Essays in Macroeconomics

by
Plamen T. Nenov

B.A., Amherst College (2007)

Submitted to the Department of Economics
in partial fulfillment of the requirements for the degree of
Doctor of Philosophy

at the
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 2012

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Abstract
This thesis examines questions in macroeconomics motivated by the 2007-2008 financial crisis and its aftermath. Chapter 1 studies the impact of a housing bust on regional labor reallocation and the labor market. I document an empirical fact, which suggests that, by increasing the fraction of households with negative housing equity, a housing bust hinders interregional mobility. I then study a multi-region economy with local labor and housing markets and worker reallocation. A housing bust creates debt overhang for some workers, which distorts their migration decisions and increases aggregate unemployment in the economy. In a calibrated version of the model, I find that the regional reallocation effect of the housing bust can account for between 0.2 and 0.5 percentage points of aggregate unemployment and 0.4 and 1.2 percentage points of unemployment in metropolitan areas experiencing deep local recessions in 2010.

Chapter 2 studies a model of endogenous fluctuations in credit market conditions. I consider an economy with productivity heterogeneity and durable capital. Entrepreneurs issue debt to buy capital but have superior information about the distribution of their future productivity and, hence, of their debt repayments. Additionally, limited pledgeability of high output realizations creates a wedge between the valuations of inside and outside investors. The combination of these two frictions leads to a new channel of interaction between the price of capital and the credit market, which in turn leads to multiple equilibria and fluctuations in output, the price of capital, and leverage across equilibria. I then use the model to analyze the effect of unconventional monetary policy by a central bank.

Financial instability is often characterized by increased uncertainty, debt rollover difficulties and asset liquidation at depressed prices. Chapter 3, which is a joint work with Felipe Iachan, studies a debt roll-over coordination game with dispersed information and market-determined liquidity conditions. We describe conditions under which an improvement in the precision of individuals’ information about financial institutions’ fundamentals leads to greater financial stability. For the limiting case of arbitrarily precise private information, that condition obtains a simple form in terms of payoff elasticities. Finally, we discuss the effects of stress tests and the "living will" mandate from the Dodd-Frank act. We conclude that given our framework, the latter policy should have a large positive impact for financial stability.

Thesis Supervisor: Daron Acemoglu
Title: Elizabeth and James Killian Professor of Economics

Thesis Supervisor: George-Marios Angeletos
Title: Professor of Economics
Acknowledgments

I am extremely grateful to my advisors Daron Acemoglu, George-Marios Angeletos and Ricardo Caballero for invaluable guidance, support, and encouragement throughout my graduate studies. Each of them dedicated substantial time and energy to me during various stages of my graduate career, and they have all been important role models.

I want to thank Bengt Holmström, Guido Lorenzoni, Robert Shimer, Jean Tirole, Iván Werning, and Bill Wheaton for helpful conversations, suggestions, and comments. Participants at the MIT macroeconomics lunch and international breakfast also provided very valuable comments on my research.

I am extremely fortunate to have benefited from the help, advice, and friendship of my MIT classmates, in particular Joaquin Blaum, JB Doyle, Laura Feiveson, Felipe Iachan, Nathan Hendren, Luigi Iovino, Amir Kermani, Marti Mestieri, Sahar Parsa, Michael Peters, Jenny Simon, Alp Simsek, Juan Pablo Xandri and especially to Stefanie Stantcheva for her encouragement and good conversations.

I want to extend my deepest gratitude to my undergraduate professors, Geoffrey Woglom, Walter Nicholson, John Gordanier, Sami Alpanda, Norton Starr, Stephen George, and especially Markus Möbius for their support and encouragement to pursue a Ph.D. in economics and for their advice in the beginning of my graduate studies. Without them a Ph.D. at MIT would have been unthinkable.

Lastly I want to thank my family for their love and unwavering support. My mother and father were with me throughout the toughest times. Without their emotional support and sacrifices, I would have accomplished nothing. And, of course, my greatest thanks to my wife Anna Mari for being by my side at all times during this journey, for her strength and support, and for all her love.

Thank you to all of you.
На мама и татко
Til Anna Mari
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Chapter 1

Labor Market and Regional Reallocation Effects of Housing Busts

1.1 Introduction

The recent recession has been characterized by significant divergence in regional economic fortunes within the U.S. In particular, in 2010, unemployment dispersion across metropolitan statistical areas (MSAs) was 2.3% compared to around 1% in the early 2000s, including the 2001 recession. At the same time, internal migration in the U.S. fell in the aftermath of the recession, with inter-state migration at an all-time low of 1.4% in 2009-2010 compared to 2% in the early 2000s. Parallel to the recession, the U.S. experienced a housing bust that has had a profound impact not only on the financial system but also on households themselves. Many were left owing more on their home mortgages than the value of the underlying houses, the so called “negative equity problem”. For example, in 2009, 5 states had 17% or more of homeowners with negative housing equity, with Nevada being at the top with almost a third of homeowners in negative equity.

A common hypothesis for the decline in mobility, which popular media and commentators have extensively discussed, involves the distortion that negative equity may create in households migration decisions.¹ This possibility has raised questions about the implications of the housing

¹There are many other important economic implications of negative equity beside the mobility effects that this paper focuses on. For example, households with negative equity invest less in maintaining their homes (Melzer
bust for the performance of the labor market.\textsuperscript{2} Despite this interest, however, the impact of a housing bust on the labor market through its effect on regional reallocation has remained largely unexplored.

This paper studies the labor market and regional reallocation effects of housing busts and addresses the likely quantitative relevance of the mobility hypothesis in the context of the recent recession. I start by documenting a novel empirical fact, which shows that the fraction of households with negative equity within a state is associated with decreases in state gross out-migration, while having no effect on gross in-migration. This observation complements previous micro-studies on the link between negative equity and household mobility and suggests that by increasing the fraction of households in negative equity, a housing bust may have an adverse effect on aggregate migration and regional labor reallocation.

I then study a multi-region economy with segmented labor markets and limited mobility. The economy consists of a continuum of islands (regions) with regional labor markets characterized by search and matching frictions, competitive housing markets and local recessions and booms. Workers reside across the regions of this economy and migrate out of a region for idiosyncratic reasons and due to regional labor and housing market conditions. Their migration is directed, i.e. they migrate to regions offering the most favorable labor and housing market conditions. Each region is endowed with a fixed supply of durable housing that can be used either by workers for housing services or for production by a local sector with a concave production function. Workers own housing by holding a mortgage, collateralized by the housing unit. However, a housing bust, which I model as an unexpected depreciation shock to some housing units, reduces the value of these units. The lack of contingency of mortgages on this shock creates debt overhang for the workers owning this housing, and the penalty incurred when defaulting distorts worker migration decisions. The higher the penalty, the lower the fraction of affected workers who out-migrate and the less the reallocation.

I characterize stationary equilibria of this economy under full immobility of workers with debt overhang and show that the model can generate the positive co-movements of relative unemployment and house prices with gross out-migration and the negative co-movement with

\textsuperscript{2}For example, as the Economist, notes: "Homeowners that are reluctant to default but unable to sell at a loss are left stuck where they are. This throws sand in the gears of America's famously fluid labour market. (Economist (2010))".
in-migration observed in the cross section of states. Three features of the model are important
for this result. First, durable housing and a downward sloping housing demand by the local
sector lead to house price differences across regions with different populations. This, combined
with idiosyncratic regional preferences by workers, which act as a migration barrier and lead to
limited arbitrage of regional differences, creates a dependence of regional house prices on the
history of labor market shocks and, consequently, a rich distribution of regional house prices.
The resulting equilibrium house price heterogeneity drives the positive co-movement between
out-migration and house prices. Limited spatial arbitrage also leads to the co-movement between
out-migration and unemployment. Last, directed migration implies that regions with booming
labor markets and lower populations and consequently, house prices, have larger population
inflows leading to the negative co-movement between house prices and unemployment with in-
migration.

In this framework, I show that a housing bust has a negative impact on aggregate unemploy-
ment and that this effect is amplified by a “regional shock”, by which I mean an aggregate shock
that causes deeper local recession, as in the recent recession. The intuition for these effects is
straightforward: the migration distortion for workers with debt overhang hinders regional re-
allocation, leaving more workers in depressed regions compared to an economy without such
a penalty. Regional reallocation, however, is more important whenever regional disparities are
larger, which results in a positive interaction.

To examine the magnitude of these effects in the recent recession, I calibrate a version of
the model economy to match pre- and post-recession facts about unemployment, unemployment
dispersion, and migration. Similarly to Shimer (2005), but in the context of a model with
regional labor markets, I find that if wages are set via Nash bargaining, then the calibrated
model cannot account for the observed regional unemployment dispersion given the volatility
of regional productivity shocks. This is no longer the case if regional wages are rigid as in Hall
(2005). Using the calibrated model with wage rigidity, I find that the mobility distortions from
“negative equity” can account for around 0.2 percentage points of the aggregate unemployment
and 0.4 percentage points of the unemployment in metropolitan areas experiencing deeper local
recessions in 2010. Furthermore, an upper bound on the regional reallocation distortions of the
housing bust corresponds to an effect of 0.5 percentage points of aggregate unemployment and
1.2 percentage points of unemployment in depressed metropolitan areas. This corresponds to
between 4 and 10% of the increase in aggregate unemployment from 2007 to 2010 and to between
7 and 20% of the increase in unemployment in depressed metropolitan areas.

The calibrated model can match some of the motivating cross-state empirical facts. It can also account for the observed relation between regional unemployment dispersion and shifts of the Beveridge curve. In particular, I show that holding the aggregate vacancy-to-unemployment ratio (market tightness) constant, larger regional shocks in the model lead to both higher unemployment and larger unemployment dispersion. Given the constant aggregate vacancy-to-unemployment ratio, this implies that unemployment for a given level of vacancies is increased, which corresponds exactly to a shift of the Beveridge curve.

Lastly, I use the calibrated model to evaluate the labor market effects of two policies, proposed for solving the mortgage crisis. The first is a monthly mortgage payment reduction proposal, in the spirit of the “Home Affordable Modification Program”, while the second is a mortgage principal reduction proposal (Pozner and Zingales (2009)). The former leads to a marginal increase in aggregate unemployment since it eliminates involuntary default, which effectively increases the default penalty that workers with debt overhang face, which slows down reallocation additionally. The latter policy proposal decreases unemployment since it eliminates the default penalty.

**Related literature.** This paper spans several strands of literature. On the empirical side my paper is related to microdata studies dealing with the effects of negative equity on household mobility (Henley (1998), Chan (2001), Ferreira, Gyourko, and Tracy (2010), Schulhofer-Wohl (2010), Ferreira, Gyourko, and Tracy (2011)). These studies have to deal with different issues arising from the general absence good quality data on both household balance sheets and migration decisions that spans a sufficiently large number of households with negative equity. Henley (1998) uses the first four waves of the British Household Panel Survey and finds a strong adverse effect of negative equity on mobility but with a very small sample of households in negative equity. Similarly, Chan (2001) documents a negative effect using mortgage data from the U.S. with mortgage pre-payments serving as proxy for house moves. More recently, Ferreira, Gyourko, and Tracy (2010) examine data from the American Housing Survey, which tracks a sample of housing units in the U.S.. Identifying moves from housing ownership changes, these authors also find an adverse effect of negative equity. However, using a dataset that tracks housing units instead of households themselves creates potentially serious mismeasurement of household moves.\textsuperscript{3} The

\textsuperscript{3}In particular, Schulhofer-Wohl (2010) re-examines the data and observes that Ferreira, Gyourko, and Tracy (2010) drop observations for some uncertain housing tenure transitions, which if treated as moves, reverse their
empirical fact I document, though not at the micro level, complements these studies by side-stepping some of these issues as well as any compositional effects, which may make household level mobility distortions of negative equity irrelevant for aggregate migration.

The theoretical model is close in spirit to previous work on regional reallocation (Lucas and Prescott (1974) and more recently Alvarez and Shimer (2011), Carrillo-Tudela and Visschers (2011), Lkhagvasuren (2011), Coen-Pirani (2010), and Shimer (2007)). In these papers, dispersion in economic conditions across islands induces worker reallocation, which has implications for aggregate unemployment. However, these papers do not investigate the effects of the housing market on mobility and none of them addresses the co-movement between gross migration and local labor and housing market conditions.

The paper is also related to studies of the effect of homeownership on mobility and unemployment (Oswald (1996), Head and Lloyd-Ellis (2010)). Head and Lloyd-Ellis (2010) investigate the impact of home ownership on mobility and aggregate unemployment and build a theory that accounts for the reduced mobility of homeowners versus renters observed in micro-data and the negative correlation between regional unemployment and homeownership. In their models, housing markets affect the regional allocation of workers, although their model cannot account for the co-movement between gross migration rates and local labor and housing market conditions. Also, their modeling focus is more on the implications of the liquidity of the market for owner-occupied housing on migration decisions. As a result, their treatment of labor market frictions cannot allow for investigating the regional reallocation effects of a housing bust or its interaction with regional shock. Another paper that deals with the interactions between the housing market and the labor market is Nieuwerburgh and Weill (2010). The authors, however, address a very different issue, documenting the secular increase in dispersion in regional house prices and wages and building a model that explains and quantitatively matches this increase.4

Examining the quantitative implications of a model with regional wage rigidity connects this paper to the growing literature on rigid wages in search and matching models of aggregate unemployment (Hall (2005), Shimer (2010), Gertler and Tigari (2009))

Recent work that also deals with the labor market implications of the housing bust includes

---

4Notowidigdo (2010) and Winkler (2010) are recent empirical contributions into the effect of the housing market on labor mobility. The first one deals with how house prices affect the mobility of workers with different skill levels, while the second examines the link between homeownership and mobility.
Estevao and Tsounta (2011), Sterk (2010) and Karahan and Rhee (2011). Estevao and Tsounta (2011) look at the effects of house price declines on state structural unemployment and use those to infer the effect of housing market deterioration on aggregate unemployment. I use an alternative strategy to address this issue by quantitatively evaluating the aggregate unemployment implications of the housing bust through the lense of a structural model. Sterk (2010) builds a DSGE model with housing frictions and a reduced form migration decision by workers to show the dynamic impact of a housing bust on unemployment and the Beveridge curve. However, unlike my modeling framework, this reduced form migration approach cannot account for the observed shifts of the Beveridge curve due to regional mismatch. This would overstate the quantitative effects of decreased mobility from the recent housing bust given the parallel increase in regional unemployment dispersion. Finally, in related but independent work, Karahan and Rhee (2011) uncover an effect of house price declines on aggregate unemployment that is slightly higher in magnitude to the effects in my model.

The rest of the paper is organized as follows. Section 1.2 presents the motivating empirical facts. In Section 1.3, I present the basic model. In Section 1.4, I provide a characterization of stationary equilibria. In particular, I show how the model can account for the co-movement of state gross flows and labor and housing market conditions, and also show the housing bust effects. Finally, Section 1.5 contains the calibration results and counterfactual experiments for the housing bust, and Section 3.5 concludes. Additionally, the Appendix contains data description, proofs of results omitted from the main text, as well as details of the calibrated model and computational procedures used.

1.2 Empirical Facts

Do housing busts and the resulting negative equity problem for some households affect mobility? There are several economic mechanisms that can lead to the distortion of a household’s decision to move due to negative equity. For example, such households may face pecuniary or non-pecuniary default penalties. Another possible mechanism comes from the combination of low homeowner wealth, together with a down payment requirement on new housing purchases, which would also affect mobility. 

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5 There is an extensive empirical literature dealing with homeowner default decisions that uncovers such default penalties, see for example recent work by Bhutta, Dokio, and Shan (2010), Guiso, Sapienza, and Zingales (2009) and Foote, Gerardi, and Willen (2008).

6 Such a mechanism is in the spirit of Stein (1995).
As discussed in the Introduction, conducting micro studies on the mobility distortion effect of negative equity has been hindered by data quality and availability. Furthermore, although ideal for identifying an effect at the household level, these micro studies may not be able to tell whether such an effect is important for inter-regional migration, which is the channel that is important for the labor market. For example, it may be that negative equity does not affect long-distance migration decisions even if it affects a household’s decision to change houses. Alternatively, there may be compositional issues as negative neighborhood peer effects from households with negative equity may actually stimulate migration for other households thus increasing aggregate migration.

In this section, I take a complementary approach and examine state level aggregated data. I find suggestive evidence that a housing bust may have an adverse effect on inter-regional migration and regional labor reallocation. In particular, I look at the co-movements of the fraction of households with negative housing equity on their primary residence with the gross in- and out-migration rate of households across states.

The data I use is an unbalanced panel of annual household state out- and in-migration rates and the estimated fraction of households with negative equity for 45 of the 50 states plus the District of Columbia, from 1993-2008.\(^7\) I obtain data on annual state gross migration rates from the IRS U.S. Population Migration Database. I construct state level estimates of the fraction of households in negative equity using household level information from the Interview Survey section of the Consumer Expenditure Survey (CE). I compute the housing equity for a household’s primary residence, using information on the balance outstanding on all mortgages and equity lines of credit that the property collateralizes as well as the reported subjective property value, according to the equation:

\[
E_{ist} = \frac{\tilde{V}_{ist} - D_{ist}}{\tilde{V}_{ist}}
\]  

Here, \(E_{ist}\) is the housing equity of household \(i\) living in state \(s\) in year \(t\), \(D_{ist}\) is the total balance outstanding on all mortgages and home equity lines of credit, and \(\tilde{V}_{ist}\) is the subjective property value. I then construct an estimate of the fraction of homeowners with negative equity in state \(s\) and year \(t\) by counting the number of sampled homeowners in state \(s\) and year \(t\) with \(E_{ist} < 0\).

\(^7\)The states that are not included because of missing observations on fraction of households in negative equity for all years between 1993 and 2008 for them are Mississippi, Montana, New Mexico, North Dakota, South Dakota, and Wyoming. Appendix A contains information on how I construct all the relevant variables and controls that comprise the panel as well as details on the data sources. Here, I just provide a brief description.
and dividing by the total number of sampled homeowners in that state and year. Multiplying by the homeownership rate for state $s$ and year $t$, I get the estimated fraction of households with negative equity in state $s$, $\hat{\text{neg}}_{st}$.

I focus on the following panel regressions:

$$\log(out_{st}) = \alpha_s + \beta_{out} \hat{\text{neg}}_{st} + \mathbf{x}'_{out} \gamma_{out} + \epsilon_{out}$$

$$\log(in_{st}) = \alpha_s + \beta_{in} \hat{\text{neg}}_{st} + \mathbf{x}'_{in} \gamma_{in} + \epsilon_{in}$$

where $out_{st}$ is the gross out-migration rate for state $s$ and year $t$, $in_{st}$ is the gross in-migration rate, $\hat{\text{neg}}_{st}$ is the estimated fraction of households with negative equity in state $s$ at time $t$, $\alpha_s$ and $\zeta_t$ are state and year fixed effects and $\mathbf{x}_{st}$ is a vector of other covariates.

I control for state economic and housing market conditions by including the log of the state unemployment rate, and the log of state house prices relative to the national level. In addition, I include a measure of the relative wage as the log of the ratio of state average hourly wage in manufacturing to the national counterpart and the log of state income per capita relative to national income. I control for mortgage credit conditions proxied by the average debt-to-value ratio (computed from the CE) and the home ownership rate.

There are two potential issues with using the estimate $\hat{\text{neg}}_{st}$ rather than the true fraction $\text{neg}_{st}$. The first issue comes from potential misclassification problems of households, since I use subjective property values, and these are noisy estimates of the actual property price (Ferreira, Gyourko, and Tracy (2010)). To address this, I compare my estimates of the fraction of negative equity by state with estimates from First American CoreLogic for 2009. The CoreLogic estimates are based on much more precisely measured house price data and hence are prone to much lower misclassification problems but are available for a much shorter time period (2008-2010) (see CoreLogic (2009) for details). Comparing the two series reveals a very high cross-sectional correlation of $\rho = 0.731$. Hence, misclassification does not appear to be a problem for the state-level variation within a given year. As an additional robustness check, I also run the panel regression using the estimated fraction of households with equity away from zero, i.e. the number of sampled homeowners in state $s$ and year $t$ with $E_{ist} \leq c < 0$ divided by the

---

8Figure 1-3 in Appendix A plots the two series and a linear regression line. This also highlights the high co-movement between the two series with any discrepancy likely due to measurement error. The only difference is that the CE estimates predict lower values for all states.
total number of sampled homeowners in that state and year. The second issue is classical measurement error that comes from using an estimate rather than the true value. I address this issue below.

The estimation results for the panel regressions are shown in Table 1.1. The first and third columns present panel regression results with no additional controls, while the second and fourth column show results with year fixed effects and state level controls. The coefficients on households with negative equity for the regressions with out- and in-migration and no controls are negative and statistically significant at the 5% level. With the controls, the coefficient on households with negative equity is still negative and significant at the 10% level for the regression with state out-migration rate as the dependent variable. In contrast the coefficient in the regression with state in-migration as a dependent variable is insignificant and close to 0. Using other cutoffs produces similar results as Table 1.10 in Appendix A shows.

The asymmetric effect of neg on out- versus in-migration is very intriguing. One possible explanation, which I explore in the model in Section 1.3 and in the rest of the paper, is that negative equity affects household regional reallocation decisions, which ultimately leads to these co-movements. Studying a multi-region equilibrium model with worker reallocation will also allow me to see how the regional reallocation effects of a housing bust map into an effect on both local and aggregate labor markets, since it is hard to empirically establish a direct link between a housing bust and the labor market.

According to the coefficient estimates, a 10 percentage point increase in the fraction of households in negative equity decreases the out-migration rate by around 2.5%. However, as mentioned above, it is important to note that measurement error in the estimated negative equity fractions, \( \hat{neg}_{st} \), biases down the regression coefficient estimates. Having a notion of the size of attenuation bias is necessary for assessing the quantitative importance of this channel, as I do in Section 1.5. Fortunately, the equity data, used to construct \( \hat{neg}_{st} \), can provide information on that bias.

Since \( \hat{neg}_{st} \) is an estimate of the true fraction, neg_{st}, it contains estimation error. This implies that the relationship between neg_{st} and \( \hat{neg}_{st} \) can be represented by:

\[ \hat{neg}_{st} = neg_{st} + v_{st} \]  

(1.4)

In particular, Table 1.10 in Appendix A shows results for \( c = -0.1 \) and \( c = -0.2 \).
Table 1.1: State panel regressions

<table>
<thead>
<tr>
<th>Dep. Variable:</th>
<th>out-migration rate, 100 ln(out)</th>
<th>in-migration rate, 100 ln(in)</th>
</tr>
</thead>
<tbody>
<tr>
<td>households with negative equity, (%)</td>
<td>-0.521** (0.198)</td>
<td>-0.502** (0.195)</td>
</tr>
<tr>
<td>relative unemployment</td>
<td>0.168*** (0.0359)</td>
<td>-0.237*** (0.0388)</td>
</tr>
<tr>
<td>relative house price</td>
<td>0.200*** (0.0679)</td>
<td>-0.170*** (0.0592)</td>
</tr>
<tr>
<td>relative wage rate</td>
<td>0.0140 (0.0829)</td>
<td>-0.0115 (0.165)</td>
</tr>
<tr>
<td>relative income</td>
<td>-0.0895 (0.302)</td>
<td>0.647** (0.246)</td>
</tr>
<tr>
<td>ave. debt-to-value ratio</td>
<td>0.0396 (0.0496)</td>
<td>0.0838 (0.0521)</td>
</tr>
<tr>
<td>home ownership rate</td>
<td>0.424 (0.394)</td>
<td>-0.346 (0.321)</td>
</tr>
<tr>
<td>Year FE</td>
<td>No 625</td>
<td>Yes 606</td>
</tr>
</tbody>
</table>

Notes: Robust standard errors with clustering on state in parenthesis; * = significant at 10%; ** = significant at 5%; *** = significant at 1%. Source: Own calculations from BLS, IRS, CE, FMHPI, and US Census Bureau. See Data Appendix for detailed description. Relative house price is constructed from the CMHPI index and median house value by state from the 2000 Census. Relative unemployment is the log of the unemployment rate for that state and year divided by national unemployment rate for that year. Relative wage rate is the log of the average hourly average manufacturing wage for that state and year divided by the national average hourly manufacturing wage for that year. Relative income is the log of income per capita for that state and year divided by national income per capita for that year. Housing equity is defined as \( E = \frac{V - D}{V} \) where \( D \) is the total debt balance outstanding on all mortgages and home equity loans that a property collateralizes and \( V \) is the subjective property value. Households with negative housing equity is the fraction of the population who are homeowners with \( E < 0 \). Average debt-to-value ratio is defined as the average of \( \frac{D}{V} \). Home ownership rate is the percent of households in the state that own a house.
where $\nu_{st}$ is a random variable with mean zero and variance $\sigma^2$, distributed i.i.d. over $s$ and $t$ and independent of $neg_{st}, \forall s, t$. I can then estimate $\sigma^2$ by using estimates of the sampling variance of $\bar{neg}_{st}$. Using the estimates for $\sigma^2$ and $Var(\bar{neg}_{st})$, and independence of $\nu_{st}$ and $neg_{st}$, I compute $Var(neg_{st})$. I derive a lower bound on the effect of attenuation bias by constructing a reliability ratio, $\lambda$, for a univariate regression, which is given by $\lambda = \frac{Var(neg_{st})}{Var(\bar{neg}_{st})} = 0.43$. This means that $|\beta_{neg}| > \frac{1}{\sqrt{\lambda \beta_{neg}}} = 0.565$. This, however, is a lower bound on the effect of attenuation bias given the high $R^2$ of 0.53 from regressing $\bar{neg}_{g}$ on the other controls in the panel regression and also given the panel nature of the data. Therefore, attenuation bias appears very important, and the true coefficient, $\beta_{neg}$, is substantially larger in magnitude than the estimate, $\beta_{neg}$. This in turn implies that the regional reallocation distortions of a housing bust may be large. I investigate this further in Section 1.5.

Another salient set of results is the positive co-movement of relative unemployment and house prices with out-migration and negative co-movement with in-migration. More specifically, a ten percent increase in state relative unemployment is associated with an increase in out-migration of around 1.7% and a decrease in in-migration of 2.4%. Overall, these estimates point to unemployment being an important driver of regional reallocation and, conversely, migration being an important adjustment mechanism in response to local recessions (Blanchard and Katz (1992)). Note, however, that both out- and in-migration co-move with changes in unemployment, which is an important observation that restricts the set of models of regional reallocation that can account for them as opposed to accounting for net migration responses only. For example, as I discuss in Section 1.3, models with undirected mobility, though accounting for net migration cannot generate the observed in- and out-migration patterns.

The co-movement of relative house prices with out- and in-migration shows that even after controlling for labor market conditions, house price changes are associated with variation in migration. One possible explanation for this observation is through perfect spatial arbitrage and large variations in amenities or construction costs at annual frequency as in the classical spatial equilibrium framework (Roback (1982)). In Section 1.3, I show an alternative explanation, which relies on limited regional mobility and a durable housing stock.

Therefore, one can summarize the results of this section as follows:

---

10 More specifically, I construct estimates of the sampling variance for each observation $\bar{neg}_{st}$, using a bootstrap procedure on the equity data for each state and year, and then average over all observations to obtain $\hat{\sigma}^2$.

11 Saks and Wozniak (2007) and Jackman and Savouri (1992) have obtained similar observations as side results but do not provide any implications they may have.
1. A higher fraction of households with negative equity within a state correlates with decreases in migration out of that state but is not correlated with migration into the state;

2. Higher unemployment in a state relative to the national level correlates with increased migration out of the state and decreased migration into the state controlling for relative house price;

3. Higher relative house prices correlate with increased out-migration and decreased in-migration controlling for relative unemployment.

I turn next to a model of a multiple-region economy that can match these facts. I then use the model to analyze the labor market and regional reallocation effect of a housing bust and the likely magnitudes of these effects in the recent recession.

1.3 Basic Model of Regional Reallocation

In this section I propose a model of regional reallocation, which provides insights into how mobility distortions as a result of a housing bust affect regional reallocation and, ultimately, the labor market. I consider a discrete time economy with an infinite number of periods \( t = 0, 1, 2, \ldots \). The economy consists of a measure 2 of islands or regions. The economy is populated by a measure \( L \) of infinitely lived workers, distributed across regions of the economy. Workers are risk neutral, derive utility from consumption as well as from housing (see Section 1.3.3 below), and can supply 1 unit of labor. The initial measure of workers in each region \( j \) is given by \( l_{-1}^j \leq L \), with \( \int l_{-1}^j dj = L \). The end-of-period or post-migration measure of workers in a region \( j \) at time \( t \) is given by \( l_t^j \in [0, L] \), \( \bar{L} \geq L. \)

1.3.1 Regional labor markets, job creation, and destruction

In each region there is a representative firm that can open job vacancies at a per-period cost of \( k \) and recruit workers. For the basic model I consider in this section, I assume that jobs remain productive for one period only. The reason for this is analytical tractability, since under it there will be no agent heterogeneity in terms of the idiosyncratic employment state at the time of the migration decision. This leads to only one relevant endogenous state variable that affects...

\(^{12}\)The upper bound of \( \bar{L} \) is a technical restriction necessary for showing the analytical results. It can be thought of as a limit on the space available within a region.
migration decisions, which makes equilibrium characterization possible. However, in Section 1.5, where I look at the quantitative effect of the housing bust in the recent recession, I consider the more general set-up with stochastic job destruction.

Each job in region \( j \) has the capacity for the production of \( A^j_t \) units of the consumption good at time \( t \) if it can hire a worker. Regional productivity \( A^j_t \) can have two possible realizations \( \overline{A} \) or \( A \) (\( \overline{A} \geq A \)) and follows a Markov chain with persistence \( \rho \geq \frac{1}{2} \). Furthermore at any time \( t \) one half of regions have \( A^j_t = \overline{A} \) and the other half have \( A^j_t = A \). I refer to the former as (relatively) “booming” regions and to the latter as (relatively) “depressed” regions.\(^{13}\)

The labor market of each region is characterized by a search and matching friction as in the standard Diamond-Mortensen-Pissarides framework (Pissarides (2000)). In particular, in a region \( j \) at time \( t \), after migration, a measure \( l^j_t \) of workers and measure \( v^j_t \) of vacancies try to match with each other. Matching is described by a standard regional reduced-form constant returns to scale (CRS) matching function \( m^j \left( l^j_t, v^j_t \right) \) giving the total number of regional matches per period. I assume that matching functions are identical across regions, i.e. \( m^j \left( l^j_t, v^j_t \right) = m \left( l^j_t, v^j_t \right) = l^j_t \cdot m \left( 1, \frac{v^j_t}{l^j_t} \right) \). Letting \( \theta^j_t = \frac{v^j_t}{l^j_t} \) be the regional labor market tightness and defining \( \mu(\theta) = m(1, \theta) \), we have that \( m \left( l^j_t, v^j_t \right) = l^j_t \mu \left( \theta^j_t \right) \). This translates into a job finding probability for a worker in a given period of \( 1 - \mu(\theta^j_t) \) and a job filling probability for a vacancy of \( \frac{\mu(\theta^j_t)}{\theta^j_t} \). Workers that remain unmatched in a given period are considered unemployed and receive a period payoff of \( e \). The total measure of unemployed in region \( j \) is then given by \( \left( 1 - \mu(\theta^j_t) \right) \overline{l}_t \) and the unemployment rate is simply \( \frac{\overline{l}_t}{l_t} \).\(^{14}\)

### 1.3.2 Job creation decisions and wage determination

I allow for wages to be determined either by Nash bargaining or to be rigid as in Hall (2005). The particular wage determination rule does not affect the results of this section, so I focus on a model with the more standard assumption of Nash bargaining. However, as I discuss in Section 1.5, the process of wage determination does matter for the calibrated model.

Given one period job length, the vacancy posting decision of the representative firm is straightforward. In particular, the firm opens vacant jobs until the cost of opening a vacancy

\(^{13}\)The local productivity shocks can be considered a proxy for any shocks that cause local recessions and result in regional variations in unemployment and unemployment differences across regions.

\(^{14}\)This definition of unemployment comes from the one period job length assumption.
equals the expected payoff from posting a vacancy, or:

\[ k = \frac{\mu(\theta^j_t)}{\theta^j_t} \left( A^j_t - w^j_t \right) \]  

(1.5)

where \( w^j_t \) is the wage rate in region \( j \) at time \( t \). Letting workers' bargaining power be \( \eta \in [0, 1) \), the wage rate is:

\[ w^j_t = e + \eta \left( A^j_t - e \right) \]  

(1.6)

where \( e \) is the outside option. Hence, equations (1.5) and (1.6) pin down the labor market tightness in a region as a function of \( A^j_t \), so that one can write \( \theta(A) \) for a given region productivity \( A \). Note that labor market tightness is a function only of regional productivity and not of the labor force in a region. This is because of the representative firm's production technology has constant returns.

1.3.3 Regional housing markets, home financing and debt overhang

In every region \( j \), there is a fixed supply \( L \) of undepreciated housing units trading in a competitive market at price \( p^j_t \). Workers derive utility \( \gamma > 0 \) at the end of each period, in which they own a single unit of housing. I normalize the period utility from renting to 0.\(^{15}\) Apart from the demand by workers, there is also residual demand for housing from a sector of local firms that use housing (and housing only) for production, with concave production function \( g(h) \), where \( g(0) = 0 \), \( \lim_{h \to 0} g'(h) = \gamma \) and \( \lim_{h \to L} g'(h) = a \leq \gamma \) and \( g''(h) \leq 0 \). This pins down regional house prices and potentially creates house price differences across regions.\(^{16}\)

Workers cannot borrow against their future income and have no access to a savings technology. Instead they buy housing via an infinite period financial contract (mortgage), which a housing unit collateralizes. Through that contract, they borrow from a competitive sector of financial intermediaries that face an exogenous interest rate of \( r = \frac{1}{\beta} - 1 \). The financial contract specifies a sequence of repayments, \( \left\{ d_s^j \right\}_{s=t}^{\infty} \), and an associated sequence of debt balance levels,

\(^{15}\)This is without loss of generality, since what matters for the home-ownership decision is the difference in utilities.

\(^{16}\)The simplistic housing market assumptions permit me to derive analytical results. One could include a housing construction sector and natural depreciation of the housing stock but that would not change the model's main implications.
\[ \{b_s^j\}_{s=t}^{\infty}, \] which are linked via a promise-keeping constraint:

\[ b_s^j = d_s^j + \frac{1}{1 + r} E_s \left[ b_{s+1}^j \right], \quad s \geq t \quad (1.7) \]

Furthermore, the mortgage is fully collateralized by the value of the housing unit, i.e.

\[ b_s^j \leq p_s^j, \quad s \geq t \quad (1.8) \]

The promise keeping constraint is the recursive form of \[ b_s^j = E_s \left[ \sum_{t=s}^{\infty} d_t^j \right] \] and a no-ponzi game condition, which is automatically satisfied in this case given full collateralization. I abstract away from the possibility of repayment risk for mortgages issued in equilibrium by restricting attention to economies, in which \[ d_t^j \leq e, \quad \forall t, j \], i.e. I assume that even unemployed workers can cover mortgage repayments in every period.\(^{17}\) Finally, a homeowner is free to terminate the mortgage contract at the beginning of every period, in which case the housing unit is sold and the financial intermediary is repaid. Full collateralization and no repayment risk implies that default is not expected to occur on contracts issued in equilibrium.

Hence, this financial contract corresponds to a frictionless financing arrangement, in which repayments are free to vary over time and there is no default. At the same time, such an arrangement is sufficient to incorporate a relevance of regional house prices for worker’s out- and in-migration decisions. Given frictionless financing and equal discount rates for the worker and the financial intermediary, it follows that workers will be indifferent over any contract \[ \{d_s^{j}\}_{s=1}^{\infty}, \{b_s^{j}\}_{s=1}^{\infty} \] that satisfies the promise keeping constraints (1.7), full collateralization constraints (1.8) and the no repayment risk constraints \[ d_t^j \leq e, \forall t, j \]. Therefore, without loss of generality, I can restrict attention to a financial contract with debt balance levels \[ b_t^j = p_t^j, \forall t, j \]. Selecting this contract significantly simplifies characterization of the worker’s problem and hence equilibrium characterization as it makes the debt balance level redundant as a state variable when defining a value function for the worker. Additionally, I assume that this restriction also applies to workers who start at \[ t = 0 \] as counterparties to a financial contract.

I model a housing bust as an unexpected housing depreciation shock at the beginning of \[ t = 0 \] for a measure \[ b_{0,-1}^j \] of homeowners in region \( j \). By depreciated housing, I mean that new potential owners of the housing unit do not derive utility from it and local firms cannot use it in

\(^{17}\)Assuming that \( e \geq \gamma \) is sufficient for this.
production so its price is 0. This assumption captures in a simple way the large heterogeneity of houses across local housing markets and the heterogeneity in homeowner balance sheets existing in reality (and therefore the differential impact of house price declines on homeowners).

I assume that the depreciation shock is unforeseen, i.e. it is a state of the world, which financial contracts are not contingent on. Therefore workers with depreciated housing are bound by the terms of the frictionless financial contract, so \( b_0^j > 0 \) for them as well, and they have debt overhang. However, these workers are free to default on that contract at any time but have to incur a default penalty of \( \zeta > 0 \). Workers with a depreciated housing unit still derive ownership utility \( -y \) from it until they default.\(^{18}\)

Finally, note that given the concave production function, firms in the local sector may be making non-zero profits in equilibrium. Similarly, financial intermediaries may be making non-zero profits since they start the economy as counterparties to financial contracts already in place. In order to account for these I assume that there is a small measure of immobile risk neutral agents living on each island who own the local firms and intermediaries and consume the any profits from these.

1.3.4 Regional migration

Workers have an idiosyncratic region preference \( \epsilon \) for regions they currently reside in. At the beginning of each period a worker draws a new \( \epsilon \) from a continuous distribution \( F \) with density function \( f \) with \( E[\epsilon] = 0 \) and support over \([-B, B]\) for some \( B > 0 \). After observing his match quality, a worker decides whether to move to a different region. Moving is instantaneous and entails a fixed cost of \( c \). Upon moving, the worker terminates the mortgage debt contract, in which case the housing unit is sold if it is not depreciated and lenders are repaid. Otherwise workers default and incur the cost \( \zeta \). Assuming that a worker with debt overhang has to terminate the mortgage when moving may at first seem problematic. After all, in reality, households with negative equity are free to move to a different region, while at the same time keeping the house and not defaulting. There are two issues with this argument. First of all, such behavior is not costless as a household still has to keep mortgage payments and pay for their new residence. Such costs would also affect a household’s migration decision in the same way that forcing a household to incur a default penalty upon moving would. Secondly, such behavior by households

\(^{18}\)Upon worker default the depreciated housing unit becomes useless to all agents in the economy.
does not seem relevant in the data.\(^{19}\)

Finally, I assume that migration is directed, i.e. a worker observes regional characteristics and migrates to the region that gives him the highest expected value. After moving, the worker draws an \( \epsilon \sim F \) for the new region.

### 1.3.5 Timing

The timing within a period is as follows:

1. Agents observe the realization of regional productivity \( A \);
2. Workers draw region specific payoffs and make migration and mortgage termination decisions;
3. Housing market opens;
4. Firms make job creation decisions for new jobs;
5. Matching of workers and jobs;
6. Production occurs and wages are paid;

### 1.4 Equilibrium

I will be focusing on symmetric recursive equilibria, in which each region \( j \) is fully characterized by a vector of state variables \( X_i^j = (A_i^j, l_{0,t-1}^j, l_{1,t-1}^j, \bar{v}_t, \underline{v}_t) \). This contains the current period productivity \( A_i^j \), as well as the beginning-of-period measure of workers with and without debt overhang, \( l_{0,t-1}^j \) and \( l_{1,t-1}^j \), respectively. Lastly, \( \bar{v}_t \) and \( \underline{v}_t \) denote the beginning-of-period distributions of workers in booming and depressed regions, respectively. Also, I define \( X_i^j = (A_i^j, l_{0,t-1}^j, \bar{v}_t, \underline{v}_t) \). These variables are relevant for the worker’s problem and regional house price determination, while \( l_{0,t-1}^j \) is only relevant for determining regional populations.\(^{20}\)

\(^{19}\)The Consumer Expenditure Survey data, which I used in Section 1.2, contains information on whether a sampled household still owns their previous home, given their current housing status (renters or home-owners). It turns out that only 0.5% of households still own their previous home and have a mortgage on it for the period 2008-2010 compared to 0.3% for 1993-2007. This, however, also includes households in the process of selling or foreclosing on their previous home. Unfortunately, CE data from 2007-2010 does not include information on property values apart from the household’s primary residence, so one cannot examine what fraction of households actually still own their previous home and have negative equity on it. For 1993-2006, this fraction is effectively 0. Therefore, the actual fraction of households who keep their old home because of negative equity but move into a different one during the housing bust period of 2008-2010 is likely to be much smaller than 0.5%.

