RISK AVERSION IN THE WARRANT MARKETS

by

Herbert Frazer Ayres

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF MASTER OF SCIENCE

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

1963

Signature of Author

School of Industrial Management

Certified by

Faculty Advisor of the Thesis
Professor Philip Franklin  
Secretary of the Faculty  
Massachusetts Institute of Technology  
Cambridge 39, Massachusetts  

Dear Professor Franklin:  

In accordance with the requirements for graduation, I herewith submit a thesis entitled "Risk Aversion in the Warrant Markets".

I wish to express my thanks to my Faculty Advisor, Professor Albert Ando, for his continued guidance and assistance during the course of this research, to Professor Paul H. Cootner for many discussions and references to unpublished literature, and to Professor Martin Greenberger whose seminar first drew my attention to the application of quantitative techniques to the theory of investments.

Sincerely yours,

Herbert F. Ayres
RISK AVERTION IN THE WARRANT MARKETS

by

Herbert Frazer Ayres

Submitted to the School of Industrial Management
on April 30, 1963 in partial fulfillment
of the requirements for the degree
of Master of Science

Although men of science have been concerned with the
general problem of decision in the face of risk or uncertainty since the time of Laplace, quantitative results are
only now being achieved in the area of practical, general
investment decision. Two formidable obstacles exist that
have hampered the rate of progress. On one hand, no method
has been found to extract even the form, let alone numerical
values, of investor or market preferences outside of the
context of the marketplace. On the other hand, the number
of investment opportunities open to a given individual is
huge; a single investment position may consist of a fraction
of a very large number of individual opportunities. These
sum in no simple way to the composite position. The statistical properties of the whole depend upon the interrelationships between all the component parts. The calculation of
an optimum position for an individual, even if his objectives
were known exactly, is well beyond the power of today's computational machinery if all investment opportunities are
allowed.

Extensive normative theory has been developed for
economic decisions; the general theory of Tobin and the
theory of optimum portfolio selection of Markowitz are in
particular pertinent to security investment. In 1961 a practical application of this theory was made by Farrar to the market
behavior of investment company portfolio managers. The predictions of the optimal theory and the market behavior of the
investment fund managers are quite close to each other. The
tory may be taken as a first order description of risk
aversion behavior in the stock markets at least for this class
of investor as a result.

It is the purpose of this paper to investigate risk aversion
in the warrant markets in view of this background. Warrants
are of interest to the speculator because of their great
volatility; they are of interest to the student of market
action because, as I hope to show, their study provides direct insight into at least one aspect of risk aversion behavior. The formulation of a model of warrant price action is a prelude to the study of convertible preferred stocks and bonds. Application of the methods of this investigation may provide direct insight into risk aversion in these latter markets, where investors may have risk preferences more comparable to stock market investors.

The research method employed here is to construct a normative model of warrant prices based upon the above referenced work and upon statistical studies of security prices and price indices. The test of the theory developed is done by examining how much of the variance of rate of expected return on investment in warrants is explained by the derived hypothesis. Two points of view are taken. One results in a residual variance of one seventh of the original; the other yields a residual variance of 3%.

Major conclusions are the following: There exists an equilibrium in indifference between a warrant and its own common stock. This equilibrium permits one to study the risk aversion pertinent to that warrant directly without taking into account the vast covariance with all other members of the opportunity set. As Farrar found that growth fund managers are less risk averting than balanced fund managers; so it is found here that warrant investors are less risk averting than growth fund managers. The form the risk aversion takes is that of demanding an expected exponential rate of growth on the money invested. It is inferred that the merit of an investment is judged by such expected exponential rates of growth in both stock markets and warrant markets. Investment decisions are invariant with the time horizon of the decision.

Thesis Adviser: Albert Ando
Title: Associate Professor of Economics
# TABLE OF CONTENTS

Section I Introduction to the Problem

1.0 Definition of a Warrant and an Introduction to its Market Value ............................................. 1

2.0 A Plausibility Argument for Risk Aversion ....................................................................................... 6

3.0 Summary of Section I and an Outline of the Rest of this Paper ..................................................... 9

3.1 Outline of the Remainder of this Paper ............................................................................................ 10

Section II Development of a Model of Stock Prices ............................................................................. 13

1.0 The Lack of Serial Correlation ....................................................................................................... 13

2.0 The Concept of a Generating Process ............................................................................................... 14

3.0 The Ramifications of the Central Limit Theorem .......................................................................... 14

4.0 Normal or Lognormal? ...................................................................................................................... 14

5.0 The Exact form of the Model of Stock Prices .................................................................................. 15

6.0 Other Models .................................................................................................................................. 16

7.0 Summary ......................................................................................................................................... 17

Section III A Normative Model of Stock Market Investor Behavior in the Face of Risk and one Experimental Verification ................................................................. 18

1.0 Opportunities and Preferences in Quantitative Terms .................................................................. 19

1.1 Quantitative Preference, the Concept of Utility ......................................................................... 20

2.0 Normative Theory of Investment .................................................................................................... 23

3.0 Specific Forms of the Considerations of 2.0 .................................................................................. 25

4.0 The Markowitz Theory of Optimum Portfolio Selection ................................................................ 27

5.0 An Experimental Application of Portfolio Theory ......................................................................... 32
5.1 Experimental Test of the Model of Farrar 34
5.2 Discussion of Farrar's Utility Function 36
5.3 The Ramifications of the Lognormal Model of Stock Prices of Section II Combined with Farrar's Utility Function 37
6.0 Insight into the Structure of the Investment Decision 38
7.0 Limitations on the Scope of the Normative Theories of Markowitz and Farrar 40
8.0 Summary 41
Section IV The Mathematical Model of Warrant Prices, Derivation and Discussion 43
1.0 The Mathematical Model of Future Warrant Values 43
2.0 The Mathematical Model of Present Warrant Prices 44

2.1 The Elimination of Time from the Model 47
2.2 The Final Model 51
3.0 General Discussion of Warrant Behavior 52
4.0 The General Behavior of the Model of Equation IV-7 58
5.0 Summary 62
Section V The Avoidance of Warrant Covariance by a Market Equilibrium Argument 64
1.0 The Assumptions 64
2.0 The Inferences 66
3.0 The Covariance Eliminating Argument 67

3.1 Anticipations about A 70
Section V The Avoidance of Warrant Covariance by a Market Equilibrium Argument [continued]

4.0 The Experiment to Test the Theory

4.1 A Refinement

5.0 Summary

Section VI A Summary of the Experimental Test of the Theory

1.0 Correlation of Return and Risk

2.0 The Results of the V Test

3.0 The Results of the VV Test

4.0 Anticipated vs Actual Values of A

5.0 Discussion of the Magnitudes Risk and Return

6.0 Summary

Section VII Details of the Experimental Work

1.0 Choice of the Warrants for Study

2.0 The Development of Estimators for the Parameters of the Common Stock Model $u$ and $\sigma$

2.1 The Determination of the Estimator of $u$

2.1.1 The Lower Bound of $S/ae$

2.1.2 Correlation of $\ln C_n/C_0$

2.1.3 The Stationarity Hypothesis

2.1.4 The Preliminary $u$ Estimation Hypothesis

2.1.5 Support for the Preliminary $u$ Estimation Hypothesis

2.1.6 The Crash of '62

2.1.7 $u$ Expectation Changes in the crash
Section VII Details of the Experimental Work [continued]

2.1.8 The Final u Estimation Hypothesis

2.2 The Estimator of \( \sigma^2, s^2 \)

3.0 Obtaining Warrant t's and E's

4.0 Obtaining Values of W and \( S_0 \) from the Market Place

5.0 The Calculation of the \( r_i \)’s

6.0 The Estimation of the Dividend Rates \( d \)

7.0 The Estimation of the Leverages \( L_i \)

8.0 The Calculation of \( A^* \)

9.0 Calculation of \( V \)

10.0 The Calculation of \( VV \)

11.0 Summary

Section VII Summary and Conclusions

1.0 Conclusions

2.0 Suggestions for Future Work

Bibliography

Appendix I Possible Weaknesses of the Models

Appendix II Approximation of Farrar's mean and Variance by the Mean and Variance of the Logarithm of Price

Appendix III Parameters of the Common Stocks

Appendix IV Parameters of the Warrants
### TABLE OF FIGURES

Figures are numbered by section.

**Section I**

<table>
<thead>
<tr>
<th>Figure I-1</th>
<th>Warrant Conversion Value</th>
<th>3</th>
</tr>
</thead>
</table>

**Section III**

<table>
<thead>
<tr>
<th>Figure III-1</th>
<th>Quantitative Good and Bad</th>
<th>19</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.</td>
<td>Best Opportunity Curve</td>
<td>20</td>
</tr>
<tr>
<td>3.</td>
<td>Directions of Indifference</td>
<td>21</td>
</tr>
<tr>
<td>4.</td>
<td>Indifference Curve</td>
<td>22</td>
</tr>
<tr>
<td>5.</td>
<td>Indifference Curves</td>
<td>22</td>
</tr>
<tr>
<td>6.</td>
<td>Normative Decision</td>
<td>24</td>
</tr>
<tr>
<td>7.</td>
<td>Irrational Indifference</td>
<td>24</td>
</tr>
<tr>
<td>8.</td>
<td>Indifference Curves</td>
<td>26</td>
</tr>
<tr>
<td>9.</td>
<td>Investment Opportunities 100% Positive Correlation</td>
<td>28</td>
</tr>
<tr>
<td>10.</td>
<td>Investment opportunities, Efficient Locus</td>
<td>29</td>
</tr>
<tr>
<td>11.</td>
<td>Markowitz Efficient Locus</td>
<td>30</td>
</tr>
<tr>
<td>12.</td>
<td>Farrar Efficient Locus</td>
<td>33</td>
</tr>
<tr>
<td>13.</td>
<td>Farrar's Chart II</td>
<td>36</td>
</tr>
</tbody>
</table>

**Section IV**

<table>
<thead>
<tr>
<th>Figure IV-1</th>
<th>The Integration of Equation IV-1</th>
<th>45</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.</td>
<td>Future and Present Warrant Prices</td>
<td>44</td>
</tr>
<tr>
<td>3.</td>
<td>Relation of lnW to lnW_e(t)</td>
<td>46</td>
</tr>
<tr>
<td>4.</td>
<td>lnW_e(t) vs time</td>
<td>49</td>
</tr>
<tr>
<td>5.</td>
<td>Temporal Equilibrium</td>
<td>50</td>
</tr>
<tr>
<td>6.</td>
<td>Expiration before Temporal Equilibrium</td>
<td>52</td>
</tr>
<tr>
<td>7.</td>
<td>The W-S Plane</td>
<td>54</td>
</tr>
<tr>
<td>8.</td>
<td>The Normalized W-S Plane</td>
<td>55</td>
</tr>
<tr>
<td>9.</td>
<td>Alleghany Corp. Scatter</td>
<td>56</td>
</tr>
<tr>
<td>10.</td>
<td>Normalized W-S Plane Cross Section</td>
<td>57</td>
</tr>
<tr>
<td>11.</td>
<td>W-S Plane for General Acceptance Warrant vs Common Stock</td>
<td>59</td>
</tr>
<tr>
<td>12.</td>
<td>The General Behavior of Equation IV-7</td>
<td>60</td>
</tr>
<tr>
<td>13.</td>
<td>General Behavior of Equation IV-7 vs Cross Section Data</td>
<td>61</td>
</tr>
<tr>
<td>14.</td>
<td>Normalized W-S Plane Motion, 14 Warrants vs Equation IV-7 in the '62 Crash</td>
<td>61</td>
</tr>
</tbody>
</table>
Section V
  Figure V-
  1. lnW vs lnS
  2. Equilibrium in Indifference
     67
     68

Section VI
  Figure VI-
  1. Scatter of r on Ls
  2. Scatter Diagram, V Test
  3. Scatter Diagram VV Test
     77
     79
     80

Section VII
  Figure VII-
  1. Equilibrium in Indifference
  2. Expected Position of High R Stocks
  3. Scatter Diagram of High R Stocks
     94
     96
1.0 Definition of a Warrant and an Introduction to its Market Value

A warrant is the right to buy some number of shares of the common stock of the company issuing the warrant. The price which must be paid per share for the common stock is specified; this price is called the exercise price. The duration of this right is also stated in the published terms of the warrant. A few warrants are perpetual.

As an illustration, suppose there were a warrant of XYZ Co. which is the right to buy one share of common stock at $50 at any time between now and five years hence. Suppose further, that the last trade of the XYZ common is just at $50 at the close of business of the exchange. The warrant of course has no conversion value today; however, it will rise a dollar for each dollar the XYZ common rises. If the common falls the conversion value of the warrant naturally stays at zero. See figure I-1. For example, if the common

---

The treatment in this paper can also be applied to similar things such as stock rights, calls, and stock options. The analysis with slight modification can also be applied to the somewhat controversial restricted executive's stock option. There is considerable opinion that some value should be placed upon these options for fair reporting to stockholders. See Campbell, E.D.: Stock Options Should be Valued, Harvard Business Review, July-August 1961.
were trading at $51, I could convert a warrant into common for $50 and sell the common for $51. The warrant thus has a cash value of $1.

I pose this question to the reader: Do you expect the market value to be substantially different from the actual value? One unfamiliar with warrant markets might easily be led by intuition to answer no. I assure you, however, that if XYZ common is an average sort of stock trading in the fall of 1963, the hypothetical warrant would trade somewhere around $20 when the common is at $50. Furthermore, the warrant price would change almost a dollar for a dollar change in the common price on the average. This situation can be used to illustrate the concept of leverage. Suppose the common were to change by 5%. The warrant would change about 12%. As a result there is a greater chance of short term gain; but, there is also a greater chance of short term loss.

Why?

The answer is that in some sense a warrant price represents today's market value of the future of the common stock of a company. In the same sense the common stock price represents today's market value of the present plus the future.

It is not surprising that the market value of the future of a common stock may be higher than today's warrant value due to conversion.
Figure I-1

Warrant Conversion Value

Exercise Price

Stock Price

45°
I shall pursue this idea a little further. Five years hence there is some small chance that XYZ common will be trading for $1,000 or more. If this unlikely event were to come to pass, the warrant on the XYZ common would be worth $950 at expiration. There are also separate chances that the common will be trading in each of the ranges $900-$1,000, $800-$900, and so on down to $50-$55. Finally, there is the possibility that the stock will be trading below $50 five years from now, in which case the warrant would have no market value at all at expiration.

This line of thought leads to what might be considered a first try at accounting for warrant market price. The set of possible prices for the XYZ common on the date of warrant expiration are of course mutually exclusive; so why not take the view that the market places a probability on each possible price and take the approach of figuring out what all these probabilities are. Then it might be reasonable to assume that the warrant market price is the expected value of the warrant on expiration. Symbolically,

\[ W = \sum_{S=E}^{S=\infty} [S(t) - E(t)] \Pr[S(t)] \]

where \( W \) is the present warrant market price, \( S \) is the common stock price, \( E \) is the warrant exercise price, \( \Pr[S] \) is the probability of \( S \), and \( S \) is assumed discreet. \( S \) and \( E \) are shown as functions of time \( t \). In the discussion above \( t \) is taken as the time to expiration. (This formulation is due to
Kruizenga (11). To Kruizenga must go credit for the general method of attack in this Section.

Unfortunately, this approach does not work. It leads to warrant market prices which are much too high under certain reasonable assumptions about the $Pr(S)$. The predicted market value of the XYZ warrant could easily be over $40 on this basis.

The main reason equation I-1 does not lead to the right answers, in my opinion, is that it assumes the market is risk indifferent, that is, it accepts an expected or average value as the equivalent of a known, certain value. As a result of work done by many investigators in the past, and as a result of whatever small contribution the work described herein may represent, I for one am very strongly convinced that securities markets are not risk indifferent.

In fact I believe the situation can much better be represented:

$$W = \sum_{S=E}^{S=\infty} D[S(t)-E(t)]Pr[S]$$

where $D$ is a discount factor less than 1 resulting from risk aversion. This generalization of viewpoint opens a veritable Pandora's Box of problems. Pertinent questions are: What is an appropriate measure of risk? How does $D$ vary with $S$ or $Pr(S)$? How does $D$ depend upon $t$? How is $D$, in the case of

---

2 There is time preference as well, but it will be found to be second order by comparison in Section VII.
the XYZ warrant say, affected by the availability of other investment opportunities?

