Experimental Investigation of 2D and 3D Internal Wave Fields

by

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Abstract

The generation of 2D and 3D internal wave fields is extensively studied via planar- and stereo- Particle Image Velocimetry (PIV) flow field measurement techniques. A benchmark was provided by an experiment involving tidal flow over a 2D Gaussian ridge; this study providing a counterpart with which studies of a 3D incised Gaussian ridge could be compared with. To further benchmark the 3D wave field studies an experiment involving the canonical setup of a vertically oscillating sphere was performed and the results compared with the latest theory; the excellent agreement obtained provided confidence in the stereo-PIV method for studying fully three-dimensional internal waves. The 3D incised Gaussian ridge generates a wave field characterized by noticeable, though weak, out-of-plane forcing that evolves from a relatively strong to a weakly localized quantity as the wave field transitions from super- to subcritical, while the in-plane velocity field appears nearly identical to its 2D counterpart.

Thesis Supervisor: Thomas Peacock
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I proudly dedicate this work to my Father, Mother and Brother. Without them, this would all be meaningless.
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Chapter 1

Introduction

The focus of this thesis is the investigation of 2D and 3D internal wave fields generated by tidal flow past topography. Building on previous studies of two-dimensional internal wave generation and propagation, we present detailed analysis and the first experimental investigations of fully three-dimensional internal wave fields using Stereo Particle Image Velocimetry (SPIV). In section 1.1 of this Introduction we discuss the basics of internal waves and provide suitable background information. Section 1.2 provides a survey of relevant previous research. Finally, an overview of the thesis is presented in section 1.3.

1.1 Background

Internal waves are propagating perturbations in a density-stratified fluid; the types of density distribution, or stratification, can range from a series of discrete layers of uniform density to a continuous, and even nonlinear, variation with height. The continuous density-stratification is the most realistic from a geophysical point of view since variations in temperature and salinity result in such spatial density changes in both the atmosphere and ocean.

Perturbations of a stable stratification result in propagating oscillations that are maintained by a balance between gravity and the restoring buoyancy force; an example of internal waves propagating in a linearly stratified fluid is given in figure 1-1.
An internal wave is characterized by spatiotemporal oscillations of the density field which are also coupled to oscillations of other physical fields such as the velocity and pressure. Such waves can also exist at the interface of two fluids of different densities, though this thesis concerns internal waves in a continuously stratified fluid.

The barotropic (ocean) tide is a primary source of oceanic internal waves; it provides a well-defined time-periodic disturbance to the stratification by virtue of the ocean floor topography impeding the oscillatory background flow and perturb the density stratification from its basic state. Internal waves are also generated by surface wind forcing, though it is a more random process which does not have the harmonic regularity of the tidally generated waves [30]. Such waves, however, are not limited to the ocean; atmospheric internal waves i.e. mountain waves are generated when stratified air is forced over an obstructing topographic feature by winds [6].

Scandinavian sea-explorers are credited with the discovery of internal waves, beginning in the mid-1890’s with the polar expedition of Helland-Hansen and Nansen who were the first to observe the dead water phenomena. Helland-Hansen and Nansen discovered oscillations at the interface of two fluids of different density which would later be confirmed as interfacial internal waves [31]. However, the Swedish oceanog-
rapher Otto Pettersson is thought to be the first to recognize the existence of internal waves. Pettersson observed, over a period of many years starting in 1909, internal waves in a Swedish fjord in which he detailed their characteristics and proved their tidal periodicity [10].

Our interest is in the generation of these internal waves by deep-water topography. The motivation lies in their essential role in ocean mixing, in addition to energy and momentum transport [13]. The understanding of this energy transport and ocean mixing is critical in detailing the global energy balance and the overall planetary response. From a dynamical perspective, internal waves play an essential role in maintaining the temperature and salinity structure of the ocean [30]. Instabilities eventually lead to breaking of the internal waves which allows for mixing of the cold dense water in the ocean basins with the warmer water of the upper ocean layers.

To demonstrate the scale on which these waves act, field studies and satellite observations have shown that internal waves propagate more than 1000 km from the generation site before complete decay. Significantly, altimetry readings estimate that up to 1 TW of tidal power is converted from the background barotropic flow to the induced baroclinic motions [35]. Such findings underscore the importance of internal waves in the ocean.

Internal waves also have biological significance by virtue of their propensity for nutrient transport in the oceans. Internal waves can lift phytoplankton from the depths of the ocean to the warm upper-layer which allows photosynthesis to take place, thus affecting chlorophyll concentrations [19]; this can be especially significant for the health of coral reef systems. Furthermore, the interaction between phytoplankton and internal tides is critical regarding the distribution of these organisms which serve as the foundation of the oceanic food chain.

1.2 Previous experimental work

Laboratory experimental investigation of internal waves began with the work of Gortler [15] who was the first to report experimental observations of internal waves.
A more detailed survey of visualization techniques for stratified flows was offered by Mowbray [28] who discusses dye bands, shadowgraph and schlieren methods. The most basic method valid for any density stratification set-up is the use of bands of colored dye to observe vertical displacements of isopycnals due to the induced perturbations; a refined version of this technique has been used to great effect by Voisin et al. [43].

Other unobtrusive and effective measurement techniques have been the shadowgraph and schlieren methods which take advantage of the direction relation between variations in concentration and refractive index. It was found by Mowbray [28] that for a linearly stratified medium the schlieren technique was the most appropriate; for a medium containing a nonlinear stratification the shadowgraph was used with the inclusion of a simple calibration modification. Furthermore, the validation of the earliest linear theory was first done by Mowbray & Rarity [29] utilizing the schlieren system to observe small amplitude waves. It should be noted that Gortler [15] observed internal waves and offered the first experimental verification of the dispersion relation using the schlieren method.

With the advancement and maturation of experimental set-ups also came the need for more direct measurements systems. Initial studies of the interaction of internal waves and topographic features involved fixing conductivity probes at various points in the fluid and collecting voltage readings which, after proper calibration, provided the fluctuations in density [3]. Lee & Beardsley [22] also used conductivity probes in exploring the balance of nonlinear and dispersive effects of long nonlinear internal waves and the corresponding relation to tidally-generated internal waves observed in Massachusetts Bay.

The synthetic schlieren technique described by Sutherland et al. [37] and Dalziel et al. [5] has been a primary measurement tool for the past decade. The difference between synthetic schlieren and the previous schlieren method is substantial; the most significant development being the ability to directly obtain quantitative measurements of the buoyancy field. Synthetic schlieren measures fluctuations in density by determining the optical distortion of light rays passing through the fluid;
the medium is illuminated by a pattern of random dots positioned behind the test section. Since the propagating internal waves perturb the isopycnals the random dot pattern will become distorted in the captured images. The distortions are quantified by using the appropriate processing software to compare the images of the distorted and known random dot pattern. As an example of the progress experimental methods have made in the past four decades, figure 1-2 displays sample data obtained by schlieren (1967) and synthetic schlieren (2000’s). Draw backs of this technique are the sensitivity of the schlieren methods to thermal noise and the requirement of a two-dimensional wave field. It should be noted that the synthetic schlieren has been modified to work for axisymmetric wave fields [32]. Synthetic schlieren has been extensively used to visualize internal waves generated by oscillating bodies [12, 34] and wave beam propagation through nonuniform stratifications [26], to name a few examples.

Particle Image Velocimetry (PIV) is the most recent addition to internal waves research, though Thomas & Stevenson [39] did track the motion of neutrally buoyant oil drops to obtain amplitude measurements of an internal wave field. The ability of PIV to provide velocity field measurements has contributed greatly to the develop-
ment of internal wave research. To date, several studies of internal wave generation have been performed using PIV. Zhang et al. [46] studied internal wave generation by using a horizontally oscillating cylinder to mimic ocean bottom topography and King et al. [20] performed both experiments and simulations of stratified tidal flow past topography by observing the flow field in the upper half domain of a horizontally oscillating sphere. However, the most geophysically relevant study has been the work of Echeverri et al. [7] which provides a thorough analysis of two-dimensional internal wave generation by an ocean ridge over a range of forcing regimes.

The limitation of standard planar-PIV is the inability to extract complete velocity measurements of three-dimensional wave fields, and all prior experiments have been so restricted. In this thesis, we utilize Stereoscopic PIV (SPIV) to investigate fully 3D internal wave fields. Related previous studies have been performed only on axisymmetric, 3D wave fields: a vertically oscillating sphere observed by synthetic schlieren [12] and fluorescent dye planes [43], and PIV to measure velocity fields in vertical planes through the center of an oscillating Gaussian mountain [21]. To date, however, there has been no investigation of a fully three-dimensional wave field using SPIV. And so, in this thesis, we perform the first set of experiments investigating 3D internal wave generation by an incised Gaussian ridge. The principal scientific question to investigate is to what degree does the 3D incision of the Gaussian ridge impact the overall wave field?

1.3 Thesis overview

The primary focus of this thesis is the experimental investigation of 2D and 3D internal wave fields. A formulation of the equations governing internal waves and an overview of the latest Green function theory is presented in chapter 2. In chapter 3 we present a refined version of the modal decomposition technique detailed by Echeverri et al. [7] and various solutions to potential difficulties that may arise in experiments. Thereafter, in chapter 4, the results of a 2D Gaussian ridge experiment are presented and compared with theoretical predictions. Furthermore, the features of the wave
field are analyzed using the modal decomposition techniques discussed in chapter 3. Chapter 5 presents an overview of the SPIV measurement technique which is then applied to studies of the vertically oscillating sphere and a 3D incised Gaussian ridge. We conclude the thesis with a summary of the key findings in chapter 6.
Chapter 2

Theory

In this chapter we discuss the theoretical formulation of internal wave propagation and generation that will be used to make predictions of internal tide generation for comparison with experiments. We begin, in section 2.1, by deriving the internal wave equations and the boundary conditions encountered in generation problems. In section 2.2 we discuss the relevant features of internal waves and the various parameters that influence their behavior. Lastly, we present the Green function approach to calculating internal tide generation, also taking into account a viscous correction.

2.1 Formulation

The equations governing incompressible, stratified flow comprise the momentum equation,

\[ \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{\rho_0} \nabla p - \mathbf{g} = \nu \nabla^2 \mathbf{u}, \quad (2.1) \]

mass conservation,

\[ \frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = 0, \quad (2.2) \]

and incompressibility,

\[ \nabla \cdot \mathbf{u} = 0, \quad (2.3) \]
where for a two-dimensional system \( \mathbf{u} = (u, w) \) is the velocity field in the vertical plane \((x, z)\), \( \rho \) is the density, \( p \) is the pressure, \( \nu \) is the kinematic viscosity and \( \mathbf{g} = -g \hat{e}_z \) is the gravitational acceleration. We define \( \mathbf{u}, p \) and \( \rho \) perturbations from a basic state such that

\[
\mathbf{u} = \mathbf{u}_b + \mathbf{u}', \quad p = p_b + p', \quad \rho = \rho_b + \rho'
\]

(2.4)

where the subscript \( b \) signifies the base state and the primes signify perturbation quantities. For a barotropic tide in a stratified ocean the known background velocity field and density are

\[
\mathbf{u}_b = U \text{Re}\{e^{-i\omega t}\} \hat{e}_x,
\]

(2.5)

and

\[
\rho_b(z) = \rho_o + \bar{\rho}(z)
\]

(2.6)

where \( U \) is the barotropic velocity amplitude, \( \omega \) is the forcing frequency, \( \rho_o \) is a reference density and \( \bar{\rho}(z) \) is the static density distribution. Applying Newton’s law to a fluid element perturbed from its basic state leads to the buoyancy frequency:

\[
N^2(z) = \frac{g}{\rho_b} \frac{d\bar{\rho}}{dz'},
\]

(2.7)

which is effectively the local natural frequency of oscillation associated with the background stratification. In the case of a linear density-stratification within the Boussinesq limit \((\Delta \rho/\rho_o \ll 1)\) the density field can be written explicitly:

\[
\rho(x, z, t) = \rho_o - \frac{\rho_o N^2}{g} z + \rho'(x, z, t).
\]

(2.8)

The expression for the total pressure under the Boussinesq limit is simplified to

\[
p(x, z, t) = g \rho_o (H - z) + p'(x, z, t),
\]

(2.9)

where \( H \) is the total depth of the fluid. For completeness we rewrite the expression for total velocity:

\[
\mathbf{u}(x, z, t) = U \text{Re}\{e^{-i\omega t}\} \hat{e}_x + \mathbf{u}'(x, z, t).
\]

(2.10)
We now proceed to obtain the equations governing the perturbed quantities by substituting (2.8), (2.9) and (2.10) into (2.1), (2.2) and (2.3) and linearizing by omitting quadratically small terms. The resulting linear equations are:

\[-i\omega U \text{Re}\{e^{i\omega t}\} + \frac{\partial u'}{\partial t} + \frac{1}{\rho_o} \frac{\partial p'}{\partial x} = \nu \nabla^2 u',\]

(2.11)

\[\frac{\partial u'}{\partial t} + \frac{1}{\rho_o} \frac{\partial p'}{\partial z} + \frac{g}{\rho_o} \rho' = \nu \nabla^2 w',\]

(2.12)

\[g \frac{\partial p'}{\partial t} - \rho_o N^2 w' = 0,\]

(2.13)

and

\[\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0.\]

(2.14)

By making use of a streamfunction \((u', w') = (-\partial_z \psi', \partial_x \psi')\) and then taking the proper combination of partial derivatives to eliminate the \(p\) and \(\rho\) terms, (2.11) - (2.14) can be combined into a single equation:

\[\frac{\partial^2}{\partial t^2} \nabla^2 \psi' + N^2 \frac{\partial^2 \psi'}{\partial x^2} = \nu \frac{\partial}{\partial t} \nabla^4 \psi'.\]

(2.15)

Since the spatial and temporal variables are clearly independent we can separate the time-dependence of the problem by using

\[\psi'(x, z, t) = \text{Re}\{\phi'(x, z)e^{-i\omega t}\}\]

(2.16)

in which \(\phi'\) is a spatial description of the perturbation variables. Substituting (2.16) into (2.15) yields the viscous internal wave equation:

\[\omega^2 \nabla^2 \phi' - N^2 \frac{\partial^2 \phi'}{\partial x^2} = i\omega \nu \nabla^4 \phi'.\]

(2.17)

The dispersion relation for internal wave propagation is derived by substituting a plane-wave solution \(\psi' = \{\Psi \exp\{i(k_x + k_z - \omega t)\} + c.c.\}\) into the inviscid form of
\[ \frac{\omega}{N} = \frac{k_x}{\sqrt{k_x^2 + k_z^2}} = \sin \theta, \quad (2.18) \]

where \( k_x \) and \( k_z \) are the horizontal and vertical wavenumbers, respectively, and \( \theta \) is the angle a line of constant phase makes with the horizontal.

### 2.2 Boundary conditions

By imposing a no-normal flow boundary condition we require the streamlines to follow the boundaries at \( z = z_t(x) \) and \( H \), where \( z_t(x) \) describes the bottom topography. We define the total streamfunction describing the complete dynamics of the flow as \( \psi_T = \psi_b + \psi' \) (and by extension \( \phi_T = \phi_b + \phi' \)). The no-flux condition can be stated as follows:

\[ \left( -\frac{\partial \psi_T}{\partial z}, \frac{\partial \psi_T}{\partial x} \right) \cdot \mathbf{n} = 0, \quad (2.19) \]

which in turn requires that

\[ \frac{d\psi_T}{dx} = 0 \quad \text{at} \quad z = z_t, H. \quad (2.20) \]

In deriving the boundary conditions for the perturbation streamfunction we begin by finding the functional form of \( \phi_T \). Taking into consideration the background flow yields \( \psi_T = -Uzx e^{-i\omega t} + \psi' \) and correspondingly \( \phi_T = -z + \phi' \). Keeping in mind (2.20), we apply \( \phi_T \) at the boundaries:

\[ \phi_T(x, z = z_t) = -z_t + \phi'(x, z = z_t) = C_1, \quad (2.21) \]

\[ \phi_T(x, z = H) = -H + \phi'(x, z = H) = C_2, \quad (2.22) \]

where \( C_1 \) and \( C_2 \) are arbitrary constants.

