Homological mirror symmetry for a Calabi-Yau hypersurface in projective space

by

Nicholas Sheridan, Bachelor of Science (Degree with Honours)
University of Melbourne, 2006

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Abstract

This thesis is concerned with Kontsevich's Homological Mirror Symmetry conjecture. In Chapter 1, which is based on [1], we consider the n-dimensional pair of pants, which is defined to be the complement of $n+2$ generic hyperplanes in $\mathbb{CP}^n$. The pair of pants is conjectured to be mirror to the Landau-Ginzburg model $(\mathbb{C}^{n+2}, W)$, where $W = z_1 \ldots z_{n+2}$. We construct an immersed Lagrangian sphere in the pair of pants, and show that its endomorphism A_∞ algebra in the Fukaya category is quasi-isomorphic to the endomorphism dg algebra of the structure sheaf of the origin in the mirror,.giving some evidence for the Homological Mirror Symmetry conjecture in this case. In Chapter 2, which is based on [2], we build on these results to prove Homological Mirror Symmetry for a smooth d-dimensional Calabi-Yau hypersurface in projective space, for any $d \geq 3$.

Thesis Supervisor: Paul Seidel
Title: Professor

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Chapter 1

On homological mirror symmetry for pairs of pants

1.1 Introduction

1.1.1 Homological Mirror Symmetry context

In its original version, Kontsevich's Homological Mirror Symmetry conjecture [3] proposed that, if X and $X^\vee$ are 'mirror' Calabi-Yau varieties, then the Fukaya category of X (A-model) should be equivalent, on the derived level, to the category of coherent sheaves on $X^\vee$ (B-model), and vice-versa. Complete or partial results in this case are known for elliptic curves [4, 5], abelian varieties [6] (see [7] for the case of the four-torus), Strominger-Yau-Zaslow dual torus fibrations [8], and K3 surfaces [9].

Kontsevich later proposed an extension of the conjecture to cover some Fano varieties [10]. The mirror of a Fano variety X is a Landau-Ginzburg model $(X^\vee, W)$, i.e., a variety $X^\vee$ equipped with a holomorphic function W (called the superpotential). The definitions of the A- and B-models on X are (roughly) the same as in the Calabi-Yau case, but the definitions on $(X^\vee, W)$ must be altered. In particular, the A-model of $(X^\vee, W)$ is the Fukaya-Seidel category, see [11]. The B-model of $(X^\vee, W)$ is Orlov's triangulated category of singularities of W, see [12]. Complete or partial results in the Fano case are known for toric varieties [13, 14, 15], del Pezzo surfaces [16], and weighted projective planes [17].

More recently, Katzarkov and others have proposed another extension of the conjecture to cover some varieties of general type, see [18, 19]. The mirror of a variety X of general type is again a Landau-Ginzburg model $(X^\vee, W)$. The definition of the B-model on $(X^\vee, W)$ is as above (the definition of the A-model in this case is problematic, but does not concern us). One direction of this conjecture has been verified for X a curve of genus $g \geq 2$, see [20, 21]. Namely, the A-model of the genus g curve is shown to be equivalent to the B-model of a Landau-Ginzburg mirror. Our main result (Theorem 2) gives evidence for the same direction of the conjecture in the case that X is a 'pair of pants' of arbitrary dimension.

1.1.2 The A-model on the pair of pants

Consider the smooth complex affine algebraic variety

$$\left\{\sum_{j=1}^{n+2} z_j = 0\right\} \subset \mathbb{CP}^{n+1} \backslash \bigcup_{j=1}^{n+2} \{z_j = 0\}.$$

This is called the (n-dimensional) **pair of pants** $\mathcal{P}^n$ (see [22]). We equip it with an exact Kähler form by pulling back the Fubini-Study form on $\mathbb{CP}^{n+1}$, and with a complex volume form η. Observe that $\mathcal{P}^1$ is just $\mathbb{CP}^1 \backslash \{3 \text{ points}\}$, i.e., the standard pair of pants.

We will consider the A-model on $\mathcal{P}^n$, i.e., Fukaya's A_∞ category $\mathcal{F}uk(\mathcal{P}^n)$ (see [23, 24]). Recall that the objects of $\mathcal{F}uk(\mathcal{P}^n)$ are compact oriented Lagrangian submanifolds of $\mathcal{P}^n$, and the morphism space between transversely intersecting Lagrangians L_1, L_2 is defined as

$$CF^*(L_1, L_2) := \bigoplus_{x \in L_1 \cap L_2} \mathbb{K}\langle x \rangle,$$

where $\mathbb{K}$ is an appropriate coefficient ring. The A_∞ structure maps are

$$\mu^d : CF^*(L_{d-1}, L_d) \otimes \ldots \otimes CF^*(L_0, L_1) \to CF^*(L_0, L_d)[2-d],$$

for $d \geq 1$, and their coefficients are defined by counts of rigid boundary-punctured holomorphic disks with boundary conditions on the Lagrangians $L_0, \ldots, L_d$. Observe that, because the symplectic form on $\mathcal{P}^n$ is exact, the Fukaya category of exact Lagrangians is unobstructed (i.e., there is no μ^0).

In general, $\mathbb{K}$ must be a Novikov field of characteristic 2, and the morphism spaces of the Fukaya category are $\mathbb{Z}_2$-graded. If we require that the objects of our category be exact embedded Lagrangians, we remove the need for a Novikov parameter. If we furthermore require that our Lagrangians come equipped with a 'brane' structure (a grading relative to the volume form η, and a spin structure), we can assign signs to the rigid disks whose count defines a structure coefficient of the Fukaya category, and therefore remove the need for our coefficient ring to have characteristic 2. The grading of Lagrangians also allows us to define a $\mathbb{Z}$-grading on the morphism spaces of the Fukaya category. Thus, by restricting the objects of the Fukaya category to be exact Lagrangian branes, we can define the category with coefficients in $\mathbb{C}$, and with a $\mathbb{Z}$-grading. For more details, see [24] or [11].

We construct an exact immersed Lagrangian sphere $L^n : S^n \to \mathcal{P}^n$ with transverse self-intersections, and a brane structure. In the case $n = 1$, we obtain an immersed circle with three self-intersections in $\mathcal{P}^1$, illustrated in Figure 1.1.2.1 (ignore the additional labels for now). This immersed circle also appeared in [20].

We point out that L^n is not an object of the Fukaya category as just defined, because

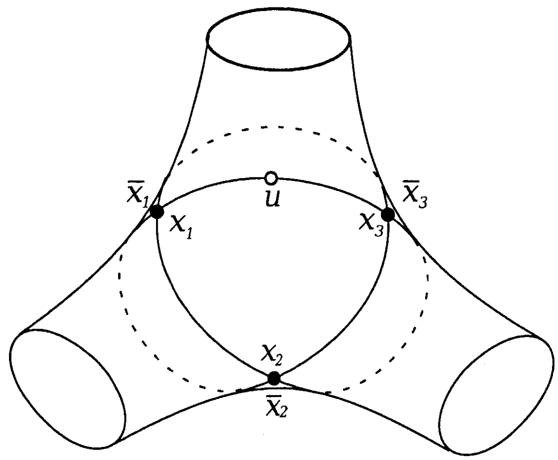


Figure 1.1.2.1: The immersed Lagrangian $L^1 : S^1 \to \mathcal{P}^1$. The image has been distorted for clarity – for L^1 to be exact, the front and back triangles should have the same area.

it is not embedded. However, we will show (in Section 1.3.1) that one can nevertheless include L^n as an 'extra' object of the Fukaya category in a sensible way.

We compute the Floer cohomology algebra of L^n:

Theorem 1.

$$HF^*(L^n, L^n) \cong \Lambda^* \mathbb{C}^{n+2}$$

as $\mathbb{Z}_2$-graded associative $\mathbb{C}$-algebras.