This entire paper is devoted to the attempt to answer these questions and to apply a model of the form of I-2 to a sample of actual market warrant prices, so as to account for market risk preferences.

It is recommended that the reader unfamiliar with warrants turn now to subsection IV 3.0 which can be read out of context.

The next subsection contains a highly simplified plausibility argument for the prevalence of risk aversion in investment markets.

2.0 A Plausibility Argument for Risk Aversion

Consider the following situation. There is one investor and one manufacturer. The investor has a net worth of twice his annual income and an annual income three times the bare subsistence level. In short, his standard of living is rather high. One objective of the investor is to build up his net worth so as to be able to continue his standard of living in his old age after retirement.

The manufacturer is in a position where he can make positive return on any amount of money he could obtain outright from the investor; his resources are large compared to the investor's.
The laws in this economy are peculiar in that only the following financial dealing between manufacturer and investor are legal. On January 1 the manufacturer offers this investment opportunity: For payment of $y$ dollars on that date the investor will receive an equal chance of the payment of $x$ dollars or zero dollars on June 30. A coin is tossed on that date to determine the outcome. A new opportunity is offered on July first of which the outcome is determined in the same manner on December 31, etc. every six months.

On the negotiation dates, the first of January and July, the manufacturer and the investor bargain over the value of $y$ and the ratio of $x$ to $y$. If $y$ is not less than one half $x$, the investor will not invest because the expected value of the transaction is zero or negative. The manufacturer will not transact if $y$ is so much less than one-half $x$ that his expected return from holding $y$ for six months is not positive. Somewhere in between these two conditions the bargain will be struck.

The risk indifferent investor will set $y$ equal to his entire net worth. If he could get the manufacturer to accept the expected present value of his lifetime income over subsistence in lieu of cash, he would throw that in as well. He would run the risk of selling himself into slavery\(^3\).

The prior probability of a man's being free drops by a factor of two for every six months of the time horizon of the

---

\(^3\) The bankruptcy laws in the USA prevent this sort of thing. As such, they are quite consistent with our national anti-slavery policy.
strategy. He stands one chance in 1024 of lasting five years. Even if the bargains can only be struck on cash, the probability of a destitute old age is high.

I appeal to the reader's own reaction as to how he would feel about adopting such a strategy; I submit that, intuitively, it is an absurd way for a man to behave. This means I am a risk averter. If the reader agrees with me, so is he. I believe the large majority of other people are as well.

Under the conditions described in this odd economy, the risk averter will protect himself by holding cash in part. The more \( y \) drops below one-half \( x \), the more he will be willing to invest.

Two simple generalizations of the hypothetical economy under discussion, I believe, give further insight into the problem.

Suppose the one investor faced ten manufacturers. Suppose he could make the same bargain as to the ratio of \( x \) to \( y \) with each that he could with the original one. In this case he has a new and powerful risk aversion strategy. Since the tosses of the ten coins will be independent, if he invests one-tenth his total investment with each manufacturer, his expected return will be unchanged; but his risk of ruin at the end of the first six months will be one in 1024 instead of one in two. This is a special case of a very important general result: the behavior of a risk averter faced with a

\[\text{Note the analogy to portfolio diversification.}\]
given investment opportunity cannot in general be deduced from an examination of that investment opportunity alone. His behavior will depend upon the inter-relationship to all the other investment opportunities he faces. Market analysis is very greatly complicated as a consequence. The theoretical formalism to handle this aspect of the problem is presented in Section III.

As a second generalisation, which supports the idea of market risk aversion, suppose ten investors faced the one manufacturer, and suppose five were risk indifferent and five were risk averters. If the nature of the risk aversion is such that no risk averter ever invests all his money, it is certain that all five will still be present after five years. The probability that at least one of the risk indifferent investors remains is only about 1/200. One might say that old risk averters are never ruined, at worst, they fade away.

3.0 Summary of Section I and an Outline of the Rest of this Paper.

In subsection 1.0 above I have defined a warrant and given some indication of likely market behavior in an assumed case. The sometimes wide difference between conversion value and market value is indicated, and the hypothesis is presented that the larger market values are associated with the possible future common stock values. It is stated but not demonstrated that expected values of future possibilities predict market prices which are too high, and it is contended that the major cause of
the discrepancy is that the market is not indifferent to investment risk.

In subsection 2.0 I present an heuristic argument which I believe supports the plausibility of the concept of risk aversion in a very idealized hypothetical situation at least. In the same situation the very important concept of risk aversion through diversification is demonstrated. It is shown that the existence of this strategy greatly complicates the analysis of the market action of individual investment opportunities.

3.1 Outline of the Remainder of this Paper.

The reader's attention is called again to equation I-2. This study represents an attempt to explain market risk preference behavior in the terms of this general model. The steps in the work as outlined here may be understood better while keeping I-2 in mind.

The first step is to obtain an estimate of the Pr (S(t)). This is done in Section II, DEVELOPMENT OF A MODEL OF STOCK PRICES. The risk aversion discounts in the warrant markets are approached in two steps, the first of which is taken in Section III, A NORMATIVE MODEL OF STOCK MARKET INVESTOR BEHAVIOR IN THE FACE OF RISK AND ONE EXPERIMENTAL VERIFICATION. This approach is taken so as to build upon the foundation laid by investigators in the stock markets.

Parts of the results of Section II and III are combined to produce the explicit mathematical model of warrant market prices, of which equation I-2 is the general form, in Section IV:
THE MATHEMATICAL MODEL OF WARRANT PRICES, DERIVATION AND DISCUSSION.

At this point in the exposition, the important problem of having to consider the interrelation of all the investment opportunities remains although the mathematical model form of the risk aversion discount factor $D$ has been settled upon.

The magnitude of this problem may be seen as follows. It is probably an underestimate to say that there are the order of $10^4$ securities traded in the United States. (There are about 1,500 stocks traded on the New York Stock Exchange alone) This means that there are the order of $10^8$ interrelationships to calculate. If weekly data were used over one year for their estimation, $5 \times 10^9$ is an approximation of the number of quantities which would be generated in the process of calculating these interrelationships. Then, the job of picking the best portfolio given one's risk preferences is a calculation task of truly astronomical proportions.

I wish to draw the particular attention of the reader to Section V. It contains a method of completely avoiding the horrible task of the consideration of all the risk interrelationships for certain special classes of investment opportunities of which, fortunately, warrants seem to be a member. The argument is essentially one of market equilibrium between the warrant and its own common stock. I think the technique may be of use in other areas. Section V is called THE AVOIDANCE OF WARRANT COVARIANCE BY A MARKET EQUILIBRIUM ARGUMENT.

The tests that are used to validate the hypothesis derived in this investigation are explained at the end of Section V.
The model of Section IV and the hypothesis of Section V together comprise a theory of risk aversion behavior in warrant markets which can be tested against market action.

Section VI, SUMMARY OF THE EXPERIMENTAL TEST OF THE THEORY, contains the proof of the pudding. In view of the number of assumptions and inferences that had to be made to reach this point the results, I believe, are rather remarkably gratifying.

The exact steps taken in the experimental work are enumerated in section VII, DETAILS OF THE EXPERIMENTAL WORK. There is a rather crucial problem in the statistical significance of the estimator of one of the parameters of the model of \( \text{Pr}(S(t)) \) which is developed in Section II. This problem is dealt with here rather than in Section II because the test of the estimation technique devised depends upon the results of Sections III and V.

Section VIII is a summary of this study.
SECTION II

DEVELOPMENT OF A MODEL OF STOCK PRICES

In order to exploit the idea that warrant prices represent today's value of part of the future of a company's common stock, one must know the probabilities associated with each value of the common stock \( S \) at some future time \( t \). These probabilities are the \( \text{Pr}(S(t)) \) which are used in equation 1-2. The derivation of this distribution is the first step towards the explanation of risk aversion in warrant markets; it is taken in this section.

It will be found below that under the assumptions enumerated \( S(t) \) is lognormally distributed.

1.0 The Lack of Serial Correlation

A great deal of work has been done in the past sixty years which involves the search for serial correlation in the first differences of prices as a time series. Reference is made to published papers by Kendall (10), Cowles and Jones (7), Osborne (13), Working (18), Alexander (1), and Cootner (6). The correlation intervals employed have varied from one week to several months. Recent unpublished work by Alexander is a study of individual stock, daily data, over thirty years.

No significant serial correlation has ever been found. Some of the work referenced reports on the first differences of prices and some of it on the first differences of the logarithm of prices.
2.0 The Concept of a Generating Process

One may think of the market as a black box containing a stochastic process which every so often generates a price change. In this paper the "every so often" is taken as weekly unless otherwise stated.

3.0 The Ramifications of the Central Limit Theorem

If one may make the following assumptions about the generating process, a very general conclusion can be reached.

The assumptions are: a. The process is stationary. b. The first two moments of the distribution of the generating process exist. And, c. The changes produced by the generating process exhibit zero correlation.

By the Central Limit Theorem the conclusion is that the distribution of the sum of the changes must approach a normal distribution with mean equal to the mean of the generating process times the number of changes which have occurred, and with variance equal to the variance of the generating process times the number of changes.

Note that here I have not stated which process is uncorrelated serially, changes in prices or changes in their logarithms. A choice is made based upon previous work reported in the literature in subsection 4.0.

4.0 Normal or Lognormal?

Bachelier (2), Remery (14), Osborne (13), Sprenkle (16), Boness (4), and Rosett (15) have applied or discussed a normal or lognormal model; the last three references involve the use of the model to discuss warrant or call prices.
Bachelier, to whom must go credit for first publishing these ideas in 1900, used a purely normal process whereas the work referenced above on options uses lognormal models. Remery suggested that the lognormal distribution might be superior in the 20's. Osborne provided a rationale for the lognormal preference by assuming the Weber-Fechner law that human beings respond approximately to the logarithm of stimulus. Discussions of tests for choosing between the normal and lognormal models are found in the references to the work of Alexander, Osborne, and Sprenkle. The lognormal is better in almost every case; often it is much better.

Finally, following Osborne, I submit that multiplication or division by a factor is a better intuitive description of a "change" than is a statement of the absolute amount of the difference in dollars with no reference to net worth.

5.0 The Exact Form of the Model of Stock Prices

It is assumed that the asymptotic form of the distribution is an adequate approximation.

Based upon the reasoning and the reference work discussed above the model chosen is that the logarithm of $S(t)/S_0^1$ is normally distributed with mean $u t$ and variance $\sigma^2 t$ where $u$ and $\sigma^2$ are the mean and variance of the weekly generating process, and $t$ is time in weeks. $S_0$ is the value of $S$ when $t=0$.

\[ \text{The sum of the differences in the logarithm exactly equals the logarithm of the ratio } S(t)/S_0. \]
This type of motion has been studied extensively in the literature. In physics it would be called one dimensional Brownian Motion. In the literature of statistics and theory of probability it is called a random walk on the continuum.

The distribution described in words above is given in equation form in II-1.

\[
f(S(t)) = (2\pi \sigma^2)^{\frac{1}{2}} \exp \left[ \frac{\ln(S(t)/S_0) - ut}{\sigma^2 t} \right] \] II-1

when \( f(S) \) is the probability density function of \( S \), and \( \ln(S) \) is the natural logarithm of \( S \).

This model has the great virtue of analytical simplicity and tractability. There is, however, a real difficulty in the application of it to actual market prices. The trouble lies in the statistical significance of the estimator of the mean, and it resulted in a considerable digression in the course of the work. The discussion of this problem is deferred to Section VII because the method of attack depends upon theoretical and experimental considerations explained in Sections III and V.

6.0 Other Models

It would be misleading to end the discussion of models of stock prices at this point. There is evidence that the

\[ \text{Here and for the remainder of this paper } S \text{ is assumed continuous.} \]
model is insufficient in richness for some stocks. See Cootner (6) for a discussion of random walks between reflecting barriers as being superior at least for some stocks.

Benoit-Mandelbrot (3) has published work that indicates that for many price indices the process does not approach a normal distribution because the second moment of the generating process does not exist. To my knowledge he has not published studies of individual stock prices.

Nevertheless, based upon the success Sprenkle had in explaining warrant prices with a similar model, and upon the principle that the simplest adequate model is best, I choose the one of subsection II-5.0.

See Appendix I for a further discussion of the possible weaknesses of the models used in this study.

7.0 Summary

In this Section the first step has been taken in the application of equation I-2 to the problem of describing risk aversion behavior in warrant markets. The historical work, the reasoning, and the assumptions which lead to the simple model of subsection 5.0 above are presented.

In the next section the attack begins on the problem of risk aversion discount factors D of equation I-2. The step taken there is the study of risk aversion behavior in the stock markets.
Section III

A NORMATIVE MODEL OF STOCK MARKET INVESTOR BEHAVIOR IN THE FACE OF RISK AND ONE EXPERIMENTAL VERIFICATION

The problem of obtaining the risk discount factors of equation I-2 is approached by first reviewing and analyzing the existing literature on risk aversion in the stock markets. This review is used as the foundation for the approach to the warrant markets.

It will be found that a normative theory does exist; it might be called the Tobin-Markowitz theory. See (17) and (12). (A normative theory is one based upon what ought to be, assuming rational behavior.) Furthermore, there exists at least one experimental, verifying application of this theory in the recent work of Farrar. See (8).

In this section the ideas of the quantification of opportunities and preferences are introduced first. Then specific parameters are introduced to represent good and bad in the case of investments. With these parameters the power of the Markowitz theory is shown. Farrar's supporting work is discussed next. The compatibility of the random walk model of stock prices of Section II is demonstrated. The combination of the model of Section II and the Farrar theory lead to an important principle of the stock market investment decision process. Finally, some limitations of scope of the Markowitz and Farrar theories are pointed out.
1.0 Opportunities and Preferences in Quantitative Terms

The idea fundamental to what is to follow is that things which are good and things which are bad can be stated in measurable terms. In the case of investments, return on investment is certainly good, and risk of loss is certainly bad. To avoid for the moment the assigning of exact quantities to these concepts, I shall continue the discussion in the simple terms of numerical good and bad. I hope the reader will bear with the very heuristic nature of this introductory subsection.

If good and bad can be measured, they can be plotted in a plane such as figure III-1.

![Figure III-1](image)

Figure III-1
Quantitative Good and Bad

Each point in this plane can be thought of as representing an opportunity such as the one at x. Of course one will not be able to find a physical real opportunity everywhere. It is clear that other points in the southeast quadrant from x
as a center are superior to the opportunity at $x$, and points in the northwest quadrant are inferior. It is natural therefore to think in terms of a ranking of opportunities in this manner: for each value of good, what is the opportunity that brings with it the least value of bad? This approach could lead to a best opportunity curve such as that of figure III-2. The upwards concave shape of this curve reflects two ideas. The first is that the greater the return, the greater the risk, to use investment parlance. The second is that bad increases faster than good beyond some point. The law of diminishing returns sets in is the way this is often phrased.

![Best Opportunity Curve](image)

**Figure III-2**
Best Opportunity Curve

1.1 Quantitative Preferences, the Concept of Utility

As the good-bad plane can be used for the discussion of opportunity, so it can be employed to consider the concepts of preference and indifference. Consider figure III-3, which is just figure III-1 with six directional vectors added. As was stated above, the southeast quadrant contains points
superior to the opportunity at x and the northwest quadrant those which are inferior. How about the southwest and northeast quadrants? Motion which is uniquely northward is purely bad; motion which is solely eastward is entirely good. It is reasonable to suppose that there is some direction between north and east, shown by I in figure III-3, in which the increased good is just balanced by the increased bad. I shall call this the direction of indifference. There should be a direction I' in the southwest quadrant in which the decreased good is balanced by the decreased bad. One should be able to move a small amount in these two directions, I and I', repeat the argument, and find new directions of indifference. Successive repetitions of this process will trace out a curve in the good-bad plane, such as that shown in figure III-4. If one were to start at a series of points east and west of x and repeat the process just described starting at x in figure III-3, one would generate a family of curves such as that shown in figure III-5.
By construction, the curves of the last two figures have the property that the person drawing them is completely indifferent to motion along any particular one of them. For this reason they are called indifference curves. Furthermore, indifference curves can be ranked. In figure III-5 the indifference curve just to the right of the one passing through x must be uniformly better than the one through x because it is reached by motion that is purely good. Similarly, the indifference curve passing through any point to the left of x must be less desirable than the one through x. By an extension of this argument one can start with the left most indif-
ference curve and rank to the right in order of an increasing something which is desirable.

