Bordered Legendrian knots and sutured Legendrian invariants

by

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Abstract

In this thesis we apply techniques from the bordered and sutured variants of Floer homology to study Legendrian knots. First, given a front diagram for a Legendrian knot K in S^3 which has been split into several pieces, we associate a differential graded algebra to each "bordered" piece and prove a van Kampen theorem which recovers the Chekanov-Eliashberg invariant $Ch(K)$ of the knot from the bordered DGAs. This leads to the construction of morphisms $Ch(K) \to Ch(K')$ corresponding to certain Legendrian tangle replacements and many related applications. We also examine several examples in detail, including Legendrian Whitehead doubles and the first known knot with maximal Thurston-Bennequin invariant for which $Ch(K)$ vanishes.

Second, we use monopole Floer homology for sutured manifolds to construct new invariants of Legendrian knots. These invariants reside in monopole knot homology and closely resemble Heegaard Floer invariants due to Lisca-Ozsváth-Stipsicz-Szabó, but their construction directly involves the contact topology of the knot complement and so many of their properties are easier to prove in this context. In particular, we show that these new invariants are functorial under Lagrangian concordance.

Thesis Supervisor: Tomasz Mrowka
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Acknowledgments

I would like to thank my advisor, Tom Mrowka, for all of his patience and encouragement throughout my graduate career. I learned a tremendous amount under his supervision, and owe not only the direction of this thesis and many of the ideas in it but also much of my overall mathematical development to his guidance over the last five years.

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I have also been fortunate to be surrounded by an active community of mathematical siblings and more distant relatives. I would like to thank them all, including Jon Bloom, Tirasan Khandawit, Thomas Koberda, Tanya Kobylyatskaya, Ben Mares, Tim Nguyen, Nikhil Savale, and Ethan Street, for many interesting conversations, seminars, and conferences over the last few years. Some would say that there's more to life than low-dimensional topology, however, and they don't just mean higher-dimensional manifolds; in that case I must also thank my friends, in particular Nikola Kamburov, Silviu Pufu, and Aaron Silberstein, with whom I have shared experiences ranging from impromptu hiking trips to my first encounter with Gergana.

Finally, I thank my family for all of their support. I am especially grateful to my wife, Ana Caraiani, who not only offered me insights into Shimura varieties (which unsurprisingly did not make it into this thesis) and more mundane algebraic questions (which did), but has also been a tremendous source of love and encouragement.

Contents

Chapter 1

Background on Legendrian knots

Let Y be a 3-manifold. A *contact structure* on Y is a 2-plane field $\xi \subset TY$ which can be written as $\xi = \ker(\alpha)$, where $\alpha \wedge d\alpha > 0$. The most basic example is

$$\xi_{\text{std}} = \ker(dz - ydx)$$

on $\mathbb{R}^3$, and a Darboux theorem states that any point in any contact manifold (Y, ξ) has a neighborhood U with a homeomorphism $\varphi : U \to \mathbb{R}^3$ such that $\xi|_U = \varphi^* \xi_{\text{std}}$. Thus just as in symplectic geometry, contact geometry only provides global information about a manifold.

The existence of contact structures on arbitrary Y was shown by Martinet [42], and a construction of Thurston and Winkelnkemper [66] associated a contact structure to any open book decomposition of Y as one direction of a correspondence that was later made precise by Giroux [28]. The first proof that a manifold could have several distinct contact structures was due to Bennequin [1]. He showed that for the standard contact structure

$$\xi = \ker\left(\sum_{i=1}^{2} x_i dy_i - y_i dx_i\right)$$

on $S^3 \subset \mathbb{C}^2$ there was no smoothly embedded disk $D \subset S^3$ with $TD|_{\partial D} = \xi|_{\partial D}$, while another contact structure on S^3 did give rise to such a disk. Eliashberg [9] completely classified *overtwisted* contact structures, i.e. contact structures containing these "overtwisted" disks, proving that there is exactly one up to contact isotopy in every homotopy class of plane fields on Y. Thus the interesting part of the classification problem is the study of *tight* contact structures, which are not overtwisted. Furthermore, he showed in [10] that ξ_{std} and ξ are the unique tight contact structures on $\mathbb{R}^3$ and S^3. For an introduction to this subject we recommend [16].

An oriented knot K in a contact manifold (Y, ξ) is called *Legendrian* if it is always tangent to ξ, meaning that $T_xK \subset \xi_x$ for each point of K. If the knot is also nullhomologous, we can use an arbitrary Seifert surface Σ to associate two "classical" numerical invariants to K. First, the normal line bundle to K in $\xi|_K$ determines a canonical framing of K, and the *Thurston-Bennequin number* $tb(K)$ measures the twisting of this framing with respect to the Seifert framing of K. Second, we can trivialize $\xi|_\Sigma$ so that $\xi|_K$ is identified with $K \times \mathbb{R}^2$, and then the *rotation number* $r(K)$ is the winding number of the oriented tangent vectors to K in $\mathbb{R}^2$ with respect to the origin. For Legendrian knots in $(\mathbb{R}^3, \xi_{\text{std}})$, Bennequin [1] showed that

$$tb(K) + |r(K)| \leq 2g(\Sigma) - 1,$$

which indeed proves that ξ_{std} is tight since an overtwisted disk would have $tb(\partial D) = 0 > 2g(D) - 1$, and also shows that $tb(K)$ is bounded above within any topological knot type.

The classical invariants of a Legendrian knot K in the standard $\mathbb{R}^3$ turn out to completely determine K up to Legendrian isotopy if K is a topological unknot [13], torus knot or figure eight [18], and so one might ask if this is true in general. The first counterexample is due to Chekanov [4], who developed a combinatorial version of what would become Legendrian contact homology and used it to distinguish two knots of type 5_2 with $tb = 1$ and $r = 0$. See [17] for more on Legendrian knots and on the closely related *transverse* knots, which satisfy $T_xK \pitchfork \xi_x$.

The Chekanov-Eliashberg invariant [4, 12], which assigns to each Legendrian knot K a differential graded algebra $(Ch(K), \partial)$ over $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$, has been a powerful tool for classifying Legendrian knots in $(\mathbb{R}^3, \xi_{\text{std}})$. In Chapter 2 of this thesis we will assign

DGAs to "bordered" knots, following ideas of Lipshitz-Ozsváth-Thurston [40, 39], and prove a van Kampen theorem which recovers $Ch(K)$ from the DGAs of the bordered pieces. This allows us to develop several interesting consequences, including morphisms between the invariants of knots related by certain tangle replacements and an analysis of the "linearized contact homology" of Legendrian Whitehead doubles.

Chapter 3 is devoted to computations of the Legendrian contact homology of various knots. We further analyze Legendrian Whitehead doubles, showing (together with results from Chapter 2) that the Whitehead double of K has particularly simple linearized contact homology unless K is nondestabilizable and $r(K) = 0$. We also provide the first known example of a Legendrian knot K with maximal Thurston-Bennequin number for which the Legendrian contact homology $LCH(K)$ vanishes, namely a representative of $m(10_{132})$ with $(tb, r) = (-1, 0)$, and show that another $m(10_{132})$ representative with $(tb, r) = (-1, 0)$ has nontrivial contact homology. Thus we show that the nontriviality of $LCH(K)$ is not determined solely by the classical invariants of K, even though for example the existence of a 1-dimensional representation $LCH(K) \to \mathbb{F}$ is known to depend only on the topological knot type and on $tb(K)$.

Finally, in Chapter 4 we use monopole Floer homology [36] and the associated monopole knot homology [37] to construct new invariants of Legendrian knots $K \subset (Y, \xi)$, and conjecturally invariants of transverse knots as well. These invariants are elements of the knot homology $KHM(-Y, \xi)$ which resemble the Heegaard Floer invariants $\hat{\mathcal{L}}(K) \in \widehat{HFK}(-Y, K)$ [41], but their construction reflects the contact topology of the complement $Y \backslash K$ in a much more direct way. We also use a functoriality result for monopole contact invariants [47, 46] to investigate the behavior of the new Legendrian invariants with respect to a relation known as Lagrangian concordance.

We note that much of the material in Chapters 2 and 3 has already appeared in the papers [61] and [60], respectively. However, the material in sections 2.6 and 3.1, as well as the entirety of Chapter 4, is presented for the first time in this thesis.

Chapter 2

A bordered Chekanov-Eliashberg algebra

2.1 Introduction

2.1.1 The Chekanov-Eliashberg invariant

Chekanov [4] defined for each Legendrian knot $K \subset (\mathbb{R}^3, \xi_{\text{std}})$ an associative unital differential graded algebra (DGA), here denoted $Ch(K)$, whose stable tame isomorphism type is an invariant of K up to Legendrian isotopy. Given a Lagrangian projection of K, i.e. a projection of K onto the xy-plane, the algebra is generated freely over $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ by the crossings of K, which correspond to Reeb chords in $(\mathbb{R}^3, \xi)$, and graded by $\mathbb{Z}/2r(K)\mathbb{Z}$, where $r(K)$ is the rotation number of K. (Etnyre, Ng, and Sabloff [21] later extended the base ring to $\mathbb{Z}[t, t^{-1}]$ and the grading to a full $\mathbb{Z}$ grading.) The differential counts certain immersed disks in the knot diagram, and although it was motivated by contact homology [12] its computation is entirely combinatorial. Thus $Ch(K)$ could be used to distinguish between two Legendrian representatives of the 5_2 knot even though their classical invariants tb and r are the same.

Legendrian knots are often specified by front projections, which are projections onto the xz-plane. A knot can be uniquely recovered from its front projection since the y-coordinate at any point is the slope $\frac{dz}{dx}$; in particular the projection has no vertical tangent lines, so at each critical point of x there is a cusp. At any crossing the segment with smaller slope passes over the one with larger slope. Ng [52] gave a construction of $Ch(K)$ for front projections, and showed that given a so-called "simple" front the DGA is very easy to describe.

Meanwhile, on the way to constructing bordered Heegaard Floer homology [39] as an invariant of 3-manifolds with marked boundary, Lipshitz, Ozsváth, and Thurston constructed a simplified model of knot Floer homology for bordered grid diagrams [40]. By cutting a grid diagram along a vertical line, they associate differential modules $CPA^-(\mathcal{H}^A)$ and $CPD^-(\mathcal{H}^D)$ over some algebra $\mathcal{A}$ to the two halves $\mathcal{H}^A$ and $\mathcal{H}^D$ of the diagram $\mathcal{H}$ so that their tensor product is the "planar Floer homology" $CP^-(\mathcal{H})$. Since the differential on CP^- counts certain rectangles in the grid diagram, the algebra $\mathcal{A}$ is constructed to remember when these rectangles cross the dividing line, and so the pairing theorem

$$CPA^-(\mathcal{H}^A) \otimes_{\mathcal{A}} CPD^-(\mathcal{H}^D) \cong CP^-(\mathcal{H})$$

is a straightforward consequence of the construction. However, the chain complex $CP^-(\mathcal{H})$ is not an invariant of the underlying knot, and a similar decomposition for the knot Floer homology complex CFK^- seems to be significantly harder.

Our goal in this chapter is to present a similar decomposition theorem for the Chekanov-Eliashberg DGA associated to a front diagram. By dividing a simple front into left and right halves K^A and K^D which intersect the dividing line in n points we will construct two DGAs, $A(K^A)$ and $D(K^D)$. These DGAs admit morphisms into them from another DGA denoted I_n, where a DGA morphism is an algebra homomorphism that preserves gradings and satisfies $\partial \circ f = f \circ \partial$. We then prove the following analogue of van Kampen's theorem.

Theorem. *The commutative diagram*

$$\begin{tikzcd} I_n \arrow[r] \arrow[d] & D(K^D) \arrow[d] \\ A(K^A) \arrow[r] & Ch(K) \end{tikzcd}$$

is a pushout square in the category of DGAs.

This theorem adds to the "algebraic topology" picture of the Chekanov-Eliashberg algebra which originated with Sabloff's Poincaré duality theorem [58] and also includes cup products, Massey products, and A_∞ product structures [6]; these previous results all apply cohomological ideas to linearizations of the DGA, whereas the van Kampen theorem suggests that the DGA should be thought of as a "fundamental group" of a Legendrian knot.

After developing the van Kampen theorem and generalizing it to further divisions of Legendrian fronts, we add to the cohomological picture by constructing a related Mayer-Vietoris sequence in linearized contact homology. We will then use these ideas to construct morphisms between the DGAs of some Legendrian knots related by tangle replacements, and in particular apply these techniques to understand augmentations of Legendrian Whitehead doubles. Finally, we make some similar observations about the closely related characteristic algebra.

2.1.2 The algebra of a simple Legendrian front

This section will review the construction of the Chekanov-Eliashberg DGA for a Legendrian front as in [50, 52]. Although it can be constructed for any front, we will restrict our attention to simple fronts, where the DGA is particularly easy to describe. Throughout this paper all DGAs will be assumed to be *semi-free* [4], that is, freely generated over $\mathbb{F}$ by a specified set of generators.

Definition 2.1.1. A Legendrian front is *simple* if it can be changed by a planar isotopy so that all of its right cusps have the same x-coordinate.

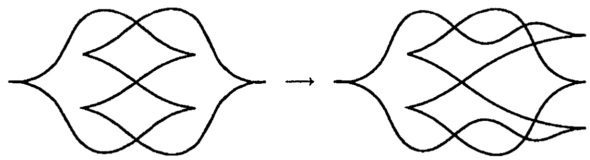
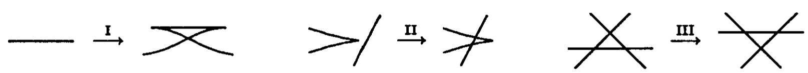


Figure 2.1.1: A front diagram of a Legendrian trefoil is made simple by pulling the two interior right cusps rightward and using Legendrian Reidemeister moves.

Remark 2.1.2. We will also describe a piece of a front cut out by two vertical lines as simple if no right cusp lies in a compact region bounded by the front and the vertical lines; this will ensure that these pieces form a simple front when glued together.

Two fronts represent the same Legendrian knot if and only if they are related by a sequence of Legendrian Reidemeister moves [64]:



Therefore every Legendrian knot admits a simple representative by taking an arbitrary front and using type II Reidemeister moves to pull each right cusp outside of any compact region, as in Figure 2.1.1, though this will increase the number of crossings.

Definition 2.1.3. The *vertices* of a simple Legendrian front are its crossings and right cusps.

The simple front on the right side of Figure 2.1.1 has ten vertices: there are seven crossings and three right cusps.

Definition 2.1.4. An *admissible disk* for a vertex v of a simple front K is a disk $D^2 \subset \mathbb{R}^2$ with $\partial D \subset K$ satisfying the following properties:

1. D is smoothly embedded except possibly at vertices and left cusps;

2. The vertex v is the unique rightmost point of D;

3. D has a unique leftmost point at a left cusp of K;

4. At any *corner* of D, i.e. a crossing $c \neq v$ where D is singular, a small neighborhood U of c is divided into four regions by $U \cap K$; we require that $U \cap D$ be contained in exactly one of these regions.

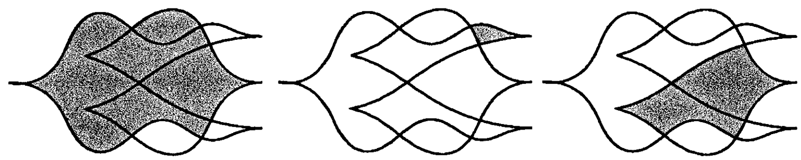


Figure 2.1.2: Disks embedded in the simple front diagram of Figure 2.1.1. The first two are not admissible – one occupies three quadrants around the top middle crossing, and one does not have its leftmost point at a left cusp – but the last one is admissible.

Let $\mathrm{Disk}(K; v)$ denote the set of admissible disks for the vertex v.

Definition 2.1.5. The Chekanov-Eliashberg algebra of a simple front K, denoted $Ch(K)$, is the DGA generated freely over $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ by the vertices of K. Its differential is given by

$$\partial c = \begin{cases} \sum_{D \in \mathrm{Disk}(K;c)} \partial D, & c \text{ a crossing} \\ 1 + \sum_{D \in \mathrm{Disk}(K;c)} \partial D, & c \text{ a right cusp,} \end{cases}$$

where ∂D denotes the product of the corners of D as seen in counterclockwise order from v.

If K has rotation number $r(K)$, we can assign a *Maslov potential* $\mu(s) \in \Gamma = \mathbb{Z}/2r(K)\mathbb{Z}$ to each strand s of K so that at any left or right cusp, the top strand s_1 and bottom strand s_2 satisfy $\mu(s_1) - \mu(s_2) = 1$. Then $Ch(K)$ admits a Γ-grading in which each right cusp has grading $|c| = 1$, and at each crossing c with top strand s_1 crossing over the bottom strand s_2 we define the grading to be $|c| = \mu(s_1) - \mu(s_2)$. (Recall that in a front projection, the strand with smaller slope always crosses over the strand with larger slope.)

Remark 2.1.6. The grading is well-defined in $\mathbb{Z}/2r(K)\mathbb{Z}$ for knots but ambiguous for links, since we may change the Maslov potential on every strand of a single component K by some constant c and thus change the gradings at every vertex where exactly one strand belongs to K by $\pm c$. In practice we will always work with an explicit choice of grading.

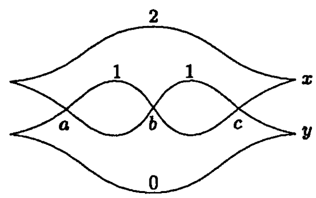


Figure 2.1.3: A simple front for another Legendrian trefoil, with vertices and Maslov potentials labeled.

Example 2.1.7. If K is the simple front of Figure 2.1.3, then $Ch(K)$ is generated freely by a, b, c, x, y satisfying

$$\begin{aligned} \partial x &= 1 + abc + a + c \\ \partial y &= 1 + cba + c + a \\ \partial a = \partial b = \partial c &= 0. \end{aligned}$$

The Maslov potentials indicated in Figure 2.1.3 give $Ch(K)$ a $\mathbb{Z}$-grading with $|x| = |y| = 1$ and $|a| = |b| = |c| = 0$.

Definition 2.1.8. A *tame isomorphism* $\mathcal{A} \to \mathcal{A}'$ of DGAs with free generators $g_1, \dots, g_n$ and $g'_1, \dots, g'_n$ is an automorphism of $\mathcal{A}$ of the form

$$g_i \mapsto g_i + \varphi(g_1, \dots, g_{i-1}, g_{i+1}, \dots g_n)$$

which fixes all other g_j, followed by the isomorphism $g_i \mapsto g'_i$ for all i. A *stabilization* of the DGA $\mathcal{A}$ preserves all generators and differentials and adds two new generators a and b, satisfying $\partial a = b$ and $\partial b = 0$, in gradings $k+1$ and k for some k. Two DGAs are said to be *stable tame isomorphic* if they are related by a sequence of tame isomorphisms, stabilizations, and destabilizations.

Theorem 2.1.9 ([4, 52])**.** *The differential ∂ on $Ch(K)$ satisfies $\partial^2 = 0$ and lowers degree by 1, and the stable tame isomorphism type of $Ch(K)$ is an invariant of K up to Legendrian isotopy.*

Finally, we will outline the proof from [50] that $\partial^2 = 0$, since we will use slight

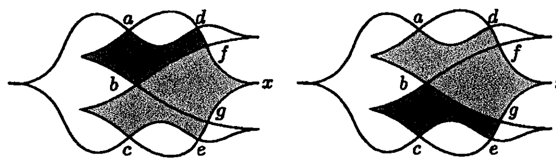


Figure 2.1.4: Two ways to split a region appearing in the proof that $\partial^2 = 0$.

variations of this argument repeatedly in the following sections. For any vertex c of K, a monomial of ∂c is the product $c_1 c_2 \dots c_k$ of corners along the boundary of a disk $D \in \mathrm{Disk}(K; c)$, and the corresponding terms of $\partial^2 c$ involve replacing some c_i in that product with ∂c_i. Since ∂c_i is the sum of terms $\partial D'$ over disks $D' \in \mathrm{Disk}(K; c_i)$, the monomials of $\partial^2 c$ are products of corners of certain regions $R = D \cup D'$. In R, the disks D and D' intersect only along a segment of a strand through c_i; at the other endpoint c' of $D \cap D'$, the region R occupies three of four quadrants; and R has two left cusps, one coming from each of D and D'.

Figure 2.1.4 shows an example of such a region R appearing in the computation of $\partial^2 x$ for the simple front of Figure 2.1.1. On the left, the lighter disk gives the monomial fce of ∂x, and differentiating this at f gives us a term $(dab)ce$ of $\partial^2 x$ via the darker disk. On the right, however, the lighter disk gives the monomial dag of ∂x, and differentiating at g contributes a term $da(bce)$ from the darker disk. Thus the term $dabce$ appears twice in $\partial^2 x$, and since $Ch(K)$ is defined over $\mathbb{Z}/2\mathbb{Z}$ these terms sum to zero.

This argument works in general: following either of the two strands through the point c' (b in Figure 2.1.4) until it intersects ∂R again (at f or g in Figure 2.1.4) gives us exactly two ways to split R into a union of disks $D \cup D'$ which contribute the same monomial to $\partial^2 c$. Since the terms of $\partial^2 c$ cancel in pairs, we must have $\partial^2 c = 0$.

2.2 The bordered Chekanov-Eliashberg algebra

2.2.1 The algebra of a finite set of points

Let n be a nonnegative integer, and suppose we have a vertical dividing line which intersects a front in n points. (Note that n will always be even in practice, but we do not need this assumption for now.) Furthermore, suppose we have a potential function $\mu : \{1, 2, \ldots, n\} \to \Gamma$, where Γ is a cyclic group such as $\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$.

Definition 2.2.1. The algebra I_n^μ is the DGA generated freely over $\mathbb{F}$ by elements $\{\rho_{ij} \mid 1 \leq i < j \leq n\}$ with grading $|\rho_{ij}| = \mu(i) - \mu(j) - 1$. It has a differential defined on these generators as

$$\partial \rho_{ij} = \sum_{i<k<j} \rho_{ik}\rho_{kj}$$

and extended to all of I_n^μ by the Leibniz rule.

Although the grading depends on μ, we will in general omit it from the notation and simply write I_n.

Proposition 2.2.2. *The differential ∂ lowers the grading by -1 and satisfies $\partial^2 = 0$.*

Proof. Both assertions follow by a straightforward calculation. □

Remark 2.2.3. This algebra appears in [45] as the "interval algebra" $I_n(n)$, where a closely related construction determines the DGA of the n-copy of a topological unknot or of a negative torus knot.

The purpose of this algebra is to remember where disks that might be counted by a differential cross the dividing line: if the boundary of a disk starts on the right side of the line and crosses it at the ith and jth points, we will use the element ρ_{ij} as a placeholder for the contribution to the boundary of the disk from the left side of the dividing line.

2.2.2 The type A algebra

Let K^A be the left half of a simple Legendrian front diagram divided along some fixed vertical line, and suppose we have a Maslov potential μ assigning an element of the

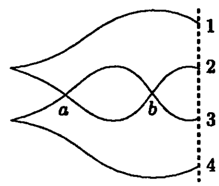


Figure 2.2.1: A half-diagram K^A constructed from the trefoil of Figure 2.1.3.

cyclic group Γ to each strand of K^A.

Definition 2.2.4. The *type A* algebra $A(K^A)$ is the DGA generated freely over $\mathbb{F}$ by the vertices of K^A. Each cusp has grading $|c| = 1$, and if a crossing c has top strand s_1 and bottom strand s_2, then its grading is $|c| = \mu(s_1) - \mu(s_2)$.

We define a differential ∂ on $A(K^A)$ exactly as in the original algebra $Ch(K)$: $\partial c = \sum \partial D$ if c is a crossing and $\partial c = 1 + \sum \partial D$ if c is a cusp, where D ranges over all disks in $\mathrm{Disk}(K^A; c)$.

The differential is clearly well-defined, since for any vertex c of K^A each term ∂D in ∂c is a monomial consisting of vertices to the left of c and these vertices are all in K^A. Furthermore, $\partial^2 = 0$ on $A(K^A)$ since the differential on $Ch(K)$ also satisfies $\partial^2 = 0$ and $A(K^A)$ is a subalgebra of $Ch(K)$.

Although $A(K^A)$ seems fairly uninteresting on its own, if the dividing line intersects it in n points, numbered in order from x_1 at the top to x_n at the bottom, then $A(K^A)$ admits a useful map from I_n^μ. By giving I_n the potential μ, we mean that the potential at x_i should equal the potential of the corresponding strand of K^A.

Definition 2.2.5. Let $\mathrm{Half}_A(K^A; i, j)$ be the set of admissible embedded left half-disks in K^A. These are defined identically to admissible disks, but instead of having a unique rightmost vertex we require the rightmost part of the boundary to be the segment of the dividing line from x_i to x_j. For such a half-disk H, we define the monomial ∂H to be the product of its corners in K^A, traversed in counterclockwise order from x_i to x_j.

We can now define an algebra homomorphism $w : I_n \to A(K^A)$ by the formula

$$w(\rho_{ij}) = \sum_{H \in \mathrm{Half}_A(K^A;i,j)} \partial H.$$

For example, in Figure 2.2.1 the algebra $A(K^A)$ is generated freely by a and b with $\partial a = \partial b = 0$, and we can compute the values of w as follows:

$$\begin{array}{lll} w(\rho_{12}) = ab + 1 & w(\rho_{14}) = 0 & w(\rho_{24}) = a \\ w(\rho_{13}) = a & w(\rho_{23}) = 0 & w(\rho_{34}) = ba + 1 \end{array}$$

Lemma 2.2.6. *The map w preserves gradings, i.e. $|\rho_{ij}| = |w(\rho_{ij})|$.*

Proof. Any half-disk $H \in \mathrm{Half}_A(K^A; i, j)$ has leftmost point at a left cusp y. As we follow the boundary of H from x_i to y, we change strands in K^A at corners $c_1, c_2, \dots, c_k$, and then while following from y to x_j we change strands at corners $c'_1, c'_2, \dots, c'_l$; by definition $\partial H = c_1 \dots c_k c'_1 \dots c'_l$. Now the difference in potential between x_i and the top strand s_1 at y is $|c_1| + \cdots + |c_k|$, and the difference between the bottom strand s_2 at y and x_j is $|c'_1| + \cdots + |c'_l|$, hence

$$(\mu(x_i) - \mu(s_1)) + (\mu(s_2) - \mu(x_j)) = \sum_l |c_l| + \sum_{l'} |c'_l|.$$

But the left hand side is $\mu(x_i) - \mu(x_j) - 1 = |\rho_{ij}|$ since $\mu(s_1) = \mu(s_2) + 1$, and the right hand side is $|\partial H|$, so we are done. $\square$

Proposition 2.2.7. *The map w is a chain map.*

Proof. We need to check that $w(\partial\rho_{ij}) = \partial w(\rho_{ij})$ for each i, j. Letting $w_{ij} = w(\rho_{ij})$ for convenience, this is the assertion that

$$\partial w_{ij} = \sum_{i<k<j} w_{ik} w_{kj}.$$

The element $w_{ij} \in A(K^A)$ is a sum of monomials corresponding to the boundaries of half-disks H, so the monomials in ∂w_{ij} are precisely those obtained by taking such

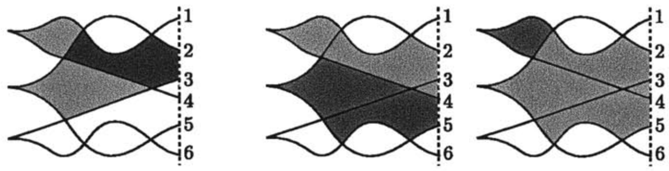


Figure 2.2.2: The region on the left can be broken into disks representing two monomials of $\partial w(\rho_{23})$, by merging the dark region with either light one to get the corresponding monomial of $w(\rho_{23})$; at center and right the region is broken into pieces representing monomials of $w(\rho_{24})w(\rho_{45})$ and $\partial w(\rho_{25})$, respectively.

an H and gluing it to full disks which start at a corner c of ∂H. The boundary of the resulting region R goes from x_i to a left cusp, back to a vertex c' where R occupies three of the four adjacent quadrants, to another left cusp, and then right to x_j and back to x_i along the dividing line; the associated monomial in ∂w_{ij} is the product of all corners of the disk except for c'.

The region R can be naturally split into a union of two admissible disks or half-disks in two ways (see Figure 2.2.2): follow either of the strands of ∂R which intersect at c' as far right as possible until they intersect ∂R again. If such a path does not end on the dividing line, this splitting contributes the related monomial to ∂w_{ij}; otherwise it ends at some point x_k strictly between x_i and x_j and so it contributes that monomial to the product $w_{ik}w_{kj}$. Therefore the monomials in the sum $\partial w_{ij} + \sum w_{ik}w_{kj}$ can be paired together as the possible splittings of these regions R, and since the two monomials in each pair are equal, the sum must be zero. $\square$

Since w is a chain map which preserves degree, it is an actual morphism $I_n \to A(K^A)$ in the category of DGAs.

2.2.3 The type D algebra

Let K^D be the right half of a simple Legendrian front diagram divided along a vertical line, with Maslov potential μ. Let I_n^μ be the algebra associated to the points on the intersection of K^D and the dividing line, again numbered from x_1 at the top to x_n at the bottom.

Definition 2.2.8. The set $\mathrm{Half}_D(K^D; c)$ consists of all admissible right half-disks

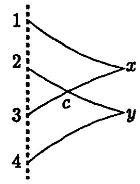


Figure 2.2.3: A half-diagram K^D constructed from the trefoil of Figure 2.1.3.

H embedded in K^D with rightmost vertex c. These are defined in the same way as admissible disks, but instead of having a unique leftmost point at a left cusp, we require the leftmost part of the boundary to be a segment of the dividing line between some points x_i and x_j. We define the word ∂H to be the product of the following in order: the corners between c and x_i on the boundary of the disk; the element $\rho_{ij} \in I_n$; and then the corners between x_j and c.

Note that the set $\mathrm{Disk}(K^D; c)$ can be defined just as in the original Chekanov-Eliashberg algebra, so in particular the left cusp of a disk $D \in \mathrm{Disk}(K^D; c)$ must lie in the half-diagram K^D.

