Mathematics for Computer Science

MIT 6.042J/18.062J

More Graphs

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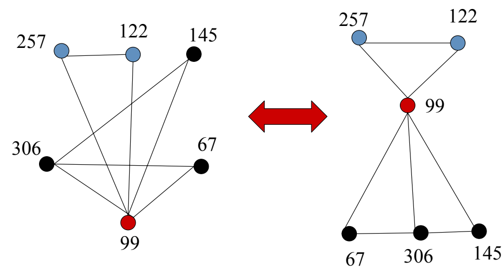
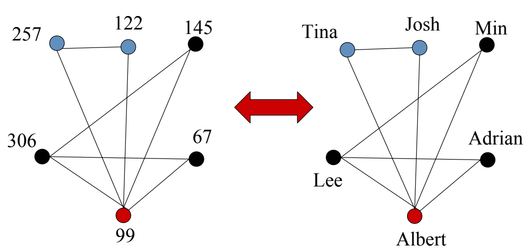
Topology, not Geometry



Equivalent (Isomorphic) Graphs



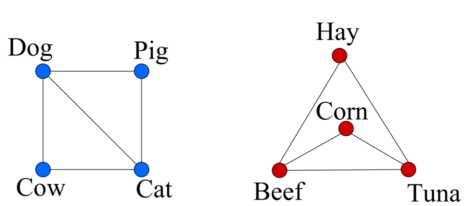
Graph Isomorphism

Graphs G_1 and G_2 are isomorphic if
there exists a bijection $f: V_1 \rightarrow V_2$ such that
for all u,v in V_1

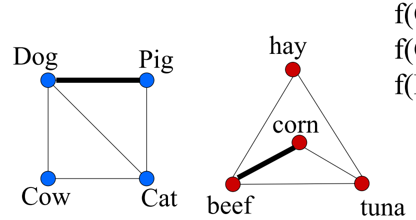
- the **edge (u,v)** is in G_1
- *iff* the **edge (f(u), f(v))** is in G_2

- If there is a one to one correspondence between the nodes of G_1 and G_2 that preserves all edge connections.

Are these Isomorphic?



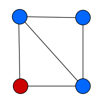
Find a Mapping

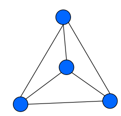


Function
f(Dog) = beef
f(Cat) = tuna
f(Cow) = hay
f(Pig) = corn

Finding the Mapping

- Not easy, can try all possible mappings
  - Roughly n! possibilities
- Can test for Invariants
  - Same number of nodes, edges
  - Same degree distributions
  - Preserves cycles, longest path, etc



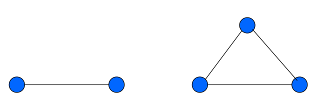


In-class Problem 1

Problem with False Proof 1

Proof (silently) assumes any 2-ended G_{n+1} can be built from a 2-ended G_n. This isn't true!

Consider the counter-example, it is two ended but I cannot construct it by adding an edge to another two ended graph.

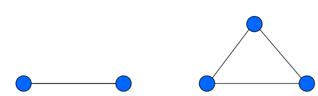


Problem with False Proof 2

After removing a vertex from G_{n+1}, the claim that G_n still has 2 vertices of degree 1 and rest degree 2

This is not true!

The same counter example works:



Revisit: Coloring with d_{max} colors

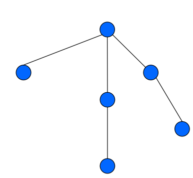
- Induction Hypothesis

  P(n) = a graph with *n* vertices and maximum degree d_{max} can be colored with d_{max} + *1* colors

- Inductive Step
  - Do you justify why your proof doesn't have the same pitfalls?

Trees

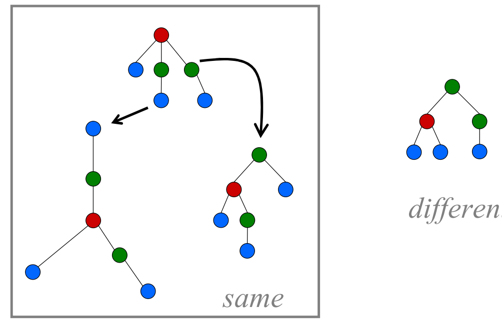
- *Definition:* A tree is simple connected graph with no cycles.



Applications of Trees

- Data structures for sorting, searching
- Spanning Trees
- Game Trees (alpha-beta trees)
- Prefix codes (Huffman encoding)
- Many algorithms based on trees (6.046)

Tree Isomorphisms



Trees

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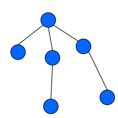


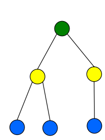
- ***Exercise***: Draw a tree with 5 vertices
- ***Question:*** How many edges does your tree have? 3,4 or 5?