What is this desirable something which increased to the right?

In terms more specific than have been introduced into the sketchy discussion so far, this quantity is called utility. Utility is constant all along a given indifference curve. An indifference curve is therefore sometimes referred to as an iso-utile.

2.0 Normative Theory of Investment

One is now in a position to answer the question: Suppose there were a rational man, with preferences described by the indifference family of figure III-5, who is faced by the best opportunity curve of figure III-2; what ought he to do?

If he is rational, he will pick that opportunity which allows him the highest possible utility. What this action corresponds to is seen in figure III-6 which is a superimposition of figure III-5 on figure III-2. He will choose that opportunity which is represented by the point P, the point of tangency between the best opportunity curve and one of the indifference curves. If the concavity of the indifference and opportunity curves is as shown, there is no indifference curve of higher utility than the one passing through P which touches the best opportunity curve unless an indifference curve to the right of P crosses the one through P.
This condition is shown in figure III-7; it is absurd. Move along PB; B is better than P. Move along BC; C is the same as B. Move along CP; P is the same as C. Thus, if this intersection of indifference curves exists, one may deduce that P is better than itself.

Thus, the point P in figure III-6 is the point of best opportunity which is also of highest utility.
3.0 Specific Forms of the Considerations of 2.0

The reader is referred to Tobin (17) for extensive rigorous discussion of the ramifications of various types of utility curves. The only one explicitly used in this thesis is a simple quadratic form for risk averters; it is introduced and discussed below in subsection III 5.0.

In quantitative investment theory the measure of good is usually taken to be the mean, or expected value of return on unit investment per unit time. This expected return rate will henceforth be called \( R \).

An individual's utility function is sometimes assumed to be of the form:\(^1\)

\[ U = R - g(m) \]

III-1

where \( U \) is utility, \( R \) is the expected return rate, \( g \) is a general function, and \( m \) is the vector of all moments the individual assigns to his distribution describing future probabilities. It is assumed that \( U = R \), and \( g = 0 \) if all moments other than the first are zero. In this formulation the function \( g(m) \) represents the measure of risk, or what is bad. It seems reasonable that, if the risk is zero, the value of the utility should be set equal to the expected return. In these terms figure III-5 may be recase as in figure III-8.

---

\(^1\) This form is obviously not perfectly general. The third moment, skew, might be desirable.
In figure III-8 it is seen that the intersections of the indifference curves and the return axis are at the values $R_1, R_2, \ldots$. The utility of a given indifference curve is equal to the value of $R$ at which it cuts the return axis, throughout.

### 3.1 A Quantitative Measure of Risk

It is also usual in investment theory to take the measure of risk to be a function of the variance (or standard deviation) alone. I shall attempt to provide a rationale for this practice.

The expected value of the investment of a unit quantity of money after a unit time passage is $1 + R$. If, in fact, the value of the investment after unit time is anything less than this, something bad may be said to have happened. It seems reasonable to examine the conditional expectation of loss from the expected value, $CEL$, given that there is such a loss. This is given by equation III-2.

$$CEL = \int_{-\infty}^{S} (S - \overline{S}) f(S) \, dS$$  \hspace{1cm} \text{III-2}
where \( N \) normalizes the truncation of the distribution; \( \bar{S} \) is 1 plus \( R \). For example, if \( f(S) \) is normal with variance \( \sigma^2 \), the value of the CEL is -0.8\( \sigma \), independent of the mean. This property of the normal distribution may be considered just an illustrative case. On the other hand, if the change over unit time (say a year) may be thought of as generated by a stochastic process in a long series of much shorter time increments (say a week), and if the reasoning of Section II3.0 is thought to apply, then the normal case almost surely approximates the true situation.

From this point of view \( \sigma \) seems quite reasonable as a measure of risk.

Under this point of view equation III-1 may be written as it

\[
U = R - g(\sigma) \tag{III-3}
\]

is in equation III-3. Figure III-8 is modified only to the extent of substituting \( \sigma \) for risk on the ordinate. Sometimes in the discussion below the variance \( \sigma^2 \) is referred to as a measure of risk. This is of no importance; all it does is to change the shape of curves in figures III-2 through 8.

In the next subsection the work of Markowitz is described.

4.0 The Markowitz Theory of Optimum Portfolio Selection

Markowitz (12) has provided an elegant exposition of which the elements are as follows. Consider Figure III-9. The
absissa shows $R$, the expected return. The ordinate is $\sigma$, the standard deviation which is the measure of expected risk. In the referenced figure two investment possibilities I and II are shown. The set of investment possibilities is called the opportunity set. If the two investments are 100% positively correlated, the situation assumed in Figure III-9, the locus of possible portfolios is a straight line joining the two individual investments. If they are not, risk aversion becomes possible through diversification. The locus of possible portfolios will often include points which have less risk than either opportunity taken individually. Such a situation is depicted in Figure III-10.

The concept of an efficient portfolio is very important in this theory. An efficient portfolio is a portfolio which has the least variance for a given expected return.

---

**Figure III-9**
Investment Opportunities 100% Positive Correlation
In Figure III-10, with only two members in the opportunity set, the locus shown is efficient. In Figure III-11 the situation for a large number of members of the opportunity set is shown. Cash, the point (0,0) ignoring inflation or deflation, is included. Markowitz showed that the problem of finding an efficient portfolio is a quadratic programming problem:

\[
\text{Minimize } \sigma_k^2 = \sum_{i,j} f_i \sigma_{ij} f_j, \quad \text{III-4}
\]

subject to the constraints

\[
R_k = \sum_i f_i R_i \\
f_i \geq 0 \quad \text{III-5} \]

\[
\sum_i f_i = 1
\]

where \(K\) is an index running over selected points on the \(R\) axis, \(i\) and \(j\) are indices running over the elements of the opportunity set \(\sigma_k^2\) is the portfolio variance associated with
Figure III-11
Markowitz Efficient Locus
return $R_k$, the $\sigma_{ij}$ are the elements of the variance-covariance matrix of the opportunity set, and $f_i$ is the fraction of the portfolio invested in the $i$th opportunity. Thus for each value of $R_k$ a minimum value is found for $\sigma^2_k$, and values for the associated $f_i$ are determined. Each efficient portfolio determines a point in the $R-\sigma^2$ plane. The locus consisting of the points representing the efficient portfolios is called the efficient locus: see figure III-11.

Thus, Markowitz has highly systematized the notion of diversification which is one of the primary tenets of professional investors and advisors.

A very significant aspect of the theory of Markowitz to this point is the fact that no use whatever has been made of the market preferences or individual utilities, beyond the assumptions that the expected return is the proper measure of that which is good and that variance is the appropriate measure of risk, that which is bad. What has been done is the distilling of the essence of investment opportunities through the concept of efficient portfolios to a form where the ramifications of individual preferences are very simply deduced once they are known. The method of accomplishing this deduction is of course that shown in Figure III-6. The efficient locus is a form of the best opportunity curve of subsections 2.0 and 1.0 above in which the points on the locus are not unique opportunities, but rather they are compound opportunities.
This is a natural point at which to discuss the work of Farrar which may be thought of as an application of Portfolio theory.

5.0 An Experimental Application of Portfolio Theory

Farrar (8) has published the results of research on the application of portfolio selection theory to explain actual macroscopic market behavior. To the best of my knowledge it is the only such study published to date.

Farrar's approach appears to be somewhat different from that of Markowitz. Farrar assumes the utility function

\[ U = R - A \sigma^2 \]  

where \( A \) is the constant coefficient of risk aversion, \( R \) is again the expected return, and \( \sigma^2 \) is as defined by Markowitz in equation III-4. Farrar's formulation of the problem is:

Maximize III-6 subject to the constraints III-5. Equation III-6 is seen to be a special case of III-3.

The maximization of III-6 results in exactly one point in the \( R-\sigma^2 \) plane for one value of \( A \). Repeated application of this prescription traces out a locus in the \( R-\sigma^2 \) plane.

Farrar then provides a formal demonstration that this locus is indeed the efficient locus of Markowitz with \( A \) as a parameter along it. This situation is shown in Figure III-12. This is exactly the result which would be obtained by solving the Markowitz quadratic programming problem and then repeatedly
Figure III-12
Farrar Efficient Locus
applying the graphical normative theory shown in figure III-6 for different values of the parameter $A$ in equation III-6.

There is an additional point to be made. Farrar's method is not exactly the same as the Markowitz procedure. An additional assumption must be made. In the Markowitz method it is completely unnecessary to know the behavior of the indifference family above the efficient locus given that the family is of the risk aversion class. Farrar's utility function must be defined throughout the opportunity set. It must in general be defined over a far wider range in $g$. This is a point which must be watched when the extensions to warrants are made, since the warrants often have much higher variances than their respective common stocks due to leverage.

5.1 Experimental Test of the Model of Farrar

As is pointed out in Section I 3.1 above, the objective functions of equations III-4 and III-6 have the order of $10^8$ terms in them if the direct approach of considering all individual securities traded in the United States as opportunities is taken. The constraints of equation III-5 have only about $10^4$ terms. The straightforward approach is far beyond the reach of the power of the computing machinery of today.

Farrar reduced the opportunity set by three orders of magnitude in two steps. First he used industry stock price indices instead of individual stocks, and he used bond price indices instead of individual bonds. This step reduced the opportunity set from about $10^4$ members to approximately $10^2$ members. In the second step he applied factor analysis to ob-
tain another reduction of a factor of ten, almost. He finally arrived at an opportunity set containing 11 members. The computational problem was reduced to one within the range of computing power of an IBM 650.

He chose the market behavior of managers of investment trusts against which to test his model. He took the portfolios of 23 funds and identified the fractions each had of his 11 factors. This permitted him to calculate an R and a σ for each fund, to determine a portfolio point in the R-σ² plane. He did the same for a large number of randomly selected portfolios. He computed the value of A most consistent with the actual portfolio of each trust. He found:

1. The portfolio points of the trusts were close to the efficient locus.
2. The ranking of the funds with respect to A produced mutually consistent results. Balanced funds had larger A's than stock funds, which in turn had larger A's than growth funds. Two of the 23 studied were out of position in this regard.
3. Randomly selected portfolios were much poorer performers than the funds. A typical one had the same risk as a one of more conservative stock funds; however, it had only half the expected return.

By way of illustration I reproduce here Farrar's Chart II, as Figure III-13, which shows the first two of these results. A corresponds to -U'' in this figure.
Based upon these results, it appears reasonable for one, who believes that the free market competitive process will cause selection of near optimal performers as investment fund managers, to conclude that this theory is adequate for a good first order description of the market place decision process.

5.2 Discussion of Farrar's Utility Function

In private discussions with Professors Raiffa, Modigliani, and Beals several reservations about this very simple utility function were expressed. Equation III-6 is not invariant under a linear transformation or a simple change of scale. The latter is easily seen. Suppose a portfolio of dollar value $P$ has expected return $R$ and variance $\sigma^2$. Then a portfolio of
value $kP$ will have expected return $kR$ and variance $k^2 \sigma^2$.
The objective functions are:

$$U = R - A \sigma^2, \text{ and } U = kR - Ak^2 \sigma^2,$$
respectively. \text{ III-7}

It is clear that the two functions will not lead to the selection of the same efficient portfolio for values of $k$ other than one.

I discussed these objections with Farrar. He had made two assumptions:

(1) The time between decisions is small, days or a very few weeks.

(2) Each time the manager approaches his portfolio he mentally normalizes so that the total value of the portfolio is considered to be unity.

In other words $k$ is always unity. It will be shown in III 5.3 below that these assumptions plus the assumption of the random walk model of Section II lead to a utility function that scales in time.

That investment company managers ignore the absolute magnitude of the portfolios over which they have authority is not intuitively unreasonable. An individual might be expected to respond to the magnitude of his own portfolio, but the portfolio manager's role is fiduciary; the money is not his.

5.3 The Ramifications of the Lognormal Model of Stock Prices of Section II Combined with Farrar's Utility Function.

Farrar makes no use of the Lognormal random walk model of future stock prices developed in Section II. If it were found to be incompatible with his formulation, obvious difficulty
would result in the attempt to use both. I shall now show that they are in fact compatible.

Farrar used actual prices, not their logarithms; however, had he used logarithms of prices and calculated their means and variances it would have made very little difference. This is shown in appendix II. If one accepts Osborne's rationale, it is the moments of the logarithm which must be substituted into the utility function, not the moments of price.

The moments of \( \ln S/S_0 \), from equation II-1 are inserted in equation III-6. This is done in equation III-8.

\[
U = (u+d)t - \sigma^2 t, \quad \text{III-8}
\]

where \( \sigma \) is the expected dividend rate. It is seen that there is no change in structure whatsoever from the original Farrar function.

Since the form of the utility function is unchanged in the above argument, it may be said that the lognormal random walk model of future stock prices and the Farrar formulation of normative investor behavior are compatible.

In fact their conjunct application leads to a very important consideration.

6.0 Insight Into the Structure of the Investment Decision

A restatement of the Farrar prescription in terms of equation III-8 yields:

\[
\text{Maximize: } U = (u+d)t - \sigma^2 t \quad \text{III-9}
\]
Subject to the constraints:

\[(u+d)t = \sum_{i} f_i (u_i + d_i) t \]  \hspace{1cm} \text{III-10}

\[f_i \geq 0\]

\[\sum_{i} f_i = 1 \]  \hspace{1cm} \text{III-11}

where \(i\) runs over all members of the opportunity set, and \(f_i\) and \(\sigma^2\) are as defined in subsection 4.0 above. But notice that the division of equations III-9 and 10 by the time \(t\) changes the problem not one whit since the problem; maximize \(U/t\), is identical to the problem: Maximize \(U\).

If the assumptions leading up to this point are valid, one may draw the following important conclusion:

Stock market investors

(1) describe the opportunity set in terms of expected rates of return, and

(2) the investment decisions they make are independent of the time horizon of those decisions.

The situation can be shown graphically. This is done

\[\text{Slope } u+d\]

\[\ln S_0 + (u+d)t\]

\[\ln S_0\]

\[0 \hspace{1cm} t \hspace{1cm} \text{time}\]

\[\text{Figure III-14}\]

Aspects of the Stock Market Investment Decision Process
in figure III-14. The slope of the rate of return line is independent of \( t \).

The stock market investment principle deduced here is of importance below because this key principle will be used as a basis for formulating investor behavior in the warrant markets.

7.0 Limitations on the Scope of the Normative Theories of Markowitz and Farrar.

It is to be pointed out that both of the theories of Farrar and of Markowitz are of the ceteris paribus type. They both provide theory of what a rational investor should do faced with an opportunity set provided he may assume that his own actions do not in any way affect the parameters describing that opportunity set.

As such, in spite of the insight provided, one is not in a position to deduce what the risk aversion behavior of the market will be in the case of a given stock, given its variance-covariance matrix, say. Stated otherwise, the whole opportunity set has to be used for research into the nature of investor utility functions.

To answer such questions a mutatis mutandis general equilibrium theory is required.

That such questions can be answered in the warrant markets, I believe, may therefore make the direct study of investor risk preferences possible through the examination of the market action of convertible securities. As such, the results of this investigation may be more useful for the light
they may shed upon the nature of the utility functions of investors than they are for the analysis of one small, rather restricted market.

8.0 Summary

This Section contains the first step in the attack upon risk aversion discount factors $D$ of equation I-2. The material contained here is a review and presentation of theoretical and empirical work that pertains to normative investor behavior in the stock markets. The first two subsections are an heuristic outline of normative behavior. Subsection 3 contains a discussion of the specific quantities which are selected to represent good and bad, return and risk. Subsection 4 summarizes the Markowitz portfolio selection theory. In subsection 5, the alternative formulation of Farrar is presented along with a description of the experimental verification he did. It is shown that the Farrar approach and the lognormal model of future stock prices worked out in Section II are definitely compatible. Furthermore, these two theories taken together permit the deduction that stock market investment decisions are based upon expected rates of return; these decisions do not depend upon the, then, time horizons.

