1:Nc
for i = 1:(Nv+1)
nodemesh(i,j+Nh1+1) = nxnode;
if i == 1
    nodepos(nxnode,:) = [j/Nc*Lc+Lh1, 0];
else
    \[w_u = ymesh(i)*Lvl;\]
    \[w_c = ymesh(i)*Lv2;\]
    \[hyp_a = Lc*w_c/(w_u-w_c);\]
    \[hyp_c = Lc*w_u*w_c/(w_u-w_c);\]
    nodepos(nxnode,:) = [j/Nc*Lc+Lh1, hyp_c/(hyp_a+j/Nc*Lc)];
end
nxnode = nxnode + 1;
end
end
for j = 1:Nh2
for i = 1:(Nv+1)
nodemesh(i,j+Nh1+Nc+1) = nxnode;
nodemesh(nxnode,:) = [j/Nh2*Lh2+Lh1+Lc, ymesh(i)*Lv2];
nxnode = nxnode + 1;
end
end
for j = 1:Nc
for i = 1:(Nv+1)
nodemesh(i,j+Nh1+Nc+Nh2+1) = nxnode;
if i == 1
    nodepos(nxnode,:) = [j/Nc*Lc+Lh1+Lc+Lh2, 0];
else
    \[w_u = ymesh(i)*Lvl;\]
    \[w_c = ymesh(i)*Lv2;\]
    \[hyp_a = Lc*w_c/(w_u-w_c);\]
    \[hyp_c = Lc*w_u*w_c/(w_u-w_c);\]
    nodepos(nxnode,:) = [j/Nc*Lc+Lh1+Lc+Lh2, hyp_c/(hyp_a+Lc-j/Nc*Lc)];
end
nxnode = nxnode + 1;
end
end
for j = 1:Nh3
for i = 1:(Nv+1)
nodemesh(i,j+Nh1+2*Nc+Nh2+1) = nxnode;
nodemesh(nxnode,:) = [j/Nh1*Lh2+Lh1+2*Lc+Lh2, ymesh(i)*Lv1];
nxnode = nxnode + 1;
end
end

% Mesh smoothing
mzwind = 9;  % Has to be odd
mzhw = (mzwind-1)/2;
for j = 2:(Nv)
for i = (mzhw):(Nv+1)+(mzhw-Nh3-mzhw)
nodepos(j+i*(Nv+1),2) = min(nodepos(j+i*(Nv+1),2),
mean(nodepos(j+i*(Nv+1),2)));
end
for j = 2:2:Nv
  for i = (1):(Nh1+2*Nc+Nh2+Nh3)
    nodepos(j+i*(Nv+1),2) = mean([nodepos(j+i*(Nv+1)+1,2),
      nodepos(j+i*(Nv+1)-1,2)]);
  end
end

ID = zeros(3, size(nodepos,1));
IDEM = zeros(1, size(nodepos,1));
IDVF = zeros(1, size(nodepos,1));
nxelem = 1;
for i = 1:((Nh1+2*Nc+Nh2+Nh3)/2)
  for j = 1:(Nv/2)
    matform = nodemesh((2*j-1):(2*j+1),(2*i-1):(2*i+1));
    elemconn(nxelem,:) = [matform(3,3)
      matform(3,1)
      matform(1,1)
      matform(1,3)
      matform(3,2)
      matform(2,1)
      matform(1,2)
      matform(2,3)
      matform(2,2)];
    ID(3,elemconn(nxelem,5:9))=1;
    nxelem = nxelem + 1;
  end
end

noelem = size(elemconn,1);
nonode = size(nodepos,1);

% Constraints - Fluid Mechanics
masknodes = [];
for i = 1:nonode
  if mod(i,Nv+1) == 0
    masknodes = [masknodes i];
    ID(1,i) = 1;
    ID(2,i) = 1;
  elseif mod(i,Nv+1) == 1
    ID(2,i) = 1;
  end
  if (i<(Nv+2)) || (nonode-i<(Nv+1))
    ID(2,i) = 1;
  end
end

% Constraints - EM
assignnodeEM = find(nodepos(:,2)==0);% Nodes for assigned non-zero poten.
for i = 1:nonode
  if mod(i,Nv+1) == 0
    IDEM(1,i) = 1;
  end
end
nodedof = 1;
for j = 1:nonode
    for i = 1:3
        if ID(i,j) == 1
            ID(i,j) = 0;
        else
            ID(i,j) = nodedof;
            nodedof = nodedof + 1;
        end
    end
end

nodedof = 1;
for j = 1:nonode
    if IDEM(1,j) == 1
        IDEM(1,j) = 0;
    else
        IDEM(1,j) = nodedof;
        nodedof = nodedof + 1;
    end
end

nodedof = 1;
for j = 1:nonode
    if IDVF(1,j) == 1
        IDVF(1,j) = 0;
    else
        IDVF(1,j) = nodedof;
        nodedof = nodedof + 1;
    end
end

entryelem = zeros(noelem,1);
exitelem = entryelem;
entryelem(1:(Nv/2)) = ones(Nv/2,1);
exitelem((noelem-Nv/2+1):noelem) = ones(Nv/2,1);

figure(1)
hold on
for i = 1:noelem
    for j = 0:3
        line([nodepos(elemconn(i,mod(j,4)+1),1),...
             nodepos(elemconn(i,mod(j+1,4)+1),1)],...
             [nodepos(elemconn(i,mod(j,4)+1),2),...
             nodepos(elemconn(i,mod(j+1,4)+1),2)]);
    end
end
axis equal
axis tight
\[ \left( -\frac{1}{2} \right) \left( -1 + r^2 \right) \left( 1 + 2s \right), \left( -\frac{1}{2} \right) \left( -1 + r^2 \right) \left( -1 + 2s \right), \left( -r \right) \left( 1 + r \right) s, \ldots \]

\[ 2 \left( -1 + r^2 \right) s \]

\[ \text{Jin} = \text{inv} \left( \left[ \text{psir}' \times \text{nodepos} \left( \text{elemconn} \left( i, : \right), 1 \right), \ldots \right. \right. \]

\[ \left. \left. \text{psir}' \times \text{nodepos} \left( \text{elemconn} \left( i, : \right), 2 \right) \right. \right. \]

\[ \left. \left. \text{psis}' \times \text{nodepos} \left( \text{elemconn} \left( i, : \right), 1 \right), \ldots \right. \right. \]

\[ \left. \left. \text{psis}' \times \text{nodepos} \left( \text{elemconn} \left( i, : \right), 2 \right) \right] \right) ; \]

\[ \text{dpsidx} = \text{Jin} \left( 1, 1 \right) \text{psir} + \text{Jin} \left( 1, 2 \right) \text{psis}; \]

\[ \text{dpsidy} = \text{Jin} \left( 2, 1 \right) \text{psir} + \text{Jin} \left( 2, 2 \right) \text{psis}; \]

\[ \text{volfrloc} = \text{psi}' \times \text{volfr} \left( \text{elemconn} \left( i, : \right) \right); \]

\[ \text{if} \ \text{strcmp} \left( \text{mumodel}, '\text{Lorentz}' \right) \]

\[ \text{mur} = \left( 1 + 2 \times \text{Beta} \times \text{volfrloc} \right) / \left( 1 - \text{Beta} \times \text{volfrloc} \right); \]

\[ \text{else} \]

\[ \text{mur} = 1 + \text{volfrloc} \times (\text{mup} - 1); \]

\[ \text{end} \]

\[ \text{K} = \text{K} + \text{alphars} \times \text{mur} \times \left( \text{dpsidx} \times \text{dpsidx}' \right) / \text{det} \left( \text{Jin} \right); \]

\[ \text{K} = \text{K} + \text{alphars} \times \text{mur} \times \left( \text{dpsidy} \times \text{dpsidy}' \right) / \text{det} \left( \text{Jin} \right); \]

\[ \text{end} \]

\[ \text{end} \]

\[ \text{D.4. assymat_antisym.m} \]

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%%% assymat_antisym.m

%%% Called out by main.m, generates finite element mesh for the

%%% fluid flow problem, using the newly developed model to describe

%%% yield component of stress in the fluid.

