Creation and Validation of a Numerical Model for the Analysis of Bending Patterns of Flexural Laparoscopic Grasper Fingers

By

Mitchell Westwood

Submitted to the Department of Mechanical Engineering in Partial Fulfillment of the Requirements for the Degree of Bachelor of Science in Mechanical Engineering at the Massachusetts Institute of Technology June 2011

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ABSTRACT

A series of analytical models are created to predict the bending behavior of novel flexural laparoscopic fingers. These models predict behavior and stresses as a function of finger geometry and actuation force. A model is first created for a blade flexure concept and is used to prove the concept impractical, eliminating unnecessary fabrication and testing. Another model is created for an initially curved flexure concept and predicts the success of the model. These fingers are prototyped and tested, confirming the analytical model. The same model predicts the benefits of a modified initially curved flexure and is the basis for the decision to move forward with that concept.

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Statement of Personal Contribution

As a member of the 4-person team of Harry O’Hanley, Matt Rosario, John Walton, and myself, in the MIT class 2.752: Development of Mechanical Products, I contributed fully to the work listed in the following thesis paper. The concept of a 3-fingered flexural laparoscopic grasper was generated by members of the team in the 2.750 class whose work is described below. However, upon joining the grasper team in 2.752, I have contributed to nearly every aspect of design, fabrication, and testing work.

It was alongside Harry O’Hanley that most of the modeling work here listed was performed. For the blade flexure concept, I performed the calculations of the area moment of inertia that led to the completion of the model. I also performed several checks to ensure our calculations were feasible. For the initially curved flexures with rectangular and chord cross sections, I evaluated the math behind the strain-energy methods we hoped to use and eliminated several errors in our calculations that would have prevented an accurate model. Credit is due to O’Hanley, who provided the framework for each of the models in MATLAB. I contributed, however, by introducing and improving the math in each of our models and by participating in the model creation process in general.

More generally, I attended weekly meetings with the team and mentors to check and improve our ideas. I introduced ideas throughout the semester that can be seen in each version of the flexural grasper. I worked with Dr. Jennifer Rosen to create concepts we hope to use in future iterations of the device. I assisted in the fabrication of a test rig and test fingers for model validation purposes. I also assisted in the fabrication and assembly of several device components for several of the prototypes created during the semester. Several concepts I generated for the device have been considered and withheld but could be used for alternate versions of the device. I also contributed to written documentation on the device. Finally, in addition to those listed above, I made more intangible contributions that made the device possible.

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Acknowledgements

This product was developed as part of the MIT courses 2.75: Precision Machine Design and 2.752: Development of Mechanical Products. I would like to thank the fall 2.75 team of Harry O’Hanley, Matt Rosario, Yuanyu Chen, Audrey Maertens, and John Walton for their creating and work on the flexural laparoscopic grasper project. I would additionally like to thank Matt Rosario and John Walton for their continuing work on the project and Harry O’Hanley for his constant ideas and support. I would like to thank Professor Alex Slocum, Dr. Julio Guerrero, Nevan Hanumara, and Nikolai Begg for their advising, countless ideas, and help with many rough spots along the way. I would also like to thank Dr. Jennifer Rosen of the Boston Medical Center who has collaborated on the flexural grasper since its conception. 2.752 receives support from the Center for Integration of Medicine and Innovative Technology (www.cimit.org) under U.S. Army Medical Research Acquisition Activity Cooperative Agreement W81XWH-09-2-0001.
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Introduction

This thesis project documents the work done to improve flexural fingers for a laparoscopic grasper through analytical modeling. The “flexural grasper” project began in the fall of 2010 as part of the MIT 2.75 course, Precision Machine Design. The grasper was designed to improve on current grasper technology by replacing the two-piece needle nose end effector, which is prone to pinching and perforation of tissue. Instead, a three-piece flexural end effector was designed and fabricated. This pseudo-rigid end effector could curl around objects through tendon-like cable actuation.

While results of the fall work were promising, a robust analytical model was needed to predict behavior of the flexural fingers in order to fine-tune their design and performance. A model was created in MATLAB to achieve this purpose and will be described in detail. A blade flexure concept was developed but deemed inadequate through the analytical model. The initially curved flexure concept used in the fall was then modeled and proved to be feasible. The model was used for material selection and was validated by bench-level testing.

Fall 2010 Work

A fall 2010 MIT 2.75: Precision Machine Design class team of Harry O’Hanley, Matt Rosario, Yuanyu Chen, Audrey Maertens, and John Walton, all MIT mechanical engineering students, designed and fabricated a working prototype for a laparoscopic grasper with enhanced finger articulation. Most current laparoscopic grasper end effectors consist of two rigid fingers pivoting about a single base joint. Like a pair of pliers, the single base joint creates a pinch point for the soft tissues the grasper most often handles, potentially causing damage to the tissue. Additionally, the non-parallel finger actuation creates a normal contact force away from the device, pushing objects out of the fingers as they close. This design is seen below in Figure 1.

FIGURE 1: A common laparoscopic grasper design.

The grasper designed by the fall 2010 team looked to address these problems with an alternative end effector design. A three-finger end effector replaced the two-finger design to constrain the effected tissue more completely. Each finger had three flexural joints along its length to allow curling behavior. This curling behavior eliminated the base pinch point seen in common designs. Also, by curling around an object instead of pinching it, the normal reaction force away from the grasper is eliminated. The end effector is seen below in Figure 2. One cable was threaded through the finger and attached at the finger tip, actuating curling behavior. Another was threaded through the first flexure only, allowing the folding behavior seen in most current graspers. Desired finger bending patterns are shown below in Figure 3. The full proof on concept prototype is shown below in Figure 4.
While the importance of flexure stiffness in curling patterns was noted in this design, no analytical model was created to assist in flexure sizing. The creation of such a model was listed as a priority for work to be continued in the spring.

Requirements

As previously mentioned, the end effector fingers are actuated by a through cable. Tension in the cable is controlled by a user-pulled trigger in the handle and the mechanical advantage of the trigger device. To ensure comfort for the user, a requirement for the device is to achieve full actuation of the fingers with a trigger input force of 20N, a value taken from a literature review of laparoscopic grasping forces. Additionally, the flexures must be designed such that the material yield stress is not reached under bending.
Blade Flexure

Although an initially-curved flexure was used for the fall grasper prototype, blade flexures were first considered for the next generation prototype for several reasons. First, the mechanism for folding of the fingers had been changed in the new design concept. Instead of a cable actuating the base joint to fold the finger to a closed position, an outer sheath would be used to slide over the fingers from the base up, folding them together. Since the outer sheath contacts the upper surface of the finger as it slides, a constantly smooth upper finger profile would promote efficient folding of the fingers. A blade flexure would enable such a profile. The proposed flexure geometry is shown below in Figure 1.

![Figure 5: Proposed blade flexure geometry.](image)

The basis of the blade flexure model is Euler-Bernoulli beam bending. Since the rigid sections of the finger, labeled 1, 2, and 3 in Figure 6 below, are much thicker than the flexures, labeled A, B, and C below, the rigid portions are assumed the undergo negligible bending compared to the flexures.

![Figure 6: Rigid zones 1, 2, and 3; and flexures A, B, and C.](image)

A cable runs along the underside of the finger, providing a tension force in the direction of its length. At each flexure the cable force is oriented into the flexure, as shown below in Figure 7, but is offset from the flexure’s neutral axis. This creates a moment at the end of the flexure, causing the observed bending.

![Figure 7: Moment created at tip of flexure.](image)
The resultant angle of the flexure tip under bending is predicted through Euler-Bernoulli beam bending,

\[ \theta = \frac{M \cdot L}{E \cdot I} \]  

where \( M \) is the moment at the flexure tip, \( L \) is the length of the flexure, \( E \) is the Young’s modulus of the flexure material, and \( I \) is the area moment of inertia. The area moment of inertia was found as a function of the finger outer radius of curvature, \( R \), and of flexure thickness, \( t \).

![Flexure cross section](image)

**FIGURE 8:** Flexure cross section. Area moment of inertia is a function of radius of curvature \( R \) and flexure thickness \( t \).