Remark 1.1.2.1. Although both $HF^*(L^n, L^n)$ and $\Lambda^*\mathbb{C}^{n+2}$ carry $\mathbb{Z}$-gradings, these gradings only agree modulo 2.

1.1.3 The B-model on the mirror

The mirror of $\mathcal{P}^n$ is conjectured to be the Landau-Ginzburg model $(\mathbb{C}^{n+2}, W)$, where

$$W = z_1 z_2 \dots z_{n+2}.$$

This paper is concerned with relating the B-model on $(\mathbb{C}^{n+2}, W)$ to the A-model on $\mathcal{P}^n$.

Recall that the B-model of $(\mathbb{C}^{n+2}, W)$ is described by Orlov's triangulated category of singularities $D^b_{\mathrm{Sing}}(W^{-1}(0))$ (see [12]). Note that 0 is the only non-regular value of W. The triangulated category of singularities is defined as the quotient of the bounded derived category of coherent sheaves, $D^bCoh(W^{-1}(0))$, by the full triangulated subcategory of perfect complexes $\mathrm{Perf}(W^{-1}(0))$. It is a differential $\mathbb{Z}_2$-graded category over $\mathbb{C}$.

Because $\mathbb{C}^{n+2} = \mathrm{Spec}(R)$ is affine (where $R := \mathbb{C}[z_1, \dots, z_{n+2}]$), the triangulated category of singularities of $W^{-1}(0)$ admits an alternative description, which is more amenable to explicit computations. Namely, it is quasi-equivalent to the category $MF(R, W)$ of 'matrix factorizations' of W, by [12, Theorem 3.9].

An object of $MF(R, W)$ is a finite-rank free $\mathbb{Z}_2$-graded R-module $P = P^0 \oplus P^1$, together with an R-linear endomorphism $d_P : P \to P$ of odd degree, satisfying $d_P^2 = W \cdot \mathrm{id}_P$. The space of morphisms from P to Q is the differential $\mathbb{Z}_2$-graded R-module of R-linear homomorphisms $f : P \to Q$, with the differential defined by

$$d(f) := d_Q \circ f + (-1)^{|f|} f \circ d_P,$$

and composition defined in the obvious way. This makes $MF(R, W)$ into a differential $\mathbb{Z}_2$-graded category over $\mathbb{C}$.

Under Homological Mirror Symmetry, our immersed Lagrangian sphere L^n should correspond to $\mathcal{O}_0$, the structure sheaf of the origin in the triangulated category of singularities of $W^{-1}(0)$. This corresponds, under the above-described equivalence, to a matrix factorization of W, which by abuse of notation we will also denote $\mathcal{O}_0$.

It follows from the computations of [25, Section 2] that, on the level of cohomology,

$$H^* \left(\mathrm{Hom}^*_{MF(R,W)}(\mathcal{O}_0, \mathcal{O}_0) \right) \cong \Lambda^* \mathbb{C}^{n+2}$$

as $\mathbb{Z}_2$-graded associative $\mathbb{C}$-algebras. Combining this with Theorem 1 establishes an isomorphism between the endomorphism algebras of the alleged mirror objects on the level of cohomology.

The Homological Mirror Symmetry conjecture predicts more: this isomorphism of cohomology algebras should extend to a quasi-isomorphism of A_∞ algebras. Namely,

$$\mathrm{Hom}^*_{MF(R,W)}(\mathcal{O}_0, \mathcal{O}_0)$$

inherits the structure of a differential $\mathbb{Z}_2$-graded $\mathbb{C}$-algebra from $MF(R,W)$, and a differential graded algebra is a special case of an A_∞ algebra.

Our main result (proved by studying the A_∞ deformations of the cohomology algebra) is that such a quasi-isomorphism does exist:

Theorem 2. *There is a quasi-isomorphism*

$$CF^*(L^n, L^n) \cong \mathrm{Hom}^*_{MF(R,W)}(\mathcal{O}_0, \mathcal{O}_0)$$

as $\mathbb{Z}_2$-graded A_∞-algebras over $\mathbb{C}$.

Remark 1.1.3.1. Of course the B-model $D^b_{\mathrm{Sing}}(\mathbb{C}^{n+2}, W)$ cannot be equivalent, in any sense, to the A-model $\mathcal{F}uk(\mathcal{P}^n)$ as we define it, because the morphism spaces in the B-model can be infinite-dimensional (even on the cohomology level) whereas the morphism space between two compact Lagrangians is always finite-dimensional. To get an A-model which has a hope of being equivalent to the B-model in some sense, we must consider the 'wrapped' Fukaya category (see [26]), which also includes non-compact Lagrangians.

1.1.4 Motivation: the A-model on the one-dimensional pair of pants

In this section, we consider the 1-dimensional case. We hope that this will aid the reader's intuition for the subsequent arguments, and provide a link with computations that have previously appeared in the literature (in [20, Section 10]), but this section could be skipped without serious harm.

Consider the immersed Lagrangian $L^1 : S^1 \to \mathcal{P}^1$ shown in Figure 1.1.2.1. We outline a description of the A_∞ algebra $\mathcal{A} = CF^*(L^1, L^1)$ up to quasi-isomorphism.

$\mathcal{A}$ has generators u, q corresponding respectively to the identity and top class in the Morse cohomology $CM^*(S^1)$, and two generators for each self-intersection point, which we label $x_1, \bar{x}_1, x_2, \bar{x}_2, x_3, \bar{x}_3$ as in Figure 1.1.2.1.

Because the homology class of L^1 is trivial in $H_1(\mathcal{P}^1)$, the generators of $\mathcal{A}$ come labeled by weights which are elements of the lattice

$$H_1\left(\mathcal{P}^1\right) \cong \mathbb{Z}\langle e_1, e_2, e_3\rangle / \langle e_1 + e_2 + e_3\rangle,$$

so that the A_∞ structure maps are homogeneous with respect to these weights. This is just because the disk contributing to such a product lifts to the universal cover, so its boundary must lift to a closed loop. See Definition 1.3.2.1 and Proposition 1.3.2.4 for the precise definition and argument. Explicitly, the weight of u, q is 0, of x_i is e_i and of $\bar{x}_i$ is $-e_i$. It follows that $\mu^1 = 0$.

The A_∞ structure maps count rigid holomorphic disks, which in this case is purely combinatorial. Our first step is to determine the cohomology algebra of $\mathcal{A}$, which has the (associative) product defined by

$$a \cdot b := (-1)^{|a|}\mu^2(a, b)$$

(using the sign conventions of [11]).

We have the following result:

Lemma 1.1.4.1. *The cohomology algebra of $\mathcal{A}$ is isomorphic (as $\mathbb{Z}_2$-graded associative $\mathbb{C}$-algebra) to the exterior algebra*

$$\Lambda^*\mathbb{C}\langle e_1, e_2, e_3\rangle$$

via the identification

$$\begin{aligned} u &\mapsto \mathbb{1} \\ x_i &\mapsto (-1)^i e_i \\ \bar{x}_i &\mapsto (-1)^{i+1} * e_i \text{ (Hodge star with respect to volume form } e_1 \wedge e_2 \wedge e_3\text{)} \\ q &\mapsto -e_1 \wedge e_2 \wedge e_3. \end{aligned}$$

Proof. *(sketch – see [20] for a more detailed proof)* The contributions of constant disks give all products involving u and q. The other products come from the two triangles on the front and back of Figure 1.1.2.1. For example, the triangle with vertices in cyclic order x_1, x_2, x_3 gives the product

$$\mu^2(x_1, x_2) = \bar{x}_3$$

corresponding to

$$e_1 \cdot e_2 = *e_3 = e_1 \wedge e_2.$$

We will not explain how to determine the signs here – see Section 1.3.4 (or [20]) for more detail. $\square$

Furthermore, we have

$$\mu^3(x_1, x_2, x_3) = -u,$$

but the corresponding product is 0 for any other permutation of the inputs. This comes from the degenerate 4-gon with vertices at u, x_1, x_2, x_3. Observe that, if we put the marked point u somewhere else on L^1, this product would again be equal to u, but possibly for a different permutation of the inputs (and would be 0 on all other permutations).

