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More on the anti-automorphism of the Steenrod algebra

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The relations of Barratt and Miller are shown to include all relations among the elements $P^r \chi P^{p^r-i}$ in the mod $p$ Steenrod algebra, and a minimal set of relations is given.

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1 Introduction

Milnor [4] observed that the mod 2 Steenrod algebra $\mathcal{A}$ forms a Hopf algebra with commutative diagonal determined by

\begin{equation}
\Delta Sq^n = \sum_i Sq^i \otimes Sq^{n-i}.
\end{equation}

This allowed him to interpret the Cartan formula as the assertion that the cohomology of a space forms a module-algebra over $\mathcal{A}$. The anti-automorphism $\chi$ in the Hopf algebra structure, defined inductively by

\begin{equation}
\chi Sq^0 = Sq^0, \quad \sum_i Sq^i \chi Sq^{n-i} = 0 \quad \text{for} \quad n > 0,
\end{equation}

has a topological interpretation too: If $K$ is a finite complex then the homology of the Spanier-Whitehead dual $DK_+$ of $K_+$ is canonically isomorphic to the cohomology of $K$. Under this isomorphism the left action by $\theta \in \mathcal{A}$ on $H^*(K)$ corresponds to the right action of $\chi \theta \in \mathcal{A}$ on $H_*(DK_+)$. In 1974 Davis [3] proved that sometimes much more efficient ways exist to compute $\chi Sq^n$; for example

\begin{equation}
\chi Sq^{2r-1} = Sq^{2r-1} \chi Sq^{2r-1},
\end{equation}

\begin{equation}
\chi Sq^{2r-r-1} = Sq^{2r-1-r} \chi Sq^{2r-1-r} + Sq^{2r-1} \chi Sq^{2r-1-r-1}.
\end{equation}

Similarly, Straffin [6] proved that if $r \geq 0$ and $b \geq 2$ then

\begin{equation}
\sum_i Sq^{2i} \chi Sq^{2(b-i)} = 0.
\end{equation}
Both authors give analogous identities among reduced powers and their images under $\chi$ at an odd prime as well. Further relations among the Steenrod squares and their conjugates appear in these articles and elsewhere (e.g. \([5]\)).

Barratt and Miller \([1]\) found a general family of identities which includes \((3), (4),\) and \((5),\) and their odd-prime analogues, as special cases. We state it for the general prime. When $p = 2$, $P^n$ denotes $\text{Sq}^n$. Let $\alpha(n)$ denote the sum of the $p$-adic digits of $n$.

**Theorem 1.1** \([1, 2]\) For any integer $k$ and any integer $l \geq 0$ such that $pl - \alpha(l) < (p - 1)n$,

\[
\sum_i \binom{k - i}{l} P^i \chi P^{n-i} = 0.
\]

The relations defining $\chi$ occur with $l = 0$. Davis’s formulas (for $p = 2$) are the cases in which $(n, l, k) = (2^r - 1, 2^r - 1 - 2, 2 - 1)$ or $(n, l, k) = (2^r - r - 1, 2^r - 2, 2^r - 2)$. Straffin’s identities (for $p = 2$) occur as $(n, l, k) = (2^r b, 2^r - 1, -1)$.

Since $\binom{k+1-i}{l} - \binom{k-i}{l-1} = \binom{k-i}{l}$, the cases $(l, k + 1)$ and $(l, k)$ of (6) imply it for $(l - 1, k)$. Thus the relations for $l = \phi(n) - 1$, where

\[
\phi(n) = 1 + \max \{ j : pj - \alpha(j) < (p - 1)n \},
\]

imply all the rest. Here we have adopted the notation $\phi(n)$ used in \([2]\); we note that it is not the Euler function $\varphi(n)$.

When $p = 2$, $\phi(2^r - 1) = 2^{r-1}$ and $\phi(2^r - r - 1) = 2^{r-1} - 1$, so Davis’s relations are among these basic relations.

Two questions now arise. To express them uniformly in the prime, let $\mathcal{P}$ denote the algebra of Steenrod reduced powers (which is the full Steenrod algebra when $p = 2$), but assign $P^n$ degree $n$. Write

\[
V_n = \text{Span}\{P^i \chi P^{n-i} : 0 \leq i \leq n\} \subseteq \mathcal{P}^n.
\]

It is natural to ask:
– Are there yet other linear relations among the $n + 1$ elements $P^i \chi P^{n-i}$ in $\mathcal{P}^n$?
– What is a basis for $V_n$?

We answer these questions in Theorem 1.4 below.

Write $e_i, 0 \leq i \leq n$, for the $i$th standard basis vector in $\mathbb{F}_p^{n+1}$. 

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Proposition 1.2  For any integers \( l, m, n \), with \( 0 \leq l \leq n \),

\[
\left\{ \sum_i \binom{k - i}{l} e_i : m \leq k \leq m + l \right\}
\]

is linear independent in \( \mathbb{F}_p^{n+1} \).