By calculating the flux through a cross-section at an arbitrary location along the topography we find that the two arbitrary constants can not be arbitrarily chosen, rather they obey the relation \( C_1 - C_2 = H \). There is no loss of generality in choosing
$C_1 = 0$ and $C_2 = -H$; as a result, the perturbation streamfunction takes on the following boundary conditions:

\[ \phi'(x, z = z_t(x)) = z_t(x) \quad \text{and} \quad \phi'(x, z = H) = 0. \] (2.23)

### 2.3 Green function solution

In formulating the Green function solution we follow the work of Balmforth & Peacock [2] and Echeverri et al. [9]. The Green function $G(x, x'|z, z')$ is found by solving the governing internal wave forced by a singularity at $(x', z')$:

\[ \cot^2 \theta \frac{\partial^2 G}{\partial x^2} - \frac{\partial^2 G}{\partial z^2} = i\delta(x - x')\delta(z - z'), \] (2.24)

where $\delta$ is the Dirac delta function. Since this particular derivation requires that both $z_t \to 0$ as $x \to \pm \infty$ and $z_t \geq 0$ be satisfied, the resulting homogeneous boundary conditions in the far field are $G(x, x'|z = 0, z') = G(x, x'|z = H, z') = 0$. Based on the procedure outlined in [2, 7, 8] one can show that the general Green function that satisfies the aforementioned boundary conditions in addition to an imposed radiation condition is:

\[ G(x, x'|z, z') = \sum_{n=1}^{\infty} \frac{1}{n\pi \cot \theta} \sin \left( \frac{n\pi z'}{H} \right) \sin \left( \frac{n\pi z}{H} \right) \exp \left\{ \frac{in\pi |x - x'|}{H \cot \theta} \right\}, \] (2.25)

in which the stratification $N(z)$ is assumed to be constant. We note that the viscosity has been removed for the Green function derivation and will be introduced later in the form of a viscous correction term.

A distribution of point-sources placed on $(x', z' = z_t(x'))$ is used to model the topography, with each delta function source satisfying the internal wave equation and the associated boundary conditions. A schematic of the problem geometry is presented in figure 2-1. The general solution for $\phi'$ is now written as a superposition
Figure 2.1: Problem geometry showing the boundaries, imposed background flow and point-source distribution.

of all the Green functions along the topography via the integral expression:

$$\phi'(x, z) = \int_a^b \gamma(x') G(x, x'|z, z_t(x')) \, dx',$$

(2.26)

where \(\gamma(x')\) is a weighting function that specifies source density. Substituting (2.25) into (2.26) and then reversing the order of integration and summation yields the expression for the perturbation stream function:

$$\phi'(x, z) = \sum_{n=1}^\infty \frac{1}{n\pi \cot \theta} \sin \left( \frac{n\pi z}{H} \right) \int_a^b \gamma(x') \sin \left( \frac{n\pi z_t(x')}{H} \right) \exp \left\{ \frac{in\pi}{H \cot \theta} |x - x'| \right\} \, dx'.$$

(2.27)

The yet unknown distribution \(\gamma(x')\) is calculated by applying the boundary condition at \(z = z_t\) which renders it a standard matrix problem solved through matrix inversion techniques (as detailed in Echeverri & Peacock [8]).

We expect viscosity to have a direct effect on the horizontal propagation of internal waves, thus we separate the two spatial coordinates in \(\phi'(x, z)\) by rewriting

$$\phi'(x, z) = \sum_{n=1}^\infty \zeta_n(x) \sin \left( \frac{n\pi z}{H} \right),$$

(2.28)

where \(\zeta_n(x)\) are the eigenfunctions of the streamfunction in \(x\). Substituting (2.28)
into (2.17) yields the linear fourth-order differential equation:

\[
\zeta_{n,xxxx} = \left( \frac{i\omega}{\nu} \cot^2 \theta + 2 \left( \frac{n\pi}{H} \right)^2 \right) \zeta_{n,xx} + \left( \left( \frac{n\pi}{H} \right)^4 - \frac{i\omega}{\nu} \left( \frac{n\pi}{H} \right)^2 \right) \zeta_n = 0. \tag{2.29}
\]

In finding the solution that satisfies the fully viscous internal wave equation, we substitute solutions of the form \( \zeta_n(x) = \exp\{f_n x\} \) into (2.29) which gives the characteristic equation:

\[
f_n^4 - \left( \frac{i\omega}{\nu} \cot^2 \theta + 2 \left( \frac{n\pi}{H} \right)^2 \right) f_n^2 + \left( \left( \frac{n\pi}{H} \right)^4 - \frac{i\omega}{\nu} \left( \frac{n\pi}{H} \right)^2 \right) = 0, \tag{2.30}
\]

where the roots \( f_n \) are all valid, though not all applicable, viscous correction terms. Since we expect the internal wave field to weaken with distance, we select a form of the viscous correction factor that includes both an inviscid and viscous term that allows wave propagation and decay, respectively:

\[
f_n = f_{n,\text{inv}} + i\nu f_{n,\text{vis}}, \tag{2.31}
\]

and to leading order

\[
f_n^2 = f_{n,\text{inv}}^2 + 2i\nu f_{n,\text{vis}} f_{n,\text{inv}}. \tag{2.32}
\]

The term \( f_{n,\text{inv}} \) is found by multiplying (2.30) by \( \nu \) and then taking the inviscid limit which produces:

\[
f_{n,\text{inv}}^2 = - \left( \frac{n\pi}{H} \right)^2 \tan^2 \theta, \tag{2.33}
\]

and then substituting (2.32) into (2.30) gives the expression for the viscous term:

\[
f_{n,\text{vis}} = - \frac{\tan \theta}{2\omega} \left( \frac{n\pi}{H} \right)^3 \sec^4 \theta. \tag{2.34}
\]

We can now include in the sum (2.27) the viscous correction term [7]:

\[
\exp \left\{ -\frac{x \nu \tan \theta}{2\omega} \left( \frac{n\pi}{H} \right)^3 \sec^4 \theta \right\}, \tag{2.35}
\]

which effectively dampens the propagating wave beams with distance. It should be
noted that the generation remains inviscid; the viscous correction simply approximates the energy dissipation of the wave field with respect to position.
Chapter 3

Modal Analysis

A clear and convenient way of representing an internal wave field in a finite depth fluid, be it a wave beam or another propagating disturbance, is by its modal content. The modal decomposition of an internal wave field provides a robust means to compare wave fields that outwardly appear different but may actually be fundamentally similar. Significant information can be drawn from a knowledge of the modal decomposition of an internal wave field; the linear energy flux of an internal wave field, for example, is readily determined by calculating the energy flux of the composite modes. Additionally, the evolution of an internal wave field can be appreciated by considering the impact of damping on the constituent modes. To this end, a reliable method is sought for obtaining an accurate modal decomposition of an internal wave field, be it theoretical or experimental.

In this chapter, we investigate different approaches to modal analysis, taking into account practical considerations such as the presence of experimental noise and incomplete data. In section 3.1 we derive the expressions for the velocity components of an internal wave field in a channel of constant depth. Then in section 3.2 the modal decomposition procedure is discussed. Section 3.3 tests the modal decomposition algorithm on a theoretical wave field and the accuracy of the procedure is analyzed. Thereafter, in sections 3.4, 3.5 and 3.6 the impact of noise, higher order harmonics and image loss on the quality of the modal analysis is studied. The application of the algorithm to experimental data in section 3.7 concludes this chapter.
3.1 Vertical modes

For two-dimensional disturbances in a fluid with constant density stratification, the linear internal wave equation for the perturbation stream function, \( \psi \), due to small amplitude internal waves is:

\[
\frac{\partial^2}{\partial t^2} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right) \psi + N^2 \frac{\partial^2 \psi}{\partial x^2} = 0.
\] (3.1)

We seek a separable solution of (3.1) that takes the form of a horizontal traveling wave with a corresponding vertical modal structure \( f(z) \), i.e.

\[
\psi(x, z, t) = f(z) \exp \left( ik_x x - \omega t \right),
\] (3.2)

where \( k_x \) is the horizontal wave number and \( \omega \) is the forcing frequency. Substituting this into (3.1) yields the modal equation:

\[
\frac{d^2 f(z)}{dz^2} + k_x^2 \left( -1 + \frac{N^2}{\omega^2} \right) f(z) = 0.
\] (3.3)

Making use of the dispersion relation for linear internal waves,

\[
\sin \theta = \frac{\omega}{N},
\] (3.4)

equation (3.3) can be rewritten:

\[
\frac{d^2 f(z)}{dz^2} + k_x^2 \cot^2 \theta f(z) = 0,
\] (3.5)

for which the solution is:

\[
f(z) = \Psi_1 \exp(i k_x \cot \theta z) + \Psi_2 \exp(-i k_x \cot \theta z),
\] (3.6)

where \( \Psi_1 \) and \( \Psi_2 \) are complex coefficients.

Assuming a flat and rigid ocean sea floor and surface, the boundary conditions at
\( z = 0, H \) require that there be zero vertical velocity. There are an infinite number of eigenfunctions satisfying these boundary conditions:

\[
f_n(z) = \Psi_n \sin(k_{z,n}z), \quad n = 1, 2, 3, ...
\]

(3.7)

where each value of the integer \( n \) corresponds to a distinct vertical mode and \( k_{z,n} = n\pi/H \) is the vertical wave number of mode \( n \). The dispersion relation (3.4) requires that \( k_x = \pm k_x \cot \theta \). Thus, for every vertical wave number \( k_{z,n} \) there is a corresponding horizontal wave number \( k_{x,n} = n\pi/H \cot \theta \).

Since (3.1) is a Sturm-Liouville type problem [25] any solution can be expressed as a superposition of the eigenfunction solutions, in which case the most general expression for the perturbation stream function is:

\[
\psi(x, z, t) = \sum_{n=1}^{\infty} \sin \left( \frac{n\pi z}{H} \right) \Re \left\{ \Psi_n \exp \left( \frac{n\pi x}{H \cot \theta} - \omega t \right) \right\}. 
\]

(3.8)

We collect the real part of (3.8) since the imaginary components are simply mathematical artifacts of the technique used to derive a solution to the governing equation. Via the definition of the stream function, \( (u, w) = (-\partial_z \psi, \partial_x \psi) \), it can be readily determined that the corresponding horizontal and vertical velocity fields are:

\[
u(x, z, t) = \sum_{n=1}^{\infty} \cos \left( \frac{n\pi z}{H} \right) \Re \left\{ U_n \exp \left( \frac{n\pi x}{H \cot \theta} - \omega t \right) \right\} 
\]

(3.9)

and

\[
w(x, z, t) = \sum_{n=1}^{\infty} \sin \left( \frac{n\pi z}{H} \right) \Re \left\{ W_n \exp \left( \frac{n\pi x}{H \cot \theta} - \omega t \right) \right\}
\]

(3.10)

where the relation between the complex constants \( U_n \) and \( W_n \) is \( U_n = i \cot \theta W_n \). The constants \( U_n \) and \( W_n \) contain information on both the amplitude and phase of the modal structure of the wave field.
3.2 Modal decomposition procedure

For a given wave field in a two-dimensional system of constant depth the complex amplitudes of (3.9) and (3.10) can be determined by taking advantage of the orthogonality relation:

\[ \int_0^H f_m(z)f_n(z)dz = \delta_{mn} \int_0^H f_m^2(z)dz, \]  

(3.11)

where \( \delta_{mn} \) is the Kronecker delta and \( f_{n,m} \) are eigenfunctions of the system. Equation (3.11) states that the inner product norm of the eigenfunctions \( f_n \) and \( f_m \) over the vertical domain \([0, H]\) is zero unless \( n = m \), i.e. the vertical mode shapes are orthogonal. This relation is exploited to obtain the \( m^{th} \) vertical mode strength and phase by projecting the vertical velocity field (3.10) onto the vertical basis mode \( m \). For example,

\[ \int_0^H w(x, z, t) \sin(k_z m z)dz = \frac{H}{2} \text{Re}\{W_m \exp \left[ i(k_{z,m}x - \omega t) \right] \}. \]

(3.12)

The same procedure applies for the horizontal velocity field as well, albeit with projection onto cosine, rather than sine, basis functions.

In order to execute a modal decomposition, equation (3.12) suggests a methodology that can be followed, which is now outlined. Vertical profiles of a velocity component are obtained for \( z = [0 : H] \) at a specific horizontal location \( x_o \) at two different times \( t_1 \) and \( t_2 \); the origins for both \( x \) and \( t \) can be arbitrarily chosen as this merely affects the definition of what is the phase of a mode. Applying (3.12) to the vertical profile of the \( w \) velocity, for example, at location \( x_o \) and times \( t_1 \) and \( t_2 \) yields values for the quantities:

\[ I_1 = \frac{H}{2} \text{Re}\{W_m \exp \left[ i(k_{z,m}x_o - \omega t_1) \right] \}. \]

(3.13)

36
and

\[ I_2 = \frac{H}{2} \Re \{W_m \exp i(k_{x,m}x_o - \omega t_2)\}, \]  

(3.14)

respectively. Having obtained values for \( I_1 \) and \( I_2 \), determining the real and imaginary components of the complex coefficient \( W_m \) is achieved by explicitly writing the real and imaginary parts of the right hand sides of (3.13) and (3.14), and rearranging to obtain the result:

\[
\begin{bmatrix}
\Re \{W_m\} \\
\Im \{W_m\}
\end{bmatrix} = \frac{2}{H \sin(\omega \Delta t)} \begin{bmatrix}
-\sin(k_{x,m}x_o - \omega t_2) & \sin(k_{x,m}x_o - \omega t_1) \\
-\cos(k_{x,m}x_o - \omega t_2) & \cos(k_{x,m}x_o - \omega t_1)
\end{bmatrix} \\
\times \int_0^H \sin(k_{x,m}z) \begin{bmatrix}
w(x_o, z, t_1) \\
w(x_o, z, t_2)
\end{bmatrix} \, dz, 
\]

(3.15)

which is well defined for all values of \( \Delta t = t_2 - t_1 \) except \( \Delta t = n \pi / \omega \) (\( n = 0, 1, 2, 3, \ldots \)).

The complex coefficient \( U_m \) can be obtained from horizontal velocity field data via the same process outlined above by alternatively using the basis function \( \cos(k_{x,m}z) \) instead of \( \sin(k_{x,m}z) \) in the orthogonality relation.

With the components of the complex coefficient in hand, finding the mode strength and phase is by virtue of the identity \( z_m = |z_m| \exp i \phi_m \), for any complex number \( z_m \). For the vertical velocity field, the mode strength and phase are, respectively:

\[
|W_m| = (\Re \{W_m\}^2 + \Im \{W_m\}^2)^{1/2},
\]

(3.16)

\[
\phi_m^{(w)} = \arctan \left( \frac{\Im \{W_m\}}{\Re \{W_m\}} \right),
\]

(3.17)

where (3.17) should return the full range numeric value of the inverse tangent of the argument. Again, the same result is used for the horizontal velocity field.