Definition 2.2.9. The *type D* algebra $D(K^D)$ is the DGA generated freely over $\mathbb{F}$ by the vertices of K^D and the generators ρ_{ij} of I_n. The cusps have grading 1 and the crossings have grading $|c| = \mu(s_1) - \mu(s_2)$, where s_1 and s_2 are the top and bottom strands through c, and the elements ρ_{ij} have grading $\mu(x_i) - \mu(x_j) - 1$ just as in I_n.

If c and c' are a crossing and cusp of K^D, respectively, then the differential on $D(K^D)$ is given by the formulas

$$\begin{aligned}
\partial c &= \sum_{D \in \mathrm{Disk}(K^D;c)} \partial D + \sum_{H \in \mathrm{Half}_D(K^D;c)} \partial H \\
\partial c' &= 1 + \sum_{D \in \mathrm{Disk}(K^D;c')} \partial D + \sum_{H \in \mathrm{Half}_D(K^D;c')} \partial H \\
\partial \rho_{ij} &= \sum_{i<k<j} \rho_{ik}\rho_{kj}.
\end{aligned}$$

Example 2.2.10. For K^D the half diagram of Figure 2.2.3, the algebra $D(K^D)$ has generators x, y, c as well as the generators ρ_{ij}, $1 \le i < j \le 4$, of I_4. The differential on the vertices is given by

$$\begin{aligned}\partial x &= 1+\rho_{12}c+\rho_{13}\\ \partial y &= 1+\rho_{24}+c\rho_{34}\\ \partial c &= \rho_{23}.\end{aligned}$$

Proposition 2.2.11. *The differential on $D(K^D)$ has degree -1, and $\partial^2=0$.*

Proof. To show that $\deg(\partial)=-1$ we only need to check that $|\partial H|=|v|-1$ for any $H\in \mathrm{Half}_D(K^D;v)$, since it is already true for full disks $D\in \mathrm{Disk}(K^D;v)$ as in the case of $Ch(K)$. Traversing the boundary of H in counterclockwise order from v, we pass through a series of corners $c_1,\dots,c_k$; a segment connecting two points x_a and x_b on the dividing line; and then some more corners $c'_1,\dots,c'_l$ on the way back to v. Since the turn at each corner c_i lowers the potential by $|c_i|$, the potential at the top strand s_1 through v satisfies $\mu(s_1)-\mu(x_a)=\sum|c_i|$, and likewise $\mu(x_b)-\mu(s_2)=\sum|c'_j|$ where s_2 is the bottom strand. Therefore

$$\begin{aligned}|c|=\mu(s_1)-\mu(s_2) &= \mu(x_a)-\mu(x_b)+\sum_{i=1}^{k}|c_i|+\sum_{j=1}^{l}|c'_j|\\ &= \sum_{i=1}^{k}|c_i|+|\rho_{ab}|+\sum_{j=1}^{l}|c'_l|+1,\end{aligned}$$

and since $\partial H=c_1\dots c_k\rho_{ab}c'_1\dots c'_l$ we have $|c|-1=|\partial H|$ as desired.

To prove that $\partial^2=0$, we proceed as in the proof of Proposition 2.2.7. For a fixed vertex v, each monomial in $\partial^2 v$ can correspond to a region R with right cusp at v and two left cusps, but now we need to consider the possibility that these cusps might lie across the dividing line; in other words, we may only see the algebra elements ρ_{ij}. If the special vertex c' between the left cusps where R occupies three of four quadrants appears to the right of the dividing line, then we map split R in two different ways just as before, by extending either strand through c' until it hits ∂R again.

The only remaining case is that of a region where the special vertex c' may be to the left of the dividing line, so that if K^D were completed to a front diagram, then c'

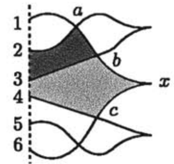


Figure 2.2.4: A region appearing in the proof that $\partial^2 = 0$ for $D(K^D)$.

would be part of the left half K^A. In this case $R \subset K^D$ is actually a half-disk which intersects the dividing line at some points x_i and x_j. For any k satisfying $i < k < j$, the strand through x_k must intersect ∂R somewhere; otherwise, following it would lead us to a right cusp in the interior of R, contradicting the assumption that K^D is simple. Then this strand together with ∂R divides R into a union of two half-disks, one half-disk H with right cusp at v whose monomial ∂H appears as a term of ∂R, and one half-disk H' with right cusp at some corner v' of ∂H. The associated monomial of $\partial^2 v$ is obtained by replacing the generator v' in ∂H with the monomial $\partial H'$, resulting in the monomial ∂R with $\rho_{ik}\rho_{kj}$ in place of ρ_{ij} since each of ρ_{ik} and ρ_{kj} appear in exactly one of ∂H and $\partial H'$. But this is also the monomial which we get from $\partial(\partial R)$ by differentiating the ρ_{ij} term and picking out the $\rho_{ik}\rho_{kj}$ term of $\partial \rho_{ij}$, so these monomials appear in pairs and their sum must be zero.

For example, Figure 2.2.4 shows such a region whose associated monomial is $a\rho_{23}\rho_{34}c$ and which appears twice in $\partial^2 x$: once from the term $\partial(b\rho_{34}c)$ using the monomial $a\rho_{23}$ of ∂b, and once from the term $\partial(a\rho_{24}c)$ using the monomial $\rho_{23}\rho_{34}$ of $\partial\rho_{24}$. $\square$

Unlike the algebra $A(K^A)$, this algebra "remembers" the interaction of disks with the boundary as part of its differential, so its differential structure is necessarily more complicated. On the other hand, the inclusion $I_n \hookrightarrow D(K^D)$ is trivially a chain map of degree 0, since the differential on elements ρ_{ij} is identical in both algebras.

2.2.4 The van Kampen theorem

Let K be a Legendrian front diagram split into a left half K^A and a right half K^D by a vertical dividing line which intersects the front in n points, and suppose we have a Maslov potential μ associated to this front. Then it is easy to see that we have a commutative diagram of algebras

$$\begin{tikzcd} I_n \arrow[r] \arrow[d, "w"'] & D(K^D) \arrow[d, "w'"] \\ A(K^A) \arrow[r] & Ch(K) \end{tikzcd} \tag{2.2.1}$$

where $I_n \to D(K^D)$ and $A(K^A) \to Ch(K)$ are inclusion maps and $w : I_n \to A(K^A)$ is the map defined in section 2.2.2, and the map $w' : D(K^D) \to Ch(K)$ sends vertices to themselves and elements ρ_{ij} to $w(\rho_{ij}) \in A(K^A) \subset Ch(K)$.

Lemma 2.2.12. *The map $w' : D(K^D) \to Ch(K)$ is a chain map of degree zero, and so the diagram above is a commutative diagram of DGAs.*

Proof. Clearly w' preserves the degrees of vertices of K^D, and it does the same for generators ρ_{ij} by Lemma 2.2.6, so w' has degree zero.

For a generator $\rho_{ij} \in D(K^D)$ we have $\partial(w'(\rho_{ij})) = \partial(w(\rho_{ij})) = w(\partial\rho_{ij}) = w'(\partial\rho_{ij})$ since w is a chain map. If instead we consider a vertex $v \in D(K^D)$, then (letting ϵ be 0 if v is a crossing and 1 if v is a cusp)

$$\begin{aligned} \partial(w'(v)) &= \epsilon + \sum_{D \in \mathrm{Disk}(K;v)} \partial D \\ &= \epsilon + \sum_{D \in \mathrm{Disk}(K^D;v)} \partial D + \sum_{i<j} \sum_{\substack{D \in \mathrm{Disk}(K;v) \\ x_i, x_j \in \partial D}} \partial D. \end{aligned}$$

The disks D with $x_i, x_j \in \partial D$ can all be obtained by gluing together a half-disk $H \in \mathrm{Half}_D(K^D; v)$ and another half-disk $H' \in \mathrm{Half}_A(K^A; i, j)$, and all such gluings give admissible disks, so

$$\begin{aligned}
\sum_{\substack{D\in\mathrm{Disk}(K;v)\\ x_i,x_j\in\partial D}} \partial D &= \sum_{\substack{H\in\mathrm{Half}_D(K^D;v)\\ x_i,x_j\in\partial H}} \sum_{H'\in\mathrm{Half}_A(K^A;i,j)} \partial(H\cup H') \\
&= \sum_{\substack{H\in\mathrm{Half}_D(K^D;v)\\ x_i,x_j\in\partial H}} \partial H|_{\rho_{ij}=w'(\rho_{ij})} \\
&= \sum_{\substack{H\in\mathrm{Half}_D(K^D;v)\\ x_i,x_j\in\partial H}} w'(\partial H)
\end{aligned}$$

where the notation in the second line means that we have replaced the unique instance of ρ_{ij} in the monomial ∂H with the expression $w'(\rho_{ij})$. But now

$$\partial(w'(v)) = \epsilon + \sum_{D\in\mathrm{Disk}(K^D;v)} w'(\partial D) + \sum_{i<j} \sum_{\substack{H\in\mathrm{Half}_D(K^D;v)\\ x_i,x_j\in\partial H}} w'(\partial H) = w'(\partial v)$$

and so w' is a chain map as desired. $\square$

Definition 2.2.13. Let $A \xrightarrow{f} B$ and $A \xrightarrow{g} C$ be morphisms in some category. Suppose that there is an object D together with morphisms $B \xrightarrow{h} D$ and $C \xrightarrow{i} D$ such that $h \circ f = i \circ g$. Then (D, h, i) is said to be the *pushout* of f and g if it satisfies the following universal property: for every commutative diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \arrow[ddr] & \\
C \arrow[r, "i"] \arrow[drr] & D & \\
 & & X
\end{tikzcd}$$

there exists a unique morphism $D \to X$ making the diagram commute.

Now that we have expended considerable effort to construct the commutative diagram (2.2.1), the following theorem is an easy consequence.

Theorem 2.2.14. *This diagram is a pushout square in the category of DGAs.*

Proof. Suppose we have another commutative diagram of DGAs as follows:

$$\begin{tikzcd}
I_n \arrow[r] \arrow[d, "w"'] & D(K^D) \arrow[d, "w'"] \arrow[ddr, "g"] & \\
A(K^A) \arrow[r] \arrow[drr, "f"'] & Ch(K) \arrow[dr, dashed, "\varphi"] & \\
& & X
\end{tikzcd}$$

Then it is easy to construct the dotted morphism $\varphi : Ch(K) \to X$. The algebra $Ch(K)$ is generated by vertices of the knot diagram K; if a vertex v is on the left side of the dividing line, then it is in the diagram K^A and we let $\varphi(v) = f(v)$, and otherwise it is in K^D and we let $\varphi(v) = g(v)$. This is clearly well-defined and makes the diagram commute, so if it is a chain map (i.e. a morphism of DGAs) then $Ch(K)$ has the universal property of a pushout.

For $v \in A(K^A) \subset Ch(K)$ we have $\partial(\varphi(v)) = \partial(f(v)) = f(\partial v) = \varphi(\partial v)$, since $v \in A(K^A)$ implies that $\partial v \in A(K^A) \subset Ch(K)$ as well.. On the other hand, for $v \in Ch(K)$ coming from the K^D side of the diagram and v_D the corresponding generator of $D(K^D)$ we have $\partial(\varphi(v)) = \partial(\varphi \circ w'(v_D)) = \partial(g(v_D)) = g(\partial v_D)$ since g is a chain map, and then $g(\partial v_D) = \varphi(w'(\partial v_D)) = \varphi(\partial(w'(v_D))) = \varphi(\partial v)$ since w' is also a chain map by Lemma 2.2.12 and so $\partial(\varphi(v)) = \varphi(\partial v)$ in this case as well. Therefore φ is a chain map, as desired. $\square$

Remark 2.2.15. The result $Ch(K) = A(K^A) \coprod_{I_n} D(K^D)$ of Theorem 2.2.14 is a noncommutative analogue of the pairing theorem $CP^-(\mathcal{H}) \cong CPA^-(\mathcal{H}^A) \otimes_{\mathcal{A}_{N,k}} CPD^-(\mathcal{H}^D)$ of [40]; even the construction of $D(K^D)$ as an algebra of the form $I_n \coprod \mathbb{F}\langle v_i \rangle$, where the v_i are the vertices of K^D, can be compared to the definition $CPD^-(\mathcal{H}^D) = \mathcal{A}_{N,k} \otimes_{I_{N,k}} \mathbb{A}\langle \mathfrak{S}(\mathcal{H}^D) \rangle$. Theorem 2.2.14 originated as an attempt to adapt the pairing theorem for CP^- to the Chekanov-Eliashberg algebra, since both $Ch(K)$ and the non-invariant CP^- are defined in terms of embedded disks in the plane rather than in the torus of combinatorial knot Floer homology.

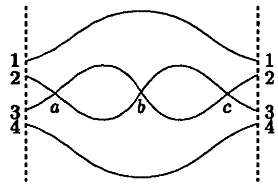


Figure 2.2.5: A Legendrian front diagram with two dividing lines.

2.2.5 Type DA algebras and the generalized van Kampen theorem

Suppose we want to divide a simple Legendrian front into multiple pieces along vertical lines, as in the bordered front K of Figure 2.2.5. We can associate a so-called DGA of *type DA* to K generalizing both the type A and type D algebras, and the analogue of the pairing theorem will follow with minimal effort.

Definition 2.2.16. The algebra $DA(K)$ is the DGA generated freely over $\mathbb{F}$ by the vertices of K and the generators of the algebra I_n corresponding to the left dividing line. The grading and differential on $DA(K)$ are defined exactly as in the type D algebra.

In Figure 2.2.5, for example, $DA(K)$ is generated by a, b, c and the elements $\rho_{ij} \in I_4$ with $1 \leq i < j \leq 4$. The differential is given by $\partial a = \rho_{23}$, $\partial b = \partial c = 0$, and $\partial \rho_{ij} = \sum_{i<k<j} \rho_{ik}\rho_{kj}$.

Lemma 2.2.17. *The differential on $DA(K)$ has degree -1 and satisfies $\partial^2 = 0$.*

Proof. We repeat the proof of Proposition 2.2.11 word for word, replacing $D(K^D)$ with $DA(K)$ as needed. □

Let I'_m be the algebra corresponding to the right dividing line, with generators denoted ρ'_{ij}. Then we can define the set of half-disks $\mathrm{Half}_{DA}(K; i, j)$ almost as in Definition 2.2.5: the right boundary of a half-disk H should still be the segment between points x'_i and x'_j on the right dividing line, but now the left boundary is allowed to be a segment on the left dividing line connecting some points x_k and x_l, in which case the monomial ∂H contains the generator ρ_{kl} in the appropriate place.

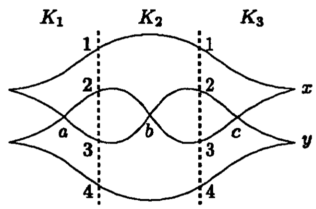


Figure 2.2.6: A trefoil diagram split into three regions by a pair of dividing lines.

Definition 2.2.18. Define an algebra homomorphism $w : I'_m \to DA(K)$ by the formula

$$w(\rho'_{ij}) = \sum_{H \in \mathrm{Half}_{DA}(K;i,j)} \partial H.$$

For example, the map $w : I'_4 \to DA(K)$ in Figure 2.2.5 is given by:

$$\begin{aligned} w(\rho'_{12}) &= \rho_{12}(abc + a + c) + \rho_{13}(bc+1) & w(\rho'_{13}) &= \rho_{12}(ab+1) + \rho_{13}b \\ w(\rho'_{14}) &= \rho_{14} & w(\rho'_{23}) &= 0 \\ w(\rho'_{34}) &= (cb+1)\rho_{24} + (cba + c + a)\rho_{34} & w(\rho'_{24}) &= b\rho_{24} + (ba+1)\rho_{34} \end{aligned}$$

Proposition 2.2.19. *The map $w : I'_m \to DA(K)$ is a morphism of DGAs.*

Proof. See the proofs of Lemma 2.2.6 and Proposition 2.2.7, with some minor changes as in the proof of Proposition 2.2.11 to account for the differentials of each ρ_{kl} that might appear in $w(\rho'_{ij})$. □

The type DA algebra generalizes both the type D algebra, by incorporating the algebra I_n of the left dividing line into the DGA structure, and the type A algebra, by admitting an analogous chain map from I'_m for the right dividing line. In fact, both the type A and type D algebras are special cases of this, with $n = 0$ and $m = 0$ respectively.

We can use this more general structure to relate overlapping pieces of a simple Legendrian front. Consider three regions K_1, K_2, and K_3 of a simple front as in Figure 2.2.6, and let K_{12}, K_{23} and K_{123} denote the larger regions $K_1 \cup K_2$, $K_2 \cup K_3$, and $K_1 \cup K_2 \cup K_3$ respectively. Then the map $w : DA(K_2) \to DA(K_{12})$ which preserves the vertices of K_2 and sends ρ_{ij} to the appropriate element $w(\rho_{ij}) \in DA(K_1) \subset DA(K_{12})$ is a chain map, as are the inclusion $DA(K_2) \hookrightarrow DA(K_{23})$ and the map

$w' : DA(K_{23}) \to DA(K_{123})$. The proofs of these facts proceed exactly as expected, as does the following theorem.

Theorem 2.2.20. *The commutative diagram*

$$\begin{tikzcd} DA(K_2) \arrow[r] \arrow[d, "w"'] & DA(K_{23}) \arrow[d, "w'"] \\ DA(K_{12}) \arrow[r] & DA(K_{123}) \end{tikzcd}$$

is a pushout square in the category of DGAs.

In the special case where K_2 is a product cobordism, so both dividing lines have the same number of points and each strand in K_2 connects x_i to x'_i without any crossings or cusps, then the inclusion $I_n \hookrightarrow DA(K_2)$ of the left dividing line of K_2 is an isomorphism and so is the chain map $w : I'_n \to DA(K_2)$ coming from the right dividing line (i.e. $w(\rho'_{ij}) = \rho_{ij}$). If furthermore the regions K_1 and K_3 have no left and right dividing lines, respectively, so that $DA(K_1) = A(K_1)$ and $DA(K_3) = D(K_3)$, then $DA(K_{123}) = Ch(K)$ and Theorem 2.2.20 reduces to the statement of Theorem 2.2.14.

2.3 Augmentations

Since it can be hard to distinguish between Legendrian knots given only a presentation of their algebras, Chekanov introduced the notion of linearization.

Definition 2.3.1. An *augmentation* of a DGA is a morphism $\epsilon : A \to \mathbb{F}$, where $\mathbb{F}$ is concentrated in degree 0 and has vanishing differential. In particular we require $\epsilon \circ \partial = 0$, $\epsilon(1) = 1$, and $\epsilon(x) = 0$ for any element x of pure nonzero degree.

Given an augmentation ϵ of the algebra A freely generated by a finite set of elements $\{v_i\}$, the differential on A turns the $\mathbb{F}$-vector space $A^\epsilon = \ker(\epsilon)/(\ker(\epsilon))^2$ with basis $\{v_i - \epsilon(v_i)\}$ into a chain complex. We can then compute the associated Poincaré polynomial $P_\epsilon(t) = \sum_{\lambda\in\Gamma} \dim(H_\lambda(A^\epsilon))t^\lambda$.

Theorem 2.3.2 ([4, Theorem 5.2]). *The set of* Chekanov polynomials $\{P_\epsilon(t) \mid \epsilon$ *an augmentation of* $Ch(K)\}$ *is invariant under stable tame isomorphisms of* $Ch(K)$ *and is therefore a Legendrian isotopy invariant.*

It is possible for $Ch(K)$ to have multiple augmentations giving the same polynomial $P_\epsilon(t)$. Melvin and Shrestha [44] constructed prime Legendrian knots with arbitrarily many Chekanov polynomials and also showed that every Laurent polynomial of the form $P(t) = t + p(t) + p(t^{-1})$ (i.e. those satisfying Sabloff's duality theorem [58]), with $p(t)$ a polynomial with positive integer coefficients, is a Chekanov polynomial of some knot. On the other hand, not every Legendrian knot even admits a single augmentation; the existence of augmentations is known to be equivalent to the existence of a normal ruling [23, 24, 57], which implies, for example, that K must have rotation number 0 [57, Theorem 1.3].

2.3.1 A Mayer-Vietoris sequence for linearized homology

Suppose that the simple Legendrian front K is divided by a vertical line into left and right halves K^A and K^D. By Theorem 2.2.14, an augmentation ϵ of $Ch(K)$ is equivalent to a commutative diagram

$$\begin{tikzcd} I_n \arrow[r] \arrow[d, "w"'] & D(K^D) \arrow[d, "\epsilon_D"] \\ A(K^A) \arrow[r, "\epsilon_A"] & \mathbb{F} \end{tikzcd}$$

of DGAs, in which case ϵ_A and ϵ_D both factor through $Ch(K)$. We can associate a Mayer-Vietoris sequence to the associated linearizations, denoted I^ϵ, A^ϵ, D^ϵ, and Ch^ϵ.

Theorem 2.3.3. *There is a long exact sequence*

$$\cdots \to H_k(I^\epsilon) \to H_k(A^\epsilon) \oplus H_k(D^\epsilon) \to H_k(Ch^\epsilon) \to H_{k-1}(I^\epsilon) \to \dots$$

of linearized homology groups.

Proof. It suffices to show that the sequence

$$0 \to I^\epsilon \xrightarrow{f} A^\epsilon \oplus D^\epsilon \xrightarrow{g} Ch^\epsilon \to 0$$

of chain complexes is exact, where $f(x) = (-w(x), x)$ and $g(x, y) = x + w'(y)$. Here we abuse notation and let w and w' refer to the linearized maps $I^\epsilon \to A^\epsilon$ and $D^\epsilon \to Ch^\epsilon$ induced by $w : I_n \to A(K^A)$ and $w' : D(K^D) \to Ch(K)$.

Clearly f is injective, since $I^\epsilon \to D^\epsilon$ is an inclusion map, and g is surjective since any generator $v - \epsilon(v)$ of Ch^ϵ is the image of either $(v - \epsilon(v), 0)$ or $(0, v - \epsilon(v))$ depending on whether v is a vertex in K^A or K^D.

To see that $\mathrm{im}(f) \subset \ker(g)$, or equivalently that $g \circ f = 0$, consider a generator $\rho_{ij} - \epsilon(\rho_{ij})$ of I^ϵ. We can compute $f(\rho) = (-w(\rho_{ij}) + \epsilon(\rho_{ij}), \rho_{ij} - \epsilon(\rho_{ij}))$, and so

$$g \circ f(\rho_{ij} - \epsilon(\rho_{ij})) = (-w(\rho_{ij}) + \epsilon(\rho_{ij})) + (w'(\rho_{ij}) - \epsilon(\rho_{ij})) = 0.$$

Conversely, given $(x, y) \in \ker(g)$, we have $x + w'(y) = 0$ in Ch^ϵ. If we write y as a sum of generators $\rho_{ij} - \epsilon(\rho_{ij})$ and $v_k - \epsilon(v_k)$ of D^ϵ, where the v_k are vertices of K^D, then y must not include any of the latter terms since they cannot be eliminated by any x in the subcomplex $A^\epsilon \subset Ch^\epsilon$. But then $y \in D^\epsilon$ is the image of some element $\rho \in I^\epsilon$ under the inclusion $I^\epsilon \hookrightarrow D^\epsilon$, and since $x + w'(y) = 0$ we have $x = -w(\rho)$, hence $(x, y) = f(\rho) \subset \mathrm{im}(f)$. ☐

Remark 2.3.4. Given a pushout $DA(K_{123}) = DA(K_{12}) \coprod_{DA(K_2)} DA(K_{23})$ as in Theorem 2.2.20 and compatible augmentations ϵ_{12} and ϵ_{23}, we get another augmentation $\epsilon : DA(K_{123}) \to \mathbb{F}$; then an identical argument gives an analogous long exact sequence

$$\cdots \to H_k(K_2^\epsilon) \to H_k(K_{12}^\epsilon) \oplus H_k(K_{23}^\epsilon) \to H_k(K_{123}^\epsilon) \to H_{k-1}(K_2^\epsilon) \to \cdots.$$

2.3.2 Connected sums

In some simple cases we can use the long exact sequence to explicitly work out the linearizations of some type A and type D algebras, reproving a result about the

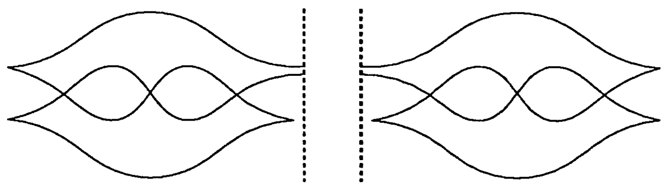


Figure 2.3.1: Diagrams K^A and K^D created by removing a single right or left cusp from a front K.

homology of connected sums which appeared in [4, 44].

Example 2.3.5. Let K^A be the left half of a diagram constructed by removing a single right cusp x from K as in the left side of Figure 2.3.1. Let ρ be the generator of I_2; then the map $w : I_2 \to A(K^A)$ sends ρ to $\partial x - 1$. An augmentation ϵ of $Ch(K)$ then immediately gives an augmentation ϵ_A of $A(K^A)$, and since $\epsilon(\partial x) = 0$ we must have $\epsilon_A(w(\rho)) = 1$.

The algebra D corresponding to the right half of K is generated by the right cusp x and the generator $\rho \in I_2$, with $|x| = 1$, $|\rho| = 0$, $\partial x = \rho + 1$, and $\partial \rho = 0$. An augmentation ϵ_D of D must satisfy $\epsilon_D(\rho) = 1$ since $\epsilon_D(\partial x) = 0$, so the linearization D^ϵ is generated by x and $\rho + 1$ with $\partial x = \rho + 1$ and thus its homology is zero. On the other hand, the corresponding augmentation of I_2 has homology $\langle \rho + 1 \rangle \cong \mathbb{F}$ in degree zero. Now by Theorem 2.3.3, the exact sequence

$$H_k(I_2^\epsilon) \to H_k(A^\epsilon) \oplus H_k(D^\epsilon) \to H_k(Ch(K)^\epsilon) \to H_{k-1}(I_2^\epsilon)$$

gives an isomorphism $H_k(Ch(K)) \cong H_k(A^\epsilon) \oplus H_k(D^\epsilon) \cong H_k(A^\epsilon)$ when $k \neq 0, 1$. Otherwise we have an exact sequence

$$0 \to H_1(A^\epsilon) \to H_1(Ch(K)^\epsilon) \to \langle \rho + 1 \rangle \xrightarrow{w_*} H_0(A^\epsilon) \to H_0(Ch(K)^\epsilon) \to 0.$$

Sabloff proved in [58, Section 5] that $Ch(K)^\epsilon$ has a fundamental class $[\kappa]$ which is nonzero in $H_1(Ch(K)^\epsilon)$, where $\kappa = \sum_{v \in V} v$ for some subset V of the vertices of K which includes all of the right cusps; in particular $x \in V$. But then $\partial \kappa = 0$ implies

$\sum_{v\in V}\partial v = 0$, and so

$$w(\rho+1) = \partial x = \sum_{v\in V\setminus\{x\}} \partial v = \partial\left(\sum_{v\in V\setminus\{x\}} v\right).$$

The right hand side is well-defined and trivial in $H_0(A^\epsilon)$ since every vertex of $V\setminus\{x\}$ is in K^A, so $w_*[\rho+1] = 0$. But applying this to the exact sequence above gives $H_0(A^\epsilon) \cong H_0(Ch(K)^\epsilon)$ and $H_1(Ch(K)^\epsilon) \cong H_1(A^\epsilon)\oplus\mathbb{F}$, so

$$P^{K^A}_{\epsilon_A}(t) = P^K_\epsilon(t) - t.$$

We note here that removing a right cusp has changed the linearized homology by removing the fundamental class, just as removing a point from a manifold will eliminate its fundamental class. We speculate in general that given a half-diagram, one might be able to define an appropriate notion of compactly supported homology which does not count disks approaching the dividing line, and use this to recover a notion of Poincaré duality analogous to $H_k(M^n) \cong H_c^{n-k}(M^n)$.

Example 2.3.6. Let K^D be constructed by removing a single left cusp from K as in the right side of Figure 2.3.1. Then $D(K^D)$ is generated by the vertices of $Ch(K)$ plus the generator ρ of I_2, and an augmentation ϵ of $Ch(K)$ gives an augmentation of $D(K^D)$ with $\epsilon_D(\rho) = 1$. The algebra A corresponding to the left half of K has no generators, since there is only a left cusp, and the map $w : I_2 \to A$ is given by $w(\rho) = 1$. As in Example 2.3.5, the long exact sequence on homology gives $H_k(Ch(K)^\epsilon) \cong H_k(D^\epsilon)$ for $k \neq 0, 1$, and then we have an exact sequence

$$0 \to H_1(D^\epsilon) \to H_1(Ch(K)^\epsilon) \to \langle \rho+1\rangle \xrightarrow{i_*} H_0(D^\epsilon) \to H_0(Ch(K)^\epsilon) \to 0$$

where $i : I_2^\epsilon \to D^\epsilon$ is the inclusion map. Therefore

$$P^{K^D}_{\epsilon_D}(t) = \begin{cases} P^K_\epsilon(t) - t, & i_*[\rho+1] = 0 \\ P^K_\epsilon(t) + 1, & i_*[\rho+1] \neq 0. \end{cases}$$

We conjecture that only the first case occurs, as in Example 2.3.5.

Proposition 2.3.7. *Let ϵ_1 and ϵ_2 be augmentations of knots K_1 and K_2. Then their connected sum $K = K_1 \# K_2$, formed by removing a right cusp from K_1 and a left cusp from K_2 and gluing them together as in Figure 2.3.1, has a canonical augmentation ϵ with Chekanov polynomial $P_\epsilon^K(t) = P_{\epsilon_1}^{K_1}(t) + P_{\epsilon_2}^{K_2}(t) - t$.*

Proof. Removing cusps from K_1 and K_2 as described, and assigning Maslov potentials so that the strands at each removed cusp have matching potentials, gives half diagrams K^A and K^D with augmentations $\epsilon_A : A(K^A) \to \mathbb{F}$ and $\epsilon_D : D(K^D) \to \mathbb{F}$ as in Examples 2.3.5 and 2.3.6. Since these satisfy $\epsilon_A(w(\rho)) = 1$ and $\epsilon_D(\rho) = 1$, they are compatible with the maps $w : I_2 \to A(K^A)$ and $i : I_2 \hookrightarrow D(K^D)$ and thus give an augmentation $\epsilon : Ch(K) \to \mathbb{F}$ by Theorem 2.2.14.

