THREE-DIMENSIONAL PHOTOELASTICITY PROBLEMS

by

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Submitted in Partial Fulfillment of the Requirements for the Degree of Master of Science in Mechanical Engineering from the Massachusetts Institute of Technology 1945

Signature of Author: ____________________________
Signature of Professor in Charge of Research: ____________________________
Signature of Chairman of Department Committee on Graduate Student: ____________________________
Cambridge, Mass.
June 10, 1945

Professor George W. Swett, 
Secretary of the Faculty, 
Massachusetts Institute of 
Technology, 
Cambridge, Massachusetts

Dear Sir:

I herewith submit a thesis entitled "THREE-
DIMENSIONAL PHOTOELASTICITY PROBLEMS" in partial fulfill-
ment of the requirements for the degree of MASTER OF 
SCIENCE, from the Massachusetts Institute of Technology.

Respectfully yours,

Minghua Lee Wu
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I. SUMMARY

The writer here proposes a new method for making photoelastic analysis of three-dimensional stress systems by taking two or more photographic pictures with polarized light, after passed a slit, illuminating on the model at different directions in the same plane. By combining the stress component determined in each picture, the stress distribution in the whole section is obtained. The writer has applied this idea to three-dimensional torsional problems with considerable success. The result obtained for a square section agrees extremely well with the calculated theoretical stress distribution over the entire range, the maximum error is only 0.8%.

In the following pages, a general discussion of three-dimensional-photoelastic methods is first presented. Then a fundamental knowledge of physical optics, stress-optic relations and the scattered light method, which the writer believe is basic to a photoelasticity investigator, follows. Finally, the writer's method is presented, with its application in torsional problems of a circular and a square section.
II. INTRODUCTION

In three dimensional stress problems, the application of photoelasticity method has been less successful than its application in two dimensional stress problems, since the stress distribution is more complicated, the direction and magnitude of principal stress changes from point to point and there is no fixed relation in such variation to follow with in most cases. Up to now, only a small part of the three dimensional problems has been solved.

Two methods has been proposed and used, the "frozen method" \(^1,2,3\) and the "scattered light method"\(^4,5,6\). Both of them have many disadvantages.

In the frozen method, only a limited number of slices can be obtained from a model, the stress pattern may change somewhat by the slicing operation unless considerable care is exercised, and there is necessarily some change in the optical and elastic constants due to annealing, and the large deformations take place with the accompanying loss of exactness in applying the usual assumptions of the theory of elasticity. Furthermore, its application is usually limited to the case where the plane of slice coincides with the plane of principal stresses. Mindlin\(^8\) and Drucker\(^9\) have suggested oblique incidence to remove the last limitation, and has been used by Frocht in torsional problem of circular shaft\(^10\). But the other disadvantages still exist.

The scattered light method proposed by Weller has not been developed to a satisfactory state yet. Drucker and Mindlin\(^11\) have pointed out that a correction must be made due to the rotation of the axes of secondary principal stresses, and in
their paper correction was given only for the case where the the magnitudes of the stresses remain fixed while their orientations, in planes perpendicular to the wave normal, change at a constant rate along the wave normal. The correction may be very complicated in other cases in general. Later, Weller stated "if the rotation of secondary principal axes is small compared to the rate of relative retardation, the correction may be neglected in practice." But even this error is neglected (which may not be justified in some cases), it is still very difficult to use his method to find the direction of principal stress in a three dimensional space by the rotation of either the light source or model.

However, the scattered light method has a distinct advantage for three dimensional photoelastic investigation in that at any point in the model the secondary principal stresses parallel to the plane normal to the light can always be determined in a simple manner no matter what are the stresses acting on the plane perpendicular to the light. The writer conceives an idea to take two or more pictures with light illuminated on the model in different directions and to combine the results obtained in each picture to give the final result. And this set of two or more pictures will give all informations to find the stress distribution in the whole section under investigation. In this paper, the writer applies this idea to the determination of the stresses in a prismatical bar of any section subjected to pure torsion only. However, she hopes to continue along this line to attack the three dimensional stress problem in general.
III. FUNDAMENTAL PHYSICAL OPTICS

From the standpoint of physical optics crystals are classified as either uniaxial or biaxial. In biaxial crystals there are two directions along which monochromatic light vibrating in any plane will travel with the same velocity. These two directions are called optic axes and the angle between them is different for every different crystal form. Uniaxial crystals may be thought of as a special case of biaxials where the angle between the axes is zero.