The area moment of inertia is found in the following integral form,

\[ \int y^2 \, dA, \]  

where \( A \) is the cross-sectional area of the flexure and \( y \) is the vertical offset from the neutral axis. The calculation was first performed about the horizontal axis of the circle with radius \( R \). The moment of inertia about the neutral axis was then found using the parallel axis theorem. These calculations are shown in detail in Appendix A.

A MATLAB model was created to predict the bending behavior of the flexural finger using the above equations. An input trigger force, \( F_s \), is multiplied by the handle’s mechanical advantage, \( TR \), and is divided by 3, the number of cables connected to the single trigger. This cable force, \( F \), offset from the neutral axis by a distance \( d \), creates a moment, \( M \), at the tip of flexure A. This causes the flexure to deflect by an angle predicted by equation (1), and the following rigid section 1 continues at that angle for its length. The tip of flexure B also sees a moment caused by the cable and deflects according to equation (1), and so on. The MATLAB model calculates the positions and angles of each of the flexures and rigid sections of the fingers and plots their resultant curling patterns in 2-D space, as viewed from above. The MATLAB model is shown in detail in Appendix C. An example of one of these plots is shown below in Figure 9.
FIGURE 9: MATLAB-generated plot of predicted finger bending.

Additionally, stresses in the flexures were calculated. Bending stress, $\sigma$, in a cantilever beam is found using

$$\sigma = \frac{M \cdot y}{I},$$

where $M$ is the moment on the flexure tip, $I$ is the area moment of inertia, and $y$ is the vertical distance from the bending neutral axis to the material in question. This stress peaks at the surface of the flexure, or

$$y = \frac{t}{2}. \quad (4)$$

The blade flexure model was tested with a range of materials and flexure geometries. The outer radius of curvature $R$ was constrained by the 12mm trocar port the grasper is designed for, and a mechanical advantage of 4 was granted by handle geometry. By altering the thickness and length of the flexures, as well as the Young’s modulus of available materials, finger bending patterns were generated for a range of potential configurations.

The MATLAB model proved the blade flexure concept to be impractical for the flexural grasper. In order for a finger material and geometry to be successful, it must reach the desired curling pattern with a 20N or less input force. At the same time, the material’s yield stress must not be reached during this bending. For every material tested, these conditions were mutually exclusive. As an example, the parameters in the below plots in Figure 10 were such that the yield stress was matched, the upper limit condition the fingers would not be designed to. However, even such a high level of stress cannot cause curling of the fingers back to the axis of the transmission shaft.
Although the benefits of the blade flexure concept were obvious, the accompanying analytical model proved the idea to be impractical to impossible. The initially curved flexure concept was taken under consideration again.

Initially Curved Flexure

The overall geometry of the initially curved flexure concept is shown below in Figure 11. As with the blade flexure concept, the initially curved flexure finger was modeled as rigid zones, labeled 1, 2, and 3, and flexural zones, labeled A, B, and C.

Instead of the moment seen at the tip of the blade flexure, the initially curved flexure sees a combination of stress and moment that changes along its length. This is still caused by a tightened cable running along the underside of the finger. The forces present in the flexure are shown below in Figure 12 below, and the relevant geometries are shown in Figure 13.
Figure 13: Relevant geometry of the initially curved flexure. Note: R is the distance to the bending neutral axis, while r is the distance to the inner radius.

The values for the moment, $M$, normal force, $N$, and shear force, $V$, are given by

$$M = FR \sin(\phi),$$  \hspace{1cm} (5)  

$$N = F \sin(\phi),$$  \hspace{1cm} (6)  

$$V = F \cos(\phi).$$  

(7)

The deflection of the initially curved flexures was modeled using strain-energy methods based on Castigliano’s theorem and modified for curved beam bending. Castigliano’s theorem calls for the use of a variable named $A_m$, defined as

$$A_m = \int \frac{dA}{r},$$  

(8)  

or, in the case of the rectangular cross section in question,

$$A_m = b \ln \frac{c}{a}.$$  

(9)

When the tensile force, $F$, acts on the flexure, the resultant angle of deflection, $\theta$, is given by

$$\theta = \int \frac{A_m \cdot F \cdot R \cdot \sin(\phi)}{A(R \cdot A_m - A)E} \, d\phi.$$  

(10)

The derivation of Equation 10 is detailed in Appendix B. As with the blade flexure model, an input force creates an angle of deflection by Equation 10, and this angle defines the angle between rigid segments on either side of the flexure. The next flexure in line deflects by another angle found with Equation 10, and so on.

Additionally, stresses must also be calculated. Radial stresses are assumed to be negligible in the model, so only circumferential stress, $\sigma_{\theta\theta}$, is found. In the initially curved case, both bending and normal stresses are present in the circumferential stress. This is normally shown by

$$\sigma_{\theta\theta} = \frac{N}{A} + \frac{M \cdot y}{I},$$  

(11)
where $N$ is the internal normal force (seen above in Figure 12), $A$ is the cross sectional area, $M$ is the internal moment (seen above in Figure 12), $I$ is the area moment of inertia, and $y$ is the distance from the neutral axis. As with the blade flexure, the value for $y$ is half the thickness of the flexure. The related equation in Castigliano’s theorem of beam bending is

$$
\sigma_{0\theta} = \frac{N}{A} + \frac{M(A-r\cdot Am)}{A\cdot r(R\cdot Am - A)}. \tag{12}
$$

Substituting Equations (5) and (6) for $M$ and $N$, and evaluating the equation where $\sigma_{0\theta}$ will be highest, at $\varphi = \pi/2$, gives

$$
\sigma_{0\theta \text{ max}} = \frac{F}{A} + \frac{F\cdot R(A-r\cdot Am)}{A\cdot r(R\cdot Am - A)}. \tag{13}
$$

This is the maximum stress seen by the flexure, so it is this stress that must be kept beneath the material yield stress. As with the blade flexure, a MATLAB file was created to model the behavior of an initially curved flexure finger using the above relationships for angle of deflection and maximum stress. The MATLAB model is given in Appendix D.

Results of the MATLAB model indicated that for several materials, the proposed initially curved flexure geometries could successfully produce desired bending patterns without exceeding yield stresses. An example of such a result is shown below in Figure 14. There, the dashed line represents the longitudinal axis of the transmission shaft, the point the finger tips must be able to return to in order to ensure complete grasping.

![Initially Curved Flexure Model Validation](image)

FIGURE 14: Finger bending plots showing successful articulation of initially curved flexures.

**Initially Curved Flexure Model Validation**

The MATLAB model for initially curved flexure finger behavior was tested against a range of materials, and several were chosen as qualified candidates for further testing. While many metals did not achieve full articulation before reaching their yield stress, as was seen with the blade flexure, several plastics were predicted to meet all stress and behavioral requirements.
Nylon 6/6, HDPE, and polycarbonate were the most promising and were chosen for further testing.

A test rig was designed and constructed to observe finger bending behavior as a function of input cable force. A rigid stand of polycarbonate constrained the end of modified fingers. A through hole allowed a cable to pass through the stand and through the holes in the test fingers. The cable was secured at the tip of the finger, and the other end of the cable was attached to a spring scale. As the cable was pulled, the input force was measured on this scale. Finger bending behavior was then compared to printed MATLAB plots of predicted behavior under various cable forces. The resulting forces and bending patterns were compared to predicted values. Examples of these patterns are shown below in Figure 15.

![Test result examples.](image)

Over the range of forces and materials, behavior of the fingers very nearly matched predicted patterns. As can be seen in Figure 15, the amount of bending predicted by the model was slightly larger than the bending experienced by the fingers, especially at high forces. However, the value of the material Young's modulus was not available from the plastics supplier. Instead, estimates for the Young's moduli were made by averaging values found in similar plastics. Additionally, the fingers were fabricated with a waterjet machine. The taper produced by this process slightly altered the geometry of the fingers. A combination of these uncertainties could easily explain the small error between the predicted and experienced bending patterns. Therefore, the model is believed to have been validated by the rig testing.

