Appendix

A Sums & Asymptotics

A.1 Summation Formulae

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2},$$

$$\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6},$$

$$\sum_{i=0}^{n} x^i = \frac{1-x^{n+1}}{1-x},$$

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x},$$

$$\sum_{i=1}^{n} ix^i = \frac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2},$$

$$\sum_{i=1}^{\infty} ix^i = \frac{x}{(1-x)^2}.$$

$$H_n ::= \sum_{i=1}^{n} \frac{1}{i} \qquad \text{(Harmonic Numbers)},$$

$$H_n \sim \ln n.$$

A.2 Factorial

$$n! ::= n \cdot (n-1) \cdots 2 \cdot 1 = \prod_{i=1}^{n} i,$$

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \qquad \text{(Stirling's Formula)}.$$

A.3 Asymptotic Notations

$$\begin{aligned}
f = o(g) &\longleftrightarrow \lim_{x\to\infty} f(x)/g(x) = 0,\\
f \sim g &\longleftrightarrow \lim_{x\to\infty} f(x)/g(x) = 1,\\
f = O(g) &\longleftrightarrow \limsup_{x\to\infty} |f(x)|/g(x) < \infty,\\
f = \Theta(g) &\longleftrightarrow f = O(g) \land g = O(f).
\end{aligned}$$

Equivalently,

$$f = O(g) \longleftrightarrow \exists c \geq 0\, \exists x_0\, \forall x \geq x_0 \;\; |f(x)| \leq cg(x).$$

B Counting

All sets $A, B, A_i \ldots$ below are assumed to be finite. The notation

$$f : A \to B$$

means that f is a total function from A to B. The set of such total functions is

$$[A \to B] ::= \{f \mid f : A \to B\}.$$

B.1 Cardinality and Functions

$$\begin{aligned}
f : A \to B \text{ is an injection} &\longleftrightarrow [\forall a, a' \in A\; f(a) = f(a') \longrightarrow a = a'],\\
f : A \to B \text{ is an injection} &\longrightarrow |A| \leq |B|.\\
f : A \to B \text{ is a surjection} &\longleftrightarrow [\forall b \in B\, \exists a \in A\; f(a) = b],\\
f : A \to B \text{ is a surjection} &\longrightarrow |A| \geq |B|.\\
f : A \to B \text{ is a bijection} &\longleftrightarrow f \text{ is an injection and a surjection},\\
f : A \to B \text{ is a bijection} &\longrightarrow |A| = |B|.\\
|[A \to B]| &= |B|^{|A|},\\
|\{f \in [A \to A] \mid f \text{ is an bijection}\}| &= |A|!,\\
|\{f \in [A \to B] \mid f \text{ is an injection}\}| &= P(|B|, |A|), \qquad (\text{for } |A| \leq |B|).
\end{aligned}$$

B.2 Pigeonhole Principle

If there are more pigeons than pigeonholes, then there must be at least two pigeons in one hole.

B.2.1 Generalized Pigeonhole Principle

If there are m pigeons and n pigeonholes, then at least one hole contains $\lceil m/n \rceil$ pigeons.

B.3 Sum Rule

$$|A_1 \cup A_2 \cup \ldots| = |A_1| + |A_2| + \cdots,$$

for disjoint sets $A_1, A_2, \ldots$.

B.4 Inclusion-Exclusion Principle

$$\begin{aligned}
|A \cup B| &= |A| + |B| - |A \cap B|, \\
|A_1 \cup A_2 \cup \cdots \cup A_n| &= \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| + \cdots \\
&\quad + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n| \\
&= \sum_{k=1}^{n} (-1)^{k+1} \sum_{S \subseteq \{1,\ldots,n\}, |S|=k} \left| \bigcap_{i \in S} A_i \right| \\
&= \sum_{\emptyset \ne S \subseteq \{1,\ldots,n\}} (-1)^{|S|+1} \left| \bigcap_{i \in S} A_i \right|.
\end{aligned}$$

B.5 Product Rule

$$|A_1 \times A_2 \times \cdots \times A_n| = |A_1| \cdot |A_2| \cdots |A_n|.$$

C Permutations and Combinations

C.1 *r*-permutations of an *n*-element set

$$P(n, r) ::= \frac{n!}{(n-r)!}$$

C.2 Division Rule

If $f : A \to B$ maps precisely k items of A to every item of B, then $|A| = k\,|B|$.