A. UNIAXIAL CRYSTALS

1. Calcite. A calcite crystal form with its optic axis is shown in Fig.1. The direction of the optic axis in calcite is determined by drawing a line like $zz'$ through a blunt
corner of the crystal so that it makes equal angles with all faces. A plane passes through the optic axis and normal to a crystal surface is called a principal section. For every point there are therefore three principal sections, one for each pair of opposite crystal faces. The principal plane of ordinary ray is defined as a plane in the crystal drawn through the optic axis and the ordinary ray. The principal plane of extraordinary ray is defined as a plane in the crystal drawn through the optic axis and the extraordinary ray. The ordinary ray always lies in the plane of incidence. This is not generally true for the extraordinary ray. The principal planes of the two refracted rays do not coincide except in special cases, where the plane of incidence is also a principal section.

Calcite is among the negative uniaxial crystals, the extraordinary index of refraction of which is less than the ordinary index. The other group is called positive uniaxial crystal whose index of the extraordinary ray is greater than that of the ordinary ray.

2. Wave Surfaces for Uniaxial Crystals. Suppose a flash of light occurs at some point P inside of a crystal, Fig. 2, and at some small interval of time later two wave surfaces are formed, one called the ordinary wave surface and the other extraordinary wave surface. The former is a sphere and the latter is an ellipsoid of revolution. The three cross sections of these surfaces are shown in Fig. 2, with the elliptic wave surfaces exaggerated. Actually it lies much closer to the spherical surfaces.
3. Propagation of Plane Waves in Uniaxial Crystals. Fig. 3 shows a beam of parallel light incident normally on the surface of a negative crystal like calcite. Starting from A, B, and C, and after a short time, the wave front of ordinary light proceeds to 00' while the extraordinary light proceeds to EE', both being planes perpendicular to the paper. The ordinary-ray velocity, which is proportional to AA', BB', and CC' is less than the extraordinary-ray velocity, which is proportional to A a, Bb, and Cc. Furthermore, the O rays are normal to 00', whereas the E rays are not.
The velocity of the extraordinary wave is defined as the velocity of $EB'$ measured in a direction normal to the wave front. In general this wave velocity $A_x, B_y, \text{ and } C_z$ is less than the ray velocity. This is also true for positive crystals.

Cross sections of negative and positive uniaxial crystals and ray-velocity surface (dotted oval) are shown in Fig. 4. The wave surface, i.e., the ellipsoid of revolution, is really

Fig 4 Ray-velocity and wave-velocity surfaces for uniaxial crystals.

a ray-velocity surface and the ray-velocity surface for the ordinary vibrations are both represented by the same circle or sphere. Hereafter the ellipsoid of revolution will be referred to as the E-wave surfaces and the oval of revolution as the E-wave-velocity surface.

In constructing Fig. 3 the optic axis is assumed to be in the plane of the paper. In case it is not in the plane of the paper, a plane drawn tangent to the ellipsoid wavelets will make contact at points in front or in back of the paper.

The directions of vibration in the $O$ and $E$ rays are most
easily specified in terms of the $O$ and $E$ principal planes. The $O$ vibrations are perpendicular to the principal plane of the $O$ ray, the tangent to the $O$-wave surface. The $E$ vibrations are in the principal plane of the $E$ ray, and tangent to the $E$-wave surface.

The $O$ wave, which vibrates everywhere perpendicular to the optic axis, has the same velocity in every direction. The vibrations of $E$ wave makes a different angle with the axis for each different ray that is drawn from $P$ (Fig. 4).

In considering parallel light incident normally on a crystal surface, two special cases arise which are of particular interest. These are the cases shown in Fig. 5, where the crystal face is cut (1) parallel to the optic axis as in (a) and (b), and (2) perpendicular to the optic axis as in (c). It should be noted that in both cases the ray velocities are

![Diagram](image)

Fig. 5 Propagation of normally incident plane waves through calcite crystals cut parallel and perpendicular to the optic axis.
equal to the wave velocities and there is no double refraction. In case (1), however, the \( E \) wave travels faster than the \( O \) wave.