**Chord Cross Section**

While handling the assembled grasper prototype with initially curved flexures, it became apparent there was an undesirable lack of lateral stiffness. All previous work had gone into optimizing bending stiffness, but the fingers were able to bend in unwanted directions. To remedy this problem, another initially curved flexure design was proposed. Instead of having a rectangular cross section, however, the flexures in this design had a chord cross section. It was thought that this cross section would increase lateral stiffness by maximizing the amount of...
material in the flexure, bounded by the circular inner diameter of the sheath the fingers must fit within. This flexure also has a constant cross section about $\varphi$, resulting in a relatively simple model. The finger and relevant flexure geometries are shown in Figures 16 and 17, respectively.

![Figure 16](image)

**FIGURE 16:** Finger with initially curved flexures with chord cross sections.

![Figure 17](image)

**FIGURE 17:** Relevant geometries of initially curved flexure with chord cross section.

A MATLAB model was created to simulate the behavior of these flexures and to compare their bending and lateral stiffness to those of the rectangular cross section flexures. Strain-energy methods were again used. The result for Equations (14), (15), and (16) are given by literature on Castigliano’s theorem and uses a separate set of geometries shown in Figure 18 below.

![Figure 18](image)

**FIGURE 18:** Relevant geometries specific to chord cross section.
The values for $A_m$, $A$, and $R$ are found to be

$$A_m = 2a \gamma - 2b \sin(\gamma) + 2 \sqrt{b^2 - a^2 \ln\left(\frac{b + a \cdot \cos(\gamma) + \sqrt{b^2 - a^2 \sin(\gamma)}}{a + b \cdot \cos(\gamma)}\right)},$$  \hspace{1cm} (14)

$$A = b^2 \gamma - \frac{b^2}{2} \sin(2\gamma),$$  \hspace{1cm} (15)

$$R = a + \frac{4b \cdot \sin^3(\gamma)}{6\gamma - 3 \cdot \sin(2\gamma)},$$  \hspace{1cm} (16)

For stiffness comparisons, bending stiffness and lateral stiffness were calculated using Equation (2). The MATLAB model for the behavior of initially curved flexures with chord cross sections is detailed in Appendix E, and the model for the comparison of bending stiffness and lateral stiffness is given in Appendix F.

As with the initially curved flexure with a rectangular cross section, the flexures with chord cross sections were predicted via the model to have adequate behavior under bending without reaching their yield stress. Therefore, they were deemed an acceptable model for further testing. Additionally, stiffness calculations show an increase in lateral stiffness of over 4 times rectangular cross section values. Since desired bending behavior is achieved without reaching yield stress and lateral stiffness is multiplied by 4, this finger geometry was pursued through additional prototyping. Initially curved flexure fingers with chord cross sections were made from nylon 6/6 and are a part of the latest generation grasper prototype.

**Chord Cross Section Model Validation**

A modified test rig was constructed to again test the analytical model against observed bending behavior. This rig again consisted of polycarbonate plates constraining the finger while a through cable provided the actuating force. Again, as the actuating cable tension force was increased, bending behavior was compared against patterns predicted by the MATLAB model. Examples of these comparisons are shown in Figure 19:

![Figure 19: Finger bending patterns under 5 Newtons (left) and 10 Newtons (right).](image-url)
As with the initially curved rectangular cross section fingers, the bending in the chord cross section fingers very closely matches the behavior predicted by the analytical model. Again, the precise value of the Young's modulus of the material is unknown and was approximated from values of other nylon plastics. However, the observed finger bending profile matched the predicted values closely enough to account for this approximation. Again, the model is believed to be validated by the rig testing.

Conclusions and Recommendations

Several analytical models were successfully created to predict the bending behavior of multiple designs of flexural laparoscopic grasper fingers. These models also predicted the failure of these flexures by determining when yield stress would be reached. Through one of these models, a blade flexure concept was found to be impractical and was discarded before fabrication and testing were performed. A model then predicted the feasibility of an initially curved flexure concept. This finger design was fabricated and tested, and the model was validated. The same model predicts the feasibility and advantages of an initially curved flexure with modified chord cross section, and these model results are the basis of the decision to move ahead with fabrication and testing of this concept.

The presented analytical models predict flexure behavior within the bounds of potential error in material property averaging and in less-than-perfect fabrication techniques. To validate the model more thoroughly, it is recommended to perform test rig experiments under more controlled conditions. Acquiring plastics with more exactly-known Young's moduli, as well as fabricating the test fingers by a method more accurate than waterjet fabrication, will provide more concrete validation data.

Additionally, several approximations were made through the creating of the analytical model. Radial stress being negligible compared to circumferential stress is one example. While the integrity of the model is not compromised by making these well-justified approximations, the model can only be improved by eliminating them. These approximations were made for simplicity, creating a more intuitive and easily-understood model. Additionally, given the unknowns of material properties and other variables earlier described, any small errors associated by approximating are negligible in the test results. However, to improve accuracy, it is recommended that a more exact model be created in the future.
References

Online version available at:
http://www.knovel.com/web/portal/browse/display?_EXT_KNOVEL_DISPLAY_bookid=2468&VerticalID=0

Appendix A
Calculation of Area Moment of Inertia for Blade Flexure

Flexural Grapser Flexure Inertia Calculations
2/24/11
Harry O' Hanley and Mitch Westwood

In this document, the area moment of inertia of the flexural finger flexure joint is calculated as a function of outer radius, R, and flexure cut out height, a (note: flexure thickness, t = R/2 - a).

First, the moment of inertia, Ix, about the horizontal axis of the circle with radius, R, must be calculated. This is not the moment of inertia through the flexure's neutral axis.

The integral form of area moment of inertia is
\[
\int y^2 \text{d}A = \int \int y^2 \text{d}y \text{d}x
\]

The inner integral is evaluated between the outer perimeter of the flexure and the upper edge of the flexure cutout
\[
\int_{R/2 + a}^{\sqrt{R^2 - x^2}} y^2 \text{d}y
\]

The indefinite outer integral of the inner integral is solved (note: indefinite integral is evaluated because Maple had difficulty computing the definite integral due to complexity)
\[
\frac{1}{3} (R^2 - x^2)^{3/2} - \frac{1}{3} \left( \frac{1}{2} R + a \right)^3
\]

The upper limit of integration is defined as x and is substituted into the solution of the above indefinite integral
\[
\frac{1}{12} x (R^2 - x^2)^{3/2} + \frac{1}{8} R^2 x \sqrt{R^2 - x^2} + \frac{1}{8} R^4 \arctan \left( \frac{x}{\sqrt{R^2 - x^2}} \right) - \frac{1}{24} x R^3 - \frac{1}{4} x R^2 a
\]

The upper limit of integration is defined as x and is substituted into the solution of the above indefinite integral
\[
x = \sqrt{R^2 - \left( \frac{R}{2} + a \right)^2}
\]

substitute into
\[
\frac{1}{24} \sqrt{3 R^2 - 4 R a - 4 a^2} \left( \frac{1}{4} R^2 + R a + a^2 \right)^{3/2}
\]
The lower limit is defined as \( x \) and is substituted into the above solution for the indefinite integral

\[
x = -\frac{1}{2} \sqrt{3 \cdot R^2 - 4 \cdot R \cdot a - 4 \cdot a^2}
\]

substitute into

\[
-\frac{1}{24} \sqrt{3 \cdot R^2 - 4 \cdot R \cdot a - 4 \cdot a^2} \left( \frac{1}{4} \cdot R^2 + R \cdot a + a^2 \right)^{3/2}
\]