Once again $H_k(I_2^\epsilon)$ vanishes for $k \neq 0$, so $H_k(Ch(K)^\epsilon) \cong H_k(A^\epsilon) \oplus H_k(D^\epsilon)$ for $k \neq 0, 1$, and we have an exact sequence

$$0 \to H_1(A^\epsilon) \oplus H_1(D^\epsilon) \to H_1(Ch^\epsilon) \to \langle \rho + 1 \rangle \xrightarrow{f} H_0(A^\epsilon) \oplus H_0(D^\epsilon) \to H_0(Ch^\epsilon) \to 0.$$

Recalling that $w_* : \langle \rho + 1 \rangle \to H_0(A^\epsilon)$ is zero, this leaves us with two cases depending on the image of the map f:

$$P_\epsilon^K(t) = \begin{cases} (P_{\epsilon_1}^{K_1}(t) - t) + (P_{\epsilon_2}^{K_2}(t) - t) + t, & f([\rho + 1]) = (0, 0) \\ (P_{\epsilon_1}^{K_1}(t) - t) + (P_{\epsilon_2}^{K_2}(t) + 1) - 1, & f([\rho + 1]) = (0, y) \end{cases}$$

where y is some nonzero homology class. In both cases this simplifies to $P_\epsilon^K(t) = P_{\epsilon_1}^{K_1}(t) + P_{\epsilon_2}^{K_2}(t) - t$, as desired. □

2.4 Tangle replacement

Suppose we want to consider the effect of a tangle replacement on the DGA of a front. We can try to isolate the tangle by placing dividing lines on either side, comparing the type DA algebras of the corresponding section of the diagram both before and

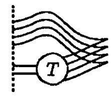
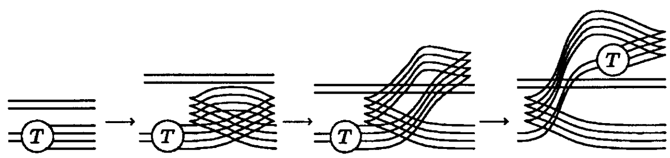


Figure 2.4.1: The half-diagram $\tilde{T}$ associated to a tangle T in Proposition 2.4.1.

after the replacement, and applying Theorem 2.2.20. This is hard in general because in addition to comparing the type DA algebras, we must ensure that the algebras of both dividing lines act compatibly on the type DA algebras.

We can avoid this problem almost completely by applying a trick from [50, Chapter 5]. Given a tangle T in the middle of the diagram, we can perform a series of Legendrian Reidemeister moves to lift it to the top of the diagram and then pull it to the right end of the front by an isotopy:



The effect of the replacement on this new front is often much easier to determine.

Proposition 2.4.1. *Let T_1 and T_2 be Legendrian tangles, and let $\tilde{T}_1$ and $\tilde{T}_2$ be ha/f-diagrams constructed from T_1 and T_2 as in Figure 2.4.1, possibly modified by some Legendrian Reidemeister moves. Then given a morphism $\varphi : D(\tilde{T}_1) \to D(\tilde{T}_2)$ which fixes ρ_{ij} for all i and j, we have a pushout diagram*

$$\begin{tikzcd} D(\tilde{T}_1) \arrow[r, "\varphi"] \arrow[d] & D(\tilde{T}_2) \arrow[d] \\ Ch(K_1) \arrow[r, "\tilde{\varphi}"] & Ch(K_2) \end{tikzcd}$$

where K_1 and K_2 are any fronts which differ only by replacing T_1 with T_2.

Proof. Use the trick mentioned above to modify each front K_i by producing $\tilde{T}_i$ on the right side of the diagram, and place a dividing line in each K_i which separates $\tilde{T}_i$ from some half-diagram K^A on the left; then K^A is independent of i, as is the map

$w : I_{m+n} \to A(K^A)$. Consider the commutative diagram

$$\begin{tikzcd} I_{m+n} \arrow[r] \arrow[d, "w"'] & D(\tilde{T}_1) \arrow[r, "\varphi"] \arrow[d] & D(\tilde{T}_2) \arrow[d] \\ A(K^A) \arrow[r] & Ch(K_1) \arrow[r] & \mathcal{A} \end{tikzcd}$$

where the left square is a pushout by Theorem 2.2.14 and $\mathcal{A}$ is some DGA making the right square a pushout as well. Since pushouts are associative, the outer rectangle of this diagram is a pushout square as well, so $\mathcal{A}$ must be isomorphic to $Ch(K_2)$ and the right square gives the desired diagram. □

Remark 2.4.2. In general, the algebras $Ch(K_1)$ and $Ch(K_2)$ of fronts in which we perform tangle replacements are not identical to the ones for which we have the morphism $\tilde{\varphi}$, since $\tilde{\varphi}$ is constructed from equivalent fronts in which we can isolate the half-diagrams $D(\tilde{T}_i)$, but they are the same up to stable tame isomorphism. Thus in applications we will write $Ch(K)$ to refer to a DGA which is stable tame isomorphic to $Ch(K)$, but this should not cause any confusion.

Corollary 2.4.3. *Let K_1 and K_2 differ by replacing tangle T_1 with T_2, and suppose we have a morphism $\varphi : D(\tilde{T}_1) \to D(\tilde{T}_2)$ as in Proposition 2.4.1. If $Ch(K_2)$ admits an augmentation, then so does $Ch(K_1)$.*

Proof. By Proposition 2.4.1 we have a morphism $\tilde{\varphi} : Ch(K_1) \to Ch(K_2)$, and since an augmentation of $Ch(K_2)$ is just a morphism $\epsilon : Ch(K_2) \to \mathbb{F}$ it follows that $\epsilon \circ \tilde{\varphi}$ is an augmentation of $Ch(K_1)$. □

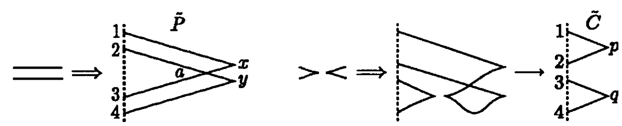
In the following subsections we will give several applications of this result. We will adopt the convention that a double arrow in any figure refers to a tangle replacement or other move that changes the Legendrian knot or tangle in question, whereas a single arrow indicates a Legendrian isotopy.

2.4.1 Breaking a pair of horizontal strands

Consider the effect of the following tangle replacement:

$$= \;\Rightarrow\; >\!<$$

where in both tangles the upper strands have Maslov potential $\mu + 1$ and the lower strands have Maslov potential μ for some μ. We will lakel the left tangle consisting of two parallel strands by P, and the right tangle consisting of two cusps by C. Construct the half-diagrams $\tilde{P}$ and $\tilde{C}$ as follows:



where in both $\tilde{P}$ and $\tilde{C}$, the strands through points $1, 2, 3$, and 4 on the dividing line have potentials $\mu + 2, \mu + 1, \mu + 1$, and μ respectively.

Construct a new DGA D' by adding an extra free generator c to the type D algebra $D(\tilde{P})$ satisfying $\partial c = 1 + \rho_{12}$. Then D' is generated by c, a, x, y, and ρ_{ij} for $1 \leq i < j \leq 4$ satisfying

$$\begin{aligned}
\partial c &= 1 + \rho_{12} \\
\partial a &= \rho_{23} \\
\partial x &= 1 + \rho_{12} a + \rho_{13} \\
\partial y &= 1 + a\rho_{34} + \rho_{24}
\end{aligned}$$

with gradings $|a| = 0$ and $|c| = |x| = |y| = 1$. On the other hand, the algebra $D(\tilde{C})$ is generated by p, q and ρ_{ij} with $\partial p = 1 + \rho_{12}$ and $\partial q = 1 + \rho_{34}$, and $|p| = |q| = 1$. (In both algebras we have $|\rho_{14}| = 1$, $|\rho_{23}| = -1$, and $|\rho_{ij}| = 0$ for all other ρ_{ij}.)

Lemma 2.4.4. *The algebra D' is stable tame isomorphic to $D(\tilde{C})$ by isomorphisms fixing all of the generators ρ_{ij}.*

Proof. Apply a sequence of tame isomorphisms to D' of the form

$$\begin{aligned}
a &\longrightarrow a + c\rho_{23} + \rho_{13} + 1, \\
x &\longrightarrow x + c(a + c\rho_{23} + \rho_{13} + 1), \\
y &\longrightarrow y + c\rho_{24} + \rho_{14} + x\rho_{34};
\end{aligned}$$

we can now easily compute that $\partial a = 0$, $\partial x = a$, and $\partial y = 1 + \rho_{34}$. Relabeling c and y by p and q, respectively, and destabilizing to remove the generators x and a sends D' to $D(\tilde{C})$, as desired. □

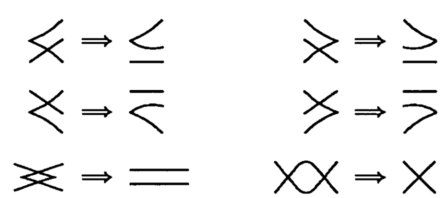
Theorem 2.4.5. *Let K' be the front obtained from a Legendrian front K by replacing the tangle P with C. Then $Ch(K)$ and $Ch(K')$ are stable tame isomorphic to DGAs $\mathcal{A}$ and $\mathcal{A}'$, where $\mathcal{A}'$ is obtained from $\mathcal{A}$ by adding a single free generator c in grading 1. Thus if $Ch(K')$ admits an augmentation. then so does $Ch(K)$.*

Proof. We have constructed an inclusion $D(\tilde{P}) \hookrightarrow D' \cong D(\tilde{U})$, so Proposition 2.4.1 gives us the induced map $Ch(K) \to Ch(K')$. □

Remark 2.4.6. Once we have the morphism $D(\tilde{P}) \hookrightarrow D(\tilde{U})$, we could just construct the map $Ch(K) \to Ch(K')$ directly by using the same sequence of tame isomorphisms and destabilizations, but replacing each ρ_{ij} with $w(\rho_{ij}) \in A(K^A) \subset Ch(K)$.

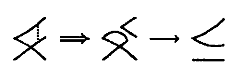
We can use this to draw similar conclusions about other tangle replacements as well. For example:

Corollary 2.4.7. *Let K' be obtained from K by any of the following tangle replacements, where the crossings removed by each replacement have grading* 0*:*

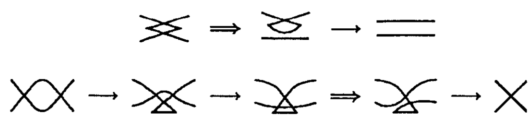


Then there are DGA maps $Ch(K) \to Ch(K')$, constructed exactly as in Theorem 2.4.5, and if $Ch(K')$ has an augmentation then so does $Ch(K)$.

Proof. We prove the first of these by applying Theorem 2.4.5 to the tangle in a small neighborhood of the dotted line, and then performing a type I Reidemeister move:

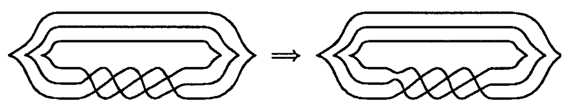


The proofs of the next three cases are identical, and the last two are proven as follows:



where in each case we use one of the first four tangle replacements together with some Legendrian Reidemeister moves. □

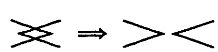
Example 2.4.8. Let K be the Legendrian closure of a positive braid in the sense of [33], so that every crossing has grading 0. If the top strand of the braid is not part of any crossing, then it belongs to a Legendrian unknot (i.e. a topological unknot with $tb = -1$ and $r = 0$) disjoint from the rest of the front and we can remove this unknot. Otherwise we can resolve the leftmost crossing on this strand:



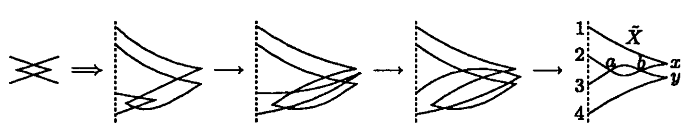
We can repeat this procedure, resolving crossings and removing unlinked Legendrian unknots, until K has become a disjoint union of such unknots. Such a front always admits an augmentation, so by Corollary 2.4.7 we see that K has an augmentation. (The set of augmentations of K was described by Kálmán [33], who also described an analogous "Seifert ruling" of K.)

2.4.2 Unhooking a clasp

Let X and C be tangles consisting of a pair of interlocking left and right cusps and a pair of disjoint left and right cusps, respectively, and consider the effect of replacing X with C in a front:



We have already computed the DGA $D(\tilde{C})$ in the previous section: it is generated freely by p, q, and ρ_{ij} with $\partial p = 1 + \rho_{12}$, $\partial q = 1 + \rho_{34}$, and $|p| = |q| = 1$. On the other hand, $\tilde{X}$ is constructed as follows:



and so $D(\tilde{X})$ is generated by x, y, a, b, and ρ_{ij} satisfying

$$\begin{aligned}\partial x &= 1+\rho_{12}(ab+1)+\rho_{13}b\\ \partial y &= 1+(ba+1)\rho_{34}+b\rho_{24}\\ \partial a &= \rho_{23}\\ \partial b &= 0.\end{aligned}$$

Let D'' be the DGA constructed by adding free generators c and d to $D(\tilde{X})$ satisfying $\partial c = b$ and $\partial d = a + \rho_{13} + (x + (\rho_{12}a + \rho_{13})c)\rho_{23}$.

Lemma 2.4.9. *The DGA D'' is stable tame isomorphic to $D(\tilde{C})$.*

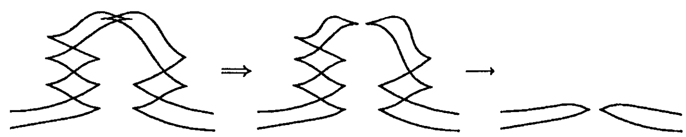
Proof. We start by applying the sequence of tame isomorphisms

$$\begin{aligned}x &\to x + (\rho_{12}a + \rho_{13})c,\\ y &\to y + c(a\rho_{34} + \rho_{24}),\\ a &\to a + \rho_{13} + x\rho_{23},\end{aligned}$$

to D''; now $\partial x = 1+\rho_{12}$, $\partial y = 1+\rho_{34}$, $\partial d = a$, and $\partial a = 0$. Next, we destabilize twice to remove the pairs of generators (d, a) and (c, b), and relabel x and y by p and q. We now have the DGA generated by p, q, and ρ_{ij} with $\partial p = 1 + \rho_{12}$ and $\partial q = 1 + \rho_{34}$, which is precisely $D(\tilde{C})$. $\square$

Proposition 2.4.10. *If K' is obtained from K by replacing the tangle X with the tangle C, then $Ch(K)$ and $Ch(K')$ are stable tame isomorphic to algebras $\mathcal{A}$ and $\mathcal{A}'$, where $\mathcal{A}'$ is obtained from $\mathcal{A}$ by adding two free generators. If $Ch(K')$ admits an augmentation, then so does $Ch(K)$.*

Example 2.4.11. Given the Legendrian Whitehead double $K_{\mathrm{dbl}}(k, l)$ of a front K as defined by Fuchs [23], we can unhook the clasp and perform $k + l$ type I Reidemeister moves to remove the extra twists from the remaining knot:



The resulting knot is a Legendrian unknot, which admits an augmentation, so by Proposition 2.4.10 we recover Fuchs's result that $K_{\mathrm{dbl}}(k,l)$ does as well for all $k,l \geq 0$.

Proposition 2.4.12. *Suppose that K has rotation number r. Then $W(K) = K_{\mathrm{dbl}}(0,0)$ admits an augmentation with Chekanov polynomial $t + t^{2r} + t^{-2r}$.*

Proof. Let U denote the Legendrian unknot. It is easy to check that if we unhook the clasp as in Example 2.4.11, then the DGA stable tame isomorphic to $Ch(U)$ is obtained from $Ch(W(K))$ by adding two extra generators c and d in degrees $1 \pm 2r$ since the corresponding crossings a and b have gradings $\pm 2r$. Any augmentation ϵ' of $Ch(U)$ gives an augmentation ϵ of $Ch(W(K))$ by composition with ι, and ϵ' must have Chekanov polynomial $P_{\epsilon'}(t) = t$. Furthermore, the inclusion ι induces an inclusion on the linearizations $A^{W(K),\epsilon} \hookrightarrow A^{U,\epsilon'}$ whose cokernel is the chain complex $C = \mathbb{F}(c + \epsilon(c)) \oplus \mathbb{F}(d + \epsilon(d))$. Note that the differential on C must be trivial since $|c| - |d|$ is even, and so $H_*(C) \cong \mathbb{F}_{1-2r} \oplus \mathbb{F}_{1+2r}$ where the subscripts denote degrees.

The short exact sequence of chain complexes

$$0 \to A^{W(K),\epsilon} \to A^{U,\epsilon'} \to C \to 0$$

gives a long exact sequence in homology, so for example the sequence

$$H_{i+1}(A^{U,\epsilon'}) \to H_{i+1}(C) \to H_i(A^{W(K),\epsilon}) \to H_i(A^{U,\epsilon'})$$

is exact, and thus when $i \neq 0, 1$ we have $H_i(A^{W(K),\epsilon}) \cong H_{i+1}(C)$. In particular, if $i \notin \{0, 1, \pm 2r\}$ it follows that $H_i(A^{W(K),\epsilon}) = 0$; and if $r \neq 0$ then $H_{\pm 2r}(A^{W(K),\epsilon}) \cong H_{1 \pm 2r}(C) \cong \mathbb{F}$. We also get an exact sequence

$$0 \to H_1(A^{W(K),\epsilon}) \to \mathbb{F} \to H_1(C) \to H_0(A^{W(K),\epsilon}) \to 0$$

since $H_2(C) \cong H_0(A^{U,\epsilon'}) \cong 0$ and $H_1(A^{U,\epsilon'}) \cong \mathbb{F}$.

By considering $W(K)$ as the closure of a long Legendrian knot, [4, Theorem 12.4] shows that the homology group $H_1(A^{W(K),\epsilon})$ must be nontrivial. Thus the injection $H_1(A^{W(K),\epsilon}) \to \mathbb{F}$ in the last exact sequence is an isomorphism, hence the map $H_1(C) \to H_0(A^{W(K),\epsilon})$ must be an isomorphism as well. But $H_1(C)$ is zero if $r \neq 0$ and $\mathbb{F}^2$ otherwise, so this determines $H_0(A^{W(K),\epsilon})$ and our computation of $H_*(A^{W(K),\epsilon})$ is complete; in particular, its Poincaré polynomial is $t + t^{2r} + t^{-2r}$, as desired. $\square$

Proposition 2.4.13. *Suppose that $r = r(K)$ is nonzero. Then every augmentation of $W(K)$ has Chekanov polynomial $t + t^{2r} + t^{-2r}$.*

Proof. Let $W(K)$ be divided into left half W^A and right half $\tilde{X}$. Then the elements of $D(\tilde{X})$ have gradings $|x| = |y| = 1$; $|a|, |b| = \pm 2r$; $|\rho_{12}| = |\rho_{34}| = 0$; $|\rho_{13}| = |\rho_{24}| = |a|$; and $|\rho_{23}| = |a| - 1$ and $|\rho_{14}| = |a| + 1$. Since ρ_{12} and ρ_{34} are the only generators in grading 0, all others must be in $\ker(\epsilon_{\tilde{X}})$ for any augmentation $\epsilon_{\tilde{X}}$; and then from $\epsilon_{\tilde{X}}(\partial x) = \epsilon_{\tilde{X}}(\partial y) = 0$ we get $\epsilon_{\tilde{X}}(\rho_{12}) = \epsilon_{\tilde{X}}(\rho_{34}) = 1$.

If we replace $\tilde{X}$ with $\tilde{C}$, so that we have a Legendrian unknot U divided into W^A and $\tilde{C}$, the Maslov potential of each strand remains unchanged, so $|\rho_{ij}|$ can still only be nonzero for ρ_{12} and ρ_{34}, and then $\epsilon_{\tilde{C}}(\partial p) = \epsilon_{\tilde{C}}(\partial q) = 0$ forces $\epsilon_{\tilde{C}}(\rho_{12}) = \epsilon_{\tilde{C}}(\rho_{34}) = 1$ as well. In particular, both $D(\tilde{X})$ and $D(\tilde{C})$ have a unique augmentation, and these take the same values on the elements ρ_{ij}, so an augmentation of $A(W^A)$ extends to an augmentation of $W(K)$ iff it extends to an augmentation of U. Thus every augmentation ϵ of $Ch(W(K))$ is the pullback of one on $Ch(U)$: construct $\epsilon' : Ch(U) \to \mathbb{F}$ by setting $\epsilon'(v) = \epsilon(v)$ for every vertex v of W^A and $\epsilon'(v) = 0$ on the vertices of $\tilde{C}$, and then ϵ is exactly the composition $Ch(W(K)) \hookrightarrow Ch(U) \xrightarrow{\epsilon'} \mathbb{F}$. But we showed in the proof of Proposition 2.4.12 that such an augmentation must have Chekanov polynomial $t + t^{2r} + t^{-2r}$, and so $W(K)$ cannot have any other Chekanov polynomials. $\square$

On the other hand, when $r(K) = 0$, we can ask the following question.

Question 2.4.14. *Suppose K is a Legendrian knot with $r(K) = 0$ which is not a topological unknot. Does the Whitehead double $W(K)$ of K has a Chekanov polynomial* *other than $t + 2$?*

In particular, this has been checked using a program written in Sage [62] for all but two of the fronts in Melvin and Shrestha's table [44], which includes one *tb*-maximizing front for each knot up through 9 crossings and their mirrors. The answer is yes for every front that admits an augmentation except the Legendrian unknot, and no for every front that does not except for $m(9_{42})$. (The unknown cases are $m(8_5)$ and $m(9_{30})$, neither of which admits an augmentation.) We note, however, that both representatives of $m(10_{140})$ in Section 3.1.2.2 admit augmentations with Chekanov polynomial t and yet the answer is no for both.

In the case of $m(9_{42})$, which is discussed further in Section 3.1.2.1, we note that the Kauffman bound on *tb* is not tight; equivalently, this knot does not admit an ungraded augmentation [56]. This is the only such knot up to nine crossings for which a *tb*-maximizing representative has $r = 0$ and the Kauffman bound on *tb* is not tight (see [51]), so we speculate that these phenomena are related. (The other knot which does not achieve the Kauffman bound is the $(4, -3)$ torus knot $m(8_{19})$, for which $\overline{tb} = -12$ and so r must be odd.) On the other hand, the Whitehead doubles of the $m(10_{132})$ representatives with $tb = -1$ and $r = 0$ in Section 3.2 have no Chekanov polynomials other than $t+2$ even though they do not admit ungraded augmentations.

2.5 Augmentations of Whitehead doubles

In this section we prove the following result, which answers Question 2.4.14 for Legendrian knots K with augmentations satisfying $P_\epsilon(t) \neq t$:

Theorem 2.5.1. *Let K be a front with rotation number 0, and suppose that K has an augmentation ϵ with Chekanov polynomial $P_\epsilon(t) = t + \sum a_i t^i$. Then its Legendrian Whitehead double $W(K)$ has an augmentation ϵ' with $P_{\epsilon'}(t) = t+2+(t+2+t^{-1})\sum a_i t^i$.*

As a sample application, we have a new proof of the following result of Melvin and Shrestha [44]:

Corollary 2.5.2. *There are prime Legendrian knots with arbitrarily many Chekanov polynomials.*

Proof. Let K_0 be any Legendrian knot with rotation number 0, and for $n \geq 1$ let K_n be the Legendrian Whitehead double of K_{n-1}; then each K_n is prime because Whitehead doubles have genus 1. We claim that K_n has at least n distinct Chekanov polynomials.

If we define a sequence of Laurent polynomials $p_1(t) = t+2$, $p_2(t) = 3t+6+2t^{-1}$, and so on by the formula

$$p_n(t) = t + 2 + (t + 2 + t^{-1})(p_{n-1}(t) - t),$$

then we can explicitly solve for $p_n(t)$ as

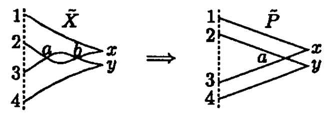
$$p_n(t) = \frac{2(t+2+t^{-1})^n + t^2 + t - 1}{t+1+t^{-1}} = t + 2\sum_{k=1}^{n} \binom{n}{k} (t+1+t^{-1})^{k-1}$$

and so the p_n are all distinct. But for any $n \geq 1$, the polynomials $p_1(t), \dots, p_n(t)$ are all Chekanov polynomials of K_n: p_1 is for all n by Proposition 2.4.12, and if $p_1, \dots, p_{i-1}$ are Chekanov polynomials of K_{i-1} then Theorem 2.5.1 guarantees that $p_2, \dots, p_i$ are Chekanov polynomials of K_i, so the claim follows by induction. $\square$

Remark 2.5.3. Since $p_n(1) = \frac{1}{3}(2 \cdot 4^n + 1)$, the ranks of the corresponding linearized homologies are all distinct as well.

If instead we take $r(K_0) \neq 0$ then $p_1, \dots, p_{n-1}$ are Chekanov polynomials of K_n by applying this argument to $K_0' = W(K_0) = K_1$ and $K_{n-1}' = K_n$, and since K_1 has Chekanov polynomial $t + t^{2r} + t^{-2r}$ we get an nth Chekanov polynomial of degree $2r+n-1$ and rank $\frac{1}{3}(2 \cdot 4^n + 1)$ for K_n by applying the same recurrence to $t + t^{2r} + t^{-2r}$ a total of $n-1$ times. Thus any n-fold iterated Legendrian Whitehead double has at least n distinct Chekanov polynomials.

The Whitehead double differs from the 2-copy, defined in [45], by a single crossing, or more precisely by replacing the tangle $\tilde{X}$ from the previous section with the tangle $\tilde{P}$:



Thus we will start by analyzing the closely related 2-copy $C(K)$, where we have fixed gradings so that for any two parallel strands of $C(K)$, the top strand has Maslov potential one greater than that of the bottom strand. (In particular, since $r(K) = 0$ we have $|a| = |b| = 0$ in $\tilde{X}$, so we may assume without loss of generality that the potentials of the strands points $1, 2, 3, 4$ on the dividing line are $1, 0, 0, -1$ in both tangles.) We will split a linearization of $C(K)$ into four subcomplexes, compute the homology of three and a half of these, and use this information to recover the linearized homology of $W(K)$.

It follows from Corollary 2.4.7 that $Ch(C(K))$ is stable tame isomorphic to a DGA obtained by adding a free generator g to $Ch(W(K))$ in grading 1. One can check that it suffices to let $\partial g = b + 1$, but we do not need this fact.

We will assume for the rest of this section that K is a fixed Legendrian knot with $r(K) = 0$ and augmentation $\epsilon : Ch(K) \to \mathbb{F}$, and that we have a simple front for K.

2.5.1 Constructing augmentations of $C(K)$ and $W(K)$

For each crossing c of K, the 2-copy $C(K)$ has four corresponding crossings, which we will label c_N, c_E, c_S, and c_W in clockwise order from the top. It is easy to check that the gradings of these crossings are $|c|$, $|c|+1$, $|c|$, and $|c|-1$ respectively. Let $K_1, K_2 \subset C(K)$ denote the upper and lower (in the z-direction) copies of K, respectively, so that both strands through c_N belong to K_1, both strands through c_S belong to K_2, and c_E and c_W involve strands from both K_1 and K_2.

Proposition 2.5.4. *Define an algebra homomorphism $\epsilon' : Ch(C(K)) \to \mathbb{F}$ as in Figure 2.5.1 by $\epsilon'(c_N) = \epsilon'(c_S) = 1$ whenever $\epsilon(c) = 1$, and $\epsilon'(v) = 0$ for all other vertices of v. Then ϵ' is an augmentation of $Ch(C(K))$.*

Proof. Since ϵ' is a "proper" augmentation in the sense of [45], meaning that $\epsilon'(v) = 0$ whenever the strands through v belong to different components of $C(K)$, this is a

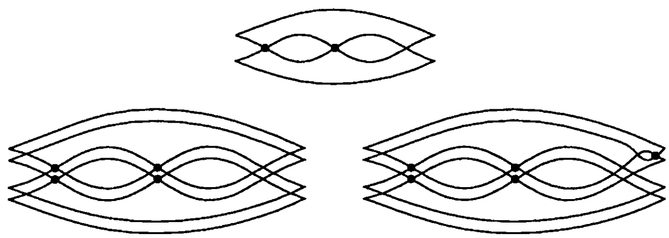


Figure 2.5.1: An augmentation ϵ of K and the corresponding augmentations ϵ' of the 2-copy and Whitehead double. The vertices v with $\epsilon(v) = 1$ are indicated by black dots.

special case of [45, Proposition 3.3c]. More generally, any augmentations ϵ_1 and ϵ_2 of K_1 and K_2 uniquely determine a proper augmentation of $C(K)$. □

Let $\mathcal{A} \cong Ch(C(K))$ be the DGA constructed by adding a generator g to $Ch(W(K))$ as in Corollary 2.4.7. The inclusion $Ch(W(K)) \hookrightarrow \mathcal{A}$ induces an augmentation of $Ch(W(K))$ which we will also call ϵ'; one can show that it satisfies $\epsilon'(a) = 0$ and $\epsilon'(b) = 1$, and is defined identically to the augmentation of $Ch(C(K))$ on all other vertices.

The inclusion $Ch(W(K)) \hookrightarrow \mathcal{A}$ induces a map on the linearized complexes which we can extend to a short exact sequence as before,

$$0 \to A^{W(K),\epsilon'} \to A^{C(K),\epsilon'} \to \mathbb{F}_1 \to 0,$$

where we are using $A^{C(K),\epsilon'}$ to mean $\mathcal{A}$ since they have the same homology, and the cokernel $\mathbb{F}_1$ is generated in degree 1 by g. The corresponding long exact sequence in homology tells us that $H_i(A^{W(K),\epsilon'}) \cong H_i(A^{C(K),\epsilon'})$ for $i \neq 0, 1$, and in particular for all $i < 0$.