Finally, in practice, to obtain the modal decomposition of either the vertical or horizontal velocity field one must specify \( m_{\text{max}} \) (the desired number of modes) and repeat the process outlined above for \( m = 1 \) through \( m_{\text{max}} \). In principle, one should be able to run this procedure on both the vertical and horizontal velocity fields for
an internal wave field, and find that the results of the two procedures are related by

\[ |U_m| = \cot \theta |W_m| \]  

(3.18)

and

\[ \phi_m^{(u)} = \phi_m^{(w)} + \frac{\pi}{2}. \]  

(3.19)

### 3.3 Theoretical validation

Having presented an analytic approach to modal decomposition in the previous section, we now proceed to investigate its effectiveness when applied to both theoretical and experimental data sets. In this section we investigate the theoretical validation by applying the method to a theoretical wave field with known properties.

A wave field was generated in MATLAB using ten input mode strengths \(|W_{1:10}| = [10\; 9\; 8\; 7\; 6\; 5\; 4\; 3\; 2\; 1]|\) and corresponding phase values \(\phi_{1:10}^{(w)} = [1\; 2\; 3\; 4\; 5\; 6\; 7\; 8\; 9\; 10]|\times \pi/5\). The vertical resolution of the analytical solution over the domain \([0, H]|\) was chosen to be 120 elements, and the resultant wave field contained a clearly defined wave beam (see figure 3-1). The periodicity of the wave field in the temporal domain is also demonstrated in figure 3-1, as is the impact of phase propagation on the wave beam structure at \(t_1\) and \(t_2\). The corresponding \(u\) velocity field (not shown) was created using the same procedure as that used to construct the \(w\) velocity, by virtue of using relations (3.18) and (3.19).

Having created an idealized wave field of known modal content, the modal decomposition method outlined in section 3.2 was applied to the \(w\) and \(u\) velocity fields at an arbitrarily chosen location \(x_o\) and the comparison between the input and the output from the algorithm for the modal amplitudes is presented in figures 3-2(a) and (b). There is excellent agreement between the modal decomposition results and the expected results for both the \(w\) and \(u\) velocity fields. The results for the \(u\) velocity field do show \(O(3\%)\) differences in the results for some of the intermediate mode numbers whereas the errors for the \(w\) velocity field were always less than \(O(1\%).\)
Figure 3-1: An idealized vertical velocity field plotted as a function of the spatial coordinates $x$ and $z$, and time $t$. The wave field was constructed in MATLAB using the known modal amplitudes $|W_{1:10}| = [10 \ 9 \ 8 \ 7 \ 6 \ 5 \ 4 \ 3 \ 2 \ 1]$ and phases $\phi_{1:10}^{(w)} = [1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10] \times \pi/5$ rad. $L$ and $H$ represent horizontal and vertical length scales, respectively; all physical quantities have arbitrary values.

We attribute this discrepancy as being due to the relevant basis functions, $\cos(k_z z)$, having maxima at the boundaries and minima at mid-domain. As a result, the inner product of the $u$ velocity profile and the $u$ basis functions are minimized which makes the algorithm highly sensitive to the numerical resolution in the vertical domain.

The phases for each mode number for both the vertical and horizontal velocity fields were also computed, and the results are displayed in figures 3-2(c) and (d). There is also excellent agreement between the known phase inputs and the algorithm outputs for both $w$ and $u$ velocity fields, again with some slight ($O(3\%)$) disagreement for the $u$ velocity field. Typically, the error in $w$ phase was no more than 0.2% whereas the $u$ phase values ranged from 0.6% to 4.6% (the higher percentage error in occurs at phase values close to zero and the absolute error is never more than $O(0.1)$ rads).

We also investigated the effect of the choice of horizontal location $x_o$ and the time interval $\Delta t$ on the results obtained by the modal decomposition algorithm. The choice of horizontal location had no effect on the results for the $w$ velocity field but it was found that for the $u$ velocity field, choosing a horizontal location that contains the
wave beam close to either boundaries yields the most accurate results. The values of $\Delta t$ have no effect provided that the condition stated in section 3.2 is met, as demonstrated by the results presented in figure 3-3.

An important parameter in accurately decomposing a velocity field into its composite modes is the vertical resolution. The effect of vertical discretization on the extraction error of the $w$ and $u$ modes, defined as

$$ \frac{|U_m|_{\text{calculated}} - |U_m|}{|U_m|} $$

is explored by increasing the resolution and calculating the error; the results are presented in figures 3-4(a) and (b). As the vertical discretization is increased there is a substantial improvement in the accuracy of the modal decomposition algorithm for both the $w$ and $u$ velocity field, though the improvement is greater for the $u$ velocity. The $w$ velocity error experiences a steep convergence to nearly zero percent error at a vertical resolution of ten elements; note that the point of convergence increases with
Figure 3-3: Strengths of the first, third and fifth modes extracted using the algorithm for $0 < \Delta t < 2\pi/\omega$. The $u$ component (dashed lines) and $w$ component (solid lines) are presented in polar form where the radial axis represents the mode strength and the azimuthal represents $\Delta t/\omega$. The hollow circles are removable singularities for which the modal analysis is undefined. Modal amplitudes for $m = 1, 3$ and 5 are represented by the outer, middle and inner circles, respectively, for both the $w$ and $u$ velocities.

increasing mode number. In contrast, the horizontal velocity components need at least 200 elements in the vertical for the algorithm to produce reliable calculations with errors less than five percent. This is to be expected since working with the $\cos(k_z z)$ basis functions effectively minimizes the projection of the velocity profile in the integrals $I_1$ and $I_2$, which diminishes the ability of the algorithm to extract modal amplitudes.

From these results we confirm the relative advantage that the $w$ velocity field has over the $u$ velocity field and we conclude that the most accurate results are obtained using the vertical velocity field data.

3.4 Impact of noise

Even in well-executed experiments a clean wave field is rarely achieved, and thus it is essential to have methods to address practical issues such as the presence of random
noise in the data. We investigate the effect of random noise on the modal decomposition results by introducing white noise to the idealized wave field investigated in section 3.3. A signal-to-noise ratio (SNR) of 5 was used, meaning the maximum amplitude of the superimposed noise is one fifth of the signal strength. This SNR value was chosen to represent a realistic, yet extreme, limit to the noise level that might be encountered in experimental settings. Based on the results in section 3.3, we henceforth choose to work with the vertical velocity field.

To remove random noise, we fit a sinusoidal curve of the form

\[ \xi(x, z, t) = A(x, z, t) \cos(\omega_0 t) + B(x, z, t) \sin(\omega_0 t) \]  

(3.21)

to the time signal at each location \((x, z)\) throughout the wave field. We begin by defining the error:

\[ E = \sum_{j=1}^{j_{\text{max}}} (A \cos(\omega_0 t_j) + B \sin(\omega_0 t_j) - w_j)^2 \]  

(3.22)

where \(j_{\text{max}}\) is the total number of sample points in time, \(A\) and \(B\) are unknown constants that are functions of position in the wave field, and \(w_j\) is the chosen velocity component at time step \(j\). Computing \(A\) and \(B\) at each spatial location requires...
obtaining and solving a linear set of equations given by $\partial_A E = 0$ and $\partial_B E = 0$:

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} \Sigma_j \cos^2(\omega_b t_j) & \Sigma_j \cos(\omega_b t_j) \sin(\omega_b t_j) \\ \Sigma_j \cos(\omega_b t_j) \sin(\omega_b t_j) & \Sigma_j \sin^2(\omega_b t_j) \end{bmatrix}^{-1} \begin{bmatrix} \Sigma_j \cos(\omega_b t_j) u_j \\ \Sigma_j \sin(\omega_b t_j) u_j \end{bmatrix}.$$  \hspace{1cm} (3.23)

Applying this process to time series obtained at each location in the wave field effectively removes random disturbances in the velocity time series.

The results presented in figure 3-5 demonstrates the noise filtering procedure. To begin, a location is chosen in the horizontal domain at $x_o$ (see figure 3-5(a)) and the temporal wave field at this location is extracted (see figure 3-5(b)). At each vertical location in this time series the filtering procedure is applied and the filtering result for a particular vertical location $z_0$ is presented in figure 3-5(c). As a general rule, the greater the number of periods contained in the temporal wave field the more accurate the signal fitting process becomes.

Henceforth, the results of fitting at every location in the wave field can be combined into the single data set:

$$\xi(x, z, t) = \text{Re}\{[A(x, z) + iB(x, z)] \exp(-i \omega_b t)\}.$$  \hspace{1cm} (3.24)

The physical meaning of the real and imaginary parts of $\xi$ are simply the amplitude of the velocity time series at times $t_1 = 0$ and $t_2 = \pi/2 \omega_b$, respectively, and this is all the information that one needs to know to determine the wave field at any other time instant.

The algorithm for wave field decomposition described in section 3.2 is readily modified to incorporate this approach. The procedure (3.15) reduces to

$$\begin{bmatrix} \text{Re}\{W_m\} \\ \text{Im}\{W_m\} \end{bmatrix} = \frac{2}{H} \begin{bmatrix} \cos(k_{x,m} x_o - \omega_b t_1) & \sin(k_{x,m} x_o - \omega_b t_1) \\ -\sin(k_{x,m} x_o - \omega_b t_1) & \cos(k_{x,m} x_o - \omega_b t_1) \end{bmatrix} \times \int_0^H \sin(k_{x,m} z) \begin{bmatrix} A(x_o, z) \\ B(x_o, z) \end{bmatrix} \, dz.$$  \hspace{1cm} (3.25)
Figure 3-5: (a) Snapshot of the wave field studied in section 3.3 with the addition of random noise (SNR = 5). (b) Temporal wave field at $x_o$. (c) Time series at $(x_o, z_o)$, with the fitted signal (thick line) superimposed on the original noisy signal (thin line).

where $A(x_o, z)$ and $B(x_o, z)$ are the coefficients of the $u$ velocity profile determined at $x_o$. The magnitude $|W_m|$ and phase $\phi_m^w$ are then calculated using (3.16) and (3.17). The relations between $W_m$ and $U_m$ still hold, and in principal one can also apply the same procedure directly to the $u$ velocity field, which has been demonstrated to be a less desirable approach.

### 3.5 Impact of harmonics

Higher order harmonics are perturbations propagating at integer multiples of the primary (forcing) frequency $\omega_o$: an $n^{th}$ order harmonic is a disturbance propagating
Figure 3-6: An idealized vertical velocity field plotted as a function of the spatial coordinates $x$ and $z$, and time $t$. The wave field was constructed in MATLAB using the known modal amplitudes $|W_{1:10}| = 0.7 |W_{1:10}|$ and phases $(\phi^{(w)}_{1:10})_1 = (\phi^{(w)}_{1:10}) + \pi/3$ rad. $L$ and $H$ represent horizontal and vertical length scales, respectively; all physical quantities have arbitrary values.

at a frequency $(n + 1)\omega_o$. These waves are generated by nonlinear effects such as wave-wave interactions at boundary reflection sites [38].

The presence of higher harmonics in experimental data typically can not be avoided, therefore it is of great importance to develop a method to filter this from experimental data. To demonstrate an effective approach for removing harmonics from a wave field, we start by considering a theoretical wave field in which higher harmonics are present. More specifically, for our test case we add first order harmonics to the wave field studied in section 3.3 with amplitudes that are 70% of the mode strengths at the fundamental frequency, i.e. $|W_{1:10}| = 0.7 |W_{1:10}|$, where the subscript indicates the harmonic order. Additionally, the phase of the first harmonic is $(\phi^{(w)}_{1:10})_1 = (\phi^{(w)}_{1:10}) + \pi/3$ rad. The resulting vertical velocity wave field is presented in figure 3-6, in which the first order harmonics are seen to generate a wave beam propagating at a larger angle with respect to the horizontal than the wave beam at the fundamental frequency. For completeness, white noise is also added to the wave field with a SNR of 5.
We begin by taking the fast Fourier transform (FFT) of the spatiotemporal wave field which is characterized by 120 elements in the vertical domain and 8 periods (250 time steps) in time. The FFT was applied to a vertical profile of the wave field at an arbitrary x-location thus producing $F(\omega, z)$. The real and imaginary components of the complex FFT algorithm results are displayed in figures 3-7(a) and (b), respectively. Making use of the property $F(\omega) = F^*(\omega)$ we can simply multiply the positive results of the FFT by 2 without any loss of information. Only the normalized frequencies $\omega/\omega_0 = 1$ and 2 contain significant velocity information whereas the other frequencies contain negligible quantities attributed to the signal noise as well as the numerical residue of the FFT algorithm. The relative phase difference of $\pi/3$ rad between the primary and first order harmonic can be identified in the vertical shift of the peaks of the two harmonics in the Fourier spectra.

While the FFT results $F(\omega, z)$ provide the frequency and phase decomposition of a wave field, for the purposes of our modal decomposition it is only the information contained in the primary harmonic that is of primary interest. The strengths of the higher order harmonics are expected to be weak in experiments performed strictly in the linear regime, and it is desirable to remove them, lest they contaminate the modal decomposition results at the fundamental frequency. Figure 3-7(c) displays the real and imaginary components of $F(\omega, z)$ at the fundamental frequency $\omega = \omega_0$, which are collected by way of performing the following integral in Fourier space:

$$F(\omega_0, z) = \int_{\omega_0 \pm \delta} F(\omega, z) d\omega$$  \hspace{1cm} (3.26)

where $\omega \pm \delta$ is a small window over which the Fourier coefficients are summed to account for any numerical leakage that may arise from the FFT algorithm. The profiles $\text{Re}\{F(\omega_0, z)\}$ and $\text{Im}\{F(\omega_0, z)\}$ can be interpreted as the vertical velocity profile at two different times separated by $\pi/2\omega_0$ seconds; when combined they form the expression of the velocity profile in time i.e.

$$w(x_0, z, t) = \text{Re}\{F(\omega_0, z)\} \cos(\omega_0 t) + \text{Im}\{F(\omega_0, z)\} \sin(\omega_0 t).$$  \hspace{1cm} (3.27)
With the complex quantity $\mathcal{F}(\omega_0, z)$ in hand we can proceed towards modal decomposition by following the procedure outlined in section 3.4. The results of the modal analysis performed on the Fourier spectra of both the vertical and horizontal velocity wave fields are presented in figure 3-8 which matched the results displayed in figure 3-2 perfectly.

In analyzing experimental data, the FFT proves to be an excellent tool in observing the frequency spectrum. However, the noise filtering presented in section 3.4 is a convenient and much simpler tool given a wave field free of harmonics. Considering the complexity of the FFT algorithm relative to noise filtering, it is reasonable to forgo the FFT analysis and directly apply the noise filtering to a linear wave field; there is added benefit to utilizing the noise filtering procedure since the resulting product can be directly handled by the modal decomposition algorithm. It should be noted that if the experimental data contains significant portions of higher harmonics then it is highly suggest that the FFT be used since noise filtering fails to sufficiently address the existence of harmonics. Most importantly, application of the FFT at the initial stages of analysis can serve to provide insight into the harmonic content of the wave field, the results of which can aid the user in determining the most appropriate method.

3.6 Impact of image loss

In a typical experiment it is to be expected that there is incomplete data over the entire vertical domain. For example, when performing velocity field studies using PIV the laser sheet produces glare near boundaries and free-surfaces, making it impractical to obtain data in those regions. As such, we consider the impact of image loss on the accuracy of the modal decomposition algorithm.