By choosing a complex volume form η on $\mathcal{P}^1$ and computing grading of the generators, one can lift the $\mathbb{Z}_2$-grading of $\mathcal{A}$ (defined by the sign of the intersection point corresponding to the generator) to a $\mathbb{Z}$-grading. See [20] for a formula for the grading that holds in the 1-dimensional case. The choice of volume form is not canonical, and hence the choice of $\mathbb{Z}$-grading is not canonical.

We have now shown that $\mathcal{A}$ lies in the set $\mathfrak{A}$ of A_∞ algebras satisfying the following conditions:

- $\mu^1 = 0$;
- The cohomology algebra is isomorphic to $\Lambda^*\mathbb{C}\langle e_1, e_2, e_3\rangle$ as $\mathbb{Z}_2$-graded associative $\mathbb{C}$-algebra;
- The A_∞ structure maps are homogeneous with respect to the weights as defined above;
- The $\mathbb{Z}_2$-grading lifts to a $\mathbb{Z}$-grading as defined above.

One can show that $\mathfrak{A}$ has a one-dimensional deformation space, in the sense of [9, Lemma

3.2]. Furthermore, the deformation class of $\mathcal{A}$ in this deformation space is given by

$$\sum_{i,j,k=1}^{3} \mu^3(x_i, x_j, x_k) = \mu^3(x_1, x_2, x_3) = -u$$

by our previous computations. In particular, it is non-zero, so $\mathcal{A}$ is versal. This determines $\mathcal{A}$ up to quasi-isomorphism, in the sense that any A_∞ algebra lying in $\mathfrak{A}$, with non-zero deformation class, is quasi-isomorphic to $\mathcal{A}$.

1.1.5 Outline of the chapter

In Section 1.2 we introduce some standing notation, and discuss the topology of the pair of pants $\mathcal{P}^n$. In particular, we introduce the coamoeba, which encodes topological information about $\mathcal{P}^n$ and is the starting point for understanding the Lagrangian immersion L^n. We give the details of the construction of the Lagrangian immersion $L^n : S^n \to \mathcal{P}^n$, and some of its properties.

In Section 1.3, we explain how to include the Lagrangian immersion L^n as an 'extra' object of the Fukaya category of embedded Lagrangians in $\mathcal{P}^n$. We define the A_∞ algebra $\mathcal{A} := CF^*(L^n, L^n)$, and establish some of its properties – namely, that it is homogeneous with respect to a certain weighting of its generators, that its $\mathbb{Z}_2$-grading lifts to a $\mathbb{Z}$-grading, and that it has a certain 'super-commutativity' property.

In Section 1.4, we give an alternative, Morse-Bott definition of the Fukaya category of embedded Lagrangians. We define the A_∞ structure coefficients by counts of objects called 'holomorphic pearly trees', which are Morse-Bott versions of the holomorphic disks usually used (and closely related to the 'clusters' of [27]). The technical parts of this section could be skipped at a first reading, but the concept of a pearly tree is important because it is the basis of our main computational technique, which is introduced in Section 1.5. This section could be read independently of the rest of the paper.

In Section 1.5, we introduce a Morse-Bott model $\mathcal{A}'$ for the A_∞ algebra $\mathcal{A}$, in which the A_∞ structure coefficients are defined by counts of objects called 'flipping holomorphic pearly trees'. We show that $\mathcal{A}'$ is quasi-isomorphic to $\mathcal{A}$. We can compute the A_∞ structure maps of $\mathcal{A}'$ by explicitly identifying the relevant moduli spaces of flipping holomorphic pearly trees. In particular, we compute that the cohomology algebra of $\mathcal{A}'$ (hence of $\mathcal{A}$) is an exterior algebra, as well as some of the higher structure maps. We use our computation of higher structure maps to show that $\mathcal{A}'$ is versal in the class of A_∞ algebras with cohomology algebra the exterior algebra, and the homogeneity and grading properties described in Section 1.3 (compare Section 1.1.4). Thus, applying deformation theory of A_∞ algebras, $\mathcal{A}'$ (and hence $\mathcal{A}$) is completely determined up to quasi-isomorphism by the coefficients and properties that we have established.

In Section 1.6, we describe the B-model of the mirror. We use the techniques of [25, Section 4] to construct a minimal A_∞ model $\mathcal{B}'$ for the differential $\mathbb{Z}_2$-graded algebra $\mathcal{B} := \mathrm{Hom}^*_{MF(R,W)}(\mathcal{O}_0, \mathcal{O}_0)$. We find that its cohomology algebra is an exterior algebra, and that it has the same grading and equivariance properties as $\mathcal{A}$. We compute higher products to show that $\mathcal{B}'$ is versal in the same class of A_∞ algebras as $\mathcal{A}'$, and hence that it is quasi-isomorphic to $\mathcal{A}'$. This completes the proof of Theorem 2.

1.2 The Lagrangian immersion $L^n : S^n \to \mathcal{P}^n$

The aim of this section is to describe the immersed Lagrangian sphere $L^n : S^n \to \mathcal{P}^n$. In Section 1.2.1 we introduce some standing notation, and describe the topology of the pair of pants $\mathcal{P}^n$. We introduce the notion of the **coamoeba** of the pair of pants, which is the starting point for visualising the Lagrangian immersion L^n.

In Section 1.2.2 we construct the Lagrangian immersion $L^n : S^n \to \mathcal{P}^n$ and establish some of its properties.

1.2.1 Topology of $\mathcal{P}^n$ and coamoebae

Let $[k]$ denote the set $\{1, 2, \ldots, k\}$. For a subset $K \subset [k]$, let $|K|$ be its number of elements and $\bar{K} \subset [k]$ its complement. Let $\widetilde{M}$ be the $(n+2)$-dimensional lattice

$$\widetilde{M} := \mathbb{Z}\langle e_1, \ldots, e_{n+2} \rangle.$$

For $K \subset [n+2]$, let e_K denote the element

$$e_K := \sum_{j \in K} e_j \in \widetilde{M}.$$

Let M be the $(n+1)$-dimensional lattice

$$M := \widetilde{M} / \langle e_{[n+2]} \rangle.$$

We will use the notation

$$M_P := M \otimes_{\mathbb{Z}} P$$

for any $\mathbb{Z}$-module P. We will not distinguish notationally between a lattice element $e_K \in \widetilde{M}$ and its image in M. We define maps

$$\begin{aligned} \widetilde{\mathrm{Log}} : \widetilde{M}_{\mathbb{C}^*} &\to \widetilde{M}_{\mathbb{R}}, \\ \widetilde{\mathrm{Log}}(z_1, \ldots, z_{n+2}) &:= (\log|z_1|, \ldots, \log|z_{n+2}|) \\ \widetilde{\mathrm{Arg}} : \widetilde{M}_{\mathbb{C}^*} &\to \widetilde{M}_{\mathbb{R}} / 2\pi\widetilde{M}, \\ \widetilde{\mathrm{Arg}}(z_1, \ldots, z_{n+2}) &:= (\arg(z_1), \ldots, \arg(z_{n+2})). \end{aligned}$$

These descend to maps

$$\begin{aligned} \mathrm{Log} : M_{\mathbb{C}^*} &\to M_{\mathbb{R}}, \\ \mathrm{Arg} : M_{\mathbb{C}^*} &\to M_{\mathbb{R}}/2\pi M. \end{aligned}$$

We can identify

$$\widetilde{M}_{\mathbb{C}^*} = \mathbb{C}^{n+2} \setminus \bigcup_j \{z_j = 0\}$$

and the quotient by the diagonal $\mathbb{C}^*$ action,

$$M_{\mathbb{C}^*} = \mathbb{CP}^{n+1} \setminus D$$

where we denote the divisors $D_j := \{z_j = 0\}$ for $j = 1, \dots, n+2$, and D is the union of all D_j. Thus we have

$$\mathcal{P}^n = \left\{ \sum_{j=1}^{n+2} z_j = 0 \right\} \subset M_{\mathbb{C}^*}.$$

Definition 1.2.1.1. The closure of the image $\mathrm{Arg}(\mathcal{P}^n)$ is called the **coamoeba** (also, sometimes, the **alga**) of $\mathcal{P}^n$, and we will denote it $\mathcal{C}^n$ (see, e.g., [28, 29]).