2.5.2 The linearized homology of $C(K)$

Mishachev showed in [45] that the DGA of $C(K)$ splits as $Ch(C(K)) = \bigoplus_{i \in \mathbb{Z}} \mathcal{A}_i$, where a vertex v is in $\mathcal{A}_{-1}$ if the top and bottom strands through v are in K_2 and K_1 respectively, in $\mathcal{A}_1$ if the top and bottom strands are in K_1 and K_2 respectively, and in $\mathcal{A}_0$ if both strands through v belong to the same component; and if $v \in \mathcal{A}_i$ and

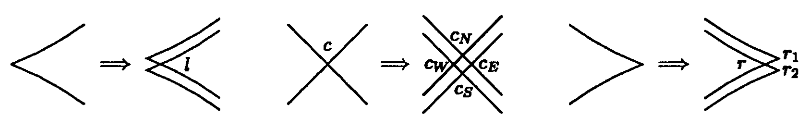
$v' \in \mathcal{A}_{i'}$, then $vv' \in \mathcal{A}_{i+i'}$. This splitting extends to the linearization with respect to the augmentation ϵ', but in fact we can split the linearized complex even more:

Proposition 2.5.5 ([52]). *There is a splitting*

$$A^{C(K),\epsilon'} \cong \bigoplus_{d\in\{N,E,S,W\}} A_d^{C(K),\epsilon'}$$

where the N, E, S, and W subcomplexes are generated by the vertices whose top and bottom strands belong to components (K_1, K_1), (K_1, K_2), (K_2, K_2), and (K_2, K_1), respectively.

The cusps and crossings of K determine several types of vertices of $C(K)$:



In this picture, the crossings l, r, r_1, and r_2 belong to the E, W, N, and S subcomplexes, respectively, and each c_d belongs to $A_d^{C(K),\epsilon'}$.

Lemma 2.5.6. *There is an isomorphism $H_i(A_W^{C(K),\epsilon'}) \cong H_{i+1}(A^{K,\epsilon})$ for all $i \in \mathbb{Z}$.*

Proof. The subcomplex $A_W^{C(K),\epsilon'}$ is generated by crossings c_W corresponding to crossings c of K, as well as crossings r adjacent to pairs of right cusps, so the generators of the ith graded component of $A_W^{C(K),\epsilon'}$ are in bijection with the generators of $(A^{K,\epsilon})_{i+1}$. Let v_w and v denote a generator of $A_W^{C(K),\epsilon'}$ and the corresponding generator of $A^{K,\epsilon}$, respectively.

No disk D contributing to $\partial^{\epsilon'} v_W$ can have more than one augmented corner, so in particular as we travel along ∂D we cannot switch between components of $C(K)$ more than once. Since ∂D leaves v_W along K_2 and returns along K_1 when traveling counterclockwise, it must switch at some crossing c'_W, which is then the unique unaugmented corner of D and so D contributes c'_W to $\partial^{\epsilon'} v_W$.

Now consider the linearized differential $\partial^{\epsilon} v \in A^{K,\epsilon}$. Each disk D' with initial vertex v and a single unaugmented corner c' corresponds to a unique disk D for v_W

with corner c'_W as described above. On the other hand, if every corner c'_j of D' is augmented then D' contributes $\sum c'_j$ to $\partial^\epsilon v$. In this case D' corresponds to one disk D_j which contributes to $\partial^{\epsilon'} v_W$ for each corner c'_j: this is the disk D_j with augmented corners $(c'_k)_S$ for all $k < j$, then an unaugmented corner at $(c'_j)_W$, and then augmented corners $(c'_k)_N$ for all $k > j$, hence D_j contributes $(c'_j)_W$ to $\partial^{\epsilon'} v_W$ for each j and the total contribution is $\sum (c'_j)_W$. Finally, if v is a right cusp, then ∂v contains an extra 1, which does not appear in ∂v_W, but this does not contribute to the linearization $\partial^\epsilon v$.

We conclude that if $\partial^\epsilon v = \sum c'_j$ then $\partial^{\epsilon'} v_W = \sum (c'_j)_W$ for all v, and the desired isomorphism follows immediately. □

Lemma 2.5.7. *Both $H_*(A_N^{C(K),\epsilon'})$ and $H_*(A_S^{C(K),\epsilon'})$ are isomorphic to $H_*(A^{K,\epsilon})$.*

Proof. See [52, Section 2.5], in particular the discussion after Definition 2.20. □

The only remaining subcomplex is $A_E^{C(K),\epsilon'}$. This complex is more complicated than the others, but it is still accessible in negative degree:

Lemma 2.5.8. *There is an isomorphism $H_i(A_E^{C(K),\epsilon'}) \cong H_{i-1}(A^{K,\epsilon})$ for all $i < 0$.*

Proof. The complex $A_E^{C(K),\epsilon'}$ is generated by crossings c_E, with $|c_E| = |c| + 1$, as well as the crossings l_i in between each pair of left cusps, satisfying $|l_i| = 0$ and $\partial l_i = 0$. For each $i < 0$, we have an isomorphism of graded components $(A_E^{C(K),\epsilon'})_i \cong (A^{K,\epsilon})_{i-1}$ matching each c_E to c, since the complexes differ only by the generators l_i in grading 0 and the right cusps of K in grading 1. The differentials are identical under this identification just as before, except we do not have to consider disks in K with all corners augmented since this can only happen for $v \in (A^{K,\epsilon})_1$. Furthermore, the image of $\partial^{\epsilon'} : (A_E^{C(K),\epsilon'})_0 \to (A_E^{C(K),\epsilon'})_{-1}$ is identical to that of $\partial^\epsilon : (A^{K,\epsilon})_{-1} \to (A^{K,\epsilon})_{-2}$ since the extra generators l_i do not contribute to $\mathrm{im}(\partial^{\epsilon'})$. Thus we have an isomorphism $H_i(A_E^{C(K),\epsilon'}) \cong H_{i-1}(A^{K,\epsilon})$ for all $i < 0$. □

Proof of Theorem 2.5.1. We have now computed $H_i(A^{W(K),\epsilon'})$ for all $i < 0$: namely, it is isomorphic to $H_i(A^{C(K),\epsilon'})$, and then the splitting of $A^{C(K),\epsilon'}$ gives an isomorphism

$$H_i(A^{W(K),\epsilon'}) \cong H_{i+1}(A^{K,\epsilon}) \oplus (H_i(A^{K,\epsilon}))^{\oplus 2} \oplus H_{i-1}(A^{K,\epsilon}).$$

If $P_\epsilon^K(t)$ and $P_{\epsilon'}^{W(K)}(t)$ are the Chekanov polynomials of ϵ and ϵ', then, it follows that $P_{\epsilon'}^{W(K)}(t) = (t + 2 + t^{-1})P_\epsilon^K(t) + f(t)$ for some actual polynomial $f \in \mathbb{Z}[t]$, since the coefficient of t^i on either side is the rank of the corresponding ith homology group for $i < 0$. By Poincaré duality [58] we can write $P_\epsilon^K(t) = t + \sum a_i t^i$ and $P_{\epsilon'}^{W(K)}(t) = t + \sum b_i t^i$, where $a_i = a_{-i}$ and $b_i = b_{-i}$ for all i; then

$$t + \sum b_i t^i = (t^2 + 2t + 1) + \sum (a_{i+1} + 2a_i + a_{i-1})t^i + f(t)$$

or

$$t^2 + t + 1 + f(t) = \sum (a_{i+1} + 2a_i + a_{i-1} - b_i)t^i.$$

The coefficients c_i on the right hand side are symmetric, i.e. they satisfy $c_i = c_{-i}$, so the left hand side must be symmetric as well, and since it is a polynomial rather than a Laurent series we must have $f(t) = n - t^2 - t$ for some $n \in \mathbb{Z}$. Therefore

$$P_{\epsilon'}^{W(K)}(t) = t + (n+1) + (t + 2 + t^{-1}) \sum a_i t^i.$$

In order to determine n, we note that $P_{\epsilon'}^{W(K)}(-1) = tb(W(K)) = 1$, and substituting $t = -1$ into the above equation leaves $n = 1$. We conclude that

$$P_{\epsilon'}^{W(K)}(t) = t + 2 + (t + 2 + t^{-1}) \sum a_i t^i,$$

as desired. $\square$

2.6 Knots which bound Lagrangian disks

Recall the $P \to C$ tangle replacement from Section 2.4.1:

$$= \;\Rightarrow\; ><$$

This tangle replacement can be realized by a Lagrangian cobordism in the symplectization of $\mathbb{R}^3$ (see [3]), and if n such replacements turn K into a Legendrian unlink U_{n+1} of $n + 1$ components, each of which is a Legendrian unknot, then we can cap each unknot with a Lagrangian disk to get a Lagrangian surface L bounded by K. It is then easy to see that L must be a disk, hence by work of Ekholm [8] we expect that L induces an augmentation ϵ of K with Chekanov polynomial t.

Proposition 2.6.1. *Suppose that a series of n tangle replacements of the form $P \to C$ turn K into a $n+1$-component unlink U_{n+1}. Then the morphism $Ch(K) \xrightarrow{\varphi} Ch(U_{n+1})$ from Theorem 2.4.5 induces an augmentation ϵ of K with $P_\epsilon^K(t) = t$.*

Proof. Up to stable tame isomorphism, the algebra $Ch(U_{n+1})$ has $n+1$ generators $x_1, \ldots, x_{n+1}$, each in grading 1 and with $\partial x_i = 0$, with unique augmentation $\epsilon_{U_{n+1}}(x_i) = 0$. The homology of this augmentation has rank $n+1$ in degree 1, and we take $\epsilon : Ch(K) \to \mathbb{F}$ to be the composition $\epsilon_{U_{n+1}} \circ \varphi$.

Since φ is equivalent to the inclusion of $Ch(K)$ into the algebra obtained by adding n free generators to $Ch(K)$, each in grading 1, we have a short exact sequence of linearized complexes

$$0 \to Ch^{K,\epsilon} \to Ch^{U_{n+1},\epsilon_{U_{n+1}}} \to \mathrm{coker}(\varphi^\epsilon) \to 0.$$

The homology of $\mathrm{coker}(\varphi^\epsilon)$ is $\mathbb{F}^n$, concentrated in degree 1, and so the long exact sequence in homology tells us that $H_i(Ch^{K,\epsilon}) \cong H_i(Ch^{U_{n+1},\epsilon_{U_{n+1}}}) \cong 0$ for $i \neq 0, 1$. Since $H_{-1}(Ch^{K,\epsilon}) \cong 0$ we know by Poincaré duality [58] that $H_1(Ch^{K,\epsilon}) \cong \mathbb{F}$. Furthermore, if $H_0(Ch^{K,\epsilon}) \cong \mathbb{F}^d$ then

$$tb(K) = \chi(H_*(Ch^{K,\epsilon})) = d - 1,$$

and by work of Chantraine [3] we know that $tb(K) = -1$ since K is Lagrangian concordant to the Legendrian unknot, hence $d = 0$. We conclude that $P_\epsilon^K(t) = t$, as desired. $\square$

There are seven known Legendrian representatives of slice knots K with at most 12 crossings such that $tb = -1$: they have topological types $m(9_{46})$, $m(10_{140})$, $m(10_{140})$, $11n_{139}$, $m(12n_{582})$, $m(12n_{768})$, and $m(12n_{838})$. The candidate topological types were determined using Knotinfo [2] (the only other possibility is $m(12_{145})$, for

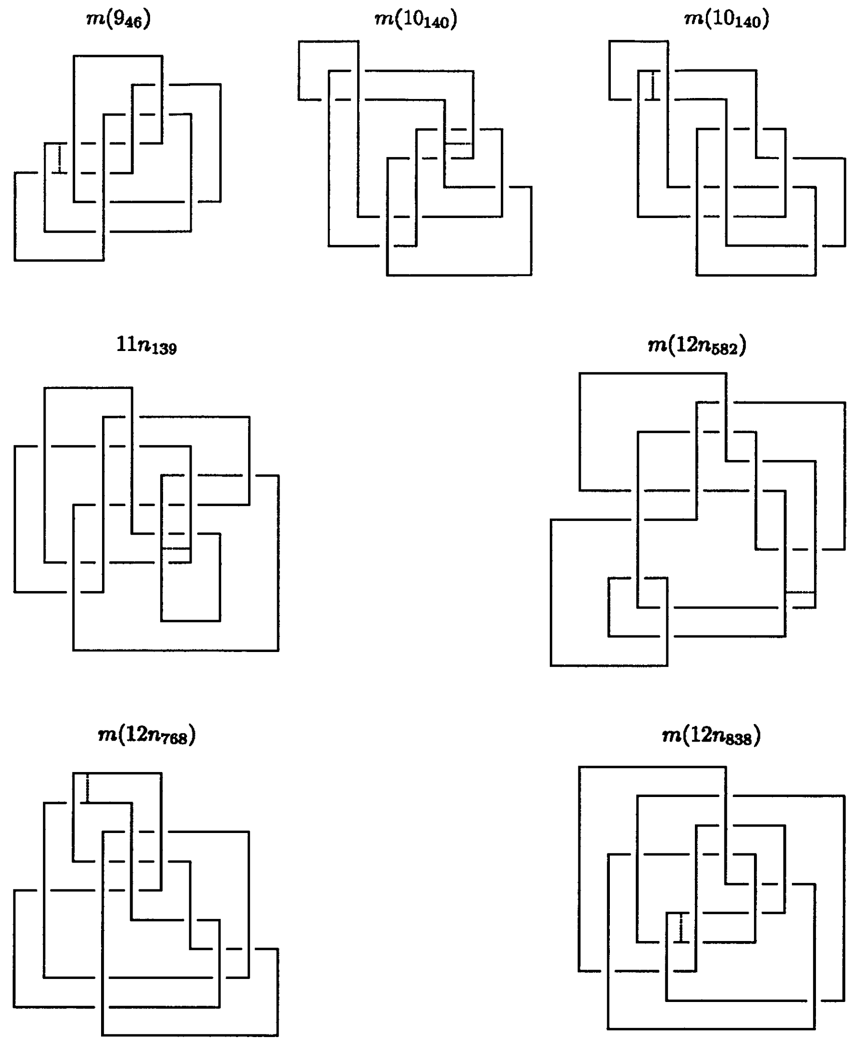


Figure 2.6.1: Seven Legendrian knots which bound Lagrangian slice disks.

which $-2 \leq \overline{tb} \leq -1$), and the representatives were determined using a combination of the Legendrian knot atlas [5] and Gridlink [7].

For each of the seven we can find a single $P \to C$ replacement resulting in a 2-component Legendrian unlink, hence they bound Lagrangian slice disks and the augmentations resulting from the morphism of Theorem 2.4.5 behave as expected. These knots are shown in Figure 2.6.1, with dashed lines indicating where the P tangle should be broken.

2.7 The characteristic algebra

2.7.1 The van Kampen theorem for the characteristic algebra

Ng [52] introduced the characteristic algebra of a Legendrian knot as an effective way to distinguish knots using the Chekanov-Eliashberg algebra when the Chekanov polynomials could not.

Definition 2.7.1. Let A be a DGA, and let $I \subset A$ be the two-sided ideal generated by the image of ∂. The *characteristic algebra* of A is the quotient $\mathcal{C}(A) = A/I$, with grading inherited from A.

Two characteristic algebras A_1/I_1 and A_2/I_2 are *stable tame isomorphic* if we can add some free generators to one or both algebras to make them tamely isomorphic.

Theorem 2.7.2 ([52, Theorem 3.4]). *The stable tame isomorphism class of the characteristic algebra $\mathcal{C}(Ch(K))$ of a Legendrian knot is a Legendrian isotopy invariant.*

There are some technicalities involved in defining stable tame isomorphisms – in particular, one must consider equivalence relations on the pair (A, I) rather than the quotient A/I – but we will ignore these since we are only concerned with the stable isomorphism class of $\mathcal{C}(Ch(K))$. For example, Ng showed that the isomorphism class (together with the gradings of the generators of $Ch(K)$) is strong enough to recover the first and second order Chekanov polynomials of K, and he conjectured that the Chekanov polynomials of all orders are determined by this information.

We can define $\mathcal{C}$ as a functor from the category of DGAs to the category of graded associative unital algebras: we have already defined it for objects of the category, and given a DGA morphism $f : X \to Y$ we note that the relation $\partial f = f\partial$ implies $f(\partial(X)) = \partial(f(X)) \subset \partial(Y)$, hence f descends to a morphism $\mathcal{C}(f) : \mathcal{C}(X) \to \mathcal{C}(Y)$. It turns out that this functor is well-behaved.

Proposition 2.7.3. *The functor $\mathcal{C}$ preserves pushouts.*

Proof. It suffices to prove that the functor $\mathcal{D} : \mathrm{GA} \to \mathrm{DGA}$ (here GA denotes graded algebras) defined by $\mathcal{D}(X) = (X, \partial_X = 0)$ is a right adjoint to $\mathcal{C}$; then, since $\mathcal{C}$ is a left adjoint it preserves colimits, which include pushouts.

Given a DGA (A, ∂) and a graded algebra X, we need to establish a natural bijection

$$\varphi : \hom_{\mathrm{GA}}(\mathcal{C}(A), X) \to \hom_{\mathrm{DGA}}(A, \mathcal{D}(X)).$$

Letting $\pi : A \to \mathcal{C}(A)$ denote the projection of graded algebras, we can define $\varphi(f) = f \circ \pi : A \to X$ for any $f \in \hom_{\mathrm{GA}}(\mathcal{C}(A), X)$. This is in fact a chain map since $(f \circ \pi) \circ \partial_A = f \circ (\pi \circ \partial_A) = 0 = \partial_X \circ (f \circ \pi)$, so $\varphi(f) \in \hom_{\mathrm{DGA}}(A, \mathcal{D}(X))$, and it is clear that φ is injective. Conversely, given a chain map $\tilde{g} \in \hom_{\mathrm{DGA}}(A, \mathcal{D}(X))$ we must have $\tilde{g}(\partial_A a) = \partial_X(\tilde{g}(a)) = 0$, and since $\tilde{g}$ vanishes on the image of ∂_A it factors through the graded algebra $\mathcal{C}(A)$, hence $\tilde{g} = \varphi(g)$ for some $g \in \hom_{\mathrm{GA}}(\mathcal{C}(A), X)$ and so φ is surjective. Since φ is also clearly natural, we conclude that $\mathcal{C}$ and $\mathcal{D}$ are adjoints, as desired. $\square$

The following version of van Kampen's theorem for characteristic algebras is now an immediate consequence of Theorems 2.2.14 and 2.2.20.

Theorem 2.7.4. *Let K be a simple Legendrian front split by a vertical dividing line into a left half K^A and a right half K^D; or, let K_1, K_2, and K_3 be adjacent regions of a simple front with $K_{12} = K_1 \cup K_2$, $K_{23} = K_2 \cup K_3$, and $K_{123} = K_1 \cup K_2 \cup K_3$. Then the diagrams*

$$\begin{tikzcd} \mathcal{C}(I_n) \arrow[r] \arrow[d, "\mathcal{C}(w)"'] & \mathcal{C}(D(K^D)) \arrow[d, "\mathcal{C}(w')"] \\ \mathcal{C}(A(K^A)) \arrow[r] & \mathcal{C}(Ch(K)) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C}(DA(K_2)) \arrow[r] \arrow[d, "\mathcal{C}(w)"'] & \mathcal{C}(DA(K_{23})) \arrow[d, "\mathcal{C}(w')"] \\ \mathcal{C}(DA(K_{12})) \arrow[r] & \mathcal{C}(DA(K_{123})) \end{tikzcd}$$

are pushout squares in the category of graded algebras.

Remark 2.7.5. Although the DGA morphism $I_n \to D(K^D)$ is an inclusion, this is generally not true of the induced map $\varphi : \mathcal{C}(I_n) \to \mathcal{C}(D(K^D))$. Suppose that K^D has some crossings but no left cusps, and let v be a leftmost crossing of K^D. If strands s_1 and s_2 pass through v, then s_1 and s_2 cannot intersect any other strands between the dividing line and v, so we must have $\partial v = \rho_{i,i+1}$ for some i. But now $\varphi(\rho_{i,i+1}) = 0$, and yet $\rho_{i,i+1} \in \mathcal{C}(I_n)$ cannot be zero since the two-sided ideal $\mathrm{Im}(\partial) \subset I_n$ is generated by homogeneous quadratic terms.

Let $\mathcal{C}'$ denote the composition of $\mathcal{C}$ with the abelianization functor from graded algebras to graded commutative algebras. Since abelianization also preserves pushouts, the abelianized characteristic algebra $\mathcal{C}'(Ch(K))$ satisfies Theorem 2.7.4 as well; in this category, pushouts are tensor products, so for example we can express this as

$$\mathcal{C}'(DA(K_{123})) \cong \mathcal{C}'(DA(K_{12})) \otimes_{\mathcal{C}'(DA(K_2))} \mathcal{C}'(DA(K_{23})).$$

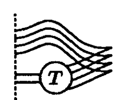
2.7.2 Tangle replacement and the characteristic algebra

The following is Conjecture 3.14 of [52].

Conjecture 2.7.6. *Let $\mathcal{K}$ be any Legendrian representative of the knot K with maximal Thurston-Bennequin number. Then the equivalence class of the ungraded abelianized characteristic algebra $\mathcal{C}'(Ch(\mathcal{K}))$ is a topological invariant of K.*

It is currently unknown whether there is a set of moves relating any pair of topologically equivalent Legendrian links L_1 and L_2 with the same tb [20]. A positive answer could provide a straightforward way to resolve Conjecture 2.7.6:

Proposition 2.7.7. *Let T_1 and T_2 be Legendrian tangles with m strands on the left and n strands on the right, and let $\tilde{T}_1$ and $\tilde{T}_2$ be constructed as in Figure 2.4.1, where to a tangle T we associate the following half-diagram:*



If there is a stable isomorphism $\varphi : \mathcal{C}'(D(\tilde{T}_1)) \to \mathcal{C}'(D(\tilde{T}_2))$ such that $\varphi(\rho_{ij}) = \rho_{ij}$ for all i and j, then replacing T_1 with T_2 (or vice versa) in a front K preserves the stable isomorphism type of $\mathcal{C}'(Ch(K))$.

Proof. Repeat the proof of Proposition 2.4.1, noting that now we are working with pushouts of commutative algebras rather than DGAs. The resulting commutative diagram

$$\begin{tikzcd} C'(D(\tilde{T}_1)) \arrow[r, "\varphi"] \arrow[d] & C'(D(\tilde{T}_2)) \arrow[d] \\ C'(Ch(K_1)) \arrow[r, "\tilde{\varphi}"] & C'(Ch(K_2)) \end{tikzcd}$$

is a pushout square, and since φ is a stable isomorphism, $\tilde{\varphi}$ must be as well. □

Remark 2.7.8. Even when the map φ isn't an isomorphism, we can still get an interesting map relating $C'(Ch(K_1))$ and $C'(Ch(K_2))$. For example, recall that the algebra $D(\tilde{C})$ of Theorem 2.4.5 was obtained by adding an extra free generator c to the algebra $D(\tilde{P})$ with $\partial c = 1 + \rho_{12}$. Thus if K' is obtained from K by the tangle replacement

$$= \;\Rightarrow\; >\!<$$

it is easy to see that $C'(Ch(K')) \cong (C'(Ch(K)) \otimes_{\mathbb{F}} \mathbb{F}[c])/\langle 1+w(\rho_{12})\rangle$, hence $C'(Ch(K'))$ is stably isomorphic to a quotient of $C'(Ch(K))$.

From now on we will abuse notation and write $C'(K) = C'(Ch(K))$, $C'(\tilde{T}) = C'(D(\tilde{T}))$, and so on whenever it is clear from the description of the (partial) front whether we are using the whole Chekanov-Eliashberg algebra or a type A, DA, or D algebra.

2.7.3 S and Z tangles

Figure 2.7.1 gives an example of two tangles, an 'S' tangle and a 'Z' tangle, which can be exchanged while preserving *tb* and the topological knot type [20].

Theorem 2.7.9. *Replacing an S tangle in a front diagram K with a Z tangle, and vice versa, preserves the abelianized characteristic algebra $C'(K)$.*

Proof. By Proposition 2.7.7 we only need to check that $C'(\tilde{S})$ and $C'(\tilde{Z})$ are stably isomorphic for the half diagrams $\tilde{S}$ and $\tilde{Z}$ of Figure 2.7.1.

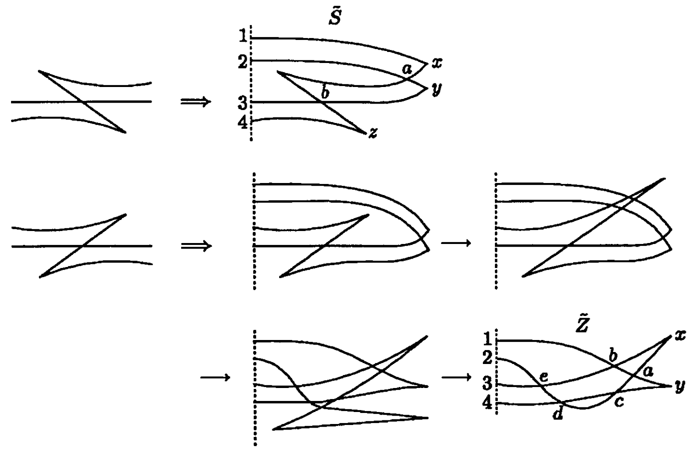


Figure 2.7.1: A pair of tangles which result in the same topological knot and value of tb but which may not preserve the Legendrian isotopy type. On the right we show the half diagrams $\tilde{S}$ and $\tilde{Z}$ of Proposition 2.7.7; note that for one we perform several Legendrian Reidemeister moves to make it simple and then eliminate some vertices.

The algebra $\mathcal{C}(\tilde{S})$ is generated by x, y, z, a, b and ρ_{ij}, modulo the elements $\partial\rho_{ij}$ and

$$\begin{aligned}
\partial x &= 1 + \rho_{12}a \\
\partial y &= 1 + \rho_{23} + ab \\
\partial z &= 1 + b\rho_{34}
\end{aligned}$$

since $\partial a = \partial b = 0$. Using $\partial\rho_{13} = \rho_{12}\rho_{23} = 0$, we see that

$$0 = \rho_{12} + \rho_{12}\rho_{23} + \rho_{12}ab = \rho_{12} + b$$

and so $b = \rho_{12}$. Then from $\partial x = 0$ we get $ba = 1$, and $\partial z = 0$ implies $a = ab\rho_{34} = (1+\rho_{23})\rho_{34} = \rho_{34}$. In particular, $\rho_{12}\rho_{34} = 1$ and the generators a and b are redundant; and then $\rho_{23} = 1 + ab = ba + ab$, which is zero in the abelianization $\mathcal{C}'(\tilde{S})$. It follows that

$$\mathcal{C}'(\tilde{S}) = \mathbb{F}[x, y, z] \otimes_{\mathbb{F}} \left(\mathcal{C}'(I_4)/\langle\rho_{12}\rho_{34} = 1\rangle\right).$$

The algebra $\mathcal{C}'(\tilde{Z})$ is generated by x, y, a, b, c, d, e and ρ_{ij}, modulo the elements

$\partial\rho_{ij}$ and

$$\begin{aligned}
\partial x &= 1+\rho_{34}c \\
\partial y &= 1+\rho_{12}d+\rho_{14}+b\rho_{34} \\
\partial a &= \rho_{12}(1+dc)+\rho_{14}c+b\rho_{34}c \\
\partial b &= \rho_{12}e+\rho_{13} \\
\partial d &= \rho_{24}+e\rho_{34} \\
\partial e &= \rho_{23}
\end{aligned}$$

with $\partial c = 0$. Applying $\rho_{34}c = 1$ to the relation $\partial a = 0$ yields

$$b = \rho_{12}(1+dc)+\rho_{14}c,$$

so the generator b is redundant. Since $\partial a = \rho_{12}+c+(\partial y)c$, we also have $c = \rho_{12}$, so once again $\rho_{12}\rho_{34} = 1$ and then $\rho_{23} = 0$ as before. Now $\partial b = 0$ implies $e = \rho_{12}\rho_{34}e = \rho_{13}\rho_{34}$ and likewise $\partial d = 0$ implies $e = \rho_{12}\rho_{24}$ (which is the same element since $\partial\rho_{14} = 0$), so e is redundant as well. We can conclude that

$$C'(\tilde{Z}) = \mathbb{F}[a,d,x,y]\otimes_{\mathbb{F}}\left(C'(I_4)/\langle\rho_{12}\rho_{34}=1\rangle\right),$$

and this is stably isomorphic to $C'(\tilde{S})$, as desired. □

Legendrian twist knots have been classified by work of Etnyre, Ng, and Vértesi [22] which allows us to verify Conjecture 2.7.6 in this case. They prove that any Legendrian representative of K_m with non-maximal tb can be destabilized, so its characteristic algebra vanishes; up to orientation, there is a unique representative maximizing tb if $m \geq -1$ (where $m = -1$ is the unknot); and for $m \leq -2$, any representative which maximizes tb can be isotoped to a front as in Figure 2.7.2, where the rectangle is filled with $|m+2|$ negative half-twists, each an S tangle or a Z tangle. Since we can replace any Z tangle with an S tangle without changing $C'(K)$, we can conclude:

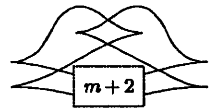


Figure 2.7.2: A front for the Legendrian twist knots K_m, $m \leq -2$.

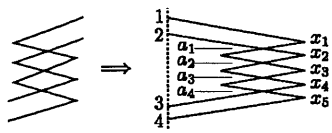


Figure 2.7.3: The $3S$ tangle and the associated partial front $\widetilde{3S}$.

