Self-shrinkers of Mean Curvature Flow and Harmonic Map Heat Flow with Rough Boundary Data
by
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Abstract

In this thesis, first, joint with Longzhi Lin, we establish estimates for the harmonic map heat flow from the unit circle into a closed manifold, and use it to construct sweepouts with the following good property: each curve in the tightened sweepout, whose energy is close to the maximal energy of curves in the sweepout, is itself close to a closed geodesic.

Second, we prove the uniqueness for energy decreasing weak solutions of the harmonic map heat flow from the unit open disk into a closed manifold, given any $H^1$ initial data and boundary data, which is the restriction of the initial data on the boundary of the disk. Previously, under an additional assumption on boundary regularity, this uniqueness result was obtained by Rivière (when the target manifold is the round sphere and the energy of initial data is small) and Freire (for general target manifolds). The point of our uniqueness result is that no boundary regularity assumption is needed. Also, we prove the exponential convergence of the harmonic map heat flow, assuming that the energy is small at all times.

Third, we prove that smooth self-shrinkers in the Euclidean space, that are entire graphs, are hyperplanes. This generalizes an earlier result by Ecker and Huisken: no polynomial growth assumption at infinity is needed.

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Chapter 1

Introduction

My thesis is focused on the existence, uniqueness and regularity of geometric flows, including the harmonic map heat flow and mean curvature flow. These flows can be thought of as the negative gradient flows of various functionals, such as the energy and area functionals. Moreover, the study of such geometric flows not only are closely related to the theory of calculus of variations and partial differential equations, but also have several beautiful applications in low dimensional geometry and topology.

The study of the harmonic map heat flow arises from finding critical points of the energy functional, i.e. harmonic maps, in certain topology classes. In [26], Grayson showed that on any closed non-simply-connected surface, there exist simple closed geodesics in each nontrivial \( \pi_1 \) homotopy class by the curve shortening flow. On the 2-sphere of which the \( \pi_1 \) homotopy group is trivial, Birkhoff used sweepouts to find non-trivial closed geodesics; see [3], [4], [13], [36] and section 2 in [17] about Birkhoff's idea. The argument works equally well on other closed Riemannian manifolds. For higher dimensions, we recommend the reader refer to [45], [7], [9] and [12, 14].

Recently, Colding and Minicozzi used Birkhoff's curve shortening process to pull each curve in the sweepout as tight as possible while preserving the sweepout (see definition 2.2.1). They showed the following useful property: each curve in the tightened sweepout whose length is close to the length of the longest curve in the sweepout must itself be close to a closed geodesic; see [13]. We call the sweepouts with the property above to be good sweepouts. Furthermore, in [14] and [12], applying the
techniques, developed by in the 1980s, for dealing with the energy concentration (see [42] and [33]), they constructed good sweepouts by 2-spheres on closed manifolds by local harmonic replacements, of which the idea is essentially the same as that of Birkhoff. The existence of good sweepouts by curves or 2-spheres not only implies the existence of non-trivial closed geodesics or harmonic 2-spheres in closed manifolds with non-trivial \( \pi_2 \) or \( \pi_3 \) homotopy group respectively, but also is a key ingredient in the proof of finite extinction of geometric flows, such as the power of mean curvature flow and Ricci flow.

The Birkhoff's curve shortening process is a discrete gradient flow of the length functional and it requires some work to show the discrete shortening process preserves the homotopy class of sweepouts. In chapter 2, joint with Longzhi Lin, we use a continuous method, i.e. the harmonic map heat flow, to tighten each curve in the sweepout by curves, which provides a natural homotopy of the sweepout. And we show that the tightened sweepout is the good sweepout defined in the previous paragraph.

For higher dimensions, as a first step, we study the weak solutions in \( H^1 \) class to the initial-boundary value problem for the harmonic map heat flow from the unit open disk into a closed manifold, given any \( H^1 \) initial data and boundary data, which is the restriction of the initial data on the boundary of the disk. In chapter 3, we obtain the results on the uniqueness and the rate of convergence of such weak solutions. Previously, under an additional assumption on the boundary regularity, this problem was investigated intensively by several mathematicians, such as Chang, Rivière, Freire and so on; see [7], [41] and [23, 22, 24]. The space \( V^T \) (see the section of background in chapter 3) plays a crucial role in the mentioned works. The lack of boundary regularity causes the weak solutions in our case will not always be in \( V^T \); see also section 2.5 in chapter 3. To get over this difficulty, we make use of the interior gradient estimate and the Hardy inequality.

In chapter 4, we discuss another independent direction of my graduate work, that is to explore the Bernstein problem for self-shrinkers under the mean curvature flow. The mean curvature flow in the Euclidean space is the negative \( L^2 \) gradient flow
of the area functional which decreases the area in the steepest way. Self-shrinkers are a special class of solutions to the mean curvature flow, in which a later time slice is the scaled down copy of an earlier time slice. The reason that self-shrinkers are interesting is that they provide the singularity models of the flow; see [30, 31] and [32]. Thus, it is important to study the classification problem for self-shrinkers. However, many numerical examples in [8] indicate that it is very difficult to classify all the self-shrinkers in general. There are many results on the classification of self-shrinkers under certain conditions, such as mean convexity, rotational symmetry and entropy stability; see [30, 31], [16] and [34].

On the other hand, self-shrinkers are hypersurfaces in $\mathbb{R}^{n+1}$ that are minimal under the conformally Gaussian changed metric; see [1], [15] and [16]. In minimal hypersurface theory, the Bernstein Theorem is one of the most fundamental theorems, and has many important applications, such as uniqueness and regularity theory for minimal hypersurfaces. Thus it is natural to ask whether there is a Bernstein type theorem for self-shrinkers. The main result of chapter 4 gives an affirmative answer to this question. Namely, we show that the only smooth entire graphical self-shrinkers are hyperplanes. This generalizes an earlier result by Ecker and Huisken [20]: no priori polynomial growth assumption is needed.

Recently, Colding and Minicozzi introduced the stability operator for self-shrinkers; see [15] and [16]. One of the key ingredients in the proof of the Bernstein theorem for self-shrinkers is to establish a Gaussian weighted stability inequality for graphical self-shrinkers. In contrast to the Bernstein theorem for minimal hypersurfaces, which is only true in the Euclidean space $\mathbb{R}^n$ with $n \leq 7$ (see [43], [5], [6] and [44]), as a consequence of the Gaussian weight in our stability inequality, the Bernstein theorem for self-shrinkers holds true for all dimensions.
Chapter 2

Existence of Good Sweepouts on Closed Manifolds

In this chapter, we present the joint work [37] with Longzhi Lin on the construction of good sweepouts by curves on closed manifolds using the harmonic map heat flow. Namely, given a minimizing sequence of sweepouts of the width (see (2.2.1)), we apply the harmonic map heat flow on each curve in the sweepout to pull it tight while preserving the sweepout. Moreover the tightened sweepout has the following good property (see Theorem 2.2.4): each curve in the tightened sweepout whose energy is close to the maximal energy of curves in the sweepout is itself close to a closed geodesic. In particular, the width is the energy of some closed geodesic.

2.1 Harmonic Map Heat Flow from a Circle

Throughout we use the subscripts $\theta$ and $t$ to denote the partial differentiations of maps with respect to $\theta$ and $t$ respectively; the map $u$ satisfies the harmonic map heat flow equation, which is defined in (2.1.1). Let $(M, g)$ be a closed Riemannian manifold. By the Nash embedding theorem, $M$ can be isometrically embedded into some Euclidean space $(\mathbb{R}^N, \langle , \rangle)$. Given a closed curve $\gamma \in H^1(S^1, M)$, we define the energy functional $E(\gamma) = \frac{1}{2} \int_{S^1} |\gamma|^2 d\theta$. The harmonic map heat flow is the negative $L^2$ gradient flow of the energy functional. Thus the equation of the harmonic map
heat flow from $S^1$ into $M$ is

$$
\begin{cases}
  u_t = u_{\theta\theta} - A_u(u_{\theta}, u_{\theta}) & \text{on } (0, \infty) \times S^1 \\
  \lim_{t \to 0^+} u(t, \cdot) = u_0 & \text{in } H^1(S^1, M)
\end{cases}
$$

(2.1.1)

where $A_u$ is the second fundamental form of $M$ in $\mathbb{R}^N$ at point $u(\theta)$.

In this section, we study the long time existence and uniqueness of the solution of (2.1.1). We show that

**Theorem 2.1.1.** Given $u_0 \in H^1(S^1, M)$, there exists a unique solution $u(t, \theta) \in C^\infty((0, \infty) \times S^1, M)$ of (2.1.1).

**Remark 2.1.2.** We would like to thank Tobias Lamm for bringing to our attention the paper [39] of Ottarsson, which has some overlap with our result and in which Theorem 2.1.1 was proved under the stronger assumption of $C^1$ initial data (and thus the $C^1$ continuity at $t = 0$). In our setting, the $C^1$ continuity at $t = 0$ may not be true. For our purpose that the harmonic map heat flow preserves the homotopy class of sweepouts, we use a different argument to show the $H^1$ continuity at $t = 0$.

### 2.1.1 Existence and regularity of solutions to (2.1.1)

First, by the corollary on page 124 of [28], given any initial data $u_0 \in C^\infty(S^1, M)$, there exists $T_0 > 0$ and a unique solution $u \in C^\infty([0, T_0) \times S^1, M)$ of (2.1.1). We show that the solution $u$ can be extended smoothly beyond $T_0$. First, note that the energy is non-increasing under the harmonic map heat flow:

**Lemma 2.1.3.** For $0 \leq t_1 \leq t_2 < T_0$,

$$
E(u(t_1, \cdot)) - E(u(t_2, \cdot)) = \int_{t_1}^{t_2} \int_{S^1} |u_t|^2 d\theta dt.
$$

(2.1.2)

**Proof.** Multiply the harmonic map heat equation by $u_t$ and integrate over $[t_1, t_2] \times S^1$,

$$
\int_{t_1}^{t_2} \int_{S^1} |u_t|^2 d\theta dt = \int_{t_1}^{t_2} \int_{S^1} \langle u_{\theta\theta}, u_t \rangle d\theta dt = -\int_{t_1}^{t_2} \int_{S^1} \langle u_{\theta}, u_{\theta t} \rangle d\theta dt
$$

$$
= E(u(t_1, \cdot)) - E(u(t_2, \cdot)).
$$

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Next, we derive the gradient bound of $u$.

**Lemma 2.1.4.** $(\partial_t - \partial^2_\theta)|u_\theta|^2 \leq 0$.

*Proof.* A straightforward computation gives that

$$
\partial_t |u_\theta|^2 = 2\langle u_\theta, u_{\theta t} \rangle = 2\langle u_\theta, u_{\theta \theta t} \rangle - 2\langle u_\theta, (A_u(u_\theta, u_\theta))_\theta \rangle
$$

$$
= 2\langle u_\theta, u_{\theta \theta t} \rangle + 2\langle u_{\theta t}, A_u(u_\theta, u_\theta) \rangle
$$

$$
= \partial^2_\theta |u_\theta|^2 - 2|u_{\theta t}|^2 + 2|A_u(u_\theta, u_\theta)|^2 \leq \partial^2_\theta |u_\theta|^2.
$$

Lemma 2.1.4 follows immediately from the above calculation. \qed

Since $u \in C^\infty((0, T_0) \times S^1)$, it follows from Lemma 2.1.3 and the local maximum principle (see Theorem 2.1 in [27] or Theorem 7.36 in [35]) that for any $\tau > 0$ and $(t, \theta) \in [\tau, T_0) \times S^1$,

$$
|u_\theta|^2(t, \theta) \leq C_0 \max \{1, \tau^{-1/2}\} E(u_0),
$$

(2.1.3)

where $C_0$ is a positive constant. Furthermore, by Proposition 7.18 in [35], $|u_{\theta t}|$ and $|u_t|$ are bounded on $[2\tau, T_0) \times S^1$. Thus, by induction on the order of differentiation of $u$, for any $\tau > 0$, the higher order derivatives of $u$ on $[2\tau, T_0) \times S^1$ are bounded uniformly by constants depending only on $M$, $E(u_0)$, $\tau$ and $T_0$. Hence, $u$ can be extended smoothly to a solution of (2.1.1) beyond $T_0$. In other word, there exists a unique solution $u \in C^\infty([0, \infty) \times S^1, M)$ of (2.1.1), if $u_0 \in C^\infty(S^1, M)$.

Next, in general, given $u_0 \in H^1(S^1, M)$, we can find a sequence $u^m_0 \in C^\infty(S^1, M)$ approaching $u_0$ in the $H^1$ topology. Let $u^m$ be the solution of the harmonic map heat flow with initial data $u^m_0$. Thus, by (2.1.3) and discussion above, for any $\tau > 0$ and $T_0 > \tau$, $u^m$ and all their derivatives are bounded uniformly, independent of $m$. Hence, by the Arzela-Ascoli theorem and a diagonalization argument, there exists a map $u \in C^\infty((0, \infty) \times S^1, M)$ solving the harmonic map heat flow with $E(u(t, \cdot)) \leq E(u_0)$. Moreover, it follows from the lemma below that $t \longrightarrow u(t, \cdot)$ is a continuous map from $[0, \infty) \longrightarrow H^1(S^1, M)$. 

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Lemma 2.1.5. Given $\epsilon > 0$, there exists $\delta > 0$, depending on $M$, $u_0$ and $\epsilon$, such that if $0 \leq t_1 < t_2$ and $t_2 - t_1 < \delta$, then $\|u(t_2, \cdot) - u(t_1, \cdot)\|_{H^1(S^1)} \leq \epsilon$.