3.6.1 Modal decomposition over reduced vertical domain

We present a modified and expanded modal decomposition method for reduced domains first introduced by Echeverri et al. [7]. Let $z = [\alpha, \beta]$ be the vertical domain
Figure 3-7: The (a) real and (b) imaginary components of the complex FFT results for the wave field in figure 3-6 as a function of normalized frequency $\omega/\omega_o$ and vertical coordinate $z$. (c) The real (solid line) and imaginary (dashed line) $F(\omega, z)$ profiles at $\omega/\omega_o = 1$. 
Figure 3-8: Modal decomposition results for the $w$ (left column) and $u$ (right column) velocity profiles extracted from the FFT results. (a) and (b) display the magnitude comparison while (c) and (d) display the phase comparisons for the idealized case. The known inputs (circle) and algorithm results (asterisk) are compared.

That contains available data, where $[\alpha, \beta] \subset [0, H]$. The basis functions $\sin(k_{z,m}z)$ and $\cos(k_{z,m}z)$ are no longer orthogonal over the cropped domain $[\alpha, \beta]$ which renders the algorithm described in section 3.2 inapplicable. This scenario therefore requires evaluating integrals over the incomplete domain, i.e.

$$\int_{\alpha}^{\beta} w(x_o, z, t) \sin(k_{z,m}z) dz = \sum_{n=1}^{N} \gamma_{mn} \Gamma_n(x_o, z, t)$$  \hspace{1cm} (3.28)

where

$$\gamma_{mn} = \int_{\alpha}^{\beta} \sin(k_{z,m}z) \sin(k_{z,n}z) dz,$$  \hspace{1cm} (3.29)

$$\Gamma_n = \text{Re}\{W_n \exp i (k_{z,n}x - \omega t)} \}.$$  \hspace{1cm} (3.30)

Again, it should be noted that this procedure is being detailed for the $w$ velocity field. Equations (3.28), (3.29) and (3.30) can in principle be formulated for the $u$ velocity by exchanging the basis functions to $\cos(k_{z,m}z)$’s and the complex constant to $U_n$, but as mentioned previously it is preferable to use vertical rather than horizontal velocity.
for modal decomposition.

Equation (3.28) is applied to values \( m = 1 \) through \( m_{\text{max}} \) at times \( t_1 \) and \( t_2 \). The first step is to solve for \( \Gamma_n(x_o, z, t_1) \) and \( \Gamma_n(x_o, z, t_2) \) by solving the linear problem:

\[
\Gamma_n(x_o, z, t) = [\mathbf{Y}_{N \times N}]^{-1} \times \int_\alpha^\beta w(x_o, z, t) \begin{bmatrix} \sin (k_{z,1}z) \\ \vdots \\ \sin (k_{z,N}z) \end{bmatrix} \, dz. \tag{3.31}
\]

Recovering the real and imaginary components of \( W_n \) is achieved via

\[
\begin{bmatrix}
\text{Re}\{W_n\} \\
\text{Im}\{W_n\}
\end{bmatrix} = \begin{bmatrix}
\cos (k_{z,n}x_o - \omega t_1) & -\sin (k_{z,n}x_o - \omega t_1) \\
\cos (k_{z,n}x_o - \omega t_2) & -\sin (k_{z,n}x_o - \omega t_2)
\end{bmatrix}
\begin{bmatrix}
\Gamma_n(x_o, z, t_1) \\
\Gamma_n(x_o, z, t_2)
\end{bmatrix} \tag{3.32}
\]

where (3.16) and (3.17) yield the \( n \)-th mode strength and phase \((n = m \text{ for a solvable matrix problem})\).

### 3.6.2 Evaluation

In order to investigate image loss, the vertical boundaries of the analytic wave field are cropped, the total image loss fraction being equally divided between the top and bottom boundaries by virtue of a parameter \( \lambda \), such that \( \alpha = \lambda/H \) and \( \beta = (1-\lambda)/H \). However, one should note that image loss in experimental data is not equally divided between the boundaries which requires \( \alpha \) and \( \beta \) to be independently determined. Modal decompositions were performed on the same analytic wave field introduced in section 3.3 for the range \( 0 \leq \lambda \leq 0.2 \).

The results for the first three modes are presented in figure 3-9, in which we compare the reduced domain method with the original approach that does not account for image loss. There is error on the order of 0.1% for the \( w \) mode strengths up to \( \lambda \simeq 0.1 \) for both of the decomposition methods used, as shown in figure 3-9(a). In general, it can be seen that for the \( w \) modes there is less than 1% error even for image loss on the order of 10% at both the top and bottom of the domain. Since image loss is typically 4–6% of the vertical domain in experiments, the results of the \( w \) velocity are
Figure 3-9: The first three modes of the idealized wave field evaluated using the analytical procedure and the procedure modified for a reduced vertical domain. The errors of the results for the (a) $w$ and (b) $u$ velocity fields; the errors attributed to the modified procedure are denoted by an asterisk.

quite promising in that the errors are low even for significant amounts of image loss. In contrast, the difference between the original method and the algorithm accounting for image loss in determining the $u$ mode strengths is significant ($O(10\%)$), as shown in figure 3-9(b). The $u$ modes are much more significantly impacted by even a small amount of image loss at the boundaries. Using the modified modal decomposition algorithm in both cases essentially eliminates this error, indicating that some image loss in experimental data should not be an issue.

### 3.7 Application to experimental data

A set of experiments producing vertical internal wave modes, performed by M. S. Mathur, was used for the experimental validation of the algorithm. Mode-1 and mode-2 wave fields were generated in a linearly stratified fluid with $N \simeq 0.85$ rad s$^{-1}$ at forcing frequency $\omega_o \simeq 0.6$ rad s$^{-1}$. The total fluid depth $H$ was 0.416 m. These distinct vertical modes were produced using the novel wave generator developed by
3.7.1 **Velocity field data**

Image snapshots of the experimental horizontal, vertical and total velocity fields for mode-1 are presented in figures 3-10(a)-(c), respectively. The spatial structure of the mode-1 $u$ (figure 3-10(a)) and $w$ (figure 3-10(b)) velocity fields is as expected for a constant stratification. The $u$ wave field contains extrema at the lower and upper boundaries whereas the $w$ wave field contains extrema in the mid-domain; both results correspond to the respective mode shapes of $\cos(k_z z)$ and $\sin(k_z z)$. The total wave field is presented in figure 3-10(c) in which the vectors indicate the local velocity direction. A traveling vortex is consistent with expectations for a mode-1 wave field in a uniform stratification [14].

Snapshots of mode-2 velocity fields are presented in figures 3-11(a)-(c). Similar to the observations of the mode-1 wave field, the mode-2 $u$ and $v$ wave fields contain extrema at the boundaries and mid-domain, respectively. Furthermore, the total velocity field shown in figure 3-11(c) is composed of two stacked, counter-rotating vortices which characterizes the spatial structure of mode-2 [27].

3.7.2 **Fourier spectra**

To determine the quality of the data, the mode-1 and 2 wave fields were analyzed using the FFT procedure outlined in section 3.5. Figures 3-12(a) and 3-12(b) present the real and imaginary components of the mode-1 and 2 wave fields ($w$ velocity), respectively, collected at the fundamental frequency.

In order to produce the profiles seen in figure 3-12 a window of size $\delta = 0.05$ rad/s was used to filter the FFT results at $\omega_0 = 0.6$ rad/s. The distinct vertical mode shapes $\sin(k_{z1})$ and $\sin(k_{z2})$ can best be observed in the real components for both mode-1 and 2, respectively; it should be noted that they do not perfectly represent the mode shapes due to a phase shift in the complex FFT output. The other frequencies of the modes-1 and 2 fourier spectra (not shown) contain negligible quantities of...
Figure 3-10: Snapshots of mode-1 (a) horizontal, (b) vertical and (c) total velocity wave fields at an arbitrary instant in time. The color scale indicates the velocity and for (c) the vectors indicate the local velocity direction.

Figure 3-11: Snapshots of mode-2 (a) horizontal, (b) vertical and (c) total velocity wave fields at an arbitrary instant in time. The color scale indicates the velocity and for (c) the vectors indicate the local velocity direction.
Figure 3-12: Profiles filtered from the FFT results of the (a) mode-1 and (b) mode-2 wave fields at the fundamental frequency; the real (solid line) and imaginary (dashed) profiles are $F(\omega_0, z)$. Vertical cross-sections of the experimental wave fields were collected at $x = 2.2$ cm. The noticeable difference between the real profiles and the corresponding sine basis functions is due to the phase shift introduced by the FFT; the real and imaginary profiles are shifted by a time period of $\pi/2\omega$.

the original vertical velocity field which leads us to conclude that the wave fields are primarily driven at the forcing frequency. This was additionally confirmed by observing time histories of various points throughout the wave field, which displayed negligible quantities of noise and verified that the data exhibited no time-varying behavior.

The reader should note that the FFT results have been included simply for the sake of completeness. Given that the wave field was found to be dominated by the fundamental forcing frequency the noise filtering technique is utilized for the modal analysis of the experimental wave fields, as described in more detail in the following section. Had the wave field contained significant quantities of higher harmonics then the results of the FFT algorithm would have been the more appropriate choice for implementation of the modal decomposition.

### 3.7.3 Modal analysis results

In presenting detailed results of the modal analysis for the mode-1 and mode-2 wave fields, we begin by assessing the amount of image loss in the experimental data.
Given that the vertical range of data covers 408 mm and is known that the total fluid height was 416 mm, there is 8 mm of unobserved motion for both experiments. The precise amount of image loss is calculated by taking images of the experimental field of view both before and after the blocking tape has been applied. For the mode-1 experimental data it was determined that the 8 mm of image loss was from the upper boundary, whereas for the mode-2 experiment the 8 mm was split between the top and bottom boundaries. The algorithm introduced in section 3.6.1 was configured accordingly in each of these two cases.

Modal decomposition of the wave fields was performed at 108 $x$-locations covering the entire horizontal domain for mode-1 and 54 $x$-locations spanning the region $0 \text{ cm} < x < 20 \text{ cm}$ for mode-2, this was due to some small but noticeable decay in quality of the mode-2 wave field for the latter half of the horizontal domain. The modal analysis procedure consisted of filtering the time-history signals of the wave fields (as was done in section 3.4) and then using the complex representation of the wave field in the reduced vertical domain modal decomposition algorithm (see section 3.6.1) to calculate the modal amplitude spectrum. This was all done with the knowledge that the wave field was dominated by the fundamental frequency.

The mode strengths for the experimental mode-1 and 2 wave fields are presented in figures 3-13 and 3-14, respectively. Clearly, the appropriate modes dominate their respective experiments. In the case of the mode-1 field, both the $u$ and $w$ mode strengths are nearly identical. This is to be expected since $\theta = 45$ degrees for both cases which means the $u$ and $w$ fields are equivalent in magnitude. Additionally, our conclusion that the $w$ velocity field is the best choice to analyze modal content is supported by these results, as we see that the detected mode-1 amplitude is much more consistent for the $w$ velocity field results than the $u$ velocity field results.

Interestingly, there is very low variability in both the $u$ and $w$ modal spectrum for the mode-2 results presented in figure 3-14. Granted, the horizontal domain is half the size used for the mode-1 analysis. The $w$ and $u$ mode strengths appear to be within the same general magnitude for both the mode-1 and 2 velocity fields. This observance ties in to the dispersion relation which, for $\theta = 45^\circ$, requires that $k_x = k_z$.
Figure 3-13: Modal analysis results for the mode-1 wave field using the (a) $u$ and (b) $w$ velocity fields. The bar graph represents the mode strengths averaged over the 108 vertical cross sections spanning the entire horizontal domain; the error bars show the standard deviation of the modal amplitudes.

Figure 3-14: Modal analysis results for the mode-2 wave field using the (a) $u$ and (b) $w$ velocity fields. The bar graph represents the mode strengths averaged over the 54 vertical cross sections spanning the first 20 cm of the horizontal domain starting from the left; the error bars show the standard deviation of the modal amplitudes.
which directly translates to $u$ and $w$ having the same maximum values. Overall, the modal decomposition produces a modal amplitude spectrum that agrees remarkably well with the expected results of the experimental mode-1 and 2 wave fields.

3.8 Conclusions

In this chapter we have discussed the theory behind the modal analysis of internal wave fields in constant-depth settings. The basic theory has been expanded to deal with natural complications that arise in experimental investigations, such as experimental noise and image loss. We have demonstrated the validity of the algorithm in an idealized test case and on experimental data that replicated discrete vertical modes. The modal analysis has proven to be a robust tool in analyzing the modal content of two-dimensional, linear internal wave fields that satisfy the constant-depth condition. This technique can now be reliably applied to linear internal wave studies, as was done in [7] and as we proceed to do in chapter 4.
Chapter 4

2D Experiments

In this chapter we investigate internal wave generation by a 2D Gaussian ridge, following the procedure detailed in Echeverri et al. [7]. In performing a systematic study of the wave field generated by a 2D Gaussian ridge we utilize the planar-PIV method and the analytical Green function solution for a thorough comparison and analysis of the results. Having established our understanding of the two-dimensional wave field we move on to the more challenging 3D experiments detailed in chapter 5. In section 4.1 we present the relevant background, and this is followed in section 4.2 by a detailed description of the experimental procedure and apparatus. In section 4.3 is a discussion of the results and comparisons with the Green function theory outlined in chapter 2.

4.1 Relevant background

With regards to topographic generation of internal waves by 2D topography, the work of Echeverri et al. [7] provides the most direct and relevant investigations for the work presented in this chapter. Their investigation focused on the generation of both sub- and super-critical wave beams for small and large excursions in a linear stratification. Good agreement between the experiment, theory and numerical simulations was observed for the visualizations of the wave fields in their study. Furthermore, in analyzing the modal content of the wave fields they found that the agreement between
the experiment, theory and numerical predictions were, with few exceptions, within the calculated error bounds.

In our studies, we investigate the wave fields generated by a 2D Gaussian ridge with a relatively high depth ratio in sub-, near- and supercritical regimes. Furthermore, we will compare our experimental results with the latest formulation of the Green function method developed by Echeverri & Peacock [8] in addition to a detailed analysis utilizing the refined modal analysis technique described in chapter 3. Most importantly, the present study will provide a base for comparison with the 3D incised Gaussian ridge experiment performed in Chapter 5.

4.2 Experiments

A 2D Gaussian ridge was used as the oscillating body that generated the internal wave field. Using an $x - z$ coordinate system centered at the base/center of the topography, the shape of the ridge is given by:

$$z_t(x) = \Lambda \exp(-x^2/2\sigma^2)$$

where $\Lambda$ is the topographic height and $\sigma$ is the parameter that directly controls the topography horizontal width and slope. The desired shape of the topography was created using a FoamLinx cutter such that the height of the ridge was $20 \pm 0.1$ cm with a maximum slope of $45 \pm 1.5^\circ$ and the total length and width of the topography were 191.5 and 34.1 cm, respectively. Figure 4-1 shows the ridge shape used in this experiment.

The experiments were performed in a 5.5 m long, 0.69 m wide and 0.51 m tall glass wave tank. A linear salt-stratification was set up using the double bucket technique [33] with a density range of 1000 - 1015 kg/m$^3$ and a volume in-flow rate of 6 L/min. A schematic of the experimental arrangement is provided in figure 4-2. The oscillatory motion of the topography was controlled by a Parker linear traverse which was slowly ramped up over 7 periods to a max peak-to-peak amplitude of 2 mm. A fixed fluid
depth of \( H = 27 \pm 0.2 \) cm resulted in a depth ratio of \( \Lambda/H = 0.74 \pm 0.01 \). The free ends of the topography were submerged \( \sim 3 \) mm below the water surface.