Corollary 2.7.10. *Let $\mathcal{K}$ be a Legendrian representative of the twist knot K_m. Then $C'(\mathcal{K})$ depends only on $tb(\mathcal{K})$ and m.*

In fact, many of these have the same abelianized characteristic algebra. Consider the tangle $3S$ in Figure 2.7.3 obtained by concatenating three S tangles. The crossings a_i of $\widetilde{3S}$ have zero differential, whereas ∂x_i takes the values $1+\rho_{12}a_1$, $1+a_1a_2$, $1+a_2a_3$, $1+a_3a_4$, and $1+a_4\rho_{34}$ for $1 \leq i \leq 5$, so $C'(\widetilde{3S})$ is generated by adjoining the elements a_i and x_i to $C'(I_4)$ together with the relations $\partial x_i = 0$. But these imply

$$\rho_{12} = a_2 = a_4 \text{ and } a_1 = a_3 = \rho_{34}$$

together with $\rho_{12}\rho_{34} = a_1a_2 = 1$, and so

$$C'(\widetilde{3S}) = \mathbb{F}[x_1, \dots, x_5] \otimes_{\mathbb{F}} (C'(I_4)/\langle \rho_{12}\rho_{34} = 1 \rangle).$$

This is stably isomorphic to the algebra $C'(\tilde{S})$ computed in the proof of Theorem 2.7.9, so we may replace three consecutive S tangles with a single one without changing $C'(K)$. This move obviously changes the topological type of K, since it removes a full negative twist, but for example we can conclude that

$$C'(\mathcal{K}_{-3}) \cong C'(\mathcal{K}_{-5}) \cong C'(\mathcal{K}_{-7}) \cong \dots$$

and

$$\mathcal{C}'(\mathcal{K}_{-4}) \cong \mathcal{C}'(\mathcal{K}_{-6}) \cong \mathcal{C}'(\mathcal{K}_{-8}) \cong \dots$$

where $\mathcal{K}_{-n}$ denotes any Legendrian representative of K_{-n} with maximal tb; these are stably isomorphic to $\mathbb{F}[x,y]/\langle (xy+1)^2 = 1\rangle$ and $\mathbb{F}[x,y,z]/\langle (xy+1)z = 1\rangle$, respectively.

Chapter 3

Some computations in Legendrian contact homology

3.1 Legendrian Whitehead doubles

The goal of this section is to investigate the Chekanov-Eliashberg algebra [4, 12] associated to the Legendrian Whitehead double $W(K)$ of an oriented Legendrian knot K. This construction is of potential interest because the algebra $Ch(W(K))$ always admits an augmentation, as in Proposition 2.4.12, and so it may be possible to extract information from $Ch(W(K))$, or even its linearized homology groups, which is not reflected in the algebra of K itself. The algebra $Ch(K)$ can detect nondestabilizability, since it collapses when K is a stabilization, but we hope that $Ch(W(K))$ can be an effective invariant of nondestabilizable knots even when $Ch(K)$ collapses or when $Ch(K_1) \cong Ch(K_2)$.

3.1.1 The double of a stabilized knot

If K is a stabilization of some other Legendrian front, then $Ch(W(K))$ carries very little information. Our main goal in this section is to prove the following theorem:

Theorem 3.1.1. *For any Legendrian knot K which is a stabilization, the DGA of $W(K)$ is determined entirely by the rotation number $r = r(K)$. More precisely,*

$Ch(W(K))$ is equivalent to the DGA generated freely by elements p, q, x, y, a in gradings $-2|r|, 2|r|, 1, 1, 0$ satisfying

$$\partial x = 1 + a(pq + 1) \qquad \partial y = 1 + (qp + 1)a$$

and $\partial p = \partial q = \partial a = 0$.

We may wish to understand some of the structure of $H_*(Ch(W(k)))$. For example, the algebra homomorphism $f : Ch(W(K)) \to \mathbb{F}[p]$ defined by $f(x) = f(y) = f(q) = 0$, $f(p) = p$, and $f(a) = 1$ is a DGA morphism if we give $\mathbb{F}[p]$ the trivial differential, and so it induces a map on homology which sends $[p]$ to the nontrivial class $p \in \mathbb{F}[p]$. Thus p is nonzero in homology, and likewise for q; and a is nonzero as well since it has $pq + 1$ as a right inverse. Now if we let $z = (qp + 1)x + y(pq + 1)$ then we can compute that

$$\begin{aligned} \partial z &= pq + qp \\ \partial(xpa + apy) &= pa + ap \\ \partial((x + az)qa + aq(y + za)) &= aq + qa, \end{aligned}$$

so a, p, and q all commute with each other in homology, and a and $pq+1$ are inverses.

Corollary 3.1.2. *Suppose that either K is a stabilized knot or $r = r(K)$ is nonzero. Then any augmentation ϵ of $Ch(W(K))$ has Poincaré polynomial $P_\epsilon(t) = t + t^{-2r} + t^{2r}$.*

Proof. The case $r(K) \neq 0$ is Proposition 2.4.13, so assume that K is a stabilization. The conditions $\epsilon(\partial x) = \epsilon(\partial y) = 0$ both imply $\epsilon(a) = 1$, and when $r \neq 0$ the augmentation must vanish on p, q, x, y since these all have nonzero grading. Then $A_\epsilon = \ker(\epsilon)/\ker(\epsilon)^2$ is generated by $p, q, x, y, a + 1$; since

$$\partial x = (a + 1) + (a + 1)pq + pq \equiv a + 1 \pmod{\ker(\epsilon)^2},$$

the induced differential ∂^ϵ on A^ϵ satisfies $\partial^\epsilon x = a + 1$, and likewise $\partial^\epsilon y = a + 1$. It follows that $\ker(\partial^\epsilon)$ has basis $\{x + y, p, q, a + 1\}$ and $\mathrm{im}(\partial^\epsilon)$ has basis $\{a + 1\}$, so the linearized homology $H^\epsilon_*(W(K))$ is generated by $x+y$, p, and q in gradings 1 and $\pm 2r$.

For $r=0$ the computation requires the consideration of several cases since we have $\epsilon(pq)=0$ rather than just $\epsilon(p)=\epsilon(q)=0$, but it is easy to check that any choice of ϵ still yields $P_\epsilon(t)=t+2$. □

Remark. The above computation of the Poincaré polynomials for the Whitehead doubles of unknots with $r>0$ and $tb=-(r+1)$ was originally done by Melvin and Shrestha [44], who called them the "twist knots" T_{2r}.

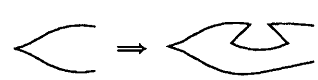
Our main tool in proving Theorem 3.1.1 is the following lemma due to Ng:

Lemma 3.1.3 ([48, Lemma 3.4]). *For any Legendrian knot K which is a stabilization, the DGAs of $W(K)$ and $W(S_+S_-(K))$ are equivalent.*

Eliashberg and Fraser [13] gave a complete classification of Legendrian representatives of the topological unknot: each one is uniquely determined by its classical invariants (tb, r), and any such pair admits a Legendrian representative $U_{tb,r}$ precisely when $tb+|r|$ is odd and negative. In this case we see that $U_{tb-2k,r}=(S_+S_-)^k(U_{tb,r})$ for all $k\geq 0$, and so when $U_{tb,r}$ is a stabilized unknot it follows that $Ch(W(U_{tb,r}))$ is independent of tb. Therefore it will suffice to construct any stabilized unknot U with $Ch(W(U))\cong Ch(W(K))$ and $r(U)=r(K)$, and then prove that $Ch(W(U))$ has the desired form.

3.1.1.1 Reduction to the case of an unknot

Suppose K is a stabilization in plat form, meaning that all left cusps of the front lie on the same vertical line and all right cusps lie on another vertical line. We apply a trick suggested by Lenny Ng and form a new front K' by stabilizing K along the top strand next to each left cusp:



Let $W_K=W(K')$; note that W_K is determined by the front diagram for K, and in particular it is not a Legendrian isotopy invariant of K.

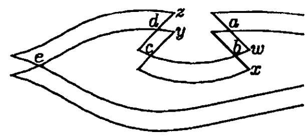


Figure 3.1.1: The front W_K near a stabilized left cusp of K.

By Lemma 3.1.3, the DGA for W_K is equivalent to the one for $W(K)$. Furthermore, it is easy to see that there are no admissible disks in the front K', since any admissible disk for K can no longer reach the intended left cusp in K'. In particular, W_K only admits what Mishachev [45] calls "thin" and "small unit" disks, which lie entirely between parallel strands of W_K coming from the same strand of K, and as a result every term of ∂v for every vertex v of W_K is either 1 (if v is a right cusp) or quadratic in the other vertices of W_K.

We will break our analysis of $Ch(W_K)$ into several parts, using local information to simplify the parts of the DGA corresponding to crossings and then constructing a related front which produces the same DGA.

3.1.1.2 The DGA near a stabilized left cusp

Consider the subalgebra A of $Ch(W_K)$ generated by the vertices near a stabilized left cusp of K, as in Figure 3.1.1. It has generators a, b, c, d, e with grading 0 and trivial differential, as well as w, x, y, z with grading 1 satisfying

$$\partial w = 1 + ab \qquad \partial y = 1 + dc$$
$$\partial x = 1 + bc \qquad \partial z = 1 + ed.$$

Note that most of the generators coming from crossings are equal in homology: for example, we have $a + c = \partial(wc + ax)$ and $c + e = \partial(zc + ey)$, so $[a] = [c] = [e]$, and

$$\begin{aligned} b + d &= \partial(yb + dw) + d(a + c)b \\ &= \partial(yb + dw + d(wc + ax)b) \end{aligned}$$

so $[b] = [d]$ as well.

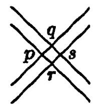


Figure 3.1.2: The front W_K near a crossing of K.

Definition 3.1.4. The vertices a and e are called the *exposed* vertices associated to this left cusp of K, since they are the only ones which can appear in the differential of some distant generator of $Ch(W_K)$.

The following lemma will be useful in simplifying $Ch(W_K)$ at distant vertices whose admissible disks have exposed corners.

Lemma 3.1.5. *If v is an exposed vertex, then there are elements i, j, k, m of A with $|i| = 0$ and $\partial i = 0$ satisfying $\partial j = 1 + vi$, $\partial k = 1 + iv$, and $\partial m = jv + vk$.*

Proof. If $v = a$, then we take $i = b$ and $j = w$ and it remains to find k and m such that $\partial k = 1+ba$ and $\partial m = wa+ak$. But $\partial(x+b(wc+ax)) = (1+bc)+b(a+c) = 1+ba$, so we take $k = x+b(wc+ax)$, and then $\partial(w(wc+ax)) = w(a+c)+(1+ab)(wc+ax) = wa + a(x + b(wc + ax)) = wa + ak$, hence $m = w(wc + ax)$.

If instead $v = e$, then we can take $i = d$ and $j = z$, and then $k = y + d(zc + ey)$ and $m = z(zc + ey)$ satisfy $\partial k = 1 + de$ and $\partial m = ze + ek = jv + vk$ as desired. ☐

3.1.1.3 The DGA near a crossing

Each crossing of K corresponds to four crossings of W_K as in Figure 3.1.2. No admissible disk for any vertex of W_K outside of this local picture can have a corner in this picture, since all admissible disks are thin, so our goal will be to apply tame isomorphisms to p, q, r, s so that we can destabilize $Ch(W_K)$ and remove them all:

Proposition 3.1.6. *The DGA for W_K is stable tame isomorphic to the one which is defined by removing the generators p, q, r, and s from the definition of $Ch(W_K)$.*

Proof. The differential for these crossings is given by

$$\partial p = 0 \qquad \partial r = p\beta$$
$$\partial q = \alpha p \qquad \partial s = \alpha r + q\beta$$

where α and β are exposed vertices attached to some left cusps. By Lemma 3.1.5 we can find i, j, k, m in the subalgebra generated by the vertices near β such that $|i| = 0$, $\partial i = 0$, $\partial j = 1 + \beta i$, $\partial k = 1 + i\beta$, and $\partial m = j\beta + \beta k$. Note that $\partial(ri + pj) = p$ and $\partial(rk + pm) = r + (ri + pj)\beta$.

Applying the sequence of tame isomorphisms

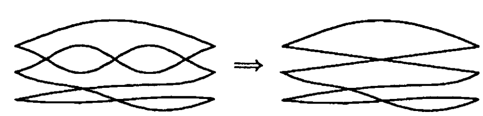
$$\begin{array}{rcl} q & \to & q + (\alpha+1)(ri+pj), \\ s & \to & s + (\alpha+1)(rk+pm), \\ r & \to & r + q\beta \end{array}$$

leaves $\partial q = p$, $\partial s = r$, and $\partial r = 0$. Thus we can destabilize twice and remove the pairs of generators (q, p) and (s, r) from the DGA, as desired. $\square$

3.1.1.4 Rearranging cusps

Proposition 3.1.7. *Let K be a stabilized Legendrian front. Then there is a stabilized topological unknot U with $r(K) = r(U)$ and $Ch(W(K)) \cong Ch(W(U))$.*

Proof. By Proposition 3.1.6, the DGA $Ch(W_K)$ depends only on the way that strands of K connect individual cusps. Thus we can change K in any way we want without changing $Ch(W_K)$ as long as it remains a stabilization and the cusps are connected in the same way as before, for example:



In particular, we can construct a new front U from K by the following procedure:

1. Fix a strand s of K which is oriented from left to right. Follow K with the given orientation, and number the cusps $1, 2, \ldots 2c$ in the order they are reached, beginning at the left cusp of s.

2. Place c left cusps numbered $1, 3, \ldots, 2c-1$ in the plane and c right cusps numbered $2, 4, \ldots, 2c$ in the plane so that cusp i is above cusp j whenever

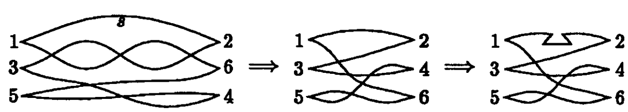
$i < j$, with all left cusps on one vertical line and all right cusps on another vertical line to the right of the first one.

3. For $1 \leq i \leq 2c-1$, connect cusp i to cusp $i+1$ by a strand which leaves the top of cusp i iff the corresponding strand of K does, and likewise at cusp $i+1$. All $2c-1$ of these strands should be mutually disjoint away from the cusps, except that the two strands through a given cusp can intersect once.

4. Connect cusp $2c$ to cusp 1 as in the previous step, except that this strand may intersect any of the others at most once each.

5. Add a positive and negative stabilization to the strand connecting cusps 1 and 2.

For example, we can apply this procedure to the stabilized trefoil above:



The front U was constructed to satisfy $Ch(W_U) \cong Ch(W_{S_+S_-(K)})$ by the above argument, where we construct $S_+S_-(K)$ by adding the stabilizations to the strand s, pushing the resulting front into plat form, and making U into a plat as well. But this last algebra is equivalent to $Ch(W(S_+S_-(K))) \cong Ch(W(K))$, and U is itself a stabilization, so in fact we have $Ch(W(U)) \cong Ch(W(K))$. Furthermore, we can see that $r(U) = r(K)$ since cusp i of K has the same orientation as cusp i of U and the double stabilization $S_+S_-(\cdot)$ preserves rotation number.

The only remaining step is to check that U is a topological unknot. But the strand connecting cusps 1 and $2c$ crosses over all of the other strands, so we may isotope it away from the rest of the diagram. Then the only remaining crossings are single crossings next to some of the cusps, such as cusps 4 and 5 in the picture above, and these single crossings can be eliminated by type I Reidemeister moves. The resulting diagram has no crossings, so it is indeed an unknot, completing the proof. $\square$

3.1.1.5 Computation for a topological unknot

By Proposition 3.1.7, we need only compute the DGA of some Whitehead doubles of the topological unknot in order to determine its value on the double of any stabilization. In this section we will explicitly compute these algebras, making repeated use of the following lemma which closely follows the proof of Lemma 3.1.3 in [48]:

Lemma 3.1.8. *Suppose that $\mathcal{A}$ is a free DGA which contains generators b, c, d in grading 0 with $\partial b = \partial c = \partial d = 0$ and generators w, x, y, z in grading 1 with*

$$\begin{aligned} \partial w &= 1 + ab & \partial y &= 1 + cd \\ \partial x &= 1 + bc & \partial z &= 1 + de, \end{aligned}$$

where a and e are some elements in grading 0 with $\partial a = \partial e = 0$ which do not involve any of b, c, d, w, x, y, z. Assume also that c, d, x, y, z do not appear in the differential of any other generator. Then $\mathcal{A}$ is equivalent to a DGA in which a, b, e are defined as in $\mathcal{A}$, generators w and z in grading 1 satisfy

$$\partial w = 1 + ab \qquad \partial z = 1 + be,$$

and the generators x, y, c, d have been eliminated but all other generators and differentials remain the same.

Proof. Since $\partial(wc + ax) = a + c$ and $\partial(xd + by) = b + d$, the sequence of tame isomorphisms

$$\begin{aligned} z &\to z + (xd + by)e, \\ y &\to y + (wc + ax)d + w, \\ x &\to x + z, \\ c &\to c + e, \\ d &\to d + b \end{aligned}$$

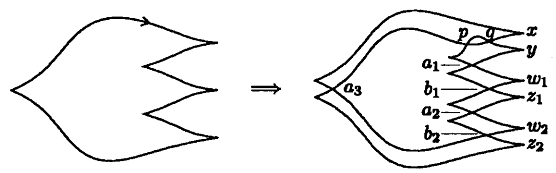


Figure 3.1.3: The Whitehead double of an unknot U with $r(U) = 2$.

gives $\partial w = 1 + ab$, $\partial z = 1 + be$, $\partial y = ad$, and $\partial x = bc$, with $\partial c = \partial d = 0$ as before.

Stabilize by adding a pair of generators p, q in gradings 0 and -1 with $\partial p = q$ and $\partial q = 0$. Performing another sequence of tame isomorphims

$$\begin{aligned} d &\to d + bp, \\ y &\to y + wp, \\ p &\to p + ad + wq \end{aligned}$$

results in $\partial d = bq$, $\partial q = 0$, $\partial y = p$, and $\partial p = 0$, so we destabilize to remove the pair (y, p).

Finally, let $s = (1+ba)z + bwe$, so that $\partial s = (1+ba)(1+be) + b(1+ab)e = 1+ba$. Apply the sequence of tame isomorphisms

$$\begin{aligned} c &\to c + ad, \\ x &\to x + sd, \\ d &\to d + bc + sbq, \\ c &\to c + wq \end{aligned}$$

to get $\partial x = d$, $\partial d = 0$, and $\partial c = q$. We have reduced $\mathcal{A}$ to the generators and relations $\partial w = 1 + ab$, $\partial z = 1 + be$, $\partial x = d$, $\partial d = 0$, $\partial c = q$, and $\partial q = 0$, so we can destabilize by removing the pairs (x, d) and (c, q) to put $\mathcal{A}$ in the desired form. $\square$

Suppose U is an unknot with $r(U) > 0$ and $tb(U) = -(r+1)$ as shown in Figure 3.1.3; the rotation number is one more than the number of right cusps. Then $Ch(W(U)) = \mathcal{A}_{2r,r}$, where $\mathcal{A}_{g,n}$ is freely generated by $x, y, p, q, w_i, z_i, b_i$ for

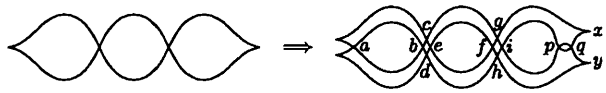
$1 \leq i \leq n$ and a_i for $1 \leq i \leq n+1$; the gradings are given by $|p| = -g$, $|q| = g$, $|x| = |y| = |w_i| = |z_i| = 1$, and $|a_i| = |b_i| = 0$; and the differentials are given by

$$\partial x = 1 + a_{n+1}(pq+1) \qquad \partial w_i = 1 + a_i b_i$$
$$\partial y = 1 + (qp+1)a_1 \qquad \partial z_i = 1 + b_i a_{i+1}.$$

If $n \geq 1$, we can use Lemma 3.1.8 to eliminate w_n, z_n, b_n, and a_{n+1} from $\mathcal{A}_{g,n}$, replacing ∂x with $\partial x = 1 + a_n(pq+1)$, and so $\mathcal{A}_{g,n} \cong \mathcal{A}_{g,n-1}$ for all $n \geq 1$, hence $Ch(W(U)) \cong \mathcal{A}_{2r,0}$.

If instead $r(U) < 0$, then we can reverse the orientation of U, since $r(-U) = -r(U)$, and note that $W(-U) = -W(U)$. Since the DGA of a Legendrian knot does not depend on its orientation, we have shown that $Ch(W(U)) \cong \mathcal{A}_{2|r|,0}$ whenever $r \neq 0$.

For $r = 0$ we use a slightly different construction, since we want our unknot U to have $tb = -3$ instead of -1. Specifically, we produce $W(U)$ as follows:



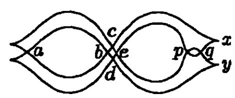
The DGA still has generators p, q, x, y, a in gradings $0, 0, 1, 1, 0$ with $\partial x = 1 + a(pq+1)$, $\partial y = 1 + (qp+1)a$, and $\partial p = \partial q = \partial a = 0$, but now there are extra generators b, c, d, e, f, g, h, i as well which satisfy

$$\partial b = 0 \qquad \partial d = 1 + ba \qquad \partial f = 0 \qquad \partial h = fa$$
$$\partial c = 1 + ab \qquad \partial e = ad + ca \qquad \partial g = af \qquad \partial i = ga + ah.$$

Just as in the proof of Proposition 3.1.6, the sequence of tame isomorphisms

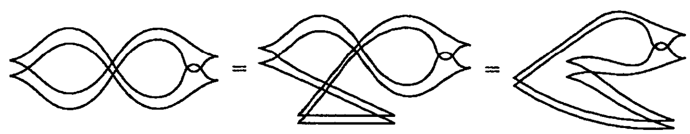
$$\begin{aligned} g &\to g + (a+1)(hb + fc), \\ i &\to i + (a+1)(hd + fe), \\ h &\to h + ga \end{aligned}$$

results in $\partial g = f$, $\partial i = h$, $\partial h = 0$, so we can destabilize and remove the pairs of generators (g, f) and (i, h). The resulting DGA is exactly $Ch(W)$ as an ungraded differential algebra, where W is the Whitehead double of an unknot with $tb = -2$ and $r = \pm 1$:



In fact, $Ch(W(U))$ and $Ch(W)$ differ only in the gradings of p and q; they are both 0 in the former but -2 and 2 in the latter.

We can perform a sequence of Legendrian Reidemeister moves to turn W into the form seen in Figure 3.1.3:



At each step we have fixed the vertices x, y, p, and q and preserved the facts that $\partial p = \partial q = 0$; that ∂x and ∂y are polynomials in pq, qp, and vertices other than p and q; and that p and q do not appear in the differentials of any other vertices. In particular, these moves give a stable tame isomorphism from $Ch(W)$ to $\mathcal{A}_{2,1}$ which is independent of the gradings of p and q as long as $|pq| = |qp| = 0$, and so the same isomorphism gives us $Ch(W(U)) \cong \mathcal{A}_{0,1}$, hence $Ch(W(U)) \cong \mathcal{A}_{0,0}$.

Proof of Theorem 3.1.1. Let K be a stabilization. We have shown that there is a stabilized topological unknot U with $r(K) = r(U)$ and $Ch(W(K)) \cong Ch(W(U))$, and that $Ch(W(U)) \cong \mathcal{A}_{2|r|,0}$, which is exactly the DGA in the statement of the theorem. $\square$

3.1.2 Whitehead doubles of some small knots

3.1.2.1 The mirror of 9_{42}

Let K be the representative of $m(9_{42})$ with $(tb, r) = (-5, 0)$ in the table of [44]; as a plat, K has three pairs of cusps and its crossings are specified by the braid word

$$2, 1, 1, 4, 5, 3, 5, 3, 2, 4, 3, 3, 2, 4.$$

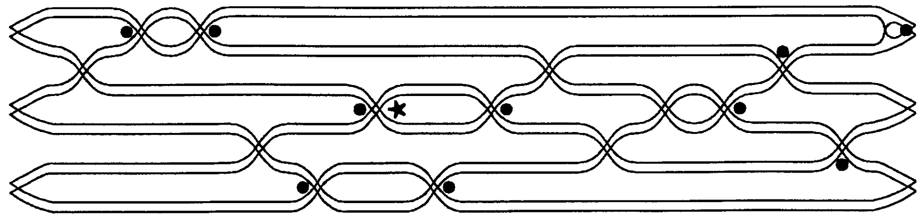


Figure 3.1.4: An augmentation ϵ of $W(K)$, for K a Legendrian knot of type $m(9_{42})$. Dots are next to vertices where $\epsilon(v) = 1$, and the star indicates a generator of $\ker(\partial_2^\epsilon) = H_2^\epsilon(W(K))$.

The Whitehead double of K has a Chekanov polynomial $t^2 + 2t + 2 + t^{-1} + t^{-2}$, hence K is not a stabilization by Corollary 3.1.2. It is easy to verify this: no vertex of $W(K)$ has grading greater than 2, so we only need to specify an augmentation ϵ for which the linearized differential ∂^ϵ has a nonzero kernel in grading 2. One such augmentation is shown in Figure 3.1.4.

On the other hand, we can also see that K is not a stabilization by showing that its characteristic algebra does not vanish, which we will do by constructing a 2-dimensional representation. (It is easy to see that any such representation lifts to a DGA morphism $Ch(K) \to \mathrm{Mat}_2(\mathbb{F})$, where $\mathrm{Mat}_2(\mathbb{F})$ has trivial differential.)

Proposition 3.1.9. *The characteristic algebra of $Ch(K)$ is nontrivial, but its abelianization vanishes.*

Proof. The characteristic algebra C is the free algebra on 17 generators $x_1, \ldots, x_{17}$

modulo the two-sided ideal generated by

$$
\begin{aligned}
\partial x_2 &= x_1 \\
\partial x_5 &= x_4 \\
\partial x_6 &= x_1 x_4 \\
\partial x_9 &= x_3 + x_4 x_8 + x_3 x_6 x_8 + x_3 x_2 x_4 x_8 \\
\partial x_{10} &= x_7 + x_8 x_1 + x_8 x_6 x_7 + x_8 x_1 x_5 x_7 \\
\partial x_{11} &= 1 + x_3 x_2 + x_4 x_8 + x_5 x_7 + x_9 x_1 + x_3 x_6 x_{10} + x_9 x_6 x_7 \\
&\quad + x_3 x_2 x_4 x_{10} + x_3 x_2 x_5 x_7 + x_9 x_1 x_5 x_7 \\
\partial x_{13} &= x_1 x_{12} + x_6 x_7 x_{12} + x_1 x_5 x_7 x_{12} \\
\partial x_{14} &= x_{12} x_4 + x_{12} x_3 x_6 + x_{12} x_3 x_2 x_4 \\
\partial x_{15} &= 1 + x_{13} + x_2 x_{12} + x_6 x_8 x_{13} + x_6 x_{10} x_{12} + x_2 x_4 x_8 x_{11} + x_2 x_4 x_{10} x_{12} + x_2 x_5 x_7 x_{12} \\
\partial x_{16} &= 1 + x_1 x_{14} + x_{13} x_4 + x_6 x_7 x_{14} + x_{13} x_3 x_6 + x_1 x_5 x_7 x_{14} + x_{13} x_3 x_2 x_4 \\
\partial x_{17} &= 1 + x_{14} + x_{12} x_5 + x_{12} x_9 x_6 + x_{14} x_8 x_6 + x_{12} x_3 x_2 x_5 + x_{12} x_9 x_1 x_5 + x_{14} x_8 x_1 x_5.
\end{aligned}
$$

Evidently $x_1 = x_4 = 0$, so we eliminate all terms involving either of them and we are left with the relations $r_i = 0$, where

$$
\begin{aligned}
r_1 &= x_3(1 + x_6 x_8) \\
r_2 &= (1 + x_8 x_6) x_7 \\
r_3 &= 1 + x_3 x_2 + x_5 x_7 + x_3 x_6 x_{10} + x_9 x_6 x_7 + x_3 x_2 x_5 x_7 \\
r_4 &= x_6 x_7 x_{12} \\
r_5 &= x_{12} x_3 x_6 \\
r_6 &= 1 + x_{13} + x_2 x_{12} + x_6 x_8 x_{13} + x_6 x_{10} x_{12} + x_2 x_5 x_7 x_{12} \\
r_7 &= 1 + x_6 x_7 x_{14} + x_{13} x_3 x_6 \\
r_8 &= 1 + x_{14} + x_{12} x_5 + x_{12} x_9 x_6 + x_{14} x_8 x_6 + x_{12} x_3 x_2 x_5.
\end{aligned}
$$

Note that $x_{12} r_1 = x_{12} x_3 + r_5 x_8$ and $r_2 x_{12} = x_7 x_{12} + x_8 r_4$, so $x_{12} x_3 = x_7 x_{12} = 0$. If we were to let $x_{12} = 0$, then the equations

$$\begin{aligned} x_3 &= x_3 r_6 + r_1 x_{13} + x_3(x_2 + x_6 x_{10} + x_2 x_5 x_7) x_{12} \\ x_7 &= r_8 x_7 + x_{14} r_2 + x_{12}(x_5 + x_9 x_6 + x_3 x_2 x_5) x_7 \end{aligned}$$

would give us $x_3 = x_7 = 0$; but then from $r_7 = 0$ we would conclude that $0 = 1$. In the abelianization of $\mathcal{C}$, however, we do have

$$x_{12} = x_{12} r_3 + (x_{12} x_3)(x_2 + x_6 x_{10} + x_2 x_5 x_7) + (x_5 + x_9 x_6)(x_7 x_{12}),$$

hence $x_{12} = 0$, and so the abelianization of $\mathcal{C}$ is trivial.