Proof. Note that by Lemma 2.1.3, $\lim_{t \to 0} u(t, \cdot) = u_0$ in the $L^2(S^1, M)$ topology. Moreover, we have

$$\int_{S^1} |u(t_2, \theta) - u(t_1, \theta)|^2 d\theta \leq \int_{S^1} \left( \int_{t_1}^{t_2} u_\theta(\theta) d\theta \right)^2 d\theta \leq (t_2 - t_1) \int_{S^1} |u_\theta|^2 d\theta dt.$$

Next, by Lemma 2.1.3 and the Cauchy-Schwarz inequality,

$$\int_{S^1} |u_\theta(t_2, \theta) - u_\theta(t_1, \theta)|^2 d\theta = \int_{S^1} |u_\theta(t_1, \theta)|^2 d\theta - \int_{S^1} |u_\theta(t_2, \theta)|^2 d\theta - 2 \int_{S^1} \langle u_\theta(t_2, \theta), u_\theta(t_1, \theta) - u_\theta(t_2, \theta) \rangle d\theta$$

$$= 2 \int_{t_1}^{t_2} \int_{S^1} |u_\theta|^2 d\theta d\theta + 2 \int_{S^1} \langle u_\theta(t_2, \theta), u(t_1, \theta) - u(t_2, \theta) \rangle d\theta$$

$$\leq 2 \int_{t_1}^{t_2} \int_{S^1} |u_\theta|^2 d\theta d\theta + 2 (t_2 - t_1)^{1/2} \left( \int_{S^1} |u_\theta(t_2, \theta)|^2 d\theta \right)^{1/2} \left( \int_{t_1}^{t_2} \int_{S^1} |u_\theta|^2 d\theta d\theta \right)^{1/2}.$$

If $C_0^2(t_2 - t_1) < 1$, then, by Lemma 2.1.3, (2.1.1) and (2.1.3),

$$\int_{S^1} |u_\theta(t_2, \theta)|^2 d\theta \leq \int_{S^1} |u(t_2, \theta)|^2 d\theta + (t_2 - t_1)^{-1} \sup_M |A|^2 \cdot E(u_0)^2. \quad (2.1.4)$$

We derive the evolution equation for $|u_\theta|^2$:

$$\partial_t |u_\theta|^2 = 2\langle u_\theta, u_{\theta \theta t} \rangle - 2\langle u_\theta, (A(u_\theta, u_\theta))_t \rangle$$

$$= \partial_\theta^2 |u_\theta|^2 - 2|u_\theta|^2 + 2\langle u_{\theta t}, A(u_\theta, u_\theta) \rangle$$

$$= \partial_\theta^2 |u_\theta|^2 - 2|u_\theta|^2 + 2(A(u, u), A(u, u)).$$

Thus by (2.1.3), if $t > t_3 = t_1 + (t_2 - t_1)/2$,

$$(\partial_t - \partial_\theta^2)|u_\theta|^2 - 4(t_2 - t_1)^{-1} \sup_M |A|^2 \cdot E(u_0)^2 \cdot |u_\theta|^2 \leq 0. \quad (2.1.5)$$
Hence
\[
\int_{S^1} |u_t|^2(t_2, \theta) d\theta \leq \inf_{t_3 \leq t \leq t_2} \int_{S^1} |u_t|^2(t, \theta) d\theta + C(t_2 - t_1)^{-1} \int_{t_1}^{t_2} \int_{S^1} |u_t|^2 d\theta dt
\]
\[
\leq (C + 2)(t_2 - t_1)^{-1} \int_{t_1}^{t_2} \int_{S^1} |u_t|^2 d\theta dt,
\]
where \( C \) depends on \( M \) and \( E(u_0) \). Combining the inequality above, (2.1.4) and Lemma 2.1.3, there exists \( \delta > 0 \) such that Lemma 2.1.5 holds true.

\[\square\]

### 2.1.2 Uniqueness of solutions to (2.1.1)

It follows from Lemma 2.1.5 and (2.1.3) that there exists \( R_0 > 0 \), depending only on \( M \) and \( u_0 \), such that for \( t \geq 0 \), \( 2\pi \cdot \sup_M |A|^2 \cdot \int_{[t] \times I_{R_0}} |u\theta|^2 d\theta < 1/64 \), where \( I_{R_0} \) is any segment on unit circle of length \( 2R_0 \). To prove the uniqueness of the solution of (2.1.1), we need the following lemma.

**Lemma 2.1.6.** Suppose that \( u \) is a solution of (2.1.1) in \( C^\infty((0, \infty) \times S^1, M) \). Then

\[
\int_0^T \int_{S^1} |u\theta|^2 d\theta dt \leq \frac{T}{4R_0^2} E(u_0) + 2 \left[ E(u_0) - E(u(T, \cdot)) \right]. \tag{2.1.6}
\]

**Proof.** The following estimate is inspired by the proof of Lemma 6.7 on page 225 of [46]. Fix \((t_1, \theta_1) \in (0, \infty) \times S^1 \). Let \( I_R(\theta_1) \) denote the arc segment on the unit circle centered at \( \theta_1 \) with length \( 2R \). And let \( \phi \) be identically one on \( I_{R_0}(\theta_1) \) and cut off linearly to zero on \( I_{R_0}(\theta_1) \setminus I_{R_0/2}(\theta_1) \). Thus,

\[
|u\theta|^4(t_1, \theta_1) = \phi^2 |u\theta|^4(t_1, \theta_1)
\]
\[
\leq \left( \int_{S^1} |\phi||u\theta||u\theta\theta|(t_1, \theta) d\theta + \int_{S^1} |\phi||u\theta|^2(t_1, \theta) d\theta \right)^2
\]
\[
\leq 8 \left( \int_{S^1} |\phi||u\theta||u\theta\theta|(t_1, \theta) d\theta \right)^2 + 2 \left( \int_{S^1} |\phi||u\theta|^2(t_1, \theta) d\theta \right)^2
\]
\[
\leq 8 \int_{I_{R_0}(\theta_1)} |u\theta|^2(t_1, \theta) d\theta \cdot \int_{S^1} |u\theta\theta|^2(t_1, \theta) d\theta + \frac{8}{R_0^2} \left( \int_{I_{R_0}(\theta_1)} |u\theta|^2(t_1, \theta) d\theta \right)^2,
\]
where the last inequality follows from Hölder's inequality and that \( \phi \) is supported in
\[ I_{R_0}(\theta_1) \text{ with } |\phi_0| \leq 2/R_0. \text{ Hence, for } 0 < t_0 \leq T, \]
\[
\int_{t_0}^{T} \int_{S^1} |u_\theta|^4 d\theta dt \leq 16\pi \cdot \epsilon(R_0) \cdot \left( \int_{t_0}^{T} \int_{S^1} |u_{\theta\theta}|^2 d\theta dt + R_0^{-2} \int_{t_0}^{T} \int_{S^1} |u_\theta|^2 d\theta dt \right),
\]
where
\[
\epsilon(R_0) = \sup_{t \geq 0, \theta_1 \in S^1} \int_{\{t\} \times I_{R_0}(\theta_1)} |u_\theta|^2 d\theta.
\] (2.1.7)

Therefore, it follows from (2.1.1) and Lemma 2.1.3 that
\[
\int_{t_0}^{T} \int_{S^1} |u_\theta|^2 d\theta dt \leq \int_{t_0}^{T} \int_{S^1} |u_t|^2 d\theta dt + \sup_M |A|^2 \cdot \int_{t_0}^{T} \int_{S^1} |u_\theta|^4 d\theta dt
\]
\[
\leq |E(u_0) - E(u(T, \cdot))| + \frac{1}{2} \int_{t_0}^{T} \int_{S^1} |u_{\theta\theta}|^2 d\theta dt + \frac{T}{8R_0^2} E(u_0).
\]
Absorbing the righthand side into the lefthand side and noting that the estimate is independent of \(t_0\), (2.1.6) follows immediately.

Now we are ready to show the uniqueness of the solution to the harmonic map heat flow.

**Lemma 2.1.7.** Given \(u_0 \in H^1(S^1, M)\), let \(u\) and \(\bar{u}\) be solutions of (2.1.1) in \(C^\infty((0, \infty) \times S^1, M)\). Then \(u = \bar{u}\).

**Proof.** Define \(v = u - \bar{u}\). Then
\[
v_t = v_{\theta\theta} - A_u(u_\theta, u_\theta) + A_u(\bar{u}_\theta, \bar{u}_\theta).
\] (2.1.8)

Multiply both sides of (2.1.8) by \(v\) and integrate over \([0, t_0] \times S^1\),
\[
\int_{\{t_0\} \times S^1} |v|^2 d\theta + 2 \int_{0}^{t_0} \int_{S^1} |v_\theta|^2 d\theta dt
\]
\[
= 2 \int_{0}^{t_0} \int_{S^1} \langle A_{\bar{u}}(\bar{u}_\theta, \bar{u}_\theta) - A_u(u_\theta, u_\theta), v \rangle d\theta dt
\]
\[
\leq C(M) \int_{0}^{t_0} \int_{S^1} |v|^2(\bar{u}_\theta|^2 + |u_\theta|^2) d\theta dt + C(M) \int_{0}^{t_0} \int_{S^1} |v||u_\theta||(\bar{u}_\theta + |u_\theta|) d\theta dt
\]

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\[
\begin{align*}
\leq C(M) \int_0^{t_0} \int_{S^1} |v|^2 (|\tilde{u}_\theta|^2 + |u_\theta|^2) d\theta dt + \int_0^{t_0} \int_{S^1} |v_\theta|^2 d\theta dt \\
\leq C(M) \int_0^{t_0} \left( \|u_\theta\|_{C^0(t)}^2 + \|\tilde{u}_\theta\|_{C^0(t)}^2 \right) \int_{S^1} |v|^2 d\theta dt + \int_0^{t_0} \int_{S^1} |v_\theta|^2 d\theta dt.
\end{align*}
\]

By Lemma 2.1.3, 2.1.6 and the Sobolev embedding theorem, there exists \( \delta > 0 \), depending on \( M \) and \( u_0 \), such that if \( t_0 \leq \delta \), then

\[
\begin{align*}
C(M) \int_0^{t_0} \|u_\theta\|_{C^0(t)}^2 + \|\tilde{u}_\theta\|_{C^0(t)}^2 dt \\
\leq C(M) \int_0^{t_0} \int_{S^1} |\tilde{u}_\theta|^2 + |u_\theta|^2 d\theta dt \\
\leq C(M) \left[ E(u_0) + \frac{t_0}{2H_0^2} + 4E(u_0) - 2E(u(t_0, \cdot)) - 2E(\tilde{u}(t_0, \cdot)) \right] \leq \frac{1}{2}.
\end{align*}
\]

Thus, absorbing the righthand side into the lefthand side,

\[
\sup_{0 \leq t \leq \delta} \int_{\tilde{t} \times S^1} |v|^2 d\theta + 2 \int_0^{\delta} \int_{S^1} |v_\theta|^2 d\theta dt \leq 0. \quad (2.1.9)
\]

Since \([0, T]\) is compact, Lemma 2.1.7 follows by iteration. \( \square \)

### 2.2 Width and Good Sweepouts

#### 2.2.1 Sweepouts and width

In [13], Colding and Minicozzi introduced the crucial geometric concepts: sweepouts and width.

**Definition 2.2.1.** A continuous map \( \sigma : [-1, 1] \times S^1 \rightarrow M \) is called a sweepout on \( M \), if \( \sigma(s, \cdot) \in H^1(S^1, M) \) for each \( s \in [-1, 1] \), the map \( s \rightarrow \sigma(s, \cdot) \) is continuous from \([-1, 1]\) to \( H^1(S^1, M) \) and \( \sigma \) maps \([-1] \times S^1 \) and \([1] \times S^1 \) to points.

The sweepout \( \sigma \) induces a map \( \tilde{\sigma} \) from the sphere \( S^2 \) to \( M \), and we will not distinguish \( \sigma \) from \( \tilde{\sigma} \). Denote by \( \Omega \) the set of sweepouts on \( M \). The homotopy class
$\Omega_\sigma$ of $\sigma$ is the path connected component of $\sigma$ in $\Omega$, where the topology is induced from $C^0([-1,1], H^1(S^1, M))$.

**Definition 2.2.2.** The width $W = W(\Omega_\sigma)$ of the homotopy class $\Omega_\sigma$ is defined by

$$W = \inf_{\delta \in \Omega_\sigma} \max_{s \in [-1,1]} E(\hat{s}(s, \cdot)). \quad (2.2.1)$$

### 2.2.2 Construction of good sweepouts

Let $\gamma : S^1 \to M$ be a smooth closed curve and $G$ be the set of closed geodesics in $M$. For $\alpha \in (0,1)$ fixed, define $\text{dist}_\alpha(\gamma, G) = \inf_{\bar{\gamma} \in G} \|\gamma - \bar{\gamma}\|_{C^{1,\alpha}(S^1)}$. We prove for the solution of (2.1.1) the following proposition, which is the key to the existence of good sweepouts, i.e. Theorem 2.2.4:

**Proposition 2.2.3.** Given $0 < \alpha < 1$, $W_0 \geq 0$, $t_0 > 0$ and $\epsilon > 0$, there exists $\delta_0 > 0$ such that if $W_0 - \delta_0 \leq E(u(t_0, \cdot)) \leq E(u_0) \leq W_0 + \delta_0$, then $\text{dist}_\alpha(u(t_0, \cdot), G) < \epsilon$.

**Proof.** If not, then there would exist $0 < \alpha < 1$, $W_0 \geq 0$, $t_0 > 0$, $\epsilon > 0$, and a sequence of solutions $u^j$ of the harmonic map heat flow satisfying that $W_0 - 1/j \leq E(u^j(t_0, \cdot)) \leq E(u^j_0) \leq W_0 + 1/j$ and $\text{dist}_\alpha(u^j(t_0, \cdot), G) \geq \epsilon$. It would follow from the evolution equation of $|u^j|^2$ (see (2.1.5)), (2.1.3), Lemma 2.1.3 and the local maximum principle (see Theorem 2.1 in [27] or Theorem 7.36 in [35]) that

$$\sup_{\theta \in S^1} |u^j|^2(t_0, \theta) \leq C \left[ E(u^j(t_0/2, \cdot)) - E(u^j(t_0, \cdot)) \right], \quad (2.2.2)$$

where $C$ depends on $M$, $t_0$ and $W_0$. Thus, $\sup_{\theta \in S^1} |u^j|(t_0, \theta) \to 0$ and it follows from (2.1.3) that $\|u^j(t_0, \cdot)\|_{C^2(S^1)}$ is uniformly bounded by constants depending on $M$, $t_0$ and $W_0$. Therefore, by the Arzela-Ascoli theorem and Theorem 1.5.1 in [29], there exists a subsequence (relabelled) $u^j(t_0, \cdot)$ converging to $u^\infty$ in $C^{1,\alpha}(S^1, M)$ and $u^\infty$ is a closed geodesic in $M$. This is a contradiction. \qed

Let $\sigma$ be a sweepout on a closed manifold $M$ and $\sigma^j$ a minimizing sequence of
sweepouts in $\Omega_\sigma$. That is

$$W \leq \max_{s \in [-1,1]} E(\sigma^j(s, \cdot)) \leq W + 1/j. \quad (2.2.3)$$

Applying the harmonic map heat flow to each slice of $\sigma^j$, we get a map $\Phi^j : [-1,1] \times [0,\infty) \times S^1 \rightarrow M$ and, for each $s \in [-1,1]$ fixed, $\Phi^j(s, t, \theta)$ solves (2.1.1) with $\Phi^j(s, 0, \theta) = \sigma^j(s, \theta)$. It follows from the proof of the long time existence and uniqueness of the solution of (2.1.1) that for any $t_0 \geq 0$, the map $s \mapsto \Phi(s, t_0, \cdot)$ is continuous from $[-1,1]$ to $H^1(S^1, M)$ and thus $\Phi^j(\cdot, t_0, \cdot)$ is still a sweepout on $M$. Since

$$\int_{-1}^1 \max_{|s| \leq 1} \int_{S^1} |\Phi^j_t(s, t, \theta)|^2 d\theta dt$$

is finite, for any $\epsilon > 0$, there exists $\delta > 0$ such that if $0 \leq t_1 < t_2 \leq t_0$ and $t_2 - t_1 < \delta$, then $\int_{t_1}^{t_2} \int_{S^1} |\Phi^j_t(s, t, \theta)|^2 d\theta dt < \epsilon$ for any $s \in [-1,1]$. Hence, by Lemma 2.1.5, for any $t_0 > 0$, $\Phi^j(\cdot, t_0, \cdot)$ is homotopic to $\sigma^j$. Therefore, it follows from Proposition 2.2.3 that the $\Phi^j(\cdot, t_0, \cdot)$ are good sweepouts on $M$. That is,

**Theorem 2.2.4.** *Given $0 < \alpha < 1$, $t_0 > 0$ and $\epsilon > 0$, there exists $\delta > 0$ such that if $j > 1/\delta$ and $s \in [-1,1]$ satisfies $E(\Phi^j(s, t_0, \cdot)) \geq W - \delta^1$, then dist$_\alpha(\Phi^j(s, t_0, \cdot), G) < \epsilon$.*

In [13], Colding and Minicozzi show that the width is positive if $\sigma$ is not the zero element in $\pi_2(M)$. In fact, assume that $W(\Omega_\sigma) = 0$, $\hat{\sigma} \in \Omega_\sigma$ and that the energy of each slice of $\hat{\sigma}$ is sufficiently small. Then each slice, $\hat{\sigma}(s, \cdot)$, is contained in a strictly convex neighborhood of $\hat{\sigma}(s, \theta_0)$ and note that $s \mapsto \hat{\sigma}(s, \theta_0)$ is a continuous curve on $M$. Hence a geodesic homotopy connects $\hat{\sigma}$ to a path of point curves and thus $\hat{\sigma}$ is homotopically trivial. Since $G$ is closed in the $H^1(S^1, M)$ topology, we have

**Corollary 2.2.5.** *If $M$ is a closed Riemannian manifold and $\pi_2(M) \neq \{0\}$, then there exists at least one non-trivial closed geodesic on $M$.*

**Remark 2.2.6.** Instead of using the unit interval $[-1,1]$ as the parameter space for the curves in the sweepout and assuming that the curves start and end in point curves, we could have used any compact space $P$ and required that the curves are constants on $\partial P$. In this case, $\Omega^P$ is the set of continuous maps $\sigma : P \times S^1 \rightarrow M$ such that

$^1$Such $s$ exists, since $W \leq \max_{s \in [-1,1]} E(\Phi^j(s, t_0, \cdot)) \leq W + 1/j.$
for each $s \in \mathcal{P}$ the curve $\sigma(s, \cdot)$ is in $H^1(S^1, M)$, the map $s \mapsto \sigma(s, \cdot)$ is continuous from $\mathcal{P}$ to $H^1(S^1, M)$, and $\sigma$ maps $\partial \mathcal{P}$ to point curves. Given $\sigma \in \Omega^\mathcal{P}$, the homotopy class $\Omega^\mathcal{P}_\sigma$ is the set of maps $\hat{\sigma} \in \Omega^\mathcal{P}$ that are homotopic to $\sigma$ through maps in $\Omega^\mathcal{P}$. And the width $W = W(\Omega^\mathcal{P}_\sigma)$ is defined by

$$W = \inf_{\hat{\sigma} \in \Omega^\mathcal{P}_\sigma} \max_{s \in \mathcal{P}} E(\hat{\sigma}(s, \cdot)). \quad (2.2.4)$$

Theorem 2.2.4 holds for general parameter space; the proof is virtually the same as when $\mathcal{P} = [-1, 1]$. 