The solution of the lower limit is substituted from the solution of the upper limit. This result is simplified and is the final equation for \( I_x(R,a) \).
\[-\frac{1}{6} \sqrt{3 R^2 - 4 Ra - 4 a^2} a^3 \right) - \left( -\frac{1}{24} \sqrt{3 R^2 - 4 Ra - 4 a^2} \left( \frac{1}{4} R^2 + Ra + a^2 \right)^{3/2} \right. \\
- \frac{1}{16} R^2 \sqrt{3 R^2 - 4 Ra - 4 a^2} \sqrt{\frac{1}{4} R^2 + Ra + a^2} \right. \\
- \frac{1}{8} R^2 \arctan \left( \frac{1}{2} \sqrt{3 R^2 - 4 Ra - 4 a^2} \sqrt{\frac{1}{4} R^2 + Ra + a^2} \right) + \frac{1}{48} \sqrt{3 R^2 - 4 Ra - 4 a^2} R^3 \\
+ \frac{1}{8} \sqrt{3 R^2 - 4 Ra - 4 a^2} R^2 a + \frac{1}{4} \sqrt{3 R^2 - 4 Ra - 4 a^2} R a^2 \\
+ \frac{1}{6} \sqrt{3 R^2 - 4 Ra - 4 a^2} a^3 \right) \\
\frac{1}{12} \sqrt{3 R^2 - 4 Ra - 4 a^2} \left( \frac{1}{4} R^2 + Ra + a^2 \right)^{3/2} \\
+ \frac{1}{16} R^2 \sqrt{3 R^2 - 4 Ra - 4 a^2} \sqrt{(R + 2 a)^2} + \frac{1}{4} R^2 \arctan \left( \frac{\sqrt{3 R^2 - 4 Ra - 4 a^2}}{\sqrt{(R + 2 a)^2}} \right) \\
- \frac{1}{24} \sqrt{3 R^2 - 4 Ra - 4 a^2} R^3 - \frac{1}{4} \sqrt{3 R^2 - 4 Ra - 4 a^2} R^2 a \\
- \frac{1}{2} \sqrt{3 R^2 - 4 Ra - 4 a^2} R a^2 - \frac{1}{3} \sqrt{3 R^2 - 4 Ra - 4 a^2} a^3 \\
simplify \rightarrow \left( \frac{1}{4} R^2 \arctan \left( \frac{\sqrt{3 R^2 - 4 Ra - 4 a^2}}{\sqrt{(R + 2 a)^2}} \right) \right. \\
- \frac{1}{24} \left( \frac{1}{8} \sqrt{4 (R + 2 a)^2} \right)^{3/2} \\
- \frac{3}{2} R^2 \sqrt{(R + 2 a)^2} + 8 a^2 + R^2 + 6 R^2 a + 12 R a^2 \right) \sqrt{3 R^2 - 4 Ra - 4 a^2} \\
Next, the cross sectional area of the flexure is calculated from the integral equation of area. Again, the double integral for area (dydx) is evaluated in two steps for computational simplicity. A similar identical procedure as above is followed, with only the integrand different. Therefore, the steps will not be as closely annotated.

\[
\int dA = \int dy \ dx
\]

\[
\int \frac{\sqrt{R^2 - x^2}}{R^2 + a}
\]

22
\[ \sqrt{R^2 - x^2} - \frac{1}{2} R - a \] (3)

\[ \int \sqrt{R^2 - x^2} - \frac{1}{2} R - a \, dx \]
\[ \frac{1}{2} x \sqrt{R^2 - x^2} + \frac{1}{2} R^2 \arctan \left( \frac{x}{\sqrt{R^2 - x^2}} \right) - \frac{1}{2} R x - a x \] (9)

\[ x = \sqrt{R^2 - \left( \frac{R}{2} + a \right)^2} \]

\[ x = \frac{1}{2} \sqrt{3 R^2 - 4 R a - 4 a^2} \] (10)

\[ \text{substitute into} \]
\[ \frac{1}{4} \sqrt{3 R^2 - 4 R a - 4 a^2} \sqrt{\frac{1}{4} R^2 + R a + a^2} + \frac{1}{2} R^2 \arctan \left( \frac{1}{2} \sqrt{\frac{3 R^2 - 4 R a - 4 a^2}{\frac{1}{4} R^2 + R a + a^2}} \right) \]

\[ - \frac{1}{4} R \sqrt{3 R^2 - 4 R a - 4 a^2} - \frac{1}{2} a \sqrt{3 R^2 - 4 R a - 4 a^2} \]

\[ x = - \sqrt{R^2 - \left( \frac{R}{2} + a \right)^2} \]

\[ x = - \frac{1}{2} \sqrt{3 R^2 - 4 R a - 4 a^2} \] (12)

\[ \text{substitute into} \]
\[ - \frac{1}{4} \sqrt{3 R^2 - 4 R a - 4 a^2} \sqrt{\frac{1}{4} R^2 + R a + a^2} - \frac{1}{2} R^2 \arctan \left( \frac{1}{2} \sqrt{\frac{3 R^2 - 4 R a - 4 a^2}{\frac{1}{4} R^2 + R a + a^2}} \right) \]

\[ + \frac{1}{4} R \sqrt{3 R^2 - 4 R a - 4 a^2} + \frac{1}{2} a \sqrt{3 R^2 - 4 R a - 4 a^2} \]

\[ \text{The equation on line 14 is the final equation for the area of the flexure. In line 15, this equation is made a function in Maple.} \]
\[ A := (R, a) \rightarrow \frac{1}{4} \sqrt{3 R^2 - 4 R a - 4 a^2} \sqrt{(R + 2 a)^2} + R^2 \arctan \left( \frac{\sqrt{3 R^2 - 4 R a - 4 a^2}}{(R + 2 a)^2} \right) \]

\[ = \frac{1}{2} R \sqrt{3 R^2 - 4 R a - 4 a^2} - a \sqrt{3 R^2 - 4 R a - 4 a^2} \]

In line 15, the area formula is made an equation in Maple

The function is tested for \( R = .008 \text{m} \) and \( a = .003 \text{m} \). A solid model of a flexure with these dimensions was made in Solidworks. The cross sectional area of the model was calculated in Solidworks to test the above area function. There was perfect agreement between the Solidworks result and the result on line 16 below.

\[ A(.008, .003) \]

\[ 0.00000523218922 \]  

Next, the location of the horizontal axis through the flexure centroid needed to be determined. In other words, this is the distance between the x-axis defined above and the horizontal axis of the centroid. This will be used to calculate the moment of inertia about the flexure's neutral axis. The equation for calculating the \( y \) component of the centroid is

\[ y_{bar} = \frac{\int y \, dA}{\int dA} \]

First, the moment integral is solved (numerator). The integrals are identical to the previous calculations and will not be closely annotated.

\[ \int_{R/2 + a}^{\sqrt{R^2 - x^2}} y \, dy \]

\[ = \frac{1}{2} R^2 - \frac{1}{2} x^2 - \frac{1}{2} \left( \frac{1}{2} R + a \right)^2 \]  

(17)
\[
\int \frac{1}{2} R^2 - \frac{1}{2} R - \frac{1}{2} \left( \frac{1}{2} R + a \right)^2 \, dx
\]

\[
x = \sqrt{R^2 - \left( \frac{R}{2} + a \right)^2}
\]

\[
x = \frac{1}{2} \sqrt{3 R^2 - 4 Ra - 4 a^2}
\]

Substitute into

\[
-\frac{1}{4} R^2 \sqrt{3 R^2 - 4 Ra - 4 a^2} + \frac{1}{48} \left( 3 R^2 - 4 Ra - 4 a^2 \right)^{3/2} + \frac{1}{4} \left( \frac{1}{2} R + a \right)
\]

\[
x = \sqrt{R^2 - \left( \frac{R}{2} + a \right)^2}
\]

\[
x = \frac{1}{2} \sqrt{3 R^2 - 4 Ra - 4 a^2}
\]

Substitute into

\[
-\frac{1}{4} R^2 \sqrt{3 R^2 - 4 Ra - 4 a^2} + \frac{1}{48} \left( 3 R^2 - 4 Ra - 4 a^2 \right)^{3/2} + \frac{1}{4} \left( \frac{1}{2} R + a \right)
\]

\[
\left( \frac{1}{4} R^2 \sqrt{3 R^2 - 4 Ra - 4 a^2} - \frac{1}{48} \left( 3 R^2 - 4 Ra - 4 a^2 \right)^{3/2} - \frac{1}{4} \left( \frac{1}{2} R + a \right) \right)^2
\]

\[
\frac{1}{2} \sqrt{3 R^2 - 4 Ra - 4 a^2} - \frac{1}{48} \left( 3 R^2 - 4 Ra - 4 a^2 \right)^{3/2} - \frac{1}{2} \left( \frac{1}{2} R + a \right)
\]