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
Kglob = sparse(max(max(ID)), max(max(ID)));  
Fglob = sparse(max(max(ID)), 1);  
dv2xldofn = zeros(nonode, 1);  
gamctdofn = zeros(nonode, 1);  

% Gauss quadrature variables (N=3)  
rval = [0, -sqrt(3/5), sqrt(3/5)];  
sval = rval;  
alphaval = [8/9, 5/9, 5/9];  

for i = 1:nelem  
    K11 = zeros(9);  
    K12 = K11;  
    K22 = K11;  
    B11 = K11;  
    B12 = K11;  
    B21 = K11;  
    B22 = K11;  
    SM11 = zeros(9,1);  
    SM12 = SM11;  
    SM21 = SM11;  
    SM22 = SM11;  
    SM01 = SM11;  
    SM02 = SM11;  
    Q1 = zeros(9,4);  
    Q2 = Q1;  
    loc2glo = spalloc(max(max(ID)), 22, 22);  
    loc2gloEM = spalloc(max(max(IDEM)), 9, 9);  
    for j = 1:9  
        if ID(1, elemconn(i,j)) == 0  
            loc2glo(ID(1, elemconn(i,j)), j) = 1;  
        end  
        if ID(2, elemconn(i,j)) == 0  
            loc2glo(ID(2, elemconn(i,j)), j+9) = 1;  
        end  
        if ID(3, elemconn(i,j)) == 0  
            loc2glo(ID(3, elemconn(i,j)), j+18) = 1;  
        end  
        if IDEM(1, elemconn(i,j)) == 0  
            loc2gloEM(IDEM(1, elemconn(i,j)), j) = 1;  
        end  
    end  
    for j = 1:length(rval)  
        for k = 1:length(sval)  
            r = rval(j);  
            s = sval(k);  
            alphars = alphaval(j)*alphaval(k);  
            psi = 1/4*[(1 + r)*(1 + s)*(r + s - 1) ...  
                       + (1 - r^2)*(1 - s^2), ...  
                       (1 - r)*(1 + s)*(-r + s - 1) + (1 - r^2)*(1 - s^2), ...  
                       (1 - r)*(1 - s)*(-r - s - 1) + (1 - r^2)*(1 - s^2), ...  
                       (1 + r)*(1 - s)*(r - s - 1) + (1 - r^2)*(1 - s^2), ...  
                       2*(1 - r^2)*(1 + s) - 2*(1 - r^2)*(1 - s^2), ...  
                       251
\[ 2*(1 - s^2)*(1 - r) - 2*(1 - r^2)*(1 - s^2), \ldots \\
2*(1 - r^2)*(1 - s) - 2*(1 - r^2)*(1 - s^2), \ldots \\
2*(1 - s^2)*(1 + r) - 2*(1 - r^2)*(1 - s^2), \ldots \\
4*(1 - r^2)*(1 - s^2)]'; \\
psir = [(1/4)*(1 + 2*r)*s*(1 + s), \ldots \\
(1/4)*(-1 + 2*r)*s*(1 + s), \ldots \\
(1/4)*(1 + 2*r)*s*(1 + s), \ldots \\
(-r)*s*(1 + s), (-(1/2))*(-1 + 2*r)*(-1 + s^2), \ldots \\
(-r)*(-1 + s)*s, \ldots \\
(-(1/2))*(1 + 2*r)*(-1 + s^2), 2*r*(-1 + s^2)]; \\
psis = [(1/4)*r*(1 + r)*(1 + 2*s), \ldots \\
(1/4)*(-1 + r)*s*(1 + 2*s), \ldots \\
(1/4)*r*(1 + r)*(1 + 2*s), \ldots \\
(-1/2)*(1 + 2*r)*(1 + 2*s), -(1 + r)*r*s, \ldots \\
(-1/2)*(-1 + 2*r)*(-1 + 2*s), \ldots \\
(-r)*(-1 + r)*s, 2*(-1 + r^2)*s]; \\
phi = 1/4*[(1 + r)*r*(1 + s) \\
(1 - r)*(1 + s) \\
(1 - r)*(1 - s) \\
(1 + r)*(1 - s)]; \\
Jin = inv([psir'*nodepos(elemconn(i,:),1), \ldots \\
psir'*nodepos(elemconn(i,:),2), \ldots \\
psis'*nodepos(elemconn(i,:),1), \ldots \\
psis'*nodepos(elemconn(i,:),2)]); \\
dpsidx = Jin(1,1)*psir + Jin(1,2)*psis; \\
dpsidy = Jin(2,1)*psir + Jin(2,2)*psis; \\
Q1 = Q1 + alphars*dpsidx*phi'/det(Jin); \\
Q2 = Q2 + alphars*dpsidy*phi'/det(Jin); \\
volfrcloc = psi'*vofr(elemconn(i,:)); \\
etav = (1-volfrloc/phim)^(-2.5*phim); \\
K11 = K11 + alphars*etav*(dpsidx*dpsidx')/det(Jin); \\
K12 = K12 + alphars*etav*(dpsidx*dpsidy')/det(Jin); \\
K22 = K22 + alphars*etav*(dpsidy*dpsidy')/det(Jin); \\
Hxloc = psi'*Hxdof(elemconn(i,:)); \\
Hyloc = psi'*Hydof(elemconn(i,:)); \\
Hsq = Hxloc^2+Hyloc^2; \\
if strcmp(mumodel,'Lorentz') \\
    murloc = (1+2*Beta*volfrloc)/(1-Beta*volfrloc); \\
    dmurdphi = 3*Beta/(1-Beta*volfrloc)^2; \\
else \\
    murloc = 1+volfrloc*(mup-1); \\
    dmurdphi = mup-1; 
end \\
SM11 = SM11 + alphars*Mni*dpsidx*murloc*Hxloc*Hxloc/det(Jin); \\
SM12 = SM12 + alphars*Mni*dpsidx*murloc*Hyloc*Hyloc/det(Jin); \\
SM21 = SM21 + alphars*Mni*dpsidy*murloc*Hxloc*Hyloc/det(Jin); \\
SM22 = SM22 + alphars*Mni*dpsidy*murloc*Hyloc*Hyloc/det(Jin); \\
SM01 = SM01 - alphars*Mni/2*dpsidx* \ldots \\
    (murloc-volfrloc*dmurdphi)*Hsq/det(Jin); \\
SM02 = SM02 - alphars*Mni/2*dpsidy* \ldots \\
    (murloc-volfrloc*dmurdphi)*Hsq/det(Jin); \\

