Massachusetts Institute of Technology
6.042J/18.062J, Fall '02: Mathematics for Computer Science
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Solutions to In-Class Problems — Week 8, Mon

Problem 1. Let S be the set of all 4-digit base-10 numbers that contain the digit 7 somewhere. (Numbers can start with a 0, and thus $0007 \in S$.) The size of S can be determined in several different ways.

(a) Determine $|S|$ by counting the number of all possible 4-digit numbers minus the number of those that do *not* contain 7.

Solution.

$$\begin{aligned} |S| &= 10^4 - 9^4 \\ &= 3439 \,. \end{aligned}$$

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(b) Determine $|S|$ by counting the set of numbers where the first 7 occurs in the first digit, where the first 7 appears as the second digit, where the first 7 appears as the third digit, and where the first 7 appears as the last digit.

Solution.

$$\begin{aligned} |S| &= 1 \cdot 10 \cdot 10 \cdot 10 \\ &\quad +9 \cdot 1 \cdot 10 \cdot 10 \\ &\quad +9 \cdot 9 \cdot 1 \cdot 10 \\ &\quad +9 \cdot 9 \cdot 9 \cdot 1 \\ &= 1000 + 900 + 810 + 729 \\ &= 3439 \,. \end{aligned}$$

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(c) Determine $|S|$ by counting the number of 4-digit numbers with exactly one 7, with exactly two 7's, with exactly three 7's and with exactly four 7's.

Solution.

$$\begin{aligned} |s| &= 4 \cdot 9^3 + 6 \cdot 9^2 + 4 \cdot 9 + 1 \\ &= 2916 + 486 + 36 + 1 \\ &= 3439 \,. \end{aligned}$$

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(d) Argue in each case that the counting method is correct by arguing that (1) everything has been counted at least once (2) nothing has been counted twice.

Solution. Part A: $|S| = |\text{all 4-digit base-10 numbers}| - |\text{4-digit base-10 numbers with no seven}|$

All 4-digit numbers with a seven is counted *exactly* once in |*all 4-digit base-10 numbers*|. 4-digit numbers without a seven are counted once in |*all 4-digit base-10 numbers*| but are subtracted away in |*4-digit base-10 numbers with no seven*|, so they are not counted.

Part B: Let P_i represent the set of 4-digit numbers with the first seven appearing in the ith digit. Then we count as follows:

$|S| = |P_1| + |P_2| + |P_3| + |P_4|$

This way of calculating $|S|$ utilizes the sum rule. Each of the sets P_i represent the numbers where the first seven appears in the ith position, with i being 1, 2, 3, or 4. These sets are disjoint because the first occurrence of the seven cannot be in more than one position. This guarantees that numbers with a seven only are counted once.

Numbers with a seven must be counted somewhere because the 7 must appear first in some position between 1 and 4, and will be included in one of the P_is. Numbers with no seven will not be counted in any of the sets.

Part C: Let S_i represent the set of 4-digit numbers with exactly i sevens. Then we count as follows:

$|S| = |S_1| + |S_2| + |S_3| + |S_4|$

This way of calculating $|S|$ again utilitizes the sum rule. Each of the sets S_i represent 4 digit numbers with *exactly* i sevens. They are disjoint since a number can not have exactly i sevens, but also have exactly j sevens, where $i \neq j$. This guarantees that all numbers with a seven are only counted once.

Numbers that have a seven must have either 1 seven, 2 sevens, 3 sevens, or 4 sevens. Therefore, they would be included in one of the above S_is and would be accounted for. 4-digit numbers cannot have more than 4 sevens, and 4-digit numbers with 0 sevens are not counted.

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Problem 2. Each of the 49 students in an MIT class understands mathematical equations, exercises regularly, or loves literature. Of these,

- 20 students understand mathematical equations,
- 20 students exercise regularly,
- 26 students love literature,
- 36 students understand mathematical equations or exercise regularly,
- 38 students understand mathematical equations or love literature, and
- 40 students exercise regularly or love literature.

(a) How many students understand mathematical equations *and* exercise regularly *and* love literature?

Solution. Let A denote the set of students who understand mathematical equations, let B denote the set of students who exercise regularly, and let C denote the set of students who love literature. We are given that $|A \cup B \cup C| = 49$, $|A| = 20$, $|B| = 20$, $|C| = 26$, $|A \cup B| = 36$, $|A \cup C| = 38$, and $|B \cup C| = 40$; whence

$$\begin{aligned}
|A \cap B \cap C| &= |A \cap B| + |C| - |(A \cap B) \cup C| \\
&= |A| + |B| - |A \cup B| + |C| - |(A \cup C) \cap (B \cup C)| \\
&= |A| + |B| - |A \cup B| + |C| - |A \cup C| - |B \cup C| + |A \cup B \cup C| \\
&= 20 + 20 - 36 + 26 - 38 - 40 + 49 \\
&= 1\,.
\end{aligned}$$

In line 1 we apply inclusion-exclusion, in line 2 we apply inclusion-exclusion and distributive laws, and in line 3 we apply inclusion-exclusion.

Ask the students to verify their answers by drawing the Venn diagram.

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(b) Suppose you don't know the number of students in the class, but the rest of the information is the same. What is the largest possible number of students who understand mathematical equations *and* exercise regularly *and* love literature? For this situation, how many students are in the class?

Solution. Since $A \cap B \cap C \subseteq A \cap B$, it follows that we must have $|A \cap B \cap C| \leq |A \cap B|$, and likewise, we have $|A \cap B \cap C| \leq |A \cap C|$ and $|A \cap B \cap C| \leq |B \cap C|$. By inclusion-exclusion, we have

$$\begin{aligned}
|A \cap B| &= |A| + |B| - |A \cup B| \\
&= 20 + 20 - 36 \\
&= 4\,,
\end{aligned}$$

and similarly,

$$\begin{aligned}
|A \cap C| &= |A| + |C| - |A \cup C| \\
&= 20 + 26 - 38 \\
&= 8
\end{aligned}$$

and

$$\begin{aligned} |B \cap C| &= |B| + |C| - |B \cup C| \\ &= 20 + 26 - 40 \\ &= 6\,. \end{aligned}$$

The strongest of the three constraints is therefore $|A \cap B \cap C| \leq |A \cap B| = 4$. Plugging the value 4 in for $|A \cap B \cap C|$, we find we can satisfy all the conditions of the problem. (Verify by Venn diagram.) The total number of students is 52.

Make sure the students understand that obtaining the constraint $|A \cap B \cap C| \leq 4$ is insufficient. They also must show that 4 actually works.

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