The formula in line 23 is for the moment equation (numerator), it is divided by the area formula solved above. Line 24 is the final formula for ybar.
\[ y_{\text{bar}} = \frac{(3 R^2 - 4 R a - 4 a^2)}{(R + 2 a)^2} - \frac{1}{2} \left( \frac{1}{2} R \sqrt{3 R^2 - 4 R a - 4 a^2} - a \sqrt{3 R^2 - 4 R a - 4 a^2} \right) \]

The location of the neutral axis was checked with Solidworks and again there was good agreement. 

\[ y_{\text{bar}}(0.008, 0.003) = 0.00740223404 \] 

Finally, the parallel axis theorem was used to calculate the moment of inertia about the flexure neutral axis.

\[ I_{\text{neutral}} = I_x - A y_{\text{bar}}^2 \] 

First, \( y_{\text{bar}}^2 \) was solved as
\[
\left( \frac{1}{2} R^2 \sqrt{3 R^2 - 4 R a - 4 a^2} - \frac{1}{24} (3 R^2 - 4 R a - 4 a^2)^{3/2} - \frac{1}{2} \left( \frac{1}{2} R + a \right)^2 \right) \sqrt{3 R^2 - 4 R a - 4 a^2} \left( \frac{1}{4} \sqrt{3 R^2 - 4 R a - 4 a^2} \sqrt{(R + 2 a)^2} + a \sqrt{3 R^2 - 4 R a - 4 a^2} \right) - \frac{1}{2} R \sqrt{3 R^2 - 4 R a - 4 a^2} - a \sqrt{3 R^2 - 4 R a - 4 a^2} \right) \]

Next, A'[ybar]^2 was solved as

\[
\left( \frac{1}{4} \sqrt{3 R^2 - 4 R a - 4 a^2} \sqrt{(R + 2 a)^2} + R^2 \arctan \left( \frac{\sqrt{3 R^2 - 4 R a - 4 a^2}}{\sqrt{(R + 2 a)^2}} \right) 
- \frac{1}{2} R \sqrt{3 R^2 - 4 R a - 4 a^2} - a \sqrt{3 R^2 - 4 R a - 4 a^2} \right) \]

\[
\left( \left( \frac{1}{2} R^2 \sqrt{3 R^2 - 4 R a - 4 a^2} - \frac{1}{24} (3 R^2 - 4 R a - 4 a^2)^{3/2} - \frac{1}{2} \left( \frac{1}{2} R + a \right)^2 \right) \sqrt{3 R^2 - 4 R a - 4 a^2} \left( \frac{1}{4} \sqrt{3 R^2 - 4 R a - 4 a^2} \sqrt{(R + 2 a)^2} + a \sqrt{3 R^2 - 4 R a - 4 a^2} \right) - \frac{1}{2} R \sqrt{3 R^2 - 4 R a - 4 a^2} - a \sqrt{3 R^2 - 4 R a - 4 a^2} \right) \]

\[
\left( \left( \frac{1}{4} \sqrt{3 R^2 - 4 R a - 4 a^2} \sqrt{(R + 2 a)^2} + R^2 \arctan \left( \frac{\sqrt{3 R^2 - 4 R a - 4 a^2}}{\sqrt{(R + 2 a)^2}} \right) 
- \frac{1}{2} R \sqrt{3 R^2 - 4 R a - 4 a^2} - a \sqrt{3 R^2 - 4 R a - 4 a^2} \right) \right) \]
\[
\left( \frac{1}{2} R^2 \sqrt{3 R^2 - 4 R \alpha - 4 \alpha^2} - \frac{1}{24} (3 R^2 - 4 R \alpha - 4 \alpha^2)^{3/2} - \frac{1}{2} \left( \frac{1}{2} R + a \right)^2 \right)^2 \left( \frac{1}{4} \sqrt{3 R^2 - 4 R \alpha - 4 \alpha^2} \sqrt{(R + 2 \alpha)^2} \right) \]

Finally \( I_x - A \gamma ybar^2 \) was solved for. The final formula for Ineutralaxis is on line 30.

\[
\left( \frac{1}{4} R^4 \arctan \left( \frac{\sqrt{3 R^2 - 4 R \alpha - 4 \alpha^2}}{\sqrt{(R + 2 \alpha)^2}} \right) \right) - \frac{1}{24} \left( \frac{1}{8} \sqrt{4 \left( (R + 2 \alpha)^2 \right)^{3/2} - \frac{3}{2} R^2 \sqrt{(R + 2 \alpha)^2}} \right) \]

\[
\left( \frac{1}{2} R^2 \sqrt{3 R^2 - 4 R \alpha - 4 \alpha^2} - \frac{1}{24} (3 R^2 - 4 R \alpha - 4 \alpha^2)^{3/2} - \frac{1}{2} \left( \frac{1}{2} R + a \right)^2 \right)^2 \left( \frac{1}{4} \sqrt{3 R^2 - 4 R \alpha - 4 \alpha^2} \sqrt{(R + 2 \alpha)^2} \right) \]

\[
\left( \frac{1}{4} R^4 \arctan \left( \frac{\sqrt{3 R^2 - 4 R \alpha - 4 \alpha^2}}{\sqrt{(R + 2 \alpha)^2}} \right) \right) - \frac{1}{24} \left( \frac{1}{8} \sqrt{4 \left( (R + 2 \alpha)^2 \right)^{3/2} - \frac{3}{2} R^2 \sqrt{(R + 2 \alpha)^2}} \right) \]

The formula for Ineutralaxis was made into a Maple equation and tested with \( R=.008 \) and \( a=.003 \). The moment of inertia of a rectangle with comparable width and thickness was also calculated. The moment of inertia of the flexure was about 60% the moment of inertia of the rectangle, which is consistent with their relative geometries.
The Ineuralaxis equation was written into MATLAB code and copied into MATLAB for integration into the finger bending model:

\[
\text{Ineuralaxis} := \begin{bmatrix}
\frac{1}{4} R^2 \arctan \left( \frac{\sqrt{3 R^2 - 4 R a - 4 R a^2}}{\sqrt{(R + 2 a)^2}} \right) - \frac{1}{24} \left( \frac{1}{4} \left( (R + 2 a)^2 \right)^{3/2} 
\end{bmatrix}
\]

\[
- \frac{3}{2} R^2 \sqrt{(R + 2 a)^2} + 8 a^2 + R^4 + 6 R^2 a + 12 R a^2 \right) \sqrt{3 R^2 - 4 R a - 4 a^2} 
\]

\[
- \left( \frac{1}{2} R^2 \sqrt{3 R^2 - 4 R a - 4 a^2} - \frac{1}{24} \left( 3 R^2 - 4 R a - 4 a^2 \right)^{3/2} - \frac{1}{2} \left( \frac{1}{2} R + a \right)^2 \sqrt{3 R^2 - 4 R a - 4 a^2} \right)
\]

\[
- a \sqrt{3 R^2 - 4 R a - 4 a^2}
\]

\[
\text{Ineuralaxis}(0.008, 0.003)
\]

\[
3.597734 \times 10^{-13}
\]

(32)
\[
cg0 = (R^4) \cdot \arctan(\sqrt{(3 \cdot R^2 - 4 \cdot R \cdot a - 4 \cdot a^2)}) \\
\quad \cdot \left(\frac{\sqrt{((R + 2 \cdot a)^2 + (0.3e1 / 0.2e1)) + 0.4e1 - \frac{\sqrt{((R + 2 \cdot a)^2 + (0.3e1 / 0.2e1))}}{0.16e2 + (a^3)} + 0.3e1 + (R^3)} + 0.24e2 + (R^2) + (0.4e1 + (R \cdot a^2) / 0.2e1) \cdot \sqrt{(3 \cdot R^2 - 4 \cdot R \cdot a - 4 \cdot a^2)} - 0.1e1 / \sqrt{(3 \cdot R^2 - 4 \cdot R \cdot a - 4 \cdot a^2)} \right) - \sqrt{(3 \cdot R^2 - 4 \cdot R \cdot a - 4 \cdot a^2)} / 0.2e1 - a \\
\quad \cdot \left(\frac{\sqrt{(3 \cdot R^2 - 4 \cdot R \cdot a - 4 \cdot a^2)} + (R^2) \cdot \sqrt{(3 \cdot R^2 - 4 \cdot R \cdot a - 4 \cdot a^2)} - \sqrt{(3 \cdot R^2 - 4 \cdot R \cdot a - 4 \cdot a^2)} / 0.2e1 - (3 \cdot R^2 - 4 \cdot R \cdot a - 4 \cdot a^2) / 0.24e2 - (R / 0.2e1 + a) / 2 \cdot \sqrt{(3 \cdot R^2 - 4 \cdot R \cdot a - 4 \cdot a^2)} / 0.2e1)^2 ;
\]
Appendix B
Derivation of Angle of Deflection Equation Using Castigliano’s Theorem

Determining \( \theta \), the deflection of an initially curved flexure under a load at the tip.
Mitch Westwood (westwood@mit.edu) and Harry O’Hanley (hohanley@mit.edu)
4/23/2011

Castigliano’s theorem calls for the following relationship to find the angle of deflection of a curved beam under load:

\[
\Theta = \frac{\partial U}{\partial M_i}.
\]

(1)

Where \( U \) is the total elastic strain energy in the curved beam, and \( M_i \) is the moment on the beam at the point of interest.

