Transverse momentum distributions inside the nucleon from lattice QCD

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Abstract. We study transverse momentum dependent parton distribution functions (TMDs) with non-local operators in lattice QCD, using MILC/LHPC lattices. Results obtained with a simplified operator geometry show visible dipole deformations of spin-dependent quark momentum densities.

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INTRODUCTION

Transverse momentum dependent parton distribution functions (TMDs) describe the longitudinal and transverse motion of quarks (or gluons) in a nucleon with large momentum \( \mathbf{P} = (0, 0, P_z) \). The quark momentum \( \mathbf{k} \) in terms of light cone coordinates \( k^\pm \equiv (k^0 \pm k^3)/\sqrt{2}, \mathbf{k}_\perp = (k_x, k_y) \) scales like \( k^+: k^\perp: P^+: 1: (P^+)^{-1} \) with the large momentum component \( P^+ \) of the nucleon. TMDs resolve the dependence on \( x \equiv k^+/P^+ \) and transverse momentum \( \mathbf{k}_\perp \), see illustration Fig. 1 a). In spin-polarized channels at leading twist, TMDs encode dipole- or quadrupole-shaped deformations of the nucleon in the \( \mathbf{k}_\perp \)-plane, see, e.g., Fig. 1 b) for a result from first explorative lattice QCD calculations [1, 2, 3]. Our studies have been motivated by a history of successful lattice computations of \( x \)-moments of generalized parton distributions (GPDs), providing images of the nucleon in the impact parameter, \( \mathbf{b}_\perp \)-, plane, see [4] for a review. A remaining theoretical problem concerns the precise form of the correlator defining TMDs in the continuum, see [5, 6] and references therein. In its basic form, it is given by [7]

\[
\Phi_q^{[\Gamma]}(x, \mathbf{k}_\perp; P, S; \mathcal{C}) \equiv \int dk^- \int \frac{d^4l}{(2\pi)^4} \, e^{-ik\cdot l} \left\{ \frac{1}{2} (P, S | \bar{q}(l) \Gamma \mathcal{U} [\mathcal{C}] q(0) | P, S) \right\}_{k^+=xP^+} \tilde{\Phi}_q^{[\Gamma]}(l, P, S; \mathcal{C})
\]

\[
= \frac{1}{P^+} \int \frac{d(l \cdot P)}{2\pi} \, e^{-i(l \cdot P)x} \int \frac{d^2k_\perp}{(2\pi)^2} \, e^{i\mathbf{k}_\perp \cdot \mathbf{k}_\perp} \tilde{\Phi}_q^{[\Gamma]}(l, P, S; \mathcal{C}) \bigg|_{l^+ = 0} (1)
\]
where $\Gamma$ is a Dirac matrix. The Wilson line $\mathcal{W}[\mathcal{G}]$ running along a continuous path $\mathcal{C}_l$ from $l$ to $0$ ensures gauge invariance of the expression. For the SIDIS and Drell-Yan scattering process, the Wilson line extends to infinity along a direction $v$ that needs to be chosen (almost) lightlike, such that the cross section factorizes into hard, perturbative parts and soft contributions, see, e.g., Ref. [8]. Based on its symmetry transformation properties, the above correlator can be parametrized in terms of TMDs [9, 10, 11], e.g.,

$$2\rho^{(q)}_{TL} = \Phi_q^{[\gamma^+ + \lambda \gamma^5]} = f_{1,q} + \lambda \frac{k_\perp \cdot S_\perp}{m_N} g_{1T,q} + \left[ \frac{S_j \epsilon_{ji}^k k_i}{m_N} f_{1T,q} \right]_{\text{out}},$$

(2)

Here $\lambda$ is the longitudinal quark polarization, and $\Lambda$ and $S_\perp$ are longitudinal and transverse nucleon polarization, respectively. The leading-twist TMDs $f_{1,q}$, $g_{1T,q}$, $f_{1T,q}$ are real-valued functions of $x$ and $k_\perp^2$.

**STRAIGHT LINK TMDS FROM THE LATTICE**

In light of the uncertainties about the precise form of the continuum correlator, and to develop our methods, our first lattice studies employ a simple operator geometry that does not relate to a specific scattering process: We connect the quark fields with a direct, straight Wilson line. For the resulting “process-independent” TMDs, the “naively time-reversal odd” functions, such as the Sivers function $f_{1T,q}$ in Eq. (2), vanish exactly.

