Massachusetts Institute of Technology
6.042J/18.062J, Fall '02: Mathematics for Computer Science
Professor Albert Meyer and Dr. Radhika Nagpal

In-Class Problems — Week 3, Mon

Problem 1. Find the flaw in the following false proof, and give a counter-example to the claim.

Claim. *Suppose R is a relation on A. If R is symmetric and transitive, then R is reflexive.*

False proof. Let x be an arbitrary element of A. Let y be any element of A such that xRy. Since R is symmetric, it follows that yRx. Then since xRy and yRx, we conclude by transitivity that xRx. Since x was arbitrary, we have shown that $\forall x \in A\ (xRx)$, so R is reflexive. □

Problem 2. In each case, say whether or not R is a equivalence relation on A. If it is an equivalence relation, what are the equivalence classes and how many equivalence classes are there?

(a) $R ::= \{(x, y) \in W \times W \mid$ the words x and y start with the same letter$\}$ where W is the set of all words in the 2001 edition of the Oxford English dictionary.

(b) $R ::= \{(x, y) \in W \times W \mid$ the words x and y have at least one letter in common$\}$.

(c) $R = \{(x, y) \in W \times W$ and the word x comes before the word y alphabetically$\}$.

(d) $R = \{(x, y) \in \mathbb{R} \times \mathbb{R}$ and $|x| \leq |y|\}$.

(e) $R = \{(x, y) \in B \times B$, where B is the set of all bit strings and x and y have the same number of 1s.$\}$

Problem 3. Let R be an equivalence relation on the set A. For an element $a \in A$, let $[a]$ denote the set $\{b \in A$ given $aRb\}$. This set is the *equivalence class of a under R* and we call a a *representative* of the set $[a]$.

Prove that the sets $[a]$ for all $a \in A$ constitute a partition of A. In other words, prove that for every $a, b \in A$, either $[a] = [b]$ or $[a] \cap [b] = \emptyset$.

Appendix

A binary relation R on a set A is

- *reflexive* if for every $a \in A$, $a \sim_R a$,
- *symmetric* if for every $a, b \in A$, $a \sim_R b$ implies $b \sim_R a$.
- *antisymmetric* if for every $a, b \in A$, $a \sim_R b$ and $b \sim_R a$ implies $a = b$.
- *asymmetric* if for every $a, b \in A$, $a \sim_R b$ implies $\neg(b \sim_R a)$.
- *transitive* if for every $a, b, c \in A$, $a \sim_R b$ and $b \sim_R c$ implies $a \sim_R c$.

for all $x, y, z \in A$.

A binary relation R on a set A is

- an *equivalence relation* iff it is reflexive, symmetric and transitive.