Planar Particle Image Velocimetry (PIV) was used to make the velocity field measurements. The motion of the seeded particles was tracked via a 2048 x 2048 pixel CCD camera. DaVis 7.2 software provided by LaVision was used to process the raw particle images. Image loss due to laser sheet reflection from the horizontal boundaries was estimated to be \( \sim 5 \) mm for both the top and bottom. Data was recorded at two fields of view by slowly shifting the topography to the side by \( \sim 20 \) cm (over the course of 2 hours) while maintaining the position of the camera, thus allowing for a new region of observation. The planar velocity fields were calculated on a grid size of \( N_x = 128 \) and \( N_z = 101 \) which translates to a spatial resolution of 7 mm\(^2\). Recording of the wave field began 8 periods into the start-up of the motion allowing for observation of time-periodic motion.

Three forcing frequencies were considered for this study: \( \omega = 0.35, 0.50 \) and 0.61 rad/s. Measurement of the wave beam angles of propagation for all three frequencies was done by finding the angle between a line of constant phase and the horizontal. The corresponding propagation angles were found to be \( \theta = 27.8^\circ, 42.0^\circ \) and 59.0\(^\circ\) (to within an error of \( \pm 1^\circ \)) and thus producing super-, near- and sub-critical wave beams, respectively. Through the dispersion relation the mean stratification value was \( N = 0.74 \) rad/s with the difference between the three calculated values being no more than 0.04 rad/s.

### 4.3 Results

Snapshots of the experimental wave fields at the same phase of oscillation for all three beam angles considered are presented in figures 4-3(a), (c) and (e), and the corresponding theoretical wave fields are plotted in figures 4-3(b), (d) and (f). The quantity plotted is the total velocity \( |u| = \sqrt{u^2 + \omega^2} \) and each wave field has been normalized by its respective far field velocity \( U = A\omega \) where \( A \) is the amplitude of oscillation (for this experiment \( A = 1 \) mm and remains unchanged). The agreement
Figure 4-1: 2D Gaussian ridge topography.

Figure 4-2: Front view of the experimental setup.
between experiment and theory is excellent; the wave field features are nicely captured by the theory though a minor difference exists in the lower propagating wave beam for the supercritical wave field (see figures 4-3(a) and (b)). For the experimental results, images from two fields of view have been stitched together to produce an enlarged domain of observation. As expected, the wave beams propagate at well-defined angles set by the forcing frequency of the system. For the super- and near-critical wave beams the normalized velocities are quite similar whereas the velocities for the subcritical regime are noticeably weaker in magnitude. The subcritical wave field experiences a localization of beam energy near the ridge and a significantly weaker beam propagating far field. In all, the three experimental wave fields provide clean beam structures in the far field.
Data was collected at three cross-sections spanning the width of the tank in order to verify the two-dimensionality of the wave field; the results are presented in figure 4-4. Cross-sectional profiles of the total velocity envelope were collected at each of the three cross-sections for the super-, near- and subcritical wave fields. There is very good agreement among all the cross-sections observed, with the typical errors being attributed to the sensitivity of the calibration procedure in determining the velocity fields. Furthermore, the consistency of the results serves as a demonstration of the repeatability of the experiments. Additional profiles collected (not shown) displayed the same level of agreement between the three cross-sections.

Insight into the harmonic content of the wave fields is obtained by performing a Fourier decomposition in time; the vertically-averaged frequency spectra of the three wave fields (collected at $x = 54.6$ cm) are presented in figure 4-5. A minimum of 7 periods were used in computing the frequency spectrum via the Fast Fourier Transform (FFT). It is apparent that the primary harmonic contained the majority of the signal strength. The strength contained in the second and third harmonics is
Figure 4-5: Vertically-averaged frequency spectrum collected at $x = 54.6$ cm for (a) $\theta = 27.8^\circ$, (b) $\theta = 42.0^\circ$ and (c) $\theta = 59.0^\circ$. The $u$ velocity (solid) and $w$ velocity (dashed) Fourier spectra have been normalized by the maximum strength value which was at the primary harmonic for all cases.

noted, though it remains at least an order of magnitude weaker for all cases. Noise levels were also negligibly low for all the collected measurements. We confirmed that the primary harmonic dominated the wave fields in our experiments by taking depth-averaged Fourier spectra at several locations, always finding results similar to those in figure 4-5.

In figure 4-6 we present velocity profiles comparing the experimental results to theory. In calculating the theoretical profiles we used the viscous Green function solution which was solved on a spatial grid outlining the topography. The number of intervals and modes used in the formulation (both of which directly affect the quality of the results) was 4000; the value of kinematic viscosity $\nu$ used was $1 \text{ mm}^2/\text{s}$. Profiles were taken at $x = 50$ cm and the image loss regions were clipped. The dominant velocity component was chosen to be the plotted quantity for each of the three wave fields; $u$ velocity component for the super- and near-critical wave beams and $w$ velocity for the subcritical. The profiles presented were taken at the phase corresponding to maximum flow in both the positive and negative $x$ directions. In general, excellent agreement is observed for all three wave beams. Noise levels remain low in general, however, a relative increase in noise is seen in the subcritical wave field. The increase in noise level for $\theta = 59.0^\circ$ is attributed to the weaker velocity values for that particular case. The velocity profile for $\theta = 27.8^\circ$ is collected at the
Figure 4-6: Velocity profile comparison between experiment (solid) and theory (dashed) for (a) $\theta = 27.8^\circ$, (b) $\theta = 42.0^\circ$ and (c) $\theta = 59.0^\circ$. The $u$ velocity is plotted for (a,b) and $w$ velocity is plotted for (c).

$x$ location where the reflected and incident wave beams interact which is a site of potential nonlinearities. The ability of the theory to accurately predict the location of interaction and, more importantly, the consequence of the interaction accurately is clear evidence that these experiments were performed in a linear regime. Interestingly, a mode-1 wave structure characterizes the subcritical wave field (figure 4-6(c)), in contrast to the well-defined wave beam profiles in figures 4-6(a) and (b).

As the forcing frequency increases, a successive decline in the magnitude of the dominant velocity component is detected. The kinetic energy contained in the baroclinic motion for $\theta = 59.0^\circ$ appears to be localized near the generation site whereas the kinetic energy generated in $\theta = 27.8^\circ$ and $42.0^\circ$ is able to dissipate away from the ridge. This free propagation and localization of energy was observed earlier in figures 4-3(a)-(b) and 4-3(c), respectively. The energy localization for $\theta = 59.0^\circ$ is explained by the restriction imposed by the topography on the motion of the fluid flow; instead of propagating unobstructed to the far field the fluid flow is confined to the region $0 \text{cm} < x < 25 \text{ cm}$. 
Modal analysis was performed on the experimental and theoretical wave fields using the method outlined in chapter 3. Figure 4-7 presents the modal spectrum for the three wave fields studied; the experimental mode strengths (solid lines) are compared with the theoretical predicted values (circles). The mode values were averaged over 45 cm $< x < 50$ cm and 45 cm $< x < 53$ cm for figure 4-7(a) and figure 4-7(b)-(c), respectively. Consistent measurements were generally observed for the first ten modes; the error bars represent the variation of the mode strengths across the range analyzed. Furthermore, the $u$ mode strengths were adjusted to match the corresponding $w$ mode strengths; this was done by plotting the product $|U_m| \tan \theta$ (red) against $|W_m|$ (black) which should, in theory, produce identical values. We indeed do observe excellent agreement between the adjusted $u$ mode strengths and the $w$ mode strengths; this further confirms the quality and consistency of the experimental wave fields. Moreover, there is notable agreement between the experimental and theoretical mode strengths for all three cases; the only exception being the minor over-prediction of the theory for the $m > 2$ modes of the subcritical wave beam. It should be mentioned that the energy flux of the internal wave field is sensitive to the mode value $a_n$ since energy flux scales as $\sim a_n^2$ [7]. Thus, the 20% disagreement observed in the mode-1 amplitude for the sub- and supercritical wave field results in a $\sim 50\%$ error for the energy flux; relatively minor differences between experiment and theory do not directly translate to quality energy flux prediction.

A transition from a mode spectrum containing energetic higher modes to a mode spectrum skewed heavily towards the first two modes is identified as criticality changes from $\epsilon < 1$ to $\epsilon \approx 1$. From super- to near-criticality we observe the expected transition from energetic high modes to energetic lower modes, with a slight increase in the overall energy content of the wave field. However, there is a significant decrease in the energy content of the subcritical wave beam in comparison to the supercritical wave beam; this serves as a quantitative confirmation of the earlier observation of energy location near the ridge for the super-critical wave beam. Not only are the modes $m > 1$ effectively suppressed in the transition from $\epsilon \approx 1$ to $\epsilon > 1$ but the strength of mode-1 is nearly halved which directly translates to a drop of 50% in
Figure 4-7: Mode strength spectrum for experiment (solid) and theoretical (circles) results for the (a) supercritical, (b) near-critical and (c) subcritical wave fields. The mean quantity is plotted and the error bars represent the variation across the analyzed range: $45 \text{ cm} < x < 50 \text{ cm}$ for $\theta = 27.8^\circ$ and $45 \text{ cm} < x < 53 \text{ cm}$ for $\theta = 42.0^\circ, 59.0^\circ$. The adjusted $u$ mode strengths $|U_m| \tan \theta$ (red) are plotted against the $w$ mode strengths $|W_m|$ (black).

4.4 Conclusions

The promising results obtained in the experimental investigation of the 2D Gaussian ridge provides the necessary background by which to address the questions pertaining to the expectedly more complicated wave field of the 3D incised Gaussian ridge. The generally excellent agreement between the Green function theory and the experiment confirm that the experiments were indeed linear and consistent. From this solid foundation, we now proceed to study 3D internal wave field generation in chapter 5.
Chapter 5

3D Experiments

Until recently internal wave experiments using Particle Image Velocimetry (PIV) have been limited to 2D and axisymmetric configurations due to the limitations in measurement technology. The recent advent of Stereo Particle Image Velocimetry (SPIV), however, provides the ability to measure detailed and accurate three-dimensional velocity field data for experimental internal wave fields. We utilize SPIV to investigate fully 3D internal wave fields for two different arrangements: an oscillating sphere and periodic stratified flow past an incised Gaussian Ridge. The goal of these studies is to demonstrate the effectiveness of SPIV in internal waves research and to explore internal tide generation problems for a more realistic test case.

In section 5.1 we present the details of the theory behind SPIV and the associated calibration procedure. Then, in section 5.2 we discuss the experimental procedure and results of the vertically oscillating sphere experiment. Lastly, in section 5.3 we investigate internal wave generation by 3D incised Gaussian ridge.

5.1 Stereoscopic particle image velocimetry

5.1.1 System set-up

The aim of SPIV is to capture the motion of particles suspended in a fluid medium and to calculate their associated velocities in three dimensions. We begin by seeding
our stratified medium with SPHERICEL hollow glass oxide particles having a mean diameter of 8 - 12 \( \mu m \) and a specific gravity range of 0.1 - 1.5. The particles are evenly dispersed prior to filling and are left to settle to their respective density levels for 4 hours after filling. Given the density range of the particles the entire stratification contains sufficient amount of seeding throughout for high-quality visualization.

A pulsed Nd:YAG laser is used to create the light sheet that illuminates the particles in a plane. The sheet optics can be modified to control the orientation, location and thickness of the laser sheet. Two Imager Pro X 4M LaVision CCD cameras with a resolution of 2042x2039 pixels are used to record images of the particles where the sampling rate is chosen such that there are 32 frames per tidal period; the frequency is typically between 1.5 and 3 Hz. A shutter time of 10 \( \mu s \) is used to insure clear images of the particle motion.

Though there are various set-ups for SPIV camera positioning, the particular configuration for these experiments had both cameras on one side facing the image plane from the front of the test-section (this set up is referred to as forward-backward scattering). The cameras have opposite viewing directions and a separation angle of 40 - 50 degrees. This arrangement allows for two separate views of the same image plane from two different positions. By capturing the motion of the tracer particles from two separate cameras the out of plane velocity component can be determined. Additionally, the accuracy of the velocity fields increases with the SPIV system since image captures in a 2D plane can possibly contain a degree of out of plane flow where these motions introduce an unrecoverable error in the calculation of 2D velocity fields.

One drawback of the forward-backward scattering configuration is that the camera facing the incoming laser is receiving forward scattered light whereas the camera facing the opposite direction is recording scattered light in the backward direction. This results in one camera receiving higher intensity level images than the other. In order to equalize the images collected from both cameras the aperture of the camera receiving backward scatter is increased.
5.1.2 Scheimpflug principle

Typically, for planar PIV, the camera is positioned directly in front of and parallel to the object plane which allows for a simple camera configuration for an in-focus view since the object plane and the plane of focus coincide. However, given that the camera configuration employed in our SPIV does not allow for parallel positioning of the cameras relative to the object plane, a set of Scheimpflug adapters are attached to the cameras which sit in between the lenses and camera body. The adapters can be adjusted so that the image plane, camera plane and object plane meet at an intersection point some distance away in the direction opposite of the camera viewing, thus fulfilling the Scheimpflug criterion [36].

SPIV relies on the Scheimpflug principle which states that if the image, lens and object planes coincide at a single point the object plane can be in focus even if unaligned with the image plane. The Scheimpflug arrangement is visualized in figure 5-1. We define a coordinate system centered on the lens and set the angles dividing the laser sheet, lens and image planes as $\theta$ and $\phi$, respectively. Figure 5-1 demonstrates the set up of a single camera; the second camera is identically situated on the other side of the vertical line of symmetry passing through the intersection of the object plane and laser sheet.

The width of the lens employed in this set up is much less than its focal length which allows us to use the thin lens approximation. Since a no-zoom lens is used the focal length $f$ maintains a constant value regardless of the $x$ position; the expression for $f$ is given by

$$f = \frac{d_\theta d_i}{d_\theta + d_i} \quad (5.1)$$

where $d_\theta$ and $d_i$ are the distances between the object/lens plane and the lens/image planes at $x = 0$, respectively. Based on the classical thin lens equation we know that

$$\frac{1}{y_i} - \frac{1}{y_\theta} = \frac{1}{f}, \quad (5.2)$$

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Figure 5-1: Visualization of the stereoscopic set up for a single camera. The red $x-y$ coordinate system is centered in the middle of the lens. The laser sheet, image and lens planes meet at an intersection point $P$, satisfying the Scheimpflug criterion. The object-plane surface crossing the laser sheet over the field of view will appear in focus to the camera despite the angled configuration of the CCD chip with respect to the laser sheet.
where

\[ y_i = -x_i \tan \phi + d_i, \quad (5.3) \]

\( y_o \) and \( y_i \) are the \( y \)-coordinates lying on the object and image planes, respectively, and \( x_i \) is the associated \( x \) location of the image plane. We find the expression describing the position of the object plane relative to the lens plane to be:

\[ y_o(x_o) = \frac{f y_i(x_i)}{f - y_i(x_i)} \quad (5.4) \]

where \( y_i \), for this particular set up, is a linear function describing the position of the CCD chip relative to the lens and \( x_o \) is the corresponding \( x \) location of the calculated object-plane. Finding \( x_o \) is facilitated by exploiting the geometry of the Scheimpflug set up to yield the following expression:

\[ x_o = -\frac{|y_o x_i|}{|y_i|} \text{sgn}(x_i). \quad (5.5) \]

By virtue of (5.4) and (5.5) the only geometric configuration that maps \( y_o(x_o) \) as a plane in focus with respect to the common coordinate system is when all three planes intersect at a common point, thus verifying the Scheimpflug condition.