Another Tree Definition

- No matter how you draw it, you get 4 edges.
- *Definition 2:* A tree is a connected graph with n vertices and $n - 1$ edges.
- In fact, *a tree is the smallest connected graph on n vertices!*

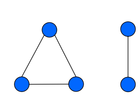
Equivalent Definitions of Trees

- A connected graph with no cycles
- A connected graph where $|E| = |V| - 1$
- A connected graph where removing any edge leaves a disconnected graph
- A graph such that there exists a unique simple path between any two vertices

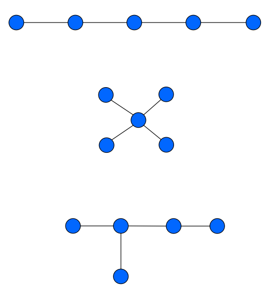


Be careful with these definitions

- What is wrong with this definition?
  - A tree is a graph with n vertices and $n - 1$ edges.
- Counter-example



Different Trees with 5 vertices



Vertex degrees

1 2 2 2 1
4 1 1 1 1
1 3 1 2 1

Sum is always 8
(2 × edges)

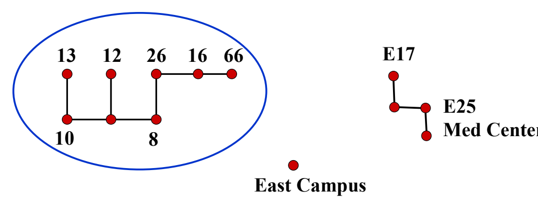
MIT Building Connections

Connectivity and Paths

- Can you get from building 10 to 36 without crossing more than 5 other buildings
  - Is there a path of length *k* from *u* to *v*?
- How many different ways are there to get from building 10 to building 36?
  - How many different paths are there from *u* to *v?*

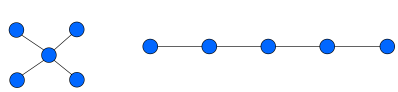
Connected Components

- Can we get from building 10 to building E17?
  - Is there a path between *u* and *v* ?
  - Are *u* and *v* connected?



Smallest Connected Graph

- MIT administration wants the number of physical connections between buildings to be minimum but still have everything connected.
  - What is the smallest connected graph I can construct? ANY TREE

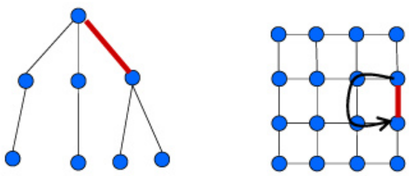


Cut Edge

- *Definition:* An edge is a **cut edge** if removing it from the graph disconnects two connected components

Problem with our smallest connected graph
- any edge disruption disconnects the graph!

Fault Tolerant Design



Robustness versus Cost

(Networks, Highways, etc)

Newark Airport Shutdown

January 9, 1995

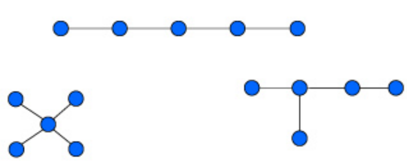
….Among the hundreds receiving citations so far are a contractor and two sub-contractors who were working on a parking garage at Newark airport when **three 26,000-volt power cables** were cut. This shut down the airport and affected travel throughout the world. Incoming domestic and international flights were waved off to other airports for nearly 24 hours, and outbound planes couldn't get off the ground, leaving gaping holes in the schedules of virtually every major airline.

In-class Problems

Extra slides

In-class Problem 0

Prove that all trees with five vertices are isomorphic to these three:



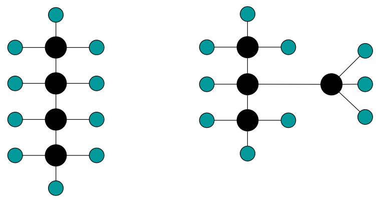
Solution to Problem 0

Use the following three facts:

1. sum of degrees = 2•edges
2. tree has $n - 1$ edges, 5 vertex tree has 4 edges
   – therefore sum of degrees is 8
3. tree is connected
   – therefore each vertex must have degree at least 1.
   – 3 degrees left over to distribute
4. The only possibilities are:
   • 4 1 1 1 1
   • 3 2 1 1 1
   • 2 2 2 1 1

Graphs with Same Degrees

- Example: isomers of butane
  - Same degree distribution, but not isomorphic



Why are some problems easier than others?

- 2 colorable (no odd cycles)
- 3 colorable (NP- complete)

- Euler circuits (connected and all even degree)
- Hamiltonian circuits (NP complete)