We will use the dummy load method, described in the literature referenced here. Although only a force, but no moment, is applied at the beam tip, we will create a dummy moment at that point and call it \( M_0 \). The value of \( M_0 \) is 0. We will also employ a tool that will make future integration much easier. We have:

\[
\Theta = \frac{\partial U}{\partial M_0} = \frac{\partial U}{\partial M_a} \cdot \frac{\partial M_a}{\partial M_0},
\]

(2)

where \( M_a \) is defined as the internal moment of the beam at point \( \Phi = 0 \) at the point where the force is applied.

According to the literature, the value for \( U \) can be approximated as:

\[
U = \frac{A_m M_i^2}{2 A (R - A_m - A) E} \sin \Phi.
\]

(3)

Summing the moments about the center of rotation, and knowing we have a force \( F \) applied at \( \Phi = 0 \),

\[
\sum M = 0 = M_0 - M_a + FR \cdot \sin \Phi,
\]

(4)

or,

\[
M_a = M_0 + FR \cdot \sin \Phi.
\]

(5)

Combining equations (2), (3), and (5), and seeing that

31
\[
\frac{\partial M_x}{\partial M_o} = 1,
\]  
(6)

\[
\Theta = \frac{\partial U}{\partial M_x} \cdot \frac{\partial M_x}{\partial M_o} = \left[ \frac{A_m\left( M_o + FR\cdot \sin(\Phi) \right)}{A(R \cdot A_m - A)E} \right] \, d\Phi.
\]  
(7)

Since we know \( M_o = 0 \), we can find the angle of deflection.

\[
\Theta = \left[ \frac{A_m\left( FR \cdot \sin(\Phi) \right)}{A(R \cdot A_m - A)E} \right] \, d\Phi. \quad (8)
\]
Appendix C
MATLAB Code for Blade Flexure Bending Analytical Model

% Flexural finger bending pattern and stress calculations
% Harry O'Hanley (hohanley@mit.edu) and Mitch Westwood
% (westwood@mit.edu)

% Vary the inputs and the program will output a graphical representation of
% the finger curling pattern as well as the maximum stress in each flexural
% joint. This model assumes the fingers to be 120 degree arcs of circles in
% cross section.
% The flexure segments are displayed in red.
% The rigid segments are displayed in blue.

clear
clc
close all

% Geometric Inputs
ta = 0.002; % Maximum flexure thickness [meters]. Read values of h for
design purposes.
tb = 0.0015;
tc = 0.0015;

Ltotal = 0.06; % Total finger length [meters]
La = 0.01; % Flexure lengths [meters]
Lb = 0.01;
Lc = 0.01;

L1 = 0.5 * (Ltotal - La - Lb - Lc); % Rigid segment lengths [meters]
L2 = 0.3 * (Ltotal - La - Lb - Lc);
L3 = 0.2 * (Ltotal - La - Lb - Lc);

R = 0.008; % Overall finger radius [meters]
d = 0.001 + R/2 - tc; % Vertical offset of cable from flexure neutral axis
[meteors]

% Material Properties
E = 2e9 * 10^9; % [Pascals]

% Force Input
Fs = 20; % Surgeon's finger gripping force [Newtons]

% Handle transmission
TR = 4; % Trigger mechanical advantage [unitless]

% Moments of Inertia
aa = R/2 - ta % Lower limit of flexure area [meters]. In other words,
this is the height of what is cut out to form the flexure
\[ ab = R/2-tb \]
\[ ac = R/2-tc \]

\[ Ia = \left( R^4 \right) \ast \tan\left( \sqrt{\left( 3 \ast R^2 - 4 \ast R \ast (R/2-ta) - 4 \ast (R/2-ta)^2 \right)} \ast \left( \left( R + 2 \ast (R/2-ta) \right)^2 \right) \left(-0.1e1 \/ 0.2e1\right) \left( R + 2 \ast (R/2-ta) \right)^2 \left(-0.1e1 \/ 0.2e1\right) \right) / 0.96e2 - (R \ast (R/2-ta)^2) \ast \left(-0.1e1 \/ 0.2e1\right) \left( R + 2 \ast (R/2-ta) \right)^2 \left(-0.1e1 \/ 0.2e1\right) \right) / 0.96e2 - (R \ast (R/2-ta)^2) \left(-0.1e1 \/ 0.2e1\right) \]"
ra = La;
rb = Lb;
rc = Lc;

r1 = L1+La;
theta1 = -(M*La)/(E*Ia));

r2 = L2+Lb;
theta2 = -(M*Lb)/(E*Ib));

r3 = L3+Lc;
theta3 = -(M*Lc)/(E*Ic));

xa = ra*cos(theta1);
ya = ra*sin(theta1);

x1 = r1*cos(theta1);
y1 = r1*sin(theta1);

xb = r1*cos(theta1)+rb*cos(theta2)*cos(theta1)-
rb*sin(theta2)*sin(theta1);
yb = r1*sin(theta1)+rb*cos(theta2)*sin(theta1)+rb*sin(theta2)*cos(theta1);

x2 = r1*cos(theta1)+r2*cos(theta2)*cos(theta1)-
r2*sin(theta2)*sin(theta1);
y2 = r1*sin(theta1)+r2*cos(theta2)*sin(theta1)+r2*sin(theta2)*cos(theta1);

xc = r1*cos(theta1)+r2*cos(theta2)*cos(theta1)-
r2*sin(theta2)*sin(theta1) + rc*cos(theta3)*cos(theta2)*cos(theta1)-
rc*cos(theta3)*sin(theta2)*sin(theta1)-
rc*sin(theta3)*sin(theta2)*cos(theta1)-
rc*sin(theta3)*cos(theta2)*sin(theta1);
yc = r1*sin(theta1)+r2*cos(theta2)*sin(theta1)+r2*sin(theta2)*cos(theta1) +
rc*cos(theta3)*cos(theta2)*sin(theta1)+rc*cos(theta3)*sin(theta2)*cos(theta1)-
rc*sin(theta3)*sin(theta2)*sin(theta1)+rc*sin(theta3)*cos(theta2)*cos(theta1);

x3 = r1*cos(theta1)+r2*cos(theta2)*cos(theta1)-
r2*sin(theta2)*sin(theta1) + r3*cos(theta3)*cos(theta2)*cos(theta1)-
r3*cos(theta3)*sin(theta2)*sin(theta1)-
r3*sin(theta3)*sin(theta2)*cos(theta1)-
r3*sin(theta3)*cos(theta2)*sin(theta1);
y3 = r1*sin(theta1)+r2*cos(theta2)*sin(theta1)+r2*sin(theta2)*cos(theta1) +
r3*cos(theta3)*cos(theta2)*sin(theta1)+r3*cos(theta3)*sin(theta2)*cos(theta1)-
r3*sin(theta3)*sin(theta2)*sin(theta1)+r3*sin(theta3)*cos(theta2)*cos(theta1);

%plot([0,xa,x1,x2,x3],[0,ya,y1,y2,y3])
%Calculate stress in flexures

sigmaa = M*(ta/2)/(10^6*Ia) %[MPa]
sigmab = M*(tb/2)/(10^6*Ib)
sigmac = M*(tc/2)/(10^6*Ic)
Appendix D
MATLAB Code for Initially Curved Flexure with Rectangular Cross Section Bending Analytical Model

%Analytical Model for Flexural Laparoscopic Fingers
%Harry O'Hanley (hohanley@mit.edu)

%This model predicts flexure displacement as a function of flexure
%geometry, material, and cable tensioning force. The graphical output
%is representative of a finger curling, viewed from above.