A consequence of the Scheimpflug adapters is in the variable magnification of the object plane across the image plane. Solving (5.4) to find the dependency of the magnification on \( y_i \) yields:

\[ M = \frac{x_i \tan \phi}{f} - M_o, \quad (5.6) \]

where the magnification \( M \) is defined as the ratio \( y_i/y_o \) and \( M_o \) is the nominal magnification at \( x = 0 \). The magnification spanning a collected image increases as the pixel position increases in \( x \) and thus introducing image distortion.

### 5.1.3 Calibration

To remedy the distortion to the images caused by variable magnification we use a calibration plate and a commercial software package (DaVis 7.2). The camera posi-
tion, viewing direction and the magnification across the image plane for each camera are determined from the calibration plate images. We seek a functional relationship between each pixel coordinate in the image plane and the corresponding object plane reference frame, which will be referred to as the global coordinate system. The process of obtaining such mapping functions is split into two parts: a standard calibration and a so-called self-calibration procedure.

In principle, one could find such mapping functions geometrically by measuring the dimensions of the set up and the optical properties of the cameras, but this is not the ideal option given the more advanced empirical approaches developed recently [4, 24, 44]. The geometric approach would require exact knowledge of the lens focal length, the angles between the various planes, the actual position of the lens plane and the nominal magnification factor [45]. For our calibration procedure we use a two-level plate with a known distribution of markers that the computer program can identify which we place as close as possible to the plane of the laser sheet. In the initial calibration procedure the program identifies the markers which are used to find the local magnification factor between the image plane and the object plane. A least-squares method is employed in determining the coefficients of a third-order polynomial that de-warps (maps) the images into a set of images with uniform magnification throughout the field of view; the procedure is exemplified in figure 5-2. This process is more effective than the geometric approach in that it accounts for any potential nonlinearities in the lens and image captures. Each measurement plane has its own set of mapping functions for both cameras, thus independent calibrations must be performed for each plane at which measurements are collected.

Following the initial calibration procedure is the self-calibration process which corrects for misalignment errors. Ideally, during the calibration procedure, the calibration plate is placed directly in the position of the light sheet. However, exact alignment is not possible, thus the mapping functions need to be modified to coincide with the plane of the laser sheet and not that of the calibration plate. To account for the potential misalignment, a self-calibration algorithm is used to modify the mapping functions found in the initial calibration procedure [44]. Ideally, if no misalignment
occurs, a distribution of particles will be located at the exact same global reference frame coordinates in both the left and right camera images. In the case of misalignment, a particle mapped to the object plane will not be located at the same locations for both cameras. Thus, the purpose of self-calibration is to measure the amount of misalignment between the object plane at which the initial calibration procedure was performed and the plane of the laser sheet.

To begin, a set of raw particle images from both cameras are selected and de-warped using the previously determined mapping functions. The two sets of images are then compared and a disparity map (vector field representing shifts in particle distributions between the two cameras) is computed using standard PIV interrogation. Since the particle images are considerably different in a single image between both cameras this procedure is applied to a range of raw images; this serves to capture the true misalignment which becomes apparent when averaged over time. The accuracy of this initial self-calibration step typically increases with the number of raw images available for analyzing. Finally, the disparity vectors are then used to adjust the original mapping functions such that the image plane for both cameras coincide with the laser sheet.

In the case where there is a sizeable difference between the laser sheet and calibration plate positions, a refining step is necessary for finer correction. A standard
triangulation method is utilized to find a common point for each disparity vector where a line from both cameras intersects at that point. Such a common point is rather difficult to find so the refine step seeks to find the point closest to the measured positions between both cameras using a suitable threshold value on the order of a sub-pixel. The refine step is repeated until results have converged and the average deviation from plane has a value of $O(1)$ pixel or equivalently $O(10^{-2})$ mm.

Correct implementation of the self-calibration procedure is essential in insuring accurate mapping functions. In all, the initial calibration procedure corrects for nonuniform magnification throughout the field of view while self-calibration modifies the mapping functions to relate the raw particle images to a global coordinate system centered on the laser sheet i.e. object plane.

5.1.4 Stereoscopic reconstruction

A modified standard cross-correlation method is utilized in computing the $(u, v, w)$ velocity fields; a commercial SPIV processing software package is used (DaVis 7.2). The reconstruction algorithm makes use of the standard cross-correlation procedure used in planar PIV in addition to incorporating the geometry of the stereoscopic set up to determine the three velocity components of the flow field.

For each camera, the raw images are first mapped onto the global coordinate system using the mapping functions found in section 5.1.3 and the 2D particle displacement fields are then computed. De-warping the images at the initial step has the added benefit of computing the displacement vectors in the global frame and thus improving the accuracy of the processing. There are other reconstruction options that de-warp the interrogations windows in the cross-correlation of the raw images; these have the disadvantages of using different interrogation window sizes for each camera view as well as compounded errors affecting the final result [44]. In evaluating the images we set the initial computational step to use an interrogation window size of $64 \times 64$ pixels with 50% overlap and the second step to use a reduced window size of $32 \times 32$ pixels with no change in overlap. The computation of the two-dimensional displacement field undergoes a total of four passes (two passes at each computational
step) which is typically sufficient for convergence of the processed two-dimensional vector field. Following the initial procedure, a smoothing procedure is applied that replaces any of the erroneous 2D vectors that are greater than a specified error-threshold value with interpolated vectors. This procedure independently constructs the 2D displacement vector field for each camera view in the reference frame of the object plane.

The crux of SPIV lies in the images capturing two projections of the same particle displacement; each camera captures a particle displacement (in the object plane) normal to its viewing direction. Following the initial 2D processing of the images, the displacement fields are interpolated onto a common coordinate system set on the object plane; this is due to the existent misalignment of the two cameras with respect to the common field of view. Akin to the inability of cross-correlation in detecting suitable peaks with increasing out-of-plane particle shifts, the out-of-plane velocity movement weakens the cross-correlation of the particle images in successive frames leading to the generation of spurious velocity vectors. However, the adverse effects of crossing particles is minimized by a preliminary reconstruction pass that eliminates spurious vectors with high reconstruction errors [44].

Reconstructing the out-of-plane displacement vectors involves following a well-established algorithm that uses both the mappings functions and the 2D displacement vectors to geometrically calculate the 3D displacement vector, as outlined in [45]. The laser sheet contains the x- and z-axes of global coordinate system while the y-axis lies normal to the laser sheet pointing towards the cameras; the coordinate system and overall geometry is visualized in figure 5-3. The origin of the global coordinate system has the coordinates \((x_o, y_o, z_o)\) and the two cameras are positioned at \(C_1 = (x_1, y_1, z_1)\) and \(C_2 = (x_2, y_2, z_2)\), respectively. We now take an arbitrary displacement vector \(\vec{V}\) positioned at a point \(P = (x_p, y_p, z_p)\) in the laser sheet with the displacements in the x-z plane given by \((\Delta x_1, \Delta z_1)\) and \((\Delta x_2, \Delta z_2)\), where the subscript denotes the camera number. From the calibration procedure, the angles describing the viewing rays of the displacements vectors are known as \(\alpha_1, \alpha_2, \beta_1\) and \(\beta_2\); \(\alpha\) and \(\beta\) are the angles between the y-axis and the viewing ray projected onto
Figure 5-3: Sketch demonstrating the out-of-plane displacement recovery. Each of the two cameras visualizes projections of the 2D particle displacements, namely $\Delta x_1$, $\Delta x_2$, $\Delta z_1$ and $\Delta z_2$. The 3D displacement vector $\vec{V} = (\Delta x, \Delta y, \Delta z)$ (shown in red) can be reconstructed by geometric considerations.
the $x$-$y$ and $y$-$z$ planes, respectively. Additionally, given that the distances between the cameras and the laser sheet are so much greater than the displacement vector we can assume negligible change in $\alpha$ and $\beta$ across the span of the displacement vector. The three components describing the total particle displacement $\vec{V} = (\Delta x, \Delta y, \Delta z)$ at point $P$ are computed via

\[
\Delta x = \frac{\Delta x_2 \tan \alpha_1 - \Delta x_1 \tan \alpha_2}{\tan \alpha_1 - \tan \alpha_2},
\]

\[
\Delta y = \frac{\Delta x_2 - \Delta x_1}{\tan \alpha_1 - \tan \alpha_2},
\]

\[
\Delta z = \frac{\Delta z_2 \tan \beta_1 - \Delta z_1 \tan \beta_2}{\tan \beta_1 - \tan \beta_2},
\]

where

\[
\tan \alpha_1 = \frac{x_p - x_1}{y_p - y_1},
\]

\[
\tan \alpha_2 = \frac{x_p - x_2}{y_p - y_2},
\]

\[
\tan \beta_1 = \frac{z_p - z_1}{y_p - y_1},
\]

\[
\tan \beta_2 = \frac{z_p - z_2}{y_p - y_2}.
\]

The final step in finding the velocity vector is simply dividing the displacement vectors by the corresponding time differences between successive images.

### 5.2 Vertically oscillating sphere: conical wave beam generation

Following the earlier work of oscillating bodies in a linearly stratified fluid we seek to confirm the SPIV measurement technique and its application to experimental internal wave studies. The oscillating sphere provides an excellent arrangement to study the generation of conical wave beams and their respective features.

Analytic formulation of oscillating bodies has been a primary area of research
regarding internal gravity waves. Hurley [18] performed the earliest work on this subject by describing a method for solving steady-state generation problems which offered a general procedure for determining the Green's function solution to the governing equations. This work was furthered by Appleby & Crighton [1] who extended the previous work to obtain solutions to the oscillating sphere. We use the latest theory of Voisin et al. [43] to compare and evaluate the SPIV measurements.

In the realm of experiments, regarding the sphere in particular, synthetic schlieren has been the primary tool in visualizing conical wave beam structures [12, 34] though fluorescein dye planes have also been utilized [11, 43]. In this study, we provide the first experimental results of 3D internal gravity wave fields and use existing theory to validate the SPIV measurement technique.

5.2.1 Theory

Great strides have been made in the theoretical developments of oscillating bodies in stratified fluids, with early works focusing on general steady-state solutions and the later works producing more accurate, problem-specific formulations. Hurley [18] developed a general method for solving steady-state propagation problems by first focusing on the much simpler \( \omega > N \) case using conformal mapping to solve the elliptic boundary value problem and then extending the solution for \( \omega < N \) by analytic continuation. The first solution for an oscillating spherical body was put forth by Appleby & Crighton [1] in which they adapted and extended the work of Hurley [18] to solving the elliptic problem for \( \omega > N \) and then utilized analytic continuation to the hyperbolic case \( \omega < N \). The outcome of these efforts was a near-field expression under the Boussinesq expression as well as the recovery of the non-Boussinesq far-field solution using matched expansions.

It should be noted that all of the solutions described up to this point have been in the inviscid limit where viscosity has been neglected. A consequence of the inviscid assumption is the unboundedness of the solution along the lines defining the wave beam. Flynn et al. [12] remedy this by adopting the boundary layer approximation of Thomas & Stevenson [40] and the method of Lighthill [23] to derive an analytic
solution to the viscous problem. A direct outcome is the viscous attenuation i.e. viscous damping of the across-beam amplitude envelope which leads to bi- to uni-modal transition; these results were experimentally verified in [12].

For our experimental verification we utilize the linear theory of Voisin et al. [43] which provides a three-dimensional formulation of the problem, in addition to accounting for viscosity and applying both near and far field of the sphere. The sphere is characterized by its radius \( a \), oscillation amplitude \( A \) and frequency \( \omega \). Figure 5-4 shows the expected conical wave beam structure of the internal waves in addition to the coordinate system employed in the analysis. We introduce a coordinate system \((s, \sigma)\) aligned with the downward propagating wave cone in addition to the cartesian coordinate system \((x, z)\) fixed at the point about which the sphere oscillates vertically. The relation between \((s, \sigma)\) and \((x, z)\) is as follows:

\[
s = -x \cos \theta - z \sin \theta, \tag{5.14}
\]

\[
\sigma = -x \sin \theta + z \cos \theta, \tag{5.15}
\]

where \( \theta = \sin^{-1}(\omega/N) \) is the angle the wave cones make with the horizontal.

Voisin [42] concluded that the wave beam structure of internal waves depended on three factors, namely: size of the disturbance, the initial motion and the fluid viscosity. The work of [42] considered each of these mechanisms in isolation to obtain an integral equation for the emitted wave field. To begin, the structure of the source term is described via its Fourier transform; an unsteadiness parameter is defined to be \( \alpha = (\omega t \tan \theta)^{-1} \); and a viscosity parameter is introduced as \( \beta = \nu/2\omega \tan \theta \). The result of this approach, the details of which are described in [42], is the formulation of a boundary integral problem.

In the case of the sphere, the boundary integral problem has been solved analytically (as detailed in [41]). The expressions for the cylindrical velocity components
Figure 5-4: Wave cone structure of a sphere oscillating vertically at forcing frequency \( \omega \). Due to the experimental set up we must resolve the wave field in quadrant 3, thus we introduce a coordinate system \((s, \sigma)\) which is tangential and perpendicular, respectively, to the wave cone cross-section profiles.

\((u_r, w)\) are:

\[
\begin{align*}
    u_r &= \omega A \sin \theta \cos \theta \int_0^{a \sin \theta} \exp \left( \frac{-\beta k^3|z|}{\sin \theta} - ik|z| \cos \theta \right) \\
    &\quad \times \frac{a^2k \cos \theta}{1 - B(\sin \theta)} j_1(ak) J_1(kr \sin \theta) \, dk, \\
    u_z &= \omega A \sin^2 \theta \int_0^{a \sin \theta} \exp \left( \frac{-\beta k^3|z|}{\sin \theta} - ik|z| \cos \theta \right) \\
    &\quad \times \frac{ia^2k \cos \theta}{1 - B(\sin \theta)} j_1(ak) J_0(kr \sin \theta) \, dk,
\end{align*}
\]

(5.16)

(5.17)

where

\[
B(\sin \theta) = \sin^2 \theta \left[ 1 - \cos \theta \left( \arctanh(\cos \theta) + \frac{i\pi}{2} \right) \right],
\]

(5.18)

and

\[
j_n = \sqrt{\frac{\pi}{2a}} J_{n+\frac{1}{2}}(x)
\]

(5.19)

is the relation between the spherical Bessel function \( j_n \) and the cylindrical Bessel
function $J_n$. In evaluating the integrals in (5.16) and (5.17) we select a sufficiently large upper limit such that the integrand has converged to zero in addition to choosing a discretization in $k$ such that the value of the integral converges as well.

To convert the theoretical solutions from the cylindrical to the cartesian coordinate system the following transformation was used:

\[
\begin{bmatrix}
    u \\
    v \\
    w
\end{bmatrix} = \begin{bmatrix}
    \sin \phi & 0 \\
    \cos \phi & 0 \\
    0 & 1
\end{bmatrix} \begin{bmatrix}
    u_r \\
    u_x
\end{bmatrix}
\]  

(5.20)

where $\phi = \arctan(-x/y)$. The time dependency of the solutions is introduced via $U = \text{Re} \left( (u, v, w)e^{-i\omega t} \right)$ to produce the final form of the solution.

