Solutions to Problem Set 8

Problem 1. Write closed form formulas which may involve factorials for the following functions:

(a) $G(n)$::= the number of directed graphs with vertices $\{1, 2, \ldots, n\}$ (note, self-loops are permitted in directed graphs).

Solution. 2^{n^2}. There are n^2 possible directed edges, each of which may or may not be in a graph. ■

(b) $U(n)$::= the number of directed graphs with vertices $\{1, 2, \ldots, n\}$ such that if the edge (i, j) is in the graph then (j, i) can not be in the graph, where i and j are distinct vertices.

Solution. $(2^n)(3^{n(n-1)/2})$. Without the possibility of self-loops, there are $3^{n(n-1)/2}$ graphs because there are $n(n-1)/2$ different pairs of vertices, each of which may have an edge in one direction, an edge in the other direction, or no edge at all. For each of these graphs there are 2^n times as many graphs with the possibility of self-loops since there are n nodes, each of which may or may not have a self-loop. ■

(c) $M(n)$::= the number of functions from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, 2n\}$.

Solution. $(2n)^n$. For each of the n arguments of a function, there are $2n$ possible values. ■

(d) $B(n)$::= the number of *injections* from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, 2n\}$.

Solution. $P(2n, n) = \dfrac{2n!}{n!}$. ■

(e) Order the above functions so that each function is O() of the function to its right. Also indicate whether the $\Theta()$ relationship holds between any successive pair of functions. No explanation necessary.

Solution. The order is B, M, U, G, and each is o() of the next, and therefore is O() but not Θ() of the next.

$$B(n) = \frac{2n!}{n!} \sim \frac{\sqrt{2\pi 2n}(\frac{2n}{e})^{2n}}{\sqrt{2\pi n}(\frac{n}{e})^n} = \sqrt{2}\frac{(2n)^{2n}}{(en)^n} = \sqrt{2}(\frac{4}{e})^n n^n = o(2^n)n^n = o((2n)^n).$$

Note that Stirling's formula is not really needed here, since we can also reason in a simple way as follows:

$$\begin{aligned}
\frac{2n!}{n!} &= 2n(2n-1)\cdots\lceil(3/2)n\rceil\left(\lceil(3/2)\rceil n-1\right)\cdots(n+1) \\
&\leq (2n)^{\lfloor n/2\rfloor}((3/2)n)^{\lceil n/2\rceil} \\
&= (2n)^{n/2}\cdot o((2n)^{n/2}) \\
&= o((2n)^n).
\end{aligned}$$

$$M(n) = (2n)^n = 2^n n^n = 2^n 2^{n\log n} = 2^{n(1+\log n)} = 2^{o(n^2)}.$$

$$U(n) = 2^n 3^{n(n-1)/2} = O(\sqrt{3}^{n^2})2^n = O(\sqrt{3}^{n^2}2^n).$$

$$G(n) = 2^{n^2}.$$

■

Problem 2. How many sequence of 6-digits are there that do not contain "123" or "456"? For example, "000456," "112397," "456123," do contain one or both of these 3-digit patterns, but "645111" and "112233" do not.

Solution. Let

- A be the set of all six-digit strings, using the digits $\{0, 1, \ldots, 9\}$.
- B be the set of six-digit strings that contain the substring 123.
- C be the set of all six-digit strings that contain the substring 456.
- D be the set of all six-digit strings that do not contain 123 or 456.

We're interested in figuring out $|D|$. Since

$$D = A - (B \cup C), \quad \text{and}$$
$$B \cup C \subseteq A,$$

we can use the inclusion-exclusion formula:

$$|D| = |A| - |B \cup C| = |A| - |B| - |C| + |B \cap C|.$$

First, $|A| = 10^6$, since we can put any of ten digits in each of six places.