C.3 Binomial Coefficients

The number of combinations of r distinct elements from an n-element set is

$$\binom{n}{r} ::= \frac{n!}{(n-r)!\,r!}$$

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1} \qquad \text{for } 1 \leq r \leq n-1 \text{ (Pascal)}$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \qquad \text{(Binomial Theorem)}$$

C.4 Multinomial Coefficients

Let $r_1, r_2, \ldots, r_n$ be non-negative integers. The number of permutations with repetition of an n-element set where the ith element of the set is repeated exactly r_i times is:

$$\binom{(r_1 + r_2 + \cdots + r_n)}{r_1 \quad r_2 \quad \ldots \quad r_n} ::= \frac{(r_1 + r_2 + \cdots + r_n)!}{r_1! \quad r_2! \quad \ldots \quad r_n!}.$$

C.4.1 Multinomial Theorem

$$(x_1 + x_2 + \cdots + x_k)^n = \sum_{n_1 + n_2 + \cdots + n_k = n} \binom{n}{n_1\ n_2\ \ldots n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}.$$

C.5 Stars & Bars

The number of r-combinations with repetition from an n-element set is

$$\binom{n+r-1}{r}.$$

C.6 Hall's Theorem

Definition. A *bipartite graph*, $G = (V_1, V_2, E)$, is a simple graph whose vertices are the disjoint union of V_1 and V_2 and whose edges go between V_1 and V_2, *viz.*,

$$E \subseteq \{\{v_1, v_2\} \mid v_1 \in V_1 \text{ and } v_2 \in V_2\}.$$

A *perfect matching* in G is an injection $f : V_1 \to V_2$ such that $\{v, f(v)\} \in E$ for all $v \in V_1$.

For any set, A, of vertices, define the neighbor set,

$$N(A) ::= \{v \mid \exists a \in A \ \{a, v\} \in E\}.$$

A set $A \subseteq V_1$ is called a *bottleneck* if $|A| > |N(A)|$.

Theorem (Hall). *A bipartite graph has a perfect matching iff it has no bottlenecks.*

D Probability

D.1 Probability Spaces

A *sample space*, $\mathcal{S}$, is a nonempty set whose elements are called *outcomes*. The *events* are subsets of $\mathcal{S}$.

A *probability space* consists of a sample space, $\mathcal{S}$, and a *probability function*, $\Pr\{\} : \mathcal{S} \to [0,1]$ satisfying the Sum Rule and such that:

$$\Pr\{\mathcal{S}\} = 1.$$

D.2 Events

$$\Pr\left\{\bigcup_{n\in\mathbb{N}} A_n\right\} = \sum_{n\in\mathbb{N}} \Pr\{A_n\} \text{ for pairwise disjoint } A_n \qquad \text{(Sum Rule)}$$

$$\Pr\{A - B\} = \Pr\{A\} - \Pr\{A \cap B\} \qquad \text{(Difference Rule)}$$

$$\Pr\left\{\overline{B}\right\} = 1 - \Pr\{B\} \qquad \text{(Complement Rule)}$$

$$\Pr\{A \cup B\} = \Pr\{A\} + \Pr\{B\} - \Pr\{A \cap B\} \qquad \text{(Inclusion-Exclusion)}$$

$$\Pr\{A \cup B\} \le \Pr\{A\} + \Pr\{B\} \qquad \text{(Boole's inequality)}$$

$$\Pr\{A\} \le \Pr\{A \cup B\} \qquad \text{(Monotonicity)}$$

D.3 Conditional Probability

$$\Pr\{A \mid B\} ::= \frac{\Pr\{A \cap B\}}{\Pr\{B\}}$$

$$\Pr\{A \cap B\} = \Pr\{A \mid B\} \Pr\{B\} \qquad \text{(Product Rule)}$$

D.3.1 Law of Total Probability

Suppose the sample space is the disjoint union of $B_0, B_1, \ldots$. Then for all events A,

$$\begin{aligned}\Pr\{A\} &= \sum_{i\in\mathbb{N}} \Pr\{A \cap B_i\} \\ &= \sum_{i\in\mathbb{N}} \Pr\{A \mid B_i\} \Pr\{B_i\}.\end{aligned}$$