Chapter 3

Harmonic Map Heat Flow with Rough Boundary Data

Let $B_1$ be the unit open disk in $\mathbb{R}^2$ and $M$ a closed Riemannian manifold. Suppose that $u \in H^1([0,T] \times B_1, M)$ is a weak solution of the initial-boundary value problem for the harmonic map heat flow, given initial data $u_0 \in H^1(B_1, M)$ and boundary data $\gamma = u_0|_{\partial B_1}$. In this chapter, we present the work [48] on the uniqueness and the rate of convergence of the weak solution $u$; see Theorem 3.3.1 and Theorem 3.4.1.

Although the main theorems are stated for the unit open disk, the proof could be modified to apply to any bounded open set of $\mathbb{R}^2$ and even of a general two dimensional Riemannian manifold.

3.1 Background

3.1.1 Notation

Throughout, we use subscripts $t$, $x_1$, $x_2$ and $r$ to denote taking derivatives with respect to $t$, $x_1$, $x_2$ and $r$; $\nabla$. and $\nabla^2$. denote gradient and Hessian operator respectively; "sup" in this note is "esssup" in the usual literature; constants in proofs are not preserved in passing from one statement to another.
3.1.2 Weak solutions of the initial-boundary value problem for the harmonic map heat flow

By the Nash embedding theorem, $M$ can be isometrically embedded in some Euclidean space $(\mathbb{R}^N, \langle \cdot, \cdot \rangle)$. Given $w \in H^1(B_1, M)$, we define the energy functional $E(w) = \frac{1}{2} \int_{B_1} |\nabla w|^2$. The harmonic map heat flow is the negative $L^2$ gradient flow of the energy functional. Thus, given $u_0 \in H^1(B_1, M)$ and $\gamma = u_0|_{\partial B_1}$, $u \in H^1([0,T] \times B_1, M)$ is a weak solution of the initial-boundary value problem for the harmonic map heat flow, if

$$
\begin{align*}
\begin{cases}
    u_t - \Delta u &= -A_u(\nabla u, \nabla u) & \text{on} & (0,T) \times B_1 \\
    u(t,x) &= \gamma(x) & \text{for} & t \geq 0, x \in \partial B_1 \\
    \lim_{t \to 0^+} u(t,\cdot) &= u_0 & \text{in} & L^2(B_1, M) \text{ topology}
\end{cases}
\end{align*}
$$

(3.1.1)

where $A$ is the second fundamental form of $M$ in $\mathbb{R}^N$ at the point $u$. Also, we recall that

$$
A_u(\nabla u, \nabla u) \text{ is perpendicular to } M \text{ at } u. \tag{3.1.2}
$$

We say that $u \in H^1([0,T] \times B_1, M)$ is a weak solution of the first equation in (3.1.1) if

$$
\int_0^T \int_{B_1} \langle u_t, \xi \rangle + \langle \nabla u, \nabla \xi \rangle + \langle A_u(\nabla u, \nabla u), \xi \rangle \, dx \, dt = 0, \tag{3.1.3}
$$

for $\forall \xi \in C_c^\infty((0,T) \times B_1, \mathbb{R}^N)$. Since $H^1_0 \cap L^\infty(B_1, \mathbb{R}^N)$ is separable, this definition is equivalent to saying that, for a.e. $t \in [0,T]$,

$$
\int_{\{t\} \times B_1} \langle u_t, \zeta \rangle + \langle \nabla u, \nabla \zeta \rangle + \langle A_u(\nabla u, \nabla u), \zeta \rangle \, dx = 0, \quad \forall \zeta \in H^1_0 \cap L^\infty(B_1, \mathbb{R}^N). \tag{3.1.4}
$$

Note that the condition given by equation (3.1.4) allows us to freeze the time and is therefore more convenient for our proofs of Theorems 3.3.1 and 3.4.1.

The restriction in the second equation of (3.1.1) is taken in the sense of the trace operator acting on $u(t,\cdot)$. Note that $\gamma$ is independent of time. Thus, $\gamma$ is the restriction of $u_0$ on the boundary of the unit open disk.
Also, by Theorem 3 on page 287 of [18], if \( w \in H^1([0,T] \times B_1) \), then there exists \( \tilde{w} \in C^0([0,T], L^2(B_1)) \) and \( \tilde{w}(t, \cdot) = w(t, \cdot) \) for a.e. \( t \). Thus, in this chapter, we always choose \( \tilde{w} \) representing \( w \) in \( H^1([0,T] \times B_1) \). In other words, we always assume that functions in \( H^1([0,T] \times B_1) \) are actually also in \( C^0([0,T], L^2(B_1)) \), and thus the third equation of (3.1.1) makes sense.

3.1.3 Previous work

Under the additional assumption that \( \gamma \in H^{3/2}(\partial B_1, M) \), the initial-boundary value problem for the harmonic map heat flow has been investigated extensively by several mathematicians, such as Chang, Rivièrè and Freire; see [7], [41], [23], [22] and [24]. Define

\[
V^T = H^1([0,T] \times B_1, M) \cap L^\infty([0,T], H^1(B_1, M)) \cap L^2([0,T], H^2(B_1, M)).
\]  

(3.1.5)

The space \( V^T \) plays a crucial role in these papers. However, unless \( \gamma \in H^{3/2}(\partial B_1) \), we are not able to show as Freire did in [22] that \( u \) with non-increasing energy is in \( V^{T'} \) for some \( T' > 0 \). Indeed, if we could, it would imply that \( \gamma \in H^{3/2}(\partial B_1, M) \) by the Sobolev trace theorem. But the image of trace operator on \( H^1(B_1) \) is exactly \( H^{1/2}(\partial B_1) \) and \( H^{3/2}(\partial B_1) \) is a proper subset of \( H^{1/2}(\partial B_1) \).

Instead, given small energy initial data \( u_0 \in H^1 \cap C^0(\bar{B}_1) \) and boundary data \( \gamma = u_0|_{\partial B_1} \) (see section 2.5), we construct weak solutions of the harmonic map heat flow in \( \cap_{T>0} H^1([0,T] \times B_1, M) \) whose energy is non-increasing in time. By the Sobolev trace theorem, such weak solutions need not be in \( V^T \) in general.

3.2 Interior Gradient Estimate for the Harmonic Map Heat Flow

In this section, we derive the interior gradient estimate for small energy solutions of the harmonic map heat flow. This is one of the key ingredients in the proofs of Theorem 3.3.1 and Theorem 3.4.1. First, using Hélein's existence result for the
Coulomb frame, we show that $u(t, \cdot) \in H^2(B_{1/2}, M)$ for a.e. $t$. Next, we follow Struwe's method in [45] to conclude that $u \in L^2([0, \bar{T}], H^2(B_{1/4}, M))$ for $1 < \bar{T} < 2$ and obtain the gradient estimate for $u$ at $(1,0,0)$.

The following elementary geometric fact is obtained in [13] and will be used frequently in this chapter. For self-containedness, we include the proof in Appendix B.

**Lemma 3.2.1.** (Lemma A.1 in [13]) There exists $C > 0$, depending on $M$, such that: if $x, y \in M$, then $|(x - y)^\perp| \leq C|x - y|^2$, where $(x - y)^\perp$ is the normal component to $M$ at $y$.

### 3.2.1 Integral bounds

First, we derive the local integral bounds for $|\nabla^2 u|$ and $|\nabla u|$:

**Lemma 3.2.2.** Suppose that $1 < \bar{T} < 2$ and $u \in H^1([0, \bar{T}] \times B_1, M)$ satisfies

$$u_t - \Delta u = -A_u(\nabla u, \nabla u)$$  \hspace{1cm} (3.2.1)

on $(0, \bar{T}) \times B_1$. Then there exists $\varepsilon > 0$, depending only on $M$, such that: if $E(u(t, \cdot)) \leq \varepsilon$ for a.e. $t \in [0, \bar{T}]$, then

$$\int_0^{\bar{T}} \int_{B_{1/4}} |\nabla^2 u|^2 dxdt \leq 10^5 \sup_{0 \leq t \leq \bar{T}} E(u(t, \cdot)), \hspace{1cm} (3.2.2)$$

$$\int_0^{\bar{T}} \int_{B_{1/4}} |\nabla u|^4 dxdt \leq 10^8 \sup_{0 \leq t \leq \bar{T}} E(u(t, \cdot))^2. \hspace{1cm} (3.2.3)$$

**Proof.** First, note that $|u_t| \in L^2(B_1)$ for a.e. $t$. Fix such a $t$. Following the proof of Theorem 4.1.1 in [29], there exists a $\delta_1 > 0$, depending only on $M$, such that: if $E(u(t, \cdot)) \leq \delta_1$, then there is a finite energy harmonic section (the so called "Coulomb frame") $e(t) = (e_1(t), \ldots, e_n(t))$ of the bundle of orthonormal frames for $u(t, \cdot)^*(TM)$, and one can construct $\beta(t) \in L^\infty(B_1, GL(n, \mathbb{C}))$ satisfying that $|\beta(t)| \leq \lambda_1$, $|\beta(t)^{-1}| \leq \lambda_1$, and

$$\partial_\beta(\beta^{-1} \alpha(t)) = \frac{1}{4} \beta(t)^{-1} f,$$  \hspace{1cm} (3.2.4)
where \( \lambda_1 \) depends only on \( M \) and the upper bound of the energy of \( u(t, \cdot) \), \( z = x_1 + ix_2 \), \( \alpha = (\partial_x u, e_1), \ldots, (\partial_x u, e_n) \) and \( f = (u_t, e_1), \ldots, (u_t, e_n) \). Thus, by the elliptic regularity for \( \partial_x \) operator (see the theorem on page 80 of [28]), \( \beta^{-1} \alpha(t) \in H^1(B_{3/4}) \). It follows from the Sobolev embedding theorem (see Theorem 2 on page 265 of [18]) and \( |\beta(t)| \leq \lambda_1 \) that \( \alpha(t) \in L^p(B_{3/4}) \) for \( 1 < p < \infty \). In particular, \( |\nabla u(t, \cdot)| \in L^4(B_{3/4}) \).

Therefore, by Theorem 8.8 in [25], \( u(t, \cdot) \in H^2(B_{1/2}) \). Next, let \( \phi \) be a smooth cut-off function, which is one in \( B_{1/4} \), compactly supported in \( B_{1/2} \), \( 0 < \phi \leq 1 \) and \( |\nabla \phi| \leq 8 \). Then, by Lemma 6.7 in Chapter III of [46] and equation (3.2.1),

\[
\int_{\{t\} \times B_1} |\Delta u|^2 \phi^2 dx \leq \int_{\{t\} \times B_1} |u_t|^2 \phi^2 dx + \sup_M |A|^2 \int_{\{t\} \times B_1} |\nabla u|^4 \phi^2
\]

\[
\leq \int_{\{t\} \times B_1} |u_t|^2 \phi^2 dx + \lambda_2 E(u(t, \cdot)) \left( \int_{\{t\} \times B_1} |\nabla^2 u|^2 \phi^2 dx + \int_{\{t\} \times B_1} |\nabla u|^2 dx \right),
\]

where \( \lambda_2 = 512 \sup_M |A|^2 \). On the other hand, approximating \( u(t, \cdot) \) by smooth functions in \( H^2(B_{1/2}) \) and integration by parts, one has

\[
\int_{\{t\} \times B_1} |\Delta u|^2 \phi^2 dx \geq \frac{1}{2} \int_{\{t\} \times B_1} |\nabla^2 u|^2 \phi^2 dx - 8 \int_{\{t\} \times B_1} |\nabla u|^2 |\nabla \phi|^2 dx. \tag{3.2.5}
\]

If \( 4\lambda_2 E(u(t, \cdot)) \leq 1 \), then

\[
\int_{\{t\} \times B_1} |\nabla^2 u|^2 \phi^2 dx \leq 32 \int_{\{t\} \times B_1} \left[ |u_t|^2 \phi^2 + |\nabla u|^2 (1 + |\nabla \phi|^2) \right] dx. \tag{3.2.6}
\]

Thus, integrating over \([0, \bar{T}]\), we have

\[
\int_0^{\bar{T}} \int_{B_1} |\nabla^2 u|^2 \phi^2 dx dt \leq 32 \int_0^{\bar{T}} \int_{B_1} \left[ |u_t|^2 \phi^2 + |\nabla u|^2 (1 + |\nabla \phi|^2) \right] dx dt, \tag{3.2.7}
\]

if \( E(u(t, \cdot)) \leq \min\{\delta_1, \lambda_2^{-1}/4\} \) for a.e. \( t \), and it follows from the proof of Lemma 3.4 in [45] (replacing the test function \( u \) by \( u\phi^2 \)) that

\[
\int_0^{\bar{T}} \int_{B_1} |u_t|^2 \phi^2 dx dt \leq 1026 \sup_{0 \leq t \leq \bar{T}} E(u(t, \cdot)). \tag{3.2.8}
\]

\[27\]
Therefore,
\[ \int_{0}^{T} \int_{B_{1}} |\nabla^{2}u|^{2} \phi^{2} dx dt \leq 10^{5} \sup_{0 \leq t \leq T} E(u(t, \cdot)), \quad (3.2.9) \]
and it follows from Lemma 6.7 in Chapter III of [46] that
\[
\int_{0}^{T} \int_{B_{1}} |\nabla u|^{4} \phi^{4} dx dt \leq 8 \sup_{0 \leq t \leq T} E(u(t, \cdot)) \int_{0}^{T} \int_{B_{1}} (|\nabla \phi|^{2} |\nabla u|^{2} + |\nabla^{2}u|^{2} \phi^{2}) dx dt \\
\leq 10^{8} \sup_{0 \leq t \leq T} E(u(t, \cdot))^{2}.
\]

\[\square\]

### 3.2.2 Pointwise gradient bound

Now we are ready to prove the interior gradient estimate:

