Solutions to In-Class Problems — Week 8, Fri

Problem 1.

(a) In how many ways can 10 customers line up at a supermarket checkout?

Solution. $10!$, the number of permutations. ■

(b) In how many ways can 10 customers line up at two supermarket checkouts?

Solution. $11!$. Drawing a tree helps. The first student has 2 choices of checkout. The second student has 3 choices: in the other checkout from the first student, in the same checkout ahead of the first student, or in the same checkout behind the first student. No matter where the second student goes, however, notice that the third student has exactly 4 choices.

An alternative way to see the answer is to add a dummy customer who acts as a separator between the customers in one checkout and those in the other. Each permutation of the 10 real customers plus the dummy corresponds to a different way that the customers can line up, where those customers to the left of the dummy go to checkout #1 and those to the right of the dummy go to checkout #2. (Later in the lecture, we'll formalize this argument using bijections.)

Students should understand both approaches. ■

(c) In how many ways can 10 customers line up at three supermarket checkouts?

Solution. $12!/2$. The tree argument is a simple extension of part (b): 3 choices for the first, 4 for the second, etc.

The argument with two dummies is trickier. There are $12!$ permutations of the 10 customers and two dummies, but we don't need to distinguish the two dummies from each other. Indeed, for each permutation where dummy #1 occurs before dummy #2 in the order, there is exactly one permutation where dummy #2 occurs before dummy #1. Thus, we overcount by a factor of 2.

Once again, students should understand both approaches. ■

(d) (Optional.) What is the general case for n customers and m supermarket checkouts?

Solution. $(n+m-1)!/(m-1)!$. ■

Problem 2. An n-input, m-output *boolean* function is a mapping $f : \{0,1\}^n \to \{0,1\}^m$.

(a) How many n-input, 1-output boolean functions are there? *Hint:* Two boolean functions are different if there exists an n-bit input on which they output different values.

Solution. There are 2^n possible inputs to the boolean function. Make a truth table listing all 2^n inputs and the corresponding 2^n outputs. Two boolean functions are different if they have different truth tables (even if they agree on all rows except one). There are 2^{2^n} different possibilities for the set of outputs, and thus 2^{2^n} n-input boolean functions. ■

(b) How many n-input, m-output boolean functions are there?

Solution. Since the truth table now has $m2^n$ entries, there are 2^{m2^n} n-input, m-output boolean functions. ■

Problem 3. On a set S of n elements, how many of the following types of relations are there? (An appendix is included if you need a reminder of the definitions.)

(a) binary relations

Solution. 2^{n^2}. We look at the matrix that corresponds to the relations such the element of the $n \times n$ matrix is 1 if the corresponding pair is in the relation and it is 0 otherwise. There are 2^{n^2} many different possible boolean $(0-1)$ $n \times n$ matrices. Another way of reasoning about this is to say that there are n^2 possible ordered pairs, and any subset of those pairs is a possible relation. If A is the set of all pairs, then all possible relations are just all possible subsets of A, and we know that $\mathrm{P}(A) = 2^{|A|} = 2^{n^2}$. ■

(b) symmetric binary relations

Solution. In a symmetric binary relation, if $(a,b) \in R$, then $(b,a) \in R$. The matrix associated with a symmetric relation is symmetric (the entries in the triangle above the main diagonal are the same as the entries in the triangle below the diagonal). The total number of entries is n^2. The number of entries on the diagonal is n and the number of entries on the upper side of the diagonal is $(n^2-n)/2$. Therefore, there are $n(n-1)/2+n = n(n+1)/2$ entries that can be set. Hence, there are $2^{n(n+1)/2}$ different symmetric matrices/relations. ■

(c) reflexive binary relations

Solution. In a reflexive binary relations, $(a, a) \in R$ for all $a \in S$, and thus the matrix associated with a reflexive relation must have 1′s along the main diagonal. These n entries are fixed, and hence the number of entries that can be set is $n^2 - n$. The number of matrices corresponding to reflexive relations is 2^{n^2-n}, and hence so is the number of reflexive binary relations.

