Mathematics for Computer Science

MIT 6.042J/18.062J

Asymptotics & Stirling's Approximation

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Integral Method

Factorial defines a product:

$$n! \;::=\; 1\cdot 2\cdot 3\cdots(n-1)\cdot n \;=\prod_{i=1}^{n} i$$

Turn product into a sum taking logs:

$$\ln(n!) = \ln(1\cdot 2\cdot 3\cdots(n-1)\cdot n)$$

$$= \ln 1 + \ln 2 + \cdots + \ln(n-1) + \ln(n)$$

$$= \sum_{i=1}^{n} \ln(i)$$

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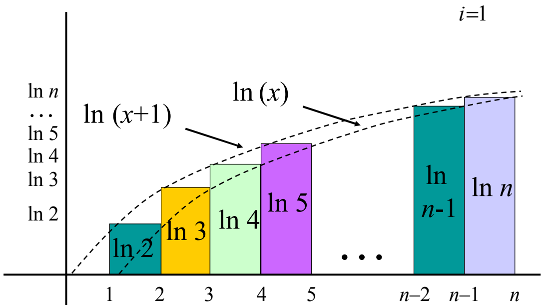
Integral Method

Integral Method to bound $\sum_{i=1}^{n} \ln i$



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Integral Method

Reminder:

$$\int \ln x\, dx = x \ln\left(\frac{x}{e}\right)$$

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Integral Method

Bounds on $\ln(n!)$

$$\int_1^n \ln(x)\,dx \;\le\; \sum_{i=1}^{n}\ln(i) \;\le\; \int_0^n \ln(x+1)\,dx$$

$$n\ln\left(\frac{n}{e}\right)+1 \;\le\; \sum_{i=1}^{n}\ln(i) \;\le\; (n+1)\cdot\ln\left(\frac{n+1}{e}\right)+1$$

$$e\cdot\left(\frac{n}{e}\right)^{n} \;\le\; n! \;\le\; e\cdot\left(\frac{n+1}{e}\right)^{n+1}$$

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Stirling's Formula

$$e\cdot\left(\frac{n}{e}\right)^{n} \le n! \le e\cdot\left(\frac{n+1}{e}\right)^{n+1}$$

So guess:

$$n! \;\approx\; \sqrt{n}\cdot\left(\frac{n}{e}\right)^{n}$$

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Stirling's Formula

A precise approximation:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

Stirling's Formula

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In-Class Problem

Problem 1

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Asymptotic Equivalence

$$\mathrm{f}(n) \sim \mathrm{g}(n)$$

$$\lim_{n\to\infty}\left(\frac{\mathrm{f}(n)}{\mathrm{g}(n)}\right) = 1$$

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Little Oh

Asymptotically smaller:

$$\mathrm{f}(n) = \mathrm{o}(\mathrm{g}(n))$$

$$\lim_{n\to\infty}\left(\frac{\mathrm{f}(n)}{\mathrm{g}(n)}\right) = 0$$

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Big Oh

Asymptotic Order of Growth:

$$\mathrm{f}(n) = \mathrm{O}(\mathrm{g}(n))$$

$$\limsup_{n\to\infty}\left(\frac{\mathrm{f}(n)}{\mathrm{g}(n)}\right) < \infty$$

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The Oh's

If $f = \mathrm{o}(g)$ or $f \sim g$ then $f = \mathrm{O}(g)$

$\lim = 0$ $\lim = 1$ $\lim < \infty$

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The Oh's

If $f = o(g)$, then $g \neq O(f)$

$$\lim \frac{f}{g} = 0 \qquad \lim \frac{g}{f} = \infty$$

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Big Oh

Equivalently,

$$f(n) = O(g(n))$$

$$\exists c, n_0 \geq 0 \ \forall n \geq n_0 \ \ |f(n)| \leq c \cdot g(n)$$

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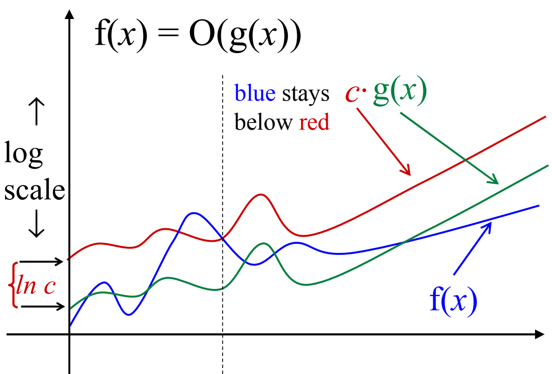
Big Oh



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Little Oh

Lemma: $x^a = o(x^b)$ for $a < b$

Proof: $\frac{x^a}{x^b} = \frac{1}{x^{b-a}}$ and $b - a > 0$.

So as $x \to \infty$,

$$x^{b-a} \to \infty \text{ and } \frac{1}{x^{b-a}} \to 0.$$

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Little Oh

Lemma: $\ln x = o(x^\delta)$ for $\delta > 0$.

Proof: $\frac{1}{z} \leq z$ for $z \geq 1$.

$$\int_1^z \frac{1}{z} dz \leq \int_1^z z\, dz$$

$$\ln z \leq \frac{z^2}{2}$$

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Little Oh

Lemma: $\ln x = o(x^\delta)$ for $\delta > 0$.

Proof: $\ln z \leq \frac{z^2}{2}$ Let $z ::= \sqrt{x^\varepsilon}$

$$\frac{\varepsilon \ln x}{2} \leq \frac{x^\varepsilon}{2}$$

$$\ln x \leq \frac{x^\varepsilon}{\varepsilon} = o(x^\delta) \quad \text{for } \delta > \varepsilon.$$

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Albert R. Meyer:
Spring '02: Added after lecture.

...ttle Oh

Other proofs:
L'Hopital's Rule,
McLaurin Series
(see a Calculus text)

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Theta

Same Order of Growth:

$$f(n) = \Theta(g(n))$$

$f(n) = O(g(n))$ and $g(n) = O(f(n))$

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In-Class Problem

Problems 2 & 3

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Big Oh Mistakes

False Lemma: $\sum_{i=1}^{n} i = O(n)$

Of course really $\sum_{i=1}^{n} i = \theta(n^2)$

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Big Oh Mistakes

False Lemma: $\sum_{i=1}^{n} i = O(n)$

False Proof:

$0 = O(1),\ 1 = O(1),\ 2 = O(1), \ldots$

So each $i = O(1)$. So

$$\sum_{i=1}^{n} i = O(1) + O(1) + \cdots + O(1)$$
$$= n \cdot O(1) = O(n).$$

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In-Class Problem

Problem 4

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