**Lemma 3.2.3.** Under the assumption in Lemma 3.2.2, there exist \( \epsilon_{1} \in (0, \epsilon] \) and \( C_{1} > 0 \), depending only on \( M \), such that: if \( E(u(t, \cdot)) \leq \epsilon_{1} \) for a.e. \( t \in [0, T] \), then

\[ |\nabla u|^{2}(1, 0, 0) \leq C_{1} \sup_{0 \leq t \leq T} E(u(t, \cdot)). \quad (3.2.10) \]

**Proof.** We will follow the suggestion in the remark after Lemma 3.10 in [45] to obtain the interior gradient estimate for \( u \). Let \( \phi \) be a smooth cut-off function, which is one in \( B_{1/8} \), compactly supported in \( B_{1/4} \), \( 0 \leq \phi \leq 1 \) and \( |\nabla \phi| \leq 16 \). Also, we define \( D^{h}w(t, x) = (w(t + h, x) - w(t, x))/h \) for \( 0 < h < h_{0} \ll 1 \), where \( w \) takes value in \( \mathbb{R} \) or \( \mathbb{R}^{N} \). Thus, for \( 0 < t_{1} \leq t_{2} \leq T - h_{0} \), using equation (3.2.1) and integration by parts, we get

\[
\int_{t_{1}}^{t_{2}} \int_{B_{1}} \partial_{t}|D^{h}u|^{2} \phi^{2} dx dt + 2 \int_{t_{1}}^{t_{2}} \int_{B_{1}} |\nabla D^{h}u|^{2} \phi^{2} dx dt \\
\leq 4 \int_{t_{1}}^{t_{2}} \int_{B_{1}} |D^{h}u||\phi||\nabla D^{h}u||\nabla \phi| dx dt + 2h^{-1} \int_{t_{1}}^{t_{2}} \int_{B_{1}} (A_{u}(\nabla u, \nabla u), D^{h}u) \phi^{2} dx dt \\
-2h^{-1} \int_{t_{1}}^{t_{2}} \int_{B_{1}} (A_{u(t+h,\cdot)}(\nabla u, \nabla u), D^{h}u) \phi^{2} dx dt
\]
\[
\leq 4 \int_{t_1}^{t_2} \int_{B_1} |D^h u| \phi \|\nabla D^h u\| \|\nabla \phi\| dx dt + \lambda_1 \int_{t_1}^{t_2} \int_{B_1} |D^h u|^2 \|\nabla u\|^2(t, x) \phi^2 dx dt \\
+ \lambda_1 \int_{t_1}^{t_2} \int_{B_1} |D^h u|^2 \|\nabla u\|^2(t + h, x) \phi^2 dx dt \\
\leq \int_{t_1}^{t_2} \int_{B_1} \|\nabla D^h u\|^2 \phi^2 dx dt + 4 \int_{t_1}^{t_2} \int_{B_1} |D^h u|^2 \|\nabla \phi\|^2 dx dt \\
+ \lambda_1 \int_{t_1}^{t_2} \int_{B_1} |D^h u|^2 (|\nabla u|^2(t, x) + |\nabla u|^2(t + h, x)) \phi^2 dx dt,
\]

where \( \lambda_1 = C \sup_M |A| \), and we use (3.1.2) and Lemma 3.2.1 in the second inequality. Thus, absorbing the first term on the right hand in the left, gives

\[
\int_{\{t_2\} \times B_1} |D^h u|^4 \phi^4 dx + \int_{t_1}^{t_2} \int_{B_1} |\nabla D^h u|^2 \phi^2 dx dt \\
\leq \int_{\{t_1\} \times B_1} |D^h u|^2 \phi^2 dx + 4 \int_{t_1}^{t_2} \int_{B_1} |D^h u|^2 \|\nabla \phi\|^2 dx dt \\
+ \lambda_1 \int_{t_1}^{t_2} \int_{B_1} |D^h u|^2 (|\nabla u|^2(t, x) + |\nabla u|^2(t + h, x)) \phi^2 dx dt. 
\]

By Lemma 6.7 in Chapter III of [46], we get

\[
\int_{t_1}^{t_2} \int_{B_1} |D^h u|^4 \phi^4 dx dt \\
\leq 8 \sup_{t_1 \leq t \leq t_2} \int_{\{t\} \times B_1} |D^h u|^2 \phi^2 dx \cdot \int_{t_1}^{t_2} \int_{B_1} (|D^h u|^2 \|\nabla \phi\|^2 + |\nabla D^h u|^2 \phi^2) dx dt.
\]

If \( 8\lambda_1 \sup_{0 \leq t \leq T} E(u(t, \cdot)) < 10^{-4} \), then, by Lemma 3.2.2 and Hölder’s inequality, we have

\[
\lambda_1 \int_{t_1}^{t_2} \int_{B_1} |D^h u|^2 (|\nabla u|^2(t, x) + |\nabla u|^2(t + h, x)) \phi^2 dx dt \\
\leq 2\lambda_1 \left( \int_{t_1}^{t_2} \int_{B_1} |D^h u|^4 \phi^4 dx dt \right)^{1/2} \left( \int_{t_1}^{t_2} \int_{B_1/4} (|\nabla u|^4(t, x) + |\nabla u|^4(t + h, x)) dx dt \right)^{1/2} \\
\leq \left( \sup_{t_1 \leq t \leq t_2} \int_{\{t\} \times B_1} |D^h u|^2 \phi^2 dx \right)^{1/2} \left( \int_{t_1}^{t_2} \int_{B_1} (|D^h u|^2 \|\nabla \phi\|^2 + |\nabla D^h u|^2 \phi^2) dx dt \right)^{1/2} \\
\leq \frac{1}{2} \sup_{t_1 \leq t \leq t_2} \int_{\{t\} \times B_1} |D^h u|^2 \phi^2 dx + \frac{1}{2} \int_{t_1}^{t_2} \int_{B_1} (|D^h u|^2 \|\nabla \phi\|^2 + |\nabla D^h u|^2 \phi^2) dx dt.
\]
Hence, (3.2.11) gives

\[
\begin{align*}
\int_{(t_2) \times B_1} & |D^hu|^2 \phi^2 \, dx - \int_{(t_1) \times B_1} |D^hu|^2 \phi^2 \, dx \\
\leq & \frac{9}{2} \int_{t_1}^{t_2} \int_{B_1} |D^hu|^2 |\nabla \phi|^2 \, dx \, dt + \frac{1}{2} \sup_{t_1 \leq t \leq t_2} \int_{(t) \times B_1} |D^hu|^2 \phi^2 \, dx.
\end{align*}
\]  
(3.2.12)

We conclude from (3.2.12) that

\[
\int_{(t_2) \times B_1} |D^hu|^2 \phi^2 \, dx \leq 2 \inf_{0 \leq t \leq t_2} \int_{(t) \times B_1} |D^hu|^2 \phi^2 \, dx + \lambda_2 \int_0^{t_2} \int_{B_{1/4}} |D^hu|^2 \, dx \, dt,
\]

where \(\lambda_2 > 0\) is a universal constant. Therefore,

\[
\int_{(t_2) \times B_1} |D^hu|^2 \phi^2 \, dx \leq 2(t_2^{-1} + \lambda_2) \int_0^{t_2} \int_{B_{1/4}} |D^hu|^2 \, dx \, dt.
\]  
(3.2.13)

Note that \(u\) is in \(H^1([0, T] \times B_1)\) and thus

\[
\int_0^{T-h_0} \int_{B_1} |D^hu - u_t|^2 \, dx \, dt \leq \int_0^1 \int_0^{T-h_0} \int_{B_1} |u(t + sh, x) - u_t(t, x)|^2 \, dx \, dt \, ds.
\]  
(3.2.14)

Since elements in \(L^2\) are continuous in the mean,

\[
\lim_{h \to 0} \int_0^{T-h_0} \int_{B_1} |D^hu - u_t|^2 \, dx \, dt = 0.
\]  
(3.2.15)

Therefore, letting \(h \to 0\), because \(h_0\) is arbitrary, (3.2.13) and (3.2.8), we conclude that for a.e. \(t\),

\[
\int_{(t) \times B_1} |u_t|^2 \phi^2 \, dx \leq 10^4(t^{-1} + \lambda_2) \sup_{0 \leq s \leq T} E(u(s, \cdot)).
\]  
(3.2.16)

By the same argument used to derive (3.2.6), for \(\frac{1}{2} < t < T\),

\[
\int_{(t) \times B_1} |\nabla^2 u|^2 \phi^2 \, dx \leq \lambda_3 \sup_{0 \leq s \leq T} E(u(s, \cdot))
\]  
(3.2.17)

where \(\lambda_3 > 0\) is a universal constant. Hence, by the Sobolev embedding theorem, for
1 < p < \infty,
\int_{1/8}^{T} \int_{B_{1/8}} |u|^p dx dt \leq \lambda_4 \sup_{0 \leq t \leq T} E(u(t, \cdot))^\frac{p}{2},
(3.2.18)
where \lambda_4 depends only on \( p \). Thus, by inserting cut-off functions and the theorem on page 72 of [28], \( |u_t| \) and \( |\nabla^2 u| \) are in \( L^p([1/4, T_1] \times B_{1/16}) \) for \( 1 < T_1 < T \). Furthermore, using the Bochner formula and the Gauss equation, one can derive the evolution equation for \( g = |\nabla u|^2 \) (see page 128 of [28]), that is,
\[ g_t - \Delta g = -2|\text{Hess}_u|^2 + 2\langle A_u(u_{x1}, u_{x1}), A_u(u_{x2}, u_{x2}) \rangle - 2|A_u(u_{x1}, u_{x2})|^2. \] (3.2.19)
Thus, \( g \in W^{1,p}([1/8, T_2] \times B_{1/32}) \) and \( |\nabla g| \in L^p([1/8, T_2] \times B_{1/32}) \) for \( 1 < T_2 < T_1 \). Therefore, by the local maximum principle (see Theorem 7.36 in [35]),
\[ g(1, 0, 0) = |\nabla u|^2(1, 0, 0) \leq \lambda_5 \sup_{0 \leq t \leq T} E(u(t, \cdot)), \] (3.2.20)
where \( \lambda_5 > 0 \) depends only on \( M \), assuming that \( E(u(t, \cdot)) \leq \varepsilon_1 \) for a.e. \( t \) and \( \varepsilon_1 = \min\{\varepsilon, 10^{-4}\lambda_1^{-1}/8\}. \)

\section{3.3 Uniqueness of Weak Solutions to the Harmonic Map Heat Flow from a Disk}

In this section, we show the uniqueness of weak solutions to (3.1.1) whose energy is non-increasing.

\textbf{Theorem 3.3.1.} If \( u \) and \( v \) are weak solutions of (3.1.1) in \( H^1([0, T] \times B_1, M) \) satisfying \( E(u(t_2, \cdot)) \leq E(u(t_1, \cdot)), E(v(t_2, \cdot)) \leq E(v(t_1, \cdot)) \) for \( t_1 \leq t_2 \), and having the same initial data \( u_0 \in H^1(B_1, M) \) and boundary value \( \gamma = u_0|_{\partial B_1} \), then \( u = v \) on \([0, T] \times B_1\).

\textbf{Remark 3.3.2.} In [2], Bertsch, Dal Passo and van der Hout proved that there exist initial data \( u_0 \in H^1(B_1, S^2) \) and boundary data \( \gamma = u_0|_{\partial B_1} \) such that (3.1.1) has infinitely many weak solutions which do not satisfy the non-increasing energy condi-
tion. Thus, Theorem 3.3.1 appears to be the optimal uniqueness statement for weak solutions of the harmonic map heat flow with time independent boundary data.

### 3.3.1 Idea of the proof for Theorem 3.3.1

The main difficulty comes in dealing with the $L^2$ inner product of $|\nabla u(t, \cdot)|^2$ and $h^2$ on $B_1$ for $\forall h \in H^1_0(B_1)$ and $t > 0$, which arises from the non-linear term in the harmonic map heat flow equation. In [22] and [24], Freire first constructed the optimal tangent frames for each fixed time and rewrote equation (3.2.1) under these frames. Then, he used a parabolic perturbation argument to show that given initial data $u_0 \in H^1(B_1, M)$ and the boundary data $\gamma = u_0|_{\partial B_1} \in H^{3/2}(B_1, M)$, any weak solution $u \in H^1([0, T] \times B_1, M)$ of (3.1.1), satisfying that $E(u(t, \cdot)) \leq E(u_0)$ for $a.e. t \in [0, T]$, is in $V^{T'}$ for some $T' \in (0, T)$; see Theorem 1.1 in [22]. Hence, he could use the Cauchy-Schwarz inequality to bound this inner product. However, without his assumption on boundary regularity, Freire’s argument does not apply (as explained in the subsection of previous work). Instead, we make use of the interior gradient estimate and Hardy’s inequality to get around this difficulty. Moreover, we can bound this inner product by some multiple of the energy of $h$, where the multiple depends only on time.

### 3.3.2 Hardy’s inequality

We start by deriving Hardy’s inequality for the unit open disk. This turns out to be the other key ingredient. Such Hardy inequalities also hold for general domains in $\mathbb{R}^2$; see [38].

**Lemma 3.3.3.** For $h \in H^1_0(B_1, \mathbb{R})$,

$$\int_{B_1} \frac{h^2}{(1 - \sqrt{x_1^2 + x_2^2})^2} dx \leq 4 \int_{B_1} |\nabla h|^2 dx. \quad (3.3.1)$$

**Proof.** First, we prove the lemma for $h \in C_c^\infty(B_1, \mathbb{R})$. Rewriting the left hand of
inequality (3.3.1) in polar coordinates and using integration by parts, we get

\[
\int_{B_1} \frac{h^2}{(1 - \sqrt{x_1^2 + x_2^2})^2} dx = \int_0^1 \int_0^{2\pi} \frac{h^2 r}{(1 - r)^2} d\theta dr
\]

\[
= - \int_0^1 \int_0^{2\pi} \frac{h^2}{1 - r} d\theta dr - \int_0^1 \int_0^{2\pi} \frac{2hh_r r^2}{1 - r} d\theta dr
\]

\[
\leq 2 \left( \int_0^1 \int_0^{2\pi} \frac{h^2 r}{(1 - r)^2} d\theta dr \right)^\frac{1}{2} \left( \int_0^1 \int_0^{2\pi} h^2 r^2 d\theta dr \right)^\frac{1}{2}
\]

\[
\leq 2 \left( \int_{B_1} \frac{h^2}{(1 - \sqrt{x_1^2 + x_2^2})^2} dx \right)^\frac{1}{2} \left( \int_{B_1} |\nabla h|^2 dx \right)^\frac{1}{2}.
\]

Thus, inequality (3.3.1) follows by absorbing the second term of the product on the right hand side in the left.

Since \( C_c^\infty(B_1) \) is dense in \( H_0^1(B_1) \), there exists a sequence of \( h_n \in C_c^\infty(B_1, \mathbb{R}) \) such that \( h_n \to h \) in \( H^1(B_1) \) topology and \( h_n \to h \) a.e. in \( B_1 \). By Fatou's Lemma (see Theorem 3 on page 648 of [18]),

\[
\int_{B_1} \frac{h_n^2}{(1 - \sqrt{x_1^2 + x_2^2})^2} dx \leq \liminf_{n \to \infty} \int_{B_1} \frac{h_n^2}{(1 - \sqrt{x_1^2 + x_2^2})^2} dx
\]

\[
\leq \liminf_{n \to \infty} 4 \int_{B_1} |\nabla h_n|^2 dx = 4 \int_{B_1} |\nabla h|^2 dx.
\]

\[ \square \]

### 3.3.3 Stability lemma

Next, to avoid repeating the computation in section 7, we will prove a general stability lemma below, i.e. Lemma 3.3.4. Suppose that \( u \) and \( v \) are weak solutions of (3.1.1) in \( H^1([0,T] \times B_1, M) \) whose energy is non-increasing and with initial data \( u_0 \) and \( v_0 \) respectively. For the moment, \( u_0 \) may not be equal to \( v_0 \). Set \( \varepsilon_2 = \min\{\varepsilon_1, C^{-1}C_1^{-1} \sup_M |A|/32\} \).