% Obtain local strain rate from nodal variables
\[ dv2dx1 = \psi' \times dv2dx1dof(elemconn(i,:)); \]
\[ \text{gamdot} = \psi' \times \text{gamdotdof}(elemconn(i,:)); \]

```matlab
if Bi == 0
    B0 = Bi * volfrloc * Hsq^0.75 * (1 - exp(-n * gamdot)) / gamdot;
    B11 = B11 + alphars * (-B0) / Hsq*dpsidy*... 
        (Hxloc*Hyloc*dpsidx'-Hyloc^2*dpsidy') / det(Jin);
    B12 = B12 + alphars * (-B0) / Hsq*dpsidy*... 
        (Hxloc^2*dpsidx'+Hxloc*Hyloc*dpsidy') / det(Jin);
    B21 = B21 + alphars * (B0) / Hsq*dpsidy*... 
        (-Hxloc*Hyloc*dpsidx'-Hyloc^2*dpsidy') / det(Jin);
    B22 = B22 + alphars * (B0) / Hsq*dpsidy*... 
        (Hxloc^2*dpsidx'+Hxloc*Hyloc*dpsidy') / det(Jin);
end % Bi
end %s
end %r
```

\[ \text{Kglob} = \text{Kglob} + \text{loc2glo} \times \text{sparse}([2*K11+K22+B11, K12'+B12, ... \quad \text{Fglob} = \text{Fglob} + \text{loc2glo} \times \text{sparse}([-SM11-SM21-SM01; ... \quad \text{entryelem}(i)
    r = -1;
    for j = 1:length(sval)
        s = sval(j);
        psi = 1/4*[(1 + r)*(1 + s)*(r + s - 1) + ... 
                   (1 - r^2)*(1 - s^2),...
                   (1 - r)*(1 + s)*(-r + s - 1) + (1 - r^2)*(1 - s^2),...
                   (1 - r)*(1 - s)*(-r - s - 1) + (1 - r^2)*(1 - s^2),...
                   (1 + r)*(1 - s)*(r - s - 1) + (1 - r^2)*(1 - s^2),...
                   2*(1 - r^2)*(1 + s) - 2*(1 - r^2)*(1 - s^2),...
                   2*(1 - s^2)*(1 - r) - 2*(1 - r^2)*(1 - s^2),...
                   2*(1 - s^2)*(1 - s) - 2*(1 - r^2)*(1 - s^2),...
                   2*(1 - s^2)*(1 + r) - 2*(1 - r^2)*(1 - s^2),...
                   4*(1 - r^2)*(1 - s^2)];
        psis = [(1/4)*r*(1 + r)*(1 + 2*s), ...
              (1/4)*(-1 + r)*r*(1 + 2*s),...
              (1/4)*(1 - r)*r*(-1 + 2*s),...
              (1/4)*r*(1 + r)*(-1 + 2*s),...
              (-1/2)*(1 + r)*r*s,...
              (-1/2)*(-1 + r^2)*(-1 + 2*s),...
              (-r)*[1 + r]*s, 2*(-1 + r^2)*s]';
        dyds = psis'*nodepos(elemconn(i,:),2);
        Hxloc = psi'*Hxdof(elemconn(i,:));
        Hyloc = psi'*Hydof(elemconn(i,:));
        Hsq = Hxloc^2+Hyloc^2;
        volfrloc = psi'*volfr(elemconn(i,:));
        if strcmp(mumodel, 'Lorentz')
            murloc = (1+2*Beta*volfrloc)/(1-Beta*volfrloc);
            dmurdphi = 3*Beta/(1-Beta*volfrloc)^2;
        else
            murloc = 1+volfrloc*(mup-1);
            dmurdphi = mup-1;
        end
    end
end
```

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end
Ptot = Papl + Mni*Hsq/2*(murloc-volfrloc*dmurdphi);
Fglob = Fglob + alphaval(j)*loc2glo(:,1:9)*...
sparse(psi*Ptot*dyds);
end %for
end %if
if exitelem(i)
  r = 1;
  for j = 1:length(sval)
    s = sval(j);
    psi = 1/4*[(1 + r)*(1 + s)*(r + s - 1) ...
      + (1 - r^2)*(1 - s^2),...
        (1 - r)*(1 + s)*(-r + s - 1) + (1 - r^2)*(1 - s^2),...
        (1 - r)*(1 - s)*(-r - s - 1) + (1 - r^2)*(1 - s^2),...
        (1 + r)*(1 - s)*(r - s - 1) + (1 - r^2)*(1 - s^2),...
        2*(1 - r^2)*(1 + s) - 2*(1 - r^2)*(1 - s^2),...
        2*(1 - s^2)*(1 - r) - 2*(1 - r^2)*(1 - s^2),...
        2*(1 - r^2)*(1 - s) - 2*(1 - r^2)*(1 - s^2),...
        2*(1 - s^2)*(1 + r) - 2*(1 - r^2)*(1 - s^2),...
        4*(1 - r^2)*(1 - s^2)]';
    psis = [(1/4)*r*(1 + r)*(1 + 2*s), ...n
      (1/4)*(-1 + r)*r*(1 + 2*s), ...n
      (1/4)*r*(1 + r)*(-1 + 2*s), ...n
      (-1/2)*(-1 + r^2)*(-1 + 2*s), ...n
      (-1 + r)^r*s,...
      (-1/2)*(-1 + r^2)*(-1 + 2*s), ...n
      (r)*(1 + r)*s, 2*(-1 + r^2)*s]';
    dyds = psis'*nodepos(elemconn(i,:),2);
    Hxloc = psi'*Hxdof(elemconn(i,:));
    Hyloc = psi'*Hydof(elemconn(i,:));
    Hsq = Hxloc^2+Hyloc^2;
    volfrloc = psi'*volfr(elemconn(i,:));
    if strcmp(mumodel,'Lorentz')
      murloc = (1+2*Beta*volfrloc)/(1-Beta*volfrloc);
      dmurdphi = 3*Beta/(1-Beta*volfrloc)^2;
    else
      murloc = 1+volfrloc*(mup-1);
      dmurdphi = mup-1;
    end
    Ptot = -Mni*Hsq/2*(murloc-volfrloc*dmurdphi);
    Fglob = Fglob + alphaval(j)*loc2glo(:,1:9)*...
      *sparse(psi*Ptot*dyds);
  end %for
end %if

D.5. assymat_Bingham.m
assymat_Bingham.m

Called out by main.m, generates finite element mesh for the fluid flow problem, using the isotropic Bingham model to describe yield component of stress in the fluid.

\[ K_{\text{glob}} = \text{sparse}(\text{max}(\text{max}(\text{ID})), \text{max}(\text{max}(\text{ID}))) \]
\[ F_{\text{glob}} = \text{sparse}(\text{max}(\text{max}(\text{ID})), 1) \]

\[ dV^{2}dx^{1}dofn = \text{zeros}(\text{nonode}, 1) \]
\[ gammadotdofn = \text{zeros}(\text{nonode}, 1) \]

% Gauss quadrature variables (N=3)
\[ r_{\text{val}} = [0, -\sqrt{3/5}, \sqrt{3/5}] \]
\[ s_{\text{val}} = r_{\text{val}} \]
\[ \alpha_{\text{val}} = [8/9, 5/9, 5/9] \]

for \( i = 1: \text{noelem} \)
    \[ K_{11} = \text{zeros}(9); \]
    \[ K_{12} = K_{11}; \]
    \[ K_{22} = K_{11}; \]
    \[ B_{11} = K_{11}; \]
    \[ B_{12} = K_{11}; \]
    \[ B_{21} = K_{11}; \]
    \[ B_{22} = K_{11}; \]