\(^{20}\)This block recursive structure is standard for models with search frictions (Carrillo-Tudela and Visschers (2011)).
A symmetric recursive equilibrium will then be defined by laws of motion for the endogenous variables, \( \tilde{t}_0, \tilde{t}_1 \), \( \tilde{t}_0 \left( \tilde{X}_i^j \right) \) and \( \tilde{t}_1 \left( \tilde{X}_i^j \right) \), and \( \tilde{\nu}_t = \tilde{\nu}_t \left( \tilde{t}_t \nu_t \right) \), \( \tilde{\nu}_t = \tilde{\nu}_t \left( \tilde{t}_t \nu_t \right) \), and by functions, \( \theta \left( A_i^j \right), w \left( A_i^j \right), p \left( X_i^j \right) \) giving regional market tightness and wages and regional house price as a function of the regional state such that (i) workers migration decisions are optimal given the laws of motions for \( A \) and the endogenous state variables, (ii) laws of motions for the endogenous state variables are consistent with workers migration decisions and with population constancy in the economy.

1.4.1 Regional house prices

I first characterize regional house prices. The demand for non-depreciated housing of a representative firm from the local sector in region \( j \) solves:

\[
\max_{h_i^j} \left\{ g \left( h_i^j \right) + \beta E_t \left[ p \left( X_{i+1}^j \right) \right] h_i^j - p \left( X_i^j \right) h_i^j \right\}
\]

which immediately implies that:

\[
\begin{align*}
\text{if } & p \left( X_i^j \right) < \gamma + \beta E_t \left[ p \left( X_{i+1}^j \right) \right] \\
\text{then } & h_i^j > 0 \\
\text{if } & p \left( X_i^j \right) \geq \gamma + \beta E_t \left[ p \left( X_{i+1}^j \right) \right] \\
\text{then } & h_i^j = 0
\end{align*}
\] (1.9)

since \( g'(0) = \gamma \). I can also derive the housing demand by workers. In Appendix B, I show that (i) workers with no debt overhang demand housing iff \( \tilde{d} \left( X_i^j \right) \leq \gamma \), and (ii) workers with debt overhang do not demand non-depreciated housing. The first result, immediately implies that the equilibrium house price satisfies \( p \left( X_i^j \right) \leq \gamma + \beta E_t \left[ p \left( X_{i+1}^j \right) \right] \) and therefore, in equilibrium, all workers with no debt overhang buy housing. Defining

\[
d \left( X_i^j \right) = \begin{cases} 
\frac{g' \left( L - l_1 \left( X_i^j \right) \right)}{\gamma} & l_1 \left( X_i^j \right) < L \\
\frac{\gamma}{\gamma} & l_1 \left( X_i^j \right) = L 
\end{cases}
\] (1.10)

it follows that

\[
p_i^j = p \left( X_i^j \right) = d \left( X_i^j \right) + \beta E_t \left[ p \left( X_{i+1}^j \right) \right]
\] (1.11)
This, together with a transversality condition on \( p_j^l \), \( \lim_{T \to \infty} \beta^T E_t \left[ p_j^l \right] = 0 \) \( \forall t, j \), implies that

\[
p_j^l = E_t \left[ \sum_{s=0}^{\infty} \beta^s d \left( X_{t+s}^j \right) \right]
\]

where \( \tilde{d}_t = d \left( X_t^j \right) \).

### 1.4.2 Migration decisions

I next turn to characterizing the worker’s migration decision. Given (1.10) and (1.11) it follows that all workers with no debt overhang buy housing in equilibrium. Furthermore, since there is no involuntary default (due to no repayment risk) all default is strategic and arises whenever a homeowner with debt overhang chooses to migrate (see Appendix B). This is because in the model the utility difference between having debt overhang and not is lower than the default penalty as the only benefit from defaulting without moving is the forgone cost of default one period later. Given these observations let \( V^h (X) \) be a worker’s end-of-period value given the regional state \( X \) and the idiosyncratic housing state \( (h = 1 \text{ for no debt overhang and } h = 0 \text{ for debt overhang}) \). Then defining

\[
\tilde{V} = \max_\tilde{x} \left\{ V^l (\tilde{x}) \right\}
\]

as the migration value, we have that:

\[
V^h (X) = \gamma - d (X) + e + \mu (\theta (A)) (w (A) - e) + \beta E_X \left[ W^h \left( X' \right) \right]
\]

where

\[
W^h (X) = \max_\tilde{\epsilon} \left\{ F (\tilde{\epsilon}) \tilde{V} + (1 - F (\tilde{\epsilon})) V^h (X) - F (\tilde{\epsilon}) (c + (1 - h) \zeta) + \int_{\tilde{\epsilon}} \epsilon dF \right\}
\]

is the beginning-of-period value function for the worker. The interpretation of these value functions is straightforward. For \( V^h \), the first part captures the per-period utility from employment/unemployment as well as utility from owning housing, net of mortgage repayment. The second is the expected value in the next period, which takes into account the migration option of the worker. In particular for a given region preference \( \tilde{\epsilon} \), a worker compares the value of staying in the region to the value of moving to a region that offers the highest expected utility,
net of the migration cost and potential default penalties. Given the structure of the problem, migration will follow a cutoff rule for \( \epsilon \), which I denote by \( \bar{\epsilon} \), and which is given by:

\[
\bar{\epsilon}(X, h) = V - V^h(X) - c - (1 - h) \zeta
\]  

(1.16)

Then for \( \epsilon < \bar{\epsilon}(X, h) \) the worker migrates and for \( \epsilon > \bar{\epsilon}(X, h) \) the worker stays, which means that the fraction of workers migrating from a region with state \( X \) is:

\[
q(X, h) = \Pr(\epsilon < \bar{\epsilon}(X, h)) = F(\bar{\epsilon}(X, h))
\]  

(1.17)

which is also the ex ante probability of worker migration prior to realization of \( \epsilon \). Note that it immediately follows that \( q(X, 0) \leq q(X, 1) \), i.e. a worker with debt overhang will migrate out of a region less often than a worker with no debt overhang.

1.4.3 Laws of motion for endogenous state variables

Given the worker migration decisions above, it follows that the end-of-period measure of workers with no debt overhang in a given region \( j \), \( l^1_{1,t} \), is:

\[
l^1_{1,t} = l^1_t(X^1_t) = \left(1 - q(X^1_t, 1)\right) l^1_{1,t-1} + \Psi\left(X^1_t\right)
\]  

(1.18)

where \( \Psi\left(X^1_t\right) \) is a function that gives the measure of workers migrating into the region at time \( t \). In particular,

\[
\Psi\left(X^1_t\right) \geq 0 \quad X^1_t \in \arg\max_x \{V^1(x)\}
\]

\[
\Psi\left(X^1_t\right) = 0 \quad \text{otherwise.}
\]

i.e. due to directed migration, some regions do not experience worker inflows. The exact form of \( \Psi(X) \) for \( X \in \arg\max_x \{V^1(x)\} \) is determined in equilibrium. Similarly, the end-of-period measure of workers with depreciated housing in a given region, \( l^0_{0,t} \) is given by:

\[
l^0_{0,t} = l^0_t(X^0_t) = \left(1 - q(X^0_t, 0)\right) l^0_{0,t-1}
\]  

(1.19)

Therefore, if \( q(X, 0) \neq 0 \), \( \forall X \), \( \lim_{t \to \infty} l^0_{0,t} = 0 \), i.e. unless such workers are completely hindered from migrating, so that \( q(X, 0) = 0 \) for some \( X \), the measure of workers with depreciated housing goes to 0 over time.
1.4.4 Stationary Equilibrium

I now turn to the first main result of this section, showing how the model can account for Facts 2 and 3 from Section 1.2. I will show this in a symmetric stationary equilibrium of this economy, in which each region has a measure \( l_0 \) of workers with debt overhang. Given that I will be looking at a stationary equilibrium, it follows that the distributions \( \bar{\nu} \) and \( \bar{\nu} \) are time invariant, and so the state vector \( X \) will contain \( A_t^j \) and \( \bar{H}_{t-1}^j \) only. Also, for notational convenience I use \( l \) to denote the beginning-of-period measure of workers with no debt overhang in a region in place of \( l_1 \).

I define a stationary equilibrium for economies that satisfy the following assumption:\(^{21}\)

**Assumption:** \( \exists \bar{B} > 0 \text{ s.t. } q(X,1) > 0, \forall X, \forall B \geq \bar{B} \text{ and } \exists \zeta, \text{ s.t. } q(X,0) = 0, \forall X, \forall \zeta \geq \zeta. \)

This assumption means that I consider economies, in which there is gross out-migration out of any region, while workers with debt overhang are completely immobile. The first part of the assumption is technical, while the second part allows me to have workers with debt overhang in a stationary equilibrium and hence to examine conceptually, the effects of a housing bust on the labor market. Therefore, for the rest of this Section, I refer to workers with debt overhang as immobile workers and to workers with no debt overhang as mobile workers. Also, I focus on equilibria in which both \( V^1(A,l) \) and \( \bar{H}^1(A,l) \) are continuous.\(^{22}\)

I first characterize the regional dynamics in this economy and the link between migration and regional characteristics resulting from the equilibrium behavior of workers. A set of technical results in Appendix B, Lemmas 11, 12, and 13, which characterize the equilibrium properties of \( V^1(A,l_1) \) and \( \bar{H}^1(A,l_1) \) allow for this. Here, I only summarize their implications. First of all, Lemma 11 implies that workers with no debt overhang weakly prefer regions with lower populations. Secondly, the law of motion \( \bar{H}^1(A,l) \) is increasing in \( l \), so that regions with a high beginning-of-period population of mobile workers can never end up with lower end-of-period populations compared to regions with low beginning-of-period population.

Lemma 12, in turn, shows that there are stable populations of mobile workers depending

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\(^{21}\)The definition of a symmetric stationary recursive equilibrium is given in Appendix B and follows the broader definition from Section 1.4.

\(^{22}\)Note that I do not show existence of such equilibria. However, Lemma 10 in Appendix B shows that having \( V^1(A,l_1) \) and \( \bar{H}^1(A,l_1) \) continuous is mutually consistent and hence possible in equilibrium.

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on regional labor productivity $A$, $l^*$ for regions with productivity $A$, and $\bar{l}^*$ for regions with productivity $\bar{A}$. Regions with productivity $A$ and with $l < \bar{l}$ will be experiencing inflows that increase their population of mobile workers to $\bar{l}^*$ or slow declines in population towards $l^*$, if $l > l^*$ and similarly for regions with productivity $A$. Therefore, Lemma 12 describes how the in-migration function $\Psi(A, l)$ looks like in equilibrium.

Lastly, Lemma 13 shows that in equilibrium, regions with high productivity weakly dominate regions with low productivity in mobile workers migration decisions for any given population $l$. This implies that, depending on parameter values, there can be two types of equilibria. The first type is a "partial compensation" equilibrium with, $l^* = 0$ and $V^1(A, l) < V^1(\bar{A}, l)$, $\forall l$, while the second is a "full compensation" equilibrium with $l^* \geq 0$ and $V^1(A, l^*) = V^1(\bar{A}, \bar{l}^*)$.

In the first type of equilibrium, house price differences cannot compensate for labor market differences for any population of mobile workers, whereas in the second house price differences do compensate fully for labor market differences as long as populations in low productivity regions fall sufficiently.

Therefore, we can make the following observations about the dynamic evolution of regions in a stationary equilibrium of this economy:

**Lemma 1.** The following hold in a stationary equilibrium of this economy

1. Regional populations of mobile workers lie in the set $[l^*, \bar{l}^*]$;

2. Transitioning from depressed to booming, a region's population of mobile workers increases to $\bar{l}$;

3. A depressed region's population of mobile workers moves down towards $l^*$ experiencing a decreasing out-migration rate as the population of mobile workers declines towards $l^*$;

4. Depressed regions experience no in-migration apart from regions with $l \in [l^*, \bar{l}]$, where $\bar{l}$ is given by equation (1.29) in Appendix B;

5. The stationary distributions $\nu^*$ and $\bar{\nu}^*$ are discrete.

**Proof.** See Appendix B.

Figure 1-1 summarizes the implications of Lemma 1 for regional net migration in "full compensation" equilibria. Regions with low productivity slowly lose population, whereas a region that experiences a positive productivity shock moves up to $l^*$.

---

23The regional evolution for "partial compensation" equilibria is similar to that of "full compensation" equilibria.
Therefore, idiosyncratic regional preferences and the mobility cost $c$ lead to limited spatial arbitrage. Depressed regions lose population more slowly than in an economy with frictionless mobility. Limited spatial arbitrage, in turn, creates a non-degenerate distribution of regions over populations. This, combined with durable housing and a downward sloping demand by the local sector create a dependence of regional house prices on the history of labor market shocks. Figure 1-2, which graphs the simulated paths for regional productivity, unemployment rate and house prices for a region, clearly shows this history dependence.

Not surprisingly, given the one-period-job-length assumption, the regional unemployment rate simply jumps with regional productivity. The behavior of house prices, however, shows how a region hit by a negative productivity shock experiences an initial large drop in house prices and subsequent smaller declines. A spatial equilibrium model with perfect spatial arbitrage in the spirit of Roback (1982) would imply a jump in house prices in response to regional productivity only, similarly to the response of unemployment. In contrast, as the Figure illustrates, limited spatial arbitrage creates very different house price dynamics with house prices movements occurring without labor market shocks. Lastly, Figure 1-2 also shows how population inflows into regions hit by a sequence of negative productivity shocks never actually reaching $I^* = 0$ but coming arbitrarily close.
regions create house price jumps.\textsuperscript{24}

The dependence of regional house prices on the history of labor market shocks also implies a rich distribution of regions over house prices. This variation, in turn, leads to the observed co-movements between regional house prices and regional out- and in-migration from Section 1.2 as I now show. First of all, I use the characterization results above to clarify how regional inflows and outflows are related to regional state variables.

**Proposition 2.** Let $\text{out}(A, l)$ and $\text{in}(A, l)$ be the out-migration and in-migration rates for a region described by $(A, l)$. Then:

1. $\text{out}(A, l) \leq \text{out}(A, l), l \in \left[l^*, \bar{l}^*\right]$ and $\text{in}(A, l) \geq \text{in}(A, l), l \in \left[l^*, \bar{l}^*\right]$ with the inequalities strict for some $l$.

2. $\text{out}(A, l)$ is increasing in $l$ and $\text{in}(A, l)$ is decreasing in $l$ for $l \in \left[l^*, \bar{l}^*\right]$.

**Proof.** See Appendix B

---

\textsuperscript{24}The large in-migration into booming regions with low population due to directed migration and the jump in population that this entails is, of course, unrealistic. However, one can smooth out these jumps by introducing a realistic convex cost of transforming housing into residential units from units used in production, for example.
that region and higher in-migration for a given population. On the other hand, for given labor productivity, a region with a smaller population experiences lower out-migration and higher in-migration. This effect is due to the difference in house prices across such regions, with workers migrating out less from regions with lower house prices and migrating more into such regions.

However, this result does not make it clear how house prices co-move with regional flows. In order to make this connection, I first characterize the behavior of house prices across regions. I will keep track of two prices, the regional house price prior to workers’ migration decisions, \( \tilde{p}(A, l) \), and the house price after migration takes place, \( p(A, l) \), i.e. \( \tilde{p} \) is the regional house price in the beginning of a period, while \( p \) is the regional house price in the end of a period. We have the following result.

**Proposition 3.** \( p(A, l) \) is increasing in \( A \) and \( l \) for \( l \in [l^*, l^*] \), and \( \tilde{p}(A, l) \) is increasing in \( l \) for \( l \in [l^*, l^*] \).

*Proof. See Appendix B* □

The proposition establishes a tight link between beginning-of-period house prices \( \tilde{p} \) and the beginning-of-period measure of mobile workers in a region, \( l \). Therefore, defining \( U(A, l) = (1 - \mu(\theta(A))) \) as the unemployment rate in region \((A, l)\), we immediately have the following result.

**Proposition 4.** Consider a cross-sectional sample of \( J \) regions from the model economy. Let \( \text{out}^j = \text{out}(A^j, l^j) \) and \( \text{in}^j = \text{in}(A^j, l^j) \) be the out-migration and in-migration rates for region \( j \in \{1, 2, ..., J\} \). Also, let \( U^j = U(A^j, l^j) \) be the unemployment rate and \( \tilde{p}^j = \tilde{p}(A^j, l^j) \) be the beginning-of-period house price for a region \( j \in \{1, 2, ..., J\} \). Then:

1. for a given \( \tilde{p}^j \), \( \text{out}^j \) is increasing in \( U^j \) and \( \text{in}^j \) is decreasing in \( U^j \);
2. for a given \( U^j \), \( \text{out}^j \) is increasing in \( \tilde{p}^j \) and \( \text{in}^j \) is decreasing in \( \tilde{p}^j \).

*Proof. First, Proposition 3 implies that \( \tilde{p}^j = \tilde{p}(A^j, l^j) \) is increasing in \( l^j \) for \( l^j \in [l^*, l^*] \). Secondly, note that \( U^j = U(A^j, l^j) \) is decreasing in \( A \). Given these two observations, fact 1 follows from fact 1 in Proposition 2 and fact 2 follows from fact 2 in that Proposition as well. □

Proposition 4 establishes that the model can account for the co-movements between relative unemployment and house prices and regional migration documented in Section 1.2. It implies that if one simulates data for many regions from the model and runs the panel regression from Section 1.2, one would obtain coefficient estimates with the same signs as in the data. The
intuition for the result is that the unemployment rate $U^j$ captures the variation in regional productivity $A^j$, while the beginning-of-period house price $\bar{p}^j$ captures the variation in $\bar{p}^j$, the measure of mobile workers. In that sense, it is directly linked to the results from Proposition 2, however it casts it into co-movements based on observable variables, like $U^j$ and $\bar{p}^j$ rather than the state variables $A$ and $l$.

It is important to discuss, which components of the model drive these results. First of all, the equilibrium house price heterogeneity arising from regional histories, drives the positive co-movement between out-migration and house prices holding current labor market conditions fixed. Limited spatial arbitrage also leads to the co-movement between out-migration and unemployment controlling for house prices. Directed migration, on the other hand, implies that regions with booming labor markets and lower populations and consequently, house prices, have larger population inflows leading to the negative co-movement between house prices and unemployment with in-migration. Therefore, models with undirected migration would have troubles accounting for the in-migration co-movements. Additionally, models with frictionless regional mobility would need high frequency variation in the value of amenities or construction costs, that is independent from variation in labor market conditions to simultaneously account for the two facts.

1.4.5 Housing bust and regional reallocation

I now turn to the second main result of this section, related to the labor market and regional reallocation effects of a housing bust. I focus on a stationary equilibrium analyzed in Section 1.4.4, switching off house price differences ($g(.)$ is linear), and on how the measure of workers, $l_0$ with debt overhang, affects the aggregate labor market. I show that increasing the measure of these immobile workers increases aggregate unemployment and that small regional shock amplifies that effect.

First of all, I define aggregate unemployment $U_{agg}$ as a function of $l_0$. It immediately follows that:

$$U_{agg} (l_0) = \left(1 - \mu (\theta (A)) \right) \left( I^* (l_0) + l_0 \right) +$$

$$+ \left(1 - \mu (\theta (A)) \right) \int_{l_0}^{I^* (l_0)} l^*_1 (A, l; l_0) d\bar{u} + l_0 \right) \right)$$

$$\left(120\right)$$

36
or, alternatively, using equations (1.18) and (1.30):

\[
U_{agg} (l_0) = (1 - \mu (\theta (A))) L - \left( \mu (\theta (\bar{A})) - \mu (\theta (A)) \right) \frac{\bar{I}^* (l_0) + l_0}{\bar{A}} 
\]

In equation (1.21), the bracketed expression \( \bar{I}^* (l_0) + l_0 \) is the end-of-period measure of workers in booming regions with \( \bar{I}^* (l_0) \) denoting the stationary equilibrium level of \( \bar{I}^* \) given \( l_0 \). The equation very clearly shows the importance of regional reallocation for aggregate unemployment. More workers in booming regions decreases unemployment as they face a higher job finding probability compared to workers in low productivity regions. Therefore, any interference with the movement of workers from low productivity to high productivity regions would increase aggregate unemployment. One can show that this is exactly what increases in \( l_0 \) do.

**Lemma 5.** The end-of-period population of workers in booming regions, \( \bar{I}^* + l_0 \), is decreasing in \( l_0 \).

**Proof.** See Appendix B.

This result is intuitive, considering that the total population in the economy is constant. Reducing the fraction of mobile workers leads to lower population dispersion over regions, which implies that the population of booming regions declines. Then we immediately have that more immobile workers have a negative impact on unemployment but also that larger local recessions amplify that effect.

**Proposition 6.** Aggregate unemployment, \( U_{agg} (l_0) \), is increasing in the measure of immobile workers, \( l_0 \). Furthermore, \( u (l_0) = -\frac{\partial U_{agg} (l_0)}{\partial A} \bigg|_{A=\bar{A}} \) is also increasing in \( l_0 \).

**Proof.** \( U_{agg} (l_0) \) increasing in \( l_0 \) follows from immediately from inspection of (1.21) given that \( \bar{I}^* + l_0 \) is decreasing in \( l_0 \) and \( U_{agg} (l_0) \) is decreasing in \( \bar{I}^* + l_0 \). The second part is also straightforward since \( -\frac{\partial U_{agg} (l_0)}{\partial A} \bigg|_{A=\bar{A}} = \frac{da}{dA} (L - \bar{I}^* (l_0) - l_0) \) and \( \bar{I}^* + l_0 \) decreasing in \( l_0 \).

The reason for this result is straightforward: worker immobility hinders regional reallocation, which results in higher aggregate unemployment. On the other hand, regional reallocation is more important whenever regional disparities are larger, i.e. when regional recessions are deeper, which gives the amplification effect. These two results point to a potentially important quantitative effect of the housing bust on the labor market in the recent recession given the
simultaneous large divergence in regional economic conditions. I turn now to addressing this question.

1.5 Quantitative Model

In this section I study the quantitative implications of my model of a housing bust. The purpose of this is to show some additional features of the model and give a sense of the potential magnitudes of the effects discussed in Section 1.4.5 in the context of the recent recession. However, the basic model with one-period job length, which I examined in Section 1.3, is not appropriate for establishing quantitative effects. Furthermore, it is silent on the effects of regional reallocation distortions for regional unemployment. Therefore, in this section I calibrate a version of the model that is richer in terms of labor market dynamics.

1.5.1 Model set-up

I first briefly describe how the calibrated model differs from the basic model. Appendix C contains a more detailed description of that model. Rather than one period long jobs, the calibrated model has stochastic job destruction, which is the standard assumption of search and matching models of the labor market. In particular, at the end of each period, after production takes place, with probability $s \in (0, 1)$ a job becomes unproductive and is destroyed. This assumption is important for generating realistic employment flows and ex ante employment heterogeneity among workers. I also assume that there’s no on-the-job search and that only unemployed workers match to vacant jobs.\footnote{Also, I count as unemployed the workers who are not employed at the beginning of a period, not the workers who are unmatched at the end of a period, as in the basic model.}

With ex ante employment heterogeneity among workers it also becomes necessary to specify migration decisions for both the employed and unemployed. I assume that only the unemployed suffer idiosyncratic region preference shocks, and migrate, while workers that are employed at the beginning of a period remain in the same region. This assumption is similar to the no on-the-job-search assumption of the standard search and matching model. However, it still allows for all important combinations of joint migration and employment flows observed in the data to be represented.\footnote{Also, CPS migration data shows that currently unemployed workers are much more likely to have migrated in the previous year than currently employed workers.}
I calibrate a model with no regional house price differences \( \varphi(\cdot) = 0 \), which implies that migration decisions are based on regional labor market conditions only. The reason for this is computational tractability, since some of the quantitative exercises below deal with simulating a transition to a steady state rather than a stationary equilibrium. Numerical simulations of the basic model, however, show that the effects of house price differences on reallocation are likely to be small.

Lastly, wages in the calibrated model are rigid in the sense of Hall (2005). As was first pointed out by Shimer (2005), the standard search model with Nash bargaining leads to a large response of wages to changes in labor productivity, unless the value of unemployment is very high (Hagedorn and Manovskii (2008)). The large sensitivity of the bargained wage implies that changes in labor productivity are mostly absorbed by changes in the wage resulting in small effects on the job finding probability and from there on unemployment. As a result, the standard search model has problems accounting for the volatility of unemployment, given the observed volatility in labor productivity.

An analogous problem arises in the environment with regional labor markets that I consider. In particular, Lkhagvasuren (2011) shows that for the period 1974-2004 regional labor productivity volatility is \( \sigma_A = 1.2\% \). Using this as the dispersion in productivities between booming and depressed regions and calibrating my model with wage determination via Nash bargaining produces regional unemployment dispersion \( \hat{\sigma}^u < 0.1\% \) for the baseline calibration versus \( \hat{\sigma}^u = 1\% \) in the data. Matching that regional unemployment dispersion entails setting regional productivity dispersion to around 6\%, that is 5 times higher than the observed one. Trying to match even higher unemployment dispersion, such as the one from the recent recession, for example, requires an even higher number.

The modification of the standard search model that Hall (2005) proposes is to have a rigid wage, arising, for example, from a social norm, which does not vary with the aggregate business cycle, thus breaking the strong link between productivity and the wage. Furthermore, the wage lies in the bargaining set of a worker-job pair for every value of productivity over the cycle and hence does not violate individual rationality\(^{27}\). This is the approach I adopt as well.\(^{28}\) A rigid wage also significantly simplifies the computation of equilibrium since the firm’s problem does not depend on the whole distribution of workers over debt overhang and no debt overhang, as

\(^{27}\)There is a large subsequent literature dealing with rigid wages in search models (see Gertler and Tigari (2009) and Shimer (2010)).
\(^{28}\)See Appendix C for the exact conditions.
would be the case with a Nash bargained wage.

1.5.2 Baseline calibration

I calibrate the model to monthly frequency. For the calibration, I consider a region in the model to correspond to a metropolitan statistical area (MSA) in the data. A metropolitan statistical area according to the US Office of Management and Budget definition is a city of at least 50,000 inhabitants and its adjacent areas that have a high degree of economic integration in terms of commuting time (OMB (2009)). Therefore, according to this definition workers within an MSA do not need to move in order to search and be matched to a vacant job in the MSA, which matches well the specification of regions in the model.29

The baseline calibration is for the low regional dispersion period of the early 2000s, when, as noted in the Introduction, MSA unemployment dispersion was around 1%. The model contains a set of fixed parameters and functional forms, as well as a set of parameters that I calibrate jointly based on matching data moments to corresponding moments simulated from the model.

I set the discount factor $\beta$ to 0.995, which gives an annual discount rate of around 6%. For the flow benefit from unemployment I set $e$ to 0.65, which lies between the values proposed by Shimer (2005) and Hagedorn and Manovskii (2008). The exogenous job destruction probability $s$ is set to be 0.034, which is consistent with the rate used in Shimer (2005) and Hall (2005). For the regional productivity process, I assume $\bar{A} = A + d$ and $A = A - d$, where $A$ is average regional productivity and $d$ parametrizes regional productivity dispersion. Given the linear production technology, I normalize the average productivity $\bar{A}$ of a worker-job pair to 1. I set the persistence of regional productivity to $\rho = 0.98$. This is consistent with the high persistence of state unemployment in the data of around 0.99.30

Turning to the matching technology, I use a Cobb-Douglas matching function, $m(\bar{u}, v) = \kappa \bar{u}^{1-\alpha} v^\alpha$, which implies that $\mu(\theta) = \kappa \theta^\alpha$ for $\theta = \frac{v}{\bar{u}}$. I estimate $\alpha$ and $\kappa$ from JOLTS using monthly data from December, 2000 to December, 2007. Note that I need an estimate for a matching function at a low level of aggregation but have only aggregate data. This would be worrying, when regional dispersion is high since mismatch would affect the estimated aggregate matching function, which will no longer correspond to the matching function at a lower level.

---

29 Note that MSAs may span one or several counties and may be contained in one or several states. However, data limitations preclude me from having the empirical facts from Section 1.2 at the MSA rather than the state level.

30 I also set the total population of workers in the economy to $L = 2$, for convenience.
of aggregation (Barnichon and Figura (2011). Therefore, I estimate the aggregate matching function for a time period when of low regional dispersion to ensure that estimate would be close to a matching function estimate at a lower level of aggregation. The estimates I obtain for the matching function are $\alpha = 0.605$ and $\kappa = 1$. The value of $\alpha$ obtained lies in the middle of the set of estimates reported by Petrongolo and Pissarides (2001).

I assume that regional preferences are drawn from a truncated normal distribution with zero mean and variance $\sigma_e^2$, where the domain is given by $[-B, B]$. The value of $B$ has a small effect on the results as long as it is larger than the mobility cost $c$, and so I set $B = 4\sigma_e$.

I calibrate the vacancy posting cost $k$, volatility of preference shock, $\sigma_e$, and regional productivity dispersion $d$ jointly by matching the following data moments using the corresponding simulated moments from the model:

1. unemployment rate of $u = 5\%$. I obtain this as the average unemployment rate for the period 2000-2007;

2. annual migration rate of $q = 5\%$. I obtain this from the CPS using aggregate data on mobility rates for people in the labor force, which corresponds to the workers in my model. Since the CPS does not track migration rate at the MSA level, I look at the average of inter-state migration rate for the period 2000-2007 and average of inter-county migration rate for the same period and take a value that lies between these two;

3. unemployment dispersion of $\sigma_u = 1\%$. I obtain this by computing

$$\hat{\sigma}_u^2 = \sqrt{\frac{\sum_{i=1}^{n} l_{i,t} (u_{i,t} - u_t)^2}{\sum_{i=1}^{n} l_{i,t}}},$$

where $l_{i,t}$ is labor force in MSA $i$ at time $t$, $l_t$ is total labor force at time $t$, $u_{i,t}$ is the unemployment rate in MSA $i$ at time $t$ and $u_t$ is national unemployment rate and taking the average over the period 2000-2007.

Note that each of the above moments roughly corresponds to the particular parameter that it identifies, $k$, $\sigma_e$, and $d$, respectively. I simulate a steady state equilibrium of this economy and

\footnote{JOLTS contains information on total hires per month, which, when divided by the total stock of unemployed, gives the job finding probability $\mu(\theta)$. The value of $\theta$ is similarly obtained as the total vacancies divided by the stock of unemployed.}
Table 1.2: Baseline Parameters

<table>
<thead>
<tr>
<th>parameter</th>
<th>description</th>
<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>discount factor</td>
<td>0.995</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>flow unemployment payoff</td>
<td>0.65</td>
</tr>
<tr>
<td>$s$</td>
<td>job destruction probability</td>
<td>0.034</td>
</tr>
<tr>
<td>$A$</td>
<td>average regional productivity</td>
<td>1</td>
</tr>
<tr>
<td>$\rho$</td>
<td>regional productivity persistence</td>
<td>0.98</td>
</tr>
<tr>
<td>$\mu(\theta)$</td>
<td>matching function</td>
<td>$\theta^{0.605}$</td>
</tr>
<tr>
<td>$\sigma_\epsilon$</td>
<td>volatility of preference shock</td>
<td>8.467</td>
</tr>
<tr>
<td>$F(\epsilon)$</td>
<td>distribution of preference shocks</td>
<td>$N(0, \sigma_\epsilon^2)$ with symmetric truncation at $B = 4\sigma_\epsilon$</td>
</tr>
<tr>
<td>$k$</td>
<td>vacancy posting cost</td>
<td>0.633</td>
</tr>
<tr>
<td>$c$</td>
<td>mobility cost</td>
<td>11.776</td>
</tr>
<tr>
<td>$d$</td>
<td>baseline regional productivity dispersions</td>
<td>0.0056</td>
</tr>
<tr>
<td>$w$</td>
<td>wage rate</td>
<td>0.9813</td>
</tr>
</tbody>
</table>
compute the moments as in the data. Additionally, I set the mobility cost to $c = 12w$, the annual wage in my model. This estimate lies in between the mobility cost estimates found in Bayer and Juessen (2008) and Kennan and Walker (2011), who report a mobility cost of around 6 and 36 monthly wages, respectively based on estimations of structural models. Lastly, the regional wage rate is set at the wage rate from a standard search and matching model with symmetric Nash bargaining and no regional productivity dispersion but otherwise parametrized as my model. This is similar to the approach from Hall (2005). I summarize this baseline calibration procedure in Table 1.2.

Before examining the effects of a housing bust and calibrating the parameters necessary for that, I first show how the model can account for the co-movement between regional unemployment dispersion and shifts in the Beveridge curve observed in the data. Showing this co-movement is independent of any housing bust effects and hence can be done without a housing depreciation shock.

1.5.3 Regional shocks and Beveridge curve shifts

In her study on the shifts of the U.S. Beveridge curve, Abraham (1987) conjectures that dispersion in regional economic conditions are associated with shifts in the curve. I confirm this by first showing that there is a positive co-movement between regional unemployment dispersion and shifts in the Beveridge curve and then showing how my model can account for it.

I look at annual data for the U.S. and construct a synthetic vacancy rate using the Conference Board HWI and JOLTS (see Appendix A for details), as well as annual employment weighted state and MSA unemployment dispersion series. I estimate a standard Beveridge curve regression, adding a cubic time trend to control for secular shifts in the curve as well as for trends in the vacancy or unemployment rates, and also include my unemployment dispersion measure to the regression. The main regressions I run are of the form:

\[ v_t = \alpha_0 + \alpha_1 u_t + \alpha_2 \hat{\sigma}_t^u + \sum_{i=1}^{3} \alpha_{2+i} t^i + \epsilon_t \]  

and

\[ u_t = \beta_0 + \beta_1 v_t + \beta_2 \hat{\sigma}_t^u + \sum_{i=1}^{3} \beta_{2+i} t^i + \epsilon_t \]  

\[ \text{(1.22)} \]
\[ \text{(1.23)} \]

Appendix C contains information on the algorithm I use to simulate this economy. For the parameters values I obtain, the simulated moments match the data moments almost exactly.
Table 1.3: Beveridge curve regressions with vacancy rate as dependent variable (a) and unemployment rate as dependent variable (b)

(a)

<table>
<thead>
<tr>
<th>Dep. Variable:</th>
<th>vacancy rate</th>
<th>( \ln(v) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>unemployment rate</td>
<td>-0.447***</td>
<td>-0.498***</td>
</tr>
<tr>
<td>( u )</td>
<td>(0.0510)</td>
<td>(0.0380)</td>
</tr>
<tr>
<td>MSA Unemployment</td>
<td>0.533**</td>
<td>0.528**</td>
</tr>
<tr>
<td>Dispersion</td>
<td>(0.235)</td>
<td>(0.189)</td>
</tr>
<tr>
<td>( u^2 )</td>
<td>0.0371**</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.0132)</td>
<td></td>
</tr>
<tr>
<td>( \ln(u) )</td>
<td></td>
<td>-0.933***</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.0664)</td>
</tr>
<tr>
<td>( R^2 )</td>
<td>0.970</td>
<td>0.977</td>
</tr>
</tbody>
</table>

(b)

<table>
<thead>
<tr>
<th>Dep. Variable:</th>
<th>unemployment rate</th>
<th>( \ln(u) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>vacancy rate</td>
<td>-2.036***</td>
<td>-1.841***</td>
</tr>
<tr>
<td>( v )</td>
<td>(0.165)</td>
<td>(0.0944)</td>
</tr>
<tr>
<td>MSA Unemployment</td>
<td>1.287*</td>
<td>0.954</td>
</tr>
<tr>
<td>Dispersion</td>
<td>(0.669)</td>
<td>(0.643)</td>
</tr>
<tr>
<td>( v^2 )</td>
<td>0.250</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.173)</td>
<td></td>
</tr>
<tr>
<td>( \ln(u) )</td>
<td></td>
<td>-1.001***</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.0763)</td>
</tr>
<tr>
<td>( R^2 )</td>
<td>0.970</td>
<td>0.982</td>
</tr>
</tbody>
</table>

Notes: Own calculations from BLS and Conference Board. Annual data from 1990-2010. Newey-West robust standard errors in parenthesis; * = significant at 10%; ** = significant at 5%; *** = significant at 1%. All regressions include a cubic time trend. \( v \) is national vacancy rate and \( u \) is national unemployment rate. Dispersion measure for variable \( x \) is defined as \( \delta_i^2 = \sum_{t=1}^{n} \frac{\varepsilon_{iT}}{\varepsilon_{iT}} (x_{i,t} - \bar{x}_t)^2 \) where \( \varepsilon_{iT} \) is labor force (employment) in MSA (sector) \( i \) at time \( t \), \( \varepsilon_{IT} \) is national labor force (employment) at time \( t \), \( x_{i,t} \) is the realization in MSA/sector \( i \) at time \( t \) and \( \bar{x}_t \) is weighted average of MSA (sector) realizations weighted by labor force (employment) weights.
Table 1.4: Regional shocks and Beveridge Curve shifts

<table>
<thead>
<tr>
<th>$\bar{A} - \bar{A}$</th>
<th>$U_{agg}$</th>
<th>$\sigma^u$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0112</td>
<td>5%</td>
<td>1%</td>
</tr>
<tr>
<td>0.0231</td>
<td>5.22%</td>
<td>2%</td>
</tr>
<tr>
<td>0.0318</td>
<td>5.56%</td>
<td>3%</td>
</tr>
</tbody>
</table>

Notes: Model simulation results. See text for details. $\bar{A} - \bar{A}$ is the productivity difference between booming and depressed regions, $U_{agg}$ is the aggregate unemployment rate and $\sigma^u$ is unemployment rate dispersion.

where $v_t$ is the aggregate vacancy rate in year $t$, $u_t$ is the national unemployment rate and $\sigma^u_t$ is regional unemployment dispersion. I also augment (1.22) and (1.23) with the square of unemployment rate and vacancy rate, respectively and also run a regression with the vacancy and unemployment rates in logs to control for non-linearities.

Table 1.3 shows the regression results for MSA unemployment dispersion.33 Not surprisingly there is a very significant negative relation between vacancies and unemployment. More interestingly, there is also a positive relation between MSA unemployment dispersion and vacancies, controlling for unemployment (or vice versa in the regressions where unemployment is the dependent variable).

This relation implies a positive co-movement between state/MSA unemployment dispersion and shifts in the Beveridge curve. The recent recession and its aftermath are examples of this positive co-movement as both state and MSA unemployment dispersion have increased significantly, and at the same time the Beveridge appears to have shifted out.34

I now show how regional shocks in my model can generate this co-movement. I calibrate regional productivities $A$ and $\bar{A}$ to match regional unemployment dispersions of $\sigma^u = 2\%$ and $\sigma^u = 3\%$, while keeping aggregate market tightness, $\theta_{agg}$, at the level of the baseline calibration. Keeping $\theta_{agg}$ constant controls for an aggregate shock that would move vacancies and unemployment in the opposite direction. Any changes in unemployment, keeping $\theta_{agg}$ constant, will then be associated with changes in vacancies in the same direction, that is shifting of the vacancy-unemployment locus. Table 1.4 contains the results for aggregate unemployment ($U_{agg}$) and

33Table 1.11 contains the results for state unemployment dispersion.
34The possibility of an outward shift in the curve has renewed interest among academics and policy makers about the importance of mismatch for the labor market. (Katz (2010) and Kocherlakota (2010)). Empirical work on the Beveridge curve and JOLTS data by Barnichon and Figura (2010) and Barnichon, Elsby, Hobijn, and Sahin (2010) provides additional evidence for the magnitude of the recent shift.
unemployment dispersion \((\hat{\sigma}^u)\). Increasing the dispersion in regional productivity, while keeping \(\theta_{ugy}\) constant, increases unemployment dispersion and aggregate unemployment, which is exactly the observed co-movement in the data. Therefore, the model generates shifts in the Beveridge curve accompanied by increases in unemployment dispersion. The quantitative effect it generates is smaller than in the data, so the model can explain only part of that co-movement.