The key to proving Lemma 3.3.4 is to bound the \( L^2 \) inner products \( \langle |\nabla u|^2, h^2 \rangle_{L^2} \) and \( \langle |\nabla v|^2, h^2 \rangle_{L^2} \) on \( B_1 \), for \( \forall h \in H_0^1(B_1) \). Such integrals arise from the non-linear terms \( A_u(\nabla u, \nabla u) \) and \( A_v(\nabla v, \nabla v) \) in equation (3.2.1). First, by the energy non-
increasing assumption and Lemma 3.2.3, we can bound $|\nabla u|$ and $|\nabla v|$ for $x_0 \in B_1$ and small time $t_0 > 0$. Namely, since the energy of $u(t, \cdot)$ is non-increasing in time, $u(t, \cdot) \rightharpoonup u_0$ weakly in $H^1(B_1)$ and strongly in $L^2(B_1)$, as $t \to 0$. Thus,

$$\lim_{t \to t_0} \int_{B_1} |\nabla u(t,x) - \nabla u_0|^2 dx = \lim_{t \to t_0} \int_{B_1} |\nabla u(t,x)|^2 dx - \int_{B_1} 2(\nabla u(t,x), \nabla u_0) dx + \int_{B_1} |\nabla u_0|^2 dx \leq 0.$$ 

Therefore, $u(t, \cdot) \rightharpoonup u_0$ strongly in $H^1(B_1)$, and by the same argument, $v(t, \cdot) \rightharpoonup v_0$ strongly in $H^1(B_1)$, as $t \to 0$. Hence, by the absolute continuity of integration, there exist $R_0 > 0$ and $T' \in (0, \min\{R_0^2, T\})$ such that, for $x_0 \in B_1$ and $t \in [0, T']$,

$$\frac{1}{2} \int_{\{t\} \times (B_{R_0}(x_0) \cap B_1)} |\nabla u|^2 dx < \varepsilon_2 \quad \text{and} \quad \frac{1}{2} \int_{\{t\} \times (B_{R_0}(x_0) \cap B_1)} |\nabla v|^2 dx < \varepsilon_2. \quad (3.3.2)$$

Note that equation (3.2.1) is invariant under the transformation $(t, x) \to (\lambda^2 t, \lambda x)$ for $\lambda > 0$, and the energy is invariant under conformal transformations of domains in $\mathbb{R}^2$. Fix $(t_0, x_0) \in (0, T') \times B_1$. Let $\lambda = \min\{\sqrt{t_0}, 1 - |x_0|\}$. Define $u_\lambda(s, y) = u(\lambda^2 s, x_0 + \lambda y)$ and $v_\lambda(s, y) = v(\lambda^2 s, x_0 + \lambda y)$. Then, $u_\lambda$ and $v_\lambda$ satisfy equation (3.2.1) on $(0, \lambda^{-2} T') \times B_1$, and for $s \in [0, \lambda^{-2} T']$, $E(u_\lambda(s, \cdot)) < \varepsilon_2$ and $E(v_\lambda(s, \cdot)) < \varepsilon_2$. Hence, by Lemma 3.2.3, for $(t_0, x_0) \in (0, T') \times B_1$,

$$|\nabla u_\lambda|^2(\lambda^{-2} t_0, 0, 0) \leq C_1 \varepsilon_2 \quad \text{and} \quad |\nabla v_\lambda|^2(\lambda^{-2} t_0, 0, 0) \leq C_1 \varepsilon_2. \quad (3.3.3)$$

Therefore,

$$|\nabla u|^2(t_0, x_0) \leq C_1 [t_0^{-1} + (1 - |x_0|)^{-2}] \varepsilon_2 \quad (3.3.4)$$

$$|\nabla v|^2(t_0, x_0) \leq C_1 [t_0^{-1} + (1 - |x_0|)^{-2}] \varepsilon_2. \quad (3.3.5)$$

Then, combining inequalities (3.3.4) and (3.3.5) with Lemma 3.3.3, we can bound the $L^2$ inner products $\langle |\nabla u|^2, h^2 \rangle_{L^2}$ and $\langle |\nabla v|^2, h^2 \rangle_{L^2}$ on $B_1$, for $\forall h \in H^1_0(B_1)$ and
Lemma 3.3.4. There exists $C_2 > 0$, depending only on $M$, such that:

$$\int_0^{T'} \int_{B_1} |\nabla u - \nabla v|^2 t^{-\frac{1}{2}} dx dt + \frac{1}{2\sqrt{T'}} \int_{\{T'\} \times B_1} |u - v|^2 dx \leq N,$$  \hspace{1cm} (3.3.6)

where

$$N = \left( \frac{1}{\sqrt{T'}} + 2\sqrt{T'} \right) \int_{B_1} (|w_0|^2 + |\nabla w_0|^2) dx$$

$$+ 8\sqrt{2T'(E(u_0) + E(v_0)) \left( \int_{B_1} |\nabla w_0|^2 dx \right)^{\frac{1}{2}}}$$

$$+ 4C_2 \int_0^{T'} \int_{B_1} (|\nabla u|^2 + |\nabla v|^2)(|w_0|^2 + |w_0|) t^{-\frac{1}{2}} dx dt.$$  \hspace{1cm} (3.3.7)

Proof. Define $w = u - v$. It is clear that

$$\int_0^{T'} \int_{B_1} |\nabla w|^2 t^{-\frac{1}{2}} dx dt$$

$$= \int_0^{T'} \int_{B_1} \langle \nabla u, \nabla w \rangle t^{-\frac{1}{2}} dx dt - \int_0^{T'} \int_{B_1} \langle \nabla v, \nabla w \rangle t^{-\frac{1}{2}} dx dt.$$  \hspace{1cm} (3.3.8)

We will estimate the first term of (3.3.8), and the second term can be estimated similarly. First, by footnote 2, $w - w_0 \in C^0([0, T], L^2(B_1))$ and the map $t \rightarrow \|w(t, \cdot) - w_0\|_{L^2(B_1)}$ is absolutely continuous, with

$$\frac{d}{dt}\|w(t, \cdot) - w_0\|^2_{L^2(B_1)} = 2 \int_{[t] \times B_1} \langle w_t, w - w_0 \rangle dx$$

$$\leq 2 \left( \int_{[t] \times B_1} |w_t|^2 dx \right)^{\frac{1}{2}} \left( \int_{[t] \times B_1} |w - w_0|^2 dx \right)^{\frac{1}{2}}$$

for a.e. $t \in [0, T]$. Thus, for $\forall t_0 \geq 0$, integrating over $[0, t_0]$ and by Hölder's inequality, we conclude that

$$\|w(t_0, \cdot) - w_0\|_{L^2(B_1)} \leq \int_0^{t_0} \left( \int_{[t] \times B_1} |w_t|^2 dx \right)^{\frac{1}{2}} dt \leq \sqrt{t_0} \left( \int_0^{t_0} \int_{B_1} |w_t|^2 dx dt \right)^{\frac{1}{2}}.$$
Therefore,
\[
\int_0^{T'} \int_{B_1} |w - w_0|^2 t^{-\frac{3}{2}} dx dt \leq 2\sqrt{T'} \int_0^{T'} \int_{B_1} |w_0|^2 dx dt < +\infty, \tag{3.3.9}
\]
\[
\lim_{t \to 0^+} t^{-\frac{3}{2}} \int_{\{t\} \times B_1} |w - w_0|^2 dx = 0. \tag{3.3.10}
\]

Next, by equation (3.2.1) and integration by parts,
\[
\int_0^{T'} \int_{B_1} \langle \nabla u, \nabla w \rangle t^{-\frac{1}{2}} dx dt
\]
\[
= \int_0^{T'} \int_{B_1} \langle \nabla u, \nabla w - \nabla w_0 \rangle t^{-\frac{1}{2}} dx dt + \int_0^{T'} \int_{B_1} \langle \nabla u, \nabla w_0 \rangle t^{-\frac{1}{2}} dx dt
\]
\[
= \int_0^{T'} \int_{B_1} \langle -u_t, w - w_0 \rangle t^{-\frac{1}{2}} dx dt + \int_0^{T'} \int_{B_1} \langle \nabla u, \nabla w_0 \rangle t^{-\frac{1}{2}} dx dt
\]
\[
- \int_0^{T'} \int_{B_1} \langle A_u(\nabla u, \nabla u), w - w_0 \rangle t^{-\frac{1}{2}} dx dt
\]
\[
\leq \int_0^{T'} \int_{B_1} \langle -u_t, w - w_0 \rangle t^{-\frac{1}{2}} dx dt + 2\sqrt{2T'E(u_0)} \left( \int_{B_1} |\nabla w_0|^2 \right)^{\frac{1}{2}}
\]
\[
- \int_0^{T'} \int_{B_1} \langle A_u(\nabla u, \nabla u), w - w_0 \rangle t^{-\frac{1}{2}} dx dt.
\]

We will bound the third term from above. In the following calculation, the energy non-increasing condition and (3.3.9) guarantee that each quantity below is finite. Since \(A_u(\nabla u, \nabla u)\) is perpendicular to \(M\) at \(u\) and \(w = u - v\), we can apply Lemma 3.2.1 to the \(L^2\) inner product of \(A_u(\nabla u, \nabla u)\) and \(w\) on \(B_1\). Note that \(w - w_0 = (u - u_0) - (v - v_0) \in H_0^1(B_1)\) for a.e. \(t\) fixed. Thus, we can apply the interior gradient estimate (3.3.6) and Lemma 3.3.3 to the \(L^2\) inner product of \(|\nabla u|^2\) and \(|w - w_0|^2\) on \(B_1\). Hence,
\[
- \int_0^{T'} \int_{B_1} \langle A_u(\nabla u, \nabla u), w - w_0 \rangle t^{-\frac{1}{2}} dx dt
\]
\[
\leq \lambda_1 \int_0^{T'} \int_{B_1} |\nabla u|^2 |w|^2 t^{-\frac{1}{2}} dx dt + \lambda_2 \int_0^{T'} \int_{B_1} |\nabla u|^2 |w_0|^2 t^{-\frac{1}{2}} dx dt
\]
\[
\leq \lambda_1 \int_0^{T'} \int_{B_1} |\nabla u|^2 |w - w_0|^2 t^{-\frac{1}{2}} dx dt + \lambda_2 \int_0^{T'} \int_{B_1} |\nabla u|^2 (|w_0|^2 + |w_0|) t^{-\frac{1}{2}} dx dt
\]
\[
\begin{align*}
\leq & \lambda_1 C_1 \varepsilon_2 \int_0^{T'} \int_{B_1} |w - w_0|^2 t^{-\frac{3}{2}} dx dt + 4 \lambda_1 C_1 \varepsilon_2 \int_0^{T'} \int_{B_1} |\nabla w - \nabla w_0|^2 t^{-\frac{1}{2}} dx dt \\
& + \lambda_2 \int_0^{T'} \int_{B_1} |\nabla u|^2 (|w_0|^2 + |w_0|) t^{-\frac{1}{2}} dx dt,
\end{align*}
\]

where \(\lambda_1 = C \sup_M |A|\) and \(\lambda_2 > 0\) (changing from line to line in the computation above) depends only on \(M\). Since \(32 \lambda_1 C_1 \varepsilon_2 < 1\),

\[
\int_0^{T'} \int_{B_1} \langle \nabla u, \nabla w \rangle t^{-\frac{3}{2}} dx dt 
\leq \int_0^{T'} \int_{B_1} \langle -u_t, w - w_0 \rangle t^{-\frac{1}{2}} dx dt + \frac{1}{32} \int_0^{T'} \int_{B_1} |w - w_0|^2 t^{-\frac{3}{2}} dx dt \tag{3.3.11}
\]

\[
+ \frac{1}{4} \int_0^{T'} \int_{B_1} |\nabla w|^2 t^{-\frac{1}{2}} dx dt + N_1,
\]

where

\[
N_1 = \lambda_2 \int_0^{T'} \int_{B_1} (|\nabla u|^2 + |\nabla v|^2) (|w_0|^2 + |w_0|) t^{-\frac{1}{2}} dx dt 
+ 2 \sqrt{2T'(E(v_0) + E(v_0))} \left( \int_{B_1} |\nabla w_0|^2 \right)^{\frac{1}{2}} + \frac{\sqrt{T'}}{2} \int_{B_1} |\nabla w_0|^2 dx.
\]

Similarly,

\[
\begin{align*}
& - \int_0^{T'} \int_{B_1} \langle \nabla v, \nabla w \rangle t^{-\frac{1}{2}} dx dt \\
\leq & \int_0^{T'} \int_{B_1} \langle v_t, w - w_0 \rangle t^{-\frac{1}{2}} dx dt + \frac{1}{32} \int_0^{T'} \int_{B_1} |w - w_0|^2 t^{-\frac{3}{2}} dx dt \tag{3.3.12} \\
& + \frac{1}{4} \int_0^{T'} \int_{B_1} |\nabla w|^2 t^{-\frac{1}{2}} dx dt + N_1.
\end{align*}
\]

Thus, combining (3.3.11) and (3.3.12), we get

\[
\int_0^{T'} \int_{B_1} |\nabla w|^2 t^{-\frac{1}{2}} dx dt \leq \frac{1}{2} \int_0^{T'} \int_{B_1} |\nabla w|^2 t^{-\frac{1}{2}} dx dt \\
- \int_0^{T'} \int_{B_1} \langle w_t, w - w_0 \rangle t^{-\frac{1}{2}} dx dt + \frac{1}{16} \int_0^{T'} \int_{B_1} |w - w_0|^2 t^{-\frac{3}{2}} dx dt + 2N_1.
\]

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Hence, by integration by parts and (3.3.10),

\[
\int_0^{T'} \int_{B_1} |\nabla w|^2 t^{-\frac{1}{2}} dx dt \leq -\frac{1}{2\sqrt{T'}} \int_{\{T'\} \times B_1} |w - w_0|^2 dx + \frac{1}{2} \int_0^{T'} \int_{B_1} |\nabla w|^2 t^{-\frac{1}{2}} dx dt + 2N_1,
\]

Therefore,

\[
\int_0^{T'} \int_{B_1} |\nabla w|^2 t^{-\frac{1}{2}} dx dt + \frac{1}{2\sqrt{T'}} \int_{\{T'\} \times B_1} |w|^2 dx \leq \frac{1}{\sqrt{T'}} \int_{B_1} |w_0|^2 dx + 4N_1, \tag{3.3.13}
\]

and \( C_2 = \lambda_2 \) in the lemma. \qed

### 3.3.4 Completion of the proof for Theorem 3.3.1

We now conclude Theorem 3.3.1 from the stability lemma, i.e. Lemma 3.3.4. Namely, under the condition of Theorem 3.3.1, that is, \( u_0 = v_0 \), we have \( N \equiv 0 \) and thus \( u(t, \cdot) = v(t, \cdot) \) in \( L^2(B_1) \) for each \( t \in [0, T'] \). Therefore, it follows from connectedness that \( u = v \) a.e. on \([0, T] \times B_1\).