4. Plane Waves at Oblique Incidence. Now consider the case of a beam of parallel light incident at an angle on the surface of a crystal whose optic axis lies in the plane of incidence and at the same time makes some arbitrary angle with the crystal surface (see Fig.6). At the point \( A \) where the light first strikes the boundary, the \( O \)-wave surface is drawn with such a radius that the ratio \( \frac{CB}{AD} \) is equal to the refractive index of the \( O \) ray. The ellipsoidal wave surface is then drawn tangent to the circle at the intersection with the optic axis \( ZZ' \). The points \( D \) and \( F \) and the new wave fronts \( DB \) and \( FB \) are located by drawing tangents from the common point \( B \) to the circle and ellipse. While the light is traveling from \( C \) to \( B \) in air, the \( O \) vibrations travel from \( A \) to \( D \) in the crystal and the \( E \) vibrations travel from \( A \) to \( F \). In the more general case where the optic axis is not in the plane of incidence, the refracted ray will not lie in the same plane.
B. BIAXIAL CRYSTALS

5. Wave Surfaces for Biaxial Crystals. For any biaxial crystal there are three particular velocities, corresponding to vibrations parallel to x, y, and z respectively (see Fig.8). The elastic-solid theory specified three different coefficients of elasticity for these three types of vibration, which give rise to these three velocities. Three cross-sectional views of the wave surfaces for biaxial crystals are drawn in Fig. 8.

Fig. 8 Cross sections of wave surfaces for a biaxial crystal.

The cross section in xz plane is the most interesting, for it contains the four singular points where the outer wave surface touches the inner surface. Redrawn in Fig.9, the rays OR₁ and OR₂ represent directions in which there is but one ray velocity. These are not the optic axes. The optic axes are located by drawing the tangent planes A₁M₁ and A₂M₂. It is difficult to show in two dimensions that these tangent planes touch the three three-dimensional outer surfaces in circles whose diameters are A₁M₁ and A₂M₂, but such is the case. Since the cross section of one surface is a circle, the line OA₁ and
OA₂ are perpendicular to the tangent planes. They therefore give the same wave velocity for both the ellipse and the circle so that OA₁ and OA₂ are the optic axes for the point O.

Ray axes do not necessarily coincide with optic axes. However, the angle between them rarely exceeds two degrees and is usually less than one degree.

6. Fresnel's Ellipsoid. The three perpendicular velocities corresponding to vibrations parallel to x, y, and z (Fig. 8) mentioned in the preceding article give three principal indices of refraction. The relations among the indices of refraction in biaxial crystals are best seen in a Fresnel's ellipsoid, a three-dimensional geometric figure which has three planes of symmetry and is constructed so that the three principal indices of refraction of light waves in their directions of vibration are equal to its three mutually perpendicular semiaxes. The three indices of refraction are designated as n_a, n_b, and n_c, and are equal respectively to the OA, OB, and OZ semiaxes of the ellipsoid (Fig. 10). n_a is the smallest index,
n_b the intermediate, and n_c the largest. In a given crystal the indices are constant for only one wave length of light, and, in general, because of dispersion the indices of refraction vary with the wave length of light and not necessarily to the same extent. Hence the accompanying diagrams are representative of conditions that exist only when monochromatic light is used.

Now consider a wave propagating along direction ON (Fig. 11). When the wave front reaches O, it cuts the ellipsoid in an ellipse ppqq. The semi-axes of the central section of the ellipsoid, perpendicular to the wave normal, give the reciprocals of the velocities for the wave normal; and the direction of a semi-axis of the section is the direction of polarization of the wave having the velocity given by the reciprocal of the other semi-axis. The semi-axes Op and Oq of the ellipse are proportional to the reciprocals of the wave velocities k_1 and k_2 respectively, for a wave normal ON. Op is the polarization
direction of $k_2$, and $Oq$ is that of $k_2$.

There are two planes (DOB and D'O'B of Fig. 10) through the intermediate principal axis (OB) of the ellipsoid which intersect the ellipsoid in circles. Since the semi-axes of each of these sections are equal, a wave front parallel to either one has only one velocity. The normals ($OQ$ and $OQ'$) to these planes are the optic axes; they lie in the $xz$ plane and make equal angles with the $z$-axis. The angle $2\alpha$ is termed the optic axial angle. When it is less than $\pi/2$, the biaxial crystal resembles, somewhat, a positive uniaxial crystal and the more acute the angle, the closer is the similarity. Such a crystal is called positive biaxial. When it is greater than $\pi/2$, the crystal resembles a negative uniaxial crystal and the closer it approaches $\pi$, the closer is the similarity. Such a crystal is called negative biaxial.
IV. STRESS-OPTIC RELATION

Experiments indicate that the same type of relation exists between stress and change of index of refraction as between stress and strain. Hence, the optical properties which an originally isotropic material assumes on the application of stress may be visualized by imagining that the Fresnel's ellipsoid is the result of the deformation of a sphere of radius \( n_0 \) subjected to principal stresses applied parallel to the axes of the ellipsoid. Thus, an element in a stressed isotropic body behaves like a refracting crystal, and since, in general, the three principal stresses at any point are different from one another, so the stressed element behaves just like the biaxial crystal mentioned in the preceding article, and the relations described there can be applied to such elements.