### 5.2.2 Experiments

Experiments were performed in an acrylic tank measuring 55 cm wide, 75 cm long and 40 cm tall. A linear salt-stratification over 38 cm depth was set up in the tank using the double bucket method. The tank was stratified from below using two Masterflex peristaltic pumps with flow rates set at 0.5 and 0.25 liters/min between the two. The resulting density stratification was measured using a calibrated PME salinity probe attached to a linear traverse for position control; the stratification measurements shown in figure 5-5 were made at two different locations after the experiment was complete. Using a linear fit to find the density gradient and then substituting that into the expression for buoyancy frequency we found $N = 0.87 \pm 0.004 \text{ rad/s}$.

Figure 5-6 shows the setup of the experiment. The area of focus for visualization was quadrant 3 of the cartesian coordinate system centered on the sphere which was suspended with its center $\sim 24 \text{ cm}$ above the tank bottom. Prior to filling the sphere was positioned in the tank such that quadrant 3 was adequately covered by the field of view of the stereoscopic set-up. Attached to the sphere (diameter $2a = 53.9 \text{ mm}$) was a thin rod (diameter of 8 mm) which was fixed to a linear traverse controlled via the LabView software; the trajectory of the sphere underwent a ramped oscillatory
motion which took 7 periods to reach the maximum amplitude of $A = 5$ mm. To reduce the negative effect of wave reflections of the boundaries Blocksom matting was placed on the bottom and side walls of the tank with only the viewing area and the wall portion of the incoming laser sheet left uncovered. Plastic wrap was placed over the top of the tank to prevent convective effects of the air circulation in the laboratory.

Based on the dispersion relation, the propagation angle can be directly controlled by setting the frequency of oscillation. In this study we observed wave cones propagating at angles of $20.3^\circ$ and $38.2^\circ$ (to within an error of $\pm0.5^\circ$). The axisymmetric waves cones were visualized at three vertical cross-sections: the first cross-section cut through the center of the sphere while the other two were positioned 4 and 8 cm away from the first, respectively. To minimize laser sheet reflection black contact paper was placed on the external tank walls. However, when observing the first cross-section there was intensity saturation at the sphere boundary in the images so those areas were taped over on the front wall of the tank to prevent damage to the CCD chips.
The two cameras of the stereoscopic set-up were positioned ~ 45 cm from the front face of the tank and the angular displacement of both cameras with respect to the tank being ~ 40°. A subtlety of SPIV post-processing is in the balance between having a large enough average particle displacement between frames and still small enough change in time so that the overall dynamics of the flow can be well-resolved in time. Given that the out-of-plane velocity component calculation is more sensitive to particle displacement size between processed images in comparison to the in-plane velocity we used a staggered processing approach. To facilitate this procedure we collected 32 images per period of oscillation so that the appropriate number of images could be skipped in processing (typically 3) and then repeating the velocity field reconstruction using the images that have been skipped. Thus we were able to obtain reliable velocity data for all three components while maintaining the original resolution in time.
5.2.3 Results

Focusing on a single quadrant of the wave field provides the opportunity for the careful comparison of experiment and theory. Given that the wave cones are symmetric about both the x- and z-axes we can extend our comparisons throughout the wave field; symmetry of the wave field in all four quadrants was confirmed in a separate experiment focusing on a larger field of view.

We begin by observing the general structure of the wave cone which can be seen in figures 5-7 and 5-8. For both experiments the angle to the horizontal is very close to that given by theory. The downward propagating wave cones are clearly visible and the wave cone structure can be distinguished over the span of \( y \). As expected, the velocity magnitude of the wave cone decreases with radial distance from the sphere center. The incongruity of the wave fields at the middle cross-section is attributed to the close proximity of the laser sheet to the sphere which rendered the particles in that corner of the field of view indistinguishable.

For the case of the \( \theta = 20.3^\circ \) wave cone the generation of second order harmonics due to nonlinearities is apparent. However, in contrast to what was discovered in Ermanyuk et al. [11], the second harmonic does not exceed the first harmonic in strength. Furthermore, we did not observe any third order harmonics in our experiments. We note that both our experiments and those of Ermanyuk et al. [11] were performed in the same regimes with similar parameters. The \( \theta = 38.2^\circ \) wave cone exhibits a clearly defined wave cone structure with no harmonic presence, though [11] did notice an evanescent second harmonic very close to the sphere. In performing a quantitative analysis, we confirmed through Fourier-filtering the wave field that waves weakly propagating at twice the forcing frequency of the system do exist (for \( \theta = 20.3^\circ \)). Such harmonics are not evident in the \( \theta = 38.2^\circ \) wave beam which leads us to choose this particular experiment to perform the comparison with theory.

The across-beam velocity profiles of the \( \theta = 38.2^\circ \) wave cone taken at the center cross section are presented in figure 5-9. It is confirmed that the axisymmetric since no out-of-plane velocity is observed. The in-plane velocities, \( u \) and \( w \), exhibit ex-
Figure 5-7: Spatial images of the conical wave cone for $\theta = 20.3^\circ$; the plotted quantity is the total velocity $|\mathbf{u}| = \sqrt{u^2 + v^2 + w^2}$. Note that the $y$ axis has been exaggerated for enhanced clarity.
Figure 5-8: Spatial images of the conical wave cone for $\theta = 38.2^\circ$; the plotted quantity is the total velocity $|\mathbf{u}| = \sqrt{u^2 + v^2 + w^2}$. Note that the $y$ axis has been exaggerated for enhanced clarity.
Figure 5-9: Across-beam profiles of the $\theta = 38.2^\circ$ wave field taken at the center plane for the $u$ (top row) and $w$ (bottom row) velocity components. The profiles cover $\sigma = [-7:7]$ cm and are at $s = 10$ cm. The theoretical solution is plotted as a solid line with the error represented by the lightly shaded area and the experimental results are shown by the discrete points. Each panel contains profiles with a phase difference $\Delta \phi = \pi/\omega$; (b and d) are $\phi = \pi/2\omega$ apart with (a and c).

Excellent agreement with theory. The experimental profiles exhibit a more pronounced asymmetrical behavior that is not captured by theory. However, the general velocity distribution is captured nicely. Consistency is also confirmed by the agreement shown at the four phases covering an entire period of oscillation.

Figure 5-10 displays the across-beam profiles collected at the cross-section 4 cm away from the sphere center. Again, excellent agreement is demonstrated between experimental and theory for all three velocity components. The in-plane velocities display the same asymmetric distribution observed at the center-sphere cross-section, however, the minor differences with theory observed in figure 5-9 have decreased slightly. Most strikingly, the $w$ velocities shown nearly perfect agreement with theory at all phases considered (figures 5-10(c) and (d)). The out-of-plane velocities are observed to be 50% weaker than the in-plane velocities, which is due to the close proximity of the observation plane to the sphere; at this location the wave field would still be dominated by in-plane motion.

In figure 5-11 the across-beam profiles collected at the plane 8 cm from the sphere
Figure 5-10: Across-beam profiles of the $\theta = 38.2^\circ$ wave field taken at a plane 4 cm off-center for the $u$ (top row), $v$ (middle row) and $w$ (bottom row) velocity components. The profiles cover $\sigma = [-8 : 6.5]$ cm and are at $s = 10$ cm. The theoretical solution is plotted as a solid line with the error represented by the lightly shaded area and the experimental results are shown by the discrete points. Each panel contains profiles with a phase difference $\Delta\phi = \pi/\omega$; (b, d and f) are $\phi = \pi/2\omega$ apart with (a,c and e).
Figure 5-11: Across-beam profiles of the $\theta = 38.2^\circ$ wave field taken at a plane 8 cm off-center for the $u$ (top row), $v$ (middle row) and $w$ (bottom row) velocity components. The profiles cover $\sigma = [-10 : 5]$ cm and are at $s = 10$ cm. The theoretical solution is plotted as a solid line with the error represented by the lightly shaded area and the experimental results are shown by the discrete points. Each panel contains profiles with a phase difference $\Delta \phi = \pi/\omega$; (b, d and f) are $\phi = \pi/2\omega$ apart with (a, c and e).

center are shown. The level of agreement observed previously exists for the third cross-section as well. There is excellent agreement with theory, more so with the $v$ and $w$ velocity components. We detect the onset of viscous damping due to the 20% drop in velocity experienced by the in-plane velocities. However, the out-of-plane velocity appears to have increased, which is to be expected since the wave field is considerably more 3D at this location.

A comparison between the theoretical solution put forth by Flynn et al. [12] and Voisin et al. [43] is displayed in figure 5-12. Both solutions capture the overall trends exhibited by a wave cone quite nicely. However, Flynn et al. [12] substantially over predict the peaks by over 50% in the $u$ and $v$ profiles and by over 35% in the $w$ profile.
Figure 5-12: Comparison of Flynn et al. [12] (green) and Voisin et al. [43] (blue) for the $\theta = 38.2^\circ$ wave cone at a plane located 8 cm away from the sphere center. The profiles cover $\sigma = [-17:17]$ cm and are at $s = 13$ cm.

Furthermore, the solution of Flynn et al. [12] does not adequately converge to zero far away from the beam region (see figure 5-12(b)). An added benefit of Voisin et al. [43] is that transient behavior is fully taken into account in their formulation.

5.3 3D incised Gaussian ridge

Experimental studies of topographic generation of internal waves have so far been limited to two-dimensional [7, 47] and axisymmetric models [20, 21]. Specifically, King et al. [20] observed a horizontally oscillating sphere to model internal tide generation by half-sphere on a plane, and in another study the same authors used a Gaussian mountain to mimic realistic ocean topography [21]. All of the aforementioned studies utilized the standard planar PIV technique to observe the complex flows of the various set-ups.

In contrast to a 2D Gaussian ridge, a 3D incised Gaussian ridge allows for fluid to flow through a gap in the ridge rather than being forced to rise up over the topography. We expect that the associated horizontal flow through this incision will be a source of nontrivial 3D wave activity. In this section we study the 3D internal wave field generated by barotropic tidal flow past an incised Gaussian ridge, this configuration being perhaps the most realistic study of 3D internal tide generation to date. We furthermore make use of Stereo-PIV (SPIV) to determine all three components of the 3D internal wave field in several planes, and investigate completely the 3D wave fields.
and see how much it differs from the results for the 2D Gaussian ridge presented in chapter 4. The experiment encompasses subcritical, critical and supercritical wave beams with variable forcing amplitudes.

### 5.3.1 Experimental arrangement and procedure

An incised Gaussian ridge, a schematic of which is presented in figure 5-13, is the central focus of this experimental study. If $x$ is the horizontal coordinate perpendicular to the axis of the ridge, the expression describing the primary Gaussian ridge is given by

$$z_t(x) = \Lambda \exp\left(-\frac{x^2}{2\sigma^2}\right), \quad (5.21)$$

where $\Lambda$ is the primary ridge height and $\sigma$ is the characteristic width of the ridge, and the subscript $t$ indicates the shape of the primary ridge. The Gaussian incision, which is in the $y$-direction along the axis of the ridge, is such that

$$z_i(y) = \Lambda - \Lambda_i \exp\left(-\frac{y^2}{2\sigma_i^2}\right), \quad (5.22)$$

where the subscript $i$ indicates the incision. For the experiments considered in this chapter we set $\Lambda = 20 \pm 0.1$, $\Lambda_i = 6.667 \pm 0.1$, $\sigma = 12.1306$ and $\sigma_i = 2.5$ cm. The maximum slope of the primary Gaussian ridge was thus $45 \pm 1.5$ degrees. The length and width of the topography used in the experiments were 191.5 and 34.1 cm, respectively. The shape was cut from an 8 ft $\times$ 2 ft $\times$ 1 ft styrofoam block using a Foamlinx foam cutter. Once the shape was cut, the sharp intersecting edges of the incision were smoothed using sandpaper to form a $\sim 15$ mm fillet.

The general experimental procedure was the same as that used for the 2D experiments presented in chapter 4: the topography was inverted and situated in the tank with the $z$-axis facing down and the $x$-axis facing left, as illustrated in figure 5-14, and a fixed linear traverse was used to control the oscillations of the topography. In order to avoid an abrupt start of oscillatory motion the amplitude of oscillation was smoothly ramped up from zero to the desired value over seven oscillation periods.
Figure 5-13: Plot of the incised ridge topography. $C_1$, $C_2$ and $C_3$ are the three cross-sections at which particle images were collected, where $C_1$ is the plane closest to the front of the tank.

Figure 5-14: Sketch of the experiment setup from the front (left) and the side (right).
The fluid depth was \( H = 26.6 \) cm giving a depth ratio \( \Lambda/H = 0.75 \pm 0.01 \), and this was confirmed by adjusting the steel supporting frame holding that topography that was measured to be level to within 0.5 degrees at its center. The two ends of the foam block were situated 3 mm below the water line.

Measurements were taken at three cross-sections across the width of the topography: \( y = -6.2 \pm 0.2 \), \( 0.5 \pm 0.2 \) and \( 6.5 \pm 0.2 \) cm, where \( y = 0 \) is the centerline of the ridge (see figure 5-13; the 3 mm difference between the two off center locations did not noticeably effect the symmetry of the measurements. Two fields of view were observed, one being close to the Gaussian bump and another being located further away, and nominally in the far field; the full water depth was contained in the field of view away from the ridge. The top and bottom boundaries of the field of view had to be blocked due to the reflections of the laser sheet, which resulted in an image loss of \( \sim 1 \) cm at the top and bottom. The image loss for the field of view near the ridge was significantly higher due to the topography blocking a greater portion of the view of one camera with respect to the other camera. The two different fields of view were obtained by using the traverse to very slowly shift the topography's position after measurements of the first field of view were complete. To assure no alteration to the stratification occurred the topography was shifted the required distance of 26.93 cm over 90 minutes and the system was then left to settle for 30 minutes.

The tank was filled using the standard double-bucket procedure described earlier in this thesis. The fluid density ranged from fresh water at the top surface to salt water of density \( 1016 \text{ kg/m}^3 \) at the base. After filling was completed the stratification was left to settle for one hour; this allowed for the glass spheres covering a range of density to settle to their appropriate density levels. The density profiles were measured before and after the experiment to detect any significant evolution of the stratification, and the results in figure 5-15 shows negligible difference between the two. The measured buoyancy frequency for the initial stratification was \( N = 0.755 \pm 0.04 \text{ rad/s} \), which decreased by no more than \( 0.003 \text{ rad/s} \) (\( \sim 0.4\% \)) over the course of the experiment.

A time resolution of 16 particle images per period was chosen for the SPIV, which resulted in collection of 17 periods worth of data for each experiment. DaVis 7.2 was
Figure 5-15: Density profiles taken (a) before and (b) after the experiment.

used to perform the vector postprocessing of the raw images. The dimensions of the fields of view at $C_1$, $C_2$ and $C_3$ were $22 \text{ cm} \times 26.1 \text{ cm}$, $20 \text{ cm} \times 25.9 \text{ cm}$ and $18.1 \text{ cm} \times 25.8 \text{ cm}$, respectively. This change in area is due to the decrease in overlap between the two cameras as the object plane is shifted further away. After processing of the images the resulting vector field grid sizes (given in number of discrete points) for $C_1$, $C_2$ and $C_3$ were $97 \times 118$, $84 \times 113$ and $74 \times 109$, respectively; the corresponding spatial resolution was on the order of $1 \text{ mm}^2$.

To obtain particle images that allowed for accurate reconstruction of the out-of-plane ($v$) velocity component the cameras were placed $\sim 80 \text{ cm}$ from $C_1$. However, since the object plane underwent a shift of $12.7 \text{ cm}$ through the experiment the quality of the reconstructed $v$ velocity field expectedly decreased for $C_3$; the same particle displacement captured onto the image plane will decrease in size as the object plane is shifted further away. The relative position of the fields of view with respect to each other, as well as the topography, were determined by capturing an image containing a reference point for each of the three cross-sections; the reference point was a needle inserted through the topography at a known location. Additionally, the laser sheet penetrates the stratified fluid from below the wave tank, thus both cameras received equal amounts of scattered light since both were similarly facing the laser sheet with
respect to its source; this allowed for equal lens settings between the two. Lastly, the laser sheet was positioned such that there would be equal illumination throughout the field of view for both cameras.

