Massachusetts Institute of Technology
6.042J/18.062J, Fall '02: Mathematics for Computer Science
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Solutions to In-Class Problems — Week 3, Fri

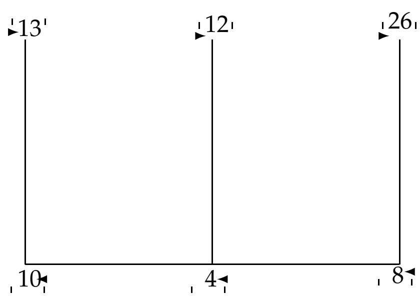
Definition: The *composition* of relations $R \subseteq A \times B$ and $S \subseteq B \times C$ is the relation

$S \circ R = \{(a, c) \mid \exists b \text{ such that } (a, b) \in R \wedge (b, c) \in S\}$.

Problem 1. Recently MIT students have been taking a hard look at the haphazard building layout, and have been asking some hard questions. As always they know they can use their superior mathematical skills to get some real answers to those hard questions.

They decide to express the MIT building layout as a relation. Let C be the set of all building numbers and let R be the relation on the set C such that $(a, b) \in R$ if building a and building b are physically adjacent and there is a door between a and b (more importantly, one doesn't have to go outside to get from a to b). Note that if $(a, b) \in R$, then (b, a) is also in R, so R is a symmetric relation. For convenience, they also define a building to be related to itself, so $(a, a) \in R$.

(a) For this part only, let C be the set of MIT buildings 10,13,12,4,8,26. Then R looks like this:



Compute $R^2 = R \circ R$.
Compute $R^3 = R \circ R^2$.

Solution. R^2 consists of all pairs of buildings that are connected via exactly one building. Since there are self-loops in the connectivity graph (each building is connected to itself), R^2 includes everything which is in R as well. For example, $(13, 10)$ which is in R must also be in R^2 because you can go from building 13 to building 10 "via one building" by just going around building 13 once and then moving on to building 10. That said, we conclude that

$$R^2 = R \cup \Big\{(13, 4), (4, 13), (10, 12), (12, 10), (10, 8), (8, 10), (12, 8), (8, 12), (4, 26), (26, 4)\Big\}$$

Similarly, R^3 contains everything which is in R^2 (because of the self-loops), plus all the extra pairs that we get by allowing connections via *two* intermediate buildings. So we have:

$$R^3 = R^2 \cup \Big\{(13,12),(12,13),(10,26),(26,10),(13,8),(8,13),(26,12),(12,26)\Big\}$$

■

(b) Let R be the map for all of MIT. What does the relation R^2 represent in terms of connectivity between numbered buildings? Can you generalize this?

Solution. R^2 is the set of all pairs of buildings that are connected via exactly 1 building. Similarly R^3 is the set of all pairs of buildings that are connected via exactly 2 buildings. This notion can be generalized to R^n, where (a,b) is in R^n iff a and b are connected via exactly $n-1$ buildings. We can recursively define R^n as $R \circ R^{n-1}$. Induction would thus be a natural way to go about proving this. It is worth looking at the proof, which can be found both in the notes and in Rosen.

It is important to notice that *in general* it is not necessarily true that R^n must include $R^{n-1}, R^{n-2}, \ldots, R$ as well. In our example, this was the case because the graph included self-loops and bidirectional edges (symmetric relation), so one can meet the requirement of going from building a to building b through $n-1$ intermedite buildings by going around building a as many times as necessary and then proceeding to building b (there are many similar time-wasting ways to meet the requirement).

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(c) One of the important questions for course 6 students was, is it possible to get from building 36 to building 10 without crossing more than 5 other buildings? Write a proposition in terms of R, using relational and set operators, which is true if this condition is satisfied and false otherwise.

Solution. The key to solving this is to first write an expression for the set of all buildings that are connected via 5 or fewer buildings. In general, this would be $(R \cup R^2 \cup R^3 \cup \ldots \cup R^6)$. As explained above, in our case this is the same as R^6. Then the predicate on R is $(36,10) \in R^6$. ■

(d) The MIT students would like to be able to get from any building to any other building, without having to go outside. Write the condition on R that must be satisfied in order for this to be true.

Solution. Let $|C|$ be n. In other words there are n different buildings. The furthest apart two buildings can be, and still be connected, is to be connected via $n-2$ other buildings (prove this to yourself). Therefore the set $(R \cup R^1 \cup R^2 \cup \ldots \cup R^{n-1})$ represents all the pairs of buildings that can reach one another without going outside. Because of the self-loops, this is just R^{n-1} since this set includes everything else as well.

However we want every pair of buildings to be connected. The set of all pairs of buildings is $C \times C$. Therefore the condition we want to satisfy is $C \times C \subseteq R^{n-1}$ ("the set of all possible pairs of buildings is a subset of the set of buildings that you can reach through $n-2$ intermediate buildings"). ■

(e) MIT administration, however, wants to keep the number of connections between building as small as possible. In other words, MIT wants the size, $|R|$, of R to be as small as possible. What is the smallest R that satisfies the requirement in part (d)? Is the smallest R unique?

Solution. The smallest graph such that all the buildings are connected would be to connect them in a straight line. The number of edges in that graph is $n - 1$. How do we know that this is the smallest number of edges? Next week we will learn a theorem that says that the smallest connected graph with vertices n has at least $n - 1$ edges. Each edge contributes two pairs therefore $|R| = 2(n - 1)$.