Finally, it is shown that the ceteris paribus nature of the Markowitz and Farrar theories do not permit of a direct examination of investor objective functions in the case of single securities, nor does it allow deduction of the equilibrium positions of individual securities if the set of objective functions were known.
In the next section the model of Section II and some of the material of this section are combined to obtain an explicit form of the model of warrant prices, which was first presented in general form in equation I-2.
SECTION IV

THE MATHEMATICAL MODEL OF WARRANT PRICES, DERIVATION AND DISCUSSION

In this section closed form expressions are obtained for the explicit mathematical model of warrant prices. Two versions were introduced in Section I; both will be used. The normative behavior of stock market investors discussed in the last section will be used to infer the general form of the risk aversion discount factors of equation I-2. The model of future stock prices derived in Section II is of course one basis for the deduction of the models of this section.

The general behavior of the model of warrant prices of this section is discussed and compared to some empirical data at the end of the section.

At the conclusion of Section IV the form of the risk aversion behavior of warrant investors will be explicit.

1.0 The Mathematical Model of Future Warrant Values.

It is said in Section I that warrant market prices are in general much higher than warrant conversion values because a warrant is in some sense representative of the future of its common stock. This idea is expressed in mathematical terms in equation I-1. This equation gives the expected value of the warrant at time $t$ in the future. Here let $W_e(t)$ be that quantity. With $S$ taken as a continuous variate, equation I-1 is rewritten as equation IV-1.
where $E$ is the exercise price and the sum of I-1 is replaced by the integral. If the value of $f(S(t))$ given by equation II-1, the lognormal model of future stock prices, is substituted into IV-1, the indicated integration yields:

$$W_e(t) = \int_E^\infty (S-E)f(S(t))dS.$$  \hspace{1cm} \text{IV-1}$$

$$W_e(t) = e^{(bt-\ln z)} C\left[\frac{bt-\ln z}{\sigma \sqrt{t}} - \frac{1}{2} \sigma \sqrt{t}\right] - C\left[\frac{\ln z - bt}{\sigma \sqrt{t}} + \frac{1}{2} \sigma \sqrt{t}\right].$$  \hspace{1cm} \text{IV-2}$$

where $b$ is $u + \frac{1}{2} \sigma^2$, $z$ is $E/S_0$, $C(x) = 1 - F_N(x)$, and $F_N(x)$ is the normal distribution function. The integration is quite straightforward. Under the assumptions made in this paper equation IV-2 is the explicit form of equation I-1.

The general situation in the integration of equation IV-1 is shown in figure IV-1. The integration is over the shaded area.

2.0 The Mathematical Model of Present Warrant Prices

Let $W$ be the present market price. The time relationship

Figure IV-2
Future and Present Warrant Prices
Figure IV-1

The Integration of Equation IV-1
between \( W \) and \( W_e(t) \) is shown in figure IV-2. If there is risk aversion and/or time preference \( W \) will be less than \( W_e(t) \) as indicated.

The reader is now referred back to subsection III 6.0 where the reasons for believing that stock market investors make their investment decisions based upon expected rates of return in the face of risk. I hereby infer that warrant market investors do the same thing.

This can be shown in graphical terms. See figure IV-3 where

\[
\begin{align*}
\text{Slope } r \\
\ln W_e(t) \\
\ln W \\
0 &\quad t &\quad \text{time}
\end{align*}
\]

Figure IV-3
Relation of \( \ln W \) to \( \ln W_e(t) \)

the logarithms of \( W \) and \( W_e(t) \) are shown instead of \( W \) and \( W_e(t) \). The inference is that there will exist a rate of return \( r \), the expected rate of return, which is independent of time \( t \); this is indicated in figure IV-3 by the slope of the dashed line joining \( W \) and \( W_e(t) \). If this analogy to stock market behavior is correct, the long sought form of the risk aversion discount factors \( D \) of equation I-2 is at hand. It is contained in relation IV-3, which is an immediate consequence
\[ \ln W + rt = \ln(W_e(t)), \text{ or} \]
\[ W = e^{-rt} W_e(t). \]  

of figure IV-3. It is seen that
\[ D = e^{-rt}. \]

From IV-4 one can conclude that D does not depend upon S in equation I-2, but there is no reason as yet to think it does not depend upon the parameters of the distribution of S, including the covariance of S with other members of the opportunity set.

In equation IV-4 it appears that D depends on time. This dependence will now be eliminated.

2.1 The Elimination of Time from the Model.

It is certainly to be expected that the present market surveys all values of \( W_e(t) \) throughout the lifetime of the warrant. If \( W_e(t) \) is plotted on semi-log paper vs. time from equation IV-2,

\[ \ln W_e(t) \]

slope b

0 0

Figure IV-4
\( \ln W_e(t) \) vs. Time
the result is as shown in figure IV-4. Note that as \( t \) gets very large, \( W_e(t) \) approaches an exponential rate of growth of \( b \) per unit time. Inspection of the model shown in equation IV-2 will reveal that this is so. The expected value of the stock price, \( S(t) \), may be obtained from \( f(S) \) given in equation II-1. It is \( S_0 e^{bt} \). In other words if the lifetime of the warrant is long enough, its rate of exponential growth approaches that of the common stock.

Since the warrant is more risky than the stock it is to be expected that the expected rate of growth demanded by the market of \( \ln W_e(t) \) must be greater than the expected rate of growth of the log of the expected value of the common stock \( b \). When \( t \) approaches zero, the rate of change of \( \ln(W_e(t)) \) approaches infinity.

It follows that if the lifetime of the warrant is long enough, there must be some value of time at which the rate of change of \( \ln(W_e(t)) \) with respect to time is just equal to \( r \) by the mean value theorem of the calculus. This situation is sketched in figure IV-5. The time at which this phenomenon occurs is called \( t_q \); I call this condition temporal equilibrium.\(^1\)

Beyond \( t_q \) the market does not obtain the demanded rate of growth \( r \); therefore, the lifetime remaining beyond \( t_q \) is ignored by the market. In temporal equilibrium \( t_q \) is the time horizon of the warrant investment opportunity.

\(^1\) The argument here is essentially due to Sprenkle (16).
Figure IV-5
Temporal Equilibrium
The meaning of this condition can be interpreted in terms of equation IV-3. Differentiate the first form with respect to time:

$$\frac{d(lnW)}{dt} = -r + \frac{d(lnW_e(t))}{dt}.$$  \hspace{1cm} \text{IV-5}

But, at temporal equilibrium, the last term is just $r$. Hence, the first derivative of $lnW$ is zero. Inspection of figures IV-5 or 4 show that the second derivative is negative.

It follows that both $lnW$ and $W$ are maxima with respect to time at temporal equilibrium.

If the remaining life of the warrant is not long enough for the rate of change of the log of $W_e(t)$ to drop so low as $r$, the situation will be as diagrammed in figure IV-6.

![Figure IV-6](image)

Expiration Before Temporal Equilibrium

Figure IV-6 shows the warrant expiring at $t_e$. There is no limit upon the required rate of growth $r$ due to the time...
behavior of $W_e(t)$. Note that since the rate of growth of $W_e(t)$ exceeds the rate of decline of $e^{-rt}$ up to $t_e$, the function $e^{-rt}W_e(t)$ is a maximum at $t_e$. It is a corner maximum, not a zero derivative maximum.

At this point one is in a position to state a final form of the mathematical model of present warrant market prices.

2.2 The Final Model

From the considerations above in 2.1 which show that $W$ is a maximum with respect to time in either case, temporal equilibrium or not, one may write:

$$W = \max_{t} e^{-rt} W_e(t), \quad 0 < t < t_e.$$  \hspace{1cm} IV-6

where $W_e(t)$ is of course given in equation IV-2. This relationship is of the form of a discounting at the rate $r$ of the expected value of the warrant at the future time $t$.

There is a further consequence of the form of IV-6. The exponential term is of the form for time preference discounting. Indeed $r$ may be thought to be partly due to risk aversion and partly due to time preference.

Finally, equation IV-6 can be reduced in dimension. It now shows $W$ as a function of $S_o/E$ with $u, \sigma, r, t_e$, and $E$ as parameters. $E$ can be eliminated as a parameter by dividing IV-6 through by $E$. The final model is:

$$\frac{W}{E} = \max_{t} e^{-rt} \frac{W_e(t)}{E}, \quad 0 < t < t_e.$$  \hspace{1cm} IV-7

This normalization permits comparison of warrants of different exercise prices in the same plane.
The situation with regards to the parameters enumerated above is thus; $E$ and $t_e$ are easily obtained from the published characteristics of the warrant. Granting the limit distribution of $\Pi - 1$, one can easily get a significant estimate of $\sigma$ from the recent stock history. The same can be done usually for $u$, but with difficulty. The expected return rate $r$ still appears to depend on the covariance with all other members of the opportunity set.

3.0 General Discussion of Warrant Behavior

A warrant is the right to buy a specified number of shares of common stock at a stated price for a named length of time. In this paper a warrant is treated as the right to buy one share for convenience. This may always be done by dividing the market price by the number of shares on which the warrant is a Call.

Some important practical considerations of warrants can be

![Figure IV-7](image)

The $W-S$ Plane
explained by the use of figure IV-7. This type of figure will be called a W-S plane. If the common stock price $S$ and the warrant price $W$ are simultaneously obtained, a single point is determined in the W-S plane. I shall show that this point will always lie in the middle region.

Points such as K and B will never be found because there is always some chance that the common stock will pay a dividend; as a result $W$ is always less than $S$. A point like C will not occur on the line $(0,0)-(0,E)$ unless the warrant is just at expiration because of the chance, however small, that $S$ will rise above $E$ before expiration. If $S$ is less than $E$, the conversion value of the warrant is zero; if $S$ is greater than $E$, the conversion value of the warrant is $S - E$. A point such as F cannot therefore occur because riskless profits could be made by buying the warrant, selling the stock short, converting the warrant and using the stock so obtained to cover the short sale. The demand for a warrant at F would force the price up until this arbitrage process were no longer profitable. Even D is most unlikely unless the warrant is at expiration because of the chance that $S$ will go higher.

The points G,H,I, and J are all reasonable and possible.

A normalized W-S plane is useful for comparing warrants of different conversion terms. Such a figure is shown as figure IV-8.

Fox (9) found that a high percentage of the variance of warrants could be accounted for by a linear regression of the warrant price on the stock price, often over 80%. I reproduce
Figure IV-8

The Normalized W-S Plane
Figure IV-9

Alleghany Corp. Scatter
Figure IV-10

Normalized W-S Plane
Cross Section
Figure IV-11
W-S Plane for General Acceptance
Warrant vs. Common Stock.
here his Figure 2 as my Figure IV-9 showing the behavior of the Alleghany warrant vs. the common stock. I also reproduce his graph 15 here as Figure IV-10. It shows 136 warrants at one point in time in the normalized W-S plane.

I also include a W-S plane scatter diagram on the General Acceptance warrant which is one of the warrants studied in the empirical work reported on in Section VII. This is figure IV-11. It may be of interest that the time over which this data was collected includes the May-June 1962 period of market turmoil. This is not apparent at all in figure IV-11.

Not all warrants are so well behaved as the ones presented here; however, the inference that a great deal of the variance in warrant prices can be accounted for by the Stock price is correct.

4.0 The General Behavior of the Model of Equation IV-7

To present an idea of the general behavior of this model several graphical figures are shown here. Because of the number of parameters in the model a comprehensive exhibit in two dimensions is impossible. Representative values of the parameters are chosen.

For Cootner's typical value of 0.03 on a weekly basis is selected. For te I took the average time to expiration, excluding perpetual warrants, for over a hundred warrants. I set r=b; this is done for convenience although it is known to be too small a value for r. It is therefore expected that the thus predicted values of W will be somewhat too high. Three loci are
Figure IV-12

General Behavior of Equation IV-7
Figure IV-13

General Behavior of Equation IV-7

vs Cross Section Data
Figure IV-114

Normalized W-S Plane Motion
14 Warrants vs Eq. IV-7
in the '62 Crash
plotted with \( u \) as a parameter. A high (Cootner's .005 on a weekly basis), average (Dow, Jones Industrial), and the value zero are selected for \( u \). The resulting plots are shown in a normalized W-S plane in figure IV-12. These same curves are then transferred to figure IV-10; this is shown in figure IV-13. It is seen that the high, low, and average curves are in general consistent with those concepts in the market place.

Finally, the 1962 high-low points are plotted for 14 warrants on the curves of figure IV-12. The results are shown in figure IV-14. The motion of these warrants, I believe, lends further credence to the efficacy of the model.

5.0 Summary

In this fourth section the general form of the warrant market price model of equation I-2 has been made explicit in equation IV-7. This is done by inferring the form of the risk aversion factor \( D \), in the warrant markets, from the behavior deduced for stock market investors in Section III.

Some practical warrant considerations and some empirical, descriptive data are introduced in subsection 3. Finally, the gross compatibility of the model of equation IV-6 with these empirical data is shown.

At this point the model is explicit; specifically, the form of the risk aversion discount factors \( D \) is known, equation IV-4, but the value of the expected exponential warrant growth rate \( r \) is not known. Its functional dependence is unknown. It is not unreasonable to expect that the value of \( r \) in any particular case will depend upon the parameters of that situation.
as well as the very complex covariance with all the other members of the opportunity set.

In the next section a market equilibrium argument is presented which, when valid, eliminates the need for consideration of the covariance problem, thus greatly reducing the magnitude of the difficulty of understanding risk aversion in the warrant markets.
SECTION V

THE AVOIDANCE OF WARRANT COVARIANCE BY A MARKET EQUILIBRIUM ARGUMENT.

The basic idea presented here is that the warrant investor does not regard a given warrant as an independent investment opportunity; he regards the warrant and its common stock together as being a single, composite investment opportunity.

When the argument below is valid, the discussion of the risk aversion appropriate to a particular warrant is greatly simplified because it is unnecessary to consider the complex covariance of that warrant with all other members of the opportunity set.

The argument depends on three assumptions and two inferences from material above. These five points are presented and discussed first. Then the covariance avoiding argument is given: an explicit expression is obtained for \( r \) of equation IV-4. This section concludes with an explanation of the experiments which will be used to test the validity of the theory developed in this investigation.

1.0 The Assumptions

There are three assumptions:

(1) The market price of a warrant is dominated by investors with a utility function of the form of equation III-6, specifically.

\[ U/t = R - \sigma^2. \]
(2) The common stock is a member of the opportunity set of the warrant investor.

(3) Market action of the warrant investors does not affect the position of the common stock in the risk-return, $\sigma$-R plane.

Assumption (1) is made by analogy with the findings of Farrar in the stock markets. It appears at first glance to be a market utility function. This is not so, however. The assumption is that it is investors with a particular value of $A$ who dominate the market price of a given warrant. All such investors see the same facts at the same time and simultaneously make the same decisions. As a result, they may be considered as a composite investor. There is no problem of scale if one accepts Farrar's assumption that investors normalize their net worth with each new decision.

The assumption of market dominating investors is nothing more than the assumption of Cootner's professional group. See again reference (6).

Assumption (2) simply says that the warrant investor may buy some of the common stock of the warrant in preference to the warrant.

Assumption (3) may be regarded as implying that the position of the common stock in the $\sigma$-R plane is the result of an equilibrium in the stock market which is controlled by stock market investors who have authority over much more money and who are more conservative than warrant market investors.
2.0 Two Inferences

The inferences are:

(1) The warrant may be regarded as having 100% positive correlation with its common stock.

(2) The warrant investor regards the measure of risk in the stock as being $\sigma$, and he regards the measure of risk in the warrant as being $L\sigma$ where

\[
L = \frac{d(\ln W)}{d(\ln S)}
\]

It has already been shown that the inference (1) is not exactly true. There is variance in warrants which can be explained by the behavior of the common stock and variance which cannot.

Please refer to Figures IV-9 and 11. Here examples of the additional variance can be seen.

When one plots a few dozen of these W-S plane charts one is struck by the fact that the imaginary central line is usually traversed many times in a matter of months. A model of the unexplained variance process might be a random walk between highly reflecting barriers. One would expect the distribution of this additional variation to approach a uniform distribution. As an element of risk, this is a very different type than that represented by the $\sigma$ of an unbounded random walk. I believe an investor with an investment horizon as long as a few weeks may well ignore this variation altogether. In any event it is small compared to the variation due to the common stock.