    \[ SM_{11} = \text{zeros}(9, 1); \]
    \[ SM_{12} = SM_{11}; \]
    \[ SM_{21} = SM_{11}; \]
    \[ SM_{22} = SM_{11}; \]
    \[ SM_{01} = SM_{11}; \]
    \[ SM_{02} = SM_{11}; \]
    \[ Q_{1} = \text{zeros}(9, 4); \]
    \[ Q_{2} = Q_{1}; \]
    \[ \text{loc2glo} = \text{spalloc}(\text{max}(\text{max}(\text{ID})), 22, 22); \]
    \[ \text{loc2gloEM} = \text{spalloc}(\text{max}(\text{max}(\text{IDEM})), 9, 9); \]
    for \( j = 1:9 \)
        if \( \text{ID}(1, \text{elemconn}(i, j)) \neq 0 \)
            \[ \text{loc2glo}(\text{ID}(1, \text{elemconn}(i, j)), j) = 1; \]
        end
        if \( \text{ID}(2, \text{elemconn}(i, j)) \neq 0 \)
            \[ \text{loc2glo}(\text{ID}(2, \text{elemconn}(i, j)), j + 9) = 1; \]
        end
        if \( \text{ID}(3, \text{elemconn}(i, j)) \neq 0 \)
            \[ \text{loc2glo}(\text{ID}(3, \text{elemconn}(i, j)), j + 18) = 1; \]
        end
        if \( \text{IDEM}(1, \text{elemconn}(i, j)) \neq 0 \)
            \[ \text{loc2gloEM}(\text{IDEM}(1, \text{elemconn}(i, j)), j) = 1; \]
        end
    end
end

for \( j = 1: \text{length}(r_{\text{val}}) \)
    for \( k = 1: \text{length}(s_{\text{val}}) \)
        \[ r = r_{\text{val}}(j); \]
        \[ s = s_{\text{val}}(k); \]
        \[ \alpha_{\text{val}} = \alpha_{\text{val}}(j) * \alpha_{\text{val}}(k); \]
    end
end
\[
\psi = \frac{1}{4}[(1 + r)^{(1 + s)}(r + s - 1) 
+ (1 - r^2)^{(1 + s^2)},
\]
\[
(1 - r)^{(1 + s)}(-r + s - 1) + (1 - r^2)^{(1 - s^2)}, 
(1 - r)^{(1 - s)}(-r - s - 1) + (1 - r^2)^{(1 - s^2)},
\]
\[
(1 + r)^{(1 - s)}(r - s - 1) + (1 - r^2)^{(1 - s^2)},
2*(1 - r^2)^{(1 + s)} - 2*(1 - r^2)^{(1 - s^2)},
2*(1 - s^2)^{(1 - r)} - 2*(1 - r^2)^{(1 - s^2)},
4*(1 - r^2)^{(1 - s^2)}]
\]

\[
\psi = [(1/4)^{(1 + 2*r)^s*(1 + s)},
\]
\[
(1/4)^{(-1 + 2*r)^s*(1 + s)},
\]
\[
(1/4)^{(*1 + 2*r)^s*(-1 + s)*s},
\]
\[
(-r)*s*(1 + s), (-1/2)^{(-1 + 2*r)^s*2*(-1 + s^2)},
\]
\[
(-r)*(-1 + s)*s,
\]
\[
(-1/2)^{((-1 + s^2) /2*r)^s, 2*r*(-1 + s^2)}
\]

\[
\psi = [(1/4)^{(1 + r)^s*(1 + 2*s)},
\]
\[
(1/4)^{(*1 + r)^r*(1 + 2*s)},
\]
\[
(1/4)^{(*1 + r)^r*(-1 + 2*s)},
\]
\[
(-1/2)^{((-1 + r^2)^s*(1 + 2*s), (-(-1 + r)^r*s,}
\]
\[
(-1/2)^{((-1 + r^2)^s*(1 + 2*s)},
\]
\[
(-r)^{(1 + r)*s, 2*(-1 + r^2)*s}]
\]

\[
\phi = 1/4*[(1 + r)^{(1 + s)}
\]
\[
(1 - r)^{(1 + s)}
\]
\[
(1 - r)^{(1 - s)}
\]
\[
(1 + r)^{(1 - s)}]
\]

\[
Jin = inv([psir'*nodepos(elemconn(i,:),1),
\]
\[
psir'*nodepos(elemconn(i,:),2),
\]
\[
psis'*nodepos(elemconn(i,:),1),
\]
\[
psis'*nodepos(elemconn(i,:),2)]
\]

\[
dpsidx = Jin(1,1)*psir + Jin(1,2)*psis;
\]
\[
dpsy = Jin(2,1)*psir + Jin(2,2)*psis;
\]
\[
Q1 = Q1 + alphas*dpsidx*phi'/det(Jin);
\]
\[
Q2 = Q2 + alphas*dpsy*phi'/det(Jin);
\]
\[
vofrloc = psi*vofr(elemconn(i,:));
\]
\[
etav = (1-vofrloc/phim)^(-2.5*phim);
\]
\[
K11 = K11 + alphas*etav*(dpsidx*dpsidx')/det(Jin);
\]
\[
K12 = K12 + alphas*etav*(dpsidx*dpsy')/det(Jin);
\]
\[
K22 = K22 + alphas*etav*(dpsy*dpsy')/det(Jin);
\]
\[
Hxloc = psi*Hxdof(elemconn(i,:));
\]
\[
Hyloc = psi*Hydof(elemconn(i,:));
\]
\[
Hsq = Hxloc^2+Hyloc^2;
\]
\[
if strcmp(mumodel,'Lorentz')
\]
\[
murlc = (1+2*Beta*vofrloc)/(1-Beta*vofrloc);
\]
\[
dmurdphi = 3*Beta/(1-Beta*vofrloc)^2;
\]
\[
else
\]
\[
murlc = 1+vofrloc*(mup-1);
\]
\[
dmurdphi = mup-1;
\]
\[
end
\]

SM11 = SM11 + alphas*Mni*dpsidx*murlc*Hxloc*Hxloc/det(Jin);
SM12 = SM12 + alphas*Mni*dpsidx*murlc*Hxloc*Hyloc/det(Jin);
SM21 = SM21 + alphas*Mni*dpsyd*murlc*Hxloc*Hyloc/det(Jin);

SM22 = SM22 + alphars*Mni*dpsidy*murloc*Hyloc*Hyloc/det(Jin);  
SM01 = SM01 - alphars*Mni/2*dpsidx...  
*(murloc-volfrloc*dmurdphi)*Hsq/det(Jin);  
SM02 = SM02 - alphars*Mni/2*dpsidy...  
*(murloc-volfrloc*dmurdphi)*Hsq/det(Jin);

% Obtain local strain rate from nodal variables
dv2dx1 = psi'*dv2dxldof(elemconn(i,:));
gamdot = psi'*gamdotdof(elemconn(i,:));

if Bi ~= 0
  BO = Bi*volfrloc*Hsq*(0.75)*(1-exp(-n*gamdot))/gamdot;
  B11 = B11 + alphars*(-BO)/Hsq*dpsidy...  
       *(Hxloc*Hyloc^2*dpsidx'-Hyloc^2*dpsidy')/det(Jin);
  B12 = B12 + alphars*(-BO)/Hsq*dpsidy...  
       *(Hxloc^2*dpsidx'+Hxloc*Hyloc*dpsidy')/det(Jin);
  B21 = B21 + alphars*(BO)/Hsq*dpsidx...  
       *(-Hxloc*Hyloc^2*dpsidx'+Hyloc^2*dpsidy')/det(Jin);
  B22 = B22 + alphars*(BO)/Hsq*dpsidy...  
       *(-Hxloc^2*dpsidx'+Hxloc*Hyloc*dpsidy')/det(Jin);

  BO = Bi*volfrloc*Hsq*(0.75)*(1-exp(-n*gamdot))/gamdot;
  B11 = B11 + alphars*BO*(dpsidx*dpsidx)/det(Jin);
  B12 = B12 + alphars*BO*dpsidx*dpsidy'/det(Jin);
  B22 = B22 + alphars*BO*(dpsidy*dpsidy')/det(Jin);

end % Bi
end % s
end % r

Kglob = Kglob + loc2glo*sparse([2*Kll+K22+2*Bll+B22, K12'+B12, -Q1;  
                              K12+B12', K11+2*K22+B11+2*B22, -Q2;...  
                              -Q1', -Q2', zeros(4)])*loc2glo';
Fglob = Fglob + loc2glo*sparse([-SM11-SM21-SM01;  
                              -SM12-SM22-SM02;zeros(4,1)]);

if entryelem(i)
  r = -1;
  for j = 1:length(sval)
    s = sval(j);
    psi = 1/4*[(1 + r)*(1 + s)*(r + s - 1) ...  