In general, the directions and magnitudes of the three principal stresses vary from point to point in the stressed body. Let the magnitudes be denoted by \( \sigma_1 \), \( \sigma_2 \), and \( \sigma_3 \). Then

\[
A ( n_a - n_o ) = E \epsilon_1 = \sigma_1 - \sqrt{\sigma_1^2 - \sigma_2^2 - \sigma_3^2}
\]

\[
A ( n_b - n_o ) = E \epsilon_2 = \sigma_2 - \sqrt{\sigma_2^2 - \sigma_1^2 - \sigma_3^2}
\]

\[
A ( n_c - n_o ) = E \epsilon_3 = \sigma_3 - \sqrt{\sigma_3^2 - \sigma_1^2 - \sigma_2^2}
\]

where \( A, E, \sqrt{ } \) are constants, \( n_c > n_b > n_a \) and \( \sigma_3 > \sigma_2 > \sigma_1 \). The axes a, b, c and 1, 2, 3 and x, y, z are coincident in the order named.

Since the differences between refractive indices are generally easier to measure than the indices themselves,
equations (1) can be rearranged into

\[ n_c - n_a = K \left( \sigma_3 - \sigma_1 \right) \]
\[ n_a - n_b = K \left( \sigma_2 - \sigma_3 \right) \]
\[ n_b - n_c = K \left( \sigma_1 - \sigma_2 \right) \]

\[ (1 + \sqrt{\lambda}) \]

where \( K = \frac{\lambda}{A} \).

When a circular polarized light (or plane polarized light making 45° with the other two principal axes) propagates through the stressed body along one principal stress direction, it divides into two components polarized at right angles to each other and vibrate along the directions of the other two principal axes, the relative retardation between them gives the difference of principal stresses in that plane. But if it propagates along a direction other than the principal axis, the two mutually perpendicular components (polarized) vibrate along the directions of the "secondary principal stresses" axes. Then the relative retardation between the two gives only the difference of the secondary principal stresses.
V. SCATTERED LIGHT METHOD

When a beam of plane polarized light (such as the one shown in Fig. 12) falls on a material particle in an isotropic medium, it will scatter in all directions. In the plane passing through the particle and perpendicular to the wave normal, the light is plane polarized, because, according to the electromagnetic theory, an electromagnetic wave is necessarily a transverse wave and the vibration is in the wave front and perpendicular to the direction of travel (Fig. 12a).

![Diagram](image)

(a) Amplitude of the light wave scattered in all directions from a single particle.  
(b) Polarization by scattering from a single particle.

Fig. 12

Fig. 12a also shows that the intensity of scattered light observed in at right angles to the wave normal will thus vary from zero, when looking along the direction of vibration, to a maximum, when looking perpendicular to that direction.

If the incident light is composed of two perpendicular components, by the same reasoning, we can conclude that the light scattered in that plane is still plane polarized (Fig. 12b).
If a whole plane of the model is illuminated by a sheet of parallel rays of polarized light formed by passage through a polarizer, collimator, and a slit, the light scattered in a given direction perpendicular to the wave normal will in general vary in intensity from point to point in the plane depending upon the projections of the ellipse of vibration associated with each point in the plane, the alternate lighter and darker regions forming a fringe pattern. This interference pattern will vary in distinctness and even in form as the direction of observation is changed, but in general it may be said that the distance between fringes in the direction of travel of the light is a function of the state of stress between the points of minimum intensity.

The space rate of formation of fringes, $\frac{dN}{dS}$, is a measure of the birefringence in successive planes normal to the observation direction, $N$ being the number of cycles of interference observed along the path and $S$ the path variable. Hence

$$\frac{dN}{dS} = \frac{1}{C} \left( \sigma_a - \sigma_b \right)$$

(3)

where $C$ is the stress-optical coefficient of the material, and $\sigma_a$ and $\sigma_b$ are the secondary principal stresses.

Let

$$d = \frac{dS}{dN},$$

then (3) becomes

$$\frac{C}{d} = \sigma_a - \sigma_b$$

(4)

If it is assumed that the stresses remain sensibly constant over a small increment of light path,

$$d = \frac{\Delta S}{\Delta N}$$

(5)

For $\Delta N = 1$, $d = \Delta S =$ fringe spacing.
VI. WRITER'S METHOD FOR TORSIONAL PROBLEMS

A. THEORY

In the case of a prismatical bar of any section subjected to pure torsion (as shown in Fig. 13), theory of elasticity shows that all stress components except $\tau_{xy}$ and $\tau_{yz}$ vanish. i.e.

$$\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0.$$ 

Thus, in such problems we need only investigate the shearing stresses $\tau_{yz}$ and $\tau_{xz}$.