### 1.5.4 Housing bust effects

I now look at the quantitative magnitudes of the labor market effects of a housing bust in my model by performing two exercises. In the first one, I show the effect through migration distortions due to negative equity and default, while the second one provides an upper bound on any regional reallocation distortions that the housing bust may have. In each exercise, there are two parameters to calibrate, the default penalty, \(\zeta\), and the fraction of the regional population that has debt overhang, which I denote by \(\lambda\). Given \(\zeta\) and \(\lambda\), I simulate the effect of the recent recession by considering a housing depreciation shock and a simultaneous permanent shock to average productivity, \(A_{2010}\), and productivity dispersion, \(d_{2010}\).\(^{35}\) I look at a transition path for the model economy, simulating 24 months of data, and use the last 12 months to set \(A_{2010}\) and \(d_{2010}\) by matching the average unemployment rate and MSA unemployment dispersion for 2010 of \(u = 9.5\%\) and \(\hat{\sigma}^u = 2.3\%\).\(^{36}\) Table 1.5 below summarizes the parameter values used in the two exercises.

**Effects through “negative equity”**

I calibrate the default penalty, \(\zeta\), by using a micro estimate of the cost of default implied by default decisions of households with negative equity. I use information from Bhutta, Dokko, and Shan (2010), who use a sample of non-prime borrowers from the states affected most deeply by the recent housing bust to estimate the negative equity threshold that is associated with

\(^{35}\)Note that the recent housing bust may have had a direct effect on the severity of the aggregate and local recessions by affecting financial intermediaries balance sheets or even through its effect on household’s consumption decisions (Iacoviello (2005), Mian and Sufi (2011), Midrigan and Philippon (2011)). However, for this paper, I want to focus on one particular channel - the indirect impact of a housing bust on the labor market through regional reallocation. Modeling the impact of the recession through these reduced form productivity shocks allows me to do that by switching off these other channels.

\(^{36}\)Since the total period length for the simulation is only 24 months, the assumption of a permanent shock is fine as long as shocks are fairly persistent at monthly frequency in the data. While, I cannot estimate the persistence of a regional shock, state unemployment dispersion data points to a regime switching process for regional shocks given the high unemployment dispersion in the pre-Great Moderation period and the low dispersion thereafter up to the recent recession. A test for multiple structural breaks (Bai and Perron (1998)) confirms this.
voluntary default. Such a threshold can be taken as the combination of pecuniary and non-pecuniary costs that are associated with default, since if default were costless, a household would default as soon as its housing equity goes below zero (Deng, Quigley, and Order (2000)).37 These authors find that if both voluntary and involuntary default are treated as observationally equivalent, as is the case in my model (since I do not allow for involuntary default), then the median household in negative equity defaults at around -30% of the value of their house (negative equity of -30%). Given a median house value of $177,000 and average monthly wage of around $3000 for 2009, it follows that this default cost translates into approximately 18 months of wage income. Therefore, I set a value of $\zeta = 18w \approx 17.7$ in my model.

I first show how the model can account for the correlation between the fraction of households in negative equity and out-migration, documented in Section 1.2. Using the baseline calibration for the early 2000s, I look at a small housing depreciation shock with the fraction of workers in debt overhang uniformly distributed across regions with a mean of 0.03 (the average fraction of homeowners with negative equity during the late 90s and early 2000s according to the CE data) and support of [0,0.06] (the dispersion across states for that period). I then simulate 10 years of data for 50 regions from the transition path of the model. Using this simulated data from the model, I create a regional panel, which I use to run a panel regression similar to that in Section 1.2. Table 1.6 compares the regression result for the simulated data to the regression from Section 1.2.

First of all, looking at the coefficient estimates on households/workers with negative equity, the model generates the negative effect observed in the data for the out-migration regression. The coefficient in the model is higher than in the data. However, this is not unexpected, given the substantial measurement error in the fraction of households with negative equity and the

37Their estimates appear to be a lower bound compared to other studies of the implied cost of default (Foote, Gerardi, and Willen (2008), Guiso, Sapienza, and Zingales (2009)).
Table 1.6: Model and data comparisons

<table>
<thead>
<tr>
<th>Dep. Variable</th>
<th>Data (EIV correction)</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>100 ln(out)</td>
<td>100 ln(out)</td>
</tr>
<tr>
<td></td>
<td>100 ln(in)</td>
<td>100 ln(out)</td>
</tr>
<tr>
<td>households/workers with negative equity (%)</td>
<td>-0.243 (0.143)</td>
<td>-0.565 (0.247)</td>
</tr>
<tr>
<td>relative unemployment 100 ln(L)</td>
<td>0.168 (0.0359)</td>
<td>-0.237 (0.0388)</td>
</tr>
</tbody>
</table>

Notes: Standard errors in parenthesis. EIV correction refers to using a reliability ratio of 0.43 to correct for errors-in-variables in the coefficient on "households with negative equity". See text for details.

resulting large attenuation bias, as discussed in Section 1.2. In particular, the third column of Table 1.6 shows the corrected coefficient, $\beta_{neg}$, using the reliability ratio of 0.43 derived in that Section. That reliability ratio is a conservative lower bound on the effect of measurement error, so the discrepancy between the model coefficient and the estimated coefficient from the data is indeed small. Secondly, the estimated coefficient on negative equity for the in-migration regression is not significant. Therefore, the calibrated model can account for the asymmetric effect of fraction of negative equity on out- and in-migration.

Comparing the coefficients on unemployment, we see that they have the same sign but the coefficients from the model are larger. Note, however, that it is hard to compare the magnitudes of the coefficients, as I am using state level data of household migration, whereas the model simulates MSA level data for individuals that are part of the labor force. Both the use of state level and household migration data will lead to weaker co-movements between unemployment and migration rates. However, the model does produce a much larger effect of unemployment on in-migration than on out-migration, while in the data that difference is much smaller. This is due to the large variation in the in-migration rate in the model as regions can vary from having no in-migration to having very large in-migration flows. This leads to the large point estimate for the effect of fraction of workers with debt overhang on in-migration as well. Nevertheless, this discrepancy between the model and data affects only how migrating workers are distributed across booming regions, and so it is not important for the labor market effect of the housing bust implied by the model, since what is relevant for that effect is how a housing bust impacts aggregate migration from depressed to booming regions.\textsuperscript{38}

\textsuperscript{38}Improving the model fit along this dimension will require a channel for migration into depressed regions as well as for lower inflow rates, for example due to convex costs of transforming housing units for production into residential units.
Table 1.7: Results for Quantitative Exercise 1 ($\zeta = 17.7$, $\lambda = 0.13$)

<table>
<thead>
<tr>
<th></th>
<th>$U_{agg}$</th>
<th>$U_A$</th>
<th>$U_{\bar{A}}$</th>
<th>$\hat{d}^n$</th>
<th>Migration rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>2010 (housing bust)</td>
<td>9.50%</td>
<td>12.12%</td>
<td>7.54%</td>
<td>2.27%</td>
<td>8.45%</td>
</tr>
<tr>
<td>2010 (no housing bust)</td>
<td>9.33%</td>
<td>11.71%</td>
<td>7.60%</td>
<td>2.03%</td>
<td>9.67%</td>
</tr>
</tbody>
</table>

Notes: Model simulation results. See text for details. $U_{agg}$ is the aggregate unemployment rate and $U_A$ and $U_{\bar{A}}$ are average unemployment rates for depressed and booming regions respectively. $\hat{d}^n$ is unemployment rate dispersion and the "migration rate" column gives the annual inter-MSA migration rate.

Returning to the quantitative exercises, I set the fraction of workers with debt overhang to $\lambda = 0.13$, which corresponds to the fraction of households in negative equity in the 5 worst affected states in 2009, based on the CE data. I use this particular number to account for a salient feature of the recent recession, that states with more severe labor market shocks in 2010 were also states, where a higher fraction of households had negative equity. For example, the two states with the largest unemployment problems during 2010 - Nevada and California (with 2010 unemployment rates of 14.3% and 12.3% respectively) - have also experienced some of the most severe negative equity problems. Using the national average would significantly understate the scope of the reallocation distortion for workers. At the same time, having booming regions with a high fraction of less mobile workers affects only the migration rate and not unemployment, as workers in booming regions migrate only to other booming regions in the model.

Table 1.7 contains the result for the counterfactual experiment (setting $\lambda = 0$, i.e. no housing depreciation shock). Through the "negative equity" channel the housing bust can account for around 0.2 percentage points on aggregate unemployment and 0.4 percentage points of unemployment in depressed metropolitan areas. Without the housing bust unemployment in booming metropolitan areas is slightly higher since such regions experience a higher inflow of unemployed workers. Also, unemployment dispersion falls by almost 0.25 percentage points. Note, however, that the migration rate is almost 8.5%, which is still substantially higher than the post-recession migration rate observed. In reality, the housing bust may have an adverse effect on reallocation not only because of negative equity and default but also because of a housing wealth shock by increasing the fraction of households who cannot afford to make a down payment on a new house and also by increasing down payment requirements because of credit market freezes. Additionally, the calibrated fraction of workers in debt overhang, $\lambda$, may be understating the true fraction of affected households since as discussed in Section 1.2 the
Table 1.8: Results for Quantitative Exercise 2 ($\zeta = 7.9$, $\lambda = 1$)

<table>
<thead>
<tr>
<th></th>
<th>$U_{agg}$</th>
<th>$U_A$</th>
<th>$U_{-A}$</th>
<th>$\hat{\sigma}^u$</th>
<th>Migration rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>2010 (housing bust)</td>
<td>9.50%</td>
<td>12.04%</td>
<td>7.44%</td>
<td>2.29%</td>
<td>3.5%</td>
</tr>
<tr>
<td>2010 (no housing bust)</td>
<td>9%</td>
<td>10.79%</td>
<td>7.70%</td>
<td>1.53%</td>
<td>9.2%</td>
</tr>
</tbody>
</table>

Notes: Model simulation results. See text for details. $U_{agg}$ is the aggregate unemployment rate and $U_A$ and $U_{-A}$ are average unemployment rates for depressed and booming regions respectively. $\hat{\sigma}^u$ is unemployment rate dispersion and the "migration rate" column gives the annual inter-MSA migration rate.

Consumer Expenditure Survey data may be misclassifying households. Nevertheless, I can use the post-recession migration rate to show an upper bound on any regional reallocation effects the housing bust may have according to my model and the implications for unemployment.

**An upper bound**

In this exercise, I use the observed post-recession migration rate to provide an upper bound on the regional reallocation and labor market effects of the housing bust according to the calibrated model. I compute the 2009 MSA migration rate as the average of inter-county and inter-state migration from the CPS and obtain a number of 3.5%. I then set the fraction of affected workers, $\lambda$, and default penalty, $\zeta$, to match this migration rate. Since there are two parameters and one target, there are multiple combinations of $\lambda$ and $\zeta$ that can achieve this. Therefore, I fix $\lambda = 1$ and set $\zeta$, i.e. I look at the increase in mobility cost for all workers that would explain the observed post-recession migration rate.

The results for this experiment are shown in Table 1.8. The effect on aggregate unemployment is 0.5%, while that on unemployment in depressed metropolitan areas is around 1.2%. Since there are other possible mechanisms operating that decrease mobility, these results are an upper bound on the labor reallocation effect of the housing bust.

**1.5.5 Discussion**

To summarize the results from the two quantitative exercises, the effect on aggregate unemployment is between 0.2 and 0.5 percentage points, while the effect on unemployment in depressed metropolitan areas is between 0.4 and 1.2 percentage points. This corresponds to between 4 and 10% of the increase in aggregate unemployment from 2007 to 2010 and to between 7 and 20% of the increase in unemployment in depressed metropolitan areas.
It is interesting to examine what the importance of the amplification effect of a regional shock is. To see this, I consider the baseline (pre-recession) calibration, and include a housing bust as the counterfactual experiment, using the parametrizations from the two exercises above. I find that for the first exercise aggregate unemployment increases by only 0.03 percentage points due to the housing bust, while for the second exercise the increase is 0.14 percentage points. Furthermore, unemployment in depressed metropolitan areas increases only by 0.05 and 0.28 percentage points, respectively. These effects are substantially smaller than the counterfactual effects from the recent recession. Therefore, hindering regional reallocation during a period when regional disparities are large as in the aftermath of the recession is particularly important. Had the housing bust somehow occurred without an increase in regional dispersion, its reallocation distortion effect on the labor market would have been substantially smaller. Consequently, diminishing of the current regional disparities would have the added effect of decreasing the impact of the housing bust on the labor market.

I also consider the robustness of the effect to the use of a different migration rate for the baseline calibration. Due to unavailability of inter-MSA migration rate data, I used an average of inter-county and inter-state migration, which implied a 5% migration rate. Here, I consider what the effects would be given a migration rate of 4% and 6%. I repeat the baseline calibration to match each of these rates and then perform the first housing bust exercise for each case. For the first case of a migration rate of 4%, the effect of the housing bust decreases slightly to 0.13 percentage points for aggregate unemployment and 0.34 percentage points for unemployment in depressed metropolitan areas. For the second case of a migration of 6%, the effect increases to 0.19 percentage points and 0.48 percentage points, respectively. Therefore, the magnitude of the effects is fairly robust to a calibration with a different migration rate.\textsuperscript{39}

Beyond the consequences for unemployment, examined up to now, it is important to examine the welfare implications of these effects. First of all, I define total welfare in the economy. Let $W_t(\zeta, \lambda)$ be total welfare at the beginning of period $t$ as a function of the default penalty $\zeta$ and fraction of affected workers $\lambda$. Then

\begin{equation}
W_t(\zeta, \lambda) = \sum_{A,h} \left[ U(A, h) V^h_U(A) + (L(A, h) - U(A, h)) V^h_L(A) \right] \tag{1.24}
\end{equation}

where $U(A, h)$ is the beginning-of-period measure of unemployed workers with housing state

\textsuperscript{39}Checking for robustness with respect to $\rho$ also produces small variations in the effects.
Table 1.9: Value functions for unemployed (a) and employed (b) workers

<table>
<thead>
<tr>
<th></th>
<th>$A = A$</th>
<th>$A = \overline{A}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>debt overhang, $h = 0$</td>
<td>189.83</td>
<td>191.18</td>
</tr>
<tr>
<td>no debt overhang, $h = 1$</td>
<td>195.5</td>
<td>195.93</td>
</tr>
</tbody>
</table>

(a)

<table>
<thead>
<tr>
<th></th>
<th>$A = A$</th>
<th>$A = \overline{A}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>debt overhang, $h = 0$</td>
<td>190.97</td>
<td>191.53</td>
</tr>
<tr>
<td>no debt overhang, $h = 1$</td>
<td>195.89</td>
<td>196.06</td>
</tr>
</tbody>
</table>

(b)

$h \in \{0, 1\}$ (0 denotes workers with debt overhang and 1 denotes workers with no debt overhang) in regions with productivity $A$, $L(A, h)$ is the beginning-of-period population of workers with housing state $h \in \{0, 1\}$, and $V^h_I(A)$ and $V^h_E(A)$ are the value functions of unemployed and employed workers, respectively. I also suppress the dependence of these objects on $\zeta$ and $\lambda$ for notational convenience. To understand the welfare effects of the regional reallocation distortion of the housing bust, similarly to the quantitative exercises above, I look at the proportional change in welfare, $\frac{W_t(\zeta(0), \lambda) - W_t(\zeta, \lambda)}{W_t(\zeta, \lambda)}$, from the counterfactual case of removing the housing depreciation shock relative to welfare under a housing depreciation shock and average that over the 24 months, for which I simulate the model economy.

First of all, Table 1.9 shows the value functions calibration used in the first quantitative exercise ($\zeta = 17.66$ and $\lambda = 0.13$) for unemployed and employed workers with and without debt overhang that reside in regions with productivity $\overline{A}$ or $A$. One can make two important observations from these values. First of all, experiencing debt overhang substantially decreases a worker’s utility irrespective of their current employment state or the region they reside in. The reason for this is that in the model workers experience idiosyncratic regional preference shocks that may induce them to migrate independently from local labor market conditions. These have a large effect on a worker’s utility as he has to either incur a large negative preference shock or the penalty for default. The second important observation, is that unemployed workers with debt overhang who reside in depressed regions have a larger discount in their lifetime expected utility relative to being employed, compared to unemployed workers without debt overhang. The reason for this is that the former are exposed to a much longer unemployment spell compared to the latter as the latter can more easily move to regions where finding a job is easier. This shows an important interaction of debt overhang and labor market outcomes at the individual level. Debt overhang leads to longer unemployment spells as unemployed workers effectively face a lower job finding probability.
Turning to the effects on welfare, I get that for the first quantitative exercise the improvement in total welfare is on the order of 0.3%, while for the second exercise it is around 1.5%. These numbers should, of course, be taken with caution given some of the assumptions in the model, most importantly, that agents in the model are risk neutral. Additionally, a thorough welfare analysis of the housing bust requires incorporating additional channels through which the housing bust affected the real economy in the recession.

Finally, I use the calibrated model to address the labor market effects of two policies proposed for dealing with the mortgage crisis. I focus on the first housing bust exercise since it has a clear channel of action of the housing bust on regional reallocation through the default penalty. The first policy I consider is the “Home Affordable Modification Program”, which has been in place since the beginning of 2009. The main objective of the program is to reduce the monthly mortgage payments of borrowers who face imminent risk of default to levels, commensurate with borrower monthly income.\footnote{See https://www.hmpadmin.com/portal/programs/hamp.jsp for details.} Therefore, the program effectively removes involuntary default by homeowners leaving only voluntary or strategic default. However, Bhutta, Dokko, and Shan (2010) find that accounting for involuntary default, the default threshold, associated with purely strategic default for the median household in negative equity is around 60% of home value. This corresponds to a doubling of the default cost that is implied when involuntary and voluntary default are not distinguished. It follows that under this policy the effective default penalty in the model would also double to $\zeta = 35.32$ from the level in Section 1.5.4. The result for the labor market is a marginal increase in aggregate unemployment and unemployment in depressed metropolitan areas of 0.01 and 0.03 percentage points, respectively.

The second policy is a proposal for broader principal reduction through a modification of personal bankruptcy law. Pozner and Zingales (2009) discuss such a policy proposal, which is exactly targeted to home owners in negative equity and calls for principal reduction to the current value of a property. Thus the proposal forces a renegotiation between borrower and lender and removes any debt overhang problem the borrower may have and any adverse consequences stemming from it. In the context of my model, this is equivalent to a removal of the default penalty on workers in debt overhang. The labor market effect of such a policy is equivalent to the effect of the counterfactual experiment in Section 1.5.4. This leads to a decrease in aggregate unemployment of 0.17 percentage points and a decrease of unemployment in depressed metropolitan areas of 0.4 percentage points. Therefore, from the perspective of the labor market,
the second policy has a more beneficial effect compared to the first.

1.6 Conclusion

This paper addresses how a housing bust may affect the labor market indirectly through its impact on regional reallocation. I document how the fraction of households with negative equity in a state correlates with state out-migration and in-migration as well as co-movements between gross migration rates and state labor and housing market conditions. I then study an multi-region model with regional labor and housing markets, which accounts for the co-movements between unemployment, house prices and gross migration due to the assumptions of directed but limited regional mobility. That model allows me to study the regional reallocation effect of a housing bust and the consequent labor market implications. A housing bust increases aggregate unemployment by hindering regional reallocation, while a regional shock amplifies that effect. Finally, I quantitatively evaluate how much of the recent rise in unemployment can be attributed to the housing bust.

There are several venues for future research arising from this work. First of all, the combination of large regional dispersion in economic conditions, together with a housing market related mobility slow-down is not a peculiarity of the recent recession only. The recession of the early 80s was characterized by an increase in regional unemployment dispersion as well, and the high interest rates of that period discouraged many households from taking on new mortgages, thus affecting their mobility. That recession was also characterized by a very high level of unemployment in the U.S. Hence, the conditions observed in the recent recession were present in that period as well, although in a slightly different form.

Second, the model economy that I studied here assumes no occupational or skill heterogeneity or equivalently, no occupational mobility costs for workers. Introducing occupational heterogeneity and limited occupational mobility for workers would lead to another channel of mismatch apart from regional mismatch and would have interesting interaction with regional reallocation decisions. Such interaction may also affect the reallocation effects of a housing bust.

Lastly, the paper shows that limited mobility has implications for regional population and house price dynamics, which are not present in the benchmark framework of frictionless mobility. Studying these implications in greater detail may lead to important insights for models with regional heterogeneity.
Data for panel regression

The data for the state panel regression results reported in Section 1.2 come from a number of sources:

1. Data on household state in- and out-migration rates I obtain from the IRS U.S. Population Migration Database, which is available up to 2008. The data is based on the year-to-year address changes of individual income tax returns. From that raw data, for each state the IRS computes the total number of tax returns, which approximates the number of households, that have migrated into and out of that state. From this data I compute state in- and out- migration rates for a given year as the ratio of the number of movers into or out of the state to sum of the number of movers (into and out of the state, respectively) and non-movers. Note that the period covered does not correspond exactly to the calendar year as it covers moves from April of a given year to April of the next year. However, I treat it as corresponding to the calendar year. The advantage of using the IRS data for tracking state migration patterns as opposed to, for example the Current Population Survey micro-data, is that it is continuously available from up to 2008, whereas CPS has a gap in 1995. Furthermore, the CPS is a survey with a limited sample of individuals so computing state inflow and outflow rates would introduce significant measurement error, which would be problematic for having precise estimates. On the other hand, the IRS data is not completely representative since it excludes the very poor and elderly. However, that should not create problems since the very poor are not homeowners and the elderly do not generally migrate for employment reasons. Also, the IRS data and looks at mobility of all households rather than mobility of individuals that are part of the labor force.

2. Data on fraction of homeowners in negative equity by state I obtain from the Interview Survey of the Consumer Expenditure Survey (CE). The Interview Survey of the CE is a quarterly survey of consumption patterns and expenditures of American consumers. The survey collects data on household characteristics, income, and major expenditures. The sample design is a rotary panel survey with data on each household available for 4 quarters. The survey includes questions on ownership of real estate, including subjective property valuation as well as principal balance outstanding on all mortgages and home equity credit
lines that a specific property collateralizes. The household characteristics include information on homeownership as well as state of residence, although the state identifier is suppressed for several states. Data on households mortgage balance outstanding, which is necessary for constructing estimates of a household’s housing equity, becomes available from 1988. However, household state identifiers are only available from 1993. Using this information for each unique homeowner in a given year I construct their housing equity \( E \) for their primary residence as \( E = \frac{V-D}{V} \), where \( D \) is the total balance outstanding on all mortgages and home equity lines of credit that a property collateralizes and \( V \) is the value of the property. Note that I remove homeowners with \( E < -2 \), as those are homeowners that report either very low home values or do not report home values. Together with the state identifier for that homeowner, I can then construct an estimate for the fraction of homeowners in negative equity for each state for the given year. As discussed in Section 1.2, for robustness to household level misclassification, I also create estimates of negative equity by using a cutoff \( c \), i.e. I count households with \( E < c < 0 \). Table 1.10 contains these results. I also use this data to construct estimates of average mortgage debt ratio \( D/V \) by state.

3. Data on homeownership rates by state I obtain from the US Census Bureau’s Housing Vacancies and Homeownership data.

4. Data on relative house prices I obtain by using the Freddie Mac House Price Index (FMHPI), formerly known as the Conventional Mortgage House Price Index (CMHPI) as well as the 2000 U.S. Census data on single family median home values by states and at the national level. In particular, the single family median house price in state \( s \) at time \( t \), \( P_{s,t} = P_{s,2000} \frac{FMHPI_{s,t}}{FMHPI_{s,2000}} \) and similarly for national house prices, \( p_t \). Relative house price in state \( s \) and time \( t \) is then \( \frac{P_{s,t}}{p_t} \).

5. Data on relative unemployment rate is constructed using data from the BLS LAUS database.

6. Data on relative wage rates is constructed using the BLS CES database. I use the average hourly manufacturing wage as it is the only wage series that spans my sample period 1993-2007.

7. Data on relative income is constructed using annual data on state level income from the BEA and population data from the US Census.
Data for the Beveridge curve regressions

I use the Conference Board Help Wanted Index and data from the BLS including the national unemployment rate (derived from the CPS), state and MSA unemployment rates and employment and labor force levels (derived from the Local Area Unemployment Statistics database), and the national vacancy rate (derived from JOLTS). Additionally I take the state unemployment dispersion data for the period 1960-1975 from Abraham (1987).

The Conference Board Help Wanted Index is the source for data on job vacancies prior to the introduction of JOLTS in December 2000. It tracks the number of help wanted advertisements in the newspapers of 51 cities, which are then aggregated into 9 Census Divison and 1 National Index. There are several well-known problems with using the HWI as a proxy for vacancies particularly for looking at shifts in the Beveridge curve. Most recently, there has been a downward secular trend in the HWI, which has been attributed to the more extensive use of online job advertising. To remedy this I construct a synthetic vacancy rate by regressing monthly vacancy data from JOLTS for the period December 2000 to December 2003 on data from the HWI and a constant, this is a period where the two series track each other well. I then use these coefficient estimates to construct a synthetic vacancy rate for the entire sample period, taking annual averages to get annual data for the period 1960 to 2000. For years 2001 to 2010 I use the JOLTS vacancy data. This approach, however, may be unsatisfactory since it does not take care of trends in the HWI in earlier years unrelated to the labor market (Abraham (1987)), which get transferred directly to the synthetic vacancy index. To address this, I include a cubic trend in all regressions, which should account for these additional secular trends.

For the national unemployment rate data, I take annual averages of the monthly seasonally adjusted series. Similarly to the HWI there may be a secular trend in the unemployment rate because of compositional changes with the aging of the baby boomer generation (Shimer (1999)). A cubic trend in the Beveridge curve regression would account for such a trend as well.

Following Abraham (1987) and Lilien (1982), I construct employment-weighted state unemployment rate dispersion according to the formula: \( \hat{\sigma_t}^2 = \sqrt{\sum_{i=1}^{n} \frac{e_{i,t}}{x_t} (x_{i,t} - x_t)^2} \), where \( e_{i,t} \) is employment in state \( i \) at time \( t \), \( e_t \) is national employment at time \( t \), \( x_{i,t} \) is the realization in state \( i \) at time \( t \) and \( x_t \) is weighted average of state realizations weighted by employment weights. For the unemployment dispersion I use annual averages of monthly seasonally adjusted state unemployment rates as well as annual averages of seasonally adjusted monthly

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Figure 1-3: Negative equity fractions from First American CoreLogic vs. CE Data

Notes: State level estimates from 2009 of fraction of mortgage holders in negative equity from Consumer Expenditure Survey versus First American CoreLogic. CE data is from own calculations. CoreLogic data is from CoreLogic (2009). Regression coefficient is 1.13 and intercept is 13.4. Correlation coefficient is $\rho = 0.731$.

state employment levels. Finally, I construct MSA unemployment dispersion according to:

$$\hat{\sigma}_t^2 = \sqrt{\frac{1}{n} \sum_{i=1}^{n} \frac{e_{i,t}}{e_t} (x_{i,t} - x_t)^2}$$

where $e_{i,t}$ is labor force in MSA $i$ at time $t$, $e_t$ is national labor force at time $t$, $x_{i,t}$ is the realization in MSA $i$ at time $t$ and $x_t$ is weighted average of MSA realizations weighted by labor force weights.
### Table 1.10: Panel regressions with cutoffs: (a) $c = -0.1$, (b) $c = -0.2$

<table>
<thead>
<tr>
<th>Dep. Variable</th>
<th>out-migration rate</th>
<th>in-migration rate</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$100 \ln(\text{out})$</td>
<td>$100 \ln(\text{in})$</td>
</tr>
<tr>
<td>households with equity $&lt; -0.1$, (%)</td>
<td>$-0.301^*$</td>
<td>$-0.0413$</td>
</tr>
<tr>
<td>relative unemployment</td>
<td>$0.167^{***}$</td>
<td>$-0.237^{***}$</td>
</tr>
<tr>
<td>$100 \ln(\frac{L_{f}}{u})$</td>
<td>$(0.0359)$</td>
<td>$(0.0388)$</td>
</tr>
<tr>
<td>relative house price</td>
<td>$0.203^{***}$</td>
<td>$-0.169^{***}$</td>
</tr>
<tr>
<td>$100 \ln(\frac{P_{f}}{P})$</td>
<td>$(0.0686)$</td>
<td>$(0.0590)$</td>
</tr>
<tr>
<td>relative wage rate</td>
<td>$0.0143$</td>
<td>$-0.00955$</td>
</tr>
<tr>
<td>$100 \ln(\frac{L_{f}}{w})$</td>
<td>$(0.0840)$</td>
<td>$(0.165)$</td>
</tr>
<tr>
<td>relative income</td>
<td>$-0.0909$</td>
<td>$0.646^{**}$</td>
</tr>
<tr>
<td>$100 \ln(\frac{L_{f}}{y})$</td>
<td>$(0.303)$</td>
<td>$(0.246)$</td>
</tr>
<tr>
<td>ave. debt-to-value ratio (%)</td>
<td>$0.0404$</td>
<td>$0.0995^*$</td>
</tr>
<tr>
<td>home ownership rate (%)</td>
<td>$0.434$</td>
<td>$-0.335$</td>
</tr>
<tr>
<td>$N$</td>
<td>606</td>
<td>606</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Dep. Variable</th>
<th>out-migration rate</th>
<th>in-migration rate</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$100 \ln(\text{out})$</td>
<td>$100 \ln(\text{in})$</td>
</tr>
<tr>
<td>households with equity $&lt; -0.2$, (%)</td>
<td>$-0.377^{**}$</td>
<td>$-0.0404$</td>
</tr>
<tr>
<td>relative unemployment</td>
<td>$0.166^{***}$</td>
<td>$-0.237^{***}$</td>
</tr>
<tr>
<td>$100 \ln(\frac{L_{f}}{u})$</td>
<td>$(0.0358)$</td>
<td>$(0.0389)$</td>
</tr>
<tr>
<td>relative house price</td>
<td>$0.204^{***}$</td>
<td>$-0.169^{***}$</td>
</tr>
<tr>
<td>$100 \ln(\frac{P_{f}}{P})$</td>
<td>$(0.0683)$</td>
<td>$(0.0588)$</td>
</tr>
<tr>
<td>relative wage rate</td>
<td>$0.0175$</td>
<td>$-0.00940$</td>
</tr>
<tr>
<td>$100 \ln(\frac{L_{f}}{w})$</td>
<td>$(0.0828)$</td>
<td>$(0.164)$</td>
</tr>
<tr>
<td>relative income</td>
<td>$-0.0970$</td>
<td>$0.645^{**}$</td>
</tr>
<tr>
<td>$100 \ln(\frac{L_{f}}{y})$</td>
<td>$(0.303)$</td>
<td>$(0.246)$</td>
</tr>
<tr>
<td>ave. debt-to-value ratio (%)</td>
<td>$0.0448$</td>
<td>$0.0984^{**}$</td>
</tr>
<tr>
<td>home ownership rate (%)</td>
<td>$0.427$</td>
<td>$-0.337$</td>
</tr>
<tr>
<td>$N$</td>
<td>606</td>
<td>606</td>
</tr>
</tbody>
</table>

Notes: Robust standard errors with clustering on state in parenthesis; $^*$ = significant at 10%; $^{**}$ = significant at 5%; $^{***}$ = significant at 1%. Source: Own calculations from BLS, IRS, CE, FMHPI, and US Census Bureau. See Data Appendix for detailed description.
Table 1.11: Beveridge curve regressions with vacancy rate as dependent variable (a) and unemployment rate as dependent variable (b)

(a) 

<table>
<thead>
<tr>
<th>Dep. Variable:</th>
<th>vacancy rate</th>
<th>( \ln(v) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>unemployment rate ( (u) )</td>
<td>-0.203***</td>
<td>-0.458***</td>
</tr>
<tr>
<td>(0.0327)</td>
<td>(0.101)</td>
<td>(0.311)</td>
</tr>
<tr>
<td>State Unemployment Dispersion</td>
<td>1.146**</td>
<td>1.146**</td>
</tr>
<tr>
<td>(0.458)</td>
<td>(0.459)</td>
<td>(0.129)</td>
</tr>
<tr>
<td>( u^2 )</td>
<td>-0.000255</td>
<td>0.411**</td>
</tr>
<tr>
<td>(0.0217)</td>
<td>(0.191)</td>
<td></td>
</tr>
<tr>
<td>( \ln(u) )</td>
<td>-0.742***</td>
<td></td>
</tr>
<tr>
<td>(0.172)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( R^2 )</td>
<td>0.801</td>
<td>0.851</td>
</tr>
</tbody>
</table>

(b) 

<table>
<thead>
<tr>
<th>Dep. Variable:</th>
<th>unemployment rate</th>
<th>( \ln(u) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v )</td>
<td>-2.294***</td>
<td>-1.044***</td>
</tr>
<tr>
<td>(0.415)</td>
<td>(0.224)</td>
<td>(1.286)</td>
</tr>
<tr>
<td>State Unemployment Dispersion</td>
<td>3.177***</td>
<td>2.803***</td>
</tr>
<tr>
<td>(0.237)</td>
<td>(0.252)</td>
<td>(0.0421)</td>
</tr>
<tr>
<td>( v^2 )</td>
<td>0.411**</td>
<td></td>
</tr>
<tr>
<td>(0.191)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \ln(v) )</td>
<td>-0.662***</td>
<td></td>
</tr>
<tr>
<td>(0.117)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( R^2 )</td>
<td>0.801</td>
<td>0.851</td>
</tr>
</tbody>
</table>

Notes: Own calculations from BLS, Conference Board and Abraham (1987). Annual data from 1960-2010. Newey-West robust standard errors in parenthesis; * = significant at 10%; ** = significant at 5%; *** = significant at 1%. All regressions include a cubic time trend. \( v \) is national vacancy rate and \( u \) is national unemployment rate. Dispersion measure for unemployment is defined as \( \delta_i^u = \sqrt{\sum_{t=1}^{n} \frac{e_{i,t}}{e_t} (u_{i,t} - \bar{u})^2} \) where \( e_{i,t} \) is employment in state \( i \) at time \( t \), \( e_t \) is national employment at time \( t \), \( u_{i,t} \) is unemployment rate state \( i \) at time \( t \) and \( u_t \) is the national unemployment rate.
1.8 Appendix B - Proofs and auxiliary results

This part of the appendix contains proofs of all results not contained in the body of the paper.

Worker value functions, homeownership and default:

Let me first define the worker’s value function generally. Let \( V^1(X) \) be the end-of-period value function of a worker who owns a non-depreciated unit of housing and \( W^1(X) \) be his beginning-of-period value function. Similarly, let \( V^0(X) \) be the end-of-period value function of a worker who owns a depreciated unit of housing and \( W^0(X) \) be his beginning-of-period value function. Also, let \( \hat{V}(X) \) be the end-of-period value function of a worker who does not own housing at the end of a period and \( \hat{W}(X) \) be his beginning-of-period value function.

Then, we have that:

\[
V^1(X) = \gamma + e - \tilde{d}(X) + \mu(\theta(A))(w(A) - e) + \beta E_X \left[ W^1(X') \right]
\]

with

\[
W^1(X) = \max_{\tilde{\tau}} \left\{ F(\tilde{\tau}) \tilde{V} + (1 - F(\tilde{\tau})) \max \left\{ V^1(X), \hat{V}(X) \right\} - F(\tilde{\tau}) c + \int_{\tilde{\tau}} \epsilon dF \right\}
\]

and

\[
\tilde{V} = \max_{\tilde{x}} \left\{ \max \left\{ V^1(\tilde{x}), \hat{V}(\tilde{x}) \right\} \right\}
\]

Also,

\[
V^0(X) = \gamma + e - \tilde{d}(X) + \mu(\theta(A))(w(A) - e) + \beta E_X \left[ W^0(X') \right]
\]

with

\[
W^0(X) = \max_{\tilde{\tau}} \left\{ F(\tilde{\tau}) \tilde{V} - F(\tilde{\tau}) (c + \zeta) + \int_{\tilde{\tau}} \epsilon dF + (1 - F(\tilde{\tau})) \max \left\{ V^0(X), \max \left\{ V^1(X), \hat{V}(X) \right\} - \zeta \right\} \right\}
\]

Finally,

\[
\hat{V}(X) = e + \mu(\theta(A))(w(A) - e) + \beta E_X \left[ \hat{W}(X') \right]
\] (1.25)
and

\[ \hat{W}(X) = \max_{\tilde{\tau}} \left\{ F(\tilde{\tau}) \tilde{V} + (1 - F(\tilde{\tau})) \max \left\{ V^1(X), \tilde{V}(X) \right\} - F(\tilde{\tau}) c + \int_{\tilde{\tau}} e dF \right\} \]

or \( \hat{W}(X) = W^1(X) \), i.e. house purchases are completely reversible. Inspection of the value functions immediately implies that

**Lemma 7.** A worker with no debt overhang prefers to buy housing iff \( \hat{d}(X) \leq \gamma \).

Secondly, I show that workers with debt overhang do not demand non-depreciated housing and default only when migrating. This follows from the following result:

**Lemma 8.** \( V^0(X) \geq V^1(X) - \beta \cdot \zeta, \forall X \).

**Proof.** First I show that \( W^1(X) - W^0(X) \leq \zeta, \forall X \). We have that:

\[ W^1(X) = F(\tilde{\tau}(X,1)) (\tilde{V} - c) + (1 - F(\tilde{\tau}(X,1))) \max \left\{ V^1(X), \tilde{V}(X) \right\} + \int_{\tilde{\tau}(X,1)} e dF \]

and

\[ W^0(X) = F(\tilde{\tau}(X,0)) (\tilde{V} - c - \zeta) + (1 - F(\tilde{\tau}(X,0))) \max \left\{ V^0(X), \max \left\{ V^1(X), \tilde{V}(X) \right\} - \zeta \right\} + \int_{\tilde{\tau}(X,0)} e dF \]

Therefore,

\[ W^0(X) = F(\tilde{\tau}(X,0)) (\tilde{V} - c - \zeta) + \int_{\tilde{\tau}(X,0)} e dF + \]

\[ + (1 - F(\tilde{\tau}(X,0))) \max \left\{ V^1(X), \max \left\{ V^1(X), \tilde{V}(X) \right\} - \zeta \right\} \]

\geq F(\tilde{\tau}(X,1)) (\tilde{V} - c - \zeta) + \int_{\tilde{\tau}(X,1)} e dF +

\[ + (1 - F(\tilde{\tau}(X,1))) \max \left\{ V^1(X), \max \left\{ V^1(X), \tilde{V}(X) \right\} - \zeta \right\} + \]

\geq F(\tilde{\tau}(X,1)) (\tilde{V} - c - \zeta) + \int_{\tilde{\tau}(X,1)} e dF +

\[ + (1 - F(\tilde{\tau}(X,1))) \left( \max \left\{ V^1(X), \tilde{V}(X) \right\} - \zeta \right) = \hat{W} \]

The first inequality comes from not using the optimal cutoff \( \tilde{\tau}(X,0) \) but rather \( \tilde{\tau}(X,1) \) and the second inequality comes from disregarding the max operator. Hence, \( W^1(X) - W^0(X) \leq \)
However, note that $W^1(X) = \bar{W} = \zeta$ and so $W^1(X) - W^0(X) \leq \zeta$. Observing that $V^1(X) - V^0(X) = \beta E_X \left[ W^1(X') - W^0(X') \right]$, we have our result.

Lemma 8 implies that if $V^1(X) \geq \bar{V}(X)$, then $\max \left\{ V^0(X), \max \left\{ V^1(X), \bar{V}(X) \right\} - \zeta \right\} = V^0(X)$, i.e. a worker with debt overhang default only when migrating and therefore does not demand non-depreciated housing from the region he currently resides in. If $V^1(X) < \bar{V}(X)$ it may be that a worker may default without migrating but he still does not demand non-depreciated housing. Therefore, we have that if $d(X) < \gamma$, a worker with debt overhang does not demand non-depreciated housing and default only when migrating.