               + (1 - r^2)*(1 - s^2), ...  
               (1 - r)*(-r + s - 1) + (1 - r^2)*(1 - s^2), ...  
               (1 - r)*(r - s - 1) + (1 - r^2)*(1 - s^2), ...  
               (1 + r)*(1 - s)*(-r - s - 1) + (1 - r^2)*(1 - s^2), ...  
               2*(1 - r^2)*(1 + s) - 2*(1 - r^2)*(1 - s^2), ...  
               2*(1 - s^2)*(1 - r) - 2*(1 - r^2)*(1 - s^2), ...  
               2*(1 - r^2)*(1 - s) - 2*(1 - r^2)*(1 - s^2), ...  
               2*(1 - s^2)*(1 + r) - 2*(1 - r^2)*(1 - s^2), ...  
               4*(1 - r^2)*(1 - s^2)]');
    psis = [(1/4)*r*(1 + r)*(1 + 2*s), ...  
            (1/4)*(-1 + r)*r*(1 + 2*s), ...];
  end
end % r
(1/4)*(-1 + r)*r*(-1 + 2*s), (1/4)*r*(1 + r)*(-1 + 2*s),...
-(1/2))*(-1 + r^2)*(1 + 2*s), (-(-1 + r))*r*s,...
(-1/2))*(-1 + r^2)*(-1 + 2*s),...
(-r)*(1 + r)*s, 2*(-1 + r^2)*s]';
dyds = psis'*nodepos(elemconn(i,:),2);
Hxloc = psi'*Hxdof(elemconn(i,:));
Hyloc = psi'*Hydof(elemconn(i,:));
Hsq = Hxloc^2+Hyloc^2;
volfrloc = psi'*volfr(elemconn(i,:));
if strcmp(mumodel,'Lorentz')
    murloc = (1+2*Beta*volfrloc)/(1-Beta*volfrloc);
dmurdphi = 3*Beta/(1-Beta*volfrloc)^2;
else
    murloc = 1+volfrloc*(mup-1);
dmurdphi = mup-1;
end
Ptot = Papl + Mni*Hsq/2*(murloc-volfrloc*dmurdphi);
Fglob = Fglob + alphaval(j)*loc2glo(:,1:9)...
*sparse(psi*Ptot*dyds);
if exitelm(i)
    r = 1;
    for j = 1:length(sval)
        s = sval(j);
        psi = 1/4*[(1 + r)*(1 + s)*(r + s - 1)... + (1 - r^2)*(1 - s^2),...]
            (1 - r)*(1 + s)*(-r + s - 1) + (1 - r^2)*(1 - s^2),...]
            (1 - r)*(1 - s)*(-r - s - 1) + (1 - r^2)*(1 - s^2),...]
            (1 + r)*(1 - s)*(r - s - 1) + (1 - r^2)*(1 - s^2),...]
            2*(1 - r^2)*(1 + s) - 2*(1 - r^2)*(1 - s^2),...]
            2*(1 - r^2)*(-1 + r) - 2*(1 - r^2)*(-1 + s^2),...]
            2*(1 - r^2)*(-1 + s) - 2*(1 - r^2)*(-1 + s^2),...]
            4*(1 - r^2)*(1 - s^2)];
        psis = [(1/4)*r*(1 + r)*(1 + 2*s),...
            (1/4)*(-1 + r)*r*(1 + 2*s),...]
            (1/4)*(-1 + r)*r*(-1 + 2*s),...]
            (1/4)*r*(1 + r)*(-1 + 2*s),...]
            (-(-1/2))*(-1 + r^2)*(1 + 2*s), (-(-1 + r))*r*s,...]
            (-(-1/2))*(-1 + r^2)*(-1 + 2*s),...]
            (-r)*(1 + r)*s, 2*(-1 + r^2)*s]';
dyds = psis'*nodepos(elemconn(i,:),2);
Hxloc = psi'*Hxdof(elemconn(i,:));
Hyloc = psi'*Hydof(elemconn(i,:));
Hsq = Hxloc^2+Hyloc^2;
volfrloc = psi'*volfr(elemconn(i,:));
if strcmp(mumodel,'Lorentz')
    murloc = (1+2*Beta*volfrloc)/(1-Beta*volfrloc);
dmurdphi = 3*Beta/(1-Beta*volfrloc)^2;
else
    murloc = 1+volfrloc*(mup-1);
dmurdphi = mup-1;
end
Ptot = -Mni*Hsq/2*(murloc-volfrloc*dmurdphi);
Fglob = Fglob + alphaval(j)*loc2glo(:,1:9)...
D.6. flowratecalc.m

Called out by main.m, calculates flow rate in and out of the channel and gives warning if these do not match.

ratein = 0;
rateout = 0;

for i = 1:length(sval)
    r = -1;
    for j = 1:length(sval)
        s = sval(j);
        psi = 1/4*[(1 + r)*(1 + s)*(r + s - 1) + ...
                (1 - r^2)*(1 - s^2), ...
                (1 - r)*(1 + s)*(-r + s - 1) + (1 - r^2)*(1 - s^2), ...
                (1 - r)*(1 - s)*(-r - s - 1) + (1 - r^2)*(1 - s^2), ...
                (1 + r)*(1 - s)*(r - s - 1) + (1 - r^2)*(1 - s^2), ...
                2*(1 - r^2)*(1 + s) - 2*(1 - r^2)*(1 - s^2), ...
                2*(1 - s^2)*(1 - r) - 2*(1 - r^2)*(1 - s^2), ...
                2*(1 - r^2)*(1 - s) - 2*(1 - r^2)*(1 - s^2), ...
                2*(1 - s^2)*(1 + r) - 2*(1 - r^2)*(1 - s^2), ...
                4*(1 - r^2)*(1 - s^2)]';
        psis = [(1/4)*r*(1 + r)*(1 + 2*s), ...
                (1/4)*(-1 + r)*r*(1 + 2*s), ...
                (1/4)*(-1 + r)*r*(-1 + 2*s), (1/4)*r*(1 + r)*(-1 + 2*s), ...
                (-1/2)*(-1 + r^2)*(1 + 2*s), (-1 + r)*r*s, ...
                (-1/2)*(-1 + r^2)*(-1 + 2*s), ...
                (-r)*(1 + r)*s, 2*(-1 + r^2)*s]';
        dyds = psis'*nodepos(elemconn(i,:),2);
        ratein = ratein + alphaval(j)*2*psi'*usol(elemconn(i,:))*dyds;
    end %if
end %if

for i = 1:length(sval)
    r = 1;
    for j = 1:length(sval)
        s = sval(j);
        psi = 1/4*[(1 + r)*(1 + s)*(r + s - 1) + (1 - r^2)*(1 - s^2), ...
                (1 - r)*(1 + s)*(-r + s - 1) + (1 - r^2)*(1 - s^2), ...
                (1 - r)*(1 - s)*(-r - s - 1) + (1 - r^2)*(1 - s^2), ...
                (1 + r)*(1 - s)*(r - s - 1) + (1 - r^2)*(1 - s^2), ...
                2*(1 - r^2)*(1 + s) - 2*(1 - r^2)*(1 - s^2), ...

        psis = [(1/4)*r*(1 + r)*(1 + 2*s), ...
                (1/4)*(-1 + r)*r*(1 + 2*s), ...
                (1/4)*(-1 + r)*r*(-1 + 2*s), (1/4)*r*(1 + r)*(-1 + 2*s), ...
                (-1/2)*(-1 + r^2)*(1 + 2*s), (-1 + r)*r*s, ...
                (-1/2)*(-1 + r^2)*(-1 + 2*s), ...
                (-r)*(1 + r)*s, 2*(-1 + r^2)*s]';
        dyds = psis'*nodepos(elemconn(i,:),2);
        rateout = rateout + alphaval(j)*2*psi'*usol(elemconn(i,:))*dyds;
    end %if
end %if
D.7. meshplots_nonuni.m