Now, a string can contain 123 in any of four positions; once this position is chosen, we have 1000 choices for the remaining positions. However, this approach counts the string 123123 twice, so the total size of B is

$$|B| = 4(1000) - 1 = 3999.$$

By the same reasoning, we get

$$|C| = 3999.$$

Finally, there are only two strings in $B \cap C$, namely 123456 and 456123. Thus,

$$|B \cap C| = 2.$$

Plugging this into our formula, we get

$$|D| = 10^6 - 3999 - 3999 + 2 = 992004.$$

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Problem 3. The Towers of Hanoi game involves stacking 32 disks onto three vertical pegs, *cf.*, Rosen §5.1, Example 5. Each disk has a hole in the middle through which a peg can slide, and all the disks have different diameters. An arrangement of the disks on the pegs is allowed only if no disk rests on a smaller disk. How many different such arrangements are allowed?

Solution. There are 3^{32} different configurations by the product rule: 3 choices of peg for the largest disk, 3 choices for the second largest disk, and so on. ■

Problem 4. Suppose we are given a list of n integers $a_1, \ldots, a_n$, which need not be distinct. Prove that there is always a set of consecutive numbers $a_{k+1}, a_{k+2}, \ldots, a_l$ whose sum $\sum_{i=k+1}^{l} a_i$ is a multiple of n.

Solution. Consider the set N of size $n+1$ of the partial sums of the a_i, starting from the left: $\{0, a_1, a_1 + a_2, \ldots, a_1 + a_2 + \ldots + a_n\}$. The elements of this set are the "pigeons" and $p \in N$ is placed in the hole $p \bmod n$. There are n holes $R = \{0, 1, \ldots, n-1\}$. Since there are $n+1$ elements of N and only n elements of R, there must be two distinct elements a_k and a_l ($k < l$ without loss of generality) of N which map to the same element of R, that is,

$$\sum_{i=1}^{l} a_i \bmod n = \sum_{i=1}^{k} a_i \bmod n$$

Subtracting, we see that

$$\left(\sum_{i=1}^{l} a_i - \sum_{i=1}^{k} a_i\right) = \sum_{i=k+1}^{l} a_i$$

is divisible by n. ■

Problem 5. A positive integer is called *square-free* if it is not divisible by the square of any positive integer greater than 1. For example $35 = 5 \cdot 7$ is square-free but $18 = 2 \cdot 3^2$ is not. 1 is square-free. Use inclusion-exclusion to find the number of square-free positive integers strictly less than 201.

Solution. We use Inclusion-Exclusion as was done for counting prime numbers in Notes and Rosen. So we compute the number of positive integers less than 200 that are *not* square-free.

Let A_2 (respectively, A_3, A_5, A_7, A_{11}, A_{13}) be the set of multiples of 2^2 (respectively, 3^2, 5^2, 7^2, 11^2, 13^2) less than 200. The union of these sets is the set of all numbers less than 150 which are not square-free, *i.e.*, have some square divisor in them. The reader should take a moment to think through why this assertion is true (why only primes? why stop at 13?).

The cardinality of this set can be computed by the Inclusion-Exclusion Principle. Since $2^2 11^2 > 200$ and $3^2 5^2 > 200$, it follows that $A_i \cap A_j = \emptyset$ unless $\{i, j\} = \{2, 3\}, \{2, 5\}$, or $\{2, 7\}$, so

$$\begin{aligned}
|A_2 \cup A_3 \cup A_5 \cup A_7| &= |A_2| + |A_3| + |A_5| + |A_7| + |A_{11}| + |A_{13}| \\
&\quad -|A_2 \cap A_3| - |A_2 \cap A_5| - |A_2 \cap A_7| \\
&= \lfloor 200/2^2 \rfloor + \lfloor 200/3^2 \rfloor + \lfloor 200/5^2 \rfloor + \lfloor 200/7^2 \rfloor + \lfloor 200/11^2 \rfloor + \lfloor 200/13^2 \rfloor \\
&\quad -\lfloor 200/2^2 3^2 \rfloor - \lfloor 200/2^2 5^2 \rfloor - \lfloor 200/2^2 7^2 \rfloor \\
&= 50 + 22 + 8 + 4 + 1 + 1 - 5 - 2 - 1 \\
&= 78
\end{aligned}$$

Therefore, the number of square-free positive integers less than 200 is

$$200 - 78 = 122.$$

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