**Law of motion for $\pi_t$ and $\mu_t$:**

The laws of motion for the distributions $\pi_t$ and $\mu_t$, $\pi_{t+1} = \mathcal{E}(\pi_t, \mu_t)$, $\mu_{t+1} = \mathcal{E}(\pi_t, \mu_t)$ are given by,

$$\pi_{t+1}(l_0, l_1) = \mathcal{E}(\pi_t((l_0, l_1)), \mu_t((l_0, l_1))) =$$

$$= \int \rho \cdot I \left\{ l_0 = l'_0(\bar{A}, l_{0,-1}, l_{1,-1}, \pi_t, \mu_t), l_1 = l'_1(\bar{A}, l_{0,-1}, l_{1,-1}, \pi_t, \mu_t) \right\} d\pi_t(l_{0,-1}, l_{1,-1}) + (1.26)$$

$$+ \int (1 - \rho) \cdot I \left\{ l_0 = l'_0(\bar{A}, l_{0,-1}, l_{1,-1}, \pi_t, \mu_t), l_1 = l'_1(\bar{A}, l_{0,-1}, l_{1,-1}, \pi_t, \mu_t) \right\} d\mu_t(l_{0,-1}, l_{1,-1})$$

and similarly,

$$\mu_{t+1}(l_0, l_1) = \mathcal{E}(\pi_t((l_0, l_1)), \mu_t((l_0, l_1))) =$$

$$= \int \rho \cdot I \left\{ l_0 = l'_0(\bar{A}, l_{0,-1}, l_{1,-1}, \pi_t, \mu_t), l_1 = l'_1(\bar{A}, l_{0,-1}, l_{1,-1}, \pi_t, \mu_t) \right\} d\mu_t(l_{0,-1}, l_{1,-1}) + (1.27)$$

$$+ \int (1 - \rho) \cdot I \left\{ l_0 = l'_0(\bar{A}, l_{0,-1}, l_{1,-1}, \pi_t, \mu_t), l_1 = l'_1(\bar{A}, l_{0,-1}, l_{1,-1}, \pi_t, \mu_t) \right\} d\pi_t(l_{0,-1}, l_{1,-1})$$

**Definition of stationary recursive equilibrium:**

**Definition 9.** A symmetric stationary recursive equilibrium for the economy described above consists of market tightness $\theta(A)$, wages $w(A)$, worker value functions $V^X(X)$, migration value $V$, migration thresholds $\bar{V}(X, \chi)$, regional house prices $p(X)$, laws of motion, $\Gamma$, for $X$, and distributions $(\pi^*, \mu^*)$ such that:
1. $\theta(A)$ satisfies (1.5) given $w(A)$;

2. $w(A)$ satisfies (1.6);

3. $V^\chi(X)$ satisfies (1.14) given $\theta(A)$, $\overline{V}$, and $\Gamma$;

4. $\overline{V}$ satisfies (1.13) given $V^\chi(X)$;

5. $\overline{\epsilon}(X,\chi)$ satisfies (1.16) given $\overline{V}$ and $V^\chi(X)$;

6. $p(X)$ satisfies (1.11) given $\Gamma$.

7. $\Gamma$ and satisfies 1.18 and law of motions for $A$ given $\overline{\epsilon}(X,\chi)$;

8. $(\overline{V}^*,\nu^*)$ are a fixed point of (1.26) and (1.27).

9. $\int' (A,1) \, d\overline{V}^* + \int' (A,1) \, d\nu^* + 2l_0 = L$ (population constancy).

**Stationary equilibrium characterization:**

**Lemma 10.** Suppose that $\overline{l}'(A,1)$ is a continuous function of $l$ for $A \in \{A, \overline{A}\}$. Then $V^1(A,l)$ is a bounded continuous function of $l$ for $A \in \{A, \overline{A}\}$. Conversely, suppose that $V^1(A,l)$ is a continuous function of $l$ for $A \in \{A, \overline{A}\}$. Then $\overline{l}'(A,1)$ is a continuous function of $l$ for $A \in \{A, \overline{A}\}$.

**Proof.** Let us define the operator

$$T[v(A,l)] = \max_{\{\overline{\epsilon}(A',l')\}} \left\{ \gamma - d(A,l) + e + \mu ((\theta(A))(w(A) - e) + \right.$$  

$$+ \beta E_A \left[ F(\overline{\epsilon}(A',l')) \overline{V} + \left( 1 - F(\overline{\epsilon}(A',l')) \right) v(A',l') - F(\overline{\epsilon}(A',l')) c + \int_{\epsilon(A',l')} d\epsilon \right] \right\}$$

with $l' = \overline{l}'(A,l)$ and a given $\overline{V}$. Note that $d(A,l) = g'(L - l')$ is a continuous function of $l'$ by assumptions on $g'$ and $\overline{l}'(A,l)$ is also a continuous function of $l$. Both are also clearly bounded. Hence $T$ maps bounded continuous functions into bounded continuous functions. Furthermore, $T$ satisfies Blackwell’s sufficient conditions for a contraction. Hence, by the Contraction Mapping Theorem (Stokey and Lucas (1989)), $T$ has a unique fixed point in the space of bounded continuous functions. Hence, $V^1(A,l)$ is a bounded continuous function.
Suppose now that \( V^1(A, l) \) is continuous in \( l \). Given this it follows immediately that \( d(A, l) \) must be continuous in \( l \) and since \( d = g'(L - l') \) is continuous in \( l' \), it follows that \( l' \) must be continuous in \( l \).

As mentioned in Section 1.4.4, I focus on equilibria, in which both \( V^1(A, l) \) and \( l'(A, l) \) are continuous. Given continuity of \( V^1(A, l) \) and the compactness of the domain of \( X \) it follows immediately that the set \( \arg \max_x \{ V(x) \} \) is nonempty. I then have the following result.

**Lemma 11.** \( V^1(A, l) \) is a non-increasing function of \( l \), and \( l'(A, l) \) is a non-decreasing function of \( l \).

**Proof.** The basic idea behind showing this result is showing that, if \( V^1(A, l) \) is a non-increasing function of \( l \), then \( l'(A, l) \) is a non-decreasing function of \( l \) and vice versa, and then showing that it is impossible for \( l' \) to be strictly decreasing in \( l \) if \( V^1(A, l) \) is strictly increasing in \( l \), i.e. the only property that is mutually consistent and hence possible in equilibrium is for \( V^1(A, l) \) to be a non-increasing function of \( l \) and \( l' \) to be a non-decreasing function of \( l \). I proceed to show this in three steps.

**Step 1:** I show that if \( l'(A, l) \) is a non-decreasing function of \( l \) then \( V^1(A, l) \), \( A \in \{ A, \overline{A} \} \) is a non-increasing function of \( l \). Define again the operator

\[
T[v(A, l)] = \gamma - d(A, l) + \epsilon + \mu (\theta(A)) \left( w(A) - e \right) + \beta E_A \left[ \omega \left( A', l' \right) \right]
\]

where

\[
\omega(A, l) = \max_{\tilde{\tau}} \left\{ F(\tilde{\tau}) \bar{V} + (1 - F(\tilde{\tau})) v(A, l) - F(\tilde{\tau}) c + \int_{\tilde{\tau}} d\epsilon \right\}
\]

We have that \( l' \) is non-decreasing in \( l \) and hence \( -d(A, l) \) is non-decreasing in \( l \). Furthermore, for \( l_1 < l_2 \):

\[
\omega(A, l_1) - \omega(A, l_2) =
\]

\[
= (1 - F(\bar{\tau}(A, l_1))) v(A, l_1) + \int_{\bar{\tau}(A, l_1)} d\epsilon + F(\bar{\tau}(A, l_1)) (\bar{V} - c) -
\]

\[
- (1 - F(\bar{\tau}(A, l_2))) v(A, l_2) + \int_{\bar{\tau}(A, l_2)} d\epsilon + F(\bar{\tau}(A, l_2)) (\bar{V} - c) \geq
\]

\[
\geq (1 - F(\bar{\tau}(A, l_2))) v(A, l_1) + \int_{\bar{\tau}(A, l_2)} d\epsilon + F(\bar{\tau}(A, l_2)) (\bar{V} - c) -
\]

\[
- (1 - F(\bar{\tau}(A, l_2))) v(A, l_2) + \int_{\bar{\tau}(A, l_2)} d\epsilon + F(\bar{\tau}(A, l_2)) (\bar{V} - c)
\]

\[
\geq (1 - F(\bar{\tau}(A, l_2))) \left( v(A, l_1) - v(A, l_2) \right)
\]

65
where the inequality comes from the fact that a different cutoff from the optimal, \( \bar{c}(A, l_1) \), is used. Hence, if \( v(A, l) \) is non-increasing in \( l \) then so is \( \omega(A, l) \). Therefore, \( \omega(A', l') \) is non-increasing in \( l \). Then it follows that \( T[V(A, l)] \) is non-increasing in \( l \). Hence, \( T \) maps non-increasing functions to non-increasing functions. Therefore, its fixed point, \( V^1(A, l) \), is non-increasing in \( l \).

**Step 2:** I show that if \( V^1(A, l) \), \( A \in \{A, \bar{A}\} \) is a non-increasing function of \( l \) then \( l'_1(A, l) \) is a non-decreasing function of \( l \). Suppose that \( V(A, l) \) is non-increasing in \( l \) but \( \exists l_1, l_2 \) with \( l_1 < l_2 \) s.t. \( l'_1 = l'_1(A, l_1) > l'_1(A, l_2) = l'_2 \). Then

\[
0 \leq V^1(A, l_1) - V^1(A, l_2) = -d(A, l_1) + d(A, l_2) + \beta E_A \left[ W^1 \left(A, l'_1\right) - W^1 \left(A, l'_2\right)\right] < 0
\]

since \(-d(A, l_1) + d(A, l_2) < 0\) given that \( l'_1 > l'_2 \) and \( E_A \left[ W^1 \left(A, l'_1\right) - W^1 \left(A, l'_2\right)\right] < 0 \) since \( V^1(A, l) \) is non-increasing in \( l \) and \( l'_1 > l'_2 \). Hence, we have a contradiction, which is due to the assumption that \( \exists l_1, l_2 \) with \( l_1 < l_2 \) s.t. \( l'_1 = l'_1(A, l_1) > l'_1(A, l_2) = l'_2 \).

**Step 3:** I show that if \( V^1(A, l) \) is strictly increasing in \( l \) then \( l'_1 \) is increasing in \( l \). Let \( l_1 < l_2 \). Then \( V(A, l_1) < V(A, l_2) \), \( A \in \{A, \bar{A}\} \) and so \( \bar{c}(A, l_1, 1) > \bar{c}(A, l_2, 1) \). Hence, \( l'_1(A, l_1) = (1 - F(\bar{c}(A, l_1, 1))) l_1 + \Psi(A, l_1) \) and \( l'_1(A, l_2) = (1 - F(\bar{c}(A, l_2, 1))) l_2 + \Psi(A, l_2) \). Now, observe that \( \Psi(A, l_1) \leq \Psi(A, l_2) \) and \( (1 - F(\bar{c}(A, l_1, 1))) l_1 < (1 - F(\bar{c}(A, l_2, 1))) l_2 \), which implies that \( l'_1(A, l_1) < l'_1(A, l_2) \).

Steps 1 and 2 show that it is mutually consistent for \( V^1 \) and \( l'_1 \) to be non-increasing and non-decreasing in \( l \), respectively. However, Step 3 shows that it is not mutually consistent for \( V^1 \) and \( l'_1 \) to be strictly increasing and strictly decreasing in \( l \), respectively. Hence, in any equilibrium \( V^1 \) and \( l'_1 \) are non-increasing and non-decreasing, respectively. \( \square \)

Next, I define

\[
\bar{l} = \begin{cases} 
\sup_l \left\{ l : (\bar{A}, l) \in \arg \max_{l} \left\{ V^1(A, l) \right\} \right\} , & \left\{ l : (\bar{A}, l) \in \arg \max_{l} \left\{ V^1(A, l) \right\} \right\} \neq \emptyset \\
0, & \text{o.w.}
\end{cases}
\]

(1.28)
for \( \bar{l} \) and similarly

\[
\bar{l} = \begin{cases} 
\sup \left\{ l : (A, l) \in \arg \max_l \left\{ V^1 (A, \bar{l}) \right\} \right\}, & \text{if } \left\{ l : (A, l) \in \arg \max_l \left\{ V^1 (A, \bar{l}) \right\} \right\} \\
0, & \text{o.w.} \end{cases}
\]

for \( \bar{l} \). Therefore, regions with populations of mobile workers above \( \bar{l} \) and \( \bar{l} \) cannot attract workers and therefore only lose population, i.e. \( \bar{l}' (A, l) < \bar{l} \), for \( \bar{l} > \bar{l} \). It follows that the functions \( \bar{l}' (A, l) \) and \( \bar{l}' (A, l) \) have a fixed point, which is unique whenever there are potential house price differences across regions, that is when \( g() \) is strictly concave.

**Lemma 12.** There exist \( \bar{l}' < \bar{l} \) and \( \bar{l}' < \bar{l} \) such that \( \bar{l}' (\bar{A}, \bar{l}) = \bar{l}' \) and \( \bar{l}' (A, \bar{l}) = \bar{l}' \). They are unique if \( g() \) is strictly concave and, furthermore, \( \bar{l}' (\bar{A}, l) = \bar{l}' \) for \( l \in [0, \bar{l}] \) and \( \bar{l}' (A, l) = \bar{l}' \) for \( l \in [0, \bar{l}] \).

**Proof.** I show this for \( A = \bar{A} \) since the other case is analogous. Note that if

\[
\left\{ l : (\bar{A}, l) \in \arg \max_l \left\{ V^1 \left( A, \bar{l} \right) \right\} \right\} \neq \emptyset
\]

then \( \bar{l} \in \left\{ l : (\bar{A}, l) \in \arg \max_l \left\{ V^1 \left( A, \bar{l} \right) \right\} \right\} \) since \( V^1 (\bar{A}, l) \) is continuous in \( l \). Hence,

\[
\left\{ l : (\bar{A}, l) \in \arg \max_l \left\{ V^1 \left( A, \bar{l} \right) \right\} \right\}
\]

is a compact set. Now \( l' \) is continuous. Hence, by Brower’s fixed point theorem there exists a \( \bar{l}' \) such that \( l' (\bar{A}, \bar{l}') = \bar{l}' \) and \( \bar{l}' < \bar{l} \). If \( \left\{ l : (\bar{A}, l) \in \arg \max_l \left\{ V^1 \left( A, \bar{l} \right) \right\} \right\} = \emptyset \) then \( \bar{l} = 0 \) and clearly \( \bar{l}' (\bar{A}, 0) = 0 \). Hence, can define \( \bar{l}' = 0 \). Noting that for \( l > \bar{l} \), \( \bar{l}' (A, l) < \bar{l} \), it follows that no fixed point of \( l' (\bar{A}, l) \) can be greater than \( \bar{l} \).

It is straightforward to show that for \( a < \gamma \bar{l}' (\bar{A}, l) \) is constant for \( l \in [0, \bar{l}] \) by using Lemma 11 and proceeding by contradiction. In particular, suppose that \( \bar{l}' (\bar{A}, l_1) > \bar{l}' (\bar{A}, l_2) \) for some \( l_1, l_2 \), such that \( \bar{l} > l_1 > l_2 \). However, this implies that \( V^1 (\bar{A}, l_1) < V^1 (\bar{A}, l_2) \) and so \( l_1 > \bar{l} \). Now, to show uniqueness, I first show that if \( (A, l) \in \arg \max \{ V^1 (A, l) \} \) then \( \left( A, l' (A, l) \right) \in \arg \max \{ V^1 (A, l) \} \) as well. I can show this by contradiction. Suppose \( A = \bar{A} \) and \( (A, l) \in \arg \max \{ V^1 (A, l) \} \) but \( \left( A, l' (A, l) \right) \notin \arg \max \{ V^1 (A, l) \} \). Hence by Lemma 11 it follows that \( l' (A, l) > l \). However, this also implies that \( l' (A, l) \geq l' (A, l) \). But if

\footnote{These bounds always exist, given that \( l_1 \leq L \).}
Lemma 12 shows that regions will either be experiencing inflows that equalize the population of mobile workers or slow declines in population if they are too big relative to regional productivity (i.e. \( l > \bar{l} \) for booming regions or \( l > l' \) for depressed regions). Lemma 12 also describes what \( \Psi(A, l) \) looks like in equilibrium. In particular, it follows that

\[
\Psi(\bar{A}, l) = \begin{cases} 
\bar{l}' - (1 - F(-c)) \bar{l} & , \bar{l} < \bar{l}' \\
0 & , \text{o.w.} 
\end{cases}
\]

(1.30)

with \( \bar{l}' = (1 - F(-c)) \bar{l} \) and similarly for \( \Psi(A, l) \).\(^{42}\) Lastly, I compare mobile workers’ value function for regions with the same populations but different productivities, \( A \):

**Lemma 13.** \( V^1(\bar{A}, l) \geq V^1(A, l) \), \( \forall l \)

*Proof.* Suppose that \( V^1(\bar{A}, l) > V^1(A, l) \) some \( l \). Hence, \( (\bar{A}, l) \notin \arg \max \{ V^1(A, l) \} \) and hence by equation Equation 1.18 and since \( \Psi(\bar{A}, l') = 0 \), \( l' \bar{A}, l) < l \) and \( l' \bar{A}, l) < l' \bar{A}, l) \). Then,

\[
V^1(\bar{A}, l) - V^1(A, l) = \gamma - d(\bar{A}, l) - \gamma + d(\bar{A}, l) + \mu(\bar{A}) (w(\bar{A}) - e) - \\
- \mu(\bar{A}) (w(\bar{A}) - e) + \beta E[W^1(A', l') | (\bar{A}, l)] - \beta E[W^1(A', l') | (A, l)]
\]

However, given that \( l' \bar{A}, l) < l' \bar{A}, l) \), it follows that \( W^1(\bar{A}, l') > W^1(A, l') \) for \( A \in \{\bar{A}, A\} \) from Lemma 11. Now, if \( W^1(\bar{A}, l') \geq W^1(A, l') \) or \( W^1(\bar{A}, l') \geq W^1(A, l') \), then \( \beta E[W^1(A', l') | (\bar{A}, l)] - \beta E[W^1(A', l') | (A, l)] > 0 \) and we have a contradiction. Hence, \( W^1(\bar{A}, l') < W^1(A, l') \) and \( W^1(\bar{A}, l') < W^1(A, l') \). If \( W^1(\bar{A}, l') \geq W^1(A, l') \), then, again we will arrive at a contradiction. Hence, \( W^1(\bar{A}, l') < W^1(A, l') \), which implies that \( V^1(\bar{A}, l') < V^1(A, l') \).

Now, define, \( l^0 = l, l^1 = l'(\bar{A}, l^0), l^2 = l'(\bar{A}, l^1), \) etc. and similarly \( l^0 = l, l^1 = l'(A, l^0), l^2 = l'(A, l^1) \), etc. Hence, by induction we have that \( V(\bar{A}, l^i) < V(A, l^i) \) for \( i = 1, 2, \ldots \).

\(^{42}\)Note that this need not be the case for a linear \( g(\cdot) \), i.e. when \( a = \gamma \), since in that case there are no house price differences across regions, so workers are indifferent between migrating to regions with any \( l \). However, for continuity with respect to \( a \), I will also look at the equilibrium, where \( \Psi \) takes this form for the case of a linear \( g(\cdot) \).
Now, clearly each sequence is either decreasing and bounded below by \( \bar{l} \) and \( \underline{l} \), respectively, or increasing and bounded above by \( \bar{l} \) and \( \underline{l} \), respectively. Hence, by continuity of \( V^1(A, l) \) and from Lemma 12 it follows that the sequences would converge to unique fixed points of \( l'_1(A, \bar{l}) \) and \( l'_1(A, \bar{l}) = \bar{l}^* \) and \( \underline{l}^* \), respectively and by continuity of \( V^1 \) then \( V^1(A, \bar{l}^*) < V^1(A, \underline{l}^*) \), which is only possible if the set \( \{ l : (A, l) \in \text{argmax}_l \{ V^1(A, \bar{l}) \} \} \) is empty, in which case \( \bar{l} = 0 \) and so \( V^1(A, 0) < V^1(A, \bar{l}) \). However, note that

\[
V^1(A, 0) = \gamma - 0 + e + (\mu(\theta(A))) (w(A) - e) + \beta E[W^1(A', 0) | A] \geq \\
\geq \gamma + e + (\mu(\theta(A))) (w(A) - e) + \beta \left( V^1(A, 0) - F(c) \cdot c + \int_c \epsilon dF \right) = \\
\gamma + e + (\mu(\theta(A))) (w(A) - e) + \beta \left( \gamma + e + (\mu(\theta(A))) (w(A) - e) + \\
+ \beta E[W^1(A', 0) | A] - F(c) \cdot c + \int_c \epsilon dF \right) \geq \ldots \geq \\
\geq \sum_{j=0}^{\infty} \beta^j \left( \gamma + e + (\mu(\theta(A))) (w(A) - e) \right) + \sum_{j=1}^{\infty} \beta^j \left( -F(c) \cdot c + \int_c \epsilon dF \right)
\]

Similarly, one can show that

\[
V^1(A, \bar{l}) = \gamma - d(A, \bar{l}) + e + (\mu(\theta(A))) (w(A) - e) + \beta E[W^1(A', \bar{l}) | A] \leq \\
\leq \gamma - d(A, \bar{l}) + e + (\mu(\theta(A))) (w(A) - e) + \beta \left( V^1(A, \bar{l}) - F(c) \cdot c + \int_c \epsilon dF \right) = \\
\gamma - d(A, \bar{l}) + e + (\mu(\theta(A))) (w(A) - e) + \beta \left( \gamma - d(A, \bar{l}) + e + (\mu(\theta(A))) (w(A) - e) + \\
+ \beta E[W^1(A', \bar{l}) | A] - F(c) \cdot c + \int_c \epsilon dF \right) \leq \ldots \leq \\
\leq \sum_{j=0}^{\infty} \beta^j \left( \gamma - d(A, \bar{l}) + e + (\mu(\theta(A))) (w(A) - e) \right) + \sum_{j=1}^{\infty} \beta^j \left( -F(c) \cdot c + \int_c \epsilon dF \right)
\]

Hence, \( V^1(A, \bar{l}) > V^1(A, 0) \) implies that \( \sum_{j=0}^{\infty} \beta^j (\gamma - d(A, \bar{l}) + e + (\mu(\theta(A))) (w(A) - e) > \\
\sum_{j=0}^{\infty} \beta^j (\gamma + e + (\mu(\theta(A))) (w(A) - e) \right), \) which is a contradiction.

\[\square\]

**Proof of Lemma 1**

Results 1 through 4 follow directly from the implication of Lemmas 11, 12, and 13 for the law of motion for \( l'_1 \). To show result 5, Lemma 12 implies that regions with productivity \( A \) move to
a population of mobile workers of $\bar{l}^*$. Furthermore, Lemma 13 implies that regions with $l^*$ that experience a sequence of negative productivity shocks always move deterministically through a set $L$ of population levels, which is countably infinite in the case equilibria with $l^* = 0$ or finite in the case of equilibria with $l^* > 0$ since out-migration rates are always bounded below by $F(-c) > 0$ (the natural out-migration rate from regions $X \in \arg\max_X \{V^1(X)\}$). Since regions with populations $l \notin L$ eventually move to $\bar{l}$ with probability 1 but conditional on being in $\bar{l}$ they never move to population states outside $L$, it follows that all states $l \notin L$ are transient, i.e. the ergodic set of the process governing regional evolutions is given by $L$. Therefore, the stationary distribution of populations across regions $\nu^*$ and $\nu^*$ are given by the set $L$.

Proof of Proposition 2

We have that $\text{out}(A, l) = \frac{\Psi(A, l) 1_{l^*}}{1 + \delta}$ and similarly $\text{in}(A, l) = \frac{\Psi(A, l) 1_{l^*}}{1 + \delta}$. To show Result 1, first of all, Lemma 13 implies that $\bar{V} - V^1(A, l) \leq V - V^1(A, l), \forall l$ and hence, $\tau(A, l, l) \leq \tau(A, l, l)$, which immediately implies that $\text{out}(A, l) \leq \text{out}(A, l)$. This inequality is of course strict for $l = l^{**}$. Similarly, from (1.30) it follows that $\text{in}(A, l, l) \geq \text{in}(A, l)$. To show Result 2, Lemma 11 implies that $q((A, l), 1)$ is increasing in $l$ and strictly so for $A = A$, which immediately implies that $\text{out}(A, l)$ is increasing in $l$. Turning to $\text{in}(A, l)$, note that $\Psi(A, l)$ is decreasing in $l$ and hence so will $\frac{\Psi(A, l) 1_{l^*}}{1 + \delta}$.

Proof of Proposition 3

We have that $p(A, l) = d(A, l) + \beta \cdot E_A[p(A', l(A, l))]$. First of all note that $d(A, l) = g' \left( \bar{l} - l(A, l) \right)$ is continuous in $l$ by the continuity of $l(A, l)$ and $g'()$. Furthermore, it is bounded and increasing in $(A, l)$ since $l(A, l)$ is increasing in $(A, l)$ by Lemma 11 and equation (1.30) and $g'()$ is decreasing. Now, clearly the operator $T[f(A, l)] = d(A, l) + \beta \cdot E_A \left[ f \left( A', l(A, l) \right) \right]$ maps bounded continuous functions into bounded continuous functions and, furthermore, satisfies Blackwell’s sufficient conditions for a contraction. Hence, by the Contraction Mapping Theorem, $T$ has a unique fixed point in the space of bounded continuous functions. Therefore, since $d(A, l)$ and $l(A, l)$ are increasing in $(A, l)$ it follows that $T$ maps increasing functions into increasing functions and therefore its unique fixed point is increasing.

To show the second part of the proposition, first of all, note that $\bar{p}(A, l^*) = p(\bar{A}, l^*)$ and
\[ \hat{p}(A, l^*) = p(A, l^*). \]
Furthermore, \[ \hat{p}(A, l) = p\left(A, l^{-1}_1(A, l)\right) \] for \( l \in \left(l^*, \bar{l}\right) \) given equation (1.30) and the inverse \( l^{-1}_1 \) is well-defined. Furthermore, given that \( l'_1(A, l) \) is increasing in \( l \) it follows that \( l'_1(A, l) \) is increasing in \( l \) as well. Hence, by the properties of \( p \) it follows that \[ p\left(A, l^{-1}_1(A, l)\right) \] is increasing in \( l \) for \( l \in \left(l^*, \bar{l}\right) \) and hence so is \( \hat{p}(A, l) \). Now, noting that \[ p\left(A, \bar{l}\right) \geq p(A, l) \] \( l < l^* \), it follows that \( \hat{p}(A, l) \) is increasing in \( l \) for \( l \in \left[l^*, \bar{l}\right] \).

**Proof of Lemma 5**

Consider the population constancy condition

\[ \int l'_1(\bar{A}, l_1) \, d\nu^* + \int l'_1(\bar{A}, l_1) \, d\nu^* + 2l_0 = L \]

It follows from Lemma 12 that \( l'_1(\bar{A}, l_1) = \bar{l} \) for every \( l_1 \) in the support of \( \bar{\nu}^* \). Hence, \( \int l'_1(\bar{A}, l_1) \, d\nu^* = \bar{l} \) and so we have that:

\[ L - 2l_0 - \bar{l} = \int_0^{\bar{l}} l'_1(\bar{A}, l_1) \, d\nu^*(l_1) \]

where the integral on the RHS is a sum over members in the support of \( \nu^* \), \( \mathcal{L} \), which is given recursively by \( \{l^n_1\}_{n=0}^{\infty} \) with \( l^n_1 = \bar{l} \) and \( l^n_1 = (1 - F\left(\bar{\nu}(A, l^{-1}_1)\right)) l^{n-1}_1 \), for \( n \geq 1 \), i.e. we have:

\[ G(l_0, \bar{l}) = L - 2l_0 - \bar{l} - \sum_{n=0}^{\infty} l'_1(A, l^n_1) \cdot \nu^*(l^n_1) = 0 \] (1.31)

where \( \nu^*(l^n_1) \) is the fraction of regions with population \( l^n_1 \). First of all, note that changes in \( l_0 \) have no direct effect on the law of motion \( l'_1(A, l_1) \) or on the stationary distribution \( \nu^* \) since \( l_0 \) does not enter directly into workers value functions and hence does not enter into \( \bar{\nu}(A, l_1) \) and neither does it enter the equation for the stationary distribution \( \nu^* \). Hence, \( \frac{\partial G}{\partial l_0} = -2 \).

Now, turning to the effect of \( \bar{l} \), first of all note that \( \bar{l} \) only affects the ergodic set of \( \nu^* \) but does not affect the equations for the distribution (1.26) and (1.27), i.e. while the set \( \{l^n_1\}_{n=0}^{\infty} \) may change, the distribution over that set does not change with \( \bar{l} \). It is actually straightforward to explicitly solve for the distributions \( \nu^* \) and \( \bar{\nu}^* \) from (1.26) and (1.27) for an equilibrium of
In particular, we have that

$$ \nu^* (l_0^n) = \rho \cdot \sum_{n=0}^{\infty} \nu^* (l_0^n) $$

$$ \nu^* (l_0^n) = (1 - \rho) \cdot \sum_{n=0}^{\infty} \nu^* (l_0^n) $$

$$ \nu^* (l_0^n) = (1 - \rho) \nu^* (l_0^{n-1}) $$

Noting that $$ \sum_{n=0}^{\infty} \nu^* (l_0^n) = 1 $$ we get that $$ \nu^* $$ follows a geometric distribution with parameter $$ 1 - \rho $$, i.e. $$ \nu^* (l_0^n) = \rho^n (1 - \rho) $$.

Turning to the effect of $$ \bar{\gamma}^* $$ on $$ \bar{\gamma} (A, l_0^n) $$, first of all observe that for $$ a = \gamma $$, $$ \bar{\gamma} (A, l_1) = \bar{\gamma} (A) $$, since there are no house price differences across regions. Therefore, $$ \sum_{n=0}^{\infty} \bar{\gamma} (A, l_1^n) = \sum_{n=0}^{\infty} (1 - F(\bar{\gamma} (A)))^{n+1} \bar{\gamma}^n (1 - \rho) = (1 - F(\bar{\gamma} (A)))^{n+1} \bar{\gamma}^n (1 - \rho) $$. Hence, $$ \frac{\partial \bar{\gamma}^n}{\partial l_0} = -1 - \frac{(1 - F(\bar{\gamma} (A)))^{n+1} \bar{\gamma}^n (1 - \rho)}{(1 - f(\bar{\gamma} (A)))^{n+1}} $$ and so by the implicit function theorem, $$ \frac{d \gamma^*}{dl_0} = -1 - \frac{2(1 - F(\bar{\gamma} (A)))^{n+1} \bar{\gamma}^n (1 - \rho)}{(1 - f(\bar{\gamma} (A)))^{n+1}} $$ Note that $$ (1 - F(\bar{\gamma} (A)))^{n+1} \bar{\gamma}^n (1 - \rho) < 1 $$ and so $$ \frac{d \gamma^*}{dl_0} < -1 $$, which immediately implies that $$ \gamma^* + l_0 $$ is decreasing in $$ l_0 $$.

1.9 Appendix C - Computational

Model for calibration

In this section I provide a description of the differences between the model used for calibration in Section 1.5 and the basic model from Section 1.3. I only include model assumptions that differ across the two models.

Regional labor markets, job creation, and destruction

As before, I consider a discrete time economy with infinite number of periods $$ t = 0, 1, 2, \ldots $$. The economy consists of a measure $$ M = 2 $$ of islands or regions. The economy is populated by a measure $$ L $$ of infinitely lived workers, residing in different regions who are risk neutral and derive utility from consumption as well as from housing services. Workers can supply 1 unit of labor.

Regional productivity follows a two state Markov chain as before. In each region there is a
representative firm that can open job vacancies at a per period cost of $k$ and recruit workers for production in the same period. At the end of each period, after production takes place, with probability $s$ a job becomes unproductive and is destroyed. The labor market of each region is characterized by a search and matching friction. After migration decisions have been made, a measure $\tilde{u}_t^j$ of unemployed workers and a measure $\tilde{v}_t^j$ of vacancies try to match with each other. Matching is described by a CRS matching function $m^j(\tilde{u}_t^j,\tilde{v}_t^j)$ giving the total number of regional matches per period. Defining $\theta_t^j = \frac{\tilde{v}_t^j}{\tilde{u}_t^j}$ as the regional market tightness and $\mu(\theta) = m(1, \theta)$, we have that the job finding probability for a worker is $\mu(\theta_t^j)$ and a job filling probability for a vacancy is $\frac{\mu(\theta_t^j)}{\theta_t^j}$. Workers that remain unmatched in a given period receive a period payoff of $e$.

**Job creation decisions**

Let $J(A)$ be the value from posting a vacant job in a region with productivity $j$. Then

$$J(A) = -k + \frac{\mu(\theta(A))}{\theta(A)}K(A) + \left(1 - \frac{\mu(\theta(A))}{\theta(A)}\right)\beta(1 - s)E_t[J(A)] \quad (1.32)$$

Similarly, let $K(A)$ be the value from a matched job. Then

$$K(A) = A - w + \beta(1 - s) \cdot E_t[K(A)] \quad (1.33)$$

where $w$ is the wage rate paid. The firm is owned by the workers in the economy and discounts payoffs at the discount rate of workers $\beta$. The firm opens vacant jobs until the cost of opening a vacancy equals the expected payoff from a filled vacancy, or:

$$J(A) = 0 = -k + \frac{\mu(\theta(A))}{\theta(A)}K(A)$$

or

$$K(A) = k \frac{\theta(A)}{\mu(\theta(A))}$$

Then, substituting in for $K(A)$, we get that:

$$\frac{\theta(A)}{\mu(\theta(A))} = \frac{A - w}{k} + \beta(1 - s) \left(\rho \frac{\theta(A)}{\mu(\theta(A))} + (1 - \rho) \frac{\theta(A)}{\mu(\theta(A))}\right) \quad (1.34)$$
Regional housing market and depreciation shock

The set-up for regional housing markets, home financing and depreciation shock are identical as before.

Worker migration

Unemployed workers have an idiosyncratic region preference $\epsilon$ for the current region that they reside in. At the beginning of each period an unemployed worker gets a new draw of $\epsilon$ from a continuous distribution $F$ with density function $f$ with $E[\epsilon] = 0$ and support over $[-B, B]$ for some $B > 0$. Upon observing his match quality, a worker decides whether to move to a different region. Moving is instantaneous and entails a fixed cost of $c$. Upon moving, the worker terminates the mortgage debt contract, in which case the housing unit is sold if it is not depreciated and lenders are repaid or the worker defaults and incurs the penalty $\zeta$. Upon moving, the worker draws a new $\epsilon \sim F$ for the new region. Migration is directed, a worker migrates to the region that gives him the highest expected value.

Regional state variables

In contrast to the one-period job model, in this model each region $j$ will be fully characterized by its current period productivity $A_j^t$ plus the beginning-of-period measure of workers with and without debt overhang, $l_{j,t}^h$, $h \in \{0, 1\}$, the beginning-of-period measure of unemployed workers with and without debt overhang, $u_{j,t}^h$, $h \in \{0, 1\}$ and the beginning-of-period distributions of workers over employment and housing states, $\bar{v}_t$ and $\bar{v}_t$. However, since there are no house prices differences across regions only the region’s current productivity, $A_{j,t}$, will be relevant for worker migration decisions. Also, I let $\bar{u}_{j,t}^h$ to be the post-migration measure of unemployed workers with housing state $h \in \{0, 1\}$ in region $j$, while $\bar{l}_{j,t}$ is the end-of-period labor force.

Worker value functions

Similarly to the one-period job model all workers (weakly) prefer to be homeowners in equilibrium. I define $V_{t \uparrow}^h (A)$ to be an unemployed worker’s value function, given regional state $A$, and

\[
\frac{\theta(A)}{\mu(\theta(A))} = \frac{A - w}{k} + \beta(1 - s) \left( \rho \frac{\theta(A)}{\mu(\theta(A))} + (1 - \rho) \frac{\theta(A)}{\mu(\theta(A))} \right)
\]
housing state \( h \). and similarly an employed worker's value function is given by \( V^h_E(A) \). Then, letting

\[
\bar{V} = \max_A \{ V^1_E(A) \} \tag{1.36}
\]

be migration value, we have that:

\[
V^h_U(A) = \mu(\theta(A)) V^h_E(A) + (1 - \mu(\theta(A))) \left( e - d + \gamma + \beta E_A \left[ W^h(A') \right] \right) \tag{1.37}
\]

and

\[
V^h_E(A) = w - d + \gamma + \beta \cdot E_A \left[ (1 - s) V^h_E(A') + s \cdot \left[ W^h(A') \right] \right] \tag{1.38}
\]

where

\[
W^h(A) = \max_{\bar{\tau}(A, X)} \left\{ F(\bar{\tau}(A, h)) \bar{V} + (1 - F(\bar{\tau}(A, h))) V^h(A) - F(\bar{\tau}(A, h))(c + (1 - h) \zeta) + \int_{\bar{\tau}(A, h)} \epsilon dF \right\} \tag{1.39}
\]

The function \( W^h(A) \) takes this form since as I now show, only unemployed workers who migrate default (this is in the spirit of Lemma 8 above).

**Lemma 14.** For \( A \in \{ A, \bar{A} \} \), \( V^h_E(A) > V^h_E(\bar{A}) - \zeta \) and \( V^0_E(A) > V^1_E(A) - \zeta \).

**Proof.** First of all note that \( W^1(A) - W^0(A) \leq \zeta \). The proof of this is identical to the proof of Lemma 8. We have that:

\[
V^1_E(A_t) = w - d + \gamma + \beta \cdot \left[ (1 - s) E_t \left[ V^h_E(A_{t+1}) \right] + s E_t \left[ W^1(A_{t+1}) \right] \right]
\]

or

\[
V^1_E(A_t) = \frac{w + \gamma - d}{1 - \beta(1 - s)} + s \beta E_t \left[ \sum_{j=0}^{\infty} \beta^j (1 - s)^j W^1(A_{t+j+1}) \right]
\]

Hence,

\[
V^1_E(A_t) - V^0_E(A_t) \leq s \beta E_t \left[ \sum_{j=0}^{\infty} \beta^j (1 - s)^j (W^1(A_{t+j+1}) - W^0(A_{t+j+1})) \right]
\]
Therefore, we have that

\[ V_E^1(A_t) - V_E^0(A_t) \leq s\beta \sum_{j=0}^{\infty} \beta^j (1 - s)^j \gamma \]

or

\[ V_E^1(A_t) - V_E^0(A_t) \leq s\beta \frac{\gamma}{1 - \beta(1 - s)} \]

Note that \( \frac{s\beta}{1 - \beta(1 - s)} < 1 \) and so \( V_E^1(A_t) - V_E^0(A_t) < \gamma \) or \( V_E^0(A_t) > V_E^1(A_t) - \gamma \). Now, consider \( V_U^0(A_t) \) and \( V_J^1(A_t) \). We have:

\[ V_U^h(A_t) = \mu(\theta(A_t)) V_E^h(A_t) + (1 - \mu(\theta(A_t))) \left( e - d + \gamma + \beta \cdot E_t \left[ W^h(A_{t+1}) \right] \right) \]

Hence,

\[ V_U^1(A_t) - V_U^0(A_t) \leq \mu(\theta(A_t)) \left( V_E^1(A_t) - V_E^0(A_t) \right) + \left( 1 - \mu(\theta(A_t)) \right) \beta E_t \left[ W^1(A_{t+1}) - W^0(A_{t+1}) \right] \]

and so:

\[ V_U^1(A_t) - V_U^0(A_t) \leq \mu(\theta(A_t)) \gamma + (1 - \mu(\theta(A_t))) \beta \cdot \gamma \]

This implies that, \( V_U^1(A_t) - V_U^0(A_t) < \gamma \) or \( V_U^0(A_t) > V_U^1(A_t) - \gamma \).

Lemma 14 implies that a worker defaults only when unemployed and migrating out of the region.