```matlab
2*(1 - s^2)*(1 - r) - 2*(1 - r^2)*(1 - s^2),...
2*(1 - r^2)*(1 - s) - 2*(1 - r^2)*(1 - s^2),...
2*(1 - s^2)*(1 + r) - 2*(1 - r^2)*(1 - s^2),...
4*(1 - r^2)*(1 - s^2)];
psis = [(1/4)*r*(1 + r)*(1 + 2*s), ... 
(1/4)*(-1 + r)*r*(1 + 2*s), ... 
(1/4)*(-1 + r)*r*(-1 + 2*s), (1/4)*r*(-1 + r)*(-1 + 2*s), ... 
(-1/2)*(-1 + r^2)*(1 + 2*s), (-1/2)*(-1 + r^2)*(-1 + 2*s), ... 
(-r)*(1 + r)*s, 2*(-1 + r^2)*s];
dyds = psis'*nodepos(elemconn(i,:),2);
rateout = rateout + alphaval(j)*2*psi...
*usol(elemconn(i,:))*dyds;
end %j
end %if
end %i

if abs((ratein-rateout)/ratein) > 1e-3
fprintf('Warning: Flowrate into and out of domain do not match\r');
end
```

```bash
Hext = Lh1;
Vext = Lvl;

if exist('Lc','var')
    Hext = Hext + Lc;
end
if exist('Lh2','var')
    Hext = Hext + Lh2;
end
if exist('Nh3','var')
    Hext = Hext + Lc + Lh1;
end

Hspc = Lhl/600;
Vspc = Lvl/100;

xpos = 0:Hspc:Hext;
ypos = 0:Vspc:Vext;
[Xpos, Ypos] = meshgrid(xpos,ypos);
xpoly = [nodepos(masknodes,1);Hext;nodepos(masknodes(1),1)];
ypoly = [nodepos(masknodes,2);Vext;nodepos(masknodes(1),2)];
[IN ON] = inpolygon(Xpos,Ypos,xpoly,ypoly);
```
Fvolfr = TriScatteredInterp(nodepos(:,1),nodepos(:,2),volfr);
FHx = TriScatteredInterp(nodepos(:,1),nodepos(:,2),Hxdof);
FHy = TriScatteredInterp(nodepos(:,1),nodepos(:,2),Hydof);
FVx = TriScatteredInterp(nodepos(:,1),nodepos(:,2),usol);
FVy = TriScatteredInterp(nodepos(:,1),nodepos(:,2),vsol);
if exist('Omegain','var')
    FOmega = TriScatteredInterp(nodepos(:,1),nodepos(:,2),Omegain);
else
    FOmega = Fvolfr;
end
FdVdx = TriScatteredInterp(nodepos(:,1),nodepos(:,2),gamdotdof);
Fp = TriScatteredInterp(nodepos(find(psol==0),1),... 
    nodepos(find(psol==0),2),psol(find(psol==0)));
volfrplot = Fvolfr(Xpos,Ypos).*~(IN&~ON);
Omegaplot = FOmega(Xpos,Ypos).*~(IN&~ON);
Hxplot = FHx(Xpos,Ypos).*~(IN&~ON);
Hyplot = FHy(Xpos,Ypos).*~(IN&~ON);
Vxplot = FVx(Xpos,Ypos).*~(IN&~ON);
Vyplot = FVy(Xpos,Ypos).*~(IN&~ON);
Pplot = Fp(Xpos,Ypos).*~(IN&~ON);
dVdxplot = FdVdx(Xpos,Ypos).*~(IN&~ON);

% Plot graphical properties
PropName(3) = {'fontsize'};
PropName(2) = {'ytick'};
PropName(1) = {'linewidth'};
PropName(4) = {'fontname'};
PropVal(3) = {24};
PropVal(2) = {[0 Vext/2 Vext]};
PropVal(1,1) = {2};
PropVal(4) = {'times new roman'};

figure(10)
contourf(Xpos,Ypos,sqrt(Hxplot.^2+Hyplot.^2),64,'linestyle','none')
colormap(hot)
hold on
hmag = streamline(Xpos,Ypos,Hxplot,Hyplot,0.1:0.1:6.9,...
    zeros(1,69),[0.1,11*size(Xpos,2)]);
set(hmag,'color', 'k','linewidth',1)
colorbar('ytick',[0.5 1],'fontsize',24)
axis equal
axis tight
fill(xpoly,ypoly,'w','linestyle','-','linewidth',2);
hold off
set(gca,PropName,PropVal,'ylim',[0 1])
title('H','fontsize',24);

figure(11)
contourf(Xpos,Ypos,sqrt(Vxplot.^2+Vyplot.^2),64,'linestyle','none')
colormap(hot)
hold on
hflowc = streamline(Xpos,Ypos,Vxplot,Vyplot,...
    zeros(1,9),0.1:0.1:0.9,[0.1,11*size(Xpos,2)]);
colorbar
set(hflowc,'color','k','linewidth',1)
set(gca,PropName,PropVal)
axis equal
axis tight
fill(xpoly,ypoly,'w','linestyle','-','linewidth',2);
hold off
title(['IVI, Bi = ',num2str(Bi,'%-5.3f'),', Mn = ',...
      num2str(MniA-,'%-5.lf')] ,'fontsize',24)
figure(12)
contourf(Xpos,Ypos,dVdxplot,64,'linestyle','none')
colorbar
axis equal
axis tight
hold on
fill(xpoly,ypoly,'w','linestyle','-','linewidth',2);
hold off
set(gca,PropName,PropVal)

figure(13)
yieldlevel = 0.001;
contourf(Xpos,Ypos,abs(dVdxplot),[0 yieldlevel],'linestyle','none')
set(gca,PropName,PropVal)
colorbar
colormap(0.5-0.5*gray)
axis equal
axis tight
hold on
fill(xpoly,ypoly,'w','linestyle','-','linewidth',2);
hold off
set(gca,PropName,PropVal)

figure(14)
contourf(Xpos,Ypos,Pplot,64,'linestyle','none')
colormap(hot)
colorbar
axis equal
axis tight
hold on
fill(xpoly,ypoly,'w','linestyle','-','linewidth',2);
hold off
set(gca,PropName,PropVal)
title('{{Itp - pjx}_x={7}}','fontsize',24);

figure(18)
c1f
n_cont = 64; % Length of custom colormap
cont_map = 1+round((0.5-0.5*cos((1:n_cont)'/n_cont*pi)).^1*(n_cont-1));
nlmap = hot(n_cont);
rectangle('position',[0,0,Hext,Vext],'facecolor',nlmap(end,:),...
  'linestyle','-','linewidth',2,'edgecolor','k');
hold on
contourf(Xpos,Ypos,volfrplot,linspace(.30,.34,64),'linestyle','none')
colormap(nlmap(cont_map(end:-1:1,:),:));
colorbar('ylim',[0.30,0.34],'ytick',[0.30,0.32,0.34]);
set(gca,PropName,PropVal)
D.8. smooth_gradv.m

```
Kglob = sparse(max(max(ID)),max(max(ID)));  
Fglob = zeros(max(max(ID)),1);  

dv2dxldof = zeros(nonode,1);  
gamdotdof = zeros(nonode,1);  