### 3.4 Rate of Convergence of the Harmonic Map Heat Flow

In this section, we study the rate of convergence of small energy weak solutions of (3.1.1). Namely, we prove that

**Theorem 3.4.1.** There exists \( \varepsilon_0 > 0 \), depending only on \( M \), such that: if \( u \) is a weak solution of (3.1.1) in \( \cap_{t > 0} H^1([0, T] \times B_1, M) \) satisfying that \( E(u(t, \cdot)) < \varepsilon_0 \) for a.e. \( t \), then there exist \( T_0 > 0 \), \( \alpha_0 > 0 \) and \( C_0 > 0 \) such that, for a.e \( t > T_0 \),

\[
\|u(t, \cdot) - u_\infty\|_{H^1} \leq C_0 e^{-\alpha_0 t}, \tag{3.4.1}
\]

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where \( u_\infty \) is some harmonic map from \( B_1 \) to \( M \) with the same boundary value \( \gamma \).

**Remark 3.4.2.** It follows from Corollary 3.3 in [14] that \( u_\infty \) is the unique harmonic map in the class of

\[
\{ w \in H^1(B_1, M) \mid w|_{\partial B_1} = \gamma \text{ and } E(w) \leq E_1 \},
\]

where \( E_1 > 0 \) is a constant depending only on \( M \).

The following is devoted to the proof of Theorem 3.4.1. First, define \( E = \sup_{t \geq 0} E(u(t, \cdot)) \). If \( E < \varepsilon_1 \), then, by the similar argument used to obtain (3.3.4), we get

\[
|\nabla u|^2(t_0, x_0) \leq C_1 E \max\{t_0^{-1}, (1 - |x_0|)^{-2}\} \leq C_1 E \left[ t_0^{-1} + (1 - |x_0|)^{-2} \right],
\]

for \( t_0 > 0 \) and \( x_0 \in B_1 \).

### 3.4.1 Decay of kinetic energy

Second, we derive two estimates of the kinetic energy: one holds for a.e. \( t_0 > 0 \) and the other holds only for \( t_0 \) large enough.

**Lemma 3.4.3.** There exists \( \varepsilon_3 \in (0, \varepsilon_1) \), depending only on \( M \), such that: if \( E \leq \varepsilon_3 \), then for a.e. \( t_0 > 0 \),

\[
\int_{\{t_0\} \times B_1} |u_t|^2 \, dx \leq \frac{4}{t_0} \int_0^{t_0} \int_{B_1} |u_t|^2 \, dx \, dt,
\]

and there exist \( T_1 > 0 \), \( \alpha_1 > 0 \) and \( C_3 > 0 \) such that for a.e. \( t_0 \geq T_1 \),

\[
\int_{\{t_0\} \times B_1} |u_t|^2 \, dx \leq C_3 \exp[-\alpha_1 (t - T_1)].
\]

**Proof.** Assume that \( E < \varepsilon_1 \). Let \( 0 < h < h_0 \ll 1 \). We define the difference quotient

\( D^h u = (u(t + h, x) - u(t, x)) / h \). Note that

\[
D^h u(t, \cdot) \in H^1_0 \cap L^\infty(B_1) \text{ for a.e. } t \text{ fixed.}
\]
Thus, by equation (3.2.1), we get
\[
\frac{d}{dt} \int_{\{t\} \times B_1} |D^h u|^2 dx
\]
\[
= -2 \int_{\{t\} \times B_1} |\nabla D^h u|^2 dx + 2h^{-1} \int_{\{t\} \times B_1} \langle A_u(\nabla u, \nabla u), D^h u \rangle dx
\]
\[
- 2h^{-1} \int_{\{t\} \times B_1} \langle A_{u(t+h,x)}(\nabla u, \nabla u), D^h u \rangle dx
\]
\[
\leq -2 \int_{\{t\} \times B_1} |\nabla D^h u|^2 dx + \lambda_1 \int_{\{t\} \times B_1} |D^h u|^2 \left( |\nabla u|^2(t, x) + |\nabla u|^2(t+h, x) \right) dx,
\]
where \( \lambda_1 = C \sup_M |A| \) and we have used (3.1.2) and Lemma 3.2.1 in the last inequality. For \( t > 0 \), by (3.4.2), (3.4.5) and Lemma 3.3.3, we have
\[
\int_{\{t\} \times B_1} |D^h u|^2 |\nabla u|^2 dx
\]
\[
\leq C_1 E \int_{\{t\} \times B_1} |D^h u|^2 \left[ t^{-1} + (1 - |x|^2)^{-2} \right] dx
\]
\[
\leq 4C_1 E \int_{\{t\} \times B_1} |\nabla D^h u|^2 dx + C_1 E t^{-1} \int_{\{t\} \times B_1} |D^h u|^2 dx.
\]
If \( 8C_1 \lambda_1 E \leq 1 \), then
\[
\frac{d}{dt} \int_{\{t\} \times B_1} |D^h u|^2 dx
\]
\[
\leq - \int_{\{t\} \times B_1} |\nabla D^h u|^2 dx + t^{-1} \int_{\{t\} \times B_1} |D^h u|^2 dx \tag{3.4.6}
\]
\[
\leq - (C_s^{-1} - t^{-1}) \cdot \int_{\{t\} \times B_1} |D^h u|^2 dx,
\]
where we have applied the Sobolev inequality to \( D^h u(t, \cdot) \in H_0^1(B_1) \) in the first inequality and \( C_s > 0 \) is the Sobolev constant of \( B_1 \). Thus, integrating over \([t_0/2, t_0]\), (3.4.6) gives
\[
\int_{\{t_0\} \times B_1} |D^h u|^2 dx \leq \frac{4}{t_0} \int_0^{t_0} \int_{B_1} |D^h u|^2 dx dt. \tag{3.4.7}
\]
On the other hand, if \( I = [2C_s, 2C_s + 1] \), then it is obvious that
\[
\inf_{t \in I} \int_{\{t\} \times B_1} |D^h u|^2 dx \leq \int_I \int_{B_1} |D^h u|^2 dx dt. \tag{3.4.8}
\]
Thus, for $t > T_1 = 2C_s + 2$, (3.4.6) also implies that

$$\int_{\{t\} \times B_1} |D^h u|^2 dx \leq \int_I \int_{B_1} |D^h u|^2 dx dt \cdot \exp[-C_s^{-1}(t - T_1)/2].$$

(3.4.9)

Letting $h \rightarrow 0$ and by (3.2.15), Lemma 3.4.3 follows with $\varepsilon_3 = \min\{\varepsilon_1, C_1^{-1}\lambda_1^{-1}/8\}$,

$$C_3 = \int_I \int_{B_1} |u_t|^2 dx dt$$

and $\alpha_1 = C_s^{-1}/2$. \hfill \Box

### 3.4.2 Completion of the proof for Theorem 3.4.1

Finally, assume that $E < \varepsilon_3$ and $32C_1 E \sup_M |A| < 1$. In the calculation below, we first apply Cauchy-Schwarz’s inequality to the $L^2$ inner product of $u_t$ and $u(t_2, \cdot) - u(t_1, \cdot)$ on $B_1$. Then, we use (3.1.2), Lemma 3.2.1 and Lemma 3.3.3 to bound the $L^2$ inner product of $A_{u(t_2, \cdot)}(\nabla u, \nabla u)$ and $u(t_2, \cdot) - u(t_1, \cdot)$ on $B_1$. Next, we deduce the last inequality from Cauchy’s inequality and the assumption on the upper bound of $E$. That is, for a.e. $T_1 < t_1 < t_2$,

$$\int_{B_1} \langle \nabla u(t_2, x), \nabla u(t_2, x) - \nabla u(t_1, x) \rangle dx$$

$$= \int_{B_1} \langle -u_t(t_2, x) - A_{u(t_2, \cdot)}(\nabla u, \nabla u), u(t_2, x) - u(t_1, x) \rangle dx$$

$$\leq \sqrt{C_s} \left( \int_{B_1} |u_t(t_2, x)|^2 dx \right)^{1/2} \left( \int_{B_1} |\nabla u(t_2, x) - \nabla u(t_1, x)|^2 dx \right)^{1/2}$$

$$+ 4C_1 E \sup_M |A| \cdot \int_{B_1} |\nabla u(t_2, x) - \nabla u(t_1, x)|^2 dx$$

$$\leq 2C_s \int_{B_1} |u_t(t_2, x)|^2 dx + \frac{1}{4} \int_{B_1} |\nabla u(t_2, x) - \nabla u(t_1, x)|^2 dx.$$

Similarly,

$$\int_{B_1} \langle \nabla u(t_1, x), \nabla u(t_1, x) - \nabla u(t_2, x) \rangle dx$$

$$\leq 2C_s \int_{B_1} |u_t(t_1, x)|^2 dx + \frac{1}{4} \int_{B_1} |\nabla u(t_1, x) - \nabla u(t_2, x)|^2 dx.$$
Summing the two inequalities above, we get

\[ \int_{B_1} |\nabla u(t_1, x) - \nabla u(t_2, x)|^2 dx \leq 4Cs \int_{B_1} (|u_t(t_1, x)|^2 + |u_t(t_2, x)|^2) dx. \quad (3.4.10) \]

Therefore, Theorem 3.4.1 follows immediately from Lemma 3.4.3 by choosing \( \epsilon_0 = \min\{\epsilon_3, C^{-1}C_1^{-1}\inf_M |A|^{-1}/32\} \).

### 3.5 Example of the Harmonic Map Heat Flow Not in \( VT \)

In this section, we construct a unique weak solution \( u \in \cap_{T>0} H^1([0, T] \times B_1, M) \) of (3.1.1) starting with small energy initial data \( u_0 \in H^1 \cap C^0(\bar{B}_1, M) \) and boundary data \( \gamma = u_0|_{\partial B_1} \), and show that in general \( u \not\in VT \).

**Proposition 3.5.1.** There exists \( \epsilon_4 > 0 \), depending only on \( M \), such that: given \( u_0 \in H^1 \cap C^0(\bar{B}_1, M) \) with \( E(u_0) < \epsilon_4 \), there exists a unique weak solution in \( \cap_{T>0} H^1([0, T] \times B_1, M) \) of (3.1.1) whose energy is non-increasing. Moreover, for \( 0 \leq t_1 < t_2 \),

\[ \frac{1}{t} \int_{B_1} |\nabla u(t_2, x) - \nabla u(t_1, x)|^2 dx \leq \int_{B_1} |\nabla u(t_1, x)|^2 dx - \int_{B_1} |\nabla u(t_2, x)|^2 dx. \quad (3.5.1) \]

**Remark 3.5.2.** Recently, Colding and Minicozzi showed that the \( H^1 \) distance between a harmonic map and a map in \( H^1(B_1, M) \) with the same boundary value can be controlled by their gap in energy, assuming energy is small; see Theorem 3.1 in [14]. This is a key ingredient in the proof of the finite extinction of Ricci flow. Our inequality (3.5.1) can be viewed as a parabolic version of their theorem.

**Proof.** First, let \( v_{m0} \in C^\infty(\bar{B}_1, \mathbb{R}^N) \) be the global approximations of \( u_0 \), constructed in Theorem 3 on page 252 of [18]. \( u_{m0} \) can be taken to the nearest point projection (onto \( M \)) of \( v_{m0} \). Moreover, the sequence of maps \( u_{m0} \in C^\infty(\bar{B}_1, M) \) approach \( u_0 \) in \( H^1 \cap C^0 \) topology. By Theorem 1.1 in [7], there exists \( \delta_1 \in (0, \epsilon_2) \), depending only on \( M \), such that: if \( E(u_{m0}) < \delta_1 \), then the weak solution \( u_m \in \cap_{T>0} W^{1,2}_p \cap \)
$C^{1+\mu/2,2+\mu}(0,T) \times \bar{B}_1, M)$ exists, where $\forall 0 < \mu < 1$ and $p > 4/(1-\mu)$, and for $0 \leq t_1 < t_2$,

$$
\int_{t_1}^{t_2} \int_{B_1} |\partial_t u_m|^2 \, dx \, dt = E(u_m(t_1, \cdot)) - E(u_m(t_2, \cdot)).
$$

(3.5.2)

Here, a map $w \in W^{1,2}_p((0,T) \times \bar{B}_1, M)$ means that $w$, $|\nabla w|$, $|\nabla^2 w|$ and $|w_t|$ are in $L^p((0,T) \times \bar{B}_1)$, and $w \in M$ for a.e. $(t,x) \in (0,T) \times \bar{B}_1$. If $E(u_0) < \delta_1/2$, then, by Lemma 3.3.4 and a diagonalization argument, there exists a subsequence (re-labeled)
of $u_m$ satisfying, for $\forall T > 0$, $\partial_t u_m \rightharpoonup \partial_t u$ weakly in $L^2([0,T] \times B_1)$, $u_m \rightarrow u$ and $\nabla u_m \rightarrow \nabla u$ strongly in $L^2([0,T] \times B_1)$. Note that the boundary data $\gamma_m \rightarrow \gamma$ in $H^{1/2} \cap C^0(S^1, M)$. Therefore, $u \in \cap_{T>0} H^1([0,T] \times B_1)$ is a weak solution of (3.1.1) with initial data $u_0$ and boundary data $\gamma = u_0|_{\partial B_1}$. Moreover, $E(u(t, \cdot)) \leq E(u_0)$ for a.e. $t$ and there exists a zero measure set $I_1 \subseteq (0,\infty)$ such that: if $t_1, t_2 \in I^c_1$ and $t_1 < t_2$, then

$$
\int_{t_1}^{t_2} \int_{B_1} |u_t|^2 \, dx \, dt \leq E(u(t_1, \cdot)) - E(u(t_2, \cdot)).
$$

(3.5.3)

Second, we will show that inequality (3.5.3) is actually equality for $a.e. 0 < t_1 < t_2$.