In inference (2) I return to the rationale of the log-normal distribution of stock prices provided by Osborne, namely,
that the proper measure of a change in price is a change in the logarithm of the price. A curve of the model of IV-6 may be replotted in terms of the logarithms as shown in figure V-1. The quantity \( L \) is the slope of this curve.

![Graph](Figure V-1
\( \ln W \) vs \( \ln S \))

I call \( L \) the leverage of the warrant. It is seen to be the same form as an elasticity. I am approximating the curve of figure V-1 over a region by fitting a curve of the form

\[
\ln W = L \ln S + \text{constant} \tag{V-3}
\]

If \( \ln W = L \ln S + \text{constant} \), and if \( \ln S \) has a standard deviation \( \sigma \), then \( \ln W \) has a standard deviation \( L \sigma \). If the decision horizon is short so that small changes are anticipated before the next decision, the failure of V-3 to apply over a very wide region should cause little difficulty.

3.0 The Covariance Eliminating Argument

As stated, the basic idea is that the warrant market
investors regard the warrant-stock combination as a single entity, not as a stock and as a warrant separately. Under the assumptions and inferences above the warrant becomes the stock's satellite in a sense.

In this discussion symbols are as defined in Section III 5.

Consider Figure V-2, a σ-R plane. The stock is at point P (σ₁, u+d) because the stock has risk measure σ₁ and expected return u+d. The warrant will have risk measure Lσ₁ by inference (2); its return r will be deduced.

The warrant price is dominated by investors with utility function V-1 with a specific value of the coefficient of risk aversion A by assumption (1). The possible investment opportunities of a combination of some of the stock and some of the warrant lie on a straight line in the plane of figure V-2 by inference (1) because this is a situation such as that shown in figure III-9. Let that specific member of the family V-1 which passes through the point P in figure V-2 have utility U₂.

Suppose the return were r₃. If that were the case, the market dominating investors would like to take the composite portfolio position C with higher utility U₃. This is impossible because there are not enough warrants available when all the market dominators want to buy. The price is bid up until there is no possibility of higher utility then U₂. This is at return r₂. It is seen that r₂ is at the intersection of PB, the line tangent to the indifference curve of Utility U₂ and risk aversion A at the point P, and the locus σ=Lσ₁.
Note that the argument here does not depend upon extension of the range of the utility function in $\sigma$ into the region above $\sigma_i$. Only the slope of the indifference at $P$ is being used.
Suppose the return were \( r_1 \). If this were so, no one with risk aversion \( A \) would prefer the warrant to the common stock. The warrant would be sold by all the market dominators, but to whom? Clearly people with lower risk aversion than \( A \). But there must be a scarcity of funds controlled by such people compared to the people with risk aversion \( A \). If that were not the case, they would have taken the warrant out of the opportunity set of the people with risk aversion \( A \) before now. The price of the warrant must fall and its return rise until the return \( r_2 \) is reached. For this market to be dominated by the investors of risk aversion \( A \), \( r_2 \) is the equilibrium condition.

Subscripts will now be dropped.

The slope of \( PB \) must be:

\[
\frac{(L\sigma - \sigma)}{(r-u-d)} ; \text{ but from eq. } V-1 \text{ it must also be:} \\
\frac{1}{A2\sigma}.
\]

This equality yields

\[
r = u + d + A(L -1)2\sigma^2.
\]

Equation \( V-5 \) is the relationship which determines the risk aversion behavior in warrant markets independent of explicit consideration of warrant covariance with other members of the opportunity set. If \( A \) is known a priori, equations IV-7 and \( V-5 \) provide a model of warrant price behavior with no remaining unknowns.

The functional form of \( V-5 \) is quite reasonable. It shows that if the leverage on the warrant approaches unity, the
expected return of the warrant approaches that on the logarithm of the common stock. If the coefficient of risk aversion \( A \) is very small, the rate of return of the warrant is essentially the same as that in \( \ln S \) despite the higher risk. If risk and leverage are high, the expected return on the warrant may be much greater than that of the stock.

At the present state of the art, there is no body of empirical or theoretical work which permits one to preselect the value of \( A \) appropriate to a given warrant. It is expected that if future work continues to demonstrate the efficacy of the application of the class of utility functions used by Farrar, such a body of knowledge will evolve.

Because of the lack of a method of obtaining \( A \) a priori, at this time, it becomes the objective of the experimental part of this study to perform an experiment which tests the theory developed here. Before describing the experiment I shall discuss the value ranges which seem intuitively reasonable for \( A \) in warrant markets.

3.1 Anticipations about \( A \).

A check of the portfolios of growth funds shows warrants are almost never held. Consultation with management members of two funds revealed that their policy is never to buy warrants. If they get them, it comes about through some sort of issue to a common stock they hold.

If this is generally true, one would expect \( A \) in the warrant markets to be less than 5 from the findings of Farrar.
See Figure III-13. The minimum value for \( A \) in the investment growth funds is 5. Notice also that a factor of 2 covers the range of \( A \) in each investment trust class, approximately. If the warrant markets represent another, less conservative, class of investments perhaps a similar factor will cover the range there.

If the warrant markets are risk averting, \( A \) will be positive. (If they are not, the theory of this section is inapplicable.)

In summary it seems reasonable to anticipate:

1. \( 0 < A < 5 \)
2. \( A_{\text{max}} / A_{\text{min}} \) will be the order of 2.

4.0 The Experiment to Test the Theory

As was stated above the objective of the experimental work of this investigation is to test the theory developed here in the absence of prior knowledge of the values of \( A \) appropriate to a particular warrant. The experiment is now described.

Select \( N \) warrants. From the history of the common stock calculate estimators of \( u \) and \( \sigma \). From the published characteristics of the warrant obtain \( t_e \) and \( E \). Obtain values of \( W \) and \( S_0 \) from the market place. Apply the model of equation IV-7; there is only one remaining unknown, \( r \). This results in \( N \) values of \( r \). Call these \( r_i, i=1,2,\cdots,N \). Define \( v(r) \) as the variance of this sample of the \( N \) \( r_i \). Calculate \( v (r) \).

Define \( V \) as in equation V-6.
\[ V = \min_A \sum_{1=1}^{N} \left[ r_i - m_i - d_i - A(L_i - 1)^2 \right] \frac{1}{N} \]

where \( m \) is the estimator of \( u \), \( s \) is the estimator of \( \sigma \), \( L \) is the leverage of the warrant, \( d \) is an estimate of the dividend rate, and \( i \) is an index running over the \( N \) warrants as in the case of the \( r_i \).

Estimate the \( d_i \). Measure the \( L_i \) from recent market action. One could attempt to get the \( L_i \) from the model of IV-7, but this just introduces more error since the estimators of \( u \) and \( \sigma \) are not perfect.

Find that value of \( A \) which minimizes the right half of equation V-6. Call it \( A^* \). Set \( dV/dA = 0 \), and solve for \( A^* \) to accomplish the minimization. Calculate \( V \).

If, as anticipated, the range of \( A \) is small, one would expect to find that the test measure \( V \) were much smaller than \( v(r) \). In effect I am regarding the theory of this section as the deduction of a special case of multiple regression function of \( r \) on \( u, \sigma, d, \) and \( L \).

4.1 A Refinement

The test described in the paragraphs above would be about as far as one could reasonably go on a least squares test if the physical considerations of the situation led one to believe that the value of \( A \) were in fact constant over a set of \( N \) warrants. But there is no reason for so believing; on the contrary, consideration of the results of Farrar in figure III-13
leads one to believe that $A$ will not be constant for all dominating investors of a class. There does not even seem to be any particular tendency towards clustering of the values of $A$ in an investor class.

Based on this small amount of information it does not seem unreasonable to suppose that the model: $A$ is uniformly distributed between $A_{\text{min}}$ and $A_{\text{max}}$, is a better choice than the model: $A=A^*$. If this be true, then the following test is a more reasonable measure of the accuracy of the theory than the ratio of $V$ to $v(r)$.

Represent the assumed uniform distribution of $A$ by two values of $A$ instead of a single one.

Define a new test measure $VV$:

$$VV = \min_{A_1, A_2} \frac{1}{N} \sum_{i,j} \left[ r_{ij} - m_{ij} - d_{ij} - A_j (L_{ij} - 1) 2s_{ij}^2 \right],$$

where the minimization picks the best two values of $A$, $A_1^*$ and $A_2^*$, and the index $j$ is added to show that the minimization also entails the decision as to which group a given warrant belongs, group 1 with coefficient of risk aversion $A_1$, or group 2 with coefficient of risk aversion $A_2$.

It is assumed that $N$ is large compared to 2 of course.

If the uniform distribution of $A$ hypothesis can be accepted I submit that the relative magnitudes of $v(r)$ and $VV$ are a fairer test of the accuracy of the theory of Section V than
the test of subsection 4.0 above.

If it is believed that A should in fact be a constant for all warrants, the construction of VV is just so much nonsense.¹

5.0 Summary

In this section a theory is presented which eliminates the need for consideration of covariance in the understanding of the risk aversion discount factors D of equation I-2. This theory requires some additional assumptions and inferences which are explicitly stated.

The theory predicts that the risk aversion behavior of warrant market investors is an explicit function of their coefficient of risk aversion and parameters of the common stock distribution of future prices when the conversion terms of the warrant are given.

Since the values of A appropriate to a given warrant are unknown a priori, tests to indicate the accuracy of the theory are designed based upon anticipations about A from the work of Farrar. These tests are the subject of the experimental phase of the investigation.

Section VI summarizes the results of the experimental work.

¹If there were experimental or theoretical grounds for expecting a correlation of A and one or more of the other parameters, the VV test could be unfair. I have been able to find neither, which does not mean that neither exist.
SECTION VI
A SUMMARY OF THE EXPERIMENTAL TEST OF THE THEORY

In this section the numerical results of the experimental part of the investigation are summarized. The details of the estimation methods, some aspects of the calculation process, the reasons for the choice of warrants in the sample, and the methods of minimization of the test quantities V and VV are included in the next section.

I attempted to apply the theory developed above to a sample of 31 warrants. It applies very well to 24 of them. It applies not at all to 4. See Appendix I. Two more had to be removed from the sample because the market action at the time of interest was too turbulent to permit an estimation of leverage. One more warrant was arbitrarily removed because it is a companion warrant to one of the 4 to which the theory does not apply. 1 The results cited here are for the group of 24.

In the subsections below first the straightforward correlation of expected return and risk is presented. Next, the results of the minimization of the test quantity V of subsection V 4.0 are discussed. The analogous results from the minimization of VV of subsection V 4.1 are then shown. The anticipations about A of subsection V 3.1 are compared to the actual findings. Finally, the magnitudes of the measures of risk and return actually found are discussed.

1 The Company was Sheraton Corp.
1.0 Correlation of Return and Risk

It is often stated that there is a strong tendency in the world of investments for higher expected return to be accompanied by higher risk. To provide a setting for the test of the hypothesis of Section V, this idea is examined first. The first order correlation coefficient between the expected return $r$ and the measure of risk $L_s$ was calculated, where $s$ is the estimator of $\sigma$, and $L$ is the leverage. The correlation coefficient was found to be 0.61. Therefore, a regression of $r$ on $L_s$ would result in a reduction in variance of about 37% over the hypothesis that $r$ equals $r$ average in this sample. A scatter diagram of $r$ vs $L_s$ is shown in figure VI-1.

It is interesting that this result is quite comparable to results found in the stock market. Yohn (18) found that the correlation coefficient between expected return and risk was 0.59 in a random sample of 45 stocks.

2.0 The Results of the V Test

The hypothesis of equation V-5 proves far superior to the hypothesis that $r$ is a linear function of $L\sigma$. After all the steps of the experiment described in subsection V 4.0 had been taken it was found that $v(r)=0.021$ and $V=0.0030$. $A^*$ was found to be 0.868.

In short the hypothesis of equation V-5 plus the assumption that $A$ is a constant for all warrants results in a
Figure VI-1

Scatter of $r$ on $L_s$
reduction of variance by a factor of seven.

If the theory is completely false and the right and left halves of equation V-5 are in fact uncorrelated, and if both halves are uniformly distributed between the observed $r_{\text{max}}$ and $r_{\text{min}}$ of the sample, the observed results have less than one chance in $2 \times 10^7$ of occurring.

A visual presentation of these results is contained in figure VI-2 where a scatter diagram of $r$ vs $A(L - 1)2s^2 + m + d$ is displayed.

Figure VI-2 is to be compared with figure VI-1.

3.0 The Results of the VV Test

If the hypothesis is accepted that $A$ is uniformly distributed in a fairly narrow range so that it is more reasonable to represent the physical facts by two optimally selected values of $A$ rather than only one, even more striking confirmation of the theory than was found above is available.

The results of the VV test are $v(r) = 0.021$, $VV = 0.000667$, $A_1^* = 0.60$, and $A_2^* = 1.09$. The reduction in variance is more than a factor of 30. The visual presentation is given in figure VI-3. The function $\text{Best} \ A_j$ simply indicates that each warrant $j$ is optimally assigned to one of two groups. One has coefficient of risk aversion $A_1^*$; in the other $A = A_2^*$.

Figure VI-3 should be compared with figures VI-1 and VI-2.

I did not actually solve the minimization problem of equation V-7. I approximated it as follows. Having done the V test, I knew $A^*$. I then calculated the $A$'s for the 24 warrants using equation V-5. Those warrants with $A$ less than $A^*$ I put
Figure VI-2

Scatter Diagram, V Test

\[ A \times (L - 1)2s^2 + m + d \]
Figure VI-3

Scatter Diagram, WV Test
in group 1; the others went into group 2.

4.0 Anticipated vs Actual Values of A.

The findings were quite in line with anticipations. It was found that

\(0.44 \leq A \leq 1.5\)

(2) A factor of 3.4 covers the range.

It is suggested above that a uniform distribution of A may not be unreasonable. The mean and standard deviation of the A's in the sample are 0.88 and 0.35, respectively. A uniform distribution between the observed \(A_{\text{max}}\) and \(A_{\text{min}}\) would have mean and variance 0.97 and 0.31 respectively.

5.0 Discussion of the Magnitudes of Risk and Return.

In the three figures which have been provided in this section one can see that the values of expected rate of return are very large indeed. (See also appendix IV.) They range from 0.21 to 0.62. Indeed five warrants exhibit rates of return greater than 0.50. I have never seen expected rates of return of this magnitude published anywhere. The average is 0.36.

It was stated in a footnote in Section I that the effect of time preference was small compared to the risk aversion effect. If the present return on an almost certain investment of about 4% can be taken as a measure of time preference, it is seen that the average warrant discount rate is almost ten times as large.
These figures are to be compared with a return of 0.04 in a savings account today and 0.13 for the Dow, Jones industrial average.

These high rates of return can be intuitively justified, perhaps, by considering the size of the measure of risk $L_s$ in figure VI-1. The range is 0.38 to 0.88. This is to be compared with Cootner's typical value of 0.21 for the standard deviation of stocks. When it is remembered that the utility function depends on the square of this quantity, it will be seen that an investor will view the highest risk warrant with almost 20 times the displeasure with which he views a typical stock.

6.0 Summary

In this section the results of the experimental phase of the investigation are summarized. To provide a basis of comparison, the naive hypothesis that return is a function of risk is investigated. It is then shown that the hypothesis of this paper, equation V-5, is far superior for the two assumptions about the nature of $A$.

The possibility that the results of this study are completely false and that the observations are due to chance is extraordinarily unlikely.

The findings, when a value of $A$ is calculated for each warrant, are shown to be compatible with reasonable anticipations about $A$ from the work of Farrar.

Attention is drawn to the very large rates of return
which were found in the sample. A rationale is suggested by considering the very large values of the risk measure.

In the next section the details of the experimental work are discussed.
SECTION VII

DETAILS OF THE EXPERIMENTAL WORK

In subsection V4.0 the steps of the experimental phase were set forth. For convenience they are reenumerated here:

1. Select N warrants.
2. Calculate estimators of u and \( \sigma \).
3. From the published warrant data obtain E and \( t_e \).
4. Obtain values of W and \( S_0 \) from the market place.
5. Apply the model of equation IV-7 to obtain \( N r_i \)'.
6. Estimate the \( d_i \).
7. Estimate the \( L_i \).
8. Find \( A^g \).
9. Calculate \( V \), calculate \( v(r) \).
10. Calculate \( VV \).