% Gauss quadrature variables (N=3)  
rval = [0,-sqrt(3/5),sqrt(3/5)];  
sval = rval;  

for i = 1:nelem  
    loc2glo = zeros(max(max(ID)),22);  

    for j = 1:9  
        if ID(1,elemconn(i,j)) ~= 0  
            loc2glo(ID(1,elemconn(i,j)),j) = 1;  
        end  
    end  
end  
```
end
if ID(2,elemconn(i,j)) ~= 0
  loc2glo(ID(2,elemconn(i,j)),j+9) = 1;
end
if ID(3,elemconn(i,j)) ~= 0
  loc2glo(ID(3,elemconn(i,j)),j+18) = 1;
end
end
for j = 1:length(rval)
  for k = 1:length(sval)
    r = rval(j);
    s = sval(k);
    psir = [(1/4)*(1 + 2*r)*s*(1 + s), ...
            (1/4)*(-1 + 2*r)*s*(1 + s), ...
            (1/4)*(-1 + 2*r)*(-1 + s)*s, ...
            (1/4)*(-1 + 2*r)*(-1 + s^2), ...
            (1/4)*(-1 + 2*r)*(-1 + s)*s, ...
            (1/4)*(-1 + 2*r)*(-1 + s^2), ...
            (1/4)*(-1 + 2*r)*(-1 + s^2), ...
            ];
    psis = [(1/4)*r*(1 + r)*(1 + 2*s), ...
            (1/4)*(-1 + r)*r*(1 + 2*s), ...
            (1/4)*(-1 + r)*r*(-1 + 2*s), (1/4)*r*(1 + r)*(-1 + 2*s), ...
            (1/4)*(-1 + r^2)*(-1 + 2*s), (1/4)*(-1 + r^2)*(-1 + 2*s), ...
            ];
    Jin = inv([psir'*nodepos(elemconn(i,:),1), ...
                psir'*nodepos(elemconn(i,:),2), ...
                psis'*nodepos(elemconn(i,:),1), ...
                psis'*nodepos(elemconn(i,:),2))];
    dpsidx = Jin(1,1)*psir + Jin(1,2)*psis;
    dpsidy = Jin(2,1)*psir + Jin(2,2)*psis;
    Hxloc = psi'*Hxdof(elemconn(i,:));
    Hyloc = psi'*Hydof(elemconn(i,:));

    % Update strain rate nodal variables
    Ul = loc2glo(:,1:9)'*soln;
    U2 = loc2glo(:,10:18)'*soln;
    dudx = dpsidx'*U1;
    dvdx = dpsidx'*U2;
    dudy = dpsidy'*U1;
    dvdy = dpsidy'*U2;
    dv2dx1 = 1/Hsq*(-Hxloc*Hyloc*dudx-Hyloc*Hxloc^2*dudy+Hxloc^2*dvdy);...
    *dvdx+Hxloc*Hyloc*dvdy);
    gamdot = sqrt(1/2*dudx^2+(dudy+dvdx)^2+1/2*dvdy^2);
    if j=1 & k=1
      dv2dx1dof(elemconn(i,9)) = dv2dx1dof(elemconn(i,9)) ...
      + ElemArea(i)/NodTotArea(elemconn(i,9))*dv2dx1;
      dv2dx1cen = dv2dx1;
      gamdotdof(elemconn(i,9)) = gamdotdof(elemconn(i,9)) ...
      + ElemArea(i)/NodTotArea(elemconn(i,9))*gamdot;
      gamdotcen = gamdot;
    else
      end
      else
D.9. smooth_h.m

%%% smooth_h.m
%%% Variable smoothing function for magnetic scalar potential
%%% gradients (the components of the magnetic field).

\[
\begin{align*}
H_{x}^{dof} & = \text{zeros}(\text{nonode},1); \\
H_{y}^{dof} & = \text{zeros}(\text{nonode},1); \\
% Gauss quadrature variables (N=3) \\
rval & = [0,-\text{sqrt}(3/5),\text{sqrt}(3/5)]; \\
sval & = rval; \\
\text{alphaval} & = [8/9,5/9,5/9];
\end{align*}
\]

for i = 1:noelem
  loc2gloEM = zeros(max(max(IDEM)),9);
  for j = 1:9
    if IDEM(1,elemconn(i,j)) ~= 0
      loc2gloEM(IDEM(1,elemconn(i,j)),j) = 1;
    end
  end
  for j = 1:length(rval)
    for k = 1:length(sval)
      r = rval(j);
      s = sval(k);
      chi = loc2gloEM'*solnEM;
      psir = [(1/4)*(1 + 2*r)*s*(1 + s), ...
              (1/4)*(-1 + 2*r)*s*(1 + s), ...
              (1/4)*(-1 + 2*r)*(-1 + s)*s, (1/4)*(1 + 2*r)*(-1 + s)*s,...
              (-r)*s*(1 + s), (-1/2)*(1 + 2*r)*(-1 + s)*s, ...
              (-r)*(-1 + s)*s, ...
              (-1/2)*(1 + 2*r)*(-1 + s^2), 2*r*(-1 + s^2)]';
      psis = [(1/4)*r*(1 + r)*(1 + 2*s), ...
              (1/4)*(-1 + r)*r*(1 + 2*s), ...
              (1/4)*(-1 + r)*r*(-1 + 2*s), ...
              (1/4)*r*(1 + r)*(-1 + 2*s), ...
              (-1/2)*(-1 + r^2)*(1 + 2*s), (-1 + r)*r*s, ...
\end{align*}
\]
\[-(1/2)*(-1 + r^2)*(-1 + 2s), (-r)(1 + r)*s,\]
\[2*(-1 + r^2)*s]\; \]

\[\text{Jin} = \text{inv}([\text{psir'}*\text{nodepos}(\text{elemconn}(i,:),1),... \]
\[\text{psir'}*\text{nodepos}(\text{elemconn}(i,:),2) \]
\[\text{psis'}*\text{nodepos}(\text{elemconn}(i,:),1),... \]
\[\text{psis'}*\text{nodepos}(\text{elemconn}(i,:),2))]); \]

\[\text{dpsidx} = \text{Jin}(1,1)*\text{psir} + \text{Jin}(1,2)*\text{psis}; \]
\[\text{dpsidy} = \text{Jin}(2,1)*\text{psir} + \text{Jin}(2,2)*\text{psis}; \]

\[\text{Hxloc} = -\text{dpsidx'*chi}; \]
\[\text{Hyloc} = -\text{dpsidy'*chi}; \]

\[\text{if} \quad j==1 \quad \& \quad k==1 \]
\[\text{Hxdof}(\text{elemconn}(i,9)) = \text{Hxdof}(\text{elemconn}(i,9)) + \ldots \]
\[\text{ElemArea}(i)/\text{NodTotArea}(\text{elemconn}(i,9))*\text{Hxloc}; \]
\[\text{Hxdofcen} = \text{Hxloc}; \]
\[\text{Hydof}(\text{elemconn}(i,9)) = \text{Hydof}(\text{elemconn}(i,9)) + \ldots \]
\[\text{ElemArea}(i)/\text{NodTotArea}(\text{elemconn}(i,9))*\text{Hyloc}; \]
\[\text{Hydofcen} = \text{Hyloc}; \]

\[\text{else} \]
\[\text{Hxdof}(\text{elemconn}(i,\text{elemnode}(j,k))) = \ldots \]
\[\text{Hxdof}(\text{elemconn}(i,\text{elemnode}(j,k))) \ldots \]
\[+ \text{ElemArea}(i)/\text{NodTotArea}(\text{elemconn}(i,\text{elemnode}(j,k)))*\text{Hxloc}+\text{Hxloc}*\text{rval}(3); \]
\[\text{Hydof}(\text{elemconn}(i,\text{elemnode}(j,k))) = \ldots \]
\[\text{Hydof}(\text{elemconn}(i,\text{elemnode}(j,k))) \ldots \]
\[+ \text{ElemArea}(i)/\text{NodTotArea}(\text{elemconn}(i,\text{elemnode}(j,k)))*\text{Hyloc}+\text{Hyloc}*\text{rval}(3); \]

\[\text{end} \]
\[\text{end} \quad %s \]
\[\text{end} \quad %r \]
\[\text{end} \quad %i \]

---

**D.10. smooth_precalc.m**