By Lemma 3.3.3, we have a presentation $\mathrm{Mat}_2(\mathbb{F}) = \mathbb{F}\langle a, b\rangle / \langle a^2 = b^2 = 0, ab+ba = 1\rangle$. We define a homomorphism $\mathcal{C} \to \mathrm{Mat}_2(\mathbb{F})$ as follows:

$$\begin{aligned} x_1, x_4, x_9, x_{10}, x_{11}, x_{15}, x_{16}, x_{17} &\mapsto 0 \\ x_6 &\mapsto 1 \\ x_3, x_7, x_{12} &\mapsto a \\ x_8 &\mapsto a+1 \\ x_2, x_5, x_{13}, x_{14} &\mapsto b. \end{aligned}$$

It is straightforward to check that each r_i is indeed sent to 0, hence the homomorphism is well-defined and $\mathcal{C}$ is nontrivial. $\square$

Remark 3.1.10. Since for example $\partial x_3 = 0$, and the map $Ch(K) \to \mathrm{Mat}_2(\mathbb{F})$ sends x_3 to $a \neq 0$, we know that $[x_3]$ is a nonzero homology class in $LCH(K) = H_*(Ch(K))$. Thus K has nontrivial Legendrian contact homology in degree -1.

We note that K is the only known knot with $r = 0$ and no augmentations which satisfies either of the following two properties:

1. There are finite-dimensional representations $Ch(K) \to \mathrm{Mat}_n(\mathbb{F})$ for some finite n.

2. The Legendrian Whitehead double $W(K)$ has Chekanov polynomials other than $t+2$.

It would be interesting to know if these properties are related.

3.1.2.2 Some Legendrian nonsimple knots with $r = 0$

We have computed the Chekanov polynomials of the Legendrian Whitehead doubles of several pairs of Legendrian knots with $r = 0$. Each pair could not be distinguished by classical invariants, Chekanov polynomials, or ruling invariants, and the Chekanov polynomials of the Whitehead doubles do not distinguish them either. All knots in this section come from the Legendrian knot atlas [5] as of October 20, 2010.

In each entry of Table 3.1 we specify braid words for plat representatives of the Legendrian knots. The Chekanov polynomials listed for each Whitehead double $W(K)$ are precisely those not of the form $t+2$ or $t+2+(t+2+t^{-1})(p(t)-t)$, where $p(t)$ is a Chekanov polynomial of K. Note that it is unknown whether two of the three knots of type 7_4 are actually distinct, and likewise for the knots of type 8_{21}.

3.2 Vanishing and nonvanishing of Legendrian contact homology

Shonkwiler and Vela-Vick [59] gave the first examples of Legendrian knots with nonvanishing contact homology which do not have maximal Thurston-Bennequin invariant, representing the knot types $m(10_{161})$ and $m(10_{145})$. Conversely, there are conjecturally nondestabilizable knots of type $m(10_{139})$, 10_{161}, and $m(12n_{242})$ with nonmaximal tb and vanishing contact homology [5, 59]. On the other hand, it is an open question whether there is a Legendrian knot K for which $tb(K)$ is maximal but the contact homology of K vanishes. In this section, we will answer this question and show that it is not determined solely by the classical invariants tb and r of K:

Knot, tb	K	$W(K)$
$m(7_2)$, 1	$t+2$	$2t+4+t^{-1}$
	2,3,3,1,1,2,2,3,1,2,2,1,3,3,2	
	4,5,3,5,3,2,1,4,4,5,3,2,1,4,2,4,3,5,4,2	
7_4, 1	$\emptyset$	$\emptyset$
	4,5,5,3,6,7,2,1,4,5,6,8,9,7,8,1,3,2,4,5,3,4,6,8,4,2	
	6,7,7,5,6,6,5,4,3,2,1,7,5,3,1,7,6,5,4,1,2	
	6,7,7,5,4,3,6,5,3,2,4,1,3,6,2,5,7,4,6,2,4,5,6,3,4	
$m(7_6)$, -1	$2t+2+t^{-1}$	$t^2+4t+6+3t^{-1}+t^{-2}$
	4,5,3,5,3,2,1,4,1,3,4,1,2,4,3,5,4,2	
	6,7,5,7,5,4,3,3,6,2,1,5,6,3,1,1,2,5,4	
$m(7_7)$, -5	$3t+2t^{-1}$	$2t^2+4t+4+3t^{-1}+2t^{-2}$
	6,7,5,5,7,4,3,6,3,5,2,1,4,1,3,6,4,3,2	
	4,5,3,3,5,2,1,4,1,3,4,1,5,2,3,2,5,4	
8_{21}, -9	$\emptyset$	$\emptyset$
	4,5,5,3,2,1,4,3,4,1,3,2,4,2	
	4,5,3,5,2,1,4,3,4,1,3,4,5,1,2,3,5,4	
9_{48}, -1	$2t+2+t^{-1}$	$t^2+4t+6+3t^{-1}+t^{-2}$
	6,7,5,7,4,3,2,1,6,5,3,6,1,3,4,2,3,5,4,1,2,6,4	
	6,7,8,9,5,7,8,4,3,2,1,9,3,2,8,6,9,7,5,2,1,8,7,6,4,8,7,3,5,6,9,8,1,2,5,4	
9_{48}, -1	$\emptyset$	$\emptyset$
	4,3,6,7,5,6,3,7,8,9,4,6,2,1,5,7,8,6,7,9,8,5,7,1,3,8,1,2,7,6,5,4	
	6,7,5,7,4,3,6,2,1,5,6,1,3,5,6,7,1,2,3,4,5,7,6	
$m(10_{132})$, -1	$\emptyset$	$\emptyset$
	6,7,4,3,7,5,3,6,4,2,5,1,3,2,5,2,4,6,2	
	4,5,3,5,3,2,4,1,3,2,4,2,5,1,3,2,4,4,3,5,4,2	
10_{136}, -3	$t^2+3t+2t^{-1}+t^{-2}$	$t^3+4t^2+5t+4+4t^{-1}+4t^{-2}+t^{-3}$
	4,5,5,6,7,3,4,2,1,5,3,2,4,7,6,3,2,3,5,2,6,3,2,5,4	
	8,9,7,6,5,9,4,3,5,4,3,8,2,1,7,8,6,4,7,1,3,4,5,8,1,2,3,4,7,6	
10_{136}, -3	$3t+2+2t^{-1}$	$2t^2+6t+8+5t^{-1}+2t^{-2}$
	6,7,5,4,3,7,2,1,6,3,2,5,1,2,4,6,1,2,3,5,1,2,6,3,4,7,5,7,6	
	4,5,3,2,5,1,4,3,4,3,1,4,3,2,4,2	
$m(10_{140})$, -1	t	$\emptyset$
	4,5,5,3,4,6,7,2,1,5,6,3,2,2,4,6,1,3,5,7,1,2,3,7,6,3,4	
	2,3,1,3,4,5,2,3,4,2,1,4,3,5,1,4,2,4,3,5,4,2	

Table 3.1: The Chekanov polynomials of the Legendrian Whitehead doubles of several pairs of knots, each with the same classical invariants and Chekanov polynomials.

Theorem 3.2.1. *There are distinct tb-maximizing Legendrian representatives K_1 and K_2 of $m(10_{132})$ with the same classical invariants such that K_1 has trivial Legendrian contact homology, even with $\mathbb{Z}[t, t^{-1}]$ coefficients, while K_2 does not.*

These Legendrian knots, found in Chongchitmate and Ng's atlas of Legendrian knots [5], can be specified as plat diagrams by the following braid words:

$$K_1: \quad 6,7,4,3,7,5,3,6,4,2,5,1,3,2,5,2,4,6,2$$
$$K_2: \quad 4,5,3,5,3,2,4,1,3,2,4,2,5,1,3,2,4,4,3,5,4,2$$

Indeed, both knots have classical invariants $tb = -1$ and $r = 0$, and Ng [49] showed that $\overline{tb}(m(10_{132})) = -1$ by bounding $\overline{tb}$ for an appropriate cable of $m(10_{132})$.

3.2.1 The vanishing $m(10_{132})$

Let K_1 be the Legendrian representative of $m(10_{132})$ in Figure 3.2.1. Its Chekanov-Eliashberg algebra is generated freely over $\mathbb{Z}[t, t^{-1}]$ by elements $x_1, \ldots, x_{23}$ with differentials specified in Appendix A.1.

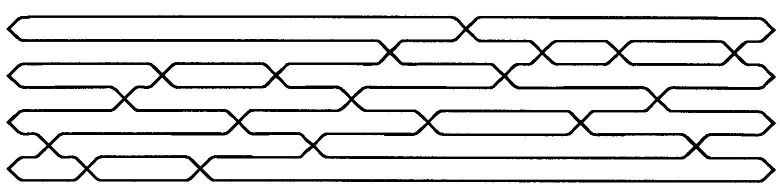


Figure 3.2.1: The representative K_1 of $m(10_{132})$, defined by the braid word 6, 7, 4, 3, 7, 5, 3, 6, 4, 2, 5, 1, 3, 2, 5, 2, 4, 6, 2.

To show that K_1 has vanishing contact homology, we need to find a relation $\partial x = 1$ in $Ch(K_1)$. Recall that $Ch(K_1)$ uses a signed Leibniz rule $\partial(vw) = (\partial v)w + (-1)^{|v|}v(\partial w)$, where $|v|$ is the grading of the homogeneous element v, and note that the generators with odd grading are

$$x_2, x_3, x_5, x_9, x_{11}, x_{12}, x_{13}, x_{15}, x_{20}, x_{21}, x_{22}, x_{23}.$$

Let

$$\begin{aligned} a &= x_{12}(x_4(1+x_2x_5) - x_8) + x_{14}x_5 \\ b &= x_{22} + x_{12} - ax_{18}; \end{aligned}$$

then $\partial a = x_{10}x_4(1+x_2x_5) - x_{10}x_8 + x_{13}x_5$, and so

$$\begin{aligned} \partial b &= 1 + x_{17}x_7 + (\partial a)x_{18} - ((\partial a)x_{18} + ax_{15}x_7) \\ &= 1 + (x_{17} - ax_{15})x_7. \end{aligned}$$

Then if $c = b(x_6 - x_4x_1) + (x_{17} - ax_{15})(x_9 + x_2)$, we can compute $\partial c = x_6 - x_4x_1$ and so

$$\partial(x_{20} - c(1 + x_{16}x_{19})) = 1.$$

Thus K_1 has trivial contact homology over $\mathbb{Z}[t, t^{-1}]$, as desired.

3.2.2 The nonvanishing $m(10_{132})$

Let K_2 be the Legendrian representative of $m(10_{132})$ in Figure 3.2.2. The algebra $Ch(K_2)$ is generated freely over $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ by $x_1, \ldots, x_{25}$ with differentials specified in Appendix A.2. In order to show that K_2 has nontrivial contact homology, it will suffice to show that the characteristic algebra $\mathcal{C}_2 = \mathcal{C}(K_2)$ is nonvanishing [59].

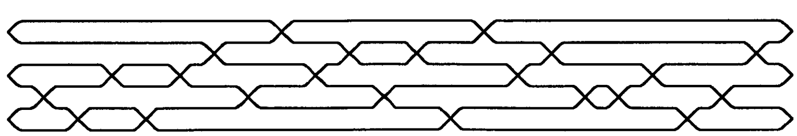


Figure 3.2.2: The representative K_2 of $m(10_{132})$, defined by the braid word $4, 5, 3, 5, 3, 2, 4, 1, 3, 2, 4, 2, 5, 1, 3, 2, 4, 4, 3, 5, 4, 2$.

The differential in $\mathcal{C}_2$ immediately gives us $x_1 = x_6 = 0$, and

$$x_{12} = \partial(x_{12}x_{23} + x_{15}x_{22} + x_{17}x_{18})$$

gives $x_{12} = 0$, hence $\partial x_{24} = 0$ becomes $(1 + x_5(x_2 + x_3))x_{20} = 1$. Then we can use $(\partial x_{13})x_{20} = 0$ and $(\partial x_{17})x_{20} = 0$ to get $x_{11} = 0$ and $x_{15} = 0$, so

$$x_1 = x_6 = x_{11} = x_{12} = x_{15} = 0.$$

Furthermore, $\partial x_{21} = 0$ becomes $x_{14} = cx_{20}$, so $\partial x_{25} = 0$ gives us $x_{14} = x_{20}$.

Consider the quotient of $\mathcal{C}_2$ by the two-sided ideal

$$\mathcal{I} = \langle x_3, x_7, x_8, x_9, x_{10}, x_{13} + 1 + x_2x_5, x_{17}, x_{19}, x_{21}, \ldots, x_{25} \rangle.$$

The quotient $\mathcal{C}_2/\mathcal{I}$ is generated by $x_2, x_4, x_5, x_{14}, x_{16}, x_{18}$, and its nontrivial relations are $c = x_2 + x_{14}(1 + x_2x_5) + x_{16}(1 + x_5x_2) = 1$ and

$$\begin{aligned} x_4 &= x_5(1 + x_2x_4) \\ x_{18} &= 1 + x_2x_4 \\ 0 &= (1 + x_5x_2)x_{18} \\ 1 &= (1 + x_2x_5)x_{18} \\ 1 &= (1 + x_5x_2)x_{14}. \end{aligned}$$

Note that the pair of relations $x_4 = x_5(1 + x_2x_4)$ and $x_{18} = 1 + x_2x_4$ are equivalent to $x_4 = x_5x_{18}$ and $(1 + x_2x_5)x_{18} = 1$, the latter of which is already known, so we can replace the pair with $x_4 = x_5x_{18}$. Furthermore, multiplying the $c = 1$ equation on the right by x_{18} gives $x_{14} = (1+x_2)x_{18}$, hence the last relation becomes $(1+x_5x_2)x_2x_{18} = 1$. Then the $c = 1$ equation becomes

$$x_{16}(1 + x_5x_2) = (1 + x_2)(1 + x_{18}(1 + x_2x_5))$$

so we multiply on the right by x_2x_{18} and get

$$x_{16} = (1 + x_2)(x_2x_{18} + x_{18}x_2(1 + x_5x_2)x_{18}) = (1 + x_2)x_2x_{18}.$$

Thus we see that x_4, x_{14}, and x_{16} can be expressed in terms of x_2, x_5, and x_{18}, and $c = 1$ can be rewritten as

$$0 = (1 + x_2)\,(1 + x_{18}(1 + x_2x_5) + x_2x_{18}(1 + x_5x_2))\,.$$

Relabeling x_2, x_5, x_{18} as a, b, c respectively, we have a homomorphism from $\mathcal{C}_2/\mathcal{I}$ to the quotient R of the free algebra $\mathbb{F}\langle a, b, c\rangle$ by the two-sided ideal generated by the relations

$$\begin{aligned}
0 &= 1 + c(1 + ab) + ac(1 + ba) \\
0 &= (1 + ba)c \\
1 &= (1 + ab)c \\
1 &= (1 + ba)ac.
\end{aligned}$$

Proposition 3.2.2. *The algebra R is nontrivial.*

Proof. We will construct an infinite-dimensional representation of R, following ideas from [59]. Let $\mathcal{H}$ be a countable-dimensional $\mathbb{F}$-vector space, with basis $\{v_0, v_1, v_2, \dots\}$, and write $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ where each $\mathcal{H}_i$ summand is isomorphic to $\mathcal{H}$. Let $f, g : \mathcal{H} \to \mathcal{H}$ be homomorphisms defined by $f(v_i) = v_{2i}$ and $g(v_i) = v_{2i+1}$, so that the diagrams

$$\begin{tikzcd}
\mathcal{H}_1 \arrow[r, "f"] & \mathcal{H}_1 \\
\oplus & \oplus \\
\mathcal{H}_2 \arrow[uur, "g"] & \mathcal{H}_2
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{H}_1 \arrow[ddr, "f"] & \mathcal{H}_1 \\
\oplus & \oplus \\
\mathcal{H}_2 \arrow[r, "g"] & \mathcal{H}_2
\end{tikzcd}$$

represent isomorphisms $\mathcal{H} \xrightarrow{\sim} \mathcal{H}_1$ and $\mathcal{H} \xrightarrow{\sim} \mathcal{H}_2$, respectively. We also define homomorphisms $p, s : \mathcal{H} \to \mathcal{H}$ by $p(v_i) = v_{i-1}$ for $i \geq 1$, $p(v_0) = 0$ and $s(v_i) = v_{i+1} + v_{2(i+1)}$. It is straightforward to check the identities

$$s \circ p = f + 1, \quad p \circ g = f, \quad p \circ s = g + 1.$$

We define a right action of a and b on $\mathcal{H} \cong \mathcal{H}_1 \oplus \mathcal{H}_2$ by the diagrams

$$\begin{tikzcd}
\mathcal{H}_1 \arrow[ddr, "p"] & \mathcal{H}_1 \\
\oplus & \oplus \\
\mathcal{H}_2 \arrow[uur, "1"'] & \mathcal{H}_2
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathcal{H}_1 \arrow[r, "g"] \arrow[ddr, "1"] & \mathcal{H}_1 \\
\oplus & \oplus \\
\mathcal{H}_2 \arrow[uur, "s"'] & \mathcal{H}_2
\end{tikzcd}$$

respectively. Then we can compute the action of ab and ba by concatenating the a and b diagrams to get

$$\begin{tikzcd}
\mathcal{H}_1 \arrow[r, "sop"] & \mathcal{H}_1 \\
\oplus & \oplus \\
\mathcal{H}_2 \arrow[r, "1"'] \arrow[uur, "g"] & \mathcal{H}_2
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathcal{H}_1 \arrow[r, "1"] \arrow[ddr, "pog"] & \mathcal{H}_1 \\
\oplus & \oplus \\
\mathcal{H}_2 \arrow[r, "pos"] & \mathcal{H}_2
\end{tikzcd}$$

respectively, hence by the above identities $1 + ab$ and $1 + ba$ are exactly the specified isomorphisms $\mathcal{H} \xrightarrow{\sim} \mathcal{H}_1$ and $\mathcal{H} \xrightarrow{\sim} \mathcal{H}_2$. Finally, let c act on $\mathcal{H}$ as the map

$$\begin{tikzcd}
\mathcal{H}_1 \arrow[dr, "\sim"] & \\
\oplus & \mathcal{H} \\
\mathcal{H}_2 \arrow[ur, "0"'] &
\end{tikzcd}$$

where the indicated isomorphism is the inverse of $\mathcal{H} \xrightarrow{\sim} \mathcal{H}_1$. Then the composition ac is the homomorphism

$$\begin{tikzcd}
\mathcal{H}_1 \arrow[dr, "0"] & \\
\oplus & \mathcal{H} \\
\mathcal{H}_2 \arrow[ur, "\sim"'] &
\end{tikzcd}$$

where the isomorphism is inverse to $\mathcal{H} \xrightarrow{\sim} \mathcal{H}_2$. It is now easy to check that $(1+ab)c = 1$, $(1+ba)c = 0$, and $(1+ba)ac = 1$. Finally, we note that $c(1+ab)$ is the projection of $\mathcal{H}$ onto $\mathcal{H}_1 \subset \mathcal{H}$ and likewise $ac(1+ba)$ is the projection onto $\mathcal{H}_2$, hence

$$1 = c(1+ab) + ac(1+ba).$$

Therefore the action which we have constructed satisfies all of the defining relations of R. $\square$

Since R is nonvanishing and we have a homomorphism $\mathcal{C}_2 \to \mathcal{C}_2/\mathcal{I} \to R$, we conclude that $\mathcal{C}_2$ (and hence the contact homology of K_2) is nonvanishing as well.

3.3 Finite-dimensional representations of $\mathcal{C}(K)$

The proof that the Legendrian knot K_2 of Section 3.2.2 has nonvanishing contact homology uses an action of $\mathcal{C}(K_2)$ on an infinite-dimensional vector space, just as the nonvanishing examples in [59] did. It turns out that this is necessary, in the sense that $\mathcal{C}(K_2)$ does not have any finite-dimensional representations.

Lemma 3.3.1. *Suppose that an $\mathbb{F}$-algebra $\mathcal{A}$ has a relation of the form $ab = 1$. If the quotient of $\mathcal{A}$ by the two-sided ideal $\langle ba - 1\rangle$ is trivial, i.e. if $0 = 1$ in $\mathcal{A}/\langle ba - 1\rangle$, then there is no representation $\mathcal{A} \to \mathrm{Mat}_n(\mathbb{F})$ for any n.*

Proof. Suppose there is a homomorphism $\varphi : \mathcal{A} \to \mathrm{Mat}_n(\mathbb{F})$, so in particular $\varphi(1) = 1$. The equation $\varphi(ab - 1) = 0$ implies that $\varphi(a)$ and $\varphi(b)$ are inverse matrices, so they commute and $\varphi(ba - 1) = 0$ as well. Then φ factors through the quotient $\mathcal{A}/\langle ba - 1\rangle$ in which $0 = 1$, hence $\varphi(1) = \varphi(0) = 0$, which is a contradiction. $\square$

Now in $\mathcal{C}_2$, we showed in Section 3.2.2 that $x_{11} = x_{12} = 0$ and $(1+x_5(x_2+x_3))x_{20} = 1$. If we impose the relation $x_{20}(1+x_5(x_2+x_3)) = 1$, then $x_{18} = x_{20}(\partial x_{22}) = 0$ as well and so $0 = \partial x_{23} = 1$, hence $\mathcal{C}_2$ has no finite-dimensional representations by Lemma 3.3.1.

Lemma 3.3.1 can also be used to prove that the characteristic algebra of the Legendrian $m(10_{161})$ studied in [59] has no finite-dimensional representations, by adding $x_{28}x_{13} = 1$ to the relations $\partial x_i = 0$ in [59, Appendix A] and showing that $0 = 1$ as a consequence, and similarly for the $m(10_{145})$ representative mentioned in the same article. Neither one of these knots has maximal Thurston-Bennequin invariant.

On the other hand, it is interesting to ask when the characteristic algebra $\mathcal{C}$ of a Legendrian knot K has n-dimensional representations. For $n = 1$ the answer depends only on tb and the topological knot type:

Proposition 3.3.2. *There is a homomorphism* $\mathcal{C} \to \mathrm{Mat}_1(\mathbb{F}) \cong \mathbb{F}$ *if and only if the Kauffman bound*

$$tb(K) \leq \text{min-deg}_a F_K(a,x) - 1$$

(see [25]) is sharp.

Proof. The Kauffman bound for K is achieved if and only if a front diagram for K admits an ungraded normal ruling [56], which happens if and only if $Ch(K)$ admits an ungraded augmentation [23, 24, 57]. An augmentation is an algebra homomorphism $Ch(K) \xrightarrow{\epsilon} \mathbb{F}$ which satisfies $\epsilon \circ \partial = 0$, and these correspond bijectively to algebra homomorphisms $\mathcal{C} \to \mathbb{F}$, so the latter exists if and only if the Kauffman bound is sharp. $\square$

In particular, the Kauffman bound is known to be sharp for all knots with at most 9 crossings except for $m(8_{19})$ and $m(9_{42})$ (see [51]); for all 10-crossing knots except $m(10_{124})$, $m(10_{128})$, $m(10_{132})$, and $m(10_{136})$ [2]; and for all alternating knots [56]. Thus the characteristic algebra of a Legendrian representative of one of these knot types has a 1-dimensional representation if and only if it is tb-maximizing.

We will now demonstrate the existence of infinitely many Legendrian knots whose characteristic algebras have n-dimensional representations for $n = 2$ but not for $n = 1$. For convenience, we will use the following presentation of $\mathrm{Mat}_2(\mathbb{F})$.

Lemma 3.3.3. *The ring* $\mathrm{Mat}_2(\mathbb{F})$ *has a presentation of the form*

$$\frac{\mathbb{F}\langle a,b\rangle}{\langle a^2 = b^2 = 0, ab + ba = 1\rangle}.$$

Proof. Let R be the $\mathbb{F}$-algebra with the given presentation, and consider a map $\varphi : R \to \mathrm{Mat}_2(\mathbb{F})$ of the form

$$\begin{aligned} a &\mapsto A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ b &\mapsto B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

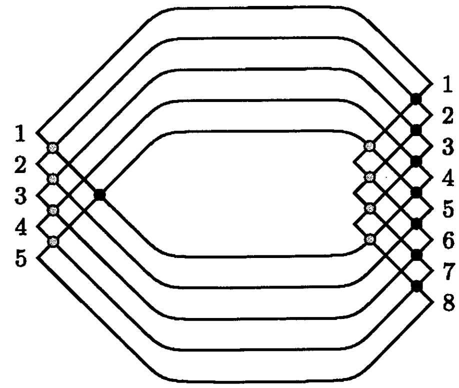


Figure 3.3.1: A Legendrian representative $T_{5,-8}$ of the $(5,-8)$-torus knot.

It is easy to check that $A^2 = B^2 = 0$ and $AB+BA = I$, so φ is a valid homomorphism, and since A, B, AB, BA form an additive basis of $\mathrm{Mat}_2(\mathbb{F})$ it is surjective. To check that φ is also injective, we note that any nonzero monomial in R is equal to one of $1, a, b, ab$, or $ba = 1 + ab$, and so $1, a, b, ab$ span R as an $\mathbb{F}$-vector space; since the image of φ has order $|\mathrm{Mat}_2(\mathbb{F})| = 16 \geq |R|$ it follows that φ is injective. $\square$

Let $T_{p,-q}$ be the Legendrian representative of the $(p,-q)$-torus knot as in Figure 3.3.1, where $q > p \geq 3$; there are p numbered left cusps at the leftmost edge of the diagram, $q - p$ left cusps in the innermost region of the diagram, and q right cusps. The algebra $Ch(T_{p,-q})$ can be computed following [52]: the front projection is simple, so $Ch(T_{p,-q})$ is generated by crossings and right cusps and the differential counts admissible embedded disks in the diagram.

We label the generators of $Ch(T_{p,-q})$ as follows. On the left half of the diagram, x_{ij} is the intersection of the strands through the numbered left cusps i and j for $1 \leq i < j \leq p$. On the right half, y_{ij} denotes the intersection of strands through the numbered right cusps i and j for $1 \leq i < j \leq \max(q, i+p-1)$, and z_i is the ith right cusp.

We define an algebra homomorphism $f : Ch(T_{p,-q}) \to \mathrm{Mat}_2(\mathbb{F})$ by sending all generators to 0 except

$$\begin{aligned} x_{i,i+1}, y_{j,j+p-1} &\mapsto a \\ x_{1,p}, y_{j,j+1} &\mapsto b. \end{aligned}$$

In Figure 3.3.1, f is equal to a on the crossings marked with gray dots, b on the crossings marked with black dots, and 0 on all other crossings and right cusps. If we can show that $f(\partial v) = 0$ for all generators v, then f is a morphism of DGAs (where $\mathrm{Mat}_2(\mathbb{F})$ has trivial differential) and it induces a representation $\mathcal{C}(T_{p,-q}) \to \mathrm{Mat}_2(\mathbb{F})$.

Proposition 3.3.4. *The homomorphism $f : Ch(T_{p,-q}) \to \mathrm{Mat}_2(\mathbb{F})$ satisfies $f(\partial v) = 0$ for all v.*

Proof. Call an admissible disk *nontrivial* if none of its corners are in $\ker(f)$. Then it is easy to see that any nontrivial disk has exactly two corners, and if both corners have the same color (in the sense of Figure 3.3.1, i.e. if they are sent to the same element of $\mathrm{Mat}_2(\mathbb{F})$) then the contribution of this disk to $f(\partial v)$ is either $a^2 = 0$ or $b^2 = 0$. Thus we can determine $f(\partial v)$ by only counting disks with initial vertex at v and having exactly one gray corner and one black corner.

If v is the right cusp z_i, then there are two nontrivial disks contributing ab and ba to the differential, so $f(\partial z_i) = 1 + ab + ba = 0$. For all crossings v, however, the only possible black corner for a nontrivial disk is $x_{1,p}$. Such a disk must include either the first or the pth numbered left cusp on its boundary depending on whether the interior of the disk is immediately above or below $x_{1,p}$, but then the boundary of the disk must pass through either z_1 or z_q, which in particular is to the right of v, and so it cannot contribute to $f(\partial v)$. We conclude that $f(\partial v) = 0$ for all generators v of $Ch(T_{p,-q})$, as desired. $\square$

We can compute $tb(T_{p,-q}) = -pq$ for all p and q, hence $T_{p,-q}$ is tb-maximizing by the classification of Legendrian torus knots [18], but for odd p the Kauffman bound is $tb(K) \leq -pq + q - p$ [14]. Using Proposition 3.3.2, we conclude:

Corollary 3.3.5. *Let $p \geq 3$ be odd and $q > p$. Then the characteristic algebra $\mathcal{C}(T_{p,-q})$ admits an n-dimensional representation for $n = 2$ but not for $n = 1$.*

Remark 3.3.6. The knots $T_{3,-4}$ and $T_{3,-5}$ are the unique *tb*-maximizing representatives of $m(8_{19})$ and $m(10_{124})$ up to change of orientation [18], so if any *tb*-maximizing Legendrian representative of a knot with at most 10 crossings has vanishing contact homology or characteristic algebra (such as the $m(10_{132})$ of Section 3.2.1) then it must represent one of $m(9_{42})$, $m(10_{128})$, $m(10_{132})$, or $m(10_{136})$. The characteristic algebra of the known *tb*-maximizing Legendrian $m(9_{42})$ is nonvanishing (in fact, it also has a 2-dimensional representation by the proof of Proposition 3.1.9), so it does not vanish.

It is not known whether there are Legendrian knots whose characteristic algebras have representations of minimal dimension $n \geq 3$, or whether this minimal dimension can be used to distingush any Legendrian knots with nontrivial characteristic algebras and the same classical invariants. We leave open the question of which Legendrian knots K admit representations $\mathcal{C}(K) \to \mathrm{Mat}_n(\mathbb{F})$ for fixed $n \geq 2$ or even for any finite n.

Chapter 4

Legendrian invariants from monopole knot homology

4.1 Background in monopole Floer homology

Our goal in this chapter is to associate an invariant $\psi(\mathcal{K})$ to each Legendrian knot $\mathcal{K}$ of topological type K in a contact manifold (Y, ξ). The desired invariant will be an element of $KHM(-Y, K)$ up to automorphism, where KHM is the monopole knot homology defined by Kronheimer and Mrowka [37, 36] with coefficients in a Novikov ring. This invariant seems to closely resemble the Heegaard Floer invariant of Legendrian knots introduced by Lisca, Ozsváth, Stipsicz, and Szabó [41], and we will also see in Section 4.5 that it behaves functorially with respect to Lagrangian concordance [3].