Proposition 1.3  The set

\[
\{ P^i \chi P^{n-i} : \phi(n) \leq i \leq n \}
\]

is linearly independent in \( \mathcal{P}^n \).

Define a linear map

\[
\mu : \mathbb{F}_p^{n+1} \to \mathcal{P}^n, \quad \mu e_i = P^i \chi P^{n-i} .
\]

Theorem 1.1 implies that if \( l = \phi(n) - 1 \) the elements in (8) lie in \( \ker \mu \), so Propositions 1.2 and 1.3 imply that (8) with \( l = \phi(n) - 1 \) is a basis for \( \ker \mu \) and that (9) is a basis for \( V_n \subseteq \mathcal{P}^n \). Thus:

**Theorem 1.4**  Any \( \phi(n) \) consecutive relations from the set (6) with \( l = \phi(n) - 1 \) form a basis of relations among the elements of \( \{ P^i \chi P^{n-i} : 0 \leq i \leq n \} \). The set \( \{ P^i \chi P^{n-i} : \phi(n) \leq i \leq n \} \) is a basis for \( V_n \).

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2 Independence of the relations

We wish to show that (8) is a linearly independent set. Regard elements of \( \mathbb{F}_p^{n+1} \) as column vectors, and arrange the \( l + 1 \) vectors in (8) as columns in a matrix, which we claim is of rank \( l + 1 \). The top square portion is the mod \( p \) reduction of the \( (l + 1) \times (l + 1) \) integral Toeplitz matrix \( A_l(m) \) with \( (i,j) \)th entry

\[
\binom{m + j - i}{l}, \quad 0 \leq i, j \leq l .
\]

**Lemma 2.1**  \( \det A_l(m) = 1 \).
**Proof.** By induction on \( m \). Since \((-1)_{\ell}^{1} = (-1)^{l}\) and \((-1)^{l+j}_{\ell} = 0\) for \( 0 < j \leq l \), \( A_{l}(-1) \) is lower triangular with determinant \((-1)^{l+1}_{l+1} = 1\). Now we note the identity

\[
BA_{l}(m) = A_{l}(m + 1)
\]

where

\[
B = \begin{bmatrix}
\binom{l+1}{1} & -\binom{l+1}{2} & \ldots & -\binom{l+1}{l+1} \\
1 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1
\end{bmatrix}.
\]

The matrix identity is an expression of the binomial identity

\[
\sum_{k} (-1)^{k} \binom{l+1}{k} \binom{n-k}{l} = 0
\]

(taking \( n = m + 1 - j \) and \( k = j + 1 \)). Since \( \det B = 1 \), the result follows for all \( m \in \mathbb{Z} \). \( \square \)

For completeness, we note that (11) is the case \( m = l + 1 \) of the equation

\[
\sum_{k} (-1)^{k} \binom{m}{k} \binom{n-k}{l} = \binom{n-m}{l-m}.
\]

To prove this formula, note that the defining identity for binomial coefficients implies the case \( m = 1 \), and also that both sides satisfy the recursion \( C(l, m, n) - C(l, m, n-1) = C(l, m+1, n) \).

## 3 Independence of the operations

We will prove Proposition 1.3 by studying how \( P^i \chi^{p_{n}-i} \) pairs against elements in \( \mathcal{P}_{s} \), the dual of the Hopf algebra of Steenrod reduced powers. According to Milnor [4], with our grading conventions

\[
\mathcal{P}_{s} = \mathbb{F}_{p}[\xi_{1}, \xi_{2}, \ldots], \quad |\xi_{j}| = \frac{p^{j} - 1}{p - 1},
\]

and

\[
\Delta \xi_{k} = \sum_{i+j=k} \xi_{i}^{p^{j}} \otimes \xi_{j}.
\]
For any finitely nonzero sequence of nonnegative integers \( R = (r_1, r_2, \ldots) \) write \( \xi^R = \xi_1^{r_1} \xi_2^{r_2} \cdots \) and let \( \| R \| = r_1 + p r_2 + p^2 r_3 + \cdots \) and
\[
| R | = | \xi^R | = r_1 + \left( \frac{p^2 - 1}{p - 1} \right) r_2 + \left( \frac{p^3 - 1}{p - 1} \right) r_3 + \cdots.
\]

The following clearly implies Proposition 1.3.

**Proposition 3.1** For any integer \( n > 0 \) there exist sequences \( R_{n,j} \), \( 0 \leq j \leq n - \phi(n) \), such that \( \| R_{n,j} \| = n \) and
\[
\langle P^i \chi^{p^n-i}, \xi^{R_{n,j}} \rangle = \begin{cases} 
\pm 1 & \text{for } i = n - j \\
0 & \text{for } i > n - j.
\end{cases}
\]

The starting point in proving this is the following result of Milnor.

**Lemma 3.2** ([4], Corollary 6) \( \langle \chi^{p^n}, \xi^R \rangle = \pm 1 \) for all sequences \( R \) with \( \| R \| = n \).