Each step will be discussed below in the subsection of the same number. Particular attention is called to subsection 2.1.5. It contains another market equilibrium in indifference argument, analogous to that of Section V, which is designed to test part of the hypothesis of u estimation. This subsection is also of importance because it describes a point of departure in basic method between previous investigators, namely Sprenkle, Rosett, and Boness, and myself.

A final subsection describes the calculation of \( VV \) of subsection V 4.1.

1.0 The Choice of Warrants for Study

The theory developed in this investigation places certain requirements on the form of the data on the common stock. It needs to be corrected for stock dividends and splits but not for cash dividends. It will be found that earnings per share
is used as an aid in \( u \) estimation below. Therefore, one wants data on per share earnings which are historically corrected for stock splits and dividends.

These characteristics of reporting are met very well by a financial service called the Value Line (20). For this reason I decided to limit the research to those warrants which have common stocks reported upon by the Value Line.

This selection resulted in 24 common stocks with 32 warrants; they are shown in appendix III. Warrant #17 was not considered from the start because it is anomalous. It is a right to buy, not only the common stock, but also another warrant. I did not think it worthwhile to develop a model for this warrant alone, particularly since I know of no other of its type.

2.0 The Development of Estimators for the Parameters of the Common Stock Model \( u \) and \( \sigma \).

The problems of obtaining statistically significant estimators for these parameters are quite different. Under the assumption of the stationary random walk of Section II, it is easy to get a good estimator of \( \sigma \). It is very hard in the case of \( u \). Define \( s \) the estimator of \( \sigma \) and \( m \) the estimator of \( u \). In a normal population the two estimators are statistically independent; \( m \) is normally distributed with mean \( u \) and standard deviation \( \sigma / n^{1/2} \). On the other hand, for the distribution of \( s \), see for example Burington and May, Handbook of Probability and Statistics, page 146 et seq. For large \( n \) (\( n > 30 \)), it is approximately correct to say that \( s \) is normally distributed
with mean $\sigma$ and standard deviation $\sigma/(2n)^{1/2}$. Consideration of the standard deviation to mean ratios of these estimators is instructive; they are:

\[
\frac{\sigma}{\sqrt{n}} \quad \text{for } m, \quad \text{and} \\
\frac{1}{\sqrt{2n}} \quad \text{for } s,
\]

where $n$ is the number of elements in the sample. It will be found below that the average value for $m$ on an annual basis is 0.167 for the common stocks of the 32 warrants considered. This is of course .0032 on a weekly basis. The average value of $s$ found is 0.353 on an annual basis, which is 0.050 on a weekly basis. These average values are now substituted into eqs. VII-1. The result is

\[
\frac{15.6}{\sqrt{n}} \quad \text{for } m, \quad \text{and still} \\
\frac{1}{\sqrt{2n}} \quad \text{for } s, \quad \text{where } n \text{ is in weeks}
\]

If one years data is used, $n$ equals 52; the standard deviation to mean ratio of $s$ is quite reasonable, less than 0.1. The same amount of data used in the computation of $m$ results in a standard deviation to mean ratio of 2.2 which is very poor indeed. To obtain the same significance in $m$, as is available with one years data in $s$, one must have 24,400 weeks of data. This is almost 470 years.

The best thing one can do, I believe, is to use as many years of data as is feasible. The assumption of stationarity of the process becomes open to great question when one considers
many years' data. There are mergers, reorganizations, loss of markets, penetration of new markets, changes of management, etc., none of which one would intuitively associate with stationarity. Clearly what is needed is a criterion for stationarity. My approach to this problem is discussed in detail below in subsection VII-2.1.

2.1 The Determination of m, the estimator of u.

The natural estimator of u is

\[ m = \sum_{i=1}^{n} (\ln S_i - \ln S_{i-1}) = \ln \left( \frac{S_n}{S_0} \right). \]

where \( n \) is the number of weeks. It would be nice and convenient to assume stationarity and to let \( n \) be large enough to get significance on paper; however, I cannot convince myself that this corresponds to the physical facts. It is my belief that \( u_t \) in the model of equation II-1 represents a straight line approximation to an exponential expected growth of earnings per share. There is doubt, however, as to just what "earnings" means. Reported earnings exclude depreciation which may be large. In our economy depreciation is a legal accounting number which may or may not correspond to economic fact. This consideration makes the use of accounting data difficult.\(^1\)

Whatever the exact meaning of earnings should be in a given business, it is clear that earnings trends often change; this sometimes happens several times in a given decade in one company.

\(^1\)The widespread practices of carrying assets at cost and expensing research do not help either.
I attempted to check this idea two ways. I examined the thought that the ratio, price to accounting earnings ae, is bounded from below, and I did a correlation of ln(S_n/S_o) to ln(C_n/C_o) where C is a five year moving average of accounting earnings plus depreciation. I chose C rather than ae to avoid the very high short term fluctuations in ae in many companies in this search for the long term trend ut.

2.1.1 The Lower Bound of S/ae.

In the period 1950 to 1959 inclusive, the price-earnings ratios of 841 stocks were observed. There were about 8,000 ratios in the sample. Only 72 were observed between 1.1 and 3.0. There were none so low as 1.0. Of the 72, 55 occurred in the first two years. There have been 4 since 1954, three of which have been the distinction of a company called Botany Industries.

One may conclude that prices have risen faster than earnings in this interval. Alternatively, since the earnings of some hundreds of stocks in the sample have quite materially advanced in this decade, one may conclude that earnings do not advance for long without concurrent, overbalancing increases in prices at least in this period.

2.1.2 Correlation of lnC_n/C_o and lnS_n/S_o=m.

I selected a sample of 55 stocks more or less at random from those in the Value Line service. I took every tenth page for a sample of 5 from 11 of their 13 books. I considered a ten year interval from 1950 to 1959. The correlation coefficient
was computed; the value found was 0.68. This means that a linear regression of either on the other would account for almost half the variance of the first. The slope of the regression line of \( m \) on \( \ln C \) is 0.75. However, the sample is poor in numbers near the origin and the average \( m \) is about 1.8 times the average \( \ln C_n / C_0 \). A least squares line forced through the origin has a slope of 1.3. Where I have to use slope of the \( \ln C \) to estimate \( u \), I shall use a coefficient of 1.0.

2.1.3 The Stationarity Hypothesis.

One cannot consider the above exactly overwhelming evidence for the hypothesis below, but one can claim consistency.

I hypothesize that the process is stationary in regions where the slope of the logarithm of the five year moving average of the cash earnings is approximately constant.

This brought me a first trial for a \( u \) estimation hypothesis.

2.1.4 The preliminary \( u \) estimation hypothesis.

1. Use as long a time interval that is consistent with stationarity to the present.

2. If price change and cash earnings changes are both positive use eq. VII-3 for \( m \).

3. If either is negative, zero, or if there is evidence of current stationarity, use expert opinion on present slope of \( \ln C \) as an estimator of \( u \).

I am forced to a step like number three because a few of the stocks selected for study have negative values for \( m \) given by equation VII-3. But Farrar found that simple risk aversion behavior accounted for market behavior to a first order. For
the market to expect that returns are zero or negative in any stock, then, is extremely difficult to accept. I took the Value Line expectations as my expert opinion. In no case did they expect negative cash earnings slope. The approach here is Bayesian; the analysis below is not because, for the interval studied, insufficient data is available to modify the prior. One needs years, not months.


2.1.5 Support for the Preliminary \( u \) Estimation Hypothesis.

It is possible to infer some rather striking evidence from Farrar's numerical results which supports the first two rules of the \( u \) estimation hypothesis.

The method is very similar to that used in Section V. It is another equilibrium in indifference argument.

The most speculative class of growth fund managers are located at the upper end of the efficient locus of figure III-13. They are uniquely classified by \( A=5 \). Although it is not shown in figure III-13, the upper end of this locus terminates in one of the factors found by Farrar in his factor analysis; he calls it \( F_1 \). It is a very large factor containing most of the classes of industrial equities. Its coordinates in the \( \sigma-R \) plane are \((0.09,0.14)\).

Consider those stocks which have an expected return on an
annual basis quite a bit greater than 14% and which belong to the industry groups of the factor $F_1$. The following assumptions are made about such individual stocks:

1. These stocks are under the observation of Cootner professionals who dominate the market in them. They will buy or sell if the stock gets too far from the proper trend.

2. The Cootner professionals can be uniquely characterized by $A=5$ in their Farrar type utility functions.

3. They estimate $u$ by the preliminary $u$ estimation hypothesis in 2.1.4.

4. Such stocks will have a high degree of positive correlation with $F_1$. Thus, portfolio possibilities exist only on the straight lines joining the stocks and $F_1$. (Farrar did not publish data on $F_1$, so I cannot check this assumption a priori.)

5. The amount of investment in $F_1$ is so large that no equilibrium disturbance such as that discussed below can move it in the $\sigma$-$R$ plane.

In Figure VII-1 a stock $S$ with standard deviation $\sigma_S$ is shown with three possible values of return $R_1$, $R_2$, and $R_3$. In this $\sigma$-$R$ plane the three parabolas are three of the family of market indifference curves for which $A=5$. The middle curve passes through $F_1$. $U_1$ is less than $U_2$, which is less than $U_3$. The line $F_1B$ is the tangent to the curve of utility $U_2$ at $F_1$.

Suppose the stock $S$ were at $(\sigma_S, R_3)$. By hypothesis portfolios are possible along the straight line joining this point and $F_1$. The fund managers would attempt to accumulate $S$ to reach the new portfolio position at $C$ with utility $U_3$ higher than $U_2$. Because of the assumed total market relative values
Figure VII-1
Equilibrium in Indifference

F_1 is at (0.09, 0.14)
of F₁ and S there is not nearly enough of S to go around. Its price is bid up and its expected return falls until it reaches \((\sigma_S, R_2)\) on the tangent line to the indifference curve of utility \(U_2\) at \(F_1\). This is the point at which portfolios of no higher utility than \(U_2\) are possible.

Suppose \(S\) were at \((\sigma_S, R_1)\). Then no combination portfolio is possible between \(F_1\) and \(S\) for the indifference family with \(A=5\). The fund managers will divest \(F_1\) of \(S\), but there will be insufficient buyers since the more conservative professionals have little interest in \(S\) because of the high value of \(\sigma_S\), and there is insufficient money in the hands of less conservative investors to support the market in \(S\), or they would have seized domination before now. The price will fall and the expected return will rise as a result. This will happen until the expected return again reaches \(R_2\). The point \((\sigma_S, R_2)\) then is the equilibrium point.

It is to be pointed out that the presence of Cootner professionals is not taken into account in the model of future stock prices developed in Section II. This difficulty is discussed in appendix I.

A search was made for stocks with the following characteristics.

(1) The expected returns must be at least 20% in the period 1950 to 1960. Here return is \(m\) plus expected dividend.

(2) The stocks must be held in growth funds in 1960.
It is felt that a period ending in 1960 will be compatible with 1958 expectations, which time is the end point in time of Farrar's work.

If all the above assumptions are exactly met and if the errors of estimation were zero the expected result would be as sketched in figure VII-2.

As a result of a search for companies meeting both requirements above, 15 companies were found. The results are shown in Figure VII-3. They are quite striking. The fifteen companies are listed on figure VII-3 in the order of ascending $\sigma$. The numbers in parentheses are the ranking of the stocks in the order of total investment by the seventy funds surveyed by the Value Line. The top 80 are reported. This ranking is as of November 7, 1960.
Figure VII-3

1. Std. Oil N. J. (8)
2. Int'l Paper (12)
3. Texaco (2)
4. Goodyear (6)
5. IBM (1)
6. GE (9)
7. MTT
9. Boeing
10. Addressograph (61)
12. Brunswick
13. Polaroid (58)
14. Xerox
15. Texas inst. (66)
The correlation coefficient of $\sigma$ and $R$ in this sample of 15 is 0.95. This means that a linear least squares regression of $\sigma$ on $R$ would exhibit a factor of ten less residual variance than $\sigma$ alone. The hypothesis that $\sigma$ varies with $R$ as the locus $F_1B$ yields a residual variance a factor of 4.5 smaller than the variance of $\sigma$ alone.

I wish to emphasize that the fifteen companies are not a selected group meeting the above requirements. They are the only stocks found meeting these requirements. I do not wish to imply that there are no more such as the search was not exhaustive. Additional companies are not easy to find.

These results are to be compared with Yohn's findings of a correlation coefficient of 0.59 in a random sample of 45 stocks, already referenced. If many other points with expected returns less than 20%, are plotted in figure VII-3, they create a formless blob above the factor $F_1$.

I consider these results strong support for the first two items in the $u$ estimation hypothesis in the test period, ending in 1960.

Between 1960 and the fall of 1962, however, there was a market period in which many stocks severely declined. It might be expected that these deleterious experiences had a very egregious effect on investor $u$ expectations.

2.1.6 The Crash of '62

The market break of 1962 had a poor effect on first line blue chip companies there is no doubt; however, most have
recovered from their lows. This is represented by the fact that the Dow, Jones Industrial Average went from a 1961 high of 735 to a 1962 low of 535 and is currently (November, 1962) around 650. This is about 12% from the top.

The picture in the low price segment of the market where small secondary, or tertiary companies are found is very different. Here, there was a real crash where the use of the words disaster and catastrophe appears conservative. For example:

On the American Stock Exchange a group of 256 stocks which have sold under $10 in 1962 and which have declined at least 60% was studied. The average decline of this group was 72%. Twenty percent of the group had declined 80% or more, and 2% of the group had declined over 90%. Furthermore, as a group these stocks are nowhere near halfway back as 1962 closes.

Intuitively, one might expect this calamity to have much reduced the $u$ expectations of the higher risk classes of investors, warrant holders included.

Thus the method of $u$ imputation which appears valid in 1960 may be no good in the fall of 1962 for application to warrants. Another contributing factor is the fact that the confidence in the $u$ estimators appears so low in view of the random walk model.

Accordingly, I decided to make a detailed study of changes of $u$ expectations in the warrant markets during the crash.

2.1.7 $u$ Expectation Changes in the Crash

Accordingly, a study was made of the motion of approximately 140 warrants in the normalized $W$-$S$ plane. The point
corresponding to the 1962 high of the common stock is connected to the point corresponding to the 1962 low of the common stock. Fourteen such plots are shown in figure IV-d4. Any motion which implies a significant ceteris paribus, downward revision of \( u \) during a drop in \( S_0 \), must include a southeastwardly component perpendicular to the curves of equation IV-7. The study of the 140 warrants on this basis reveals that there is definitely no evidence of a significant drop in \( u \) expectations.

I do not wish to imply that no warrants reveal a downwards revision; some do. Some also exhibit an upwards revision, however. The 14 plotted are from the sample of warrants studied here. If they all are presented, the picture becomes confusing.

2.1.8 The Final \( \mu \) Estimation Hypothesis

As a result of the evidence presented in Figures VII-3 and IV-d4, the hypothesis of subsection 2.1.4 is slightly modified:

(1) Use as long a time interval as is consistent with stationarity to the present.

(2) If price change and earnings change are both positive, use equation VII-3 for \( \mu \), but use geometric mean of the '61 high and the '62 low, for \( S_n \).

(3) If either is negative, zero, or if there is evidence of current astationarity use expert opinion on the slope of the logarithm of \( C \) as the estimator of \( u \).

(3a) However, take the line \( F_B \) in Figure VII-3 as an upper bound on the expected returns.
The numbers of warrants that are affected by each rule are

**Rule Number**

(1), (2) - 23 warrants
(3) - 5 warrants
(3a) - 3 warrants.

The modification of rule 2 was made because the findings of subsection 2.1.7 revealed so little drop in m's, that I believe much of the drops in the common stocks was due to a correlation with the general market. The modification is an attempt to mitigate that effect.

The modification of rule 3 is made because I am loath to accept points to the right of the line \( F_1B \) of figure VII-3.

The values of m found are shown in appendix III. The 23 available standard deviation to mean ratios (SDMR) are shown for the m's. They are not nearly so good as the SDMR for the values of s, \( 1/(2x38)^{1/2} \) approximately. As explained above, this is to be expected.