4.1.1 Sutured monopole homology

Let (M, γ) be a balanced sutured manifold. Kronheimer and Mrowka [37] defined the monopole Floer homology of (M, γ) as follows:

1. Choose an oriented, connected surface T such that the components of ∂T are in one-to-one correspondence with the components of γ. Form the product sutured manifold $(T \times I, \delta)$, where $I = [-1, 1]$, with annuli $A(\delta) = \partial T \times I$ and

$R_\pm(\delta) = T \times \{\pm 1\}$.

2. Glue the annuli $A(\delta)$ to $A(\gamma)$ by some orientation-reversing map $A(\delta) \to A(\gamma)$ sending $\partial R_+(\delta)$ to $\partial R_+(\gamma)$. The resulting 3-manifold should have boundary $\bar{R}_+ \cup \bar{R}_-$ for some connected, closed, orientable surfaces $\bar{R}_\pm = R_\pm(\gamma) \cup R_\pm(\delta)$.

3. Form a closed manifold $\bar{Y}$ by gluing the boundary along some diffeomorphism $h : \bar{R}_+ \to \bar{R}_-$, and let $\bar{R} \subset \bar{Y}$ be the image of $\bar{R}_\pm$.

We require that $\bar{R}$ has genus at least 2, and that T contains a simple closed curve c such that $c \times \{\pm 1\}$ is a non-separating curve in $\bar{R}_\pm$.

Definition 4.1.1. The sutured monopole homology of (M, γ) is defined as

$$SHM(M, \gamma) = \widetilde{HM}(\bar{Y}|\bar{R}),$$

where $\widetilde{HM}(\bar{Y}|\bar{R})$ is the direct sum of $\widetilde{HM}(\bar{Y}, \mathfrak{s})$ over all Spin^c structures $\mathfrak{s}$ satisfying $\langle c_1(\mathfrak{s}), \bar{R}\rangle = 2g(\bar{R}) - 2$.

Note that since $g(\bar{R}) \geq 2$, the class $c_1(\mathfrak{s})$ cannot be torsion if $\widetilde{HM}(\bar{Y}, \mathfrak{s})$ contributes to $\widetilde{HM}(\bar{Y}|\bar{R})$; but then $\overline{HM}(Y, \mathfrak{s}) = 0$, so $\widetilde{HM}(\bar{Y}, \mathfrak{s})$ and $\widehat{HM}(\bar{Y}, \mathfrak{s})$ are canonically isomorphic. Thus in [37] the authors drop the decoration and simply write $HM(\bar{Y}|\bar{R})$.

We can also define SHM using local coefficients. Let $\mathcal{R}$ be a ring with exponential map $\exp : \mathbb{R} \to \mathcal{R}^\times$ and write $t^n = \exp(n)$ for convenience. To any smooth 1-cycle η in $\bar{Y}$ we can associate a local system Γ_η on the Seiberg-Witten configuration space $\mathcal{B}(\bar{Y}, \mathfrak{s})$ whose fiber at any point is $\mathcal{R}$ and which assigns to any path $z : [0,1] \to \mathcal{B}(\bar{Y}, \mathfrak{s})$ the multiplication map by $t^{r(z)}$, where

$$r(z) = \frac{i}{2\pi} \int_{[0,1]\times\eta} \mathrm{tr}(F_{A_z})$$

for A_z the connection on $[0,1] \times \bar{Y}$ corresponding to the path z.

Suppose that the diffeomorphism $h : \bar{R}_+ \to \bar{R}_-$ restricts to an orientation-preserving homeomorphism $c \times \{1\} \to c \times \{-1\}$, resulting in a curve $\bar{c} \subset \bar{Y}$. If η is taken to be a curve dual to $\bar{c}$, in the sense that $\bar{c} \cdot \eta = 1$, then we can define

$SHM(M,\gamma;\Gamma_\eta) = \widetilde{HM}(\bar{Y}|\bar{R};\Gamma_\eta)$. Similarly to the case without local coefficients, the choice of η implies that if $t - t^{-1}$ is invertible in $\mathcal{R}$ then $\overline{HM}(\bar{Y}|\bar{R};\Gamma_\eta) = 0$. Hence $\widetilde{HM}(\bar{Y}|\bar{R};\Gamma_\eta) \cong \widehat{HM}(\bar{Y}|\bar{R};\Gamma_\eta)$ and the authors simply write $HM(\bar{Y}|\bar{R};\Gamma_\eta)$ without the decoration.

Proposition 4.1.2 ([37]). *If $t-t^{-1}$ is invertible in $\mathcal{R}$, then $SHM(M,\gamma;\Gamma_\eta)$ depends only on (M,γ) and $\mathcal{R}$. In this case we can allow $\bar{R}$ to have genus 1, but if $g(\bar{R}) \geq 2$ and $\mathcal{R}$ has no $\mathbb{Z}$-torsion then we also have*

$$SHM(M,\gamma;\Gamma_\eta) \cong SHM(M,\gamma) \otimes \mathcal{R}.$$

Given a knot K in a closed, oriented 3-manifold Y, we can form a sutured manifold $Y(K) = (M,\gamma)$ as in [31] by taking M to be the knot complement $Y \backslash N(K)$ and $\gamma \subset \partial M$ a pair of oppositely oriented meridians. Then monopole knot homology is defined by

$$KHM(Y,K) = SHM(M,\gamma),$$

and if we work with local coefficients we get $KHM(Y,K) \otimes \mathcal{R} \cong SHM(M,\gamma;\Gamma_\eta)$.

From now on we will fix $\mathcal{R}$ to be the Novikov ring

$$\left\{ \sum_\alpha c_\alpha t^\alpha \,\middle|\, \alpha \in \mathbb{R},\ c_\alpha \in \mathbb{Z},\ \#\{\beta < n \mid c_\beta \neq 0\} < \infty \text{ for all } n \right\},$$

with $\exp(\alpha) = t^\alpha$ and $(t-t^{-1})^{-1} = -t - t^3 - t^5 - \dots$. Although we will drop the local system Γ_η from our notation, we are always working with local coefficients over $\mathcal{R}$.

4.1.2 Contact structures in monopole Floer homology

Let (Y,ξ) be a closed contact 3-manifold. Kronheimer and Mrowka [35] associate a contact invariant

$$\psi(\xi) : \Lambda(\xi) \to \widetilde{HM}_\bullet(-Y, \mathfrak{s}_\xi, c_{\mathrm{bal}}, \Gamma_\eta)$$

where $c_{\text{bal}} = 2\pi c_1(\mathfrak{s}_\xi)$ is a balanced perturbation of the Seiberg-Witten equations and $\Lambda(\xi)$ is the set of orientations of an appropriate moduli space. In general we will ignore the orientations $\Lambda(\xi)$, since we are only concerned with the contact invariant as an element of $\widecheck{HM}$ up to automorphism, and so we will abuse notation and write $\psi(\xi) \in \widecheck{HM}_\bullet(-Y, \mathfrak{s}_\xi, c_{\text{bal}}, \Gamma_\eta)$.

Mrowka and Rollin [47, 46] investigated the behavior of the contact invariant under symplectic cobordisms.

Definition 4.1.3. A symplectic cobordism (W, ω) from (Y_-, ξ_-) to (Y_+, ξ_+) is said to be *left-exact* if ω is exact near Y_-, or equivalently if it is given in a collar neighborhood of Y_- by a symplectization $\frac{1}{2}d(t^2\eta_-)$ where $\xi_- = \ker \eta_-$. It is *right-exact* if the same holds near Y_+, and *boundary-exact* if it is both left- and right-exact.

Theorem 4.1.4 ([46, Theorem 3.5.4]). *Let W be a boundary-exact cobordism (W, ω) as above such that the map*

$$H^1(W; \mathbb{Z}) \to H^1(Y_+; \mathbb{Z})$$

is surjective, and let $W^\dagger$ denote W viewed as a cobordism from $-Y_+$ to $-Y_-$. Then $\psi(\xi_-) = \widecheck{HM}(W^\dagger, \mathfrak{s}_\omega)(\psi(\xi_+))$.

If W fails to be right-exact, meaning that we can only guarantee $\omega|_{\xi_+} > 0$, this functoriality statement will still hold for monopole Floer homology with Novikov coefficients; for boundary-exact cobordisms, however, it is true over $\mathbb{Z}$.

Corollary 4.1.5. *If W is a symplectic 2-handle cobordism corresponding to Legendrian surgery, then*

$$\psi(\xi_-) = \widecheck{HM}(W^\dagger)\psi(\xi_+)$$

and in particular $\widecheck{HM}(W^\dagger, \mathfrak{s})\psi(\xi_+)$ is zero for all Spin^c *structures except $\mathfrak{s}_\omega$.*

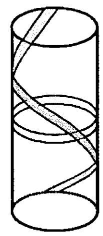


Figure 4.2.1: The convex torus $\partial(Y \backslash N(\mathcal{K}))$, cut along a meridian. The horizontal circles are sutures, while the pair of parallel arcs (or circles, once the top and bottom are identified) are dividing curves and have slope $\frac{1}{tb(\mathcal{K})}$.

4.2 The Legendrian knot invariant

Let $\mathcal{K} \subset (Y, \xi)$ be a Legendrian knot of topological knot type K. Our goal is to construct an appropriate contact structure $\bar{\xi}$ on a closure $(\bar{Y}, \bar{R})$ of the sutured knot complement $Y(K)$ so that the contact invariant $\psi(\bar{\xi})$ does not depend on any of the choices we must make. This will give us an invariant $\psi(\mathcal{K})$ of the Legendrian knot $\mathcal{K}$ which is an element of $KHM(-Y, K)$ up to automorphism.

Take a standard neighborhood $N(\mathcal{K})$ whose boundary is a convex torus; any sufficiently small tubular neighborhood of $\mathcal{K}$ will suffice, since one can be taken to any other by a contact isotopy and perturbation of the boundary. If we assign coordinates to $\partial N(\mathcal{K}) \cong \mathbb{R}^2/\mathbb{Z}^2$ so that $(\pm 1, 0)$ is a meridian and $(0, \pm 1)$ a longitude, then its dividing set Γ consists of two parallel curves of slope $\frac{1}{tb(\mathcal{K})}$, where $tb(\mathcal{K})$ is the Thurston-Bennequin invariant of $\mathcal{K}$. (See for example [19, Section 2.4].) In particular, if we view the sutured knot complement $Y(\mathcal{K})$ as the contact manifold $(Y \backslash N(\mathcal{K}), \xi|_{Y \backslash N(\mathcal{K})})$ with contact boundary, then each of the meridional sutures will intersect each dividing curve transversely in a single point as in Figure 4.2.1. Here, and in all other figures, we will color regions white and gray to represent the positive and negative regions, respectively, of a convex surface.

4.2.1 Closure of the sutured knot complement

Our construction follows the definition of the sutured invariant as in Section 4.1.1. We must first pick an auxiliary surface T whose boundary components are in one-to-

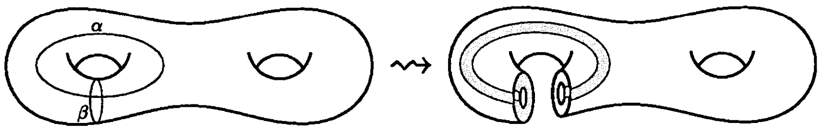


Figure 4.2.2: The construction of the auxiliary surface $(T \times I, \Xi)$.

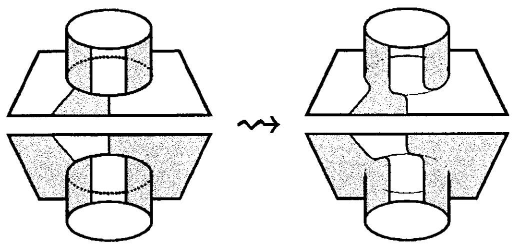


Figure 4.2.3: Gluing $T \times I$ to $Y \backslash N(\mathcal{K})$ and rounding edges in a cylindrical neighborhood of one of the sutures on $\partial N(\mathcal{K})$, as viewed from $T \times \{1\}$ on top and $T \times \{-1\}$ on the bottom.

one correspondence with the sutures of $Y(\mathcal{K})$ and glue the annuli $\partial T \times I \subset T \times I$ to neighborhoods $A(\gamma)$ of the sutures. In order to form a contact structure on this glued manifold, we must assign a contact structure to $T \times I$ whose restriction to a neighborhood of $\partial T \times I$ agrees with ξ in a neighborhood of $A(\gamma)$. By Giroux's flexibility theorem [27] it suffices to ensure that the dividing curves match, sending the positive region of $A(\gamma)$ to the negative region of $\partial T \times I$ and vice versa.

Let T_0 be a closed surface of genus at least 2, and pick a pair of dual curves $\alpha, \beta \subset T_0$ such that $\alpha \cdot \beta = 1$. Give T_0 the I-invariant contact structure Ξ_α whose dividing curves consist of two parallel disjoint copies of α on each surface $T_0 \times \{\pm 1\}$. We define $(T \times I, \Xi)$ as the contact manifold obtained by cutting $T_0 \times I$ along a convex perturbation of the annulus $\beta \times I$, as in Figure 4.2.2. Note that $T \times I$ can also be viewed as a product sutured manifold $(T \times I, \delta)$ with sutures $\partial T \times \{0\}$ in the annuli $A(\delta) = \partial T \times I$.

We now form a contact manifold $(Y', \xi') = (Y \backslash N(\mathcal{K})) \cup (T \times I)$ by gluing along some orientation-reversing diffeomorphism $A(\delta) \to A(\gamma)$ as described above. This manifold has edges, corresponding to the corners $\partial T \times \partial I$, which we smooth using edge-rounding [29], under which dividing curves turn to the left (as viewed from outside Y') as they approach an edge. See Figure 4.2.3.

Lemma 4.2.1. *The contact manifold* $Y' = (Y \backslash N(\mathcal{K})) \cup (T \times I)$ *depends only on* $\mathcal{K}$, (Y, ξ), *and the genus of* T_0.

Proof. The construction of $T \times I$ depends only on $g(T_0)$ and on the curves $\alpha, \beta \subset T_0$. Given any other pair of curves α' and β' which intersect once, there is an isomorphism $\varphi : T_0 \to T_0$ with $\varphi(\alpha) = \alpha'$ and $\varphi(\beta) = \varphi(\beta')$, and this extends to a contactomorphism $\varphi \times Id : (T_0 \backslash N(\beta)) \times I \to (T_0 \backslash N(\beta')) \times I$. □

Finally, we close up Y' to get a contact manifold $(\bar{Y}, \bar{\xi})$ with distinguished convex surface $\bar{R}$. The boundary of Y' consists of two convex surfaces $\bar{R}_+$ and $\bar{R}_-$ determined by $T \times \{\pm 1\} \subset \bar{R}_\pm$. These are split by pairs of parallel dividing curves $\Gamma_\pm \subset \bar{R}_\pm$ into positive and negative regions $(\bar{R}_+)_\pm \subset \bar{R}_+$ and $(\bar{R}_-)_\pm \subset \bar{R}_-$, and each of $(\bar{R}_+)_-$ and $(\bar{R}_-)_+$ is an annulus. Fix any diffeomorphism $h : \bar{R}_+ \to \bar{R}_-$ which sends $(\bar{R}_+)_\pm$ to $(\bar{R}_-)_\mp$, and hence also Γ_+ to Γ_-, and such that $h(x \times \{1\})$ and $x \times \{-1\}$ lie in the same component of Γ_- for any $x \times \{1\}$ in $\Gamma_+ \cap (\mathrm{int}(T) \times \{1\})$. In other words, a dividing curve $c \subset \Gamma_+$ corresponds to one of the two copies of $\alpha \subset T_0$, and we require $h(c)$ to be the dividing curve of Γ_- corresponding to the same copy of α.

Finally, we glue $\bar{R}_+$ to $\bar{R}_-$ via h. The resulting contact manifold is the desired $(\bar{Y}, \bar{\xi})$.

Definition 4.2.2. The contact invariant of the Legendrian knot $\mathcal{K}$ is $\psi(\mathcal{K}) = \psi(\bar{Y}, \bar{\xi}) \in \widetilde{HM}(-\bar{Y}, \mathfrak{s}_{\bar{\xi}}; \Gamma_\eta)$.

We can compute that

$$\langle c_1(\bar{\xi}), \bar{R} \rangle_{\bar{Y}} = \chi((\bar{R}_+)_+) - \chi((\bar{R}_+)_-) = 2 - 2g(\bar{R})$$

and so $\langle c_1(\mathfrak{s}_{\bar{\xi}}), \bar{R} \rangle_{-\bar{Y}} = 2g(\bar{R}) - 2$. This means that $\psi(\mathcal{K})$ is in fact an element of $\widetilde{HM}(-\bar{Y}|\bar{R}; \Gamma_\eta) = SHM(-Y(K); \Gamma_\eta)$, which is by definition the knot homology with local coefficients. Therefore

$$\psi(\mathcal{K}) \in KHM(-Y, K) \otimes \mathcal{R}.$$

Remark 4.2.3. The desire to arrange that $\langle c_1(\bar{\xi}), \bar{R}\rangle = 2-2g$ motivated our choice of contact structure on $T \times I$. In particular, $\chi((\bar{R}_+)_+)$ and $\chi((\bar{R}_+)_-)$ sum to $\chi(\bar{R}_+) = 2-2g$, and if we fix their difference as above then we must have $\chi((\bar{R}_+)_-) = 0$. But now $(\bar{R}_+)_-$ does not have any sphere or torus components, and if it had disk components then $\bar{R}_+$ would not have a tight neighborhood [27], so $(\bar{R}_+)_-$ is forced to be a union of annuli.

In addition to the Legendrian knot $\mathcal{K} \subset (Y, \xi)$, the construction of $\psi(\mathcal{K})$ from a closure $(\bar{Y}, \bar{\xi})$ with distinguished convex surface $\bar{R}$ potentially depends on both the choice of diffeomorphism $\bar{R}_+ \to \bar{R}_-$ and the genus $g(T_0)$. Our goal in the next few sections is to prove that this is not the case.

4.2.2 Invariance under diffeomorphism

In this section we establish that $\psi(\mathcal{K})$ is independent of the choice of diffeomorphism $\bar{R}_+ \to \bar{R}_-$.

Proposition 4.2.4. *Let $(\bar{Y}', \bar{\xi}')$ be the contact manifold obtained from $\bar{Y}$ by cutting along the convex surface $\bar{R}$ and regluing along some orientation-preserving diffeomorphism h such that $h(\gamma) = \gamma$ for each dividing curve γ of $\bar{R}$. Then there is an isomorphism $\widetilde{HM}(-\bar{Y}'|\bar{R};\Gamma_\eta) \to \widetilde{HM}(-\bar{Y}|\bar{R};\Gamma_\eta)$ which sends $\psi(\bar{Y}', \bar{\xi}')$ to $\psi(\bar{Y}, \bar{\xi})$.*

Lemma 4.2.5. *Proposition 4.2.4 holds when h is a Dehn twist along some nonseparating curve c which does not intersect the dividing curves Γ of $\bar{R}$.*

Proof. We observe that c is nonisolating, i.e. that every component of $\bar{R}\backslash(\Gamma \cup c)$ has a boundary component which intersects Γ, and thus by the Legendrian Realization Principle [34, 29] we can take c to be Legendrian. Indeed, the complement $\bar{R}\backslash\Gamma$ has two connected components; if c is nonseparating within its component then it is clearly nonisolating. Otherwise, c divides its component of $\bar{R}\backslash\Gamma$ into two components, say A and B. Since c is nonseparating in $\bar{R}$ there is a path in $\bar{R}\backslash c$ which connects A to B, and this path must pass through the other component of $\bar{R}\backslash\Gamma$. In particular, the path crosses Γ, so both ∂A and ∂B intersect Γ and thus c is nonisolating.

Suppose now that h is a positive Dehn twist along c, and that c has been realized as a Legendrian curve. Then h can be realized by (-1)-surgery on c with respect to the framing induced by $\bar{R}$, and since $tw(c,\bar{R}) = -\frac{1}{2}|c \cap \Gamma| = 0$ this is a Legendrian surgery. If W is the corresponding symplectic cobordism, and $W^\dagger$ is the oppositely oriented cobordism from $-\bar{Y}'$ to $-\bar{Y}$, then

$$\widetilde{HM}(W^\dagger)(\psi(\bar{Y}',\bar{\xi}')) = \psi(\bar{Y},\bar{\xi})$$

by Corollary 4.1.5. The fact that $\widetilde{HM}(W^\dagger)$ gives an isomorphism $\widetilde{HM}(-\bar{Y}'|\bar{R};\Gamma_\eta) \to \widetilde{HM}(-\bar{Y}|\bar{R};\Gamma_\eta)$ is an easy consequence of the surgery exact triangle for $\widetilde{HM}$ and the adjunction inequality [36].

If instead h is a negative Dehn twist, we note that $\bar{Y}$ can be obtained from $\bar{Y}$' by a positive Dehn twist along c, hence we construct a cobordism W from $\bar{Y}'$ to $\bar{Y}$ as above and then $\widetilde{HM}(W^\dagger)^{-1}$ is the desired isomorphism. □

Proof of Proposition 4.2.4. In general, we can arrange by an isotopy that the diffeomorphism h is actually the identity on each dividing curve. Then h restricts to a boundary-fixing diffeomorphism on the closure of each component of $\bar{R}\backslash\Gamma$. One component is an annulus A, so up to isotopy $h|_A$ is a composition of Dehn twists about the core of A. The other component is a surface Σ of genus $g(\bar{R}) - 1 \geq 1$ with two boundary components, and so $h|_\Sigma$ can also be expressed as a product of Dehn twists about nonseparating curves which do not intersect $\Gamma = \partial\Sigma$. Since $h = h|_A \circ h|_\Sigma$, repeated application of Lemma 4.2.5 completes the proof of Proposition 4.2.4. □

We have now shown that the construction of $\psi(\mathcal{K}) \in KHM(-Y,K) \otimes \mathcal{R}$ is independent of all choices except possibly the genus $g = g(\bar{R})$. Thus we have constructed a sequence of Legendrian knot invariants $\psi_g(\mathcal{K})$ for $g \geq 2$; we conjecture that these are all equal.

Remark 4.2.6. One can apply a similar construction in sutured Floer homology [31, 32]; indeed, Lekili [38] has shown that

$$SFH(M,\gamma) \cong HF^+(\bar{Y}|\bar{R}),$$

where again $\bar{Y}$ and $\bar{R}$ are constructed from the sutured manifold (M, γ) as above and the right hand side is the sum of $HF^+(\bar{Y}, \mathfrak{s})$ over all extremal Spin^c structures. Since $SFH(Y(K)) \cong \widehat{HFK}(Y, K)$, the Heegaard Floer contact invariant [55] of $(\bar{Y}, \bar{\xi})$ gives us an element

$$\tilde{\psi}_g(\mathcal{K}) \in \widehat{HFK}(-Y, K)/\{\pm 1\}.$$

The invariance of ψ_g under the choice of diffeomorphism h follows just as before, except we replace the monopole Floer homology argument in the proof of Lemma 4.2.5 with the Heegaard Floer homology argument in section 2 of [26].

4.2.3 The Legendrian unknot

The Legendrian representatives of the topological unknot $U \subset S^3$ were classified by Eliashberg and Fraser [13]: they are completely determined by their classical invariants tb and r, and there is a representative $\mathcal{U}$ with $(tb, r) = (-1, 0)$ so that all others are stabilizations of $\mathcal{U}$. In this subsection we will prove that the Legendrian invariant of $\mathcal{U}$ is a unit of $KHM(U) \cong \mathcal{R}$, a fact which we will use in Section 4.2.4.

Our strategy is to explicitly determine the contact structure on a particular closure $\bar{Y}$ of $S^3(\mathcal{U})$.

Lemma 4.2.7. *Let ξ be the I-invariant contact structure on $(S^1 \times I) \times I$ whose dividing curves on the annulus $S^1 \times I$ are a pair of parallel arcs $\{t_1\} \times I$ and $\{t_2\} \times I$. Then after edge-rounding, ξ is contactomorphic to the complement of $\mathcal{U}$.*

Proof. By the classification of tight contact structures on solid tori [29, Theorem 2.3], there is a unique tight contact structure Ξ on $S^1 \times D^2$ for which the dividing curves on the boundary have slope -1; since $tb(\mathcal{U}) = -1$, the complement of $\mathcal{U}$ must be $(S^1 \times D^2, \Xi)$. But if we round the edges on $((S^1 \times I) \times I, \xi)$, we get a tight contact structure on $S^1 \times D^2$ for which the dividing curves on the boundary $S^1 \times S^1$ have slope -1, and so this contact structure must be Ξ as well. $\square$

Proposition 4.2.8. *The invariant $\psi_g(\mathcal{U})$ is a unit in $KHM(U) \cong \mathcal{R}$.*

Proof. We glue a surface $T \times I$ to $(S^1 \times I) \times I$ as in Section 4.2.1, identifying the annuli $\partial T \times I$ with $(S^1 \times I) \times \partial I$, to get an I-invariant contact manifold $Y' = \Sigma_g \times I$ for some $g \geq 2$ which is universally tight by Giroux's criterion [27] and has convex boundary. Gluing $\Sigma_g \times \{1\}$ to $\Sigma_g \times \{-1\}$ via the identity map, we get the closure $\bar{Y} = \Sigma_g \times S^1$ with S^1-invariant, universally tight contact structure $\bar{\xi}$ and distinguished surface $\bar{R} = \Sigma_g \times \{*\}$. Since no component of $\Gamma \subset \Sigma_g$ is separating, Theorem 5 of [53] asserts that $\bar{\xi}$ is weakly fillable.

The claim that $KHM(U) = \widecheck{HM}(\bar{Y}|\bar{R}) \cong \mathcal{R}$ now follows from Lemma 4.7 of [37]. Furthermore, since $\bar{\xi}$ is weakly fillable we know that $\psi(\bar{\xi})$ is a unit of $\widecheck{HM}(\bar{Y})$ [35, 46], and since $\psi(\bar{\xi}) \in \widecheck{HM}(\bar{Y}|\bar{R}) \cong \mathcal{R}$ the proposition follows. □

Remark 4.2.9. Wendl [67, Corollary 2] has shown that $(\bar{Y}, \bar{\xi})$ has vanishing untwisted ECH contact invariant. By work of Taubes [65] it follows that the untwisted contact invariant $\psi(\bar{\xi}) \in \widecheck{HM}(\bar{Y}|\bar{R})$ is also zero, so we must work with twisted coefficients for $\psi_g(\mathcal{U})$ to be nonzero.

4.2.4 Invariance under choice of genus

We expect there to be a connected sum formula of the form

$$\psi_g(\mathcal{K}) \otimes \psi_{g'}(\mathcal{K}') = \psi_{g+g'-1}(\mathcal{K}\#\mathcal{K}')$$

for Legendrian knots $\mathcal{K} \subset Y$ and $\mathcal{K}' \subset Y'$. If $\mathcal{K}'$ is the Legendrian unknot in (S^3, ξ_{std}), then $\psi_{g'}(\mathcal{K}')$ is a unit of $KHM(-Y, \mathcal{K}') \cong \mathcal{R}$ by Proposition 4.2.8. By taking $g' = 2$, it would follow that $\psi_g(\mathcal{K}) = \psi_{g+1}(\mathcal{K})$ for all $g \geq 2$, hence $\psi_g(\mathcal{K})$ would be independent of g. We will instead explain another approach to show that $\psi_g(\mathcal{K})$ does not depend on g.

Conjecture 4.2.10. *Let (Y, ξ) be a (possibly disconnected) contact 3-manifold with two disjoint embedded convex tori T and T', and let $\varphi : T \times I \to T' \times I$ be a contactomorphism of two I-invariant neighborhoods of T and T' in Y. Let (Y', ξ') be the manifold obtained by cutting Y along T and T' and then gluing $T \times (-1, 0]$ and*

$T' \times (-1,0]$ to $T' \times [0,1)$ and $T \times [0,1)$, respectively, via φ. Then there is a left-exact symplectic cobordism (W,ω) from (Y,ξ) to (Y',ξ'), which is obtained topologically by taking the symplectization $Y \times I$ and gluing $(T^2 \times I) \times I$ to the neighborhoods of $(T \times I) \times \{1\}$ and $(T' \times I) \times \{1\}$ along $(T^2 \times I) \times \partial I$.

If T and T' were instead pre-Lagrangian tori and symplectic in a weak filling of Y, this would be the "splicing" construction of Niederkrüger and Wendl [53] which is used to construct a weak filling of Y'; we note, however, that convex tori are not pre-Lagrangian and so the same construction does not apply.

Remark 4.2.11. We cannot insist that the cobordism (W,ω) of Conjecture 4.2.10 be right-exact, i.e. that $\omega|_{Y'} = d\alpha'$ for a contact form α'. Indeed, let (T^3,ζ_1) be the strongly symplectically fillable contact structure $\zeta_1 = \ker(\cos(z)dx - \sin(z)dy)$, and let (Y,ξ) consist of two disjoint copies of (T^3,ζ_1). Perturb the torus $\{z=0\}$ in T^3 to be convex, and let T and T' be identical copies of that convex torus in each component of Y. Then $(Y',\xi') = (T^3,\zeta_2)$, where $\zeta_2 = \ker(\cos(2z)dx - \sin(2z)dy)$ is not strongly fillable [11]; but we can glue strong fillings of each (T^3,ζ_1) component of Y to (W,ω) to get a strong filling of (T^3,ζ_2), a contradiction.

Given a Legendrian knot $\mathcal{K} \subset Y$, we begin to construct the closure $\bar{Y}$ of $Y\backslash N(\mathcal{K})$ by gluing a product $T \times I$ to $Y\backslash N(\mathcal{K})$ along two convex annuli A and A'. Both A and A' are neighborhoods of meridians in $\partial N(\mathcal{K})$, and the dividing curves on each are a pair of arcs $\{t_0,t_1\} \times I \subset S^1 \times I$. In the final gluing step, we arrange for the two components of ∂A to be glued together, and likewise $\partial A'$, so that the image of each of these is a convex torus in $\bar{Y}$ with a pair of parallel dividing curves, and the torus intersects the surface $\bar{R}$ in a simple closed Legendrian curve.