The smallest graph is not unique, for starters we can connect the buildings in a different order in the line. But there are many other connected graphs with n vertices and $n-1$ edges (any tree with n nodes in fact satisfies this constraint, as we shall see next week). ■

Problem 2. A relation R from $A = \{a_1, a_2, \ldots, a_m\}$ to $B = \{b_1, b_2, \ldots, b_n\}$ can be represented as a boolean matrix M_R, where $M_R(i,j) = 1$ if the pair $(a_i, b_j) \in R$ and $M_R(i,j) = 0$ if the pair (a_i, b_j) is not in R.

We define boolean matrix multiplication to be the same as regular matrix multiplication except that "+" is replaced by $\vee$ (Boolean OR) and "$\times$" is replaced by $\wedge$ (Boolean AND).

(a) We have a student set $stud = \{Adrian, Min, Josh\}$, a class set $class = \{6.042, 6.046\}$ and a lecture set $lect = \{Albert, Charles, Radhi\}$. The relation K "is taking class" as a subset of $stud \times class$ is defined by the list: $\{(Adrian, 6.042), (Min, 6.046), (Josh, 6.042), (Josh, 6.046)\}$ and the relation L "is lectured by"as a subset of $class \times lect$ is defined by the list: $\{(6.042, Albert), (6.042, rahdi), (6.046, Charles)\}$. The relation T "is taught by" is the composition of relations K and L. Represent relation T in boolean matrix and compare it with the boolean matrix multiplication of relation K and relation L.

Solution. The relation T is represented by the matrix

	$Albert$	$Charles$	$Radhi$
$Adrian$	1	0	1
Min	0	1	0
$Josh$	1	1	1

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(b) Let M_P be the boolean multiplication of M_R and M_S, where $R \subseteq A \times B$ and $S \subseteq B \times C$. Write down the formula of the boolean matrix multiplication of $M_P(i,j)$ in terms of M_R and M_S. [1]

[1]Recall that the *composition* of relations $R \subseteq A \times B$ and $S \subseteq B \times C$ is the relation $S \circ R = \{(a, c) \mid \exists b$ such that $(a, b) \in R \wedge (b, c) \in S\}$

Solution. Suppose set B has n elements,

$$M_P(i,j) = [M_R(i,1) \wedge M_S(1,j)] \vee [M_R(i,2) \wedge M_S(2,j)] \vee \ldots \vee [M_R(i,n) \wedge M_S(n,j)]$$

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(c) Prove that boolean multiplication of M_R and M_S is equal to $M_{S \circ R}$.

Solution. Let M_P be the boolean product of M_R and M_S, what we want to prove is that

$$(a_i, c_j) \in S \circ R \Longleftrightarrow M_P(i,j) = 1$$

Recall that by the definition of composition, $(a_i, c_j) \in S \circ R$ iff there exists a k such that $(a_i, b_k) \in R$ and $(b_k, c_j) \in S$. (We assume that set B has n elements.) Also, we have already got

$$M_P(i,j) = \underbrace{[M_R(i,1) \wedge M_S(1,j)]}_{b_1 \text{ is the “link”}} \vee \underbrace{[M_R(i,2) \wedge M_S(2,j)]}_{b_2 \text{ is the “link”}} \vee \ldots \vee \underbrace{[M_R(i,n) \wedge M_S(n,j)]}_{b_n \text{ is the “link”}}$$

Case 1:($\Longrightarrow$) If $(a_i, c_j) \in S \circ R$, then for at least one k, $1 \leq k \leq n$, $(a_i, b_k) \in R$ and $(b_k, c_j) \in S$. Consequently, $M_R(i,k) = 1$ and $M_S(k,j) = 1$. This turns $[M_R(i,k) \wedge M_S(k,j)]$ true, and hence $M_P(i,j) = 1$.

Case 2: ($\Longleftarrow$) If $M_P(i,j) = 1$ then there is at least one k, $1 \leq k \leq n$, for which $[M_R(i,k) \wedge M_S(k,j)] = 1$. This means that both $M_R(i,k) = 1$ and $M_S(k,j) = 1$. Since M_R and M_S are the matrix representations of R and S, we can conclude that $(a_i, b_k) \in R$ and $(b_k, c_j) \in S$, and so, by the definition of composition, $(a_i, c_j) \in S \circ R$

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(d) What does the regular multiplication of M_R and M_S give you?

Solution. Let M_P be the regular product of M_R and M_S.

$$M_P(i,j) = \underbrace{[M_R(i,1) \times M_S(1,j)]}_{b_1 \text{ is the “link”}} + \underbrace{[M_R(i,2) \times M_S(2,j)]}_{b_2 \text{ is the “link”}} + \ldots + \underbrace{[M_R(i,n) \times M_S(n,j)]}_{b_n \text{ is the “link”}}$$

If $[M_R(i,k) \times M_S(k,j)] = 1$, where $1 \leq k \leq n$, both $M_R(i,k) = 1$ and $M_S(k,j) = 1$. Since M_R and M_S are the matrix representations of R and S, we can conclude that $(a_i, b_k) \in R$ and $(b_k, c_j) \in S$ and there is a path from a_i to c_j via b_k. Therefore, $M_P(i,j)$ gives you the number of paths from $a_i \in A$ to $c_j \in C$ via B.

The regular multiplication gives you the number of paths from any element in A to any element in C via B. In general many questions on relations can be framed as operations on boolean matrices which is very useful for programming. ■

Problem 3. The term *six degrees of separation* implies that everyone knows everyone else indirectly through at most 6 other people. Discuss how you would write a computer program to determine if six degrees of separation holds within our 6.042 class.

Solution. Meant to be an open ended discussion. Beyond framing the relational question correctly, things to think about are how would you collect data, what representation would you use, how would you make it efficiency, etc. ■