



Milestones of Probability

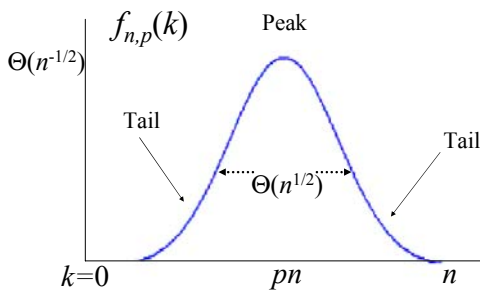


3 Milestones

- Central Limit Theorem
- Poisson Distribution
- **Strong** Law of Large Numbers



The Binomial PDF



Binomial Limits

Is there a “limiting shape” for $f_{n,p}$?

What limit? – two parameters:

- p fixed, $n \rightarrow \infty$
- ~~n fixed, $p \rightarrow 0$ or 1~~
- $n \rightarrow \infty, p \rightarrow 0$ together **how?**



Binomial Limits

Is there a “limiting shape” for $f_{n,p}$?

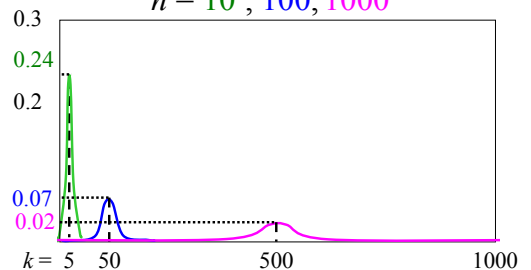
- p fixed, $n \rightarrow \infty$

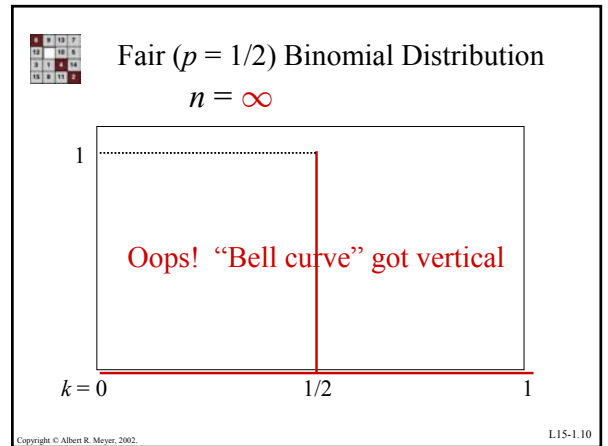
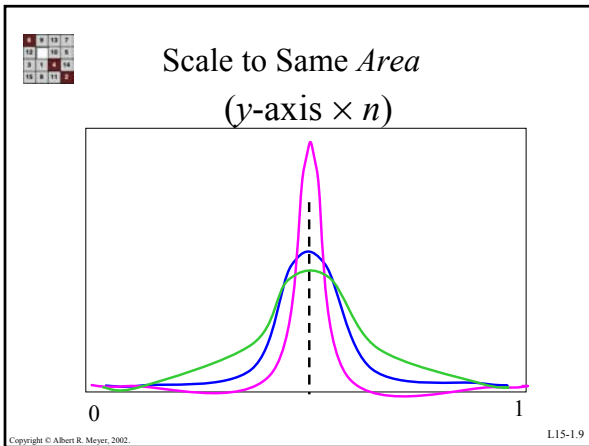
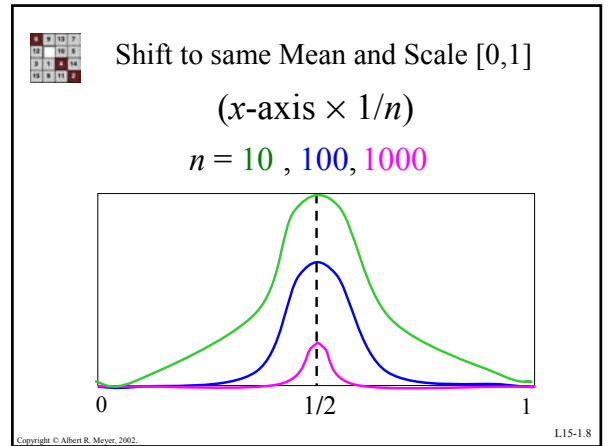
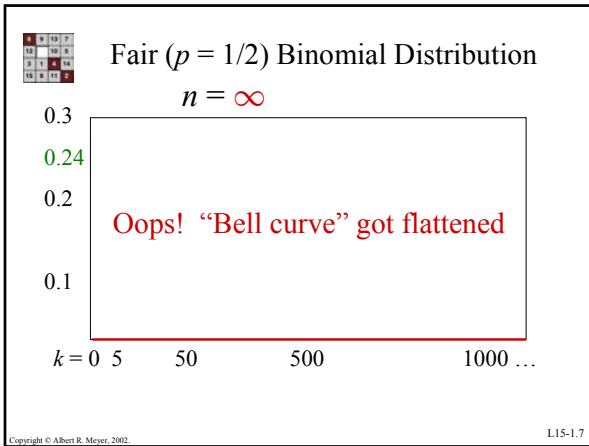
Say $p = 1/2$