In the basis of \( \mathcal{P} \) dual to the monomial basis of \( \mathcal{P}_* \), the element corresponding to \( \xi_1^j \) is \( P^j \). Since the diagonal in \( \mathcal{P}_* \) is dual to the product in \( \mathcal{P} \), it follows from (13) and Lemma 3.2 that
\[
\langle P^i \chi^{p^n-i}, \xi^R \rangle = \begin{cases} 
\pm 1 & \text{for } i = \| R \| \\
0 & \text{for } i > \| R \|.
\end{cases}
\]

So we wish to construct sequences \( R_{n,j} \), for \( \phi(n) \leq j \leq n \), such that \( \| R_{n,j} \| = n \) and \( \| R_{n,j} \| = j \). We deal first with the case \( j = \phi(n) \).

**Proposition 3.3** For any \( n \geq 0 \) there is a sequence \( M = (m_1, m_2, \ldots) \) such that

1. \( | M | = n \),
2. \( 0 \leq m_i \leq p \) for all \( i \), and
3. If \( m_j = p \) then \( m_i = 0 \) for all \( i < j \).

For any such sequence, \( \| M \| = \phi(n) \).

**Proof.** Give the set of sequences of dimension \( n \) the right-lexicographic order. We claim that the maximal sequence satisfies the hypotheses.

Suppose that \( R = (r_1, r_2, \ldots) \) does not satisfy the hypotheses. If \( r_1 > p \) then the sequence \( (r_1 - (p + 1), r_2 + 1, r_3, \ldots) \) is larger. If \( r_j > p \), with \( j > 1 \), then the sequence \( (r_1, \ldots, r_{j-2}, r_{j-1} + p, r_j - (p + 1), r_{j+1} + 1, r_{k+2}, \ldots) \) is larger. This proves (2). To prove (3), suppose that \( r_j = p \) with \( j > 1 \), and suppose that some earlier entry is nonzero. Let \( i = \min \{ k : r_k > 0 \} \). If \( i = 1 \), then the sequence \( (r_1 - 1, r_2, \ldots, r_{j-1}, 0, r_{j+1} + \ldots) \) is larger.
The function in fact determine properties (1)–(3) of Proposition 3.3.

Let \( M \) be a sequence satisfying (1)–(3), and write \( l = \|M\| - 1 \). To see that \( l = \phi(n) - 1 \) we must show that

\[
(14) \quad p(l + 1) - \alpha(l + 1) \geq (p - 1)n
\]

and

\[
(15) \quad pl - \alpha(l) < (p - 1)n.
\]

The excess \( e(R) \) is the sum of the entries in \( R \), so that \( p\|R\| - e(R) = (p - 1)|R| \).

The \( p \)-adic representation of a number minimizes excess, so for any sequence \( R \) we have \( e(R) \geq \alpha(||R||) \) and hence \( p\|R\| - \alpha(||R||) \geq (p - 1)|R| \); so (14) holds for any sequence.

To see that (15) holds for \( M \), let \( j = \min\{i : m_i > 0\} \), so that \( (p - 1)n = (p' - 1)m_j + (p' + 1 - 1)m_{j+1} + \cdots \) and \( l + 1 = p' - 1m_j + p'm_{j+1} + \cdots \). The hypotheses imply that \( l \) has \( p \)-adic expansion

\[
(1 + \cdots + p^{j-2})(p - 1) + p^{j-1}(m_j - 1) + p'm_{j+1} + \cdots,
\]

so

\[
\alpha(l) = (j - 1)(p - 1) + (m_j - 1) + m_{j+1} + \cdots
\]

from which we deduce

\[
pl - \alpha(l) = (p - 1)(n - j) < (p - 1)n.
\]

This completes the proof of Proposition 3.3. \( \square \)

**Corollary 3.4** The function \( \phi(n) \) is weakly increasing.

**Proof.** Let \( M \) be a sequence satisfying the conditions of Proposition 3.3, and note that the sequence \( R = (1, 0, 0, \ldots) + M \) has \( |R| = n + 1 \) and \( \|R\| = \|M\| + 1 = \phi(n) + 1 \).

If \( p \) does not occur in \( M \), then \( R \) satisfies the hypotheses of the proposition (in degree \( n + 1 \)) and hence \( \phi(n) \leq \phi(n + 1) \). If \( p \) does occur in \( M \), then the moves described above will lead to a sequence \( M' \) satisfying the hypotheses. None of the moves decrease \( \| - \| \), so \( \phi(n) \leq \phi(n + 1) \). \( \square \)

**Remark 3.5** Properties (1)–(3) of Proposition 3.3 in fact determine \( M \) uniquely.

**Proof of Proposition 3.1.** Define \( R_{n,\phi(n)} \) to be a sequence \( M \) as in Proposition 3.3. Then inductively define

\[
R_{n,j} = (1, 0, 0, \ldots) + R_{n-1,j-1} \quad \text{for} \quad \phi(n) < j \leq n.
\]

This makes sense by monotonicity of \( \phi(n) \), and the elements clearly satisfy \( |R_{n,j}| = n \) and \( \|R_{n,j}\| = j \). This completes the proof. \( \square \)
References


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