Now we will give a description of the coamoeba $\mathcal{C}^n$ for all n. It will be described in terms of a certain polytope, which we first describe.

Definition 1.2.1.2. Let Z_n be the zonotope generated by the vectors e_j in $M_{\mathbb{R}}$, i.e.,

$$Z_n = \left\{ \sum_{j=1}^{n+2} \theta_j e_j : \theta_j \in [0,1] \right\} \subset M_{\mathbb{R}}$$

(this is the projection of the cube $[0,1]^{n+2}$ in $\widetilde{M}_{\mathbb{R}}$).

Definition 1.2.1.3. The cells of ∂Z_n are indexed by triples of subsets $J, K, L \subset [n+2]$ such that

- $J \sqcup K \sqcup L = [n+2]$;
- $J \neq \phi$ and $K \neq \phi$.

Namely, we define the cell

$$\begin{aligned} U_{JKL} &:= \left\{ \sum_{i=1}^{n+2} \theta_i e_i : \theta_j = 0 \text{ for } j \in J,\, \theta_k = 1 \text{ for } k \in K,\, \theta_l \in [0,1] \text{ for } l \in L \right\} \\ &\subset \partial Z_n. \end{aligned}$$

We note that

$$\dim\left(U_{JKL}\right) = |L|,$$

and $U_{J'K'L'}$ is part of the boundary of U_{JKL} if and only if

$$J \subseteq J', K \subseteq K', \text{ and } L \supsetneq L'.$$

In particular, the vertices of Z_n are the 0-cells $U_{\bar{K},K,\phi} = \{e_K\}$, and are indexed by proper, non-empty subsets $K \subset [n+2]$.

Proposition 1.2.1.4. *$\mathcal{C}^n \subset M_{\mathbb{R}}/2\pi M$ is the complement of the image of the interior of πZ_n.*

Proof. $\mathcal{C}^n$ is the closure of the set of those

$$\boldsymbol{\theta} = \sum_j \theta_j e_j$$

such that there exist r_j satisfying

$$\sum_{j=1}^{n+2} \exp(r_j + i\theta_j) = 0.$$

In other words, the convex cone spanned by the vectors $\exp(i\theta_j)$ contains 0.

Therefore the complement of $\mathcal{C}^n$ consists of exactly those $\boldsymbol{\theta}$ such that the coordinates $\theta_1, \ldots, \theta_{n+2}$ are contained in an interval of length $< \pi$. By adding a common constant we may assume all θ_j lie in $[0, \pi)$. Thus the complement of $\mathcal{C}^n$ is exactly the image of the interior of πZ_n. $\square$

Remark 1.2.1.5. As we saw in Definition 1.2.1.3, the vertices of $\partial(\pi Z_n)$ are the points πe_K where $K \subset [n+2]$ is proper and non-empty. Observe that the vertices $\pi e_K, \pi e_{\bar{K}}$ get identified because

$$\pi e_K - \pi e_{\bar{K}} \in 2\pi M.$$

We can draw pictures in the lower-dimensional cases (see Figure 1.2.1.1).

Proposition 1.2.1.6. *The map* $\operatorname{Arg} : \mathcal{P}^n \to \mathcal{C}^n$ *is a homotopy equivalence. In particular,* $\mathcal{P}^n$ *has the homotopy type of an* $(n+1)$*-torus with a point removed.*

Proof. We choose to work in affine coordinates

$$\tilde{z}_j := \frac{z_j}{z_{n+2}} \text{ for } j = 1, \ldots, n+1$$

on $\mathbb{CP}^{n+1} \setminus D$. So

$$\mathcal{P}^n \cong \{1 + \tilde{z}_1 + \ldots + \tilde{z}_{n+1} = 0\} \subset (\mathbb{C}^*)^{n+1}.$$

It is shown in [30] that there exists a subset $W \subset \mathcal{P}^n$, such that the inclusion $W \hookrightarrow \mathcal{P}^n$ is a homotopy equivalence, and the projection

$$\operatorname{Arg} : W \to M_{\mathbb{R}}/2\pi M$$

is a homotopy equivalence onto its image, which is

$$\operatorname{Arg}(W) = \left\{(\tilde{\theta}_1, \ldots, \tilde{\theta}_{n+1}) : \text{ at least one } \tilde{\theta}_j = \pi\right\} \subset M_{\mathbb{R}}/2\pi M.$$

It is easy to see that the inclusion

$$\operatorname{Arg}(W) \hookrightarrow \mathcal{C}^n$$

is a homotopy equivalence (both are strong deformation retracts of $(M_{\mathbb{R}}/2\pi M)\backslash(0,0,\ldots,0)$). Hence, we have a commutative diagram

$$\begin{array}{ccc} W & \longrightarrow & \mathcal{P}^n \\ \downarrow & & \downarrow{\scriptstyle \operatorname{Arg}} \\ \operatorname{Arg}(W) & \longrightarrow & \mathcal{C}^n \end{array}$$

in which all arrows but the one labeled 'Arg' are known to be homotopy equivalences. It follows that $\operatorname{Arg} : \mathcal{P}^n \to \mathcal{C}^n$ is also a homotopy equivalence. □

Corollary 1.2.1.7. *For $n > 1$, there are natural isomorphisms*

$$\pi_1(\mathcal{P}^n) \cong H_1(\mathcal{P}^n) \cong M.$$

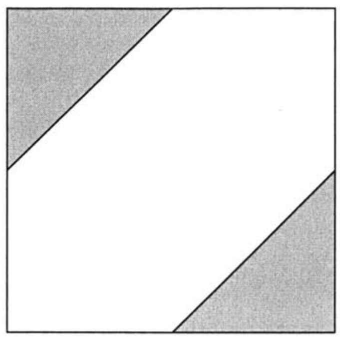
When $n = 1$, we still have a natural isomorphism $H_1(\mathcal{P}^1) \cong M$, but the fundamental group is no longer abelian. Instead, there is a natural isomorphism

$$\pi_1(\mathcal{P}^1) \cong \langle a, b, c | abc \rangle.$$

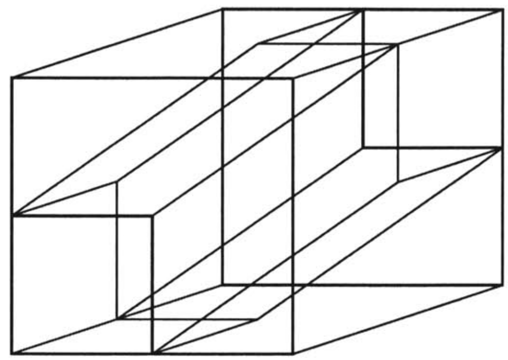
1.2.2 Construction of the Lagrangian immersion $L^n : S^n \to \mathcal{P}^n$

We observe that the Lagrangian $L^1 : S^1 \to \mathcal{P}^1$ can be seen rather simply in the coamoeba. It corresponds to traversing the hexagon which forms the boundary of the coamoeba (see Figure 1.2.2.1). The two triangles that make up the coamoeba correspond to the holomorphic triangles that give the product structure on Floer cohomology.

We will show that a similar picture exists for higher dimensions. Namely, by Propo-



(a) The coamoeba of $\mathcal{P}^1$.