![Diagram of torsion in a prismatic bar](image)

**Fig. 13**

The shearing stresses acting on any element is shown in Fig. 13(b).

Now, let the axis of the prismatic bar be in a horizontal position, and any section of the bar perpendicular to the $z$-axis be illuminated by light propagating along $x_0$ direction (as shown in Fig. 14) having passed through a vertical slit parallel to oy axis. The light is polarized at 0° to the slit
length (a circular polarized light can also be used), since, for pure shear, the secondary principal stresses make an angle of 45° with the shearing stresses acting on the xy-plane illuminated by the light.

Refering to Fig. 14b, we can see that the difference between the secondary principal stresses parallel to the yz-plane and acting on planes perpendicular to yz-plane can be determined in the ordinary scattered light method by taking a picture along the zo direction. This difference of secondary principal stresses gives the shearing stress τ_{yz}. It should be noted that from the stress ellipsoid described in IV we can see that the normal and shearing stresses acting on planes perpendicular to the wave normal need not be considered in calculating or measuring the secondary principal stresses as the former may be varied at will without affecting the latter, even though such variation do change the magnitudes and directions of the true principal stresses.

Let the secondary principal stresses be denoted by a and b. From equation (4), we know
\[ \sigma_a - \sigma_b = \frac{C}{d} \]
and since
\[ \tau_{yz} = \frac{1}{2} (\sigma_a - \sigma_b) \] (5)
so that \( \tau_{yz} \) is determined and has the value of
\[ \tau_{yz} = \frac{1}{2} \frac{C}{d_x} \] (6)
where \( d_x \) is the fringe spacing measured along x direction or the direction of light propagation, and C is the stress-optic coefficient depending on the material of model and wave.
length of light used. For Bakelite BT-61-835, and wave length of 5460 \( \AA \), \( C = 87.7 \), same as in two dimensional photoelasticity.

Then a similar picture is taken with the light propagating vertically downward in \( y \) direction (Fig. 15). By the same reasoning this gives

\[
\tau_{xz} = \frac{i}{2} \frac{C}{d_y} .
\]

(7)

where \( d_y \) denotes the fringe spacing along \( y \) direction.

Of course, in the second picture, the light can be more conveniently arranged to incident on the model from a source placed below the model. It will give the same result.

After \( \tau_{xz} \) and \( \tau_{yz} \) are determined the resultant shearing stress is

\[
\tau = \sqrt{\tau_{xz}^2 + \tau_{yz}^2}
\]

(8)

and the angle between \( \tau \) and \( \tau_{xz} \) is

\[
\theta = \tan^{-1} \frac{\tau_{yz}}{\tau_{xz}}
\]

(9)

Although the photoelasticity method can only give the magnitudes of \( \tau_{xz} \) and \( \tau_{yz} \), their directions are easily determined by the direction of the applied torque.

In general, this method requires two pictures taken in the way described above; but for bars whose section has the same shape when rotated 90° about z-axis, such as circular, square, octagonal .... section, one picture is enough, since the two will be the same. In such case, \( d_x \) is first measured along the line whose stress distribution is to be determined, then turn the picture 90° and measure \( d_y \) on a line which has the same absolute position to space as the first line has before the picture is rotated (see Fig. 16).
Unfortunately, when the writer was ready to apply this method to investigate shearing stresses in several prismatical bars, the light source was out of service. Professor W.M.Murray, my adviser, suggested to use the pictures on circular and square shafts taken by R.A.Frigon\textsuperscript{16} here in 1941. He took the picture with the light illuminated in one direction only. But, as mentioned in bottom of page 21, the fringe pattern of such section has exactly the same shape with regard to the x and y axes, the pictures will serve the present purpose. However, the boundaries of these two pictures are not well shown, which will cause some error in the following calculation.

(1) Circular section: We know that the shearing stress in a circular section under pure torsion is always perpendicular to the radius. Therefore the fringe spacing measured along a radius perpendicular to the wave front will give the resultant
Fig. 17 Fringe photograph of circular shaft in torsion.