Fair ($p = 1/2$) Binomial Distribution

$n = 10, 100, 1000$





Normalize!

- Shift to **expectation 0** (subtract μ_n)
- Scale x -axis to **SD 1** (divide by σ_n)

$$B^* ::= (B - \mu) / \sigma$$

- Scale y -axis to **area 1** (multiply by σ_n) to see Bell Curve.

Copyright © Albert R. Meyer, 2002. L15-1.11

Central Limit Theorem
 As $n \rightarrow \infty, p$ fixed,
 $B_{n,p}^* \rightarrow$ Bell-shaped Curve
 technically:

$$\eta(x) ::= \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$\lim_{n \rightarrow \infty} \sigma_n f_{n,p}(\frac{\alpha n - \mu_n}{\sigma_n}) = \eta\left(\frac{\alpha n - \mu_n}{\sigma_n}\right)$$

Copyright © Albert R. Meyer, 2002. L15-1.12



Central Limit Theorem

Could derive η from Stirling's formula (see Feller). Other proofs along lines of Chernoff bound proof (using "moment generating function").



Central Limit Theorem

More generally, for **any** independent, identically distributed

$$X_1, X_2, \dots, X_m, \dots$$

let $S_n ::= X_1 + X_2 + \dots + X_n$

As $n \rightarrow \infty$, p fixed,

$$S_n^* \rightarrow \text{Bell Curve}$$



In-Class Problem

Problem 1



Poisson Distribution

Misprints in Weekly Notes: $\lambda \approx 1/3$ per page.
30 lines \times 90 char,

$$n \approx 2700 \text{ char/page.}$$

Say each char has *independent* prob

$$p ::= \lambda/n = 1/3 \cdot 1/2700 = 1/8100$$

of misprint. pdf for errors/page would be:

$$f_{n, \lambda/n} = f_{2700, 1/8100}$$



Poisson Distribution

Packets at router: $\lambda \approx 10/\mu\text{sec}$.

$$n = 1000 \text{ cycles}/\mu\text{sec}, \lambda/n = \Pr\{\text{packet at cycle}\}$$

Say each cycle has *independent* prob of containing a packet: pdf for packets/ μsec would be:

$$f_{n, \lambda/n} = f_{1000, 1/100}$$



Poisson Distribution

Bombs/rescue team:

$$\lambda \approx 200 \text{ bombers} \times 50 \text{ bombs}/1500 \text{ rescue teams}$$
$$n = 1500$$

Say bomb has *independent* prob of landing in team area: pdf for bombs/team would be:

$$f_{n, \lambda/n} = f_{1500, 2/3}$$



Poisson Distribution

Common theme:

Moderate average, λ , arising from a large number, n , of unlikely samples



Poisson Limit Theorem

As $n \rightarrow \infty, p \rightarrow 0, np = \lambda$

$B_{n, \lambda/n}^* \rightarrow \text{Poisson var } P_\lambda$

- one parameter λ
- $\mu = \lambda$
- $\sigma = \sqrt{\lambda}$



Poisson Distribution

$$\Pr \{P_\lambda = k\} ::= \frac{\lambda^k}{k!} e^{-\lambda}$$

Routine proof using

$$\lim_{n \rightarrow \infty} (1 + x/n)^n = e^x$$



Strong Law of Large Numbers

- Random Var R , unknown μ_R
- Keep sampling R_1, R_2, \dots
- intuitively expect

$$\frac{R_n}{n} \xrightarrow{?} \mu_R$$

True – with probability 1,

but does NOT follow from Weak Law



In-Class Problem

Problem 2