(b) The coamoeba of $\mathcal{P}^2$. This picture lives in $(S^1)^3$, drawn as a cube with opposite faces identified, and we are removing the zonotope illustrated, which looks somewhat like a crystal.

Figure 1.2.1.1: $\mathcal{C}^1$ and $\mathcal{C}^2$.

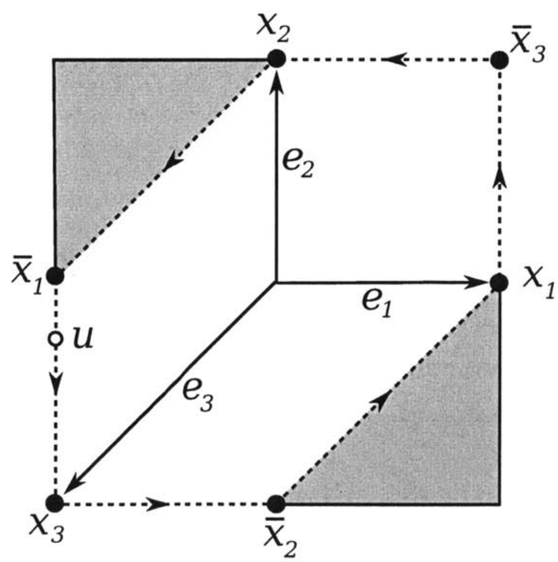


Figure 1.2.2.1: The projection of L^1 to $\mathcal{C}^1$.

sition 1.2.1.4, we know that the boundary of $\mathcal{C}^n$ is a polyhedral n-sphere that intersects itself at its vertices. In this section, we will explain how to lift this immersed polyhedral n-sphere to an immersed Lagrangian n-sphere in $\mathcal{P}^n$.

Remark 1.2.2.1. This is not the first time that the coamoeba has been used to study Floer cohomology. It appeared in [28] (with the name 'alga'), where it was used as motivation to construct Landau-Ginzburg mirrors to some toric surfaces. It was conjectured in [31] that this picture generalizes to higher dimensions. There is a connection between the 'tropical coamoeba' of the Landau-Ginzburg mirror (X, w) of projective space, introduced in [31], and our construction, but we will not go into it.

Consider the real projective space

$$\mathbb{RP}^n = \left\{ \sum_j z_j = 0, z_j \in \mathbb{R} \right\} \subset \left\{ \sum_j z_j = 0 \right\} \subset \mathbb{CP}^{n+1}.$$

Clearly it is Lagrangian and invariant with respect to the $S_{n+2} \times \mathbb{Z}_2$ action, so by an equivariant version of the Weinstein Lagrangian neighbourhood theorem, there is an $S_{n+2} \times \mathbb{Z}_2$-equivariant symplectic embedding of the radius-η disk cotangent bundle

$$D^*_\eta \mathbb{RP}^n \hookrightarrow \left\{ \sum_j z_j = 0 \right\} \subset \mathbb{CP}^{n+1}$$

for some sufficiently small $\eta > 0$. We may choose this embedding to be J-holomorphic along the zero section with respect to the almost-complex structure induced by the standard symplectic form and metric on $D^*_\eta \mathbb{RP}^n$. The $\mathbb{Z}_2$-invariance says that complex conjugation acts on $D^*_\eta \mathbb{RP}^n$ by -1 on the covector.

Our immersed sphere L^n will land inside this neighbourhood. Now consider the double cover of $\mathbb{RP}^n$ by S^n. Think of S^n as

$$S^n = \left\{ \sum_j x_j^2 = 1 \right\} \bigcap \left\{ \sum_j x_j = 0 \right\} \subset \mathbb{R}^{n+2},$$

and denote the real hypersurfaces

$$D_j^{\mathbb{R}} := \{x_j = 0\} \subset S^n.$$

Then the double cover just sends $(x_1, \ldots, x_{n+2}) \mapsto [x_1 : \ldots : x_{n+2}]$. This extends to a double cover $D_\eta^* S^n \to D_\eta^* \mathbb{RP}^n$. Composing this with the inclusion $D_\eta^* \mathbb{RP}^n \to \mathbb{CP}^n$ gives a map $i : D_\eta^* S^n \to \mathbb{CP}^n$.

Lemma 1.2.2.2. *Suppose that $f : S^n \to \mathbb{R}$ is a smooth function whose gradient vector field (with respect to the round metric on S^n) is transverse to the real hypersurfaces $D_j^{\mathbb{R}}$. Then for sufficiently small $\epsilon > 0$, the image of the graph $\Gamma(\epsilon df) \subset T^* S^n$ lies inside $D_\eta^* S^n$, and its image under the map i into $\mathbb{CP}^n$ avoids the divisors D_j.*

Proof. Note that the graph of ϵdf in $D_\eta^* S^n$ is the time-ϵ flow of the zero-section by the Hamiltonian vector field corresponding to f, which is exactly $J(\nabla f)$, where J is the standard complex structure on $\mathbb{CP}^n$ (we observe that the round metric on S^n is exactly the metric induced by the Fubini-Study form and standard complex structure). Given a point $q \in D_j^{\mathbb{R}}$, we can holomorphically identify a neighbourhood of its image in $\mathbb{CP}^n$ with a neighbourhood of 0 in $\mathbb{C}^n$, in such a way that a neighbourhood of q in S^n gets identified with a neighbourhood of 0 in $\mathbb{R}^n \subset \mathbb{C}^n$. We can furthermore arrange that the divisor D^j corresponds to the first coordinate being 0.

When we flow $\mathbb{R}^n$ by $J(\nabla f)$, the imaginary part of the first coordinate will be strictly positive (respectively negative) because ∇f is transverse to $D_j^{\mathbb{R}}$, in the positive (respectively negative) direction. Therefore the first component can not be zero, so the image avoids D_j. □

Definition 1.2.2.3. Let $g : \mathbb{R} \to \mathbb{R}$ be a smooth function so that

1. $g'(x) > 0$;
2. $g(-x) = -g(x)$;

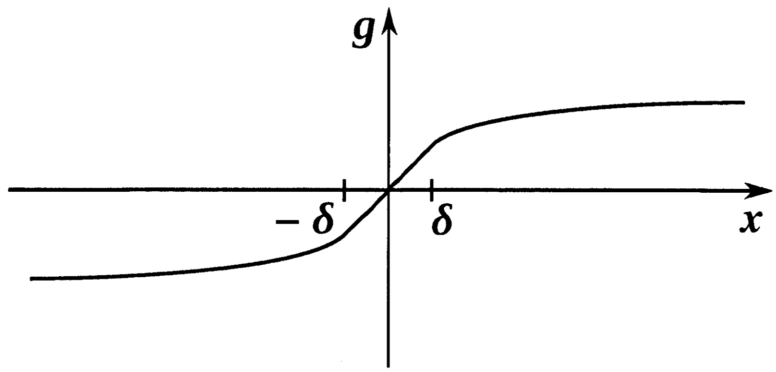


Figure 1.2.2.2: The function g.

3. $g(x) = x$ for $|x| < \delta$;

4. $g'(x)$ is a strictly decreasing function of $|x|$ for $|x| > \delta$;

5. $g'(x) < \delta$ for $|x| > 2\delta$,

where $0 < \delta \ll 1$ (see Figure 1.2.2.2). We define $f : S^n \to \mathbb{R}$ by restricting the function

$$\begin{aligned} \tilde{f} : \mathbb{R}^{n+2} &\to \mathbb{R} \\ \tilde{f}(x_1, \ldots, x_{n+2}) &= \sum_{j=1}^{n+2} g(x_j), \end{aligned}$$

recalling that S^n sits inside $\mathbb{R}^{n+2}$ as above.