Light plane: normal to axis. Incident polarization: 0° with light plane. Wave length of light: 5460 Å.
Polariscope slit: 0.3". Camera orientation: aligned with shaft axis.
Dimension of shaft: diameter = 0.875" Length = 6".
Applied moment: 333.9 in.-lb.
Model material: Bakelite BT-61-898.
Magnification of picture: about 4.57 X
shearing stress. This can also be seen from Fig. 17. For, if light is incident on the model the second time in a direction perpendicular to that in the first time, \( d_y \) will be infinite or \( \tau_{xz} = 0 \).

Fringe orders along a radius were measured from a picture of 9.6 \( X \) magnification, and listed in the following:

<table>
<thead>
<tr>
<th>Fringe order</th>
<th>Distance from center</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1.31</td>
</tr>
<tr>
<td>2</td>
<td>1.84</td>
</tr>
<tr>
<td>3</td>
<td>2.25</td>
</tr>
<tr>
<td>4</td>
<td>2.59</td>
</tr>
<tr>
<td>5</td>
<td>2.88</td>
</tr>
<tr>
<td>6</td>
<td>3.15</td>
</tr>
<tr>
<td>7</td>
<td>3.40</td>
</tr>
<tr>
<td>8</td>
<td>3.65</td>
</tr>
<tr>
<td>9</td>
<td>3.84</td>
</tr>
<tr>
<td>10</td>
<td>4.05</td>
</tr>
<tr>
<td>11</td>
<td>4.20 (at boundary)</td>
</tr>
</tbody>
</table>

This data was plotted in Fig. 18, and the value taken from a smooth curve drawn through these points was used in the evaluation of shearing stresses along the radius.

<table>
<thead>
<tr>
<th>Fringe order from center</th>
<th>Distance between fringes ( d' = \frac{d}{X} )</th>
<th>Radius in magnified picture ( r' )</th>
<th>Corresponding radius in actual shaft ( r )</th>
<th>Shearing stress ( \tau )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0.71</td>
<td>0.074</td>
<td>322</td>
</tr>
<tr>
<td>1</td>
<td>1.31</td>
<td>1.60</td>
<td>0.166</td>
<td>795</td>
</tr>
<tr>
<td>2</td>
<td>1.84</td>
<td>2.07</td>
<td>0.215</td>
<td>1,028</td>
</tr>
<tr>
<td>3</td>
<td>2.25</td>
<td>2.42</td>
<td>0.252</td>
<td>1,273</td>
</tr>
<tr>
<td>4</td>
<td>2.58</td>
<td>2.75</td>
<td>0.286</td>
<td>1,451</td>
</tr>
<tr>
<td>5</td>
<td>2.87</td>
<td>3.01</td>
<td>0.314</td>
<td>1,560</td>
</tr>
<tr>
<td>6</td>
<td>3.14</td>
<td>3.27</td>
<td>0.340</td>
<td>1,750</td>
</tr>
<tr>
<td>7</td>
<td>3.38</td>
<td>3.50</td>
<td>0.364</td>
<td>1,830</td>
</tr>
<tr>
<td>8</td>
<td>3.61</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
where \( d' \) and \( r' \) were measured in the magnified picture and

\[
\tau = \frac{C}{d'/9.6} = \frac{87.7 \times 9.6}{2 \times d'} = \frac{427}{d'}. \quad (10)
\]

This result was plotted in Fig. 19, which shows that the stress distribution is essentially a straight line and the maximum shearing stress is 2243 psi.

(2) Square section: The shearing stress distribution was calculated along a line OA (Fig. 20), passing through center 0 of the section and perpendicular to an edge. Since it makes 45° with both \( x \) and \( y \) axes, \( d_y' = d_x' \) and hence \( \tau_{xz} = \tau_{yz} \). On other lines, however, \( d_y' \) and \( d_x' \) should be measured separately as stated in VI A.

Fringe orders along OA were measured on a picture of 7.46X magnification, its length was to be denoted by a "I" on the dimensions as before. The result is listed as follows:

<table>
<thead>
<tr>
<th>Fringe order</th>
<th>Distance from center to the point between two fringes ( r' )</th>
<th>Fringe spacing ( d_x' )</th>
<th>Distance from center to the point on model ( r )</th>
<th>Shearing stress at ( r ) ( \tau )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.24</td>
<td>1.44</td>
<td>0.166</td>
<td>322</td>
</tr>
<tr>
<td>1</td>
<td>1.96</td>
<td>0.90</td>
<td>0.262</td>
<td>514</td>
</tr>
<tr>
<td>2</td>
<td>2.47</td>
<td>0.65</td>
<td>0.551</td>
<td>756</td>
</tr>
<tr>
<td>3</td>
<td>2.85</td>
<td>0.52</td>
<td>0.582</td>
<td>892</td>
</tr>
<tr>
<td>4</td>
<td>3.20</td>
<td>0.47</td>
<td>0.428</td>
<td>935</td>
</tr>
</tbody>
</table>
Fig. 19
Fig. 20 Fringe photograph of square section in torsion