```

% Variable initialization for the smoothing operation
%%%%%%%%%%%%%%%%%%%%

NodTotArea = zeros(nonode,1);
ElemArea = zeros(noelem,1);
dv2dx1dof = NodTotArea;
dv2dx1dofn = NodTotArea;
gamdotdof = NodTotArea;
gamdotdofn = NodTotArea;
Hxdof = NodTotArea;
Hydof = NodTotArea;

for i = 1: noelem
    xv = [nodepos(elemconn(i,[1,5,2,6,3,7,4,8]),1),... 
        nodepos(elemconn(i,[1,1]),1)];
    yv = [nodepos(elemconn(i,[1,5,2,6,3,7,4,8]),2),... 
        nodepos(elemconn(i,[1,1]),2)];
```
ElemArea(i) = polyarea(xv,yv);
NodTotArea(elemconn(i,:)) = NodTotArea(elemconn(i,:)) ...
   + ElemArea(i)*ones(9,1);
end

elemnode = [9 7 5;
   6 3 2
   8 4 1]; % This is used in the mapping of strain rate nodal variables

D.11. update_volfr.m

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%% update_volfr.m
%%% Calculates the distribution of the dimensionless slip parameter
%%% and updates volume fraction of the suspension.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Gauss quadrature variables (N=3)
rval = [0,-sqrt(3/5),sqrt(3/5)];
sval = rval;
alphaval = [8/9,5/9,5/9];

volfrlocconv = 1;
phicount = 1;

while (volfrlocconv > convcrit) && (phicount<1000)
    volfrn = zeros(nonode,1);
    for i = 1:noelem
        loc2glo = zeros(max(max(ID)),22);
        loc2gloEM = zeros(max(max(IDEM)),9);
        for j = 1:9
            if ID(l,elemconn(i,j)) ~= 0
                loc2glo(ID(l,elemconn(i,j)),j) = 1;
            end
        end
        if ID(2,elemconn(i,j)) ~= 0
            loc2glo(ID(2,elemconn(i,j)),j+9) = 1;
        end
        if ID(3,elemconn(i,j)) ~= 0
            loc2glo(ID(3,elemconn(i,j)),j+18) = 1;
        end
        if IDEM(l,elemconn(i,j)) ~= 0
            loc2gloEM(IDEM(1,elemconn(i,j)),j) = 1;
        end
    end
    U1 = loc2glo(:,1:9)'*soln;
    U2 = loc2glo(:,10:18)'*soln;

    for j = 1:length(rval)
        for k = 1:length(sval)
            r = rval(j);
            s = sval(k);
            alphars = alphaval(j)*alphaval(k);

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\[
\psi = \frac{1}{4} \left[ (1 + r)(1 + s)(r + s - 1) \right. \\
\left. + (1 - r^2)(1 - s^2) \right] \\
(1 - r)(1 + s)(-r + s - 1) + (1 - r^2)(1 - s^2), \ldots \\
(1 - r)(1 - s)(-r - s - 1) + (1 - r^2)(1 - s^2), \ldots \\
(1 + r)(1 - s)(r - s - 1) + (1 - r^2)(1 - s^2), \ldots \\
2*(1 - r^2)(1 + s) - 2*(1 - r^2)(1 - s^2), \ldots \\
2*(1 - s^2)(1 - r) - 2*(1 - r^2)(1 - s^2), \ldots \\
2*(1 - r^2)(1 - s) - 2*(1 - r^2)(1 - s^2), \ldots \\
2*(1 - s^2)(1 + r) - 2*(1 - r^2)(1 - s^2), \ldots \\
4*(1 - r^2)(1 - s^2) \right]; \\
\text{psir} = \left[ (1/4)*(1 + 2*r)*s*(1 + s), \ldots \\
(1/4)*(-1 + 2*r)*s*(1 + s), \ldots \\
(1/4)*(1 + 2*r)*(-1 + s)*s, \ldots \\
(-r)*s*(1 + s), (-1/2))\right. \\
\left. (-1 + s)*s, \ldots \\
(-1/2))\left. (1 + 2*r)*(-1 + s^2), 2*r*(-1 + s^2) \right] ; \\
\text{psis} = \left[ (1/4)*r*(1 + r)*(-1 + 2*s), \ldots \\
(1/4)*(-1 + r)*r*(-1 + 2*s), \ldots \\
(1/4)*r*(-1 + r)*(-1 + 2*s), \ldots \\
(-1/2)*(-1 + r^2)*(1 + 2*s), (-1/2)*r*s, \ldots \\
(-1/2)*(-1 + r^2)*(-1 + 2*s), \ldots \\
(-r)*(1 + r)*s, 2*(-1 + r^2)*s \right] ; \\
\phi = \frac{1}{4} \left[ (1 + r)^2 \right. \\
(1 - r)^2 \\
(1 - r)^2 \\
1) \right] \\
Jin = \operatorname{inv}(\text{psir}'*\text{nodepos}(\text{elemconn}(i,:),1), \ldots \\
\text{psir}'*\text{nodepos}(\text{elemconn}(i,:),2) \\
\text{psis}'*\text{nodepos}(\text{elemconn}(i,:),1), \ldots \\
\text{psis}'*\text{nodepos}(\text{elemconn}(i,:),2))); \\
dpsidx = Jin(1,1)*\text{psir} + Jin(1,2)*\text{psis}; \\
dpsidy = Jin(2,1)*\text{psir} + Jin(2,2)*\text{psis}; \\
Hx = \text{psi}'*\text{Hxdof}(\text{elemconn}(i,:)); \\
Hy = \text{psi}'*\text{Hydof}(\text{elemconn}(i,:)); \\
volfloc = \text{psi}'*\text{volfr}(\text{elemconn}(i,:)); \\
H = \sqrt{Hx^2+Hy^2}; \\
dHdx = dpsidx'*\text{Hxdof}(\text{elemconn}(i,:)); \\
dHdy = dpsidy'*\text{Hxdof}(\text{elemconn}(i,:)); \\
dHydx = dpsidx'*\text{Hydof}(\text{elemconn}(i,:)); \\
dHydy = dpsidy'*\text{Hydof}(\text{elemconn}(i,:)); \\
if \text{strcmp(\text{mumodel},'Lorentz')} \\
\text{murloc} = (1+2*\text{Beta}*\text{volfloc})/(1-\text{Beta}*\text{volfloc}); \\
\text{dmurdp} = 3*\text{Beta}/(1-\text{Beta}*\text{volfloc})^2; \\
\text{dmur2dp} = 6*\text{Beta}^2/(1-\text{Beta}*\text{volfloc})^3; \\
\text{else} \\
\text{murloc} = 1+\text{volfloc}*(\text{mup}-1); \\
\text{dmurdp} = \text{mup}-1; \\
\text{dmur2dp} = 0; \\
\text{end} \\
\text{f_hm} = (1-\text{volfloc})^4; \\
\text{V2} = 1/H*(-\text{Hy} \times \text{psi}'*\text{U1}+\text{Hx} \times \text{psi}'*\text{U2}); \]
\[
\frac{dHdx2}{dt} = \frac{-Hy/H*(Hx*dHdx+Hy*dHydx)\ldots
+Hx/H*(Hx*dHdx+Hy*dHydx);}{H2dphidx2 = \frac{-H*Hy*dpsidx'\*volfr(elemconn(i,:))\ldots
+H*Hx*dpsidy'\*volfr(elemconn(i,:));}
\]

% Slip parameter
Omega = \max(Omegalim, \frac{2*a^2*f_hm*Mni}{9/V2 \ldots
*(dmurdphi*HdHdx2+1/2*dmur2dph2i2*H2dphidx2));

phinew = \min(\phiulim, \max(\phillim, \frac{1}{(1+Omega)*volfr0}))

if j==1 && k==1
  volfrn(elemconn(i,9)) = phinew;
  volfrcen = phinew;
else
  volfrn(elemconn(i,elemnode(j,k))) = ... volfrn(elemconn(i,elemnode(j,k))) ...
+ \text{ElemArea}(i)/... \text{NodTotArea}(elemconn(i,elemnode(j,k)))... *\min(\phiulim,... \max(\phillim,(volfrcen+(phinew-volfrcen)*rval(3))));
end
end %s
end %r
end %i
volfrloccov = sum(abs(volfrn-volfr)/length(volfr);
if mod(phicount,100)==0
  meshplots_nonuni
  title(phicount);
elseif phicount == 1
  Omega = volfr0./volfrn-1;
end
phicount = phicount+1;
end %while

if phicount >=1000
  phicount
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