Lemma 1.2.2.4. *∇f is transverse to all of the hypersurfaces $D_j^{\mathbb{R}}$ in a positive sense.*

Proof. One can compute that ∇f is the projection of the vector

$$\sum_{j=1}^{n+2} f_j \frac{\partial}{\partial x_j} \in T\mathbb{R}^{n+1}$$

to TS^n, where $\mathbb{R}^{n+1} = \{\sum_j x_j = 0\} \subset \mathbb{R}^{n+2}$ and

$$f_j := g'(x_j) - \frac{\sum_{k=1}^{n+2} g'(x_k)}{n+2}.$$

By the construction of g, one can check that $f_j > 0$ whenever $|x_j| < \delta$. The result follows. $\square$

Definition 1.2.2.5. Let $L^n_\epsilon : S^n \to \mathbb{CP}^n$ be the graph of ϵdf in $\mathbb{CP}^n$, for $\epsilon > 0$ sufficiently small. Note that it lies in $\mathcal{P}^n$ by Lemmas 1.2.2.2 and 1.2.2.4, and is Lagrangian because it is the graph of an exact one-form. We will frequently fix an ϵ and write L^n.

Remark 1.2.2.6. L^n is S_{n+2}-invariant (because f and our Weinstein neighbourhood are). Furthermore, because $f(-x) = -f(x)$, df is invariant under the $\mathbb{Z}_2$-action

$$(x, \alpha) \mapsto (a(x), -a^*\alpha)$$

where $a : S^n \to S^n$ is the antipodal map. Recall that complex conjugation $\tau : \mathcal{P}^n \to \mathcal{P}^n$ induces the $\mathbb{Z}_2$-action $(x, \alpha) \mapsto (x, -\alpha)$ in $D^*_\eta S^n$, so $\tau \circ L^n = L^n \circ a$. In other words, the image of L^n is preserved by complex conjugation, but it acts via the antipodal map on the domain S^n.

Proposition 1.2.2.7. *Define the maps*

$$\begin{aligned} \iota_\epsilon : S^n &\to M_{\mathbb{R}}/2\pi M, \\ \iota_\epsilon &:= \operatorname{Arg} \circ L^n_\epsilon, \end{aligned}$$

and

$$q : \partial Z_n \to M_{\mathbb{R}}/2\pi M$$

(the standard inclusion). Then there exist homotopy equivalences $p_\epsilon : S^n \to \partial Z_n$, defined for $\epsilon > 0$ sufficiently small, such that

$$\lim_{\epsilon \to 0} \|\iota_\epsilon - q \circ p_\epsilon\|_{C^0} = 0.$$

In other words, ι_ϵ converges absolutely, modulo reparametrisation, to ∂Z_n.

Proof. We consider a cellular decomposition of S^n which is dual to the cellular decomposition induced by the hypersurfaces $D_j^{\mathbb{R}}$, and is isomorphic to the cellular decomposition of ∂Z_n defined in Definition 1.2.1.3. We will show that the image of each cell in the decomposition, under ι_ϵ, converges to the corresponding cell in ∂Z_n.

Definition 1.2.2.8. We define a cellular decomposition of S^n whose cells are indexed by triples of subsets $J, K, L \subset [n+2]$ such that

- $J \sqcup K \sqcup L = [n+2]$;
- $J \neq \phi$ and $K \neq \phi$.

Namely, we define the cell

$$V_{JKL} := \left\{(x_1, \ldots, x_{n+2}) \in S^n : x_j = \max_i\{x_i\} \text{ for all } j \in J, x_k = \min_i\{x_i\} \text{ for all } k \in K\right\}$$

(this is dual to the cellular decomposition with cells

$$W_{JKL} := \{x_j \geq 0 \text{ for } j \in J, x_k \leq 0 \text{ for } k \in K, \text{ and } x_l = 0 \text{ for } l \in L\},$$

induced by the hypersurfaces $D_j^{\mathbb{R}}$).

We now have

$$\dim(V_{JKL}) = |L|,$$

and $V_{J'K'L'}$ is part of the boundary of V_{JKL} if and only if

$$J \subseteq J', K \subseteq K', \text{ and } L \supsetneq L'.$$

Thus, this cellular decomposition is isomorphic to that of ∂Z_n by cells U_{JKL}, described in Definition 1.2.1.3. See Figure 1.2.2.3 for the picture in the case $n = 2$.

Our Lagrangian is obtained from the immersion $S^n \to \mathbb{CP}^n$ by pushing off with the vector field $J(\nabla f)$. Thus, by Lemma 1.2.2.4, it is approximately equal (to order ϵ^2) to the composition of the map

$$\begin{array}{rcl} S^n & \to & \left\{\sum_j z_j = 0, z_j \neq 0\right\} \subset \mathbb{C}^{n+2} \\ (x_1, \ldots, x_{n+2}) & \mapsto & (x_1, \ldots, x_{n+2}) + i\epsilon(f_1, \ldots, f_{n+2}) \end{array}$$

with the projection to $\mathbb{CP}^n \setminus D = \mathcal{P}^n$. Thus we have

$$\iota_\epsilon(x_1, \ldots, x_n) = (\arg(x_1 + i\epsilon f_1), \ldots, \arg(x_{n+2} + i\epsilon f_{n+2})) + \mathcal{O}(\epsilon^2).$$

Now, when $|x_l|$ is sufficiently large, we have

$$\arg(x_l + i\epsilon f_l + \mathcal{O}(\epsilon^2)) \approx \arg(x_l) = 0 \text{ or } \pi.$$

When $|x_l|$ is sufficiently small, we have

$$\arg(x_l + i\epsilon f_l + \mathcal{O}(\epsilon^2)) \in (0, \pi)$$

because $f_l > 0$ (by Lemma 1.2.2.4). More precisely, we have the following:

Lemma 1.2.2.9. *If we choose $\epsilon > 0$ sufficiently small, then we have:*

- *If $|x_l| \geq \sqrt{\epsilon}$, then $\arg(x_l + i\epsilon f_l + \mathcal{O}(\epsilon^2)) = \arg(x_l) + \mathcal{O}(\sqrt{\epsilon})$, where $\arg(x_l) = 0$ or π;*

- *If $|x_l| \leq \sqrt{\epsilon}$, then $\arg(x_l + i\epsilon f_l + \mathcal{O}(\epsilon^2)) \in (0, \pi)$, because f_l is strictly positive for $|x_l|$ sufficiently small (by Lemma 1.2.2.4).*

Observe that, on the cell V_{JKL}, we have

$$x_j \geq \sqrt{\epsilon} \text{ for } j \in J, \text{ and } x_k \leq -\sqrt{\epsilon} \text{ for } k \in K,$$

because $\sum_l x_l^2 = 1$ and $\sum_l x_l = 0$. Therefore, by Lemma 1.2.2.9,

$$\begin{aligned}
\arg(x_j + i\epsilon f_j) + \mathcal{O}(\epsilon^2) &= \mathcal{O}(\sqrt{\epsilon}) \text{ for } j \in J, \\
\arg(x_k + i\epsilon f_k) + \mathcal{O}(\epsilon^2) &= \pi + \mathcal{O}(\sqrt{\epsilon}) \text{ for } k \in K, \text{ and} \\
\arg(x_l + i\epsilon f_l) + \mathcal{O}(\epsilon^2) &\in (0, \pi) + \mathcal{O}(\sqrt{\epsilon}) \text{ for } l \in L.
\end{aligned}$$

It follows that $\iota_\epsilon(V_{JKL})$ lies in an $\mathcal{O}(\sqrt{\epsilon})$-neighbourhood of U_{JKL}.