Using this idea you can count the number of relations that are not reflexive by just substracting the number of reflexive relations from the total number of relations. ■

(d) symmetric and reflexive binary relations

Solution. The matrix associated with a symmetric and reflexive relation is symmetric and has 1′s along the main diagonal. The total number of entries is n^2 and there are $(n^2 - n)/2$ that are on the upper side of the diagonal. Hence the number of such relations is $2^{(n(n-1))/2}$. ■

(e) symmetric or reflexive binary relations

Solution. By inclusion-exclusion, the number of symmetric or reflexive relations equals the number of reflexive relations plus the number of symmetric relations minus the number of reflexive and symmetric relations, which equals $2^{n(n+1)/2} + 2^{n^2-n} - 2^{(n(n-1))/2}$. Simplify at your peril. ■

Problem 4. Consider the set of undirected graphs on the set $V = \{1, 2, \ldots, n\}$ of vertices. (Recall that undirected graphs have no self-loops.) Count the number of such graphs by exhibiting a bijection with one of the types of relations in Problem 3 . Prove that your mapping is a bijection.

Solution. There exists a bijection f from the set of undirected graphs with vertices $V = \{1, 2, \ldots, n\}$ to the set of symmetric and reflexive binary relations. We define f to be the function that maps any undirected graph G to the symmetric and reflexive binary relation $f(G) = \{(a, b) \in V \times V : a = b \text{ or } (a, b) \in G\}$.

First, we show that f is injective. For any two distinct undirected graphs G_1 and G_2 defined on V, one must contain an edge not in the other, since otherwise they wouldn't be distinct. Assume without loss of generality that $(x, y) \in G_1$ and $(x, y) \notin G_2$. I claim that $(x, y) \in f(G_1)$ and $(x, y) \notin f(G_2)$. The first part of the claim follows from the definition of f. For the second part, assume that $(x, y) \in G_2$. Then, by the definition of f, either $x = y$, which is not possible since undirected graphs have no self-loops, or $(x, y) \in G_2$, which is not possible by assumption. Thus, f is injective.

Now, we show that f is surjective. For any symmetric reflexive relation R on V, consider the undirected graph G with edge set $\{(a, b) : a \neq b \text{ and } (a, b) \in R\}$. I claim that $f(G) = R$.

Let $(a, b) \in R$. If $a \neq b$, then $(a, b) \in G$ by definition of G, and hence $(a, b) \in f(G)$ by definition of f. If $a = b$, then $(a, b) \in f(G)$, since $f(G)$ contains (x, x) for all $x \in V$. Thus, we have $R \subseteq f(G)$.

Now, let $(a, b) \in f(G)$. If $a \neq b$, then $(a, b) \in G$ by definition of f, and hence $(a, b) \in R$ by definition of R. If $a = b$, then $(a, b) \in R$, since R is reflexive. Thus, we have $f(G) \subseteq R$.

Since $R \subseteq f(G)$ and $f(G) \subseteq R$, we have $f(G) = R$, and hence f is surjective, since it maps some undirected graph to R. Consequently, the number of undirected graphs with n labeled vertices is $2^{n(n-1)/2}$. ■

1 Appendix

1.1 Relations

A binary relation R on a set A is a subset $R \subseteq A \times A$. A binary relation R is

- *reflexive* if $(a, a) \in R$ for every $a \in A$;
- *symmetric* if aRb implies bRa for every $a, b \in A$.

1.2 Functions

A function $f : A \to B$ is

- *injective (one-to-one)* if $f(x) = f(y)$ imples that $x = y$ for all x and y in the domain of f;
- *surjective (onto)* if for every element $b \in B$, there exists an element $a \in A$ such that $f(a) = b$;
- *bijective (one-to-one correspondence)* if f is both injective and surjective.