Let $0 < h < h_0 \ll 1$. Define $D^h u(t,x) = (u(t+h,x) - u(t,x))/h$. Thus, for $0 < t_1 < t_2$,

$$
\int_{t_1}^{t_2} \int_{B_1} |u_t|^2 \, dx \, dt = \lim_{h \to 0} \int_{t_1}^{t_2} \int_{B_1} \langle u_t, D^h u \rangle \, dx \, dt
$$

$$
= \lim_{h \to 0} \int_{t_1}^{t_2} \int_{B_1} \langle \nabla u, \nabla (D^h u) \rangle \, dx \, dt - \int_{t_1}^{t_2} \int_{B_1} \langle A(u, \nabla u), D^h u \rangle \, dx \, dt.
$$

We will bound the second term. In the following calculation, we use (3.1.2) and Lemma 3.2.1 in the first inequality, apply the gradient estimate (3.4.2) in the second inequality, and use (3.4.5) and Lemma 3.3.3 in the last inequality. Thus,

$$
|\int_{t_1}^{t_2} \int_{B_1} \langle A(u, \nabla u), D^h u \rangle \, dx \, dt|
$$

$$
\leq hC \sup_M |A| \int_{t_1}^{t_2} \int_{B_1} |\nabla u|^2 |D^h u|^2 \, dx \, dt
$$


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\begin{align*}
&\leq hCC_1 E(u_0) \sup_M |A| \int_{t_1}^{t_2} \int_{B_1} |t^{-1} + (1 - |x|)^{-2}| |D^h u|^2 dx dt \\
&\leq t_1^{-1} hCC_1 E(u_0) \sup_M |A| \int_{t_1}^{t_2} \int_{B_1} |D^h u|^2 dx dt \\
&\quad + 4hCC_1 E(u_0) \sup_M |A| \int_{t_1}^{t_2} \int_{B_1} |\nabla D^h u|^2 dx dt,
\end{align*}

For the first term,

\begin{align*}
\int_{t_1}^{t_2} \int_{B_1} - (\nabla u, \nabla D^h u) dx dt
&= \frac{1}{2h} \int_{t_1}^{t_2} \int_{B_1} (|\nabla u(t, x)|^2 - |\nabla u(t + h, x)|^2) dx dt + \frac{h}{2} \int_{t_1}^{t_2} \int_{B_1} |\nabla D^h u|^2 dx dt \\
&= \frac{1}{2h} \int_{t_1}^{t_1+h} \int_{B_1} |\nabla u|^2 dx dt - \frac{1}{2h} \int_{t_2}^{t_2+h} \int_{B_1} |\nabla u|^2 dx dt + \frac{h}{2} \int_{t_1}^{t_2} \int_{B_1} |\nabla D^h u|^2 dx dt.
\end{align*}

If $E(u_0) < C^{-1}C_1^{-1} \inf_M |A|^{-1}/8$, then

\begin{align*}
\int_{t_1}^{t_2} \int_{B_1} |u_t|^2 dx dt
\geq \lim_{h \to 0} \frac{1}{2h} \int_{t_1}^{t_1+h} \int_{B_1} |\nabla u|^2 dx dt - \frac{1}{2h} \int_{t_2}^{t_2+h} \int_{B_1} |\nabla u|^2 dx dt - \frac{h}{8t_1} \int_{t_1}^{t_2} \int_{B_1} |D^h u|^2 dx dt \\
\geq \lim_{h \to 0} \frac{1}{2h} \int_{t_1}^{t_1+h} \int_{B_1} |\nabla u|^2 dx dt - \frac{1}{2h} \int_{t_2}^{t_2+h} \int_{B_1} |\nabla u|^2 dx dt.
\end{align*}

Define $f(t) = \int_{B_1} |\nabla u(t, x)|^2 dx$ and $F(t) = \int_0^t f(s) ds$. Since $f \in L^1([0, T])$ for $\forall T \in (0, \infty)$, $F'(t) = f(t)$ for a.e. $t \in [0, T]$. Hence, there exists a zero measure set $I_2 \subseteq (0, \infty)$ such that: if $t \in I_2^c$, then

\begin{equation}
\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} \int_{B_1} |\nabla u|^2 dx dt = \int_{B_1} |\nabla u(t, x)|^2 dx. \tag{3.5.4}
\end{equation}

Therefore, if $t_1, t_2 \in I_2^c$ and $0 < t_1 < t_2$, then

\begin{equation}
\int_{t_1}^{t_2} \int_{B_1} |u_t|^2 dx dt \geq E(u(t_1, \cdot)) - E(u(t_2, \cdot)). \tag{3.5.5}
\end{equation}

Combining the inequality (3.5.5) with the inequality (3.5.3), we get that, if $t_1, t_2$ are
not in $I_1 \cup I_2$ and $0 < t_1 < t_2$,
\[ \int_{t_1}^{t_2} \int_{B_1} |u_t|^2 dx dt = E(u(t_1, \cdot)) - E(u(t_2, \cdot)). \]

Third, assume that $E(u_0) < \min\{\varepsilon_3/2, \delta_1/2, C^{-1}C_1^{-1} \inf_M |A|^{-1}/32\}$. Since $u_{m0} \rightarrow u_0$ in $H^1(B_1, M)$ topology, we may also assume that $E(u_{m0}) \leq 2E(u_0)$. Thus, for $0 \leq t_1 < t_2$, using (3.4.2), Lemmas 3.2.1, 3.3.3 and 3.4.3, we estimate

\[
\int_{B_1} \langle \nabla u_m(t_2, x), \nabla u_m(t_2, x) - \nabla u_m(t_1, x) \rangle dx \\
= \int_{B_1} \langle -\partial_t u_m(t_2, x) - A_{um}(t_2, x)(\nabla u_m, \nabla u_m), u_m(t_2, x) - u_m(t_1, x) \rangle dx \\
\leq \left( \int_{B_1} |\partial_t u_m(t_2, x)|^2 dx \right)^{\frac{1}{2}} \left( \int_{B_1} |u_m(t_1, x) - u_m(t_2, x)|^2 dx \right)^{\frac{1}{2}} \\
+ C \sup_{M} |A| \int_{B_1} |\nabla u_m(t_2, x)|^2 |u_m(t_1, x) - u_m(t_2, x)|^2 dx \\
\leq \sqrt{t_2 - t_1} \cdot \left( \int_{B_1} |\partial_t u_m(t_2, x)|^2 dx \right)^{\frac{1}{2}} \left( \int_{t_1}^{t_2} \int_{B_1} |\partial_t u_m|^2 dx dt \right)^{\frac{1}{2}} \\
+ CC_1 E(u_m(t_1, \cdot)) \sup_{M} |A| \int_{B_1} (t_2 - t_1)^{-1} |u_m(t_1, x) - u_m(t_2, x)|^2 dx \\
+ CC_1 E(u_m(t_1, \cdot)) \sup_{M} |A| \int_{B_1} \frac{|u_m(t_1, x) - u_m(t_2, x)|^2}{(1 - \sqrt{x_1^2 + x_2^2})^2} dx \\
\leq \frac{33}{16} \int_{t_1}^{t_2} \int_{B_1} |\partial_t u_m|^2 dx dt + \frac{1}{4} \int_{B_1} |\nabla u_m(t_2, x) - \nabla u_m(t_1, x)|^2 dx,
\]

where we use the fact that the energy of $u_m$ is non-increasing in the last inequality. Thus,

\[
\int_{B_1} |\nabla u_m(t_1, x)|^2 dx - \int_{B_1} |\nabla u_m(t_2, x)|^2 dx \\
= \int_{B_1} |\nabla u_m(t_1, x) - \nabla u_m(t_2, x)|^2 dx \\
+ 2 \int_{B_1} \langle \nabla u_m(t_2, x), \nabla u_m(t_1, x) - \nabla u_m(t_2, x) \rangle dx \\
\geq \frac{1}{2} \int_{B_1} |\nabla u_m(t_1, x) - \nabla u_m(t_2, x)|^2 dx - 5 \int_{t_1}^{t_2} \int_{B_1} |\partial_t u_m|^2 dx dt.
\]
Hence, it follows from (3.5.2) that

\[ \frac{1}{2} \int_{B_1} |\nabla u_m(t_1, x) - \nabla u_m(t_2, x)|^2 dx \leq \int_{B_1} |\nabla u_m(t_1, x)|^2 dx - \int_{B_1} |\nabla u_m(t_2, x)|^2 dx. \]

By Lemma 3.3.4, there exists a zero measure set \( I_3 \subseteq (0, \infty) \) such that: if \( t \in I_3^c \), then there exists a subsequence (relabeled) of \( u_m \) such that \( u_m(t, \cdot) \rightharpoonup u(t, \cdot) \) in \( H^1(B_1, M) \) topology. Hence, if \( 0 < t_1 < t_2 \) and \( t_1, t_2 \in (I_1 \cup I_2 \cup I_3)^c \), then

\[ \frac{1}{2} \int_{B_1} |\nabla u(t_2, x) - \nabla u(t_1, x)|^2 dx \leq \int_{B_1} |\nabla u(t_1, x)|^2 dx - \int_{B_1} |\nabla u(t_2, x)|^2 dx \]

\[ - \frac{1}{2} \int_{B_1} |\nabla u(t_1, x)|^2 dx - \frac{1}{2} \int_{B_1} |\nabla u(t_2, x)|^2 dx = \int_{t_1}^{t_2} \int_{B_1} |\nabla u(t, x)|^2 dx dt. \]

We can modify the definition of \( u \) on \((I_1 \cup I_2 \cup I_3) \times B_1\) by taking limits. Therefore, the modified map solves (3.1.1) satisfying that the energy is non-increasing, and the uniqueness for weak solutions of the harmonic map heat flow follows from Theorem 3.3.1. \( \square \)
Chapter 4

A Bernstein Type Theorem for Self-similar Shrinkers

In this chapter, we present the work on [47] on the Bernstein theorem for self-shrinkers under the mean curvature flow. Namely, we show that the only smooth entire graphical self-shrinkers in $\mathbb{R}^{n+1}$ are hyperplanes.

4.1 Background

4.1.1 Self-shrinkers under the mean curvature flow

A one-parameter family of smooth hypersurfaces, $F : (0, T) \times M^n \rightarrow \mathbb{R}^{n+1}$, moves by mean curvature, if

$$\frac{dF}{dt} = -Hn,$$  \hspace{1cm} (4.1.1)

where $n$ is the unit normal of $M_t = F(t, M)$ and $H = \text{div}(n)$ is the mean curvature.

Self-shrinkers represent a special class of solutions of (4.1.1) in which a later time slice is a scaled down copy of an earlier slice. More precisely, a hypersurface $\Sigma$ is said to be a self-shrinker if it satisfies the following equation

$$H = \frac{1}{2}(\vec{x}^2, n),$$ \hspace{1cm} (4.1.2)
where $\vec{x}$ is the position vector in $\mathbb{R}^{n+1}$.

Throughout, we assume that $\Sigma$ can be written as an entire graph of a smooth function $u(x_1, \ldots, x_n) : \mathbb{R}^n \rightarrow \mathbb{R}$, thus (4.1.2) is equivalent to

$$\text{div} \left( \frac{Du}{\sqrt{1 + |Du|^2}} \right) = \frac{x_1 u_{x_1} + \cdots + x_n u_{x_n} - u}{2\sqrt{1 + |Du|^2}}, \quad (4.1.3)$$

where we use the subscript $x_i$ to denote differentiation with respect to $x_i$, $Du = (u_{x_1}, \ldots, u_{x_n})$ and $|Du|^2 = u_{x_1}^2 + \cdots + u_{x_n}^2$.

### 4.1.2 Evolution of graphs by mean curvature

Ecker and Huisken studied the mean curvature evolution of entire graphs in a series of papers beginning with [20] in 1989. In particular, they proved in the appendix of [20] that the only smooth self-shrinkers in $\mathbb{R}^{n+1}$ which are entire graphs having at most polynomial growth are hyperplanes.

Later, in [21], they derived various interior estimates for mean curvature flow and proved the global existence of the smooth mean curvature evolution of entire graphs with only locally Lipschitz initial data. Also, in [11], Colding and Minicozzi proved sharp gradient and area estimates for graphs moving by mean curvature.

### 4.2 Main Result

In this section, we state our Bernstein type theorem for self-shrinkers:

**Theorem 4.2.1.** Suppose that the smooth function $u : \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies the self-shrinker equation (4.1.3). Then $u = a_1 x_1 + \cdots + a_n x_n$ for some constants $a_1, \ldots, a_n \in \mathbb{R}$.

The main point in Theorem 4.2.1 is that no assumption on the growth at infinity is needed and thus it generalizes the result of Ecker and Huisken in [20].

In contrast to the Bernstein theorem for minimal hypersurfaces, which is true only for $n+1 \leq 7$ (see [44], [5] and [6]), our Bernstein type theorem for self-shrinkers holds
for any $n$.

### 4.3 Proof of Theorem 4.2.1

The proof of Theorem 4.2.1 is very elementary in that does not require using gradient or curvature estimates for mean curvature flow. The idea of the proof is inspired from that of the Bernstein theorem for minimal surfaces. However, the proof here works well for any dimension, while the proof of the Bernstein theorem for minimal hypersurfaces is complicated even in relatively low dimensions; see [43]. The reason behind this is: smooth self-shrinkers in $\mathbb{R}^{n+1}$, that are entire graphs, have polynomial volume growth as minimal hypersurfaces (although the orders of volume growth are different), and the weighted stability inequality (4.3.2) with weight $\exp(-|x|^2/4)$ makes the right hand side of (4.3.2) tend to zero in any dimension by choosing appropriate cut-off functions.

#### 4.3.1 Gaussian weighted stability inequality

First, we prove a weighted stability inequality for smooth self-shrinkers in $\mathbb{R}^{n+1}$ that are entire graphs. In [16] and [15], Colding and Minicozzi introduced the operator

$$L = A_{ij} - \frac{1}{2} \langle \mathbf{x}, \nabla \Sigma \rangle + |A|^2 + \frac{1}{2},$$

(4.3.1)

which is saying the linearization of equation (4.1.2). The weighted stability inequality in Lemma 4.3.1 below is equivalent to that $-(L - \frac{1}{2}) \geq 0$ in the $L^2$ space with weight $\exp(-|x|^2/4)$.

**Lemma 4.3.1.** Let $\eta$ be a smooth compactly supported function on $\mathbb{R}^{n+1}$. Then

$$\int_\Sigma \eta^2 |A|^2 e^{-\frac{|x|^2}{4}} \leq \int_\Sigma \nabla \Sigma \eta^2 |A|^2 e^{-\frac{|x|^2}{4}},$$

(4.3.2)

where $A = (a_{ij})$ is the second fundamental form of $\Sigma$ in $\mathbb{R}^{n+1}$ and $\nabla \Sigma$ is the gradient of a function on $\Sigma$.
Proof. Let \( v_{n+1} = (0, \ldots, 0, 1) \in \mathbb{R}^{n+1} \) and \( f = \langle n, v_{n+1} \rangle \). Colding and Minicozzi showed that \( Lf = \frac{1}{2} f \) in Lemma 5.5 of [16]. For self-containedness, we include the proof here. Indeed, at any point \( \vec{x} \in \Sigma \), choose a local geodesic frame \( e_1, \ldots, e_n \), that is, \( \langle e_i, e_j \rangle = \delta_{ij} \) and \( \nabla_{e_i}e_j(\vec{x}) = 0 \). Thus,

\[
\nabla f = \sum_{i=1}^n (\nabla_{e_i}n, v_{n+1})e_i = \sum_{i,j=1}^n -a_{ij}\langle e_j, v_{n+1} \rangle e_i,
\]

\[
\Delta f = \sum_{i=1}^n (\nabla_{e_i}n, v_{n+1}) = \sum_{i,j=1}^n -a_{ij}\langle e_j, v_{n+1} \rangle - a_{ij}\langle \nabla_{e_i}e_j, v_{n+1} \rangle
\]

\[
= \langle \nabla H, v_{n+1} \rangle - |A|^2 \langle n, v_{n+1} \rangle,
\]

where \( H = -\sum_{i=1}^n a_{ii} \) is the mean curvature of \( \Sigma \). Since \( \Sigma \) is a self-shrinker,

\[
\nabla H = \frac{1}{2} \sum_{i=1}^n (\vec{x}, \nabla_{e_i}n) e_i = -\frac{1}{2} \sum_{i,j=1}^n a_{ij}\langle e_j, e_i \rangle e_i.
\]

Hence,

\[
\Delta f = \frac{1}{2} (\vec{x}, \nabla f) - |A|^2 f.
\]

Note that the upward unit normal of \( \Sigma \) is given by

\[
n = \frac{(-Du, 1)}{\sqrt{1 + |Du|^2}},
\]

and thus \( f = \langle n, v_{n+1} \rangle = 1/\sqrt{1 + |Du|^2} > 0 \). Hence, the function \( g = \log f \) is well defined and \( g \) satisfies the following equation

\[
\Delta g - \frac{1}{2} (\vec{x}, \nabla g) + |\nabla g|^2 + |A|^2 = 0.
\]

Multiplying by \( \eta^2 e^{-\frac{|\vec{x}|^2}{4}} \) on both sides of equation (4.3.6) and integrating over \( \Sigma \), give

\[
0 = \int_{\Sigma} \eta^2 \text{div}_\Sigma \left( e^{-\frac{|\vec{x}|^2}{4}} \nabla g \right) + \int_{\Sigma} \eta^2 (|\nabla g|^2 + |A|^2) e^{-\frac{|\vec{x}|^2}{4}}
\]
\[ = - \int_\Sigma 2\eta (\nabla_\Sigma \eta, \nabla_\Sigma g) e^{-\frac{|\gamma|^2}{4}} + \int_\Sigma \eta^2 (|\nabla_\Sigma g|^2 + |A|^2) e^{-\frac{|\gamma|^2}{4}} \]
\[ \geq - \int_\Sigma (\eta^2 |\nabla_\Sigma g|^2 + |\nabla_\Sigma \eta|^2) e^{-\frac{|\gamma|^2}{4}} + \int_\Sigma \eta^2 (|\nabla_\Sigma g|^2 + |A|^2) e^{-\frac{|\gamma|^2}{4}} \]
\[ \geq \int_\Sigma (-|\nabla_\Sigma \eta|^2 + \eta^2 |A|^2) e^{-\frac{|\gamma|^2}{4}}. \]

4.3.2 Rate of volume growth

Second, we study the volume growth of entire graphical self-shrinkers. Let \( B_R \) be the open ball in \( \mathbb{R}^n \) centered at the origin with radius \( R \) and \( \omega_n \) be the volume of the unit \( n \)-sphere in \( \mathbb{R}^{n+1} \). Also, we define \( M_R = \sup_{B_R} |u| \). We show that

**Lemma 4.3.2.** There exists a constant \( C_0 > 0 \), depending only on \( n \), such that

\[
\text{Vol}(\Sigma \cap B_R \times \mathbb{R}) \leq C_0 R^{n-1}(M_R + R)(RM_R + 1),
\]

where Vol stands for volume.