We are now able to define the map

$$p_\epsilon : S^n \to \partial Z_n$$

to be a cellular map which identifies the cellular decompositions V_{JKL} and U_{JKL} (hence is a homotopy equivalence), and such that

$$\|\iota_\epsilon - q \circ p_\epsilon\|_{C^0} = \mathcal{O}(\sqrt{\epsilon}).$$

We assume inductively that a map with these properties has been defined on all cells of dimension $< d$, then extend it to the cells of dimension d relative to their boundaries. □

Now observe that, because $f(-x) = -f(x)$, $df(-x) = -df(x)$ (identifying tangent spaces by the antipodal map), so the only points where L^n has a self-intersection are where $df = 0$, i.e., critical points of f. A self-intersection point looks locally like the intersection of the graph of df with the graph of $-df$, which is transverse because $f - (-f) = 2f$ is Morse. We will now describe the critical points and Morse flow of f.

Lemma 1.2.2.10. *If $x_j > x_k \geq 0$, then*

$$(\nabla f)\left(\frac{x_k}{x_j}\right) > 0.$$

Similarly, if $x_j < x_k \leq 0$, then

$$(\nabla f)\left(\frac{x_k}{x_j}\right) < 0.$$

Proof. We prove the first statement. If $x_j > x_k \geq 0$ then, by the construction of g, $g'(x_j) < g'(x_k)$. It follows that $f_j < f_k$, and hence that $f_j x_k < f_k x_j$, using the notation from the proof of Lemma 1.2.2.4. Thus,

$$\begin{aligned}
(\nabla f)\left(\frac{x_k}{x_j}\right) &= \sum_{l=1}^{n+2} f_l \frac{\partial}{\partial x_l}\left(\frac{x_k}{x_j}\right) \text{ (since } x_k/x_j \text{ is constant in the radial direction)} \\
&= \frac{1}{x_j^2}(f_k x_j - f_j x_k) \\
&> 0.
\end{aligned}$$

The proof of the second statement is similar. □

Corollary 1.2.2.11. *There is one critical point p_K of f for each proper, non-empty subset $K \subset [n+2]$, defined by*

$$V_{\bar{K},K,\phi} = \{p_K\}.$$

Explicitly, p_K has coordinates (recalling $\sum_j x_j = 0$)

$$x_j = \begin{cases} -\frac{1}{|K|} & j \in K, \\ +\frac{1}{|\bar{K}|} & j \in \bar{K}, \end{cases}$$

up to a positive rescaling so that $\sum_j x_j^2 = 1$. Observe that Arg *maps p_K to the vertex πe_K of ∂Z_n.*

Proof. Critical points of f cannot lie on the hypersurfaces $D_j^{\mathbb{R}}$, since ∇f is transverse to the hypersurfaces. Suppose that $x_j > x_k > 0$. Then by Lemma 1.2.2.10,

$$(\nabla f)\left(\frac{x_k}{x_j}\right) > 0,$$

so $\nabla f \neq 0$. Hence, at a critical point of f, all positive coordinates x_j are equal. By a similar argument, all negative coordinates are equal. It follows that the points p_K are the only possiblities for critical points of f.

To prove that each p_K is indeed a critical point, observe that by S_{n+2} symmetry, the Morse flow of f must preserve the equalities

$$\begin{aligned} x_k &= x_l \qquad \text{for all } k, l \in K, \text{ and} \\ x_k &= x_l \qquad \text{for all } k, l \in \bar{K}. \end{aligned}$$

The set of points satisfying these equalities is exactly $\{p_K, p_{\bar{K}}\}$, hence the Morse flow preserves these points. Thus each p_K is a critical point of f. $\square$

Lemma 1.2.2.12. *Let $\phi : \mathbb{R} \times S^n \to S^n$ denote the flow of ∇f with respect to the round metric on S^n, so that $\phi(0, \cdot) = id$. Given a proper, non-empty subset $K \subset [n+2]$, we define*

$$\mathcal{S}(p_K) := \left\{ q \in S^n : \lim_{t \to \infty} \phi(t, q) = p_K \right\} \subset S^n,$$

the stable manifold of p_K, and

$$\mathcal{U}(p_K) := \left\{ q \in S^n : \lim_{t \to -\infty} \phi(t, q) = p_K \right\} \subset S^n,$$

the unstable manifold of p_K. Then

$$\mathcal{S}(p_K) = \{(x_1, \ldots, x_{n+2}) \in S^n : \{k \in [n+2] : x_k = \min_l \{x_l\}\} = K\}$$

and

$$\mathcal{U}(p_K) = \{(x_1, \ldots, x_{n+2}) \in S^n : \{k \in [n+2] : x_k = \max_l \{x_l\}\} = \bar{K}\}.$$

Proof. We prove the first statement. Suppose we are given $q = (x_1, \ldots, x_{n+2}) \in S^n$. Let

$$\lim_{t \to \infty} \phi(t, q) := p_K,$$

and

$$K' := \{k \in [n+2] : x_k = \min_l \{x_l\}\}.$$

We will show that $K = K'$.

First observe that, by S_{n+2} symmetry, any equality of the form $x_j = x_k$ is preserved under the forward and backward flow of ∇f. Consequently any inequality of the form $x_j > x_k$ is also preserved under the (finite-time) flow. It follows that $K' \subset K$.

We prove that $K \subset K'$ by contradiction: suppose that $j \notin K'$ but $j \in K$. After flowing for some time, x_j would have to be negative (in order to converge to p_K). Then for any $k \in K'$ we would have $x_k < x_j < 0$, so by Lemma 1.2.2.10 we have

$$(\nabla f)\left(\frac{x_j}{x_k}\right) < 0.$$

Thus, the ratio x_j/x_k is bounded above away from 1, so even in the limit $t \to \infty$, x_j can not approach the minimum value $x_k = \min_l\{x_l\}$. This is a contradiction, hence $K \subset K'$.

Therefore $K = K'$. This completes the proof of the first statement. The proof of the second statement is analogous. $\square$

Corollary 1.2.2.13. *The critical point p_K of f has Morse index*

$$\mu_{\text{Morse}}(p_K) = n + 1 - |K|.$$

Proof. The Morse index of p_K is the dimension of the stable manifold of p_K, which by Lemma 1.2.2.12 is $n + 1 - |K|$. $\square$

Remark 1.2.2.14. Observe that, as a consequence of Lemma 1.2.2.12,

$$V_{JKL} = \overline{\mathcal{U}(\bar{J}) \cap \mathcal{S}(K)}$$

(see Figure 1.2.2.3).