With these poor estimators of \( u \), the consistency of the results summarized in Section VI is surprising. There are two reasons why the estimates of \( u \) may be more accurate than one would expect. First, this is a descriptive work, and the market may be in the same boat I am. That is as estimators of how the market estimates m these numbers may be much more accurate than they are as actual estimators of the stochastic processes themselves. Second, to the extent that Cootner
barriers are a general phenomenon, applying to all stocks, the estimates of \( u \) will be much more accurate than the assumption of an unbounded walk would lead one to believe.

The periods I accepted as stationary are shown in column 4 "Time of \( u \) Est." of appendix III.

2.2 The Estimator of \( \sigma^2, s^2 \)

A forty week period from mid-February to mid-November 1962 was chosen. This period encompasses the great market turbulence of the end of May and the beginning of June. Study of the data revealed that there were two weeks in that period which resulted in extraordinary price changes for some of the common stocks which had not been duplicated for many years. On the grounds of the very large volume in total trading of all stocks, I decided to identify this period with an overall marketplace transient. These two weeks were eliminated from the sample for all the common stocks studied for consistency, although they did not all show extraordinary changes.

The estimator used is \( n s^2 / n-1 \) where \( s^2 \) is given by

\[
\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} \left( \frac{\left| \sum_{i}^{n} z_{i} \right|}{s_{i-1}^2} - u \right)^2
\]

VII-4

The values found for this estimator are given in Appendix III.

3.0 Obtaining Warrant t's and E's

I used the publications of the financial service The Warrant Convertible Compass (21) as sources of these data, specifically, Today's Common Stock Warrants, summer edition 1962. A point to be watched is that many warrants have E's
which are a function of time; that is, on certain dates in the life of the warrants the value of $E$ changes discontinuously.

4.0 Obtaining Values of $W$ and $S_0$ from the Market Place.

I took a single representative point for each warrant as close to November 1, 1962 as market action would permit. I required a number of points in the same general location so I could select a point near their centroid. See figure IV-11 which shows the point selected for the General Acceptance Corp. warrant.

5.0 The Calculation of the $r_i$'s.

At this point one can calculate the $r_i$'s by solving the model of equation IV-7 for $r$.

One must take care to check for temporal equilibrium. The test is: $r$ must be equal to or less than the time derivative of $\ln[W_e(t)]$. If $r$ is greater than this derivative, an iterative calculation is used to find that value of time $T$ for which $r$ equals the time derivative of $\ln[W_e(t)]$. Usually four or five iterations are sufficient for better than one percent accuracy.

The application of this process yields 31 values of $r$; they are shown in appendix IV. Those warrants which are in temporal equilibrium are shown as such. $T$ is that value of time for which $W$ is a maximum; it is also shown. For warrants in temporal equilibrium, $T$ is less than $t_e$. 
6.0 The Estimation of the Dividend Rates $d_i$.

It would be nice to apply a simple rule to all warrants such as choose the current rate or the ten year average. I was unable to find such a rule that I really felt fit each company. I took the position that history is likely to repeat and set up the following general rule:

If $T$ is 3 years or less, take the current rate. If $T$ is greater than 3, take the ten year average.

Out of the 24 warrants in the final sample there were thirteen to which I felt the rule did not apply. They are:

#3 Atlas. This company has just changed its business. I chose 1% to represent some small payment in the 10 year horizon.

#4 General Acceptance. There was an obvious change in dividend policy in 1958. I used the five year average.

#6,23,24 Mack Trucks. Similar situation to #4. I used the current rate for the short term warrant, #6, and the five year average rate for the longer term ones.

#8 McCrory. Structure has recently change due to merger. I used the current rate.

#11 Symington Wayne. Same as #8.

#14 TWA. The market is predicting recovery for this company apparently. I chose 1% to represent a small average payment in the 8 year horizon.

#27,28 Seaboard World Airlines. Current rate for the short term warrant. Same as #14 for the long term one.

#30,31 United Air Lines. Current for the short term, Some improvement for the longer term one.

The dividend rates are shown in appendix III, implicit in the $m + d$ column.
7.0 The Estimation of the Leverages \( L_i \).

These are estimated from the market. W-S plane scatter diagrams were made using data from the December 1961 to November 1962 period. Please refer to the diagram in figure IV-II. It is one such. Leverage \( L \) was obtained by calculating \( S_o/W \) times the slope of the best straight line through the point \( W, S_o \). These lines are not least squares lines. They were drawn by eye to minimize departure not departure squared. I think least squares would place too much emphasis on short term transients. If there was any difficulty in estimating the slope, I estimated the maximum and the minimum it might be and took the average.

The values of \( L \) are given in appendix IV. As can be seen, two are not given. This is because no satisfactory estimate of slope could be reached by this method. The period of interest was one of market instability. I might speculate that there was at stationarity in the generating process of the price changes of the common stock.

8.0 Calculation of \( A^* \).

\( A^* \) is found by differentiating \( V-6 \) with respect to \( A \), setting the result to zero and solving for \( A^* \). It is found

\[
A^* = \frac{\sum x_i y_i}{\sum y_i^2}, \quad N=24, \quad \text{VII-5}
\]

where \( x_i = r_i - d_i - m_i \), and \( y_i = 2(L_i - 1)s_i^2 \). It is found that \( A^* = 0.868 \).
9.0 Calculation of V.

If VII-5 is substituted in the equation V-6, it is found that

\[
V = \frac{1}{N} \left[ \sum_{i=1}^{N} x_i^2 - \frac{\left( \sum_{i=1}^{N} x_i y_i \right)^2}{\sum_{i=1}^{N} y_i^2} \right], \quad N=24. \tag{VII-6}
\]

This yields \( V = 0.0030 \).

10.0 Calculation of \( VV \)

\( VV \) was approximated by calculating the A's by equation V-5, assigning all with A less than 0.868 to group one and the others to group two. The \( A_1^* \) and \( A_2^* \) were obtained by applying VII-5 to the two groups. The values are 0.60 and 1.09 respectively. \( VV \) was then approximated by

\[
VV = \frac{1}{N} \left[ \sum_{i=1}^{n_1} x_i^2 \left( \frac{\sum_{i=1}^{n_1} x_i y_i}{\sum_{i=1}^{n_1} y_i^2} \right)^2 + \sum_{i=2}^{n_2} x_i^2 \left( \frac{\sum_{i=1}^{n_2} x_i y_i}{\sum_{i=1}^{n_2} y_i^2} \right)^2 \right]. \tag{VII-7}
\]

where \( n_1 \) is the number in group 1, \( n_2 \) is the number in group 2, and the extra subscript 1 or 2 shows which group contains the warrant. \( VV \) was found to be 0.00067.

The values of A are shown in appendix IV.

11.0 Summary.

In this section the steps leading to the results explained in the previous section are explained in detail. In Section VIII the entire investigation is summarized and the conclusions are stated and discussed.
Section VIII
SUMMARY AND CONCLUSIONS

In this report Section I contains a brief introduction to warrant price behavior. The general form of the theoretical relationship to account for the market prices of warrants is given in equation I-2. The risk aversion discount factors D are contained in this relationship in an unspecified form. The remainder of this study is devoted to making the D's explicit.

Section I also contains an heuristic argument in support of risk aversion as a general practice of investors.

Section II contains a derivation of the normative probability distribution of future stock prices. It is found to be lognormal under the assumptions that the generating process of weekly price changes meets the requirements of the Central Limit Theorem and that investors respond to changes in the logarithm of prices rather than to changes in price directly.

Section III contains an outline of normative decision theory appropriate to the investment problem. The theory of Markowitz is summarized as is the work of Farrar. It is shown that Farrar's experimental work indicates that the predictions of the normative theory and the behavior of investment company managers are quite consistent.

It is then shown that there is nothing inconsistent in Farrar's utility function and the lognormal process of
Section II; indeed, their joint application permits a deduction about the structure of the investment problem.

In Section IV it is assumed that the warrant investment problem has the same structure; the model of equation I-2 is integrated using the distribution of Section II to obtain an explicit model of warrant prices. This is shown in equation IV-7. The expected return rates of the warrant are still unknown although the functional form of the discount factors \( D \) is explicit.

In Section V an equilibrium in indifference argument is presented which results in an explicit form for this expected rate of return in terms of the constants of the distribution of the common stock, warrant leverage, and the coefficient of risk aversion of the market dominating professionals. Since at this stage in the practical application of investment theory the coefficients are not known a priori, Section V concludes with a discussion of the experiments used to test the theory derived here. These are based upon anticipations about the coefficients of risk aversion in the warrant markets from the work of Farrar.

Section VI contains a summary of the results of the experimental phase of the work. The hypothesis of this paper is found to be much superior to either the hypothesis that expected return is a constant for all warrant or the hypothesis that expected return is a simple linear function of the measure of risk.
Section VII contains a detailed, step by step, discussion of the experimental work. Much attention is paid to the problem of estimating $u$, the mean of the generating process of stock prices, because of the low statistical significance of the estimator. A test of the estimation method is deduced by another equilibrium in indifference argument using the numerical results of Farrar.

Appendix I contains a discussion of possible weakness of the models used here. Appendix II contains a demonstration of the fact that the reported numerical results of Farrar would change very little if he had used changes in logarithms of prices rather than changes in prices. The appendices III and IV exhibit the numerical values of the parameters examined in the work of this investigation.

1.0 Conclusions

The conclusions are numbered according to the section which contains the material on which they are based.

I.1 The assumption of risk indifference leads to normative investor behavior which is intuitively appalling to this investigator at least.

I.2 Warrant market prices may be accounted for by taking the view that today's price is the discounted future expected value of the warrant.

II.1 If the generating process of changes of the logarithm of stock prices is uncorrelated from one period to another, and if the first two moments of the process exist, and if the process is stationary, future stock prices are asymptotically lognormally distributed.
III.1 Normative investment theory provides a first order description of the behavior of professional stock market investors, at least investment company managers.

III.2 When Farrar's formulation and the model of Section II are simultaneously applicable, investment decisions are based upon expected rates of return and variances of the generating process. The decisions are independent of the time horizon of those decisions.

In other words, once all the covariance has been taken into account and the efficient locus is known, risk averting investors demand higher expected exponential growth of their portfolios for higher risk in a known manner. This is the mechanism of risk aversion in the stock market.

IV.1 If the structure of the decision process is the same in the warrant markets as it is in the stock markets, an explicit model of warrant prices may be deduced as in Section IV; it has one unknown, the expected rate of return on the warrant.

V.1 Under the assumptions of Section V it is possible to derive an explicit expression for the expected rates of return in the warrant markets.

VI.1 The hypothesis derived in Section V is statistically much preferable to either the hypothesis that the expected rate of return is a constant for all warrants or the hypothesis that the expected rate of return is a simple function of expected risk.

2.0 Suggestions for Future Work

I have not studied the motion in the W-S plane over extended ranges. It would be interesting to make a study to see if the hypothesis of motion at constant $A$ is a good one compared to motion at constant $r$, say. I would speculate that it is.
Perhaps more significant than just the study of warrant markets per se, is the fact that the methods derived appear to avoid the problem of the consideration of covariance. These techniques may have value in research into the nature and structure of investor indifference curves.

Furthermore, other markets can be considered beyond just the ones in warrants. A similar theory can be worked out for convertible bonds. Such a theory is more complicated than the one here. The process is compound since the future value of the interest bearing aspect of the investment is also a random variable, not necessarily independent of the future price of the common stock. There is the risk of call to be considered in addition. I would guess that higher values of A will be found applicable to convertible bonds than is the case with warrants.

I should expect the same formalism to apply to convertible preferred stocks.

Finally, I should like to point out that whenever one can find an investment opportunity which is 100% positively correlated with another investment opportunity which will be unmoved by equilibrium perturbations due to the first, one can use the type of ceteris paribus equilibrium in indifference argument that I have to examine the slope of the indifference curves of the assumed market dominating investors. This is a technique which may be of general interest for penetrating the cloud of smoke and fire raised by the covariance
problem. I see no reason why mutatis mutandis equilibrium arguments could not also be applied, but I have done no work of that type.
BIBLIOGRAPHY

Books and Papers


14 Remery, see comment and reply in 13 above.


Financial Services

20 The Value Line, 3-5 East 44th St., New York 17, New York.

21 The Warrant Convertible Compass, 413 Franklin National Bank Building, Garden City, N. J.
Appendix I

POSSIBLE WEAKNESSES OF THE MODELS

In this appendix four areas of possible weakness are discussed. These are the simple unbounded random walk model, the statistical significance of the estimator of u, the ramifications of Benoit-Mandelbrot's Pareto-Levy contentions, and the warrants for which the theory did not work.

1.0 Objections to the Simple Unbounded Random Walk Model.

In Cootner's already referenced study a test of hypothesis between the simple model and a random walk between reflecting barriers is described. He found that the latter was preferable statistically. A major method of the test was to look for negative correlation. One would only expect a small amount since the stock would be in the presence of a barrier a small fraction of the time. This is what was found.

Furthermore, he studied stocks which had exhibited more steady growth than the average stock. I should have preferred to use such a model; however, to do so would have greatly complicated the research. I should have had to answer such questions as: How far are the barriers from the central trend line ut? What are their shapes? What are their reflectivities? What types of events cause changes in position of the barriers? What kind of changes?
I assumed that the barriers would be fairly far from the trend line so that investors' future expectations would be fairly well approximated by the model of equation II-1.

2.0 The Significance of m

It is seen that the SDMR's of the m's in appendix III are still very high in spite of the effort to improve them. The average of the SDMR's is about 0.7. In view of this it is surprising to me that the results of this study are as good as they are. See Section VI. Errors in m should enter directly into errors in r by equation V-5.

The SDMR's for the stocks examined in figure VII-3 are less than is the case for the warrant common stocks but not by a factor of 2. This suggests that the results of this figure are unlikely.

I can think of two reasons to account for this. First, there may be Cootner barriers nearer to the ut line than I assumed above; but, second, I may have obtained a good approximation of how investors in fact form estimates of u.

If the first reason is right, the real SDMR's could be much smaller than the model of equation II-1 would lead one to believe.

If the second reason is true, my estimators will give better predictions of warrant and stock market behavior than one might expect because market action depends on the investors'
estimators of parameters of the stochastic processes, rather than upon the true parameters of those processes.

I have assumed that the second reason is the case. If I am wrong, the values of \( r \) found in this investigation will tend to be too high because the upper barrier would truncate the distribution of equation II-2, for short duration warrants. If the lifetime of the warrant is long enough, the distribution will tend to approach a uniform distribution.

3.0 The Pareto-Levy Distributions

Benoit-Mandelbrot (3) has studied many different kinds of price indices, and prices. He has concluded that the Central Limit Theorem cannot be applied to the process as is done here in Section II because the second moment of the generating process does not exist. (Sometimes he finds processes in which the first moment does not exist.) Foma in his forthcoming doctoral dissertation at the University of Chicago will report on the divergence of the variance of 25 stocks over a five year period, daily data considered. He finds that the divergence is very slow. It corresponds to an alpha (defined as in (3)) range of 1.9 to 1.99. An alpha of 2 corresponds to the normal case.

If this is the situation in the stocks of my sample, the values of \( r \) I found will in fact be too low because the tail of the real distribution will be longer than that of equation II-1.
There is an even more alarming consequence of the Pareto-Levy distribution. Classical utility theory has as a fundamental postulate that decisions are made based upon expected value of utility of wealth. Farrar gets his utility function by expanding utility of wealth in a Taylor series. He takes the expected value of that series ignoring all terms but the first two. This brings him to \( U = R - AO^2 \). If the process is really Paretian, his \( U \) is always minus infinity because the variance does not exist.\(^1\)

It appears at first glance that normative decision theory is in great trouble if one cannot speak of moments. I do not believe it is. I do not see that the concept of indifference is in any way compromised.

I do not see why indifference families of the form of figure III-8 cannot be constructed with some risk measure other than variance. I shall suggest four:

1. The expected value of loss, given that there is a loss.
2. The expected value of the range.
3. The negative semi-interquartile range.
4. \( 1/\alpha \)

All four of these possible measures exist if \( \alpha \) is greater than one, I believe. (My guess is that it is number 2 which is really used; that is why the year's high and low values for stock prices appear in the financial pages of newspapers every day.)

\(^1\) The expected value of utility may exist; the Taylor Series approximation cannot be used.
I see no reason why a normative decision such as that in figure III-6 is not possible although calculation of the position of a composite portfolio in this plane will have to be worked out if the notion of covariance is inapplicable. If number 4 is a reasonable risk measure, this should not be too hard because a linear combination of Paretian variates is itself distributed Pareto-Levy.