**Wage determination**

Rather than through Nash bargaining, the wage rate in each region is pinned down at \( w \). However, \( w \) lies within the bargaining set of a worker-job match. In particular, at bargaining stage, in region \( j \), the outside option of a worker is \( V_U^h(A) = e + \beta A \left[ W^h(A') \right] \). Hence, the minimum wage a worker would accept, \( w^h(A) \), leaves him indifferent between employment and unemployment, i.e. it solves:

\[ V_E^h(A, w^h) = V_U^h(A) \quad (1.40) \]

or

\[ w^h(A) = e + \beta E_t \left[ W^h(A_{t+1}) \right] - \beta \left[ (1 - s)E_t \left[ V_E^h(A_{t+1}) \right] + sE_t \left[ W^h(A_{t+1}) \right] \right] \quad (1.41) \]
which implies that

$$w^h(A) = e + (1 - s)\beta \left[ E_t \left[ W^h(A_{t+1}) \right] - E_t \left[ V^h_{E_t} (A_{t+1}) \right] \right]$$  \hspace{1cm} (1.42)

Similarly, the maximum wage a firm would accept, $\bar{w}(A)$, leaves it indifferent between employing the worker and keeping the job vacant, i.e. it solves:

$$K(A, \bar{w}) = 0$$  \hspace{1cm} (1.43)

or

$$\bar{w}^l(A) = A + \beta (1 - s) \cdot E_t [K(A_{t+1})]$$  \hspace{1cm} (1.44)

Using (1.34) and (1.35) we get that:

$$\bar{w}(A) = \bar{A} + k \beta (1 - s) \left( \rho - \frac{\theta(A)}{\mu(\theta(A))} + (1 - \rho) \frac{\theta(A)}{\mu(\theta(A))} \right)$$  \hspace{1cm} (1.45)

$$\bar{w}(A) = A + k \beta (1 - s) \left( \rho - \frac{\theta(A)}{\mu(\theta(A))} + (1 - \rho) \frac{\theta(A)}{\mu(\theta(A))} \right)$$  \hspace{1cm} (1.46)

Therefore,

$$w \in \left[ \max \{ w^0(A), w^1(A) \} , \bar{w}(A) \right]$$  \hspace{1cm} (1.47)

Note that with wages determined via Nash bargaining there will be two wages depending on a worker’s housing state $h$, with the wage rate being a weighted average of the two boundaries.

**Laws of motion for regional unemployment and labor force**

We have the following laws of motion:

$$\tilde{u}_t^0(A, u_t^0) = (1 - q^0(A)) u_t^0$$  \hspace{1cm} (1.48)

$$\tilde{p}_t^0(A, p_t^0, u_t^0) = l_t^0 - q^0(A)u_t^0$$  \hspace{1cm} (1.49)

$$\tilde{u}_t^1(A, u_t^1, l_t^1) = \begin{cases} (1 - q^1(A)) u_t^1 & \text{if } A = \bar{A} \\ (1 - q^1(\bar{A})) u_t^1 + \Phi_t (u_t^1, l_t^1) & \text{if } A = \bar{A} \end{cases}$$  \hspace{1cm} (1.50)
I describe \( \Phi_t \left( u^1, l^1 \right) \) below. Using these laws of motion one can derive laws of motion for the distributions \( u_t \) and \( l_t \). However, I am more interested in deriving laws of motions for the first moments of these distributions, i.e. for the total (beginning-of-period) unemployment and total labor force in booming and depressed regions conditional on housing state \( h \). Denote these by \( \bar{U}^h_t \) and \( \bar{L}^h_t \) for booming regions and \( \bar{U}^h_t \) and \( \bar{L}^h_t \) for depressed regions, where \( h \in \{0, 1\} \). Then,

\[
\begin{align*}
\bar{U}^h_{t+1} &= \rho \left[ \int_{(\mathcal{A}, u^h, l^h)} (1 - \mu (\mathcal{A})) (1 - s) \bar{u}^h (\mathcal{A}, u^h, l^h) + s l^h (\mathcal{A}, u^h, l^h) d\nu_t \right] \\
&\quad + (1 - \rho) \left[ \int_{(\mathcal{A}, u^h, l^h)} (1 - \mu (\mathcal{A})) (1 - s) \bar{u}^h (\mathcal{A}, u^h, l^h) + s l^h (\mathcal{A}, u^h, l^h) d\nu_t \right]
\end{align*}
\]

and similarly for \( \bar{L}^h_{t+1} \) and \( \bar{L}^h_{t+1} \). Additionally, we have a population constancy condition:

\[
\sum_h \bar{L}^h + \bar{L}^h = L
\]

Given that agents are indifferent between migrating to any booming regions, there will be some indeterminacy in the in-migration function \( \Phi_t \left( u^1, l^1 \right) \). Similarly to the one-period job model I will focus on a migration function where all booming regions have the same end-of-period population of workers with no debt overhang in a given time period, which I denote by \( \bar{l}^1_t \). Therefore,

\[
\Phi_t \left( u^1, l^1 \right) = \bar{l}^1_t - l^1 + q^1 (\mathcal{A}) \cdot u^1
\]

and, solving for the above laws of motion, we get that:

\[
\begin{align*}
\bar{U}^1_{t+1} &= \rho \left[ \int_{(\mathcal{A}, u^1, l^1)} (1 - \mu (\mathcal{A})) (1 - s) \left( u^1 + \bar{l}^1_t - l^1 \right) + s l^1 d\nu_t \right] \\
&\quad + (1 - \rho) \left[ \int_{(\mathcal{A}, u^1, l^1)} (1 - \mu (\mathcal{A})) (1 - s) \left( u^1 + s l^1 - q^1 (\mathcal{A}) u^1 \right) d\nu_t \right]
\end{align*}
\]

or

\[
\begin{align*}
\bar{U}^1_{t+1} &= \rho \left[ (1 - \mu (\mathcal{A}) (1 - s) \left( \bar{U}^1_{t+1} + \bar{L}^1_t - \bar{L}^1_t \right) + s l^1 \right] \\
&\quad + (1 - \rho) \left[ (1 - \mu (\mathcal{A}) (1 - s) (1 - q^1 (\mathcal{A})) l^1 + s (L^1 - q^1 (\mathcal{A}) L^1) \right]
\end{align*}
\]

Similarly,

\[
\begin{align*}
\bar{L}^1_{t+1} &= \rho \left[ (1 - \mu (\mathcal{A}) (1 - s) (1 - q^1 (\mathcal{A})) \bar{L}^1_{t+1} + s (L^1 - q^1 (\mathcal{A}) \bar{L}^1_{t+1}) \right] \\
&\quad + (1 - \rho) \left[ (1 - \mu (\mathcal{A}) (1 - s) \left( \bar{U}^1_{t+1} + \bar{L}^1_t - \bar{L}^1_t \right) + s l^1 \right]
\end{align*}
\]
with an equivalent expression for $\bar{U}^0_{t+1}$ and $\bar{U}^0_{t+1}$. Furthermore,

$$L^1_{t+1} = \rho L^1_t + (1 - \rho) \left( L^1_t - q^1(A) L^1_t \right)$$

$$L^0_{t+1} = \rho \left( L^0_t - q^0(A) L^0_t \right) + (1 - \rho) \bar{U}^0_t$$

$$T^1_{t+1} = \rho \left( T^1_t - \phi^0(A) T^0_t \right) + (1 - \rho) \left( T^0_t - \phi^0(A) T^0_t \right)$$

$$L^0_{t+1} = \rho \left( L^0_t - \phi^0(A) L^0_t \right) + (1 - \rho) \left( L^0_t - \phi^0(A) L^0_t \right)$$

Summing up $T^1_{t+1}$, $L^1_{t+1}$, $L^0_{t+1}$ and $T^0_{t+1}$ and using the population constancy condition, we have that:

$$\bar{U}^1_t = T^1_t + q^1(A) L^1_t + q^0(A) L^0_t + \phi^0(A) U^0_t$$

Hence, finally we get:

$$\bar{U}^1_{t+1} = \rho \left[ (1 - \mu(\theta(A))) (1 - s) \left( \bar{U}^1_t + q^1(A) L^1_t + q^0(A) L^0_t + \phi^0(A) U^0_t \right) \right] + (1 - \rho) \left[ (1 - \mu(\theta(A))) (1 - s) \left( \bar{U}^1_t + q^1(A) L^1_t + q^0(A) L^0_t + \phi^0(A) U^0_t \right) \right]$$

$$\bar{U}^0_{t+1} = \rho \left[ (1 - \mu(\theta(A))) (1 - s) \left( \bar{U}^0_t + q^0(A) L^0_t + \phi^0(A) U^0_t \right) \right] + (1 - \rho) \left[ (1 - \mu(\theta(A))) (1 - s) \left( \bar{U}^0_t + q^0(A) L^0_t + \phi^0(A) U^0_t \right) \right]$$

$$\bar{U}^0_{t+1} = \rho \left[ (1 - \mu(\theta(A))) (1 - s) \left( \bar{U}^0_t + q^0(A) L^0_t + \phi^0(A) U^0_t \right) \right] + (1 - \rho) \left[ (1 - \mu(\theta(A))) (1 - s) \left( \bar{U}^0_t + q^0(A) L^0_t + \phi^0(A) U^0_t \right) \right]$$

and for the labor force measures:

$$\bar{L}^1_{t+1} = \rho \left( \bar{L}^1_t + q^1(A) L^1_t + q^0(A) L^0_t + \phi^0(A) U^0_t \right) + (1 - \rho) \left( \bar{L}^1_t - q^1(A) L^1_t \right)$$

$$\bar{L}^0_{t+1} = \rho \left( \bar{L}^0_t - q^0(A) L^0_t \right) + (1 - \rho) \left( \bar{L}^0_t + q^1(A) L^1_t + q^0(A) L^0_t + \phi^0(A) U^0_t \right)$$

$$\bar{T}^1_{t+1} = \rho \left( \bar{T}^1_t - \phi^0(A) U^0_t \right) + (1 - \rho) \left( \bar{T}^0_t - \phi^0(A) U^0_t \right)$$

$$\bar{T}^0_{t+1} = \rho \left( \bar{T}^0_t - q^0(A) U^0_t \right) + (1 - \rho) \left( \bar{T}^0_t - q^0(A) U^0_t \right)$$

**Computational Algorithms**

Here I describes the algorithms used for simulating the stationary equilibrium for the one-period job model as well as the equilibrium of the calibrated model.
Simulating the one-period job model

Computing the stationary equilibrium of the one-period job model requires a nested fixed point approach. The inner fixed point requires computing value function $V^1$ and law of motion $l^1$ for a given value of $l^*$ and $\ell^*$. In the outer loop, I vary $l^*$ and $\ell^*$ to satisfy $V^1 \leq V^1$ and the population constancy condition, respectively. I use the following algorithm:

1. I construct a grid $\mathcal{G}$ for $l$;
2. Guess a value of $\ell^*$ on the grid and set $l_0^* = 0$:
3. Pick initial value functions $\overline{V}_0^1$ and $\underline{V}_0^1$ on $\mathcal{G}$. Iterate the following steps until convergence of $V^1$:
   
   - Given $\overline{V}_i^1$ and $\underline{V}_i^1$ derive the implied law of motion for $l_i^1$;
   - Given law of motion for $l_i^1$ and solve $\overline{V}_i^1$ and $\underline{V}_i^1$ for $\overline{V}_{i+1}^1$ and $\underline{V}_{i+1}^1$. Note that I use linear interpolation for values of $V_n^1$ that are not on the grid, when solving for $V_{i+1}^1$.
4. Check whether $V^1 - V^1 \leq \epsilon_V > 0$ for all $l \in \mathcal{G}$. If not, increase $l_0^*$ and go back to Step 3;
5. Given $\overline{V}^1$ and $\underline{V}^1$, compute $l^1$ and use it to simulate $B$ regions for $T$ periods starting from random initial conditions (for productivity state and labor force levels);
6. Using the empirical distribution at time $T$, check population constancy condition. If population constancy condition is sufficiently close to being valid terminate, otherwise increase or decrease $\ell^*$ so that the population constancy condition holds and go back to Step 1.

Simulating the model for calibration

- Given the block-recursive nature of the equilibrium in this case (due to the assumption of no house price differences across regions), I first solve for the worker value functions (which depend on the productivity of the region and the housing and employment state of the worker) as well as migration cutoffs. I also solve for regional market tightness;
- Using the solved value functions and market tightness I check whether the wage rate lies in the worker-job bargaining set; If not, then equilibrium does not exist for this wage rate;
If the wage rate lies in the bargaining set, I proceed to solve for the average measure of unemployed in booming and depressed regions, $U^h$, $U^h$, $h \in \{0, 1\}$ as well as the total labor force in booming and depressed regions $L^h$ and $L^h$. Based on these I can solve for aggregate unemployment rate, aggregate market tightness and migration rate.

To determine unemployment dispersion for the stationary equilibrium, I simulate the stationary distribution of unemployment and labor force over regions. I simulate $B = 2000$ regions for $T = 1000$ periods starting from random initial conditions (productivity state, labor force and unemployment level) using the outflow probabilities from workers problems and job finding probabilities for booming and depressed regions. I then take the resulting distribution at time $T$ and compute unemployment dispersion and average unemployment rate in booming and depressed regions from it.

Simulating the non-stationary equilibrium is identical apart from using the dispersion in unemployment across booming and depressed regions as the measure of unemployment dispersion in this case because of computational issues in simulating the non-stationary distribution over regions.
Chapter 2

Debt Capacity and Asset Prices - an Investment Quality Channel

2.1 Introduction

The recent financial crisis and its aftermath were characterized by a significant deterioration in credit market conditions. There were disruptions across a broad set of debt markets, such as markets for new syndicated lending (Ivashina and Scharfstein (2010)), unsecured commercial paper (Brunnermeier (2009)), and mortgage and asset backed securities (Gorton (2009)). Short term collateralized borrowing against asset backed securities was also severely affected, with repo haircuts increasing substantially relative to pre-crisis levels (Krishnamurthy (2010)). These events and the subsequent sharp fall in output and investment have renewed interest in understanding how shocks arising from within the financial system and financial frictions affect the broader economy. Motivated by these questions, in this paper I study a model of endogenous fluctuations in credit market conditions and their effect on capital reallocation and aggregate output.

I start with a standard production economy with productivity heterogeneity, a fixed supply of durable capital and no technological externalities. A spot market allows for capital reallocation across entrepreneurial firms. Entrepreneurs buy capital from this market by issuing debt securities collateralized by future output produced with that capital. Nevertheless, they have superior information about the distribution of their future productivity and hence repayment to outside investors. Within this framework, using a variant of the collateral equilibrium concept
proposed by Geanakoplos and Zame (2009), I first show that the economy has a unique equilibrium characterized by high equilibrium leverage, a high price of capital in the secondary market and reallocation of capital from low productivity to high productivity producers, which results in high aggregate output and productivity.

However, this result is not robust to the introduction of a standard financial friction, which limits the pledgeability of high output realizations to outside investors. In particular, the combination of limited pledgeability with asymmetric information in the standard economy I consider creates a new channel of interaction between the spot market for durable capital and the credit market. This two-way feedback between the two markets, which is shown schematically in Figure 2-1, creates a complementarity in lenders’ debt purchase decisions, which operates as a pecuniary externality. In particular, a lender’s decision to buy debt contracts with low or high face value affects the borrowing constraints that entrepreneurs face and hence their debt capacity. Debt capacity, however affects how much entrepreneurs can bid up the price of capital. On the other hand, the price of capital affects the pool of entrepreneurs who are willing to buy capital and produce. A higher price of capital allows high productivity borrowers to separate from low productivity risky borrowers when issuing debt, thus improving the quality of investment. However, when the price of capital is low, good borrowers cannot separate from bad borrowers who are also more likely to issue high face value debt. Hence, the price of capital indirectly affects lenders’ expected debt payoffs.

The complementarity is sufficiently strong to lead to multiple equilibria. In the “high leverage” equilibrium the price of capital is high, leverage is high and capital is reallocated from less productive to more productive firms. In the “low leverage”, capital reallocation is depressed with lower productivity entrepreneurs delaying exit, the price of capital is low and so is leverage. As a result, this “investment quality” channel of credit market fluctuations leads to a positive co-movement between credit volume, asset prices and aggregate output.

I then study policies that can improve credit market conditions and capital reallocation in the context of a central bank that uses unconventional monetary policy. In particular, if a central bank can commit to offering a high price for debt contracts with high face value, or equivalently low haircuts on such debt, the economy never falls into a “low leverage” equilibrium, capital is reallocated optimally and aggregate output is high. Moreover, the central bank need not buy any debt in equilibrium as its commitment only acts as a floor for debt prices. This implies that a policy such as the TALF, through which the central bank directly lends in asset backed
Figure 2-1: Investment quality channel of credit market fluctuations

Borrowing Constraints/Debt Prices

Borrower pool

Debt Capacity/Leverage

Price of Capital

securities markets, can have a beneficial effect on these markets and on aggregate output. This happens without a significant holding of debt by the central bank in equilibrium, which was indeed the case with the TALF program.

The channel of fluctuations in credit conditions investigated here is distinct from the standard mechanisms operating in models with endogenous collateral constraints, in which an asset price directly effects a borrower’s debt capacity (Kiyotaki and Moore (1997)). In particular, as in these models, borrowing constraints affect the price of capital through a cash-in-the-market pricing effect (Shleifer and Vishny (1992), Allen and Gale (1994)). However, unlike those models, in my model the price of capital, does not directly affect borrowing constraints but rather affects the quality of the borrower pool through the ability of good borrowers to separate from bad borrowers.

It is important to emphasize that the mechanism of interaction between the price of capital and borrowing constraints, though related, is distinct from the standard mechanism of adverse selection models of market liquidity (Eisfeldt (2004), Kurlat (2009), Malherbe (2010), and Tirole (2010)). In particular, it comes from a general equilibrium effect, which influences collateral quality by affecting whether there can be separation on debt issuance or pooling only. In my model, if the price of capital were exogenously fixed at a high (low) price there would be only separation (pooling) in the credit market. This importance of the price of capital, which is determined in general equilibrium, for the ability of borrowers to separate and, hence, for

the set of financial contracts traded in equilibrium is distinct from other models that study corporate finance and financial contracting issues in general equilibrium, which do not consider such feedbacks (Dubey and Geanakoplos (2002), Dubey, Geanakoplos, and Shubik (2005), Bisin and Gottardi (2006), Bisin, Gottardi, and Ruta (2009) and Zame (2006)).

The analysis of unconventional monetary policy brings the paper close to a growing recent literature, which examines the effects of direct intervention by a central bank in credit markets (Gertler and Karadi (2011), Ashcraft, Garleanu, and Pedersen (2010), and Gertler and Kiyotaki (2010)). Finally, on the applied side, the paper provides some insight into the effect of financial crises and fluctuations in credit market conditions on capital reallocation. Factor reallocation across establishments has been shown to be an important component of aggregate productivity growth (Foster, Haltiwanger, and Krizan (2000)). Through this process of creative destruction low productivity firms shrink and exit, while high productivity firms enter and expand. The majority of this research, however, has focused on the effects of financial development on factor reallocation and TFP investigating processes that take place at low frequency (Buera and Shin (2009)). Nevertheless, financial market imperfections can affect factor reallocation at business cycle frequencies as well (Eisfeldt and Rampini (2006)). In fact, credit crunches can actually slow down the exit of low productivity firms (Caballero and Hammour (2005)) and lead to sluggish TFP growth as the experience of Japan during the late 90s has shown (Caballero, Hoshi, and Kashyap (2008)). One would expect that this effect should be particularly enhanced in the aftermath of a financial crisis (Estevao and Severo (2009)). Goldberg (2011) address some of these issues and finds quantitative important TFP losses from decreases in firms’ ability to borrow. However he works with a model, in which entrepreneur debt capacity is subject to exogenous shocks, while in my model debt capacity fluctuations arise endogenously.

The rest of the paper is organized as follows: Section 2.2 describes the set-up for the model with asymmetric information, defines the equilibrium notion used and shows uniqueness. Section 2.3 contains the main result of the “investment quality” channel. I add limited pledgeability to the model and show how equilibrium multiplicity can arise. I also investigate the reason for the multiplicity and show that it is the result of a strategic complementarity in lenders’ debt purchase decisions. Section 2.4 discusses unconventional monetary policy and Section 2.5 provides concluding comments.
2.2 Model Set-up and Equilibrium

2.2.1 Preferences and Technology

I consider a two period economy with \( t = 0, 1 \). There is a measure 2 of agents with utility function given by

\[
U (c_0, c_1) = c_0 + c_1
\]  

(2.1)

where \( c_0 \) and \( c_1 \) is \( t = 0 \) and \( t = 1 \) consumption, respectively. One half of these agents have access to a production technology, and so I refer to them as Entrepreneurs. Entrepreneurs can use their production technology together with a durable good, which I call capital, to produce \( t = 1 \) consumption. Capital cannot be consumed either at \( t = 0 \) or \( t = 1 \). However, similarly to a Lucas tree, a unit of capital delivers \( \epsilon \geq 0 \) units of \( t = 1 \) consumption regardless of whether it is used in production or not.

Each Entrepreneur \( i \) has access to a linear production technology \( f^i(k^i) = a^i \cdot k^i \), where \( k^i \) is the capital that Entrepreneur \( i \) operates and \( a^i \) is an idiosyncratic productivity that is realized at \( t = 1 \). Even though \( a^i \) is realized at \( t = 1 \) so there is ex post heterogeneity among Entrepreneurs, there is also ex ante heterogeneity at \( t = 0 \). In particular, Entrepreneurs have a type \( \theta \in \{B, G\} \) at \( t = 0 \), with a distribution in the population of \( Pr(\theta^i = G) = \phi \) and \( Pr(\theta^i = B) = (1 - \phi) \). Ex post productivity \( a^i = a(\theta^i) \) is distributed as follows:

\[
a(\theta) = \begin{cases} 
\bar{a} & Pr. \, \eta^\theta \\
\lambda^\theta \bar{a} & Pr. \, 1 - \eta^\theta
\end{cases}
\]  

(2.2)

where \( \bar{a} > a \), \( \eta^G > \eta^B \), and \( 1 = \lambda^G > \lambda^B = \lambda \).\footnote{Also, I define \( \Delta a = \bar{a} - a \).} I define the expected productivity at \( t = 0 \) for an Entrepreneur of type \( \theta \) as

\[
E^\theta[a] = \eta^\theta \bar{a} + (1 - \eta^\theta) \lambda^\theta \bar{a}
\]  

(2.3)

Note that \( E^G[a] > E^B[a] \), so Good Entrepreneurs strictly dominate Bad Entrepreneurs in terms of production efficiency. Therefore, given the linear preferences, the welfare maximizing allocation of capital involves the G-type Entrepreneurs holding all the capital and producing.

The productivity distribution for \( a(\theta) \) has two important features. First of all, \( a(\theta) \) domi-
nates a (B) in the first order stochastic dominance sense, and, secondly, a (G) is more compressed compared to a (B), so a (G) dominates a (B) in the sense of second order stochastic dominance as well. Given the binary outcome structure, one can thinking of $\eta^\theta$ as the probability that the Entrepreneur’s project is successful, in which case it delivers $\bar{a}$, while in case it is unsuccessful, the project delivers $\lambda^\theta a$.

Entrepreneur types $\theta$ are private information at $t = 0$. Entrepreneurs are endowed with $k^E$ units of capital, and $e^E$ units of the $t = 0$ consumption good. The rest of the population are agents without access to a production technology, which I call Consumers. Consumers are endowed with $k^C$ units of capital and $e^C$ units of $t = 0$ consumption good. Finally, $K = k^C + k^E$ denotes the aggregate capital stock in the economy.

There is a resale market for capital open at $t = 0$ with capital trading at a price of $p$. Lastly, all agents in the economy have access to a linear storage technology, which can transfer 1 unit of the consumption good from $t = 0$ to $t = 1$.

### 2.2.2 Credit market

The credit market arrangement I work with is similar to Geanakoplos (2003) and Geanakoplos and Zame (2009). Agents cannot commit to repaying unsecured debt and instead only borrow by issuing debt contracts collateralized by durable capital and its $t = 1$ output. Collateralized debt cannot be issued contingent on the possible $t = 1$ output realization of capital and hence is potentially risky. Given the linear production functions of Entrepreneurs, such collateralized debt contracts will be given by pairs $\{(A_i, C_i)\}_i$ of a promised payoff $A_i$ and capital collateralizing it $C_i$. Additionally, for simplicity, I assume that debt is non-recourse, i.e. an agent can default on some debt obligations while meeting others.

Without loss of generality, I restrict attention to debt contracts $\{(\gamma, 1)\}_{\gamma \in [0, \infty)}$ since any debt contract $(A_i, C_i)$ is equivalent to $C_i$ units of contract $(\frac{A_i}{C_i}, 1)$. Also, for notational convenience I drop the reference to a unit of capital and denote debt contracts by their face value $\{\gamma\}_{\gamma \in [0, \infty)}$.

Therefore, a debt contract with face value $\gamma$ issued by a borrowing Entrepreneur, will have a $t = 1$ payoff given by:

$$d = \min\{\gamma, a(\theta) + \epsilon\}$$

where $a(\theta)$ is the Entrepreneur’s realized productivity at $t = 1$ and $\epsilon$ is the payoff of capital that is independent from the identity of the Entrepreneur utilizing it. Since there is no recourse,
default occurs whenever the face value of debt exceeds the total production of the Entrepreneur.

I will look at symmetric collateral equilibria (Geanakoplos and Zame (2009)), competitive equilibria, in which each debt contract with face value $\gamma$ trades in an anonymous competitive market at a price of $D(\gamma)$. All such debt securities will be priced even if they are not traded in equilibrium, and the set of contracts traded in equilibrium will be determined endogenously. Note that the identity of a borrower will be irrelevant in a symmetric collateral equilibrium with \textit{ex ante} asymmetric information, so focusing on anonymous debt markets, which removes the dependence of the price of debt $D(\gamma)$ on the identity of the borrower is without loss of generality.\footnote{However, in Section 2.3.1, where I consider the case of no \textit{ex ante} information asymmetry, debt contracts depend on the type of the borrower, i.e. we will have $D(\gamma, \theta)$.}

Given the \textit{ex ante} information asymmetry at $t = 0$, Entrepreneurs of any type can sell debt in the same security markets. Furthermore, buyers of debt will hold fully diversified pools of debt securities across idiosyncratic Entrepreneur production risk, and a single Entrepreneur can potentially be a seller of debt securities and a buyer of the (pooled) debt of other Entrepreneurs.\footnote{Thinking of traded securities as fully diversified pools is similar to Dubey, Geanakoplos, and Shubik (2005). In that paper heterogeneous default creates adverse selection and as in my model there is an equilibrium price for every asset but some assets are not traded in equilibrium.}

### 2.2.3 Agent’s Problem

I first define the problem of each agent in this economy. In particular, a Consumer’s problem is given by

$$
V^C(e^C, k^C; p, \{D(\gamma)\}_{\gamma}) = \max_{c_0^C, z_1^C, \{k_1^C(\gamma)\}_{\gamma}, \{b_1^C(\gamma)\}_{\gamma}} \frac{1}{\alpha} \left( c_0^C + c_1^C \right)
$$

s.t.

$$
\begin{align*}
&c_0^C + z_1^C + \int_0^\infty (p - D(\gamma)) k_1^C(\gamma) d\gamma + \int_0^\infty D(\gamma) b_1^C(\gamma) d\gamma = e^C + p \cdot k^C \\
&c_1^C = z_1^C + \int_0^\infty \max\{0, \epsilon - \gamma\} k_1^C(\gamma) d\gamma + \int_0^\infty \tilde{D}(\gamma) b_1^C(\gamma) d\gamma \\
&c_0^C \geq 0, \quad z_1^C \geq 0, \quad k_1(\gamma) \geq 0, \quad b_1^C(\gamma) \geq 0, \quad \forall \gamma \in [0, \infty)
\end{align*}
$$

where $k_1^C(\gamma)$ is a Lebesgue measurable function that gives the measure of capital that collateralizes debt with face value $\gamma$, which the agent buys. Note that this is also the measure of debt
with face value $\gamma$ issued by the agent. Similarly, $b^C(\gamma)$ is the holding of debt with face value $\gamma$ that the Consumer buys at $t = 0$, and $\bar{D}(\gamma)$ is the $t = 1$ payoff of the (pooled) debt with face value $\gamma$.\(^5\) The Entrepreneur’s problem is similar apart from the idiosyncratic payoff uncertainty from production. In particular we have

$$V^E(e^E, k^E; \theta, p, \{D(\gamma)\}) = \max_{c^E_0, z^E_1, \{k^E_1(\gamma)\}, \{b^E_1(\gamma)\}} \ c^E_0 + E \left[ c^E_1 | \theta \right]$$

$$\text{s.t. } c^E_0 + z^E_1 + \int_0^\infty (p - D(\gamma))k^E_1(\gamma) d\gamma + \int D(\gamma)b^E_1(\gamma) d\gamma = e^E + p \cdot k^E$$

$$c^E_1 = \begin{cases} 
    z^E_1 + \int_0^\infty \max \{0, \lambda(\theta^1)a + \epsilon - \gamma\} k^E_1(\gamma) d\gamma + \int_0^\infty \bar{D}(\gamma)b_1(\gamma) d\gamma, & a = \lambda(\theta^1)a \\
    z^E_1 + \int_0^\infty \max \{0, a + \epsilon - \gamma\} k^E_1(\gamma) d\gamma + \int_0^\infty \bar{D}(\gamma)b^E_1(\gamma) d\gamma, & a = a
  \end{cases}$$

$$c^E_0 \geq 0, \ z^E_1 \geq 0, \ k^E_1(\gamma) \geq 0, \ b^E_1(\gamma) \geq 0, \ \gamma \in [0, \infty)$$

where I have suppressed the dependence of $c^E_0$, $c^E_1$, $z^E_1$, $\{k^E_1(\gamma)\}$, $\{b^E_1(\gamma)\}$ on $\theta$ for notational convenience. However, I make this dependence explicit in the definition of equilibrium below.

### 2.2.4 Equilibrium Definition

I next define the competitive equilibrium for this economy. As mentioned above the definition is based on the notion of collateral equilibrium of Geanakoplos and Zame (2009). I parametrize the economy by the agents initial endowments, the fraction of Good Entrepreneurs in the economy, the production technologies of the two types and the payoff of capital $\epsilon$, i.e. $\nu = (e^E, k^E, e^C, k^C, \phi, \bar{\alpha}, q, \lambda, \eta^G, \eta^B, \epsilon)$. I look at symmetric equilibria, in which agents of the same type choose the same consumption and asset holdings. In the definition below allocations with superscript $E$ refer to Entrepreneurs and allocations with superscript $C$ refer to Consumers. Also I use the individual agent allocation and aggregate allocation for that group interchange-
ably.

Given that I restrict attention to debt contracts collateralized by 1 unit of capital, \( k^E_1(\gamma, \theta) \) will also be the debt with face value \( \gamma \) issued by Entrepreneurs of type \( \theta \) and similarly for \( k^C_1(\gamma) \). \( D(\gamma) \) is the price of debt contract with face value \( \gamma \) and \( p \) is the resale price of capital. I define the equilibrium as follows:

**Definition 15.** A collateral equilibrium \( CE(\nu) \) for a given \( \nu \) consists of \( t = 0 \) asset holdings and \( t = 0 \) and \( t = 1 \) consumption allocations \( \{c_0^C, c_1^C, z_1^C, \{k^C_1(\gamma)\}_\gamma, \{b^C_1(\gamma)\}_\gamma\}, \{c_0^E(\bar{a}, \theta), c_1^E(a, \theta), z_1^E(\theta), \{k^E_1(\gamma, \theta)\}_\gamma, \{b^E_1(\gamma, \theta)\}_\gamma\}_{\theta \in \{B, G\}} \) for all agents in the economy, prices of capital \( p \), and of debt contracts \( \{D(\gamma)\}_\gamma \) such that:

1. consumption allocations and asset holdings solve agents' optimization problems (2.4) and (2.5) given prices.
2. \( k^C + k^E = \int k_1^C(\gamma, \theta)d\gamma + \phi \int k_1^E(\gamma, G)d\gamma + (1 - \phi) \int k_1^E(\gamma, G)d\gamma \) (asset market clearing);
3. \( b_1^C(\theta) + \phi b_1^E(\gamma, G) + (1 - \phi) b_1^E(\gamma, B) = k_1^C(\gamma, \theta) + \phi k_1^E(\gamma, G) + (1 - \phi) k_1^E(\gamma, B), \forall \gamma \) (debt market clearing);

Note that there will be price indeterminacy, for example, for debt that is not traded in equilibrium\(^6\). In that case, however, all other prices and allocations are the same. Hence, I will not pay specific attention to that type of multiplicity. Therefore, in the results below, I will call an equilibrium unique, whenever the price of capital and equilibrium allocations are the same across equilibria. However, there may be genuine equilibrium multiplicity in terms of different allocations and the price of capital.

For the results below I will be focusing on economies that satisfy the following parameter restrictions

**Assumption 1 (A1):** \( \frac{\phi E^E + (\lambda a + \epsilon) K}{K - \phi K} \leq E^B[\alpha] + \epsilon \) and \( \frac{\epsilon E^E + (\lambda a + \epsilon) K}{K - \phi K} \geq E^B[\alpha] + \epsilon \)

and

**Assumption 2 (A2):** \( \frac{\phi E^E + (\lambda a + \epsilon) K}{K - \phi K} > \eta^B \Delta a + \alpha + \epsilon \)

\(^6\) The indeterminacy of prices of debt contracts not traded in equilibrium is characteristic of collateral equilibria (see Simsek (2010)) and is not an artifact of the asymmetric information assumption.
Assumption 1 ensures that Good Entrepreneurs do not have enough wealth to push up the price of capital above the valuation of Bad Entrepreneur whenever they issue debt with face value up to \( \gamma = \lambda a + \epsilon \). Assumption 2, on the other hand, states that Good Entrepreneurs have enough wealth to push up the price of capital above the level that Bad Entrepreneurs would find profitable to buy capital whenever debt with face value \( \gamma = a + \epsilon \) is valued as issued by Good Entrepreneurs only.

### 2.2.5 Simplifying the agent’s problem

First of all, it is straightforward to see from the Consumer's problem (2.4) that for \( p > \epsilon \), \( k^C(\gamma) = 0 \), \( \forall \gamma \), i.e. Consumers sell their capital holdings if the price of capital is sufficiently high. Therefore, if \( p > \epsilon \) Consumers are lenders only and buy debt issued by Entrepreneurs. Furthermore, I will assume that \( e^C \) is sufficiently large so that \( \frac{\hat{D}(\gamma)}{D(\gamma)} \leq 1 \), \( \forall \gamma \) that is Consumers have sufficiently “deep pockets” to accommodate all demand for borrowing by Entrepreneurs so \( t = 0 \) debt prices are never above the expected debt payoff at \( t = 1 \).\(^7\)

Turning to the Entrepreneurs’ problem, since \( \frac{\hat{D}(\gamma)}{D(\gamma)} \leq 1 \), \( \forall \gamma \) in equilibrium given the “deep pocket” assumption for Consumers, the equilibrium expected return to buying debt with any face value will be the same as that of the storage technology or lower. Hence, I will define

\[
Z = z^E_1 + \int D(\gamma) b^E_1(\gamma) d\gamma
\]

(2.6)

to be the total investment of an Entrepreneur into safe storage, and debt of other Entrepreneurs, where the composition of \( Z \) is pinned down in equilibrium. Then, the Entrepreneur’s problem simplifies to:

\[
V^E(e^E, k^E; \theta, p, \{D(\gamma)\}_\gamma) = \max_{c^E_0, Z, \{k^E(\gamma)\}_\gamma} c^E_0 + E[c^E_1 | \theta]
\]

(2.7)

\[
s.t. \quad c^E_0 + Z + \int_0^\infty (p - D(\gamma)) k^E(\gamma) d\gamma = e^E + p \cdot k^E
\]

Note that \( \frac{e^E + e^C}{E^G(a) + \epsilon} \geq E^G(a) + \epsilon \) will be sufficient to guarantee this, that is as long as there is sufficient liquidity or \( t = 0 \) consumption endowment to bid up the price of capital to the valuation of Good Entrepreneurs all demand for borrowing will be met by Consumers’ endowments in equilibrium.
\begin{equation}
    c_0^E = \begin{cases} 
        Z + \int_0^{\lambda(\theta)\alpha + \epsilon (\lambda(\theta)a + \epsilon - \gamma)k_1^E(\gamma)d\gamma} , & a = \lambda(\theta)a \\
        Z + \int_0^{\bar{a} + \epsilon (\bar{a} + \epsilon - \gamma)k_0^E(\gamma)d\gamma} , & a = \bar{a} 
    \end{cases}
\end{equation}

\[ c_0^E \geq 0, \ Z \geq 0, \ k_1^E(\gamma) \geq 0, \ \gamma \in [0, \infty) \]

I now define

\[ R(\gamma; \theta, p, D(\gamma)) = \begin{cases} 
    Pr(a + \epsilon > \gamma|\theta)E \left[ \frac{a + \epsilon - \gamma}{p - D(\gamma)} | a + \epsilon > \gamma, \theta \right] , & \gamma < \bar{a} + \epsilon \\
    0 , & o.w. 
\end{cases} \quad (2.8) \]

This corresponds to the expected leveraged return of an Entrepreneur from buying capital and simultaneously selling debt with face value \( \gamma \). Note that in equilibrium \( p - D(\gamma) > 0 \) for \( \gamma < \bar{a} + \epsilon \) as otherwise an Entrepreneur would be able to derive infinite utility by selling unlimited amounts of debt with some face value \( \gamma < \bar{a} + \epsilon \), which would violate some market clearing condition. Therefore, \( R(\gamma; \theta, p, D(\gamma)) \) is well defined.

Then we have the following partial characterization result for an Entrepreneur’s investment decision:

**Lemma 16.** Let \( R(\theta, p, \{D(\gamma)\}_\gamma) \equiv \max_{\gamma} R(\gamma; \theta, p, D(\gamma)) \) and \( \Gamma(\theta) \equiv \arg \max_{\gamma} \{R(\gamma; \theta, p, D(\gamma))\} \). Then:

- If \( R(\theta, p, \{D(\gamma)\}_\gamma) < 1 \) then \( k_1^E(\gamma) = 0, \ \gamma \in [0, \infty) \)
- If \( R(\theta, p, \{D(\gamma)\}_\gamma) > 1 \) then \( a_0^E = 0 \) and \( Z = 0 \);
- If \( R(\theta, p, \{D(\gamma)\}_\gamma) \geq 1 \) then \( k_1^E(\gamma) \geq 0 \) for \( \gamma \in \Gamma \).
- \( V^E(e^E, k^E; \theta, p, \{D(\gamma)\}_\gamma) = \nu(\theta, p, \{D(\gamma)\}_\gamma) \cdot (e^E + p \cdot k^E) \), where
  \[ \nu(\theta, p, \{D(\gamma)\}_\gamma) = \max \{1, R(\theta, p, \{D(\gamma)\}_\gamma)\} \]

is the shadow value of Entrepreneur’s \( t = 0 \) wealth.

**Proof.** See Appendix

Lemma 16 implies that there is a separation between the decision of an Entrepreneur of how much to allocate between consumption and purchase of productive capital and his financing
decision of what face value debt to issue. Therefore, in order to characterize the equilibria of this economy one can focus on the equilibrium determination of the Entrepreneurial shadow value of wealth, \( \nu(\theta, p, \{D(\gamma)\}_\gamma) \), which depends on the financing decisions of the Entrepreneur only, with consumption and capital investment decisions depending on \( \nu(\theta, p, \{D(\gamma)\}_\gamma) \).