Suppose Conjecture 4.2.10 holds. We can take the genus g closure $(\bar{Y},\bar{R})$ of $Y\backslash N(\mathcal{K})$ as above and the closure $(\Sigma_{g'} \times S^1, \Sigma_{g'} \times \{*\})$ of the Legendrian unknot complement $S^3\backslash N(\mathcal{U})$ as in Section 4.2.3, and then splice them together along one of the convex tori in $\bar{Y}$ and the corresponding torus in $\Sigma_{g'} \times S^1$. See Figure 4.2.4 for an illustration. The resulting manifold $(\bar{Y}',\bar{R}')$ is not in general an allowed closure of a knot complement, but in this case the only interesting part of it comes from $Y\backslash N(\mathcal{K})$

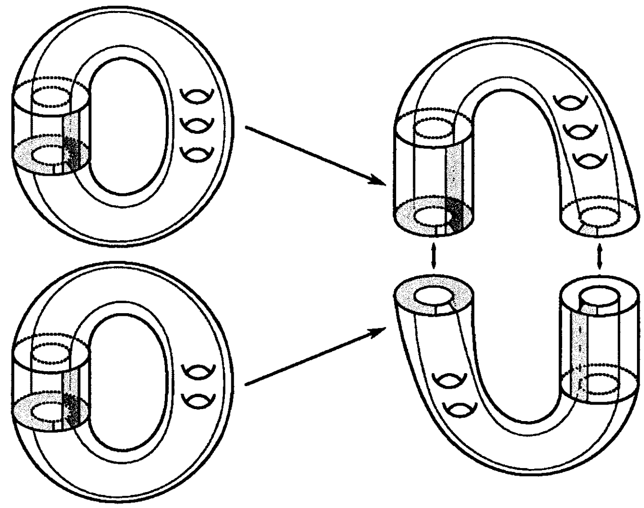


Figure 4.2.4: Splicing closures of knot complements along the indicated convex tori, all of which are oriented so that the normal direction is upward (whether or not this matches the boundary orientation on the right). We also indicate part of the negative annulus on the boundary of each knot complement for reference.

and it is easy to see that $\bar{Y}'$ is a genus $g+g'-1$ closure of $Y\backslash N(\mathcal{K})$. Thus we have an map

$$\widetilde{HM}(\bar{Y} \cup (\Sigma_{g'} \times S^1) \mid \bar{R} \cup (\Sigma_{g'} \times \{*\}); \Gamma_\eta) \to \widetilde{HM}(\bar{Y}'|\bar{R}'; \Gamma_{\eta'})$$

induced by the cobordism W described in Conjecture 4.2.10; we remark that the cobordism certainly exists, although the symplectic structure ω is conjectural. Theorem 3.2 of [37] asserts that this map is an isomorphism.

Remark 4.2.12. Since $\widetilde{HM}(\Sigma_{g'} \times S^1 \mid \Sigma_{g'} \times \{*\}) \cong \mathcal{R}$, the Künneth formula tells us that the source of $\widetilde{HM}(W)$ is isomorphic to $\widetilde{HM}(\bar{Y}|\bar{R};\Gamma_\eta) \otimes \mathcal{R}$, hence W induces an isomorphism $\widetilde{HM}(\bar{Y}|\bar{R};\Gamma_\eta) \xrightarrow{\sim} \widetilde{HM}(\bar{Y}'|\bar{R}';\Gamma_{\eta'})$. The symplectic structure ω then implies that $\widetilde{HM}(W^\dagger) : \widetilde{HM}(-\bar{Y}'|\bar{R}';\Gamma_{\eta'}) \to \widetilde{HM}(-\bar{Y}|\bar{R};\Gamma_\eta)$ is an isomorphism carrying $\psi_{g+g'-1}(\mathcal{K})$ to $\psi_g(\mathcal{K})$, hence if we take $g' = 2$ it follows that $\psi_g(\mathcal{K}) = \psi_{g+1}(\mathcal{K})$ as elements of $KHM(-Y,K)$ up to automorphism.

From now on we will drop the g subscript and simply write $\psi(\mathcal{K})$ to mean $\psi_g(\mathcal{K})$ for some fixed $g \geq 2$.

4.3 Stabilization

Let $S_+(\mathcal{K})$ and $S_-(\mathcal{K})$ denote the positive and negative stabilizations of a Legendrian knot $\mathcal{K}$, which may also be thought of as the connected sums $\mathcal{K}\#\mathcal{U}_\pm$ where $\mathcal{U}_\pm \subset S^3$ is the topologically trivial knot with $tb = -2$ and $r = \pm 1$. We expect the following to be true of $\psi(\mathcal{K})$:

Conjecture 4.3.1. *For any Legendrian knot $\mathcal{K} \subset Y$ we have $\psi(S_-(\mathcal{K})) = \psi(\mathcal{K})$ and $\psi(S_+(\mathcal{K})) = 0$.*

A theorem of Epstein, Fuchs, and Meyer [15] characterizes transverse knots as pushoffs of Legendrian knots up to negative stabilization, and so ψ would give a transverse knot invariant as well. Namely, let $\mathcal{T}$ be a transverse pushoff of $\mathcal{K}$, and define $\tau(\mathcal{T}) = \psi(\mathcal{K})$ as an element of $KHM(-Y, K)$. Then by Conjecture 4.3.1 it would follow that $\tau(\mathcal{T})$ is an invariant of $\mathcal{T}$ up to transverse isotopy, and if $\mathcal{T}$ is a stabilization of some other transverse knot then $\tau(\mathcal{T}) = 0$.

4.3.1 Multiple stabilizations

We can give a direct geometric proof of a slightly weaker result than the desired $\psi(S_+(\mathcal{K})) = 0$ of Conjecture 4.3.1.

Proposition 4.3.2. *If $\mathcal{K}$ is any Legendrian knot, then $\psi(S_+S_-(\mathcal{K})) = 0$.*

Proof. We will construct a closure $\bar{Y}$ of $\mathcal{K}' = S_+S_-(\mathcal{K})$ with an overtwisted disk, so that the vanishing follows from Corollary B of [47]. Stabilization of a Legendrian knot $\mathcal{K}$ corresponds to attaching a bypass to its complement: if we stabilize to get $S_\pm(\mathcal{K})$ inside a standard neighborhood $N(\mathcal{K}) \subset Y$ and fix a standard neighborhood $N(\mathcal{K}^\pm) \subset N(\mathcal{K})$, then $Y\backslash N(\mathcal{K}^\pm)$ is obtained from $Y\backslash N(\mathcal{K})$ by a bypass attachment. See [18] or the proof of Theorem 1.5 in [63] for discussion.

In the leftmost part of Figure 4.3.1 we have indicated the attaching arcs c_+ and c_- of bypasses corresponding to positive and negative stabilizations on $\partial(Y\backslash N(\mathcal{K}'))$, as in Figure 10 of [63], with a single meridional suture μ between them. The dividing curves are shown with orientation for convenience, so that they have the same orientation

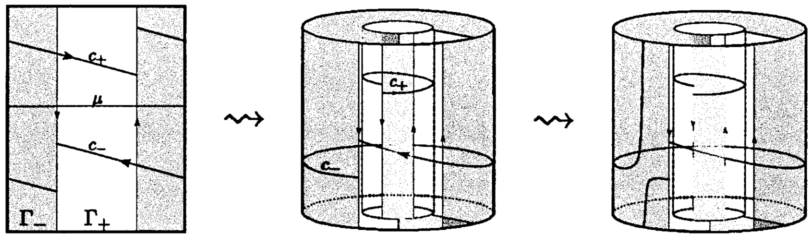


Figure 4.3.1: Attaching curves for bypasses in the complement of $S_+S_-(\mathcal{K})$ and its closure.

as the positive region $\partial\Gamma_+$. If we start to form the closure of $Y\backslash N(\mathcal{K}')$ by attaching a surface $T \times I$ to neighborhoods of the sutures and then rounding edges, we may then cut out $T \times I$ to get a contact manifold with corners as in the middle figure; this indicates the positions of the arcs $c_\pm$ on the boundary components $\bar{R}_\pm \subset Y'$.

We wish to glue $\bar{R}_+$ to $\bar{R}_-$ so that the arcs c_+ and c_- are glued together, but as shown in the middle of Figure 4.3.1 we cannot do this by identifying the inside and outside regions in the obvious way. Indeed, we must identify the white component $(\bar{R}_+)_+$ on the outside with the gray component $(\bar{R}_-)_-$ on the inside, identifying the left dividing curve on the outside with the left dividing curve on the inside and likewise for the right dividing curves, but then c_+ and c_- cannot be made parallel so that they end up identified. The problem is that as we follow them leftward and around the back of the cylinder from the leftmost dividing curves, the arc c_- ends "above" its starting point whereas c_+ ends "below" its starting point. However, we can glue c_+ to c_- by applying a Dehn twist to the outer gray annulus $(\bar{R}_+)_-$ along its core as shown on the right side of Figure 4.3.1. We can then "untwist" c_- by reparametrizing $(\bar{R}_+)_-$, sliding the lower endpoint of c_- downward along its dividing curve until it has nearly traversed the entire curve and lies just above the other endpoint; this allows us to identify it with $(\bar{R}_-)_+$ so that c_- is sent to c_+. Now we can glue $\bar{R}_+$ to $\bar{R}_-$ so that c_- and c_+ are identified, and the union of their respective bypasses is an overtwisted disk in the closure $(\bar{Y}, \bar{\xi})$. We conclude that $\psi(\bar{Y}, \bar{\xi})$, and hence $\psi(\mathcal{K}')$, is zero. $\square$

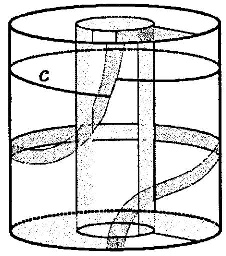


Figure 4.3.2: The contact structure on a partial closure of the complement of $S_-(\mathcal{U})$ near the negative bypass.

4.3.2 Negative stabilization

The complement of the negative stabilization $S_-(\mathcal{U})$ of the Legendrian unknot differs from the complement of $\mathcal{U}$ by a single negative bypass attachment. Thus when we construct a closure of $S^3\backslash N(S_-(\mathcal{U}))$ as in the proof of Proposition 4.2.8, before the final gluing step we have a contact structure ξ' on $Y' = \Sigma_g \times I$ whose dividing curves on either boundary component differ by a single Dehn twist about a meridian μ of $S_-(\mathcal{U})$. Figure 4.3.2 shows the dividing curves in $N_\mu \times I$, where $N_\mu \subset \Sigma_g$ is a neighborhood of that meridian, as well as the arc of attachment $c \subset N_\mu \times \{1\}$ for the bypass, which lies inside $N_\mu \times I$. Note that ξ' is I-invariant on $(\Sigma_g\backslash N_\mu) \times I$, and that if we dig out the bypass the resulting contact structure is I-invariant on all of $\Sigma_g \times I$, so (Y', ξ') is a basic slice in the terminology of [30].

Let $(\bar{Y}, \bar{\xi})$ be the closure of (Y', ξ') constructed by gluing $\Sigma_g \times \{1\}$ to $\Sigma_g \times \{-1\}$ via a Dehn twist τ around the meridian μ from Figure 4.3.2, where $\Sigma_g \times \{1\}$ is the outside region and $\Sigma_g \times \{-1\}$ is the inside one; we will write $\bar{Y} = M_\tau$ for convenience, since it is the mapping torus of τ. Then $\bar{\xi}$ is universally tight, since basic slices are by [30, Theorem 1.1].

Remark 4.3.3. If we perform the same construction for $S_+(\mathcal{U})$, we get a contact structure on $\Sigma_g \times I$ with the same dividing curves before the final gluing step. This is the other basic slice with the given dividing curves, and can be distinguished from ξ' by the sign of its relative Euler class $\tilde{e}$.

We conjecture that $\bar{\xi}$ is weakly fillable, which would imply that $\psi(S_-(\mathcal{U}))$ is a unit of $KHM(S^3, U) \cong \mathcal{R}$. Together with a splicing argument as in Section 4.2.4, it would follow that $\psi(S_-(\mathcal{K})) = \psi(\mathcal{K})$ for all $\mathcal{K}$. From Proposition 4.3.2 and the fact that $S_+S_-(\mathcal{K}) = S_-S_+(\mathcal{K})$ we could then conclude that $\psi(S_+(\mathcal{K})) = \psi(S_+S_-(\mathcal{K})) = 0$, completing the proof of Conjecture 4.3.1.

4.4 Properties of the invariant

4.4.1 Loose knots

Recall that a Legendrian knot $\mathcal{K} \subset (Y, \xi)$ is said to be *loose* if the complement of $\mathcal{K}$ is overtwisted.

Proposition 4.4.1. *If $\mathcal{K} \subset Y$ is loose, then $\psi(\mathcal{K}) = 0$.*

Proof. By assumption $Y \backslash \mathcal{K}$ has an overtwisted disk, so any closure $(\bar{Y}, \bar{\xi})$ does as well. Then $\psi(\bar{Y}, \bar{\xi})$ vanishes (see [47, Corollary B]), hence so does $\psi(\mathcal{K})$. □

4.4.2 Connected sums

Proposition 4.4.2. *Let (Y, ξ) and (Y', ξ') be contact manifolds and $\mathcal{K} \subset Y$ a Legendrian knot. Then under the obvious identification*

$$KHM(-Y, K) \otimes \widecheck{HM}(-Y') \cong KHM(-(Y \# Y'), K)$$

coming from the Künneth formula for $\widecheck{HM}(-(Y \# Y'))$, we have $\psi(\mathcal{K} \subset Y) \otimes \psi(Y') = \psi(\mathcal{K} \subset Y \# Y')$.

Proof. Fix a closure $(\bar{Y}, \bar{\xi})$ of $Y \backslash N(\mathcal{K})$. Then $(\bar{Y} \# Y', \bar{\xi} \# \xi')$ is a closure of $Y \# Y' \backslash N(\mathcal{K})$, and so the claim is immediate. □

Corollary 4.4.3. *Let $\mathcal{U}_Y \subset (Y, \xi)$ be a Legendrian unknot with $(tb, r) = (-1, 0)$ contained in a Darboux ball of Y. Then $\psi(\mathcal{U}_Y) \in KHM(-Y, K) \cong \widecheck{HM}(-Y)$ is equal to the contact invariant of (Y, ξ).*

Proof. Write $(Y, \xi) = (Y, \xi) \# (S^3, \xi_{\text{std}})$ with $\mathcal{U}_Y$ in the S^3 summand. By the previous proposition we have $\psi(\mathcal{U}_Y) = \psi(\mathcal{U} \subset S^3) \otimes \psi(Y)$, and $\psi(\mathcal{U} \subset S^3) = 1$ as an element of $KHM(-S^3, U) \cong \mathcal{R}$ up to automorphism by Proposition 4.2.8. □

4.4.3 Contact $(+1)$-surgery

The following is a direct analogue of Theorem 1.1 of [54], which concerns the Heegaard Floer invariant $\hat{\mathcal{L}}(\mathcal{K}) \in \widehat{HFK}(-Y, K)$ (or more generally $\mathcal{L}(\mathcal{K}) \in HFK^-(-Y, K)$) but is much harder to prove.

Theorem 4.4.4. *Let $\mathcal{K}$ and $\mathcal{S}$ be disjoint Legendrian knots in (Y, ξ), and let $(Y_{\mathcal{S}}, \xi_{\mathcal{S}})$ denote the contact manifold obtained by performing contact $(+1)$-surgery along $\mathcal{S}$. Let $\mathcal{K}_{\mathcal{S}}$ be the Legendrian knot $\mathcal{K}$ as viewed in $Y_{\mathcal{S}}$, and suppose that both $\mathcal{K}$ and $\mathcal{K}_{\mathcal{S}}$ are nullhomologous. Then there is a map $KHM(-Y, K) \to KHM(-Y_{\mathcal{S}}, K_{\mathcal{S}})$ such that $\psi(\mathcal{K}) \mapsto \psi(\mathcal{K}_{\mathcal{S}})$.*

Proof. We may obtain (Y, ξ) by performing contact (-1)-surgery on $\mathcal{S} \subset Y_{\mathcal{S}}$. Since $\mathcal{S}$ and $\mathcal{K}$ are disjoint it is easy to see that we can fix a closure $\bar{Y}_{\mathcal{S}}$ of the complement $Y_{\mathcal{S}}(\mathcal{K}_{\mathcal{S}})$ so that contact (-1)-surgery on $\mathcal{S} \subset \bar{Y}_{\mathcal{S}}$ gives a closure $\bar{Y}$ of $Y(\mathcal{K})$, and the surface $\bar{R}$ is the same in both closures. The symplectic cobordism (W, ω) from $\bar{Y}_{\mathcal{S}}$ to $\bar{Y}$ coming from this handle attachment gives a map

$$\widecheck{HM}(W^{\dagger}) : \widecheck{HM}(-\bar{Y}) \to \widecheck{HM}(-\bar{Y}_{\mathcal{S}})$$

carrying $\psi(\mathcal{K})$ to $\psi(\mathcal{K}_{\mathcal{S}})$ by Corollary 4.1.5, and $\widecheck{HM}(W^{\dagger}, \mathfrak{s})(\psi(\mathcal{K}))$ is zero for all Spin^c structures $\mathfrak{s} \neq \mathfrak{s}_{\omega}$. If we restrict $\widecheck{HM}(W^{\dagger})$ to the Spin^c structures on W which restrict to $\bar{\xi}_{\mathcal{S}}$ and $\bar{\xi}$ on the boundary, then we have a map

$$F_{W^{\dagger}} : KHM(-Y, K) \to KHM(-Y_{\mathcal{S}}, K_{\mathcal{S}})$$

such that $F_{W^{\dagger}, \mathfrak{s}}(\psi(\mathcal{K}))$ is $\psi(\mathcal{K}_{\mathcal{S}})$ for a unique Spin^c structure (again, $\mathfrak{s}_{\omega}$) and zero for all others. $\square$

4.5 Lagrangian concordance

Chantraine [3] defined an interesting notion of concordance on the set of all Legendrian knots in a contact 3-manifold Y.

Definition 4.5.1. Let $\mathcal{K}_0$ and $\mathcal{K}_1$ be Legendrian knots parametrized by embeddings $\gamma_i : S^1 \to Y$, and let $Y \times \mathbb{R}$ be the symplectization of Y. We say that $\mathcal{K}_0$ is *Lagrangian concordant* to $\mathcal{K}_1$, denoted $\mathcal{K}_0 \prec \mathcal{K}_1$, if there is a Lagrangian embedding $L : S^1 \times \mathbb{R} \hookrightarrow Y \times \mathbb{R}$ and a $T > 0$ such that $L(s,t) = (\gamma_0(s), t)$ for $t < -T$ and $L(s,t) = (\gamma_1(s), t)$ for $t > T$.

Theorem 4.5.2 ([3]). *The relation $\prec$ descends to a relation on Legendrian isotopy classes of Legendrian knots. If $\mathcal{K}_0 \prec \mathcal{K}_1$ then $tb(\mathcal{K}_0) = tb(\mathcal{K}_1)$ and $r(\mathcal{K}_0) = r(\mathcal{K}_1)$.*

Our goal in this section is to investigate the behavior of $\psi(\mathcal{K})$ under Lagrangian concordance:

Theorem 4.5.3. *Let $\mathcal{K}_0, \mathcal{K}_1$ be Legendrian knots in a contact homology 3-sphere Y satisfying $\mathcal{K}_0 \prec \mathcal{K}_1$. Then there is a homomorphism $KHM(-Y, K_1) \to KHM(-Y, K_0)$ sending $\psi(\mathcal{K}_1)$ to $\psi(\mathcal{K}_0)$.*

We compare this with the remarks in [3, Section 5.2], where it is observed that Lagrangian concordance induces a map $LCH(\mathcal{K}_1) \to LCH(\mathcal{K}_0)$ on Legendrian contact homology.

Proof. We fix a particular closure $(\bar{Y}_i, \bar{R}_i)$ of the sutured knot complements $Y(\mathcal{K}_i)$: place the meridional sutures close together so that in $\partial(Y \backslash \mathcal{K}_i)$ they bound an annulus A in which the dividing curves are parallel to a longitude. In the other annulus A' bounded by the sutures, the dividing curves twist around the meridional direction a total of $tb(\mathcal{K}_i)$ times; recall that $tb(\mathcal{K}_0) = tb(\mathcal{K}_1)$. We glue a surface $T \times I$ to each complement and round edges, resulting in a manifold with boundary $\bar{R}_+ \sqcup \bar{R}_-$ and $\text{int}(A) \subset \bar{R}_+$. Finally, we glue $\bar{R}_+$ to $\bar{R}_-$ by identifying $(x, 1) \in T \times \{1\}$ to $(x, -1) \in T \times \{-1\}$ for all $x \in \text{int}(T)$, and identifying A to A' by a homeomorphism composed of enough Dehn twists around the core of A to make the dividing curves match.

This construction guarantees that $Z_0 = \bar{Y}_0 \backslash \text{int}(Y \backslash \mathcal{K}_0)$ and $Z_1 = \bar{Y}_1 \backslash \text{int}(Y \backslash \mathcal{K}_1)$ are contactomorphic as 3-manifolds with torus boundary. In the symplectization $Y \times \mathbb{R}$, the cylinder $\mathcal{K}_0 \times \mathbb{R}$ is Lagrangian, hence it has a standard neighborhood symplectomorphic to a neighborhood N of the 0-section in $T^*(S^1 \times \mathbb{R})$. Then a neighborhood of the boundary $T^2 \times \mathbb{R}$ of the symplectization $Z_0 \times \mathbb{R}$, can be identified with the complement of the 0-section in N.

Now consider the Lagrangian cylinder $\mathcal{L} \subset Y \times \mathbb{R}$ defining the concordance from $\mathcal{K}_0$ to $\mathcal{K}_1$. Once again, $\mathcal{L}$ has a neighborhood symplectomorphic to N; if we remove a sufficiently small neighborhood of $\mathcal{L}$, then there is a collar neighborhood of $\partial((Y \times \mathbb{R})\backslash\mathcal{L})$ which is orientation-reversing symplectomorphic to N with the 0-section removed. Thus we can glue $(Y \times \mathbb{R})\backslash\mathcal{L}$ to $Z_0 \times \mathbb{R}$ to get a symplectic manifold W with two infinite ends. One of these ends is a piece $\bar{Y}_0 \times (-\infty, T]$ of the symplectization of $\bar{Y}_0$, and since Z_0 is contactomorphic to Z_1 the other end is $\bar{Y}_1 \times [T, \infty)$. Thus W is a boundary-exact symplectic cobordism from $\bar{Y}_0$ to $\bar{Y}_1$.

Finally, we wish to show that the map $i^* : H^1(W, \bar{Y}_1) \to H^1(\bar{Y}_0)$ is zero. By Poincaré duality it suffices to show that $H_3(W, \bar{Y}_0) \to H_2(\bar{Y}_0)$ is zero, or equivalently (by the long exact sequence of the pair $(W, \bar{Y}_0)$) that the map $H_2(\bar{Y}_0) \to H_2(W)$ is injective. But there is a natural isomorphism $H_2((Y \times \mathbb{R})\backslash\mathcal{L}) \cong H_2(Y\backslash\mathcal{K}_0)$ by Alexander duality, hence by the Mayer-Vietoris sequence and the five lemma it follows that $H_2(\bar{Y}_0) \to H_2(W)$ is an isomorphism as well, and so i^* is indeed zero.

Since W is a boundary-exact symplectic cobordism and $H^1(W, \bar{Y}_1) \to H^1(\bar{Y}_0)$ is zero, we apply Theorem 4.1.4 to conclude that

$$\psi(\bar{Y}_0, \bar{\xi}_0) = \widetilde{HM}(W^\dagger, \mathfrak{s}_\omega)\psi(\bar{Y}_1, \bar{\xi}_1).$$

Therefore $\widetilde{HM}(W^\dagger, \mathfrak{s}_\omega)$ induces a map $f : KHM(-Y, K_1) \to KHM(-Y, K_0)$ satisfying $f(\psi(\mathcal{K}_1)) = \psi(\mathcal{K}_0)$, as desired. □

Corollary 4.5.4. *If $\mathcal{K}_0 \prec \mathcal{K}_1$ and $\psi(\mathcal{K}_0)$ is nonzero, then so is $\psi(\mathcal{K}_1)$.*

Corollary 4.5.5. *If a Legendrian knot $\mathcal{K} \subset (S^3, \xi_0)$ bounds a Lagrangian disk in D^4, then $\psi(\mathcal{K})$ is a unit of $KHM(-S^3, K)$.*

Proof. The Legendrian unknot $\mathcal{U}$ is Lagrangian cobordant to $\mathcal{K}$, and $\psi(\mathcal{U})$ is a generator of $KHM(-S^3, U) \cong \mathcal{R}$ by Proposition 4.2.8, so by Theorem 4.5.3 there is a map $KHM(-S^3, K) \to \mathcal{R}$ such that the image of $\psi(\mathcal{K})$ is a unit. □

Thus although monopole invariants are generally hard to compute, we have shown that $\psi(\mathcal{K})$ is nonvanishing (in fact, a unit) for each of the seven Legendrian knots of Figure 2.6.1.

Appendix A

The Legendrian $m(10_{132})$ knots

A.1 The differential of the vanishing $m(10_{132})$

Let K_1 be the representative of $m(10_{132})$ with braid word

$$6, 7, 4, 3, 7, 5, 3, 6, 4, 2, 5, 1, 3, 2, 5, 2, 4, 6, 2.$$

Then $Ch(K_1)$ has generators $x_1, \ldots, x_{23}$ over $\mathbb{Z}[t, t^{-1}]$ with the following nonzero differentials [43]:

$$\begin{aligned}
\partial x_2 &= -x_1 \\
\partial x_4 &= x_3 \\
\partial x_6 &= x_3 x_1 \\
\partial x_8 &= x_3 + x_3 x_2 x_5 - x_6 x_5 \\
\partial x_9 &= x_1 + x_7 x_4 x_1 - x_7 x_6 \\
\partial x_{11} &= 1 + x_2 x_5 + x_7 x_4 + x_7 x_4 x_2 x_5 - x_7 x_8 + x_9 x_5 \\
\partial x_{12} &= x_{10} \\
\partial x_{13} &= x_{10} x_4 x_1 - x_{10} x_6 \\
\partial x_{14} &= -x_{12} x_4 x_1 + x_{12} x_6 + x_{13} \\
\partial x_{17} &= x_{10} x_4 x_{15} + x_{10} x_4 x_2 x_5 x_{15} - x_{10} x_8 x_{15} + x_{13} x_5 x_{15} \\
\partial x_{18} &= -x_{15} x_7 \\
\partial x_{20} &= 1 - x_4 x_1 + x_6 - x_4 x_1 x_{16} x_{19} + x_6 x_{16} x_{19} \\
\partial x_{21} &= 1 - x_{12} x_4 x_{15} - x_{12} x_4 x_2 x_5 x_{15} + x_{12} x_8 x_{15} - x_{14} x_5 x_{15} + x_{17} \\
&\quad - x_{19} x_5 x_{15} - x_{19} x_{16} x_{12} x_4 x_{15} - x_{19} x_{16} x_{12} x_4 x_2 x_5 x_{15} \\
&\quad + x_{19} x_{16} x_{12} x_8 x_{15} - x_{19} x_{16} x_{14} x_5 x_{15} + x_{19} x_{16} x_{17} \\
\partial x_{22} &= 1 - x_{10} + x_{17} x_7 + x_{10} x_4 x_{18} + x_{10} x_4 x_2 x_5 x_{18} - x_{10} x_8 x_{18} + x_{13} x_5 x_{18} \\
\partial x_{23} &= t^{-1} + x_{15} x_2 + x_{15} x_7 x_4 x_2 + x_{15} x_9 - x_{18} x_3 x_2 + x_{18} x_6.
\end{aligned}$$

A.2 The differential of the nonvanishing $m(10_{132})$

Let K_2 be the representative of $m(10_{132})$ with braid word

$$4,5,3,5,3,2,4,1,3,2,4,2,5,1,3,2,4,4,3,5,4,2.$$

Then $Ch(K_2)$ has generators $x_1,\ldots,x_{25}$ over $\mathbb{Z}/2\mathbb{Z}$ with the following nonzero differentials [43]:

$$\begin{aligned}
\partial x_2 = \partial x_3 &= x_1 \\
\partial x_7 &= x_4 + x_5(1 + (x_2 + x_3)x_4) \\
\partial x_8 &= x_6 \\
\partial x_9 &= x_6(1 + (x_2 + x_3)x_4) \\
\partial x_{10} &= x_9 + x_8(1 + (x_2 + x_3)x_4) \\
\partial x_{13} &= x_6(x_2 + x_3) + x_{11}(1 + x_5(x_2 + x_3)) \\
\partial x_{14} &= (1 + (x_2 + x_3)x_4)x_{12} \\
\partial x_{15} &= x_{12}x_{11} \\
\partial x_{16} &= x_{14}x_{11} + (1 + (x_2 + x_3)x_4)x_{15} \\
\partial x_{17} &= x_{12}(x_{13} + x_8(x_2 + x_3)) + x_{15}(1 + x_5(x_2 + x_3)) \\
\partial x_{19} &= (1 + (x_2 + x_3)x_4) + cx_{18} \\
\partial x_{20} &= x_{18}x_{12} \\
\partial x_{21} &= x_{14} + x_{19}x_{12} + cx_{20} \\
\partial x_{22} &= (1 + x_5(x_2 + x_3))x_{18} \\
\partial x_{23} &= 1 + x_{11}x_{22} + (x_{13} + x_8(x_2 + x_3))x_{18} \\
\partial x_{24} &= 1 + x_{22}x_{12} + (1 + x_5(x_2 + x_3))x_{20} \\
\partial x_{25} &= 1 + c
\end{aligned}$$

where

$$c = x_2 + x_3 + (1 + (x_2 + x_3)x_4)x_{17} + x_{14}(x_{13} + x_8(x_2 + x_3)) + x_{16}(1 + x_5(x_2 + x_3)).$$

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