Light plane: normal to axis. Incident polarization: 0°.
Wave length of light: 5460 Å. Polariscope slit: 0.3".
Camera orientation: aligned with shaft axis.
Magnification of photograph: 3.84x
Dimension of shaft: side = 0.9375" length = 8"
Material of model: Bakelite BT-61-895
Applied moment: 199.7 in.-lb.
The result was plotted in Fig. 21, which shows that the stress along line OA increases slightly faster than the distance from 0 increases, and the maximum stress at the boundary is 1,155 p.s.i.

C. COMPARISON WITH THEORETICAL STRESS DISTRIBUTION

(1) Circular section: The theoretical distribution is a straight line varying from zero at center to a maximum at boundary of the value \[ \tau_{\text{max}} = \frac{16 \times 333.9}{\pi \times 0.375} = 2,530 \text{ p.s.i.} \]

Comparing the photoelastic result with this theoretical value, we see that the former always has a lower value than the latter with a maximum discrepancy about 11%. The error is mostly due to the inaccuracy in the determination of the center point and the boundary of the photograph, since they are not clearly shown in the photograph, and the magnification of the picture was not stated. If a line is drawn not passing 0 but most of the points in Fig. 19, it will intersect the radius scale at 0'. And if 0' is shifted to 0, the line will be very close to the theoretical one, and the maximum discrepancy will be less than 5%.

(2) Square section: The theoretical value in this case needs a little more calculation. In theory of elasticity,
Fig. 21

Shearing Stress $\tau$, lbf/in.$^2$

- Theoretical
- Photoelastic

$r'$, in.
a stress function $\psi$ of $x$ and $y$ is introduced for torsional problems such that

$$
\tau_{xz} = \frac{\partial \psi}{\partial y} \quad \tau_{yz} = \frac{\partial \psi}{\partial x}
$$

For rectangular section, the stress function is

$$
\psi = - \frac{32}{\pi^3} G \theta \beta \sum_{n=0,1,2}^\infty \frac{(-1)^n}{(2n+1)^3} \cos \left( \frac{(2n+1)\pi y}{b} \right) \frac{\cosh \left( \frac{2n+1}{2} \right) \pi - \frac{x}{a}}{b} \left[ \cosh \left( \frac{2n+1}{2} \right) \pi - 1 \right]
$$

where $G$ is the modulus of rigidity, $\theta$ is the angle of twist per unit length.

For square section, $a=b$,

$$
\psi = - \frac{32}{\pi^3} G \theta a^2 \sum_{n=0,1,2}^\infty \frac{(-1)^n}{(2n+1)^3} \cos \left( \frac{(2n+1)\pi y}{b} \right) \frac{\cosh \left( \frac{2n+1}{2} \right) \pi - \frac{x}{a}}{a} \left[ \cosh \left( \frac{2n+1}{2} \right) \pi - 1 \right]
$$

To facilitate the calculation, OA is placed on y-axis, so that $\tau_{xz}(0,y)$ gives the stress distribution along OA.

Now $\tau_{yz}(0,y) = - \frac{\partial \psi(0,y)}{\partial x} = 0$,

and $\tau_{xz} = - \frac{32}{\pi^3} G \theta a^2 \sum_{n=0,1,2}^\infty \frac{(-1)^n}{(2n+1)^3} \sin \left( \frac{(2n+1)\pi y}{2a} \right) \frac{\sinh \left( \frac{2n+1}{2} \right) \pi - \frac{1}{a}}{a} \left[ \sinh \left( \frac{2n+1}{2} \right) \pi - 1 \right]

= - \frac{16}{\pi^3} G \theta a \left[ \frac{1}{2\pi} \sin \frac{\pi y}{2a} \left( \frac{1}{\cosh \frac{\pi y}{2a}} - 1 \right) - \frac{1}{2} \sin \frac{5\pi y}{2a} \left( \frac{1}{\cosh \frac{5\pi y}{2a}} - 1 \right) - \ldots \right]