### 2.2.6 Assumptions on lenders’ beliefs

I can now introduce two refinements on lenders’ beliefs, under which I will characterize the equilibrium. Given the unobserved Entrepreneur heterogeneity, the prices of debt contracts not traded in equilibrium will not be irrelevant for equilibrium behavior but rather will play an important role for determining what equilibria exist. In fact, the prices of debt contracts not traded in equilibrium will play the same role that out-of-equilibrium beliefs play in dynamic games of incomplete information.\(^8\) Therefore, I make the following belief assumptions:

**Belief Consistency 1 (BC1):** Let \( \Gamma \) be the set of debt contracts traded in equilibrium, and \( p \) be the equilibrium price of capital. For debt contract with face value \( \gamma' \notin \Gamma \), let \( \mu' \in \Delta \Theta \) be the lenders’ equilibrium belief about the types of borrowers who issue debt with face value \( \gamma' \) and let \( \hat{D}(\gamma', \Theta) = \max_{\mu \in \Delta \Theta} E[d(\gamma')|\mu], T \subset \Theta \). Lenders’ equilibrium debt valuations fail BC1 iff \( \mu'(\theta) \neq 0 \) for \( \theta \in \Theta \), s.t. \( \forall \gamma \in \Gamma, R(\gamma; \theta, p, D(\gamma)) > R\left(\gamma'; \theta, \hat{D}(\gamma', \Theta)\right) \) and for \( \theta' \in \Theta / \theta, R\left(\gamma'; \theta', p, \hat{D}(\gamma', \Theta)\right) > R(\gamma; \theta', p, D(\gamma)), \forall \gamma \in \Gamma \).

This assumption corresponds to the intuitive criterion assumption in signaling games. It means that if an agent were to observe the issuance of a debt contract that is not expected to be traded in equilibrium he should not assume that the borrower issuing it is of a type, for whom selling such a debt contract is not optimal, given the prices of capital and debt traded in equilibrium, provided there is a borrower, for whom selling such debt contract is optimal when lenders adjust their beliefs. The second belief assumption looks similar but has different implications.

**Belief Consistency 2 (BC2):** Let \( \Gamma \) be set of debt contracts traded in equilibrium, and \( p \) be the equilibrium price of capital. For debt contract with face value \( \gamma' \notin \Gamma \), let \( \mu' \in \Delta \Theta \) be lenders’ equilibrium belief about the types of borrowers who issue debt with face value \( \gamma' \).

---

\(^8\)I also implicitly assume that lenders have the same beliefs.
Let $\mu$ be lenders’ prior distribution over $\Theta$ and $\hat{D}(\gamma') = E[d(\gamma')|\mu]$. Lenders’ equilibrium debt valuations fail BC2 iff $\mu'(\theta) = 0$ for $\theta \in \Theta$, s.t. $\forall \gamma \in \Gamma$ $R(\gamma; \theta, p, D(\gamma)) < R(\gamma'; \theta, p, \hat{D}(\gamma'))$.

The second assumption removes “superstitions” in the sense of Fudenberg and Levine (2009), which are off equilibrium beliefs that are false but support an equilibrium since they cannot be disconfirmed. It means that if agents were to observe the issuance of a debt contract that is not expected to be traded in equilibrium, they should not exclude borrower types, for which it is optimal to issue such a debt given the prices of capital and debt traded in equilibrium.

Given these two refinements, I can now derive the main result of this section, the uniqueness of a collateral equilibrium, in which the price of capital equals the Good Entrepreneur’s valuation and consequently there is output maximizing capital reallocation and production.

2.2.7 Equilibrium Characterization

I now show that under BC1 and BC2 there exists an essentially unique symmetric collateral equilibrium, in which $p = E^G[a] + \epsilon$, and all capital is held by Good Entrepreneurs. This will be in stark contrast to the result in Section 2.3, where I show that introducing limited pledgeability leads to multiple equilibria with very interesting properties.

**Proposition 17.** Suppose that A1, A2, BC1 and BC2 hold. Then there exists an essentially unique symmetric collateral equilibrium with price of capital $p = E^G[a] + \epsilon$, in which all the capital in the economy is held by Good Entrepreneurs.

**Proof.** See Appendix.

Proposition 17 shows that asymmetric information on its own need not affect the optimal allocation of capital in this economy. In particular, the equilibrium price of capital in the economy is always high enough so that Bad Entrepreneurs find it unprofitable to buy any capital and instead sell their capital holdings. This in turn means that only Good Entrepreneurs issue debt in equilibrium, with lenders valuating debt contracts at high prices that reflect this. Facing fair debt prices, Good Entrepreneurs borrow to push up the price of capital to their valuation. Any other price of capital cannot be supported by the borrowing behavior of Entrepreneurs. A low price of capital means that both types of Entrepreneurs borrow heavily leading to excess demand for capital at that price. A higher price of capital but lower than $p = E^G[a] + \epsilon$ is not consistent with equilibrium either as Entrepreneurs issue only low face value debt in that case so
they can push up the price only to the Bad Entrepreneur's valuation. Note that this uniqueness result depends crucially on the belief refinements BC1 and BC2. Both of them are necessary for a unique equilibrium, and if either of them is removed then one can construct other equilibria, in which the price of capital and the allocation of capital are different. I next move to the main result of the paper and show how introducing limited pledgeability into this framework interacts with asymmetric information and breaks the essential uniqueness result, leading to equilibrium multiplicity even under the two belief refinements.

2.3 Limited pledgeability and multiplicity

I now add an assumption of limited pledgeability. In particular, I make the following addition to the basic model from Section 2.2. Entrepreneurs cannot credibly pledge all output when productivity is high. More specifically, I assume that Entrepreneurs can only pledge up to $\hat{a} = a$ in any idiosyncratic state. If he were to promise a payoff higher than this, an Entrepreneur can simply renege on it ex post with lenders only able to collect up to the pledgeable limit $\hat{a}$. Working with moral hazard problems like this is common in the literature on financial frictions (Kiyotaki and Moore (1997)). They have the realistic implication of creating a wedge between inside and outside investors' valuations of project payoffs.

Given limited pledgeability, debt prices at $t = 0$ would reflect the ex post reneging possibility. As an example, suppose that only Good Entrepreneurs issue debt of any face value in equilibrium. Then equilibrium debt prices will take the following form:

$$D(\gamma) = \begin{cases} 
\gamma & , \gamma \leq a + \epsilon \\
\hat{a} + \epsilon & , \gamma \geq a + \epsilon 
\end{cases}$$

Therefore, any debt promise above $a + \epsilon$ is valued only at $\hat{a} + \epsilon$ as lenders expect that it is not credible. Similarly, suppose that both Good and Bad Entrepreneurs issue debt (of any face value) in equilibrium. Then

9 Alternatively, one can think of a limited pledgeability constraint of similar form arising from an agency problem, that results in nontransferable output (Holmstrom and Tirole (2008)), because of a hidden income problem (Townsend (1979)) and high monitoring costs or because of differences in beliefs (Landier and Thesmar (2009)). The main results of this paper follow through with such alternative assumptions.
In this case, promises between \( \lambda a + \epsilon \) and \( a + \epsilon \) are valued by taking into account the possibility that some debt is issued by Bad Entrepreneurs even though a Good Entrepreneur meets such a promise with certainty.

### 2.3.1 Limited Pledgeability without Asymmetric Information

It is important to show that equilibrium multiplicity arises from the interaction of limited pledgeability and asymmetric information. To this end I first show that if there is no asymmetric information about Entrepreneur type there is still a unique equilibrium. Furthermore, if Assumption A1 holds the equilibrium is characterized by optimal reallocation of capital to Good Entrepreneurs although the equilibrium price of capital may be lower than in Proposition 17.10

**Proposition 18.** Suppose that A1 holds and there is no asymmetric information about Entrepreneur type at \( t = 0 \) but there is limited pledgeability. Then there exists an essentially unique equilibrium.

**Proof.** See Appendix.

Therefore, on its own limited pledgeability does not generate equilibrium multiplicity, i.e. it is the interaction between limited pledgeability and asymmetric information that matters.

### 2.3.2 Multiple Equilibria

I will show that there is equilibrium multiplicity under the following parameter assumption:

**Assumption 3 (A3):** \( E^B[a] > (\phi + (1 - \phi) \eta^B) E^G[a] + (1 - \phi) (1 - \eta^B) \lambda a \)

Then we have the following result:

\[ D(\gamma) = \begin{cases} 
\gamma, & \gamma \leq \lambda a + \epsilon \\
(\phi + (1 - \phi) \eta^B) \gamma + (1 - \phi) (1 - \eta^B) (\lambda a + \epsilon), & \lambda a + \epsilon < \gamma \leq a + \epsilon \\
a + \epsilon, & \gamma \geq a + \epsilon 
\end{cases} \]  

(2.10)
Proposition 19. Suppose that $A1, A2, A3, BC1$ and $BC2$ hold. Then there exists an equilibrium, in which the price of capital is $p > \eta^B \Delta a + a + \epsilon$ and all capital is held by Good Entrepreneurs. Additionally, there exists an equilibrium, in which the price of capital is $p = E^B[a] + \epsilon$ and all capital is held by both Good and Bad Entrepreneurs.

Proof. See Appendix.

The above result shows that limited pledgeability and asymmetric information interact in an important way that can affect the equilibrium price of capital and its allocation across Entrepreneurs. In particular, it shows that there is a “high leverage” equilibrium, in which the price of capital is high, capital is held exclusively by Good Entrepreneurs and aggregate debt is high. However, there can also exist a “low leverage” equilibrium, in which the price of capital is depressed, capital is held by both Good and Bad Entrepreneurs and aggregate debt is low. In that second equilibrium, equilibrium debt valuations are so low that issuing high face value debt is blocked, and as a result capital reallocation towards high productivity Entrepreneurs is stifled as they cannot “muster up” enough resources to acquire all the capital in the economy. As a result aggregate output is lowered. This follows immediately from the fact that Good Entrepreneurs have higher expected productivity. Furthermore, debt issued in the “high leverage” equilibrium has face value $\gamma = a + \epsilon$ and only $\gamma = \lambda a + \epsilon$ in the “low leverage” equilibrium. Also equilibrium leverage, which I define in a standard way as the ratio of the price of capital to the internal funds used to purchase it, i.e. its margin requirement (Geanakoplos (2010)) is

$$\frac{p_1}{p_1 - a} \geq \frac{E^B[a] + \epsilon}{\eta^B \Delta a}$$ \hspace{1cm} (2.11)

in the “high leverage” equilibrium, whereas in the “low leverage” equilibrium it is

$$\frac{p_2}{p_2 - \lambda a} = \frac{E^B[a] + \epsilon}{\eta^B(a - \lambda a)}$$ \hspace{1cm} (2.12)

The right hand side of (2.11) is clearly higher than the right hand side of (2.12).

2.3.3 Mechanism

To better understand the mechanism responsible for the equilibrium multiplicity, it is instructive to look at indifference curve maps for Good and Bad Entrepreneurs in the two equilibria. In particular, for the given equilibrium price, I plot indifference curves for the two types in the
space of debt face value, $\gamma$, versus debt price, $D(\gamma)$. I also plot equilibrium debt prices for the two equilibria. Figure 2-2 shows these curves for the “high leverage” (a) and “low leverage” (b) equilibrium. The dashed blue line is the relevant indifference curve of a Bad Entrepreneur that delivers the same utility as the utility from his equilibrium allocation, and, similarly, the dashed red line is the relevant indifference curve of a Good Entrepreneur. These curves also represent the debt prices that would make each type of Entrepreneur indifferent between issuing debt with any face value and the debt they issue in equilibrium. The solid black line represents equilibrium debt prices.\footnote{Note that since each Entrepreneur type prefers higher priced debt for the lowest possible promise (face value) Entrepreneur utility is increasing in the north-west direction.}

A decrease in the price of capital has two effects. First, it leads to a downward shift in the Bad Entrepreneur’s indifference curve. This implies that for a given equilibrium price of debt, $D(\gamma)$, a Bad Entrepreneur is more likely to issue debt with any face value. The second effect is a downward tilt in the Good Entrepreneur’s indifference curve. These effects have two consequences. First of all they affect the Good Entrepreneur’s ability to separate from the Bad Entrepreneur. When the price of capital is high, as in Figure 2-2(a), the Bad Entrepreneur is worse off when he issues debt with the face value that a Good Entrepreneur issues. In contrast, when the price of capital is low, as in Figure 2-2(b), the Good Entrepreneur can only pool with the Bad Entrepreneur. The second consequence is that for a low price of capital as in Figure 2-2(b) the Good Entrepreneur’s indifference curve at the equilibrium allocation lies above the Bad Entrepreneur’s indifference curve. As a result, as long as Assumption A3 holds, there is adverse selection in debt markets with face value $\gamma \in (\lambda_a + \epsilon, a + \epsilon]$. This means that if the price of debt with this face value is set using the prior distribution over Entrepreneur types, only Bad Entrepreneurs would issue high face value debt. Therefore, the equilibrium price of debt has to reflect this.

Therefore, the price of capital affects the quality of investment and expected debt repayments and from there the debt price schedule that Entrepreneurs face. A given debt price schedule in turn affects Entrepreneurs’ equilibrium choice of debt issuance. Therefore their debt capacity is endogenous and depends on the price of capital but only indirectly. This is in stark contrast to the standard endogenous borrowing constraint channel (Kiyotaki and Moore (1997)) in which debt capacity depends directly on either the contemporaneous or future price of an asset. In fact, given limited pledgeability this channel is present in my model too, if one interprets the
Figure 2-2: Good borrower separates when the price of capital is high
$t = 1$ capital payoff $\epsilon$ as the liquidation price of capital at $t = 1$. Higher value of $\epsilon$ relaxes $t = 0$ borrowing constraints for Entrepreneurs but is directly reflected in the $t = 0$ price of capital, i.e. it does not interact with the mechanism explored here.

Since, the shape of the equilibrium price schedule for debt affects Entrepreneur’s equilibrium debt capacity, it also affects the price of capital indirectly. In particular, higher debt prices induce Entrepreneurs to borrow by issuing higher face value debt. This in turn increases the aggregate demand for capital at any given price of capital, which implies a higher market clearing price. This “cash-in-the-market pricing” channel (Shleifer and Vishny (1992), Allen and Gale (1994)) is standard in models with borrowing constraints, in which relaxing the borrowing constraint pushes up asset prices that agents bid for.

To summarize, higher debt prices, imply higher equilibrium leverage, which pushes up the price of capital. A higher price of capital, on the other hand, allows high productivity borrowers to separate from low productivity risky borrowers when issuing debt, thus validating the high debt prices. However, tight borrowing constraints (low debt prices), lead to a low price of capital. When the price of capital is low, good borrowers cannot separate from bad borrowers and, furthermore, only bad borrowers are likely to issue riskier debt, which validates the low debt prices. This two-way feedback between the credit market and asset market leads to what I call an “investment quality” channel of interaction.

It is important to emphasize that the equilibrium multiplicity that arises though the product of asymmetric information and adverse selection is not the result of the standard adverse selection models of asset market liquidity. Though the market for debt securities in my model can suffer from an adverse selection problem, if the price of capital were exogenously fixed at a high (low) level there will be only separation (pooling). In that sense, there would be a unique equilibrium if the price of capital were exogenously fixed. Hence, the reason for the multiplicity is a general equilibrium effect that operates through the price of capital. Another way to see that the mechanism is different is to compare how the borrower pool changes across equilibria. In the standard adverse selection model, moving from the separating to the pooling equilibrium entails an increase in the fraction of good types. In my model, moving from the low capital price to the high capital price equilibrium entails a decrease in the fraction of Bad Entrepreneurs borrowing, i.e. the borrower pool improves in two ways, with both the fraction of good borrowers increasing and the fraction of bad borrowers falling. Therefore the above mechanism is different from the standard adverse selection models applied to study fluctuations in asset market liquidity (Eisfeldt
(2004), Kurlat (2009), Malherbe (2010), and Tirole (2010)).

2.3.4 Pecuniary externality

The fact that there is a multiplicity of equilibria even though there are no technological externalities, suggests that there is a pecuniary externality that agents fail to internalize. It turns out that the externality operates through debt prices. Lenders in the model fail to internalize the effect of setting a higher debt price schedule on the price of capital and from there on the borrower pool and expected debt repayments. Therefore, there is a complementarity in lenders’ debt pricing decisions. To see this more clearly, in this section I look at a reduced form game-theoretic representation of the above economy.\textsuperscript{12} I first consider a highly stylized version of the above economy and then re-cast it as a Bayesian game.

Modified economy and agents’ payoffs

First, consider the following simplified version of the model economy. There are two periods \( t = 0, 1 \). A measure 1 of Entrepreneurs and Consumers have \( e^E \) and \( e^C \) units of \( t = 0 \) consumption good, respectively and derive utility from \( t = 1 \) consumption. Additionally there is a small measure of capital holders who hold \( K \) units of capital and derive utility from \( t = 0 \) consumption. The production technology is the same as before, as is the financial frictions and limited pledgeability assumption. However, I restrict the set of possible debt contracts to just \( \Gamma = \{\lambda a + \epsilon, a + \epsilon\} \).

We have the following sequence of events at \( t = 0 \):

1. A Consumer and an Entrepreneur from the population are randomly matched with one another and stay matched until the end of \( t = 1 \).

2. The Consumer announces debt prices, at which he would lend to the Entrepreneur. For simplicity, I directly restrict the action space of the Consumer to abstract from any monopoly power that arises from the random matching assumption. The price of debt with face value \( \gamma = \lambda a + \epsilon \) is fixed at \( D(\lambda a + \epsilon) = \lambda a + \epsilon \) (borrowing at this face value is always risk-free). The price of debt with face value \( \gamma = a + \epsilon \) is \( D(a + \epsilon) \in \{\eta^B a + (1 - \eta^B) \lambda a + \epsilon, a + \epsilon\} \), i.e. the Consumer has a binary choice over the price of

\textsuperscript{12}Such an analysis of a game-theoretic representation of a Walrasian model is similar to that in Angeletos and La’O (2009).
debt with face value $\gamma = a + \epsilon$. The first is a more conservative action, where the Consumer demands a high interest rate for such a debt contract. In the second is a more lax one, where the Consumer lends risk free. The Consumer can commit to his announcement in all subsequent stages of the game.

3. After the Consumer's announcement the Entrepreneur chooses buys capital in a decentralized market from capital holders, borrowing from the Consumer at the debt prices announced by the Consumer.

At $t = 1$ production takes place for Entrepreneurs that hold capital and Entrepreneurs repay their debt to Consumers if possible.

**Game actions and payoffs**

I now consider the following game that corresponds to the stylized economy above. There is a measure one of pairs of Consumers and Entrepreneurs. Entrepreneurs have productivity type $\theta \in \{B, G\}$, which is private information, with $\Pr(\theta = G) = \phi$ in the population. A Consumer’s action is given by $s_C \in \{0, 1\}$. Note that $s_C = 0$ will correspond to setting $D(a) = \eta^B a + (1 - \eta^B) \lambda a + \epsilon$, and $s_C = 1$ corresponds to $D(a) = a + \epsilon$ above. An action for the Entrepreneur is given by $y$, which corresponds to the debt he chooses to issue. His action space is $S_E = \{0, a + \epsilon, a + \epsilon\}$. The sequence of actions is as given above, with Consumers moving first and Entrepreneurs following within each Consumer-Entrepreneur pair $i$.

The payoff function of an Entrepreneur with type $\theta$ as a function the action of the Consumer in the pair and the action profile of all other Entrepreneurs and Consumers in the economy is:

$$V^E(\gamma, s_C, p; \theta) = e^E \left[ I(\gamma = 0) + (1 - I(\gamma = 0)) \frac{\eta^B (\delta + \epsilon - \gamma) + (1 - \eta^B) \max \{0, \lambda^B a + \epsilon - \gamma\}}{p - g(\gamma, s_C)} \right]$$  

(2.13)

where

$$g(\gamma, s_C) = \begin{cases} 
0 & \text{if } \gamma = 0 \\
\lambda a + \epsilon & \text{if } \gamma = \lambda a + \epsilon \\
(1 - s_C)(\eta^B a + (1 - \eta^B) \lambda a) + s_C \cdot a + \epsilon & \text{if } \gamma = a + \epsilon 
\end{cases}$$

(2.14)

$p$, on the other hand, is a function of all the Consumers’ and Entrepreneurs’ actions defined
implicitly by:

\[ K = \int I(\gamma > 0) \frac{e^E}{p - g(\gamma, s_C^i)} \, \text{d}i \]  

(2.15)

with the property that \( p = E^G[a] + \epsilon \) if \( \gamma = a + \epsilon, s_C^i = 1 \) for \( \theta^i = G \) and \( \gamma = 0 \) for \( \theta^i = B \), and \( p = E^B[a] + \epsilon \) if \( \gamma = \lambda a + \epsilon \) and \( s_C^i = 0 \) for \( \theta \in \{B, G\} \).

The payoff function of Consumer is given by:

\[
V^C(s_C, \gamma, p) = \varepsilon^C + \frac{e^E}{p - g(\gamma, s_C)} \left( -g(\gamma, s_C) + \Pr(\theta = G|\gamma, p) \cdot \gamma + \Pr(\theta = B|\gamma, p) \cdot \min\{a, \gamma\}\theta = B \right)
\]  

(2.16)

Lastly, Assumption A3 above holds in this case as well, i.e. \( E^B[a] > (\phi + (1 - \phi) \eta^G) E^G[a] + (1 - \phi)(1 - \eta^H) \lambda a \).

One can then define a symmetric Perfect Bayesian Equilibrium for that game, which will involve actions for Entrepreneurs and Consumers and beliefs for Consumers about the type of Entrepreneur they face conditional on the Entrepreneur’s action \( \gamma \) and the actions of all other Entrepreneurs and Consumers, such that there is sequential optimality and Consumer beliefs are determined via Bayesian updating wherever possible (plus additional restrictions on belief consistency corresponding to BC1 and BC2).

**Endogenous Complementarity in Consumers’ Actions**

In order to see how the endogenous complementarity between Consumers arises, suppose that Consumer \( i \) and Entrepreneur \( i \) anticipate that all other Consumers set \( s_C = 0 \) and Entrepreneurs set \( \gamma = \begin{cases} \lambda a + \epsilon, & \theta = B \\ \lambda a + \epsilon, & \theta = G \end{cases} \), which implies that \( p = E^B[a] + \epsilon \). Consider now the sub game within the Consumer-Entrepreneur pair \( i \), conditional on \( p = E^B[a] + \epsilon \) and \( s_C^i = 0 \), Entrepreneur \( i \)'s optimal action is

\[
\gamma^i(p = E^B[a] + \epsilon, s_C^i = 0) = \begin{cases} \{0, \lambda a + \epsilon\}, & \theta^i = B \\ \lambda a + \epsilon, & \theta^i = G \end{cases}
\]  

(2.17)

The Entrepreneur’s optimal action conditional on \( p = E^B[a] + \epsilon \) and \( s_C^i = 1 \) is
\[ \gamma^i (p = E^B[a] + \epsilon, s_C^i = 1) = \begin{cases} \bar{a} + \epsilon, & \theta^i = B \\ \bar{a} + \epsilon, & \theta^i = G \end{cases} \]  

(2.18)

The Consumer's payoff from choosing \( s_C^i = 0 \) is then simply

\[ V^C = e^C \]  

(2.19)

and his payoff from choosing \( s_C^i = 1 \) is

\[ V^C = e^C + \frac{e^E}{E^B[a] - \bar{a}} \left( -\bar{a} + \eta^B \bar{a} + (1 - \eta^B) \lambda \right) < e^C \]  

(2.20)

Hence, it is optimal for Consumer \( i \) to choose \( s_C^i = 0 \), which implies that these action profiles constitute an equilibrium.

On the other hand, suppose that Consumer \( i \) and Entrepreneur \( i \) anticipate that all other Consumers set \( s_C = 1 \) and Entrepreneurs set

\[ \gamma = \begin{cases} 0, & \theta = B \\ \bar{a} + \epsilon, & \theta = G \end{cases} \]  

(2.21)

which implies that \( p = E^G[a] + \epsilon \). Then in the subgame of Consumer-Entrepreneur pair \( i \), Entrepreneur \( i \)'s optimal action, conditional on \( p = E^G[a] + \epsilon \) and \( s_C^i = 0 \) is to set

\[ \gamma^i (p = E^G[a] + \epsilon, s_C^i = 0) = \begin{cases} 0, & \theta^i = B \\ \{0, \lambda \bar{a} + \epsilon\}, & \theta^i = G \end{cases} \]  

(2.22)

The Entrepreneur's optimal action, conditional on \( p = E^G[a] + \epsilon \) and \( s_C^i = 1 \) is

\[ \gamma^i (p = E^G[a] + \epsilon, s_C^i = 1) = \begin{cases} 0, & \theta^i = B \\ \bar{a} + \epsilon, & \theta^i = G \end{cases} \]  

(2.23)

The Consumer's payoff from \( s_C^i = 0 \) is

\[ V^C = e^C \]  

(2.24)
and his payoff from $s_C^1 = 1$ is

$$V^C = e^C + \frac{e^E}{EG[a]}(-a + a) = e^C$$

(2.25)

that is the Consumer is indifferent between $s_C^1 = 0$ and $s_C^1 = 1$. Therefore, action profiles $s_C = 0, \gamma = \begin{cases} \lambda a + \epsilon, & \theta = B \\ \lambda a + \epsilon, & \theta = G \end{cases}$ constitute an equilibrium.

These two equilibria correspond to the two equilibria identified in Proposition 19. Comparing the two clearly shows the endogenous complementarity that arises in Consumers' payoffs. In particular, considering the best response function of a Consumer $i$ as a function of the action profile of other consumers $s_C$, we get that:

$$s_i^1(s_C) = \begin{cases} 0 & s_C = 0 \\ (0,1) & s_C = 1 \end{cases}$$

(2.26)

i.e. Consumer $i$'s best response is (weakly) increasing in the average action of other Consumers.

### 2.4 Unconventional monetary policy

The previous Section showed that in the economic environment I consider there can be aggregate output decreasing deterioration in credit market conditions. In that "low leverage" equilibrium capital is not optimally reallocated to high productivity producers and as a result aggregate output falls below its level in the "high leverage" equilibrium. The natural question that arises then is what are the potential policy responses in this framework that can alleviate a deterioration in credit market conditions. It turns out that a form of unconventional monetary policy, in which a central bank participates directly in the credit market can completely remove the credit crunch in the model. Analyzing such forms of unconventional monetary policy, in which a central bank lends directly in credit markets, has received much attention recently after such tools were extensively utilized by the Federal Reserve in the aftermath of the financial crisis (Gertler and Karadi (2011), Ashcraft, Garleanu, and Pedersen (2010), and Gertler and Kiyotaki (2010)).

I now introduce a central bank in the model economy. I assume that its objective function is to maximize total $t = 1$ output in the economy. For simplicity, I put aside issues of lump
sum transfers and redistribution that have to be achieved to ensure that a “high leverage” equilibrium Pareto dominates a “low leverage” equilibrium. Additionally, the central bank is endowed with \( e \) units of the \( t = 0 \) consumption good, with \( e \geq e^C \), so that it can unilaterally meet all borrowing needs in this economy.

Then the following policy by the central bank can maximize \( t = 1 \) output by removing the “low leverage” equilibrium: the central bank bids in the debt markets of this economy by using the following debt valuation:

\[
\bar{D}(\gamma) = \begin{cases} 
\gamma, & \gamma \leq a + \epsilon \\
\gamma, & \gamma > a + \epsilon
\end{cases}
\]  

(2.27)

We then have the following result:

**Proposition 20.** Suppose that \( A1, A2, A3, BC1 \) and \( BC2 \) hold. Then in the presence of a central bank that participates in debt markets with debt valuation given by (2.27) there exists a unique equilibrium in which the price of capital is \( p > \eta B \triangle a + a + \epsilon \) and all capital is held by Good Entrepreneurs.

Proof. See Appendix.

Therefore, by committing to buying higher face value debt at a high price, the central bank effectively unfreezes the credit market and increases aggregate output. Offering a higher price for this higher face value debt is equivalent to lending to the real sector at a lower margin requirement or haircut than private creditors are willing to lend at. As discussed in Section 2.3.2, the margin requirement on an asset equals the fraction of the asset’s price that has to be financed via internal funds. Therefore, by increasing the price of high face value debt, the central bank...
bank lowers the equilibrium haircut that Entrepreneurs accept when borrowing against capital. In this sense, the optimal policy of the central bank in the model bears similarity to the Term Asset-Backed Securities Loan Facility (TALF) that the Federal Reserve introduced in late 2008 in an attempt to increase the availability of credit to consumers and small businesses during the most severe credit contraction episodes of the financial crisis. The essence of the TALF was that the Federal Reserve would make non-recourse loans against newly issued AAA-rated asset backed securities (ABS), which were secured by auto loans, student loans, credit card loans, and small business loans at a lower haircut than private creditors in that period would offer.\(^{14}\) Though not lending directly to consumers and small businesses but rather buying newly issued securities from financial intermediaries, the aim of the TALF was to support the former group’s access to credit. This is equivalent in my model, since in my model there is no layer of financial intermediation between borrowers and lenders.

It is important to note that a central bank can achieve the same outcome by setting lower debt prices \(\bar{D}(\gamma)\) than (2.27) as well. In fact, as long as debt prices are greater than

\[
\bar{D}(\gamma) = \begin{cases} 
\gamma, & \gamma \leq \lambda a + \epsilon \\
(\phi + (1 - \phi) \eta^B) \gamma + (1 - \phi) (1 - \eta^B) \lambda a + \epsilon, & \lambda a + \epsilon < \gamma \leq a + \epsilon \\
(\phi + (1 - \phi) \eta^B) a + (1 - \phi) (1 - \eta^B) \lambda a + \epsilon, & \gamma > a + \epsilon
\end{cases}
\tag{2.28}
\]

only the “high leverage” equilibrium will be selected. Whenever the debt price the central bank sets is lower than (2.27), the central bank ends up holding zero debt in equilibrium and its actions are only to provide a floor in debt prices. Interestingly, this roughly corresponds to the actual results from the TALF. While the facility is credited with unfreezing markets for securitized credit, it required a relatively small commitment by the Federal Reserve along the equilibrium path since from the $200 to $1 trillion billion authorized for the program, the only around $50 billion worth of credit was extended (Sack (2010)). This particular observation of small outlay in equilibrium despite a large commitment by the central bank should hold more generally for other environments and credit markets, in which a market disruption is the result of a complementarity in lender debt valuations, as for example in the case of sovereign debt (Calvo (1988)).

2.5 Conclusion

The present paper considers a model of endogenous fluctuations in credit market conditions and haircuts. I identify a novel mechanism that comes from an interaction between the price of capital and the credit market due to a selection effect of the price of capital that improves or worsens an asymmetric information problem in the credit market. As a result there is equilibrium multiplicity that connects aggregate borrowing, capital reallocation, aggregate productivity and aggregate output. Identifying this “investment quality” channel is the novel contribution of this paper relative to the existing literature on financial market imperfections and their macroeconomic and asset pricing implications.

The observation that the equilibrium multiplicity can be thought of as arising from a coordination problem is important. In particular, one can show that depending on parameters there are dominance regions, in which there is a unique equilibrium. This means that one can use standard global games techniques that select a unique equilibrium (Carlsson and van Damme (1993), Morris and Shin (1998)). Given a unique equilibrium, one can look at robust predictions as a function of model primitives such as entrepreneurial net worth or the distribution of Entrepreneur types.

Finally, the “investment quality” channel does not depend on having a fixed supply of capital but is also present in a more general framework with investment in new capital under convex adjustment costs. In that environment there is again equilibrium multiplicity, with investment, borrowing, aggregate productivity and aggregate output co-moving positively across equilibria.

2.6 Appendix

Proof of Lemma 16

The results follow directly from the optimization problem (2.7) of the Entrepreneur. Let \( \nu \) be the Lagrange multiplier on the budget constraint and \( \kappa_c, \kappa_z \) and \( \kappa_y \) be the multipliers on the corresponding inequality constraints. Taking first order conditions (which are necessary and sufficient because of linearity), we have:

\[
E^0: \quad 1 - \nu + \kappa_c = 0
\]

\[
Z: \quad 1 - \nu + \kappa_z = 0
\]
$k_1^E(\gamma) : \Pr(a + \epsilon > \gamma|\theta)E[a + \epsilon - \gamma|a + \epsilon > \gamma, \theta] - \nu(p - D(\gamma)) + \kappa_\gamma = 0 \ \gamma \in [0, \infty)$

The first two conditions imply that, $\nu \geq 1$. The third condition implies that $k_1^E(\gamma)$ may be indeterminate for $\gamma \geq \bar{a} + \epsilon$ and be pinned down only in equilibrium. However, this will never be the case for the economies I consider given that $e^E > 0$, so issuing debt with $\gamma < \bar{a} + \epsilon$ is always possible in equilibrium. Therefore, we have $k_1^E(\gamma) = 0$ for $\gamma \geq \bar{a} + \epsilon$. For $\gamma < \bar{a} + \epsilon$, the last condition can be rewritten as

$$\left( \Pr(a + \epsilon > \gamma|\theta)E\left[\frac{a + \epsilon - \gamma}{p - D(\gamma)}|a + \epsilon > \gamma, \theta\right] - \nu + \frac{\kappa_\gamma}{p - D(\gamma)} \right) = 0$$

or

$$\left( R(\gamma, \theta, p, D(\gamma)) - \nu + \frac{\kappa_\gamma}{p - D(\gamma)} \right) = 0$$

Then, $R(\theta, p, \{D(\gamma)\}_\gamma) < 1$ implies that $R(\theta, p, \{D(\gamma)\}_\gamma) < 1$ for $\gamma \in [0, \infty)$ and hence $k_1^E(\gamma) = 0$ for $\gamma \in [0, \infty)$. Similarly, $R(\theta, p, \{D(\gamma)\}_\gamma) > 1$ implies that $R(\gamma; \theta, p, D(\gamma)) > 1$ for some $\gamma \in [\bar{a} + \epsilon)$ and hence $\nu > 1$, which implies that $\kappa_\epsilon = \kappa_z > 0$ and $c_0 = z = 0$. Finally, $R(\theta, p, \{D(\gamma)\}_\gamma) \geq 1$ implies that $k_1^E(\gamma) \geq 0$ for $\gamma \in \Gamma$, i.e. for $\gamma$ s.t. $R(\gamma; \theta, p, D(\gamma)) \geq 1$.

Finally, note that $\nu = \max \{1, R(\theta, p, \{D(\gamma)\}_\gamma)\}$ is the shadow value of wealth for an Entrepreneur given his type and equilibrium prices, so in equilibrium at $t = 0$, $V^E(\theta) = \nu \cdot (e^E + p \cdot k^E)$.

**Proof of Proposition 17**

I proceed in two steps. I first show that there always exists an equilibrium with price of capital $p = E^G[a] + \epsilon$, in which all capital is held by Good Entrepreneurs. I then show that no equilibrium with different price of capital can exist.

I proceed by constructing a collateral equilibrium with these properties. Suppose that $p = E^G[a] + \epsilon$ and we have the following debt prices:

$$D(\gamma) = \begin{cases} 
\gamma, & \gamma \leq \lambda a + \epsilon \\
\eta^B \gamma + (1 - \eta^B) (\lambda a + \epsilon), & \lambda a + \epsilon < \gamma < a + \epsilon \\
\eta^G \gamma + (1 - \eta^G) (a + \epsilon), & \gamma \geq a + \epsilon 
\end{cases}$$
and capital holdings by Entrepreneurs given by \( k^E_1 (\gamma, B) = 0, \gamma \in [0, \bar{a}] \) and \( k^F_1 (\gamma, G) = \begin{cases} \frac{K}{\bar{a}}, & \gamma = \gamma^* \\ 0, & \text{otherwise} \end{cases} \), where

\[
\gamma^* = \min \left\{ \bar{a} + \epsilon, \frac{(K - \phi k^E)}{\eta^G K} (E^G[a] + \epsilon) - \frac{1 - \eta^G}{\eta^G} (a + \epsilon) \right\}
\]

I first verify that these capital holdings by Entrepreneurs are consistent with the equilibrium prices. First of all a Bad Entrepreneur solves

\[
\max \gamma R(\gamma, B, p, D(\gamma)) = \max \gamma \frac{E^B[a] + \epsilon - \eta^B \gamma - (1 - \eta^B) \min \{\lambda_2 + \epsilon, \gamma\}}{p - D(\gamma)}
\]

Note that for \( \gamma \geq \bar{a} + \epsilon \), we have:

\[
R(\gamma, B, p, D(\gamma)) = \frac{\eta^B (\bar{a} + \epsilon - \gamma)}{p - \eta^G \gamma - (1 - \eta^G)(\bar{a} + \epsilon)} = \frac{\eta^B (\bar{a} + \epsilon - \gamma)}{\eta^G (\bar{a} + \epsilon - \gamma)} = \frac{\eta^B (\bar{a} + \epsilon - \gamma)}{\eta^G (\bar{a} + \epsilon - \gamma)} = \eta^G < 1
\]

Similarly, for \( \gamma \leq \lambda_2 + \epsilon \), we have:

\[
R(\gamma, B, p, D(\gamma)) = \frac{E^B[a] + \epsilon - \gamma}{p - \gamma} = \frac{E^B[a]}{E^G[a]} + \epsilon - \gamma < 1
\]

and for \( \lambda_2 + \epsilon < \gamma < \bar{a} + \epsilon \), we have:

\[
R(\gamma, B, p, D(\gamma)) = \frac{\eta^B (\bar{a} + \epsilon - \gamma)}{p - \eta^B \gamma - (1 - \eta^B)(\lambda_2 + \epsilon)} < 1
\]

as

\[
\eta^B (\bar{a} + \epsilon - \gamma) < p - \eta^B \gamma - (1 - \eta^B)(\lambda_2 + \epsilon) \iff
\]

\[
\iff E^B[a] + \epsilon < p = E^G[a] + \epsilon
\]

Therefore \( R(B, p, \{D(\gamma)\}) \) \( \gamma \) \( < 1 \) and by Lemma 16, \( k^E_1 (\gamma, B) = 0, \gamma \in [0, \bar{a}] \). Turning to the problem of a Good Entrepreneur, note that for \( \gamma \leq \lambda_2 + \epsilon \) and for \( \gamma \geq \bar{a} + \epsilon \), \( R(\gamma, G, p, D(\gamma)) = 1 \),

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while for $\lambda a + \epsilon < \gamma < a + \epsilon$,

$$R(\gamma, G, p, D(\gamma)) = \frac{E^G[a] + \epsilon - \gamma}{p - \eta^B\gamma - (1 - \eta^B)(\lambda a + \epsilon)} < 1$$

since $-\gamma < -\eta^B\gamma - (1 - \eta^B)(\lambda a + \epsilon)$. Therefore, by Lemma 16, having $k^E(\gamma, G) = \begin{cases} \frac{K}{\phi}, & \gamma = \gamma^* \\ 0, & \text{o.w.} \end{cases}$

where $\gamma^* = \min\left\{ \frac{\alpha}{\alpha - (K - \phi k^E)((E^G[a] + \epsilon) - E^E)} \right\}$ is consistent with the Good Entrepreneur’s optimization problem. Finally, note that the prices of capital and of debt are consistent with market clearing given the capital holding decisions of Entrepreneur types. Also, debt prices satisfy both BC1 and BC2. To see consistency with BC1, note that if $\bar{D}(\gamma) = \gamma$ for $\lambda a + \epsilon < \gamma < a + \epsilon$, which is debt valuation is only Good Entrepreneurs borrow, then $R(\gamma, G, p, \bar{D}(\gamma)) = 1$ for $\lambda a + \epsilon < \gamma < a + \epsilon$, i.e. Good Entrepreneurs are not better off issuing debt with face value $\lambda a + \epsilon < \gamma < a + \epsilon$, so $D(\gamma)$ does not fail BC1. Similarly, if $\bar{D}(\gamma) = \phi \gamma + (1 - \phi)(\eta^B\gamma + (1 - \eta^B)(\lambda a + \epsilon))$, $R(\gamma, G, p, \bar{D}(\gamma)) < 1$ for $\lambda a + \epsilon < \gamma < a + \epsilon$, so $D(\gamma)$ does not fail BC2.