If the indifference curves intersect the absissa, the value of return at the intersections provide a nice measure for order ranking of the indifference family. I do not see why the maximization of this measure does not represent a perfectly good normative theory of behavior.

In other words the prescription may become

Maximize the expected value of:

\[ U = \text{return-risk measure}. \]

This is still a maximization of the expected value of an objective function, which can be called utility, but it is not obvious that it is always possible to deduce a unique function, utility of wealth, for any possible risk measure.

I have assumed that if the stocks in my sample are Paretian, they have alphas near 2 and the variance of the sample observed will correlate quite well with the real risk measure, whatever it may be.

4.0 The Warrants for which the Theory Did Not Work

The basic model used for stock prices of equation II-1
assumes that there is a uniform long term growth, that is the expected value of \( \ln S \) is \( u_t \). Suppose, however, that the real market expectations were for a sudden increase in the mean. In terms of my identification of \( u \) with the rate of growth of earnings, this would mean an expectation that the earnings of the company would soon jump to a higher point and would continue to grow from there. If this were really the case, the value of \( r \) found by the methods of this investigation would come out too low. There were four warrants for which the value of \( r \) was so low as to show risk loving behavior. All of these have had statements published about them speaking of a large change in earning power. They are:

- **#9 Molybdenum Corp.** There has been speculation that the reserves of rare earth metals the company owns will find new commercial applications.

- **#10 Sperry Rand.** There have been predictions that the Univac division will stop showing large losses and start showing a profit.

- **#12 Telegroset.** There has been speculation that the company's start in the special purpose data processing field will result in big breakthroughs in new industries. Its Reservoir system for airlines is an example.

- **#29, 30 Sheraton Corp.** The company announced that it would attempt to qualify under the real estate trust laws. This would sharply reduce taxes.

In the case of Sheraton Corp. #29 did not exhibit risk loving; I threw it out anyway as being damned by association with #30.
There were two more warrants excluded from the sample because their market action in the fall of 1962 was such that I could not get a good estimate of leverage. They are #13 and #26. I did not attempt to consider #17 because it is not of the same class as the others. It is a warrant on common stock and also the other Universal American warrant, #16.
Appendix II

APPROXIMATION OF FARRAR'S MEAN AND VARIANCE

BY THE MEAN VARIANCE OF THE LOGARITHM

OF PRICE

It is shown in this appendix that Farrar's reported numerical results would be very little changed had he calculated the mean and variance of changes in the logarithm of price instead of changes in price.

1.0 The Mean

Farrar made various assumptions about his R's all of which lead to similar results. The one closest to the model under consideration here is his constant rate of growth assumption. Numerical values of his expected returns will be on the average

\[ R_f = 1 + u + d + \frac{1}{2} \sigma^2 \]  

since he used \( R_f = 1 \) for no change. The subscript \( f \) emphasizes his usage. Whereas \( R \) (my usage) is \( u + d \). But \( u + d \) are the order of \( 0.1 \); \( \sigma^2 \) is the order of \( .01 \). Therefore,

\[ R_f = 1 + R \]  

2.0 The Variance

His definition of variance is

\[ \sigma_f^2 = \frac{1}{n} \sum_{i=1}^{n=140} (X_i - R_f)^2 \]  

AII-2
where the index \( i \) runs over the 140 months of the period '46 to '58, \( R_i \) is defined as in equation AII-1, and, I believe, \( X_i \) is \( S_i/S_{i-1} \). Making use of the fact that the ratios of adjacent monthly prices are close to 1, one may write

\[
\sigma_f^2 = \frac{1}{n} \sum_{i=1}^{140} \left( \frac{S_i}{S_{i-1}} - \frac{S_{i-1}}{S_{i-1}} - R_i \right)^2
\]

\[
= \frac{1}{n} \sum_{i=1}^{140} \left[ \ln \left( 1 + \frac{S_i - S_{i-1}}{S_{i-1}} \right) - R_i \right]^2
\]

\[
= \frac{1}{n} \sum_{i=1}^{140} \left[ \ln \left( \frac{S_i}{S_{i-1}} \right) - R_i \right]^2 \approx S^2
\]

since \( R \) on a monthly basis is small. Thus,

\[
\sigma_f^2 = S^2.
\]

AII-5
Appendix III
PARAMETERS OF THE COMMON STOCKS

<table>
<thead>
<tr>
<th>#</th>
<th>Warrant</th>
<th>s</th>
<th>m</th>
<th>SMDR_u</th>
<th>T_u</th>
<th>m + d</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Allegheny</td>
<td>.35</td>
<td>.11</td>
<td>.1</td>
<td>9</td>
<td>.11</td>
</tr>
<tr>
<td>2</td>
<td>Armour</td>
<td>.34</td>
<td>.14</td>
<td>.7</td>
<td>13</td>
<td>.18</td>
</tr>
<tr>
<td>3</td>
<td>Atlas</td>
<td>.45</td>
<td>.13</td>
<td>na</td>
<td>na</td>
<td>.14</td>
</tr>
<tr>
<td>4</td>
<td>Gen'l Acceptance</td>
<td>.18</td>
<td>.12</td>
<td>.4</td>
<td>13</td>
<td>.17</td>
</tr>
<tr>
<td>5</td>
<td>Hilton Hotels</td>
<td>.21</td>
<td>.13</td>
<td>.4</td>
<td>13</td>
<td>.18</td>
</tr>
<tr>
<td>6</td>
<td>Mack Trucks '56</td>
<td>.27</td>
<td>.14</td>
<td>.5</td>
<td>13</td>
<td>.20</td>
</tr>
<tr>
<td>7</td>
<td>Martin Marietta</td>
<td>.34</td>
<td>.12</td>
<td>.8</td>
<td>13</td>
<td>.17</td>
</tr>
<tr>
<td>8</td>
<td>McCrory Corp.</td>
<td>.26</td>
<td>.14</td>
<td>na</td>
<td>na</td>
<td>.18</td>
</tr>
<tr>
<td>9</td>
<td>Molybdenum Corp.</td>
<td>.57</td>
<td>.16</td>
<td>na</td>
<td>na</td>
<td>.16</td>
</tr>
<tr>
<td>10</td>
<td>Sperry Rand</td>
<td>.31</td>
<td>.13</td>
<td>na</td>
<td>na</td>
<td>.04</td>
</tr>
<tr>
<td>11</td>
<td>Symington Wayne</td>
<td>.24</td>
<td>.15</td>
<td>.6</td>
<td>8</td>
<td>.20</td>
</tr>
<tr>
<td>12</td>
<td>Telegro Register</td>
<td>.56</td>
<td>.14</td>
<td>na</td>
<td>na</td>
<td>.14</td>
</tr>
<tr>
<td>13</td>
<td>Textron</td>
<td>.24</td>
<td>.08</td>
<td>1.0</td>
<td>10</td>
<td>.13</td>
</tr>
<tr>
<td>14</td>
<td>TWA</td>
<td>.34</td>
<td>.23</td>
<td>na</td>
<td>na</td>
<td>.24</td>
</tr>
<tr>
<td>15</td>
<td>Tri-Continental</td>
<td>.23</td>
<td>.13</td>
<td>.5</td>
<td>14</td>
<td>.18</td>
</tr>
<tr>
<td>16</td>
<td>Universal Am. '62</td>
<td>.41</td>
<td>.19</td>
<td>.8</td>
<td>8</td>
<td>.19</td>
</tr>
<tr>
<td>17</td>
<td>...</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>General Tire</td>
<td>.37</td>
<td>.26</td>
<td>.4</td>
<td>13</td>
<td>.29</td>
</tr>
<tr>
<td>19</td>
<td>Kerr-McGee Oil '64</td>
<td>.39</td>
<td>.16</td>
<td>.7</td>
<td>14</td>
<td>.19</td>
</tr>
<tr>
<td>20</td>
<td>...</td>
<td>.39</td>
<td>.16</td>
<td>.7</td>
<td>14</td>
<td>.18</td>
</tr>
<tr>
<td>21</td>
<td>Ling Temco Vought $30</td>
<td>.44</td>
<td>.28</td>
<td>.6</td>
<td>6</td>
<td>.28</td>
</tr>
<tr>
<td>22</td>
<td>...</td>
<td>.44</td>
<td>.28</td>
<td>.6</td>
<td>6</td>
<td>.28</td>
</tr>
<tr>
<td>23</td>
<td>Mack Trucks '59</td>
<td>.27</td>
<td>.14</td>
<td>.5</td>
<td>13</td>
<td>.20</td>
</tr>
<tr>
<td>24</td>
<td>...</td>
<td>.27</td>
<td>.14</td>
<td>.5</td>
<td>13</td>
<td>.20</td>
</tr>
<tr>
<td>25</td>
<td>National General</td>
<td>.36</td>
<td>.14</td>
<td>.9</td>
<td>9</td>
<td>.15</td>
</tr>
<tr>
<td>26</td>
<td>National Homes B</td>
<td>.59</td>
<td>.09</td>
<td>2.0</td>
<td>10</td>
<td>.09</td>
</tr>
<tr>
<td>27</td>
<td>Seab. World AL $5</td>
<td>.47</td>
<td>.29</td>
<td>na</td>
<td>na</td>
<td>.29</td>
</tr>
<tr>
<td>28</td>
<td>...</td>
<td>.47</td>
<td>.29</td>
<td>na</td>
<td>na</td>
<td>.30</td>
</tr>
<tr>
<td>29</td>
<td>Sheraton Corp. $10</td>
<td>.19</td>
<td>.6</td>
<td>.3</td>
<td>13</td>
<td>.21</td>
</tr>
<tr>
<td>30</td>
<td>...</td>
<td>.19</td>
<td>.6</td>
<td>.3</td>
<td>13</td>
<td>.21</td>
</tr>
<tr>
<td>31</td>
<td>United AL '66</td>
<td>.39</td>
<td>.11</td>
<td>1.0</td>
<td>13</td>
<td>.13</td>
</tr>
<tr>
<td>32</td>
<td>...</td>
<td>.39</td>
<td>.11</td>
<td>1.0</td>
<td>13</td>
<td>.13</td>
</tr>
</tbody>
</table>

Average  

SMDR_u is the standard deviation to mean ratio.  
T_u is the time of the estimate of u, in years.  
na means not available.  
Other symbols are defined as before.  
All the data above is on an annual basis.
Appendix IV

PARAMETERS OF THE WARRANTS

<table>
<thead>
<tr>
<th>#</th>
<th>r</th>
<th>T</th>
<th>temp. eqlb?</th>
<th>L</th>
<th>Ls</th>
<th>A</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.25</td>
<td>2.0</td>
<td>y</td>
<td>1.55</td>
<td>.54</td>
<td>1.02</td>
</tr>
<tr>
<td>2</td>
<td>.24</td>
<td>2.1</td>
<td>y</td>
<td>1.5</td>
<td>.50</td>
<td>.59</td>
</tr>
<tr>
<td>3</td>
<td>.30</td>
<td>9.6</td>
<td>y</td>
<td>1.31</td>
<td>.59</td>
<td>1.26</td>
</tr>
<tr>
<td>4</td>
<td>.23</td>
<td>1</td>
<td>y</td>
<td>2.32</td>
<td>.42</td>
<td>.65</td>
</tr>
<tr>
<td>5</td>
<td>.24</td>
<td>8.9</td>
<td>c</td>
<td>2.52</td>
<td>.53</td>
<td>.44</td>
</tr>
<tr>
<td>6</td>
<td>.30</td>
<td>3.8</td>
<td>y</td>
<td>2.30</td>
<td>.62</td>
<td>.53</td>
</tr>
<tr>
<td>7</td>
<td>.28</td>
<td>4.0</td>
<td>y</td>
<td>2.08</td>
<td>.71</td>
<td>.52</td>
</tr>
<tr>
<td>8</td>
<td>.31</td>
<td>5.2</td>
<td>y</td>
<td>1.84</td>
<td>.48</td>
<td>1.20</td>
</tr>
<tr>
<td>9</td>
<td>-.18</td>
<td>(.1)</td>
<td>.9</td>
<td>1.53</td>
<td>.89</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>-.16</td>
<td>(.1)</td>
<td>4.8</td>
<td>1.11</td>
<td>.34</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>.26</td>
<td>5.5</td>
<td></td>
<td>2.22</td>
<td>.54</td>
<td>.44</td>
</tr>
<tr>
<td>12</td>
<td>-.06</td>
<td>(.1)</td>
<td>2.4</td>
<td>1.19</td>
<td>.69</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>.15</td>
<td>15.4</td>
<td>y</td>
<td>(.2)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>.36</td>
<td>8.0</td>
<td>y</td>
<td>1.4</td>
<td>.48</td>
<td>1.30</td>
</tr>
<tr>
<td>15</td>
<td>.22</td>
<td>3.2</td>
<td>y</td>
<td>1.67</td>
<td>.38</td>
<td>.57</td>
</tr>
<tr>
<td>16</td>
<td>.38</td>
<td>4.3</td>
<td></td>
<td>1.94</td>
<td>.79</td>
<td>.63</td>
</tr>
<tr>
<td>17</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>.55</td>
<td>.3</td>
<td>y</td>
<td>1.62</td>
<td>.60</td>
<td>1.50</td>
</tr>
<tr>
<td>19</td>
<td>.61</td>
<td>1.6</td>
<td>c</td>
<td>2.26</td>
<td>.88</td>
<td>1.10</td>
</tr>
<tr>
<td>20</td>
<td>.47</td>
<td>3.0</td>
<td>y</td>
<td>1.76</td>
<td>.69</td>
<td>1.22</td>
</tr>
<tr>
<td>21</td>
<td>.53</td>
<td>3.8</td>
<td>c</td>
<td>1.53</td>
<td>.67</td>
<td>1.21</td>
</tr>
<tr>
<td>22</td>
<td>.57</td>
<td>3.8</td>
<td></td>
<td>1.80</td>
<td>.79</td>
<td>.93</td>
</tr>
<tr>
<td>23</td>
<td>.34</td>
<td>6.6</td>
<td>y</td>
<td>2.14</td>
<td>.58</td>
<td>.89</td>
</tr>
<tr>
<td>24</td>
<td>.31</td>
<td>6.0</td>
<td>y</td>
<td>1.76</td>
<td>.47</td>
<td>1.26</td>
</tr>
<tr>
<td>25</td>
<td>.29</td>
<td>10.8</td>
<td>y</td>
<td>1.77</td>
<td>.63</td>
<td>.71</td>
</tr>
<tr>
<td>26</td>
<td>.30</td>
<td>6.9</td>
<td>c</td>
<td>(.2)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>27</td>
<td>.53</td>
<td>2.6</td>
<td>y</td>
<td>1.54</td>
<td>.72</td>
<td>.92</td>
</tr>
<tr>
<td>28</td>
<td>.43</td>
<td>7.7</td>
<td>c</td>
<td>1.30</td>
<td>.61</td>
<td>.89</td>
</tr>
<tr>
<td>29</td>
<td>.24</td>
<td>1.9</td>
<td></td>
<td>1.40</td>
<td>.27</td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>.09</td>
<td>(.1)</td>
<td>3.8</td>
<td>1.37</td>
<td>.27</td>
<td></td>
</tr>
<tr>
<td>31</td>
<td>.32</td>
<td>3.5</td>
<td></td>
<td>1.79</td>
<td>.69</td>
<td>.86</td>
</tr>
<tr>
<td>32</td>
<td>.32</td>
<td>5.0</td>
<td>y</td>
<td>2.07</td>
<td>.80</td>
<td>.61</td>
</tr>
</tbody>
</table>

Avg. .36 .61 .88

temp eqlb? means temporal equilibrium? y means yes; c means close. All other symbols are defined as before.

(1) Return appears greater in the common stock than in the warrant, anomalous.

(2) Insufficient data to estimate leverage.

(3) Excluded from all reported results.
It is to be emphasized that the values of A shown above are not used in any way to obtain \( A^* \). They were used in the approximation of \( VV \) only to the extent that the two groups for the two optimal values of \( A, A_1^* \) and \( A_2^* \), were chosen on the basis that group one has values of \( A \) less than \( A^* \); the others were assigned to group two.

The actual values may be of some interest in themselves. They were calculated from equation V-5.