From page 249 in reference 15,

$$
Mt = 2 \int \int \psi dx \ dy = 0.1406 \ G \ \theta \ (2a)^4
$$
\[ \tau_{xz}(0, y) = -\frac{M_t}{1406\pi^2a^3}\left\{ \frac{\sin\theta}{2a(\cot\theta - 1)} - \frac{1}{2a} \sin\frac{3\pi y}{6} \cos\frac{\pi y}{6} \right\} \]

\[ = 199.7 \times \frac{8}{0.1406\times 0.32757} \left[ \sin\frac{\pi y}{2a} - \frac{1}{2.508} \frac{1}{2a} \sin\frac{3\pi y}{6} \left( \frac{1}{\frac{5}{2}} \right)^2 \right] \]

\[ = 1393 \left\{ \sin\frac{\pi y}{2a} - \frac{1}{2a} \sin\frac{3\pi y}{6} + \frac{1}{5^2} \sin\frac{5\pi y}{6} \right\} \]

\[ = 1393 \left\{ \sin\frac{\pi y}{2a} - \frac{1}{2a} \sin\frac{3\pi y}{6} + \frac{1}{5^2} \sin\frac{5\pi y}{6} \right\} \]

\[ \tau_{xz}(0, a) = 1393 \left\{ \sin\frac{\pi y}{2a} - \frac{1}{2a} \sin\frac{3\pi y}{6} + \frac{1}{5^2} \sin\frac{5\pi y}{6} \right\} \]

\[ = 1393 \left\{ \sin\frac{\pi y}{2a} - \frac{1}{2a} \sin\frac{3\pi y}{6} + \frac{1}{5^2} \sin\frac{5\pi y}{6} \right\} \]

\[ = 1393 \left\{ \sin\frac{\pi y}{2a} - \frac{1}{2a} \sin\frac{3\pi y}{6} + \frac{1}{5^2} \sin\frac{5\pi y}{6} \right\} \]

\[ = 1393 \left\{ \sin\frac{\pi y}{2a} - \frac{1}{2a} \sin\frac{3\pi y}{6} + \frac{1}{5^2} \sin\frac{5\pi y}{6} \right\} \]

\[ \tau_{xz}(0, \frac{1}{4}a) = \frac{\sin\frac{\pi}{8} - \frac{1}{3^2} \sin\frac{3\pi}{8} + \frac{1}{5^2} \sin\frac{5\pi}{8} - \frac{1}{7^2} \sin\frac{7\pi}{8}}{3^2} \]

\[ = 1393 \left\{ \sin\frac{\pi y}{8} - \frac{1}{3^2} \sin\frac{3\pi y}{8} + \frac{1}{5^2} \sin\frac{5\pi y}{8} - \frac{1}{7^2} \sin\frac{7\pi y}{8} \right\} \]

\[ = 1393 \left\{ \sin\frac{\pi y}{8} - \frac{1}{3^2} \sin\frac{3\pi y}{8} + \frac{1}{5^2} \sin\frac{5\pi y}{8} - \frac{1}{7^2} \sin\frac{7\pi y}{8} \right\} \]

\[ \tau_{xz}(0, \frac{3}{4}a) = \frac{\sin\frac{3\pi}{8} - \frac{1}{3^2} \sin\frac{9\pi}{8} + \frac{1}{5^2} \sin\frac{15\pi}{8} - \frac{1}{7^2} \sin\frac{21\pi}{8}}{3^2} \]

\[ = 1393 \left\{ \sin\frac{3\pi y}{8} - \frac{1}{3^2} \sin\frac{9\pi y}{8} + \frac{1}{5^2} \sin\frac{15\pi y}{8} - \frac{1}{7^2} \sin\frac{21\pi y}{8} \right\} \]

\[ = 1393 \left\{ \sin\frac{3\pi y}{8} - \frac{1}{3^2} \sin\frac{9\pi y}{8} + \frac{1}{5^2} \sin\frac{15\pi y}{8} - \frac{1}{7^2} \sin\frac{21\pi y}{8} \right\} \]
The theoretical values plotted in Fig.21 agree extremely well with the ones calculated from the fringe pattern over the whole line. This proves the validity of the method proposed by the writer. She believes that this very simple method should be able to solve all pure torsional problems of prismatic bar.

(The writer, perhaps, should point out that the result obtained by R.A. Frigon for the square section is wrong. The maximum theoretical stress is only 1165 psi, not 1452 psi as stated in his paper, page 47. He made the mistake in substituting the length of the side, 0.9375, for $a$, which is half of the length of the side, in the formular. Of course, his interpretation from the fringe photograph is also not correct.)

$+^*$ see first page in his appendix. By this substitution, he should only get 145.2 psi. $2^8 \times 145.2 = 1163$ psi, which is the correct value.
VII. BIBLIOGRAPHY


15. "Theory of Elasticity" by Timoshenko.