To show that no equilibrium with price of capital $p < E^G[a] + \epsilon$ can exist (it is clear that no equilibrium with $p > E^G[a] + \epsilon$ can exist), first of all notice that there cannot be an equilibrium with $p < (\phi\eta^G + (1 - \phi)\eta^B)\bar{a} + \phi(1 - \eta^G)(\bar{a} + \epsilon) + (1 - \phi)(1 - \eta^B)(\lambda a + \epsilon)$, with debt prices that do not fail BC2. To see this note that if this is the case, then if $\bar{D}(\gamma) = \phi \gamma + (1 - \phi)(\eta^B\gamma + (1 - \eta^B)(\lambda a + \epsilon))$ for $\gamma > \bar{a}$, then $R(\gamma, G, p, \bar{D}(\gamma))$ can be made arbitrarily large for $\gamma > \bar{a}$. However, if $D(\gamma) = \bar{D}(\gamma)$ for $\gamma > \bar{a}$, then $k^E(\gamma, \theta) = \infty$ for some $\gamma > \bar{a}$, which is not consistent with market clearing. Therefore, $p \geq (\phi\eta^G + (1 - \phi)\eta^B)\bar{a} + \phi(1 - \eta^G)\bar{a} + (1 - \phi)(1 - \eta^B)\lambda a + \epsilon$ in any equilibrium, with debt prices that do not fail BC2.

Now suppose that we have an equilibrium with $E^G[a] + \epsilon > p \geq (\phi\eta^G + (1 - \phi)\eta^B)\bar{a} + \phi(1 - \eta^G)\bar{a} + (1 - \phi)(1 - \eta^B)\lambda a + \epsilon$. Then it must be the case that

$$D(\gamma) \leq (\phi\eta^G + (1 - \phi)\eta^B)\gamma + \phi(1 - \eta^G)(\bar{a} + \epsilon) + (1 - \phi)(1 - \eta^B)(\lambda a + \epsilon)$$

for $\gamma \geq \bar{a} + \epsilon$. To show this, suppose that $E^G[a] + \epsilon > p$ and $D(\gamma) = \eta^G\gamma + (1 - \eta^G)(\bar{a} + \epsilon)$ for $\gamma \in [\bar{a} + \epsilon, \hat{\gamma}]$, where $\gamma^* > \gamma = \bar{a} + \epsilon \geq \bar{a} + \epsilon$. For this to be an equilibrium, it must be that

$$p(\hat{a}) = \frac{\phi e^C + K(\eta^G(\hat{a} - \bar{a}) + \bar{a} + \epsilon)}{K - \phi k^E}$$

(2.30)
and
\[ R(\gamma, B, p, D(\hat{\gamma})) = 1 \] (2.31)

The first equality comes from market clearing. The second equality comes from the fact that if \( \hat{D}(\gamma) = \eta^G \gamma + (1 - \eta^G) (a + \epsilon) \) for \( \gamma \geq a + \epsilon \) then \( R(\gamma, B, p, \hat{D}(\gamma)) \) is increasing in \( \gamma \) for \( p < E^G[a] + \epsilon \), so if \( R(\hat{\gamma}, B, p, D(\hat{\gamma})) < 1 \) then if \( D(\gamma) \) is to satisfy BC1, there is a \( \hat{\gamma} > \gamma \), such that \( D(\hat{\gamma}) = \eta^G \hat{\gamma} + (1 - \eta^G) (a + \epsilon) \) as well. \( R(\hat{\gamma}, B, p, D(\hat{\gamma})) = 1 \) implies that we can implicitly define
\[ \hat{p}(\hat{a}) = \eta^B (\bar{a} - \hat{a}) + \eta^G (\hat{a} - a) + a + \epsilon \] (2.32)

Note, however, that \( p(\hat{a}) > \hat{p}(\hat{a}) \) since for \( \hat{a} = a \), A2 implies that \( p(a) > \eta^B \Delta a + a + \epsilon = \hat{p}(a) \). Hence, equations (2.30) and (2.31) cannot be satisfied simultaneously. More generally, if

\[ D(\gamma) = \left( \frac{\phi}{(1 - \phi)\psi + \phi} + \frac{1 - \phi}{(1 - \phi)\psi + \phi} \eta^G \right) \hat{\gamma} + \frac{\phi}{(1 - \phi)\psi + \phi} (1 - \eta^G) (a + \epsilon) + \frac{1 - \phi}{(1 - \phi)\psi + \phi} (1 - \eta^B) (\lambda a + \epsilon) \]

for some \( \hat{\gamma} = \hat{a} + \epsilon \geq a + \epsilon \), where \( \psi \) is the fraction of Bad Entrepreneurs who issue debt with that face value, then \( p(\hat{a}) > \frac{\phi e^{\gamma} + K(n^G(\hat{a} - a) + a + \epsilon)}{K - \delta \epsilon} \), while \( R(\hat{\gamma}, B, p, D(\hat{\gamma})) < 1 \), i.e. \( p(\hat{a}) > \hat{p}(\hat{a}) \) for that case as well. Therefore, in an equilibrium with \( E^G[a] + \epsilon > p \), (2.29) must hold for \( \gamma \geq a + \epsilon \).

Next, suppose that
\[ E^G[a] + \epsilon > p \geq (\phi \eta^G + (1 - \phi) \eta^B) \bar{a} + \phi (1 - \eta^G) a + (1 - \phi) (1 - \eta^B) \lambda a + \epsilon \] (2.33)

and
\[ D(\gamma) \leq (\phi \eta^G + (1 - \phi) \eta^B) \gamma + \phi (1 - \eta^G) (a + \epsilon) + (1 - \phi) (1 - \eta^B) (\lambda a + \epsilon) \] (2.34)

Then, we have that
\[ R(\gamma, G, p, D(\gamma)) = \frac{\eta^G (\bar{a} + \epsilon - \gamma)}{p - D(\gamma)} \leq \frac{\eta^G (\bar{a} + \epsilon - \gamma)}{(\phi \eta^G + (1 - \phi) \eta^B) \bar{a} + \phi (1 - \eta^G) a + (1 - \phi) (1 - \eta^B) \lambda a + \epsilon - D(\gamma)} = \frac{\eta^G}{(\phi \eta^G + (1 - \phi) \eta^B)} \]

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for $\gamma \geq a+\epsilon$ with equality iff $p = (\phi \eta^G + (1 - \phi) \eta^B) \bar{a} + \phi (1 - \eta^G) a + (1 - \phi) (1 - \eta^B) \lambda a + \epsilon$
and $D(\gamma) = (\phi \eta^G + (1 - \phi) \eta^B) \gamma + \phi (1 - \eta^G) (\bar{a} + \epsilon) + (1 - \phi) (1 - \eta^B) (\lambda a + \epsilon)$. Furthermore,

$$
\frac{d}{d\gamma} R(\gamma, \eta^G, \eta^B, p, D(\gamma)) = \eta^G \frac{-(p - D(\gamma)) + (\phi \eta^G + (1 - \phi) \eta^B) (\bar{a} + \epsilon - \gamma)}{(p - D(\gamma))^2} \\
= \eta^G \frac{p + (\phi \eta^G + (1 - \phi) \eta^B) \bar{a} + \phi (1 - \eta^G) a + (1 - \phi) (1 - \eta^B) \lambda a + \epsilon}{(p - D(\gamma))^2}
$$

for $\gamma \geq a + \epsilon$ and so

$$
R(\lambda a + \epsilon, \eta^G, \eta^B, p, D(\gamma)) > R(a + \epsilon, \eta^G, \eta^B, p, D(\gamma)) \iff \\
p > (\phi \eta^G + (1 - \phi) \eta^B) \bar{a} + \phi (1 - \eta^G) a + (1 - \phi) (1 - \eta^B) \lambda a + \epsilon
$$

Let $D(\gamma) = (\phi + (1 - \phi) \eta^B) \gamma + (1 - \phi) (1 - \eta^B) (\lambda a + \epsilon)$ for $\gamma \in [\lambda a + \epsilon, a + \epsilon]$. Observe

that

$$
\frac{d}{d\gamma} R(\gamma, \eta^G, \eta^B, p, D(\gamma)) = \frac{-(p - D(\gamma)) + (\phi + (1 - \phi) \eta^B) (E^G[a] + \epsilon - \gamma)}{(p - D(\gamma))^2}
$$

Then

$$
\frac{d}{d\gamma} R(\gamma, \eta^G, \eta^B, p, D(\gamma)) < 0 \iff -(p - (1 - \phi) (1 - \eta^B) \lambda a) + (\phi + (1 - \phi) \eta^B) E^G[a] + \epsilon < 0
$$

which is equivalent to:

$$p > (\phi + (1 - \phi) \eta^B) E^G[a] + (1 - \phi) (1 - \eta^B) \lambda a + \epsilon$$

Therefore,

$$R(\lambda a + \epsilon, \eta^G, \eta^B, p, \lambda a + \epsilon) > R(a + \epsilon, \eta^G, \eta^B, p, a + \epsilon) \iff \\
p > (\phi + (1 - \phi) \eta^B) E^G[a] + (1 - \phi) (1 - \eta^B) \lambda a + \epsilon \quad \text{(2.35)}$$

Noting that

$$(\phi \eta^G + (1 - \phi) \eta^B) \bar{a} + \phi (1 - \eta^G) a + (1 - \phi) (1 - \eta^B) \lambda a + \epsilon > \\
(\phi + (1 - \phi) \eta^B) E^G[a] + (1 - \phi) (1 - \eta^B) \lambda a + \epsilon \iff \\
E^B[a] > \eta^B E^G[a] + (1 - \eta^B) \lambda a$$

which always holds, it follows that $R(\lambda a + \epsilon, \eta^G, \eta^B, p, \lambda a + \epsilon) > R(\gamma, \eta^G, \eta^B, p, D(\gamma))$ for all $\gamma \geq a + \epsilon$ and for $p$ and $D(\gamma)$ such that (2.33) and (2.34) hold. Therefore in any equilibrium, in which
(2.33) and (2.34) hold, Good Entrepreneurs issue debt with face value \( \gamma = \lambda a + \epsilon \). This implies that \( D(\gamma) = \eta B \gamma + (1 - \eta B)(\lambda a + \epsilon) \) for any face value \( \gamma > \lambda a \), that is sold in equilibrium. Note, however, that assumption A1 then implies that \( p = E^B[a] + \epsilon \) in any such equilibria. This, however, is a contradiction as

\[
E^B[a] + \epsilon < (\phi \eta^G + (1 - \phi) \eta^B) \bar{a} + \phi (1 - \eta^G) a + (1 - \phi) (1 - \eta^B) \lambda a + \epsilon
\]

Therefore, there cannot exist an equilibrium with price of capital \( p < E^G[a] \).

Proof of Proposition 18

I will prove this Proposition as well as Proposition 19 using the following two Lemmas.

**Lemma 21.** Suppose that the equilibrium price of debt faced by a Bad Entrepreneur is

\[
D(\gamma) = \begin{cases} 
\gamma, & \gamma \leq \lambda a + \epsilon \\
\kappa \gamma + (1 - \kappa)(\lambda a + \epsilon), & \gamma \in (\lambda a + \epsilon, \bar{a} + \epsilon) \\
\kappa (\bar{a} + \epsilon) + (1 - \kappa)(\lambda a + \epsilon), & \gamma > \bar{a} + \epsilon
\end{cases}
\]

where \( \kappa \in [0, 1] \). Let \( \Gamma(B) = \arg \max_\gamma \{ R(\gamma; B, p, D(\gamma)) \} \) and let \( \gamma^*(B) \in \Gamma(B) \). Then:

- If \( p < \kappa \bar{a} + (1 - \kappa) \lambda a + \epsilon \), then (i) if \( R(a + \epsilon; B, p, D(a + \epsilon)) > 1 \), then \( \Gamma(B) = [a + \epsilon, \bar{a} + \epsilon] \); (ii) if \( R(a + \epsilon; B, p, D(a + \epsilon)) < 1 \), then \( \gamma^*(B) = 0 \), (iii) if \( R(a + \epsilon; B, p, D(a + \epsilon)) = 1 \), then \( \Gamma(B) = [0, \bar{a} + \epsilon] \);

- If \( p \geq \kappa \bar{a} + (1 - \kappa) \lambda a + \epsilon \), then (i) if \( p \leq E^B[a] + \epsilon \), then \( \Gamma(B) = [0, \bar{a} + \epsilon] \), (ii) if \( p > E^B[a] + \epsilon \), then \( \gamma^*(B) = 0 \).

**Proof.** Let \( R(\gamma; B) = R(\gamma; B, p, D(\gamma)) = \frac{E^B[a] + \epsilon - \min_\gamma \{ \lambda a + \epsilon, \gamma \}}{p - D(\gamma)} \) be the return of a Bad Entrepreneur if he issues debt with face value \( \gamma \). I consider several cases:

1. \( a + \epsilon \geq \gamma \geq \lambda a + \epsilon \). Taking F.O.C. w.r.t. \( \gamma \), we get:

\[
\frac{-(p - D(\gamma)) + \kappa (\bar{a} + \epsilon - \gamma)}{(p - D(\gamma))^2} \leq 0
\]

which is equivalent to \( p \leq \kappa \bar{a} + (1 - \kappa) \lambda a + \epsilon \).
2. \( \lambda a \geq \gamma \geq 0 \). Taking F.O.C. w.r.t. \( \gamma \) we get:

\[
\frac{-(p - D(\gamma)) + (E^B[a] + \epsilon - \gamma)}{(p - D(\gamma))^2} \geq 0
\]

which is equivalent to \( p \leq E^B[a] + \epsilon \).

The results then follow from comparing \( R(a + \epsilon; B) \) and \( R(0; B) \) for different cases of \( \kappa \) and \( p \).

\[ \square \]

**Lemma 22.** Suppose that the equilibrium price of debt faced by a Good Entrepreneur is

\[
D(\gamma) = \begin{cases} 
\gamma, & \gamma \leq \lambda a + \epsilon \\
\kappa \gamma + (1 - \kappa)(\lambda a + \epsilon), & \gamma \in (\lambda a + \epsilon, a + \epsilon] \\
\kappa(a + \epsilon) + (1 - \kappa)(\lambda a + \epsilon), & \gamma > a + \epsilon
\end{cases}
\]

where \( \kappa \in [0, 1] \). Let \( \Gamma(G) = \arg \max \gamma \{R(\gamma; G, p, D(\gamma))\} \) and let \( \gamma^*(G) \in \Gamma(G) \). Then:

- If \( p < \kappa E^G[a] + (1 - \kappa)\lambda a + \epsilon \), then \( \Gamma(G) = [a + \epsilon, a + \epsilon] \).
- If \( p > \kappa E^G[a] + (1 - \kappa)\lambda a + \epsilon \) and \( p < E^G[a] \) then \( \gamma^*(G) \) is unique and \( \gamma^*(G) = \lambda a \).

**Proof.** Let \( R(\gamma, G) = R(\gamma; G, p, D(\gamma)) = \frac{E^G[a + \epsilon - \min\{a, \gamma\}]}{p - D(\gamma)} \) be the return of a Good Entrepreneur if he issues debt with face value \( \gamma \). I consider several cases:

1. \( a + \epsilon \geq \gamma \geq \lambda a + \epsilon \). Taking F.O.C. w.r.t. \( \gamma \), we get:

\[
\frac{-(p - D(\gamma)) + \kappa(E^G[a] + \epsilon - \gamma)}{(p - D(\gamma))^2} \geq 0
\]

which is equivalent to:

\[-(p - (1 - \kappa)(\lambda a + \epsilon)) + \kappa(E^G[a] + \epsilon) \geq 0
\]

and hence

\[ p \leq \kappa E^G[a] + (1 - \kappa)\lambda a + \epsilon \quad (2.36) \]
2. \( \lambda a \geq \gamma \geq 0 \). Taking F.O.C. w.r.t. \( \gamma \) we get:

\[
(p - D(\gamma)) + (E^G[a] + \epsilon - \gamma) > 0 \\
\frac{(p - D(\gamma))^2}{(p - D(\gamma))^2} > 0
\]

which is equivalent to:

\[
p \leq E^G[a] \tag{2.37}
\]

The Lemma follows immediately upon inspection of (2.36) and (2.37).

I show existence by constructing an equilibrium for this economy. I will construct an equilibrium for the following debt prices:

\[
D(\gamma, B) = \begin{cases} 
\gamma, & \gamma \leq \lambda a + \epsilon \\
\eta^B \gamma + (1 - \eta^B) (\lambda a + \epsilon), & \lambda a + \epsilon < \gamma \leq a + \epsilon \\
\eta^B (a + \epsilon) + (1 - \eta^B) (\lambda a + \epsilon), & \gamma > a + \epsilon 
\end{cases}
\]

and

\[
D(\gamma, G) = \begin{cases} 
\gamma, & \gamma \leq a + \epsilon \\
a + \epsilon, & \gamma > a + \epsilon 
\end{cases}
\]

Note that debt issued by different Entrepreneurs is traded in separate markets.

I conjecture that the equilibrium price of capital is:

\[
p = \begin{cases} 
E^B[a] + \epsilon + \frac{\phi \epsilon + (a + \epsilon) \cdot K}{K - \phi k^E} \leq E^B[a] + \epsilon \\
\frac{\phi \epsilon + (a + \epsilon) \cdot K}{K - \phi k^E} \leq E^G[a] + \epsilon \\
E^G[a] + \epsilon, & \text{if } p = E^G[a] + \epsilon 
\end{cases}
\]

To determine the Entrepreneurs’ asset holding decisions we need to determine \( R(\theta, p, D(\gamma, \theta)) \) given asset prices.

For the asset holding decisions of a Good Entrepreneur, from Lemma 22, setting \( \kappa = 1 \), implies that \( \gamma^*(G) = a + \epsilon \). Hence, \( k^E_1(a + \epsilon, G) > 0 \) and \( k^E_1(\gamma, G) = 0 \) for \( \gamma < a + \epsilon \). Lemma 16 then implies that \( k^E_1(a + \epsilon, G) = \frac{\epsilon + p k^E}{p - a - \epsilon} \) if \( p < E^G[a] \) or \( k^E_1(a + \epsilon, G) \in [0, \frac{\epsilon + p k^E}{p - a - \epsilon}] \) if \( p = E^G[a] \).

For the asset holding decisions of Bad Entrepreneur, given \( D(\gamma, B) \), note that if \( p > E^B[a] + \epsilon, \)
then $\gamma^*(B) = 0$ and $k_1^E(\gamma, B) = 0$. If $p = E^B[a] + \epsilon$, then $k_1^E(\gamma^*(B), B) = K - k_1^E(a + \epsilon, G)$, for some $\gamma^*(B) \in [0, a + \epsilon]$. Given asset holding decisions, the conjectured market clearing price for capital follows immediately.

I show essential uniqueness for an equilibrium with debt prices as above. Define the demand for capital as a function of $p$ as:

$$k(p) = \begin{cases} 
\frac{\phi(e^E + p + k^E)}{p - a - \epsilon} + \frac{(1 - \phi)(e^E + p + k^E)}{p - a - \epsilon}, & \frac{\phi(e^E + p + k^E)}{p - a - \epsilon} < E^G[a] \\
\frac{\phi(e^E + p + k^E)}{p - a - \epsilon}, & E^B[a] + \epsilon < p < E^G[a] + \epsilon \\
0, & p = E^G[a] + \epsilon 
\end{cases}$$

Note that this defines a decreasing relation between $p$ and $k^G$. Therefore $k(p) = K$ has a unique solution.

**Proof of Proposition 19**

I proceed in two steps. I first show the existence of an equilibrium with price of capital of $p \geq \eta^B \Delta a + a + \epsilon$. I then show that there exists an equilibrium with price of capital of $p = E^B[a] + \epsilon$.

**Step 1.**

I conjecture the following equilibrium prices:

1. The price of capital is:

   $$p = \begin{cases} 
\frac{\phi e^E + (a + \epsilon) K}{K - \phi k^E}, & \frac{\phi e^E + (a + \epsilon) K}{K - \phi k^E} \leq E^G[a] \\
E^G[a], & o.w. 
\end{cases}$$

2. The price of debt is:

   $$D(\gamma) = \begin{cases} 
\gamma, & \gamma \leq a + \epsilon \\
a + \epsilon, & \gamma > a + \epsilon 
\end{cases}$$

For the asset holding decisions of a Good Entrepreneur, setting $\kappa = 1$ in Lemma 22 implies that $\gamma^*(G) = a + \epsilon$. $k_1^E(a + \epsilon, G) > 0$ and $k_1(\gamma, G) = 0$ for $\gamma \neq a + \epsilon$. Lemma 16 then implies that $k_1^E(a + \epsilon, G) = \frac{\phi e^E + p + k^E}{p - a - \epsilon}$ if $p < E^G[a]$ or $k_1^E(a, G) \in \left[0, \frac{\phi e^E + p + k^E}{p - a - \epsilon}\right]$ if $p = E^G[a]$. 

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For the asset holding decisions of a Bad Entrepreneur, setting \( \kappa = 1 \) in Lemma 21 implies that \( k^{B}_1(\gamma, B) = 0 \) as \( R(\alpha + \epsilon; B, p, D(\alpha + \epsilon)) < 1 \) since \( p > \eta B \Delta a + a + \epsilon \) given A2. Given asset holding decisions, the conjectured market clearing price for capital follows immediately. The conjectured prices of debt are consistent with BC1 and BC2 as no Entrepreneur type strictly prefers to issue debt with any other face value.

**Step 2.**

I conjecture the following equilibrium prices:

1. The price of capital is:
   
   \[ p = E^B[a] + \epsilon \]

2. The price of debt is:
   
   \[
   D(\gamma) = \begin{cases} 
   \gamma & , \gamma \leq \lambda a + \epsilon \\
   \lambda a + \epsilon & , \gamma > \lambda a + \epsilon 
   \end{cases}
   \]

For the asset holding decisions of a Good Entrepreneur, setting \( \kappa = 0 \) in Lemma 22 implies that \( \gamma^*(G) = \lambda a + \epsilon \) given A3 and so \( k^{B}_1(\lambda a + \epsilon, G) = \frac{\epsilon + p \cdot k}{p - \lambda a - \epsilon} \).

For the asset holding decisions of a Bad Entrepreneur, setting \( \kappa = 0 \) in Lemma 21, implies that

\[
\gamma^*(B) = \begin{cases} 
\lambda a + \epsilon & p < E^B[a] + \epsilon \\
[0, \lambda a + \epsilon] & p = E^B[a] + \epsilon 
\end{cases}
\]

is consistent with optimization by that Entrepreneur. Lemma 16 then implies that

\[
k^{B}_1(\lambda a + \epsilon, B) = \begin{cases} 
\frac{\epsilon + p \cdot k}{p - \lambda a - \epsilon} & p < E^B[a] + \epsilon \\
[0, \frac{\epsilon + p \cdot k}{p - \lambda a - \epsilon}] & p = E^B[a] + \epsilon 
\end{cases}
\]

Given asset holding decisions, the conjectured market clearing price for capital follows immediately given A1. Furthermore, given A1, it follows that Bad Entrepreneurs hold positive capital in equilibrium.

To show that the debt prices \( D(\gamma) \) satisfy BC1, first of all note that if \( \tilde{D}(\gamma) = \gamma \) for some \( \gamma \leq \alpha + \epsilon \), then by Lemma 21 a Bad Entrepreneur would issue debt with face value \( \gamma \) as \( R(\lambda a + \epsilon; B, p, D(\lambda a + \epsilon)) = 1 \) and so \( R(\alpha + \epsilon; B, p, \tilde{D}(\alpha + \epsilon)) > 1 \) as well given the equilibrium price. Hence, issuing debt with face value \( \lambda a + \epsilon < \gamma \leq \alpha + \epsilon \) is not dominated for a Bad
Entrepreneurs no matter what beliefs lenders have. Secondly, consider the price of debt for \( \gamma > a + \epsilon \) and suppose that \( \tilde{D}(\gamma) = a + \epsilon \). However, given limited pledgeability this is equivalent to the case of \( \gamma = a + \epsilon \). To show that \( D(\gamma) \) satisfies BC2, suppose that \( \tilde{D}(\gamma) = \phi \gamma + (1 - \phi) (\lambda a + \epsilon) \) for some \( \gamma \leq a + \epsilon \). Lemma 21 a Bad Entrepreneur would issue debt with face value \( \gamma \) as \( R (\lambda a + \epsilon; B, p, D(\lambda a + \epsilon)) = 1 \) and so \( R \left( a + \epsilon; B, p, \tilde{D}(a + \epsilon) \right) > 1 \) as well given the equilibrium price. However, Lemma 22 implies that \( R (\lambda a + \epsilon; G, p, D(\lambda a + \epsilon)) > R \left( \gamma; G, p, \tilde{D}(\gamma) \right) \) given A3. Therefore, debt prices are consistent with BC2 as well.

**Proof of Proposition 20**

Clearly the “high leverage” equilibrium still exists. Suppose also that there exists an equilibrium with price of capital \( p \leq \eta B a + a + \epsilon \). Given the central bank’s debt valuation (2.27) and that it has sufficient large \( t = 0 \) endowment, it follows that equilibrium debt prices always equal that valuation. Lemma 22 then implies that \( \gamma^* (G) = a + \epsilon \) and so \( k^{Eb}_{E, E} (a + \epsilon, G) = \frac{p_B G k^{Eb}_{E}}{p - a - \epsilon} \). However, this demand is not consistent with a price \( p \leq \eta B a + a + \epsilon \) given A2.
Chapter 3

Runs on Debt and The Role of Transparency

*Joint with Felipe Iachan

3.1 Introduction

Financial crises are commonly associated with an increase in uncertainty about asset quality\(^1\). The episode of 2007-2009 is no exception, given the substantial increase in uncertainty about the quality of various asset-backed securities combined with opacity of financial firms' portfolios\(^2\). A leading narrative for the cause of financial crises, including the most recent one, relates the resulting lower quality of information about portfolios to the incentives of depositors and short-term lenders to run on them by refusing to roll over their demand deposits or loans\(^3\). This triggers asset sales in markets with limited absorption capacity, leading to fire sale effects\(^4\), which, in turn, deepen the crisis by spreading the runs onto healthier financial institutions, leading to a potential collapse of the whole financial system. Such a view of the nature of financial crises would call for a policy response aimed at increasing transparency of financial intermediaries' balance sheets.

\(^1\)Mishkin (1991)
\(^2\)Flannery, Kwan, and Nimalendran (2010).
\(^3\)See, for example, Gorton (2010). As Gorton (2010) notes about previous banking crises: “The problem was that no one outside the banking system knew which banks were the weak banks, which banks were risky. Even other banks might not have known. Without knowing which specific banks were the riskiest, depositors were cautious and withdrew their cash from all banks.”
\(^4\)Shleifer and Vishny (2011).
This paper theoretically evaluates whether deterioration in information quality can indeed lead to a higher failure rate of financial intermediaries and examines how fire sales can amplify such shocks. We study a model of the financial system with the risk of coordination failures by short-term lenders that leads to bank runs and failure. In the model, short-term lenders make rollover decisions but face a complementarity in their actions through the ability of their respective financial intermediary to withstand a run and bring long-term projects to completion. However, short-term lenders have imperfect information about the quality of their respective financial institution, which makes them uncertain about the actions of other lenders but also directly affects their own expected payoffs conditional on bank survival or failure. Additionally, an asset market in which banks liquidate assets creates a systemic link between the rollover decisions of short-term lenders in different institutions.

We derive conditions, under which more precise information by lenders improves financial stability, by reducing the likelihood of rollover difficulties. If differences in financial intermediaries’ portfolios lead to differences in the “upside” risk associated with a higher probability of repayment of long-term lenders conditional on bank survival, then increasing the uncertainty about banks’ portfolios lowers bank failure rates. In this case, lenders effectively become more optimistic about the value of each bank’s promises conditional on survival. Conversely, if portfolio differences mostly drive differences in “downside” risk associated with lower repayment of long-term relative to short-term lenders in the case of bank failure, more precise information improves financial stability. In this case, higher uncertainty about portfolio qualities increases the likelihood of intermediary failures. In the limiting case of arbitrarily precise information, we show that the condition boils down to a simple rule relating the relative sensitivities of the payoffs from rollover and running with respect to changes in bank fundamentals.

After clarifying when increases in transparency can be beneficial for improving financial stability we proceed to examine how limited demand for liquidated assets interacts with shocks to portfolio uncertainty. In particular, the adverse effects of increases in portfolio uncertainty on bank failure are amplified endogenously through asset markets. The dependence of a bank’s ability to survive a run of short-term creditors on asset prices creates a downward sloping supply curve for bank assets, which combined with an elastic demand curve, leads to fire sales and amplification. Thus, increases in uncertainty can have important systemic effects.

The issues of transparency, complexity, and quality of information in the context of the recent financial crisis have attracted much recent interest both among academics, aiming to understand
the causes of the crisis, as well as among policymakers. Two particularly important recent papers on this topic are Dang, Gorton, and Holmstrom (2009) and Caballero and Simsek (2012). The first paper examines the effects of asymmetric information on aggregate liquidity provision. It shows that more information actually reduces welfare as it reduces trade between agents. Debt contracts are optimal for the provision of liquidity as they minimize the incentives of agents to become privately informed and maximizes trade. Therefore, increasing transparency in this setting or increasing agents’ incentives to acquire more precise information reduces welfare. The second paper shows how complexity, defined as a financial intermediary’s uncertainty about its cross-exposure and counterparty risk amplifies the effect of shocks in the financial system and interacts with secondary asset markets to create fire sale events. In that context, reduction in uncertainty has a beneficial impact for the financial system. Our paper complements this growing literature by studying how changes in the quality of individual information can influence the coordination problem implicit in any run or roll over crisis episode. In this way, it clarifies when increasing portfolio transparency, for example, through stress testing is desirable as a tool for improving financial stability. Additionally, it provides an analysis of fire sales in intermediation models featuring coordination as a central element, showing how fire sales can endogenously amplify uncertainty shocks in these models and exacerbate any deleterious effect of uncertainty.

The paper is also related to models of bank runs (Diamond and Dybvig (1983)) and the effect of idiosyncratic information on the coordination problem of depositors (Goldstein and Pauzner (2005) and Rochet and Vives (2004)). However, it differs in its modeling approach and in the focus of our analysis, which studies the effects of quality of information and transparency, particularly in the presence of an asset market. The model is similar to Morris and Shin (2004), who examine the effect of coordination risk on the price of corporate debt has a different focus and conclusions. In its emphasis on the effect of information on equilibrium behavior in a coordination game, the paper is closely related to the work by Morris and Shin (2002) and Angeletos and Pavan (2007) who consider the welfare effects of more precise public and private information in economies with strategic complementarities.

The paper is most closely related to recent work by Moreno and Takalo (2011), who investigate the effect of transparency on a model with bank runs and find that increasing transparency has an unambiguously negative effect on the probability of bank runs in their framework. In contrast to them, we find that this effect is ambiguous and that increasing transparency during a financial crisis may in fact reduce the probability of bank runs and the severity of a crisis.
Additionally, we examine the amplification effect that fire sales have on increases in uncertainty in this framework.

The paper is organized as follows. Section 3.2 studies the basic model of a financial system subject to the risk of run by short-term lenders. Section 3.3 presents the main result of the paper, which states conditions under which transparency reduces the probability of bank runs. Section 3.4 extends the model by including an asset market that determines liquidity conditions and shows the amplification effect that fire sales have on shocks to information quality in the model. Section 3.5 provides a discussion of optimal transparency policy in our framework, as well as brief concluding remarks.

3.2 Model with exogenous liquidity conditions

3.2.1 Set-up

The economy lasts for two periods: \( t = 1, 2 \). In this economy there exists a set of financial intermediaries, which are indexed by \( i \in [0,1] \). Each of these has originally invested in an independent project, that delivers random returns and only fully matures at \( t = 2 \).

Each intermediary has a relationship with a continuum unit measure of lenders, which hold claims on the project. These claims are in the form of short-term debt, which gives each lender the option to roll it over or not. Formally, a debt holder has two actions available at \( t = 1 \). She can either refuse \( (a = 0) \) or agree \( (a = 1) \) to roll over the current debt. Because of the mismatch in the maturity structure of assets and liabilities, intermediaries are potentially illiquid: too many refusals to roll over their current debt lead to failure at \( t = 1 \).

For each intermediary, there are three relevant events: failure at \( t = 1 \) due to rollover difficulties, failure at \( t = 2 \) due to project failure, or successful completion at \( t = 2 \). The probabilities of these events are influenced by the strength of the bank’s fundamentals, \( \theta_i \in [0,1] \), with \( \theta_i \sim U [0,1] \), and by a measure of aggregate market liquidity, \( l \). The index \( l \) identifies the facility in liquidating a limited volume of the project’s assets, generating revenues for the payment of agents who refuse to rollover, or in finding other sources of funding.\(^5\)

There is imperfect information about the fundamentals of projects. Each debt holder \( j \) of intermediary \( i \), receives a signal \( \theta_{ij} = \theta_i + \eta_{ij} \), where \( \eta_{ij} \sim U [-\epsilon, \epsilon] \), with \( \epsilon > 0 \) and relatively

\(^5\)For the model in this section \( l \) is held fixed. However, in Section 3.4 we allow for a feedback from bank’s asset liquidations in to \( l \) to capture fire sale events.
small. This particular structure is helpful in disciplining both beliefs about fundamentals and about the information obtained by other agents and their likely actions.

We proceed to the formal description of the production technology held by intermediaries. There is limited reversibility at \( t = 1 \), which means that the technology is only capable of generating a limited amount of resources without leading to early termination. Formally, each intermediary fails at \( t = 1 \) if the proportion of agents choosing to allow the rollover of its debt \((x)\) is less than the cut-off \( g(\theta_i, \Gamma) \). This cut-off is jointly affected by fundamentals \((\theta_i)\) and by a vector \( \Gamma \equiv (R, K, l) \). This vector specifies both aggregate liquidity conditions \((l)\) and elements of the previously designed debt contract, soon to be described. Therefore, failure occurs whenever \( x < g(\theta_i, \Gamma) \). The cut-off function \( g(\theta_i, l) \) is continuously differentiable and naturally satisfies \( \frac{\partial g(\theta_i, \Gamma)}{\partial \theta_i} \leq 0, \quad \frac{\partial g(\theta_i, \Gamma)}{\partial l} \leq 0, \quad \text{and} \quad \frac{\partial g(\theta_i, \Gamma)}{\partial R} \geq 0 \), where \( K \) is the payoff that a lender collects if she refuses to roll over and the bank does not fail (see Table 3.1 below).

If the intermediary fails at \( t = 1 \), lenders who choose to roll over receive a liquidation payoff \( \chi_{a=1}(\theta) \) and lenders who refuse to roll over receive a payoff \( \chi_{a=0}(\theta) \). \( \chi_{a=1}(\theta) \) and \( \chi_{a=0}(\theta) \) are continuously differentiable and satisfy \( \frac{d\chi_{a=0}}{d\theta} - \frac{d\chi_{a=1}}{d\theta} < 0 \), that is, the higher the bank’s fundamentals the lower the net payoff difference across lender types.

A project that survives to period \( t = 2 \) succeeds and generates a return \( R \) for each remaining debt holder with probability \( p(\theta) \). It fails to deliver any output with probability \( (1 - p(\theta)) \). These payoffs as functions of relevant events are represented in Table 3.1.

<table>
<thead>
<tr>
<th></th>
<th>Refuse</th>
<th>Accept</th>
</tr>
</thead>
<tbody>
<tr>
<td>Project Fails at ( t = 1 )</td>
<td>( \chi_R(\theta) )</td>
<td>( \chi_A(\theta) )</td>
</tr>
<tr>
<td>Project Fails at ( t = 2 )</td>
<td>( K )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>Project is successful</td>
<td>( K )</td>
<td>( R )</td>
</tr>
</tbody>
</table>

Table 3.1: Payoffs

We assume that \( \chi_{a=1}(\theta) \leq \chi_{a=0}(\theta) \leq K < R \). Since what ultimately determines a lender’s decisions is the difference in payoffs between the two available actions, we define \( k(\theta) = \chi_{a=0}(\theta) - \chi_{a=1}(\theta) > 0 \) as the net payoff from running in case of bank liquidation. Given the properties of \( \chi_{a=1} \) and \( \chi_{a=0} \) it follows that \( k(\cdot) \) is continuously differentiable and decreasing in \( \theta \). The assumption of a \( 0 \) payoff for agents that roll over when the project fails at \( t = 2 \) can be obtained from normalizations and does not lead to any loss of generality. Then, the net payoff from rolling over versus running is given by:
3.2.2 Dominance regions

We make the following set of assumptions about the properties of the underlying economy.

A1. There exists $\theta$, such that for all $\theta < \theta$ and all $\Gamma$, $g(\theta, \Gamma) = 1$.

A2. There exists $\bar{\theta}$, such that for all $\theta > \bar{\theta}$ and all $\Gamma$, $g(\theta, \Gamma) = 0$ and $R_p(\theta) - K > 0$.

A3. $\theta > \epsilon$ and $(1 - \bar{\theta}) > \epsilon$.

(A1) ensures the existence of a lower dominance region, i.e., a region in which fundamentals are so weak that, in a perfect information benchmark, refusing to roll over debt becomes a strictly dominant action. (A2) ensures that an upper dominance regions exists: for sufficiently high fundamentals rollover is a strictly dominant action. In principle, both dominance regions can be made arbitrarily small as long as assumption (A3) is satisfied. We impose assumption (A3) to ensure that whenever a signal $\theta_i \in [\bar{\theta}, \bar{\theta}]$ is received, the support of the posterior about the
fundamental is bounded away from zero and one. This avoids unnecessary complications in the Bayesian updating process that would emerge close to the extremes of the unit interval.

3.2.3 Examples

In this section, we provide two model environments that satisfy the conditions described previously. These illustrate different illiquid investment technologies, with specific failure thresholds, which when funded with short-term liabilities might lead to liquidity crises and inefficient liquidation.

**Example 1.** A bank holds a set of assets, including a resalable portfolio and intangibles with limited transferability, that jointly offer a return of $R$ with probability $p(\theta)$. Of this set, $a(\theta)$ are liquid assets that can be sold at $t=1$ at a price of $l$, where $\theta \in [0,1]$ parametrizes the strength of the bank’s portfolio. Note that $a(\theta)$ is a smooth increasing function of $\theta$. The bank has a measure one of short-term lenders, who, in case of refusal to roll over, receive payoff $K$ as long as the bank has not failed. Otherwise, they receive $0 < k(\theta) < K$, where $k(\theta)$ is decreasing in $\theta$. If they roll over, they receive $R$ in case the project is successful and 0 otherwise. The bank pays off creditors that refuse to roll over by selling liquid assets from its portfolio. The bank fails whenever it runs out of liquid assets given the demand of creditors for repayment, i.e., $K(1-x) \geq l \cdot a(\theta)$ which implies $g(\theta, \Gamma) = 1 - \frac{1}{K} a(\theta)$. Note that $g_0 < 0, g_1 < 0$, and $g_K > 0$.

**Example 2.** Similarly to Example 1, each bank has a productive asset that pays off $R$ with probability $p(\theta)$ at $t=2$ and delivers an $l=1$ cash-flow of $f(\theta)$, and a measure $a$ of liquid assets that can be sold at $t=1$ at a price of $l$. Note that $f(\theta)$ is a smooth increasing function of $\theta$. The bank has a measure one of short-term lenders that choose to roll over their short-term debt at $t=1$ up until $t=2$. If they refuse to roll over they receive payoff $K$ as long as the bank has not failed, otherwise they receive $0 < k(\theta) < K$, where $k(\theta)$ is decreasing in $\theta$. If they roll over they receive $R$ in case the project is successful and 0 otherwise. The bank pays off creditors that refuse to roll over by selling liquid assets from its portfolio. The bank fails whenever it runs out of liquid assets given the demand of creditors for repayment, i.e., $K(1-x) \geq f(\theta) + l \cdot a$, which defines the threshold $g(\theta, \Gamma) = 1 - \frac{f(\theta) + l \cdot a}{K}$. Note that $g_0 < 0, g_1 < 0$, and $g_K > 0$.

---

6 Allowing for negative cash-flows, interpreted as additional financial distress resource requirements, is necessary to allow the existence of a lower dominance region.
3.2.4 Equilibrium

We first study the properties of the Bayesian Nash Equilibrium in the rollover game, taking as given aggregate liquidity conditions. This allows us to characterize outcomes and the extent of inefficient intermediary failure in a partial equilibrium setting. In Section 3.4.1, we study the consequences from the endogenous determination of aggregate liquidity conditions.

We define a Bayesian Nash Equilibrium for the rollover game, in which the vector $\Gamma$ is taken as given, as follows.

