Massachusetts Institute of Technology
6.042J/18.062J, Fall ’02: Mathematics for Computer Science
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In-Class Problems — Week 4, Fri

Definition: The graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are **isomorphic** iff there is a bijection $f : V_1 \to V_2$ such that for all $u, v \in V_1$, the edge $(u, v) \in E_1 \longleftrightarrow (f(u), f(v)) \in E_2$.

Problem 1. Let L_n be the $n + 1$-vertex simple "line" graph consisting of a single simple path of length n. For example, L_4 is shown in the figure below:

The line graph L_4

Let's say a simple graph has "two ends" if it has exactly two vertices of degree one, and all its other vertices have degree two. In particular, for $n \geq 1$, the graph L_n has two ends. Consider the following false theorem.

False theorem: Every simple graph with two ends is isomorphic to L_n for some $n \geq 1$.

(a) Draw a diagram of the smallest simple graph with two ends which is not isomorphic to any line graph.

(b) Explain briefly, but clearly, where the following proof goes wrong:

False proof: We prove by induction on the number, $n \geq 1$, of edges in a simple graph, that every two-ended graph with n edges is isomorphic to L_n.

(Base case $n = 1$): A simple graph with one edge can only consist of the two vertices connected by that edge and some number of vertices not attached to any edge, i.e., vertices of degree zero. Since a two-ended graph cannot have vertices of degree zero, the only two-ended graph with one edge must consist solely of the two vertices connected by the edge, which makes it isomorphic to L_1.

(Induction case): Assume that $n \geq 1$ and every two-ended graph with n edges is isomorphic to L_n. Now let G_n be any two-ended graph with $n \geq 1$ edges. By hypothesis, G_n is isomorphic to L_n. Suppose an edge is added to G_n to form a two-ended graph G_{n+1}.

Since G_n is isomorphic to L_n, it consists of a simple path of length n. The only way to add an edge to the path and preserve two-endedness is to have that edge go from one end—that is, one of the degree-one vertices—to a new vertex, lengthening the path by one. That is, the resulting $n + 1$-edge graph must be a simple path of length $n + 1$, so it is isomorphic to L_{n+1}. Q.E.D.

(c) Here is another argument for the same False Theorem. Explain exactly where it goes wrong.

False proof 2: Same induction hypothesis and base case as in part (b).

(Induction case): For any $n \geq 1$, let G_{n+1} be any two-ended graph with $n+1$ edges. Let G_n be the graph which results from removing one of the degree-one vertices v of G_{n+1} and the edge $\{v, w\}$ attached to it. So G_n no longer has the vertex, v, of degree one. But the degree of w is one less in G_n than it was in G_{n+1}, so G_n still has two vertices of degree one and one fewer vertex of degree two. Therefore G_n is also two-ended. By induction G_n consists of a simple path of length n. But G_{n+1} is obtained by attaching an edge from one end of the path to a vertex v not on the path, thereby lengthening the path by one. So G_{n+1} is isomorphic to a simple path of length $n+1$; that is, it is isomorphic to L_{n+1}. Q.E.D.

(d) Describe how to make a small revision to one of the false proofs above so that it becomes a correct proof of the theorem "Every *connected* simple graph with two ends is isomorphic to L_n for some $n \geq 1$."

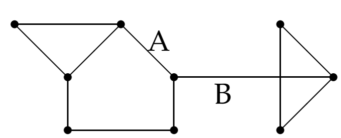
Problem 2. Let G_1 and G_2 be two graphs that are isomorphic to each other. Argue why if there is a cycle of length k in G_1, there must be a cycle of length k in G_2.

Problem 3. Prove that definition 1 implies definition 2.

Definition 1: A tree is an acyclic graph of n vertices that has $n-1$ edges.

Definition 2: A *tree* is a connected graph such that $\forall u, v \in V$, there is a unique path connecting u to v.

Problem 4. An edge of a connected graph is called a *cut-edge* if removing the edge disconnects the graph.



(a) In the above figure, are either A or B cut-edges? Explain.

(b) Prove that in an undirected connected graph, an edge e is a cut-edge *if and only if* no simple cycle contains e.

(c) Using the previous part, argue that in a tree every edge is a cut-edge but in an $n \times n$ mesh no edge is a cut-edge. Why might this be important?