%Note: the geometric values listed below are not necessarily those used in the final
design. They should be treated as placeholder variables.

clc
clear all
clf

%Inner radii of flexures [meters]
ra = 0.003;
rb = 0.003;
rc = 0.003;

%Thickness of flexures [meters]
ta = 0.0015;
tb = 0.0012;
tc = 0.0013;

%Width of flexures [meters]
ba = 0.003175;
bb = 0.003175;
bc = 0.003175;

%Total finger length (including rigid sections and flexures) [meters]
Ltotal = 0.06;

%Fractional length of rigid sections
L1frac = .4;
L2frac = .3;
L3frac = .3;

%Length of rigid sections [meters]
L1 = L1frac*(Ltotal-2*(ra+rb+rc));
L2 = L2frac*(Ltotal-2*(ra+rb+rc));
L3 = L3frac*(Ltotal-2*(ra+rb+rc));

%Young's Modulus [Pa]
E = 2*10^9;

%Surgeon gripping force [Newtons]
Fs = 20;
%Trigger Transmission Ratio [unitless]
TR = 3;

%Force applied to individual finger [Newton]
F = (1/3)*TR*Fs;

%Fictitious moment applied for calculation only (Dummy Load Method)
M0=0;

%Flexure A Geometry
aa = ra; %Inner radius
cia = ra+ta; %Outer radius
phia = pi; %Flexure arc

Aa = ba*(ca-aa); %Cross section area
Ra = (aa+ca)/2; %Radius to flexure neutral axis
Ama = ba*log((ca/aa)); %Area to radius correlation

Va = -F*cos(phia); %Sheer force
Na = F*sin(phia); %Normal force
Ma = F*Ra*sin(phia)+M0; %Moment

%Flexure B Geometry
ab = rb; %Inner radius
cb = rb+tb; %Outer radius
phib = pi; %Flexure arc

Ab = bb*(cb-ab); %Cross section area
Rb = (ab+cb)/2; %Radius to flexure neutral axis
Amb = bb*log((cb/ab)); %Area to radius correlation

Vb = -F*cos(phib); %Sheer force
Nb = F*sin(phib); %Normal force
Mb= F*Rb*sin(phib)+M0; %Moment

%Flexure C Geometry
ac = rc; %Inner radius
cc = rc+tc; %Outer radius
phic = pi; %Flexure arc

Ac = bc*(cc-ac); %Cross section area
Rc = (ac+cc)/2; %Radius to flexure neutral axis
Amc = bc*log((cc/ac)); %Area to radius correlation

Vc = -F*cos(phib); %Sheer force
Nc = F*sin(phib); %Normal force
Mc= F*Rc*sin(phic)+M0; %Moment

%Plotting
%Flexure A angular deflection
r1 = L1;
thetal = -(2*Ama*F*Ra)/(Aa*(Ra*Ama-Aa)*E);

%Flexure B angular deflection
\( r_2 = L_2; \)
\( \theta_2 = -(2*Amb*F*Rb)/(Ab*(Rb*Amb-Ab)*E); \)

\%Flexure C angular deflection
\( r_3 = L_3; \)
\( \theta_3 = -(2*Amc*F*Rc)/(Ac*(Rc*Amc-Ac)*E); \)

\%Location of Flexure A to be at (0,0)
\%Location of Flexure B
\( x_1 = r_1*cos(\theta_1); \)
\( y_1 = r_1*sin(\theta_1); \)

\% Location of Flexure C
\( x_2 = r_1*cos(\theta_1)+r_2*cos(\theta_2)*cos(\theta_1)-r_2*sin(\theta_2)*sin(\theta_1); \)
\( y_2 = r_1*sin(\theta_1)+r_2*cos(\theta_2)*sin(\theta_1)+r_2*sin(\theta_2)*cos(\theta_1); \)

\% Location of Finger Tip
\( x_3 = r_1*cos(\theta_1)+r_2*cos(\theta_2)*cos(\theta_1)-r_2*sin(\theta_2)*sin(\theta_1)+r_3*cos(\theta_3)*cos(\theta_2)*cos(\theta_1)-r_3*cos(\theta_3)*sin(\theta_2)*sin(\theta_1)-r_3*sin(\theta_3)*cos(\theta_2)*cos(\theta_1)-r_3*sin(\theta_3)*sin(\theta_2)*sin(\theta_1); \)
\( y_3 = r_1*sin(\theta_1)+r_2*cos(\theta_2)*sin(\theta_1)+r_2*sin(\theta_2)*cos(\theta_1)+r_3*cos(\theta_3)*cos(\theta_2)*sin(\theta_1)+r_3*cos(\theta_3)*sin(\theta_2)*cos(\theta_1)-r_3*sin(\theta_3)*cos(\theta_2)*sin(\theta_1)-r_3*sin(\theta_3)*sin(\theta_2)*cos(\theta_1); \)

\%Plot
\%Note: the dashed line represents the longitudinal axis of the transmission
\%shaft.

\texttt{plot([0,x1,x2,x3],[0,y1,y2,y3])}
\texttt{hold on}
\texttt{plot(0,0,'ro',x1,y1,'ro',x2,y2,'ro');}
\texttt{hold on}
\texttt{plot([0 .03], [0 -.03], '--')}
\texttt{title('Initially Curved -- Nylon 6')}

\%Calculate stresses [MPa]
\%Flexure A stress [MPa]
\( \sigma_A = ((F*sin((pi/2)))/Aa)+((F*Ra*sin((pi/2))+M0)*(Aa-ra*Ama))/(Aa*Ra*(Ra*Ama-Aa))/10^6 \)

\%Flexure B stress [MPa]
\( \sigma_B = ((F*sin((pi/2)))/Ab)+((F*Rb*sin((pi/2))+M0)*(Ab-rb*Amb))/(Ab*Rb*(Rb*Amb-Ab))/10^6 \)

\%Flexure C stress [MPa]
\( \sigma_C = ((F*sin((pi/2)))/Ac)+((F*Rc*sin((pi/2))+M0)*(Ac-rc*Amc))/(Ac*Rc*(Rc*Amc-Ac))/10^6 \)
Material Yield Stress [MPa]
\[ \sigma_Y = 70 \]
Appendix E
MATLAB Code for Initially Curved Flexure with Chord Cross Section Bending Analytical Model

%Analytical Model for Flexural Laparoscopic Fingers
%Curling Pattern With Initially Curved Chord Cross Section Flexures
%Harry O'Hanley (hohanley@mit.edu) and Mitch Westwood
(westwood@mit.edu)

%This model predicts flexure displacement as a function of flexure
%geometry, material, and cable tensioning force. The graphical output is
%representative of a finger curling, viewed from above.

clear all
clc
clf

R = .005;
t = 0.0012;
c_i = R - t;
h = c_i;
a = 0;

phi = acos(h/R);

A = R^2 * phi - R^2 * sin((2 * phi)) / 0.2e1;

A_m = (2 * a * phi) - 0.2e1 * R * sin(phi) + 0.2e1 * sqrt(R^2 - (a^2)) * log((R + a * cos(phi) + sqrt(R^2 - (a^2)) * sin(phi)) / (a + R * cos(phi)));

R_m = (a + 0.4e1 * R * sin(phi)^3) / (0.6e1 * phi - 0.3e1 * sin(0.2e1 * phi));

%%Angle subtended by each flexure (degrees)
Tm1 = 30;
Tm2 = 135;
Tm3 = 45;
Tm4 = 135;
Tmin = 30;
Tmaxb = 150;
Tminc = 30;
Tmaxc = 150;

%%Vertical offset of cable from center of rotation [meters]
n = 0.0011;
np = 0.0011;
nc = 0.0011;
%Total finger length (including rigid sections and flexures) [meters]
L_{\text{total}} = 0.071;