In our approach, we calculate matrix elements $\langle P,S|O|P,S\rangle$ from ratios of three- and two-point functions using the same techniques as GPD calculations by the LHP collaboration in Ref. [12]. We also use the same sequential propagators and quark propagators, calculated by LHPC with domain-wall valence fermions on top of asqtad-improved staggered MILC gauge configurations [13, 14, 15] with $2+1$ quark flavors at a lattice spacing $a \approx 0.12$ fm. The difference with respect to GPD calculations is that we directly insert the non-local operator $O \equiv \bar{q}(l)\Gamma\mathcal{W}[\mathcal{G}]q(0)$ in our three-point function. The Wilson line $\mathcal{W}[\mathcal{G}]$ is approximated as a step-like product of HYP-smeared link-variables as illustrated in Fig. 2 a). To renormalize, we use the renormalization condition
FIGURE 2.  a) Representation of a straight Wilson line (dashed line) as a step-like product of link variables. b) Amplitude $A_2(l^2,0)$ for up quarks at a pion mass $m_\pi \approx 500$ MeV, using straight gauge links.

employed in Refs. [16, 17], see Ref. [2] for details. At present, we lack an interpretation of this renormalization condition in terms of a physical scale in the context of TMD factorization.

The connection between the matrix elements $\Phi^{[F]}$ and TMDs is established through a parametrization in terms of Lorentz-invariant amplitudes $\tilde{A}_i(l^2,l \cdot P)$. For straight Wilson lines, we obtain in analogy to the parametrization in terms of amplitudes $A_i(k^2,k \cdot P)$ [9]:

$$\Phi^{[F^\mu]} = 2 P^\mu \tilde{A}_2 + 2 i m_N^2 l^\mu \tilde{A}_3,$$

$$\Phi^{[F^\mu T^5]} = -2 m_N S^\mu \tilde{A}_6 - 2 i m_N P^\mu (l \cdot S) \tilde{A}_7 + 2 m_N^3 l^\mu (l \cdot S) \tilde{A}_8.$$

The TMDs are then obtained by

$$f_1(x,k^2_\perp) = 2 \int \mathcal{M} \tilde{A}_2(l^2,l \cdot P), \quad g_{1T}(x,k^2_\perp) = 4 m_N^2 \partial_{k^2_\perp} \int \mathcal{M} \tilde{A}_7(l^2,l \cdot P).$$

Our Euclidean lattice approach is restricted to the determination of amplitudes $\tilde{A}_i$ for $l^0 = -i l_4 = 0$, i.e., to the region $l^2 < 0$, $|l \cdot P| \leq \sqrt{-l^2} |P|$, where $P$ is the selected three-momentum of the nucleon on the lattice. Nevertheless, $x$-integrated TMDs and densities are directly accessible: Integrating Eq. (1) with respect to $x$ removes $\mathcal{M}$ and sets $l \cdot P$ to zero. Correspondingly, the $x$-integral of, e.g., $f_1$ becomes $\int_{-1}^{1} dx f_1(x,k^2_\perp) = f_1^{[1]}(k^2_\perp) = 2 \mathcal{M} \tilde{A}_2(l^2,0)$.

In Fig. 2 b), we show unrenormalized lattice data for $\tilde{A}_2(l^2,0)$ as open symbols and renormalized data as solid points. Studies of the Wilson line operator at different lattice spacings indicate that discretization errors become large for link lengths $l \lesssim 0.25$ fm, i.e., in the gray shaded region of Fig. 2 b). The curve and statistical error band correspond to a Gaussian fit to the renormalized data in the range $\sqrt{-l^2} \geq 0.25$ fm. Similar fits for $\tilde{A}_7$ enable us to calculate the “worm-gear” function $g_{1T}^{[1]}$, and correspondingly, the dipole deformed $x$-integrated density $\rho_{TL}^{(q)}[1]$ defined in Eq. (2) and shown in Fig. 1 b). For the Gaussian distributions, the strength of the deformation in
Fig. 1 b) can be expressed as an average transverse quark momentum shift [1]

\[
\langle k_x \rangle_{TL} = m_N \int \frac{d^2 k_{\perp} k_{\perp}^2 / (2m_N^2) \ g_{1T}^{[1]}(k_{\perp})}{\int d^2 k_{\perp} f_{1T}^{[1]}(k_{\perp})} = \left\{ \begin{array}{l} 67(5) \text{ MeV (up)} \\ -30(5) \text{ MeV (down)} \end{array} \right.,
\]

which we find to be rather insensitive to the renormalization condition used. Reference [18] reveals a remarkable similarity of our results for \(\langle k_x \rangle_{TL}\) with a light-cone constituent quark model [19], despite the unphysically large quark masses employed in our lattice calculation.

An ongoing project with staple-shaped gauge links can potentially address TMDs specific to SIDIS or the Drell-Yan process, including T-odd functions responsible for single-spin asymmetries.

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**REFERENCES**