Massachusetts Institute of Technology
6.042J/18.062J, Fall '02: Mathematics for Computer Science
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In-Class Problems — Week 5, Wed

Problem 1. We apply the following operation to a simple graph G: pick two vertices $u \neq v$ such that either

1. there is an edge of G between u and v and there is also a path from u to v which does *not* include this edge; in this case, delete the edge $\{u, v\}$.
2. there is no path from u to v; in this case, add the edge $\{u, v\}$.

Keep repeating these operations until it is no longer possible to find two vertices $u \neq v$ to which an operation applies.

Assume the vertices of G are the integers $1, 2, \ldots, n$ for some $n \geq 2$. This procedure can be modelled as a state machine whose states are all possible simple graphs with vertices $1, 2, \ldots, n$. G is the start state, and the final states are the graphs on which no operation is possible.

(a) Let G be the graph with vertices $\{1, 2, 3, 4\}$ and edges

$$\{\{1, 2\}, \{3, 4\}\}$$

What are the possible final states reachable from start state G? Draw them.

(b) Below are three derived variables. Indicate the *strongest* property from the list below that each variable is guaranteed to satisfy, no matter what the starting graph G is, and justify your answer. The properties are: *constant*, *strictly increasing*, *strictly decreasing* ,*weakly increasing*, *weakly decreasing*, *none of these*

For any state, let e be the number of edges in it, and let c be the number of connected components it has. For example, since e may increase or decrease in a transition, it does not have any of the first four properties, so it would be classified *none of these*. The derived variables are

i) c,

ii) $c + e$,

iii) $2c + e$.

(c) Conclude that the procedure terminates.

(d) Prove that any final state must be a tree on the vertices.

Problem 2. The table below shows the preferences of each girl and boy in decreasing order.

boys	*girls*
$1 : CBEAD$	$A : 35214$
$2 : ABECD$	$B : 52143$
$3 : DCBAE$	$C : 43512$
$4 : ACDBE$	$D : 12345$
$5 : ABDEC$	$E : 23415$

(a) We saw that using a "boy greedy" strategy, where each boy in turn got his favorite available girl led to marriages in which Boy 4 and Girl C are a rogue couple. Which other boys are in rogue couples in these marriages? (You should reconstruct the greedy marriages from the table.)

(b) Find a stable set of marriages. (You can do this by trial and error; you're not expected to remember the Mating Procedure from the Notes, which we'll review later in class.)

Problem 3. Four Students want separate assignments to four VI-A Companies. Here are their preference rankings:

Student	Companies
Albert:	HP, Bellcore, AT&T, Draper
Carole:	AT&T, Bellcore, Draper, HP
Eric:	HP, Draper, AT&T, Bellcore
Radhi:	Draper, AT&T, Bellcore, HP

Company	Students
AT&T:	Radhi, Albert, Eric, Carole
Bellcore:	Eric, Carole, Albert, Radhi
HP:	Radhi, Eric, Albert, Carole
Draper:	Carole, Radhi, Eric, Albert

Use the Mating Algorithm to find *two* stable assignments of Students to Companies.

Problem 4. Prove that the Mating Algorithm produces stable marriages. (Don't look up the proof in the Course Notes.)