The parameters that were systematically varied in this study were the amplitude of oscillation, \( A \), and the forcing frequency, \( \omega \). Two forcing amplitudes of \( A = 1 \) and 2 mm were used in generating sub-, near- and super-critical wave beams with propagation angles of \( \theta = 18.55^\circ, 27.74^\circ, 41.18^\circ \) and \( 53.74^\circ \) (to within an error of \( \pm 0.5^\circ \)). The criticalities for these experiments were \( \varepsilon = 2.98, 1.90, 1.14 \) and 0.73 (to within an error of \( \pm 0.05 \)), respectively.

In order to characterize the type of flow regimes encountered in these experiments, in addition to the criticality we must consider the excursion parameter \( \chi \) and the vertical Froude number \( Fr_z \). Potential nonlinearity is predicted by the value of \( \chi \), which is formally defined as the ratio \( Uk_t/\omega \) where \( U \) is the induced flow velocity amplitude \( U = A\omega \) and \( k_t \) is the inverse horizontal wavelength of the topography. For the range of values utilized in our study \( \chi \) is \( O(10^{-3}) \), thus the small excursion allows us to expect a linear wave field. The vertical Froude number \( Fr_z \), describing the balance of the fluid inertia to the restoring buoyancy force i.e. \( U/NA \) where \( \bar{A} \) is the averaged ridge height in \( y \). When \( Fr_z > 1 \) the inertia of the fluid overcomes the local buoyancy which forces separation of the flow from the topography. We find that \( Fr_z \) is \( O(10^{-3}) \) for our experiments, however, which effectively rules out flow separation as a significant issue.

### 5.3.2 Results: Linearity, repeatability and periodicity

We first seek to confirm three important properties of the wave field: linearity, repeatability and periodicity. The linearity of the wave field is verified by observing that the wave fields generated by forcing with amplitudes \( A = 1 \) and 2 mm are such that one is simply twice the strength of the other. Periodicity is confirmed by observing the time histories of various vertical cross-sections and confirming that the amplitude of periodic oscillation at the forcing frequency achieves a steady value.

Two snapshots of the wave fields for \( \theta = 27.74^\circ \) at the same phase of oscillation
are presented in figure 5-16, with the only difference being that the forcing amplitude is \( A = 1 \) mm for figure 5-16(a) and (b), and \( A = 2 \) mm for figure 5-16(c) and (d). From the snapshots we see that the wave fields normalized by their respective barotropic tidal velocities present nearly identical results. Moreover, vertical profiles of the normalized total-velocity, \( |\mathbf{u}|/U \), were taken at three \( x \)-locations; the results of which are presented in figure 5-17. There is excellent agreement between the \( A = 1 \) and 2 mm profiles for all three locations observed. In these comparisons the variations of the two different forcing amplitudes is negligible. We note that these results also reasonably confirm the repeatability of the experiments by virtue of the consistency of the normalized results. It is worth mentioning the noticeable level of noise in the \( v \) velocity measurements for \( A = 1 \) mm (figure 5-16(b)) which is as a consequence of processing images that exhibit smaller displacements of the seed particles compared to the \( A = 2 \) mm wave field.

To confirm periodicity of the wave field, figure 5-18 displays vertical time series
plots taken at five equally spaced points spanning \(x = 40 - 60\) \(\text{cm}\) for the \(\theta = 27.74^\circ\) wave field. As expected, the achievement of a time-periodic state occurs at later times as the location distance increases from the generate site. Significantly, we see that the majority of the data is contained in the steady state regime.

Finally, to further confirm linearity of the wave field the existence of higher harmonics in the measurements was investigated by inspecting Fourier spectra of various points in the wave field. Figure 5-19 presents sample frequency spectrums of vertically-averaged \(u, v,\) and \(w\) profiles for all four of the forcing frequencies studied. As can be seen the wave fields are highly linear as the spectrum is dominated by the forcing frequency of the system. The existence of second harmonics is noted in the \(\theta = 18.55^\circ\) wave field though it is imperceptible; for the remaining forcing frequencies higher order harmonics are nonexistent. However, the noise level of the \(v\) velocity field is, on average, an order of magnitude higher than the in-plane velocities, albeit the \(v\) noise level is still an order of magnitude weaker than the fundamental frequency. Spectra of several \(x\)-locations in the wave field were observed for completeness and it can be concluded that the wave fields are dominated by their respective fundamental frequencies except perhaps for localized regions where boundary reflections take
Figure 5-18: Temporal evolution of the normalized $w$ velocity at $C_1$ for locations (a) $x = 40$, (b) $x = 45$, (c) $x = 50$, (d) $x = 55$ and (e) $x = 60$ cm. The thick dashed lines represent the achievement of a periodic state.
When reasonable, for the experimental results presented in the next subsection, we performed further tests of repeatability, linearity and periodicity. In all reported cases, when investigated, these fundamental requirements were all satisfied.

5.3.3 Results: 3D wave field structure

A total of 30 experiments were performed encompassing a combination 4 forcing frequencies and 2 forcing amplitudes over 3 cross-sections. For each experiment containing the same forcing parameters the PIV data was matched across the three cross-sections; this was facilitated by referencing the data from each cross-section to the same phase of oscillation. In this section we present the results of the 3D sub-

Figure 5-19: Vertically-averaged Fourier spectra for (a) $\omega_o = 0.24$ rad/s ($\theta = 18.55^\circ$), (b) $\omega_o = 0.35$ rad/s ($\theta = 27.74^\circ$), (c) $\omega_o = 0.5$ rad/s ($\theta = 41.18^\circ$) and (d) $\omega_o = 0.61$ rad/s ($\theta = 53.74^\circ$) taken at $x = 50$ cm (a,c and d) and $x = 60$ cm (b). The $u$, $v$ and $w$ velocities are represented as blue, green and black, respectively.
near- and super-critical wave beams and discuss their key features.

Figure 5-20 displays a snapshot of the $\theta = 18.55^\circ$ wave field where the in-plane velocities $u_p$ and the out-of-plane velocity component $v$ have been separated. It is apparent that the $u_p$ fields exhibit the same behavior as their 2D Gaussian ridge counterparts (see figure 4-3). The in-plane velocity associated with the wave field maintains its structure for all three cross-sectional cuts, with only minor differences perceptible. The striking feature of this case concerns the $v$ velocity component. Near the ridge the $v$ wave forms appear similar to the curved conic-section type wave forms observed from the oscillating sphere becoming seemingly level in the constant depth region of the far field (see figures 5-7 and 5-8). Since we known that waves oscillating with the forcing frequency must be inclined at a certain $18.55^\circ$ with respect to the horizontal for this experiment, the observed structure must be a slice through a more complex wave field with three-dimensional structure. Furthermore, the symmetry of the $v$ wave structure is evident by comparing in figures 5-20(b) and (f), the flow direction reverses between the two figures which is precisely what is expected from symmetry. Furthermore, we note that the $v$ wave field in figure 5-20(d) is significantly weaker than for the two planes either side of the mid-plane, consistent with our expectation of a wave field that is symmetric about the mid plane. The symmetry is not perfect however, and this can perhaps be attributed to a slight misalignment of the laser sheet with the center-plane or some other form of asymmetry that is beyond reasonable experimental control.

The visualization of a wave field propagating at $\theta = 27.74^\circ$ is presented in figure 5-21. Akin to the $\theta = 18.55^\circ$ wave field, the $u_p$ velocity fields are nearly identical in $y$. As expected the velocities are greater in comparison to the earlier case of lower forcing frequency. The existence of wave-wave interactions, as well as wall horizontal-boundary reflections, is brought to the attention of the reader; these are evident in the second field of view ($x = 40 - 60$ cm) of figures 5-21(a), (c) and (e). Interestingly, in contrast to the results for $\theta = 18.55^\circ$, the $v$ velocity field is quite clearly asymmetric about the central mid-plane. There is strong wave activity in the lower half of the vertical domain in $C_3$ as opposed to the out-of-plane waves throughout the vertical in
Figure 5-20: Snapshots of the $\theta = 18.55^\circ$ wave field at $C_1$ (a-b), $C_2$ (c-d) and $C_3$ (e-f). Plots in the left column represent the in-plane velocities and plots on the right represent the out-of-plane $v$ velocities (velocities normalized by $U$).
Figure 5-21: Snapshots of the $\theta = 27.74^\circ$ wave field at $C_1$ (a-b), $C_2$ (c-d) and $C_3$ (e-f). Plots in the left column represent the in-plane velocities and plots on the right represent the out-of-plane $v$ velocities (velocities normalized by $U$).

$C_1$. Also in contrast to the previous $\theta = 18.55^\circ$ wave field, the $v$ wave activity at the mid-plane $C_2$ shifts from the far field to the near field which appears to be originating from the ridge incision. Finally, we note that the $v$ velocity field is somewhat weaker for these experiments than for the experiments with $\theta = 18.55^\circ$.

A near-critical ($\theta = 41.18^\circ$) wave field is shown in figure 5-22. Again the $u_p$ velocity fields display very similar wave fields from $C_1$-$C_3$. Thoough not clearly seen in $C_2$ and $C_3$ the primary, downward propagating wave beam from the topography is reflected back upwards in the far field. There is also a much broader and weaker wave beam initially propagating upwards from the ridge, which is then reflected downwards in the far field. Some semblance of a wave field exists for the $v$ velocity fields though the structures have become less defined and patchy than for the more supercritical regimes. Noticeable $v$ activity is absent in the mid-plane (figure 5-22(d)) and there is
some evidence of symmetry by virtue of the reversal of the $v$ velocity flow directions between $C_1$ and $C_3$.

The subcritical ($\theta = 53.74^\circ$) wave field is presented in figure 5-23. There is no longer a well defined $u_p$ wave beam that propagates over the total length of the horizontal, and the wave field structure transforms from a beam to a rolling vortex (characteristic of a mode-1 wave). Additionally, the in-plane velocities are substantially weaker than in the super- and near-critical wave fields. Transition from near- to subcritical has resulted in a significant diminishing of the $v$ velocity field everywhere except for the area near the ridge at $C_1$. This behavior is suggestive of an evolution from a fully 3D to a quasi-2D internal wave field in the far field since no noticeable wave activity is observed.
Figure 5-23: Snapshots of the $\theta = 53.74^\circ$ wave field at $C_1$ (a-b) and $C_2$ (c-d). Plots in the left column represent the in-plane velocities and plots on the right represent the out-of-plane $v$ velocities (velocities normalized by $U$).
Figure 5-24: Comparison of the wave fields generated by the 2D Gaussian ridge (left column) and the 3D Gaussian ridge (middle and right columns). The color intensity represents the total velocity normalized by the barotropic tidal velocity, $|u|/U$, and the vectors signify local velocity direction; the top, middle and bottom rows represent the super-, near- and subcritical wave fields.
Finally, a comparison between the wave fields generated by the 2D and 3D Gaussian ridge is given in figure 5-24. The most striking difference is contained in the 3D supercritical wave field: the total velocity $|u|$ wave beams seemingly appear curved in comparison to the straight beams observed in the 2D wave beams. Furthermore, the reflection dynamics appear more complex and distorted for the 3D wave field, which could be a part of a more complex three-dimensional interaction process. Interestingly, both the 2D and 3D subcritical wave beams (figures 5-24(g) and (h)) show an upward propagating wave beam, though there is no evidence of the strong internal wave activity immediately over the slope in the 2D subcritical wave field. Overall, there is a modest yet noticeable, weakening of the total velocity field for the 3D ridge. This suggests that the ridge incision may serve to weaken the oscillatory tidal flux by providing a relief for the flow, which would normally be forced to travel over the ridge through higher density fluid which in turn generated greater flow velocities.

5.4 Conclusions

In this chapter the theory behind SPIV was introduced in detail and the integration of this new measurement technique to stratified flows experimental research was discussed and demonstrated. The theoretical formulation and the experimental procedure for an oscillating sphere in a uniformly stratified fluid were investigated, and excellent agreement was found. In particular, we found excellent agreement with the latest theory of Voisin et al. [43] for all three velocity components in three different cross-sectional planes.

We have also presented an experimental investigation of a fully three-dimensional internal wave field generated by an incised Gaussian ridge. With respect to the wave fields, we find that as the system becomes more subcritical the magnitude of the out-of-plane velocity components becomes significantly weaker even though the subcritical topographies in this experiment have higher tidal forcing velocities. We observed symmetric looking wave fields with strong out-of-plane velocity components for super- and near-critical regimes. However, this symmetry seemed to be broken as
we increased the wave beam angle. This could be a result of the complex dynamics of
the three-dimensional wave field or some experimental imperfections contained in the
study; further investigations would be needed to study this further. Most importantly,
our results confirm that weakening effect the incision has on the internal wave fields;
in comparison to two-dimensional counterpart experiments performed in chapter 4,
the 3D Gaussian ridge generates a slightly weaker total velocity field as a whole.

Overall, we demonstrated through systematic laboratory experiments utilizing the
robust SPIV measurement technique that fully 3D wave fields can indeed be observed.
As a result of this study, we have established confidence in SPIV as a valid and reliable
measurement technique in the investigations of internal wave fields.
Chapter 6

Conclusion

Given the importance of internal waves to the global energy balance and the unanswered questions concerning their generation and energy dissipation in three dimensions, we have performed a series of laboratory experiments to help improve their understanding. The purpose of this thesis was to present a systematic study of 2D and 3D internal wave generation via an oscillating sphere and 2D/3D Gaussian ridges.

A mathematical formulation of internal wave propagation was presented in chapter 2. Starting from the basic equations governing fluid flow we derive the internal wave equation and the corresponding boundary conditions. A discussion of the Green function theory with viscous correction concluded the chapter.

Chapter 3 presents a refined algorithm to decompose a two-dimensional wave field in a constant depth channel into its constitutive modes. An analytic expression for the mode content was derived which was then modified to account for various experimental issues such as noise and image loss. The modal decomposition technique was validated by applying the algorithm and modifications to data obtained from experiments recreating mode-1 and 2 wave fields.

A benchmark experiment was performed in chapter 4 to study the wave field generated by an oscillating 2D Gaussian ridge. Planar-PIV was used to obtain the velocity fields of the super-, near and subcritical wave beams. The experimental results were then compared to the Green function theory and further analyzed using modal decomposition. Overall, we find that there is good agreement between theory
and experiment. Most importantly, we obtain data with which to compare the wave fields generated by 3D incised ridge.

In chapter 5 the basics of SPIV are discussed and the experimental results of the vertically oscillating sphere and the 3D incised Gaussian ridge are presented. The three dimensional wave fields generated by the oscillating sphere were observed using SPIV and the results compared with the theoretical formulation of Voisin et al. [43]. The excellent agreement between the experiment and theory for all three velocity components affirmed SPIV as an effective tool in studying internal wave fields. We found that the 3D incised ridge produced relatively strong three-dimensional wave fields at frequencies corresponding to supercritical wave beams; the out-of-plane velocity components diminished significantly as the wave field transitioned to near and subcriticality. However, the in-plane velocity fields looked nearly identical to the wave fields generated by the 2D Gaussian ridge. Overall, we found that the total velocity of the wave fields became slightly weaker due to the incision.
Bibliography