%Fractional length of rigid sections
L_{1\text{frac}} = .3;
L_{2\text{frac}} = .3;
L_{3\text{frac}} = .4;

%Length of rigid sections [meters]
L_1 = L_{1\text{frac}}*(L_{\text{total}}-6*R);
L_2 = L_{2\text{frac}}*(L_{\text{total}}-6*R);
L_3 = L_{3\text{frac}}*(L_{\text{total}}-6*R);

%Young's Modulus [Pa]
E = 2*10^{9};

%Surgeon gripping force [Newtons]
for F_s = 5:5:20;

%Trigger Transmission Ratio [unitless]
TR = 3;

%Force applied to individual finger [Newtons]
F = (1/3)*TR*F_s;

%Ficticious moment applied for calculation only (Dummy Load Method)
M_0=0;

%Plotting
A_m = A_m;
Amb = A_m;
Amc = A_m;

A_a = A;
Ab = A;
Ac = A;

R_a = R_m;
R_b = R_m;
R_c = R_m;

%Angular deflections are evaluated between their max and min angles.
%Flexure A consists of two regions. These regions are modeled separately
%and their deflections are added. For lengths r_1, r_2, and r_3, the
%lengths
%of the flexures are added to the lengths of the rigid segments near
%them
%in order to have an accurate total length. This assumes flexure a
%spans 3R
%in width, flexure b spans 2R, and flexure c spans 2R.

%Flexure A angular deflection
\[ r_1 = L_1 + 4R; \]
\[ \theta_1 = -((-A_m^*F^*(R_a \cos(T_{maxal}/180\pi) + n_a*(T_{maxal}/180\pi)))/(A_a*(R_a*A_m - A_a)*E)) + \]
\[ (-A_m^*F^*(R_a \cos(T_{minal}/180\pi) + n_a*(T_{minal}/180\pi)))/(A_a*(R_a*A_m - A_a)*E))\];

% Flexure B angular deflection
\[ r_2 = L_2 + 2R; \]
\[ \theta_2 = -((-A_m^*F^*(R_b \cos(T_{maxb}/180\pi) + n_b*(T_{maxb}/180\pi)))/(A_b*(R_b*A_m - A_b)*E)) + \]
\[ (-A_m^*F^*(R_b \cos(T_{minb}/180\pi) + n_b*(T_{minb}/180\pi)))/(A_b*(R_b*A_m - A_b)*E))\];

% Flexure C angular deflection
\[ r_3 = L_3 + R; \]
\[ \theta_3 = -((-A_m^*F^*(R_c \cos(T_{maxc}/180\pi) + n_c*(T_{maxc}/180\pi)))/(A_c*(R_c*A_m - A_c)*E)) + \]
\[ (-A_m^*F^*(R_c \cos(T_{minc}/180\pi) + n_c*(T_{minc}/180\pi)))/(A_c*(R_c*A_m - A_c)*E))\];

% Location of Flexure A to be at (0, 0)
% Location of Flexure B
\[ x_1 = r_1 \cos(\theta_1); \]
\[ y_1 = r_1 \sin(\theta_1); \]
% Location of Flexure C
\[ x_2 = r_1 \cos(\theta_1) + r_2 \cos(\theta_2) \cos(\theta_1) - \]
\[ r_2 \sin(\theta_2) \sin(\theta_1); \]
\[ y_2 = r_1 \sin(\theta_1) + r_2 \cos(\theta_2) \sin(\theta_1) + r_2 \sin(\theta_2) \cos(\theta_1); \]
% Location of Finger Tip
\[ x_3 = r_1 \cos(\theta_1) + r_2 \cos(\theta_2) \cos(\theta_1) - \]
\[ r_2 \sin(\theta_2) \sin(\theta_1) + r_3 \cos(\theta_3) \cos(\theta_2) \cos(\theta_1) - \]
\[ r_3 \cos(\theta_3) \sin(\theta_2) \sin(\theta_1) - \]
\[ r_3 \sin(\theta_3) \sin(\theta_2) \cos(\theta_1) - \]
\[ r_3 \sin(\theta_3) \cos(\theta_2) \sin(\theta_1); \]
\[ y_3 = r_1 \sin(\theta_1) + r_2 \cos(\theta_2) \sin(\theta_1) + r_2 \sin(\theta_2) \cos(\theta_1) + \]
\[ r_3 \cos(\theta_3) \cos(\theta_2) \sin(\theta_1) + r_3 \cos(\theta_3) \sin(\theta_2) \cos(\theta_1) - \]
\[ r_3 \sin(\theta_3) \sin(\theta_2) \sin(\theta_1) + r_3 \sin(\theta_3) \cos(\theta_2) \cos(\theta_1); \]
% Plot
% Note: the dashed line represents the longitudinal axis of the transmission
% shaft.

plot([0, x_1, x_2, x_3], [0, y_1, y_2, y_3])
hold on
plot(0, 0, 'ro', x_1, y_1, 'ro', x_2, y_2, 'ro');
hold on
plot([0 .07], [0 -.07], '--')
title('Initially Curved Chord -- Nylon 6')

%Calculate stresses [MPa]
%Flexure A stress [MPa]
sigmaA = (((F/Aa)+((F*Ra)*(Aa-c_i*Ama))/(Aa*Ra*(Ra*Ama-Aa))))/10^6
end
Appendix F
MATLAB Code for Bending and Lateral Stiffness Comparison of Initially Curved Flexures with Rectangular and Chord Cross Sections

clc
clear all

%%Stiffness Comparison Between Rectangular and Chord Initially Curved %Flexures
%%Harry O'Hanley (hohanley@mit.edu) and Mitch Westwood (westwood@mit.edu)
%For each flexure type, find the area moment of inertia at the "top" of %each flexure (phi = pi/2).
%Although they actually have a chord cross section, the chord flexures are %called "circular" here.

%%Finding Bending Stiffness of Chord Flexure
R = .005; %Outer radius of chord flexure
t = .0012; %Greatest thickness of chord flexure.
c_i = R-t; %Distance between bending axis and bottom of chord flexure.
h_max = c_i*sin(pi/2);
b_max = sqrt(R^2 - h_max^2);
I_bend_circ = (R^2 - b_max^2) * (0.3e1 / 0.2e1) * b_max / 0.6e1 + R^2 * sqrt(R^2 - b_max^2) * b_max / 0.4e1 + R^4 * atan(b_max * (R^2 - b_max^2)) / 0.4e1 - 0.2e1 / 0.3e1 * h_max^3 * b_max; %Chord Bending Stiffness

%%Finding Bending Stiffness of Rectangular Flexure
%Upper and lower limits of x and y for rectangular cross section.
u_x = .0016;
l_x = -.0016;
u_y = .004;
l_y = .003;
I_bend_rect = u_y^3 * (u_x - l_x) / 0.3e1 - l_y^3 * (u_x - l_x) / 0.3e1 %Rectangular Bending Stiffness

%%Finding Lateral Stiffness of Chord Flexure
C = sqrt(R^2-c_i^2);
I_lat_circ = -sqrt(R^2 - c_i^2) * (c_i^2) * (0.3e1 / 0.2e1) / 0.2e1 + R^2 * sqrt(R^2 - c_i^2) * sqrt(c_i^2) / 0.4e1 + R^4 * atan(sqrt(R^2 - c_i^2) * (c_i^2) / 0.4e1 - 0.2e1 / 0.3e1 + (R^2 - c_i^2) * (0.3e1 / 0.2e1) * c_i; %Chord Lateral Stiffness

%%Finding Lateral Stiffness of Rectangular Flexure

45
\[ I_{\text{lat\_rect}} = u_x^3 \times (u_y - l_y) / 0.3el - l_x^3 \times (u_y - l_y) / 0.3el \] %Rectangular Lateral Stiffness

%%Ratio of Bending Stiffnesses
\[ I_{\text{bend\_ratio}} = I_{\text{bend\_circ}} / I_{\text{bend\_rect}}; \]

%%Ratio of Lateral Stiffnesses
\[ I_{\text{lat\_ratio}} = I_{\text{lat\_circ}} / I_{\text{lat\_rect}}; \]