**Definition 23.** A Bayesian Nash Equilibrium (BNE) for the rollover game consists of a strategy $a : \Theta \to \{0, 1\}$ and a fraction $x : \Theta \to [0, 1]$ of lenders that roll over at $t = 1$ s.t.

1. $a(\theta_i)$ solves $a = 1$ if $E_{\theta|\theta_i} [\pi (\theta, x(\theta), \Gamma)] > 0$, $a = 0$ if $E_{\theta|\theta_i} [\pi (\theta, x(\theta), \Gamma)] < 0$ and $a \in \{0, 1\}$ if $E_{\theta|\theta_i} [\pi (\theta, x(\theta), \Gamma)] = 0$;

2. $x(\theta) = E_{\theta_i|\theta} [a(\theta_i)]$.

We focus on equilibria involving cutoff strategies such that $a(\theta_i) = 1$ iff $\theta_i \geq \theta^*$. Equilibria in cut off strategies can be characterized by making use of Proposition 24 below, which ensures the existence of a unique equilibrium threshold.

**Proposition 24.** Every BNE in cutoff strategies of this economy can be described by a unique threshold $\theta^*$ which solves

$$\int_0^1 \pi (\theta^* + \epsilon - 2\epsilon \cdot x, x, \Gamma) dx = 0. \quad (3.3)$$

**Proof.** In the Appendix.

The proof of Proposition 24 follows the intuition for uniqueness results in global games (Morris and Shin (2001)). A lender at the cutoff $\theta^*$ has Laplacian beliefs\(^7\) about the fraction of other lenders who roll over. Since that lender is indifferent between rolling over and running, her expected net payoff given these posterior beliefs equals zero, which gives Equation (3.3). Lenders who observe lower signals than $\theta^*$ are more pessimistic about both the bank’s fundamentals and the expected fraction of other lenders who roll over and, in turn, prefer to run. The opposite holds for lenders who observe signals greater than $\theta^*$.

A related way to characterize equilibria is to look at the threshold, $\theta^f$, at which banks fail. Given an equilibrium with a cutoff $\theta^*$, the share of agents willing to roll over debt is a smooth

\(^7\)That is, agents with $\theta_i = \theta^*$ believe $x(\theta, \Gamma)$ to be uniformly distributed on the $[0, 1]$ interval.
function of the state, as given by

\[
x(\theta, \theta^*(\Gamma)) = \begin{cases} 
1, & \theta > \theta^*(\Gamma) + \epsilon, \\
\frac{g(\theta^*(\Gamma) - \epsilon)}{2\epsilon}, & \theta \in [\theta^*(\Gamma) - \epsilon, \theta^*(\Gamma) + \epsilon], \\
0, & \theta < \theta^*(\Gamma) - \epsilon.
\end{cases}
\]

Therefore, \( \theta^f \) satisfies

\[
\frac{\theta^f - (\theta^* - \epsilon)}{2\epsilon} = g(\theta^f, \Gamma), \tag{3.4}
\]

or, equivalently

\[
\theta^f = 2\epsilon g(\theta^f, \Gamma) + \theta^* - \epsilon. \tag{3.5}
\]

A lender that observes a signal \( \theta^* \) believes that \( \theta \sim U[\theta^* - \epsilon, \theta^* + \epsilon] \). Since such a lender is indifferent between rolling over or not, we have that

\[
E_{\theta^*} [k(\theta) | \theta < \theta^f] \cdot Pr_{\theta^*} (\theta < \theta^f) + K Pr_{\theta^*} (\theta > \theta^f) = E_{\theta^*} [R_p(\theta) | \theta \geq \theta^f] Pr_{\theta^*} (\theta > \theta^f), \tag{3.6}
\]

in which both expectations and probabilities are taken with respect to the posterior belief of the agent that received the cutoff signal. The left-hand side represents the payoffs of refusing to roll over, while the right-hand side represents the payoffs from rolling over debt. Expressions (3.4) and (3.6) jointly determine two equilibrium cutoffs: a failure state and a rollover trigger.\(^8\)

Substituting for \( \theta^* \) in equation (3.6), we get the following equation for the failure cutoff \( \theta^f \):

\[
\int_{\theta^f}^{\theta^f + 2\epsilon(1-g(\theta^f, \Gamma))} \frac{1}{2\epsilon} R_p(\theta) d\theta = \int_{\theta^f}^{\theta^f + 2\epsilon(1-g(\theta^f, \Gamma))} \frac{1}{2\epsilon} k(\theta) d\theta + K \left(1 - g(\theta^f, \Gamma)\right). \tag{3.7}
\]

Note that \( g(\theta^f, \Gamma) \) has an important interpretation as a probability. If we consider the probability that a lender observing the threshold signal \( \theta^* \) assigns to his bank failing we have that

\[
Pr_{\theta^*} (\theta < \theta^f) = \int_{\theta^* - \epsilon}^{\theta^f} \frac{1}{2\epsilon} d\theta = \int_{\theta^* - 2\epsilon(1-g(\theta^f, \Gamma))}^{\theta^f} \frac{1}{2\epsilon} d\theta = g(\theta^f, \Gamma), \tag{3.8}
\]

\(^8\)Note that from equation (3.5) it follows that \( \theta^f < \theta^* \iff g(\theta^f, \Gamma) < \frac{1}{2} \).
This probability is distinct from the prior probability that a bank fails,

\[ Pr(\theta < \theta^f) = \theta^f. \]  

(3.9)

In fact, the two probabilities move in opposite directions with \( \theta^f \) with the former decreasing in \( \theta^f \) and the latter increasing in \( \theta^f \). This distinction will be important for the discussion of the effects of transparency in Section 3.3 below.

Additionally, note that in a limit economy, where \( \epsilon \to 0 \), the two strategic and failure thresholds converge, so that \( \theta^*_{t=0} = \theta^f_{t=0} \). They are determined from

\[ 0 = \left[ R_p \left( \theta^f_{t=0} \right) - K \right] \left( 1 - g \left( \theta^f_{t=0}, \Gamma \right) \right) - k \left( \theta^f_{t=0} \right) g \left( \theta^f_{t=0}, \Gamma \right), \]  

(3.10)

which can be interpreted in the following way. The agent who receives the cut off signal knows the type of his intermediary to be arbitrarily close to \( \theta^*_{t=0} \). However, there is still residual uncertainty about where she ranks in the distribution of posteriors about such fundamental, so that strategic uncertainty about the action of other agents is still present in the limit. That agent therefore believes that the probability of failure of his intermediary is \( g \left( \theta^*_{t=0}, \Gamma \right) \), according to equation (3.8), so that the term on the right-hand side is simply the net payoff difference between rolling over debt, which has an expected payoff of \( R_p \left( \theta^*_{t=0} \right) \left( 1 - g \left( \theta^*_{t=0}, \Gamma \right) \right) \), and refusing to do so, which has an expected payoff of \( K \left( 1 - g \left( \theta^*_{t=0}, \Gamma \right) \right) + k \left( \theta^*_{t=0} \right) g \left( \theta^*_{t=0}, \Gamma \right) \).

### 3.3 Understanding the role of noise

How does the quality of lenders’ information about the institution’s portfolio affect their rollover decisions? When does more transparency decrease the probability of bank failure and can it ever increase it? In this section we address these issues. We look at the effect of private information precision on the failure threshold, \( \theta^f \) in our general framework. The following proposition provides an answer to this question for the case of small amounts of idiosyncratic uncertainty.

**Proposition 25.** Consider the above model of investment financing with short-term debt and let \( \theta^f \) be the threshold of fundamentals for which failure occurs at \( t = 1 \). Then \( \frac{\partial \theta^f}{\partial \epsilon} > 0 \) iff

\[
\left( 1 - g \left( \theta^f, \Gamma \right) \right) \left\{ R_p \left( \theta^f + 2\epsilon \left( 1 - g \left( \theta^f, \Gamma \right) \right) \right) - E_{\theta^f} \left[ R_p \left( \theta \right) | \theta > \theta^f \right] \right\} < g \left( \theta^f, \Gamma \right) \left\{ k \left( \theta^f - 2\epsilon g \left( \theta^f, \Gamma \right) \right) - E_{\theta^f} \left[ k \left( \theta \right) | \theta < \theta^f \right] \right\}.
\]  

(3.11)
Furthermore, $\lim_{\epsilon \to 0} \left( \frac{\partial \theta^I}{\partial \epsilon} \right) > 0$ iff

$$
(1 - g(\theta^I_{\epsilon=0}, \Gamma))^2 R \cdot p'(\theta^I_{\epsilon=0}) < g(\theta^I_{\epsilon=0}, \Gamma)^2 || k'(\theta^I_{\epsilon=0}) ||
$$

(3.12)

Proof. See Appendix.

Proposition 25 gives a clear condition under which more transparency lowers the liquidation threshold. What is the intuition for this condition? First, note that lenders have rational expectations about the bank failure cutoff $\theta^I$, but are uncertain whether their bank is a failing or a surviving bank. Consider the payoff of a lender at the strategic threshold $\theta^*$. Such agent is indifferent between running and rolling over, as seen in equation (3.6). A marginal increase in signal uncertainty has two countervailing effects for such a lender. On the one hand, it increases the payoff from rolling over conditional on bank survival, as the lender now expects her bank to have a higher expected type $\theta$ when it survives. Nevertheless, it also increases the payoff from running conditional on bank failure, as the lender expects the bank to have a lower type $\theta$ conditional on failure.

If the expected increase in the payoff from rolling over given the increase in uncertainty is lower than the expected increase in the payoff from running, in order for a lender to be indifferent, she must rationally expect the failure probability to be lower. The probability of bank failure, is simply given by the probability that an insufficient fraction of lenders roll over, as Figure 3-2 shows, i.e.,

$$
Pr_{\theta^*} \left( \theta < \theta^I \right) = Pr_{\theta^*} \left( x(\theta) < g(\theta^I, \Gamma) \right).
$$

(3.13)

Since an indifferent lender has Laplacian beliefs about the fraction of other lenders that roll over, that probability is just $g(\theta^I, \Gamma)$. Therefore, in order for a lender to be indifferent, she must rationally expect the failure cutoff $\theta^I$ to increase. Conversely, if the expected increase in the payoff from rolling over given the increase in uncertainty is higher than the expected increase in the payoff from running, in order for a lender to be indifferent, she must rationally expect the failure probability for the bank to be higher.

In other words, as idiosyncratic uncertainty increases, if the expected increase in payoffs from rolling over is lower than the expected increase in payoffs from running, then the marginal failing bank must be able to withstand a run by more lenders, as more lenders are better off
running, and vice versa. That means that a bank at the failure threshold should have stronger fundamentals, indicating that the equilibrium $\theta^f$ should increase.\(^9\)

It is even clearer to see the two countervailing effects when looking at the condition for the limiting case with $\epsilon \to 0$. From equation (3.12) we have that $\lim_{\epsilon \to 0} \left( \frac{\partial \theta^f}{\partial \epsilon} \right) > 0$ iff

\[
\frac{(1 - g(\theta^f_{\epsilon=0}, \Gamma)) R \cdot p'(\theta^f_{\epsilon=0})}{g(\theta^f_{\epsilon=0}, \Gamma) \|k'(\theta^f_{\epsilon=0})\|} < \frac{g(\theta^f_{\epsilon=0}, \Gamma)}{1 - g(\theta^f_{\epsilon=0}, \Gamma)}
\]

(3.14)

Combining this with the condition for determination of the failure cutoff in the limiting case as $\epsilon \to 0$ from equation (3.10), we get that

\[
\frac{(1 - g(\theta^f_{\epsilon=0}, \Gamma)) R \cdot p'(\theta^f)}{g(\theta^f_{\epsilon=0}, \Gamma) \|k'(\theta^f)\|} < \frac{g(\theta^f_{\epsilon=0}, \Gamma)}{1 - g(\theta^f_{\epsilon=0}, \Gamma)} = \frac{R \cdot p(\theta^f_{\epsilon=0}) - K}{k(\theta^f_{\epsilon=0})}
\]

(3.15)

\(^9\)It is important to note that in the case with uniform prior, studied here, changes in the variance of idiosyncratic noise lead to changes in the posterior variance without changing the posterior mean. This is not the case with more general priors, where changes in the variance of the idiosyncratic signals would change other moments of the posterior distribution, as well. Furthermore, unlike the uniform case, with a more general prior, changes in the variance of idiosyncratic signals also affect the uncertainty about the actions of others of the marginal agent observing a signal at the strategic cutoff. Therefore, the uniform prior case serves as an important benchmark where increases in the variance of idiosyncratic noise only affect the posterior uncertainty about the bank's fundamentals without affecting the uncertainty about the actions of other. Decomposing the effects of changes in payoff uncertainty and uncertainty about others' actions is then an important issue that arises when one considers the case of a more general prior. Understanding the effects of both types of uncertainty would be an interesting question to pursue in future research.
or equivalently
\[
\frac{\eta_{a=1} (\theta_L^{f=0})}{\eta_{a=0} (\theta_L^{f=0})} < \frac{g (\theta_L^{f=0}; \Gamma)}{1 - g (\theta_L^{f=0}; \Gamma)} = \frac{Pr_{\theta^{f}} (\theta_L^{c=0} < \theta_L^{f})}{1 - Pr_{\theta^{f}} (\theta_L^{c=0} < \theta_L^{f})}
\]  
(3.16)

where \( \eta_{a=1} (\theta) = \frac{R (\theta) \theta}{\theta (\theta^{c=0} - K)} \) and \( \eta_{a=0} (\theta) = \left| \frac{k^\prime (\theta) \theta}{k (\theta)} \right| \) are the elasticities of the payoff from rolling over with respect to \( \theta \) conditional on bank survival and the payoff from running conditional on bank failure, respectively, evaluated at \( \theta = \theta_L^{f=0} \). Then a marginal increases in uncertainty about bank quality will increase the failure threshold if the ratio of the sensitivity of the payoff from rolling over conditional on survival to the sensitivity of the payoff from running conditional on failure is less than the bank failure odds ratio that a lender who observes the cutoff signal assigns to his banks. If portfolio differences mostly lead to differences in the “upside” risk associated with a higher probability of repayment of long term lenders conditional on bank survival, then increasing the uncertainty about individual portfolios lowers bank failure threshold, as lenders effectively become more optimistic about the value of each bank’s promises. If, however, portfolio differences mostly drive differences in “downside” risk associated with lower repayment of long-term relative to short-term lenders in the case of bank failure and restructuring, then increasing transparency is what lowers bank failure.

3.4 The effects of market liquidity and short-term debt

As discussed in Section 3.3, under some conditions increases in lender uncertainty about bank portfolio quality may increase the likelihood and magnitude of banking crises. In this section we examine the systemic effects of changes in market liquidity and excessive reliance on short-term debt on the size of a banking crisis. The former effect is of particular interest given the potential role of fire sales for the exacerbation of financial crises (Shleifer and Vishny (2011), Duffie (2010)).

**Proposition 26.** \( \theta_L^{f} \) is a decreasing function of \( l \) and an increasing function of \( K \).

**Proof.** See Appendix. \( \square \)

As Proposition 26 shows, a decrease in market liquidity affects bank failure adversely: it increases the fragility of each bank by lowering its ability to survive a run by short-term lenders. A similar result obtains for increases in short-term debt obligations, \( K \). Therefore, whenever market liquidity responds to asset liquidation volumes, a crisis contagion mechanism emerges.
In that case, it can have important amplification effects of shocks to lender uncertainty. We turn next to the study of such amplification effects.

### 3.4.1 Endogenous Liquidation Value - an Example

To illustrate our ideas about how market liquidity may endogenously respond to bank failure we first describe an extension of Example 1. Consider a case in which liquidity conditions are given by a price, , which is determined endogenously in an asset market. Asset supply is determined by the liquidation required to repay short-term lenders who choose not to roll over debt. On the other side of the market, there is an asset demand which is not perfectly elastic, indicating some limited absorption capacity by other market participants.

Asset demand is then given by

\[ l = h(a) \tag{3.17} \]

with \( h'(\cdot) < 0 \). Asset supply, on the other hand is given by

\[
\begin{align*}
    a^s &= \int_0^1 \max \left\{ a(\theta) \cdot \frac{(1-x)K}{l} \right\} \, d\theta = \\
    &= \int_0^{\theta_f} a(\theta) \, d\theta + \frac{K}{l} \int_{\theta_f}^{\theta_f + \epsilon} \left( 1 - \frac{\theta - (\theta_f + \epsilon)}{2\epsilon} \right) \, d\theta = \\
    &= \int_0^{\theta_f} a(\theta) \, d\theta + \epsilon \left( 1 - g(\theta, \Gamma) \right) a\left( \theta_f \right). 
\end{align*}
\]

Therefore, \( l = h \left( \int_0^{\theta_f} a(\theta) \, d\theta + \epsilon \left( 1 - g(\theta, \Gamma) \right) a\left( \theta_f \right) \right) \),

which implies that \( l \) is a decreasing function of \( \theta_f \) and \( \epsilon \). In the next section, we look at a general relation between \( l \) and \( \theta_f \) motivated by this example and endowed with these properties.

### 3.4.2 General equilibrium determination of liquidity conditions

We assume that the locus \( H(\theta_f, \Gamma, \epsilon, l) = h(\theta_f, \Gamma, \epsilon) - l = 0 \) describes equilibrium conditions in an asset market, with \( \frac{\partial H}{\partial \theta_f} < 0 \) and \( \frac{\partial H}{\partial \epsilon} < 0 \). As discussed in the previous section, these properties arise because of limited absorption capacity on the demand side of an asset market, which would lead to a fire sale effect. We first provide a definition for an equilibrium in the model augmented with an asset market.
Definition 27. An equilibrium for the rollover game augmented with an asset market consists of a bank failure cutoff $\theta^f$ and an asset price $l$ s.t.

1. $\theta^f$ satisfies:

$$
\psi \left( \theta^f, \Gamma \right) = \int_{\theta^f}^{\theta^f + 2c - 2c \varrho(\theta^f, \Gamma)} \frac{1}{2c} \text{R} \varphi(\theta) \, d\theta - \int_{\theta^f - 2c \varrho(\theta^f, \Gamma)}^{\theta^f} \frac{1}{2c} \kappa(\theta) \, d\theta - K \left( 1 - g(\theta^f, \Gamma) \right) = 0
$$

(3.19)

2. $l$ satisfies an asset market equilibrium condition:

$$
H \left( \theta^f, \Gamma, \epsilon, l \right) = 0
$$

(3.20)

The next result clarifies the conditions under which an endogenous asset price leads to a fire sale and an amplification of the effect of increasing uncertainty in lenders beliefs. First, we characterize the equilibrium effect of $\epsilon$ in the next proposition.

Proposition 28. Let $\frac{\partial \theta^f}{\partial \epsilon}$ denote the general equilibrium effect of $\epsilon$. Then

$$
\frac{\partial \theta^f}{\partial \epsilon} = \frac{\frac{\partial \theta^f}{\partial \epsilon} |_{\psi} + \frac{\partial \theta^f}{\partial \epsilon} |_{\psi} \frac{\partial l}{\partial \theta^f} |_{H}}{1 - \frac{\partial \theta^f}{\partial \epsilon} |_{\psi} \frac{\partial l}{\partial \theta^f} |_{H}}.
$$

(3.21)

Proof. See Appendix. \qed

Therefore, as long as $\frac{\partial \theta^f}{\partial \epsilon} |_{\psi} \frac{\partial l}{\partial \theta^f} |_{H} < 1$, we can write equation (3.21) as

$$
\frac{\partial \theta^f}{\partial \epsilon} = \sum_{i=0}^{\infty} \left( \frac{\partial \theta^f}{\partial \epsilon} |_{\psi} \frac{\partial l}{\partial \theta^f} |_{H} \right)^i \left( \frac{\partial \theta^f}{\partial \epsilon} |_{\psi} + \frac{\partial \theta^f}{\partial \epsilon} |_{\psi} \frac{\partial l}{\partial \theta^f} |_{H} \right). \tag{3.22}
$$

The condition $\frac{\partial \theta^f}{\partial \epsilon} |_{\psi} \frac{\partial l}{\partial \theta^f} |_{H} < 1$ holds if the asset market equilibrium condition, $H$, is sufficiently flat in $(\theta, l)$. That occurs as long as the demand function is not too inelastic, so that variations in the liquidation threshold generate moderate price changes and the term $\sum_{i=0}^{\infty} \left( \frac{\partial \theta^f}{\partial \epsilon} |_{\psi} \frac{\partial l}{\partial \theta^f} |_{H} \right)^i$ converges. We can interpret the equilibrium change in the failure threshold in the following way.

First, notice that $\frac{\partial \theta^f}{\partial \epsilon} |_{\psi}$ is the direct impact of the increased noise on the failure threshold, taking $l$ as fixed, as given by the equilibrium of the game described in Section 3.2.4. In addition to that effect, as the asset market equilibrium condition, $H (\cdot) = 0$, potentially depends on the support of noise as well, an increase in noise has a direct impact on $l$ through the asset market, given by $\frac{\partial l}{\partial \epsilon} |_{H}$. In a first iteration, that should lead to a change in the cut-off of $\frac{\partial \theta^f}{\partial \epsilon} |_{\psi} \frac{\partial l}{\partial \theta^f} |_{H}$. 

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However, any increase in the failure cut-off leads to more assets being sold and is reflected into a lower price in asset markets, which feeds back towards a bigger change in the failure cutoff. The i-th feedback interaction gives rise to the term \( \left( \frac{\partial f}{\partial t} \mid \psi \frac{\partial l}{\partial t} \mid H \right)^i \), which adds to the direct consequences of an increase in noise, so that any direct impact is amplified by a factor of

\[
\sum_{i=0}^{\infty} \left( \frac{\partial f}{\partial t} \mid \psi \frac{\partial l}{\partial t} \mid H \right)^i = \frac{1}{1 - \frac{\partial f}{\partial t} \mid \psi \frac{\partial l}{\partial t} \mid H}.
\]

Therefore, provided that the feedback effect is bounded, any direct impact of an increase in uncertainty in financial fragility is amplified through asset markets in a loop, in which more debt rollover crises lead to lower liquidation prices, which lead to rollover difficulties, which in turn lead to more liquidation and further price depression. Indeed, the amplification mechanism discussed in this section applies not only to decreases in the precision of noise but to any effects of exogenous variables on financial fragility. For example, an increase in the return upon success \( R \) leads to a reduction in the failure threshold, which is endogenously amplified through a positive impact on asset prices.

As an additional thought exercise, we can imagine a decrease in the precision of information that applies only to a limited (positive measure) set of intermediaries. Under the conditions previously discussed, that leads to an increase in the failure thresholds for those institutions. In turn, this leads to more asset liquidation among this group, which impacts negatively market prices for the liquidated asset. The effects of this reduction in prices impact all intermediaries in the economy and generate a feedback loop. Once the repercussions are intermediated by and amplified through asset markets, they are no longer restricted to the original set affected by the lower precision of signals and propagate to the whole set of intermediaries in the economy.

### 3.5 Policy Discussion and Concluding Remarks

In this paper, we study how information quality affects short-term lenders’ debt rollover decisions and bank failure. When liquidation payoffs are sufficiently sensitive to the bank fundamentals, relative to the sensitivity of rollover payoffs, a decrease in the precision of individual information leads to more frequent rollover crises. An endogenous asset market can serve as an amplification mechanism for this reaction, as rollover difficulties lead to more asset liquidation, lowering prices and precipitating further deterioration of liquidity conditions. Therefore, affecting the level of bank portfolio transparency can be an important policy tool for dealing with runs and systemic events by short-term lenders but whether this entails increasing or decreasing transparency...
depends on the effect of a bank’s portfolio quality on lenders’ payoffs.

Examining the relative sensitivities of liquidation versus rollover payoffs conditional on survival in order to understand the role of transparency is ultimately a matter for empirical investigation. However, considering the nature of banks balance sheets, we conjecture that the former should dominate the latter substantially. Debt is limited in its upside conditional on full repayment and banks have equity buffers to absorb losses and protect debt holders from losses. On the other hand, whenever a bank goes bankrupt and its equity is wiped out, debt holders become residual claimants and bank asset quality together with being the first in line for the proceeds from liquidation starts mattering considerably.

Consider, for simplicity, the extreme case where rollover payoffs do not depend on the quality of the bank’s portfolio. Therefore, increases in uncertainty affect only the expected payoffs from liquidation and hence affects adversely bank failure probabilities. In that case, a policy that aims to reduce uncertainty about portfolio quality such as a stress test clearly helps stabilize the financial system. There is indeed a common view the stress tests of major U.S. financial institutions of 2009, the so called Supervisory Capital Assessment Program (SCAP), helped stabilize the financial system by providing information about bank portfolios to financial markets (Peristian, Morgan, and Savino (2010)). This reduced uncertainty, decreased bank CDS premia and increased stock prices prompting banks to seek new equity financing from financial markets (Greenlaw, Kashyap, Schoenholtz, and Shin (2012)). In fact, the success of the 2009 stress tests for stabilizing the financial system has made annual stress tests of important financial institutions an import component of the Dodd-Frank act.

However, stress tests may not be informative enough regarding the payoffs to debt holders in case of bank failure, as they primarily provide information about future capital shortfalls, which can serve as a low precision signal about portfolio quality and an even lower precision signal about repayments conditional on liquidation. Another possible policy, which is substantially more informative and aims to reduce uncertainty precisely regarding liquidation payoffs is the so called “Living Will” requirement mandated by the Dodd-Frank act and which regulators of financial intermediaries in the US and Europe have recently begun implementing. A “Living Will” effectively forces a financial intermediary to disclose how its liquidation payoffs would look like conditional on insolvency and failure. In particular, banks are required to produce information on winding up trading books and to arrange potential buyers of their assets in case of failure (FT (2011)). In the context of our paper, this requirement can be rationalized as
aiming to stabilize the financial system by reducing the incentives of short term lenders to run on banks during a crisis episode.

Nevertheless, a thorough welfare evaluation of these and other policy interventions requires the addition of a contracting stage in the environment we study, as our analysis was conducted with a fixed pay-off structure for the debt-repayment game, taking previously determined contracts as given. This additional stage can help shed light on the relevant ranges for pay-off parameters and the magnitude of the essential comparative statics and interactions we study. In the presence of an endogenous asset liquidation market and mis-coordination in rollover decisions, individual contracts and strategies do lead to externalities. For example, the presence of fire sales and the increased financial fragility that more short-term debt creates as shown by Proposition 26 imply that there would be a fire sale externality in short-term debt contracts similarly to Stein (2012). As such, properly designed policies targeting contracts, asset markets and taxing pay-offs from investment decisions can lead to welfare gains.

Appendix

Proof of Proposition 24

Let us first define \( x(\theta, \hat{\theta}) \) as the fraction of lenders that roll over debt for a bank of type \( \theta \) when lenders follow a strategy with cutoff \( \hat{\theta} \). Secondly, we define \( v(\theta, \hat{\theta}, \Gamma) = \int_{\theta_{\hat{\theta}-\epsilon}}^{\theta_{\hat{\theta}+\epsilon}} \pi \left( \theta, x(\theta, \hat{\theta}), \Gamma \right) d\theta \) as the expected net payoff from rollover for a lender that observes a signal \( \theta \), and expects other lenders to follow cutoff strategies with cutoff \( \hat{\theta} \).

We first show several important properties of \( v(\theta, \hat{\theta}, \Gamma) \). Firstly, \( v(\theta, \hat{\theta}, \Gamma) \) is continuous in \( \theta \), as \( \pi \left( \theta, x(\theta, \hat{\theta}), \Gamma \right) \) is bounded and the limits of integration are continuous functions of \( \theta \). Secondly, the function \( \tilde{v}(\theta, \Gamma) \equiv v(\theta, \hat{\theta}, \Gamma) \) is continuous, non-decreasing in \( \hat{\theta} \) and strictly increasing for \( \hat{\theta} > \theta \). To see this, note first that \( x(\theta, \hat{\theta}) \) is continuous in \( \hat{\theta} \) and \( \pi(\theta, x, \Gamma) \) is continuous in \( x \), so \( \pi \left( \theta, x(\theta, \hat{\theta}), \Gamma \right) \) is continuous in \( \hat{\theta} \). It is also bounded and the limits of integration are continuous in \( \hat{\theta} \), so \( \tilde{v}(\theta, \Gamma) \) is continuous in \( \hat{\theta} \). Lastly, let \( \hat{\theta}_1 < \hat{\theta}_2 \). Then, for \( \hat{\theta} \) we have that \( x(\theta, \hat{\theta}) = 1 - \frac{\theta - (\hat{\theta} - \epsilon)}{2\epsilon} \) for \( \theta \in [\hat{\theta} - \epsilon, \hat{\theta} + \epsilon] \) or inverting this function, \( \theta = \hat{\theta} - \epsilon + 2\epsilon(1 - x) \). Hence, we can do a change of variables in the integral \( \tilde{v}(\theta, q) \) and rewrite it as

\[
\tilde{v}(\theta, \Gamma) = \frac{1}{2\epsilon} \int_{0}^{1} (2\epsilon) \pi \left( \theta + \epsilon - 2\epsilon \cdot (1 - x), (1 - x), \Gamma \right) dx \tag{3.23}
\]
or equivalently
\[ \nu(\bar{\theta}, \Gamma) = \int_0^1 \pi(\bar{\theta} + \varepsilon - 2\varepsilon \cdot x, x, \Gamma) \, dx \quad (3.24) \]

Then, looking at \( \nu(\bar{\theta}_2, \Gamma) - \nu(\bar{\theta}_1, \Gamma) \) we have \( \int_0^1 \left[ \pi\left(\bar{\theta}_2 + \varepsilon - 2\varepsilon \cdot x, x, \Gamma\right) - \pi\left(\bar{\theta}_1 + \varepsilon - 2\varepsilon \cdot x, x, \Gamma\right) \right] \, dx \). Note, however, that \( \pi(\theta, x, \Gamma) \) is weakly increasing in \( \theta \), \( \forall x \). Therefore, \( \nu(\bar{\theta}_2, \Gamma) - \nu(\bar{\theta}_1, \Gamma) \geq 0, \forall \bar{\theta}_1 < \bar{\theta}_2 \). Furthermore, note that for \( \bar{\theta} < \bar{\theta}_2 \) this holds with strict inequality.

Given the above properties, \( \nu(\bar{\theta}, \Gamma) > 0 \) and \( \nu(\bar{\theta}, \Gamma) < 0 \), it follows that
\[ \nu(\bar{\theta}, \Gamma) = 0 \quad (3.25) \]

has a unique solution \( \theta^*(\Gamma) \) and \( \theta^*(\Gamma) \in (\bar{\theta}, \bar{\theta}) \). Note that \( \theta^*(\Gamma) \) describes the cutoff for an equilibrium in cutoff strategies if
\[ v(\theta_i, \theta^*(\Gamma), \Gamma) > 0, \ \forall \theta_i > \theta^*(\lambda) \quad (3.26) \]

and
\[ v(\theta_i, \theta^*(\Gamma), \Gamma) < 0, \ \forall \theta_i < \theta^*(\lambda) . \quad (3.27) \]

Notice that \( \frac{\partial v}{\partial \theta_i}(\theta_i, \theta^*(\Gamma), \Gamma) = \frac{1}{2\varepsilon} [\pi(\theta_i + \varepsilon, x(\theta_i + \varepsilon, \theta^*(\Gamma), \Gamma) - \pi(\theta_i - \varepsilon, x(\theta_i - \varepsilon, \theta^*(\Gamma), \Gamma), \Gamma)] \geq 0, \) as \( \pi \) is increasing in \((\theta, x)\). Using

\[ x(\theta, \theta^*(\Gamma)) = \begin{cases} 1 & , \ \theta > \theta^*(\Gamma) + \varepsilon \\ \frac{\theta - (\theta^*(\Gamma) - \varepsilon)}{2\varepsilon} & , \ \theta \in [\theta^*(\Gamma) - \varepsilon, \theta^*(\Gamma) + \varepsilon] \\ 0 & , \ \theta < \theta^*(\Gamma) - \varepsilon \end{cases} \quad (3.28) \]

and \( \pi(\theta, x, \Gamma) = -k(\theta) I_{x < g(\theta, \Gamma)} + (1 - I_{x < g(\theta, \Gamma)})(p(\theta) R - K) \), at \( \theta^*(\Gamma) \), we obtain a strict inequality since
\[ \frac{\partial v}{\partial \theta_i}(\theta_i, \theta^*(\Gamma), \Gamma) \bigg|_{\theta_i = \theta^*} = \frac{1}{2\varepsilon} (p(\theta^* + \varepsilon) R - K - k(\theta)) > 0 \]

That inequality is also strict in a neighborhood of \( \theta^* \), from the continuity of \( x(\theta, \theta^*(\Gamma)) - g(\theta, \Gamma) \), which guarantees that there exists a neighborhood of \( \theta^* \), in which \( \frac{\partial v}{\partial \theta_i}(\theta_i, \theta^*(\Gamma), \Gamma) \) is continuous. As a consequence, (3.26) and (3.27) follow.

Therefore, the solution to:
\[ \int_0^1 \pi(\theta^* + \varepsilon - 2\varepsilon \cdot x, x, \Gamma) \, dx = 0 \quad (3.29) \]
describes the unique cutoff for the equilibrium in cutoff strategies.
Proof of Proposition 25

Let

$$\psi \left( \theta^f, \Gamma \right) = \int_{\theta^f}^{\theta^f + 2\epsilon - 2\epsilon g(\theta^f, \Gamma)} \frac{1}{2\epsilon} R_p(\theta) d\theta - \int_{\theta^f - 2\epsilon g(\theta^f, \Gamma)}^{\theta^f} \frac{1}{2\epsilon} k(\theta) d\theta - K \left( 1 - g \left( \theta^f, \Gamma \right) \right)$$

(3.30)

so that \( \psi = 0 \) implicitly defines \( \theta^f \). We can then compute

$$\frac{\partial \psi}{\partial \theta^f} = \frac{R}{2\epsilon} \left[ p \left( \theta^f + 2\epsilon - 2\epsilon g(\theta^f, \Gamma) \right) - p(\theta^f) \right] + g \left[ \frac{1}{2\epsilon} \left( K \theta^f - k \left( \theta^f - 2\epsilon g(\theta^f, \Gamma) \right) \right) \right] > 0$$

and so \( \lim_{\epsilon \to 0} \frac{\partial \psi}{\partial \theta^f} = R' \left( \theta^f \right) \left( 1 - g \right) \left( \theta^f \right) g - g_0 \left[ R_p(\theta^f) - K + k(\theta^f) \right] > 0 \). Similarly,

$$\frac{\partial \psi}{\partial \epsilon} = \frac{1}{\epsilon} \left[ \left( 1 - g \left( \theta^f, \Gamma \right) \right) R_p \left( \theta^f + 2\epsilon - 2\epsilon g(\theta^f, \Gamma) \right) - \int_{\theta^f}^{\theta^f + 2\epsilon - 2\epsilon g(\theta^f, \Gamma)} \frac{1}{2\epsilon} R_p(\theta) d\theta \right]$$

$$- \frac{1}{\epsilon} \left[ g \left( \theta^f, \Gamma \right) k \left( \theta^f - 2\epsilon g(\theta^f, \Gamma) \right) - \int_{\theta^f - 2\epsilon g(\theta^f, \Gamma)}^{\theta^f} \frac{1}{2\epsilon} k(\theta) d\theta \right]$$

or

$$\frac{\partial \psi}{\partial \epsilon} = \frac{1}{\epsilon} \left( 1 - g \left( \theta^f, \Gamma \right) \right) \left\{ R_p \left( \theta^f + 2\epsilon \left( 1 - g \left( \theta^f, \Gamma \right) \right) \right) - E \left[ R_p(\theta) | \theta > \theta^f \right] \right\} -$$

$$- \frac{1}{\epsilon} g \left( \theta^f, \Gamma \right) \left\{ k \left( \theta^f - 2\epsilon g(\theta^f, \Gamma) \right) - E \left[ k(\theta) | \theta < \theta^f \right] \right\}$$

By the implicit function theorem, \( \frac{\partial \theta^f}{\partial \epsilon} = -\frac{\partial \psi}{\partial \theta^f} \) and so \( \frac{\partial \theta^f}{\partial \epsilon} > 0 \) iff

$$\left( 1 - g \left( \theta^f, \Gamma \right) \right) \left\{ R_p \left( \theta^f + 2\epsilon \left( 1 - g \left( \theta^f, \Gamma \right) \right) \right) - E \left[ R_p(\theta) | \theta > \theta^f \right] \right\} <$$

$$g \left( \theta^f, \Gamma \right) \left\{ k \left( \theta^f - 2\epsilon g(\theta^f, \Gamma) \right) - E \left[ k(\theta) | \theta < \theta^f \right] \right\}$$

Furthermore, we have that \( \lim_{\epsilon \to 0} \frac{\partial \theta^f}{\partial \epsilon} = (1 - g \left( \theta^f, \Gamma \right))^2 R_p' \left( \theta^f \right) + g \left( \theta^f, \Gamma \right)^2 k' \left( \theta^f \right) \), leading to

$$\lim_{\epsilon \to 0} \left( \frac{\partial \theta^f}{\partial \epsilon} \right) = \frac{(1 - g \left( \theta^f, \Gamma \right)) R_p' \left( \theta^f \right) + g \left( \theta^f, \Gamma \right) k' \left( \theta^f \right)}{R_p' \left( \theta^f \right) \left( 1 - g \right) - k' \left( \theta^f \right) g - g_0 \left[ R_p(\theta^f) - K + k(\theta^f) \right]}$$
Therefore, \( \lim_{\epsilon \to 0} \left( \frac{\partial \psi}{\partial \epsilon} \right) > 0 \) iff \( (1 - g (\theta', \Gamma))^2 R \psi' (\theta') + g (\theta', \Gamma)^2 k' (\theta') < 0 \).

**Proof of Proposition 26**

Using the function \( \psi \) defined by equation (3.30), we have that

\[
\frac{\partial \psi}{\partial l} = -g_l \left[ R \psi (\theta' + 2 \epsilon - 2 \epsilon g (\theta', l)) - K + k (\theta' - 2 \epsilon g (\theta', \Gamma)) \right] > 0
\]

which by the implicit function theorem and given that \( \frac{\partial \psi}{\partial \epsilon} > 0 \), implies that \( \frac{\partial \psi}{\partial \epsilon} < 0 \). Similarly, we have that

\[
\frac{\partial \psi}{\partial K} = - \left( 1 - g (\theta', \Gamma) \right) - g_K \left[ R \psi (\theta' + 2 \epsilon - 1 - g (\theta', \Gamma)) - K + k (\theta' - 2 \epsilon g (\theta', \Gamma)) \right] < 0
\]

which implies that \( \frac{\partial \psi}{\partial K} > 0 \).

**Proof of Proposition 28**

We have

\[
\psi (\theta', \gamma, \epsilon) = 0
\]
\[
H (\theta', \gamma, \epsilon) = 0
\]

The linearized system is given by

\[
\begin{bmatrix}
\psi_0 & \psi_1 \\
H_0 & H_1
\end{bmatrix}
\begin{bmatrix}
d\theta' \\
dl
\end{bmatrix}
= -\begin{bmatrix}
\psi_\epsilon \\
H_\epsilon
\end{bmatrix} \, d\epsilon.
\]

Therefore, the partial derivatives are given by

\[
\begin{bmatrix}
\frac{\partial \psi_\epsilon}{\partial \epsilon} \\
\frac{\partial H_\epsilon}{\partial \epsilon}
\end{bmatrix} = \frac{1}{\psi_1 H_0 - \psi_0 H_1} \begin{bmatrix}
\psi_\epsilon \\
H_\epsilon
\end{bmatrix} \begin{bmatrix}
H_1 & -\psi_1 \\
-H_0 & \psi_0
\end{bmatrix} \, d\epsilon.
\]

As a consequence,

\[
\frac{\partial \theta'}{\partial \epsilon} = \frac{H_1 \psi_\epsilon - \psi_1 H_\epsilon}{\psi_1 H_0 - \psi_0 H_1} = \frac{\psi_\epsilon}{\psi_{\theta'}} - \frac{\psi_0}{\psi_{\theta'}} \frac{H_1}{H_0} \left( 1 - \frac{\psi_{\theta'}}{\psi_{\theta'}} \right) \frac{\partial \psi}{\partial l} \, dH.
\]
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