**Proof.** First, using the pull back of \( \eta \) induced by the projection \( \pi : B_{2R} \times \mathbb{R} \rightarrow B_{2R} \), we extend the vector field \( \eta \) to the cylinder \( B_{2R} \times \mathbb{R} \). Let \( \omega \) be the \( n \)-form on the cylinder \( B_{2R} \times \mathbb{R} \) such that for any \( X_1, \ldots, X_n \in \mathbb{R}^{n+1} \),

\[
\omega(X_1, \ldots, X_n) = \det(X_1, \ldots, X_n, \eta).
\]

Then, in coordinates \( (x_1, \ldots, x_{n+1}) \), we have

\[
\omega = \frac{dx_1 \wedge \cdots \wedge dx_n + \sum_{i=1}^{n} (-1)^{n+i} u_{x_i} dx_1 \wedge \cdots \wedge \hat{dx_i} \wedge \cdots \wedge dx_{n+1}}{\sqrt{1 + |Du|^2}},
\]

and

\[
d\omega = (-1)^{n+1} \text{div} \left( \frac{Du}{\sqrt{1 + |Du|^2}} \right) dx_1 \wedge \cdots \wedge dx_{n+1}
\]
\[= (-1)^{n+1} \frac{x_1 u_{x_1} + \cdots + x_n u_{x_n} - u}{2 \sqrt{1 + |Du|^2}} dx_1 \wedge \cdots \wedge dx_{n+1}.\]
By the Cauchy-Schwarz inequality,

\[ |d\omega| \leq \frac{1}{2} \sqrt{x_1^2 + \cdots + x_n^2 + u^2}, \tag{4.3.11} \]

and by (4.3.8), given any orthogonal unit vectors \( X_1, \ldots, X_n \) at a point \((x_1, \ldots, x_{n+1})\),

\[ |\omega(X_1, \ldots, X_n)| \leq 1, \tag{4.3.12} \]

where the equality holds if and only if

\[ X_1, \ldots, X_n \in T_{(x_1, \ldots, x_n, \omega(x_1, \ldots, x_n))} \Sigma. \tag{4.3.13} \]

For minimal graphs, where \( d\omega = 0 \), such an \( \omega \) is called a calibration; see page 3 in [10]. Let \( \Omega \) be the region enclosed by \( \Sigma, \partial B_R \times \mathbb{R} \) and \( B_R \times \{-M_R\} \). Hence, by Stokes’ Theorem,\(^1\)

\[
\text{Vol}(\Sigma \cap B_R \times \mathbb{R}) = \int_{\Sigma \cap \tilde{\Omega}} \omega = -\int_{\partial \Sigma \cap \tilde{\Omega}} \omega + \int_{\tilde{\Omega}} d\omega \\
\leq \text{Vol}(\partial B_R \times [-M_R, M_R]) + \text{Vol}(B_R \times \{-M_R\}) + \frac{1}{2} \int_{\tilde{\Omega}} \sqrt{x_1^2 + \cdots + x_n^2 + u^2}.
\]

Since \( \Omega \) is contained in the cylinder \( \tilde{\Omega} = B_R \times [-M_R, M_R] \), we conclude that

\[
\text{Vol}(\Sigma \cap B_R \times \mathbb{R}) \leq 2\omega_{n-1} R^{n-1} M_R + n^{-1} \omega_{n-1} R^n + \frac{1}{2} \int_{\tilde{\Omega}} (R + M_R) \\
\leq 2\omega_{n-1} R^{n-1} M_R + n^{-1} \omega_{n-1} R^n + n^{-1} \omega_{n-1} R^n M_R (R + M_R) \\
\leq \omega_{n-1} R^{n-1}(2M_R + R + R^2 M_R + RM_R^2) \\
\leq 2\omega_{n-1} R^{n-1}(M_R + R)(RM_R + 1).
\]

Therefore, Lemma 4.3.2 follows immediately with \( C_0 = 2\omega_{n-1} \). \( \Box \)

\(^1\)we choose the orientation of \( \Sigma \) to be compatible with the upward unit normal, the orientation of \( \partial B_R \times \mathbb{R} \) to be compatible with the outward unit normal and the orientation of \( B_R \times \{-M_R\} \) to be compatible with the downward unit normal. Thus the orientation of \( \Omega \) is chosen such that the orientation of \( \partial \Omega \) induced from \( \tilde{\Omega} \) coincides with that we just defined above.
4.3.3 Height estimate

Third, we use the maximum principle for mean curvature flow to bound the $L^\infty$ norm of $u$ on $B_R$.

**Lemma 4.3.3.** Suppose that $R > 1$. Then there exists a constant $C_1 > 0$, depending on $n$ and $M_{2\sqrt{n}}$, such that $M_R \leq C_1 R$. In particular, entire graphical self-shrinkers have polynomial volume growth.

**Proof.** Define

$$w(x_1, \ldots, x_n, t) = \sqrt{R^2 + 1-t} \cdot u \left( \frac{x_1}{\sqrt{R^2 + 1-t}}, \ldots, \frac{x_n}{\sqrt{R^2 + 1-t}} \right), \quad (4.3.14)$$

where $t \in [0, R^2]$. We derive the evolution equation for $w$:

$$\frac{d w}{dt} = \sqrt{1 + |Dw|^2} \cdot \text{div} \left( \frac{Dw}{\sqrt{1 + |Dw|^2}} \right). \quad (4.3.15)$$

Thus $\{\Sigma_t = \text{Graph}_{w(.,t)}\}_{t>0}$ is a one-parameter family of smooth hypersurfaces in $\mathbb{R}^{n+1}$ moving by mean curvature (after composing with appropriate tangential diffeomorphisms). Using arguments like those in Lemma 3 in [11] and [19], we construct suitable open balls as barriers. Let $\rho > 0$ be some constant to be chosen later and $a^+ = \sup_{B_{\rho R}} w(x_1, \ldots, x_n, 0) + \rho R + 1$. Consider the open ball $B_0^+$ centered at $(0, \ldots, 0, a^+)$ with radius $\rho R$. It is easy to check that $x_1^2 + \cdots + x_n^2 + (w(x_1, \ldots, x_n, 0) - a^+)^2 > \rho R$, for any $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Thus $B_0^+$ and $\Sigma_0$ are disjoint. Let $B_t^+$ be the open ball in $\mathbb{R}^{n+1}$ centered at $(0, \ldots, 0, a^+)$ with radius $R_t^+ = \sqrt{\rho^2 R^2 - 2nt}$. Then $\{\partial B_t^+\}_{t>0}$ is a one-parameter family of smooth hypersurfaces moving by mean curvature and it shrinks to its center at $T = \rho^2 R^2/2n$. We choose $\rho^2 > 2n + 1$ to guarantee that $\partial B_t^+$ is not contained in the cylinder $B_R \times \mathbb{R}$, for any $t \in [0, R^2]$. By the maximum principle for mean curvature flow, $\Sigma_t$ and $\partial B_t^+$ are always disjoint for all $t \in [0, R^2]$. Indeed, assume that $T_0$ is the first time that $\Sigma_{T_0}$ and $\partial B_{T_0}^+$ are not disjoint. Note that at every time $t$, the distance between $\Sigma_t$ and $\partial B_t^+$ can be achieved by a straight line segment perpendicular to both $\Sigma_t$ and $\partial B_t^+$. And outside $B_{2\rho R} \times \mathbb{R}$, the distance
between $\Sigma_t$ and $\partial B^+_t$ is larger than $\rho R$. Thus at $T_0$, $\Sigma_{T_0}$ touches $\partial B^+_t$ at some point $\overrightarrow{x}_0 \in \mathbb{R}^{n+1}$, and for $t$ close to $T_0$, $\Sigma_t$ and $\partial B^+_t$ can be written as graphs over the tangent hyperplane of $\Sigma_{T_0}$ at $\overrightarrow{x}_0$ in a small neighborhood of $\overrightarrow{x}_0$. Thus for $t$ close to $T_0$, the evolution equations of the corresponding graphs are locally uniformly parabolic. Hence, the assumption violates the maximum principle for uniformly parabolic partial differential equations. This is a contradiction. Therefore, we get the upper bound for $w$ at time $t = R^2$:

$$\sup_{B^+_R} w(x_1, \ldots, x_n, R^2) \leq a^+ - \sqrt{\rho^2 - 2n - 1} \cdot R$$

$$\leq \sup_{B^+_R} w(x_1, \ldots, x_n, 0) + \rho R + 1 - \sqrt{\rho^2 - 2n - 1} \cdot R$$

$$\leq \sup_{B^+_R} w(x_1, \ldots, x_n, 0) + \sqrt{2n + 1} R + 1.$$  

Similarly, define $a^- = \inf_{B^+_R} w(x_1, \ldots, x_n, 0) - \rho R - 1$ and compare $\Sigma_t$ with $\partial B^-_t$, which is centered at $(0, \ldots, 0, a^-)$ with radius $R^-_t = \sqrt{\rho^2 R^2 - 2nt}$. Therefore,

$$\inf_{B^-_R} w(x_1, \ldots, x_n, R^2) \geq \inf_{B^-_R} w(x_1, \ldots, x_n, 0) - \sqrt{2n + 1} R - 1. \quad (4.3.16)$$

In sum,

$$\sup_{B^+_R} |w|(x_1, \ldots, x_n, R^2) \leq \sup_{B^+_R} |w|(x_1, \ldots, x_n, 0) + \sqrt{2n + 1} R + 1. \quad (4.3.17)$$

Note that $w(x_1, \ldots, x_n, R^2) = u(x_1, \ldots, x_n)$ and

$$w(x_1, \ldots, x_n, 0) = \sqrt{R^2 + 1} \cdot u(x_1/\sqrt{R^2 + 1}, \ldots, x_n/\sqrt{R^2 + 1}). \quad (4.3.18)$$

Thus, by inequality (4.3.17) and the assumption that $R > 1$, we conclude that

$$\sup_{B^+_R} |u| \leq 2(\sup_{B^+_p} |u| + \sqrt{2n + 1}) R. \quad (4.3.19)$$

Hence, choosing $\rho = 2\sqrt{n}$ and $C_1 = 2(\sup_{B^+_R} |u| + \sqrt{2n + 1})$, gives $M_R \leq C_1 R$. 

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4.3.4 Completion of the proof for Theorem 4.2.1

Finally, we choose a sequence of \( R_j \rightarrow \infty \) and a sequence of smooth cut-off functions \( \eta_j \), which satisfies that \( 0 \leq \eta_j \leq 1 \), \( \eta_j \) is 1 inside \( B_{R_j} \) and vanishes outside \( B_{R_j+1} \), and \( |\nabla \Sigma \eta_j| \leq |D \eta_j| \leq 2 \). By the polynomial volume growth of \( \Sigma \), we get that as \( j \rightarrow \infty \),

\[
\int \Sigma |\nabla \Sigma \eta_j|^2 e^{-\frac{|x|^2}{4}} \leq \int_{\Sigma \cap (B_{R_j+1} \setminus B_{R_j})} 2e^{-\frac{|x|^2}{4}} \rightarrow 0. \tag{4.3.20}
\]

Hence, by the weighted stability inequality for \( \Sigma \) and the monotone convergence theorem, we conclude that

\[
\int \Sigma |A|^2 e^{-\frac{|x|^2}{4}} = 0. \tag{4.3.21}
\]

Therefore, \( |A| = 0 \) and \( u = a_1 x_1 + \cdots + a_n x_n \) for some constants \( a_1, \ldots, a_n \in \mathbb{R} \).
Appendix A

Proof of Lemma 3.2.1

**Proposition A.0.4.** There exists $\varepsilon_5 > 0$, depending on $M$, so that: if $x, y \in M$ and $|x - y| < \varepsilon_5$, then $\text{dist}_M(x, y) < 2|x - y|$, where $\text{dist}_M(x, y)$ is the intrinsic distance between $x$ and $y$ on $M$.

**Proof.** If not, then there exists a sequence of $(x_j, y_j) \in M \times M$ such that $|x_j - y_j| \to 0$ but $\text{dist}_M(x_j, y_j) \geq 2|x_j - y_j|$. Since $M$ is compact, there exist $x_0 \in M$ and a subsequence (relabeled) of $(x_j, y_j)$ satisfying that $x_j \to x_0$ and $y_j \to x_0$. There exists $0 < \delta_1 < \sup_M |A|/4$ such that the geodesic ball $B_{\delta_1}^M(x_0)$ centered at $x_0$ with radius $\delta_1$ is strictly geodesically convex. If $j$ is sufficiently large, then $x_j$ and $y_j$ are in $B_{\delta_1}^M(x_0)$. Let $l_j$ be the geodesic distance between $x_j$ and $y_j$, and $\gamma_j : [0, l_j] \to B_{\delta_1}^M(x_0)$ be the unit speed minimizing geodesic joining $x_j$ and $y_j$. Thus,

$$|y_j - x_j|^2 = \int_0^{l_j} 2(\gamma_j'(s) - x_j, \gamma_j'(s)) ds$$

$$= \int_0^{l_j} \int_0^s (2|\gamma_j'(\tau)|^2 + 2(\gamma_j(\tau) - x_j, \gamma_j''(\tau))) d\tau ds$$

$$\geq \int_0^{l_j} \int_0^s 2 \left(1 - 2\delta_1 \sup_M |A|\right) d\tau ds$$

$$\geq \frac{l_j^2}{2}.$$

Therefore, $\text{dist}_M(x_j, y_j) \leq \sqrt{2}|x_j - y_j|$ and this is a contradiction. $\square$

**Proof.** (of Lemma 3.2.1) If $|x - y| \geq \varepsilon_5$, then $|(x - y)^\perp|/|x - y|^2 \leq \varepsilon_5^{-1}$. Otherwise,
\( \gamma : [0, l] \rightarrow M \) be the minimizing geodesic joining \( y \) to \( x \) with length \( l \leq 2|x - y| \).

\[
\begin{align*}
|\langle x - y \rangle^\perp| &= \int_0^l \langle \dot{\gamma}'(s), V \rangle ds = \int_0^l \int_0^s \langle \gamma''(\tau), V \rangle d\tau ds \\
&\leq \sup_M |A| \cdot \frac{l^2}{2} \leq 2\sup_M |A| \cdot |x - y|^2,
\end{align*}
\]

where \( V = (x - y)^\perp / |(x - y)^\perp| \). Therefore, Lemma 3.2.1 follows immediately with \( C = \max\{\varepsilon_5^{-1}, 2\sup_M |A|\} \). \qed
Bibliography