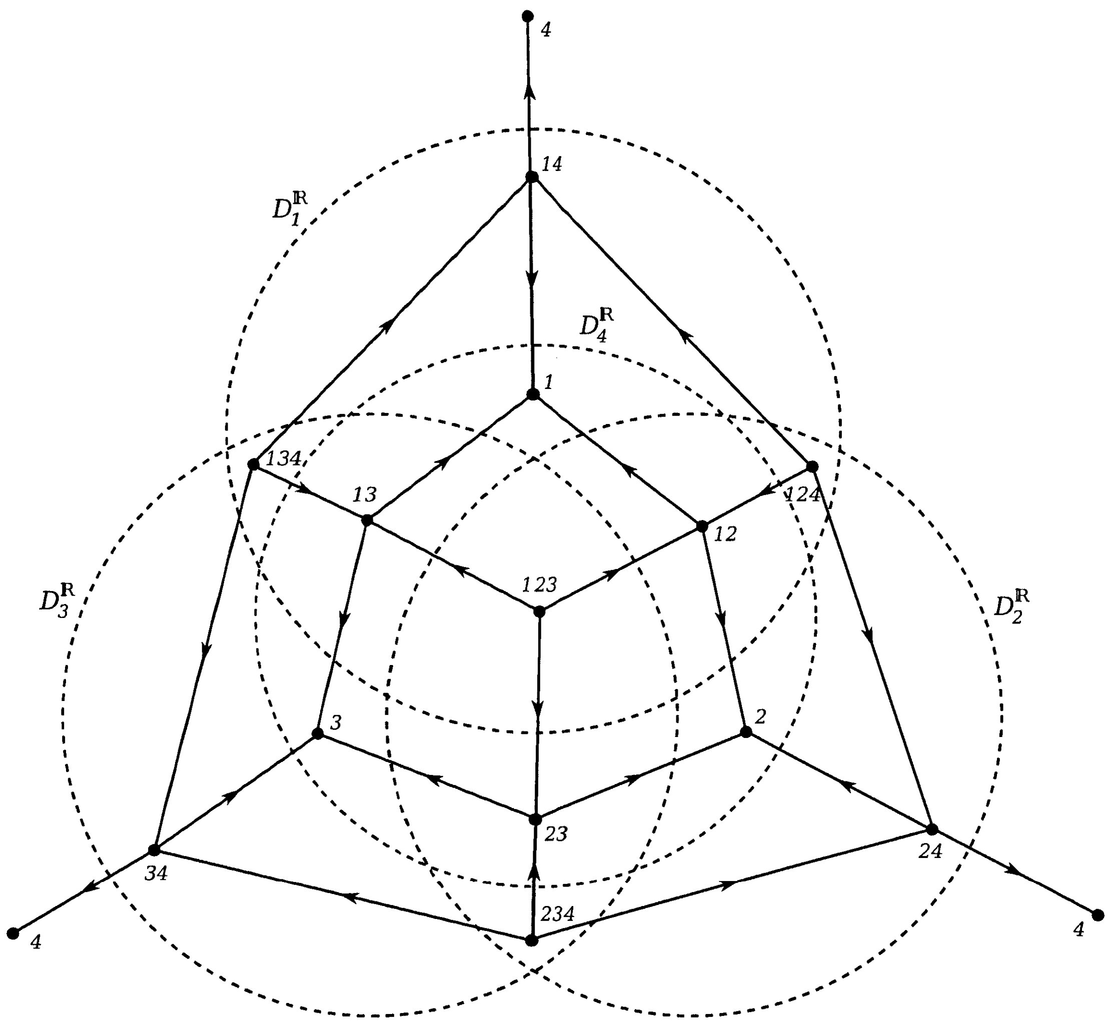


Figure 1.2.2.3: The dual cell decompositions for $n = 2$. The dashed circles represent the hypersurfaces $D_j^{\mathbb{R}}$ as labeled. Each region is labeled with the list of coordinates that are negative in that region (e.g., the label '124' means that $x_1 < 0, x_2 < 0, x_3 > 0, x_4 < 0$ in that region). The arrows represent the index-1 Morse flow lines of ∇f. The dots represent critical points of f. The picture really lives on a sphere, and the three points labeled '4' should be identified (at infinity). Observe that the flowlines correspond to the edges of the polyhedron ∂Z_2, illustrated in Figure 1-1(b).

1.3 The A_∞ algebra $\mathcal{A} := CF^*(L^n, L^n)$

This section is concerned with the definition and properties of the A_∞ algebra $\mathcal{A}^n := CF^*(L^n, L^n)$. We will simply write '$\mathcal{A}$' rather than '$\mathcal{A}^n$' unless we wish to draw attention to the dimension.

In Section 1.3.1, we will explain why L^n, despite being an immersion rather than an embedding, can be regarded as an 'extra' object of the Fukaya category of $\mathcal{P}^n$, as defined in [11, Chapters 8 – 12]. This section can **not** be read independently of that reference. In Sections 1.3.2 – 1.3.4, we establish certain properties of $\mathcal{A}$.

1.3.1 Including L^n as an 'extra' object of $\mathcal{F}uk(\mathcal{P}^n)$

In [11, Chapters 8 – 12], it is shown how to define the Fukaya category of a symplectic manifold (X, ω) with the following properties and structures:

- $\omega = d\theta$ is exact;
- X is equipped with an almost-complex structure J_0 in a neighbourhood of infinity, compatible with ω;
- X is convex at infinity, in the sense that there is a bounded below, proper function $h : X \to \mathbb{R}$ such that
$$\theta = -dh \circ J_0.$$

These assumptions are actually not quite the same as those in [11], but the arguments and definitions work in the same way.

In particular, $X = \mathcal{P}^n$ has these properties: we equip it with the standard (integrable) complex structure J_0, then the restriction of the Fubini-Study form to $\mathcal{P}^n$ is given by

$\omega = d\theta$, where $\theta = -dh \circ J_0$, and

$$\begin{aligned} h : \mathcal{P}^n &\to \mathbb{R}, \\ h([z_1 : \ldots : z_{n+2}]) &= \log \left(\frac{\sum_{j=1}^{n+2} |z_j|^2}{\left(\prod_{j=1}^{n+2} |z_j|^2\right)^{\frac{1}{n+2}}} \right) \end{aligned}$$

is proper and bounded below.

With this data, the Fukaya category of compact, exact, embedded, oriented Lagrangians L can be defined over a field of characteristic 2, and with $\mathbb{Z}_2$ gradings (the 'preliminary' Fukaya category of [11, Chapters 8, 9]). If X is furthermore equipped with a complex volume form η (note: we will not take a quadratic complex volume form as in [11], because we assume our Lagrangians to be oriented), then the Fukaya category of compact, exact, embedded, oriented Lagrangian branes $L^{\#}$ can be defined over $\mathbb{C}$, and the $\mathbb{Z}_2$ grading can be lifted to a $\mathbb{Z}$ grading.

We define the Fukaya category of $\mathcal{P}^n$ to include an 'extra' object corresponding to the Lagrangian immersion $L^n : S^n \to \mathcal{P}^n$.

Remark 1.3.1.1. A theory of Lagrangian Floer cohomology for immersed Lagrangians has been worked out in [32] using Kuranishi structures, but we will give a definition that is compatible with the definition of [11] using explicit perturbations, with the aim of using it to make computations in Section 1.5.

First, we note that $H^1(S^n) = 0$ for $n > 1$, so L^n is automatically exact (this is an additional restriction in the case $n = 1$ – see the caption to Figure 1.1.2.1).

Now we explain the modifications necessary to the definition of the (preliminary) Fukaya category given in [11, Chapters 8, 9], to include the object L^n.

Remark 1.3.1.2. We will not mention brane structures, orientations and gradings for the purposes of this Section 1.3.1, because they work exactly the same as in [11, Chapters

11, 12]. We observe that $H^1(S^n) = 0$ for $n > 1$, so L^n admits a grading (the case $n = 1$ is easily checked). S^n is also spin, so L^n admits a brane structure. These observations, together with the modifications described in this section that show we can include L^n as an extra object of the preliminary Fukaya category, allow us to include L^n as an extra object in the 'full' ($\mathbb{Z}$-graded, with $\mathbb{C}$ coefficients) Fukaya category of $\mathcal{P}^n$.

Definition 1.3.1.3. We define an object L of the (preliminary) Fukaya category to be an exact Lagrangian immersion

$$L : N \to \mathcal{P}^n$$

of some closed, oriented n-manifold N into $\mathcal{P}^n$, which is **either** an embedding **or** the Lagrangian immersion $L^n : S^n \to \mathcal{P}^n$.

Definition 1.3.1.4. We define

$$\mathcal{H} := C_c^\infty(\mathcal{P}^n, \mathbb{R}),$$

the space of smooth, compactly supported functions on $\mathcal{P}^n$ (the space of Hamiltonians), and $\mathcal{J}$, the space of smooth almost-complex structures on $\mathcal{P}^n$ compatible with ω, and equal to the standard complex structure J_0 outside of some compact set.

Definition 1.3.1.5. For each pair of objects (L_0, L_1), we define a Floer datum (H_{01}, J_{01}) consisting of

$$H_{01} \in C^\infty([0,1], \mathcal{H}) \text{ and } J_{01} \in C^\infty([0,1], \mathcal{J})$$

satisfying the following property: if ϕ^t denotes the flow of the Hamiltonian vector field of the (time-dependent) Hamiltonian H_{01}, then the image of the time-1 flow $\phi^1 \circ L_0$ is transverse to L_1. One then defines a **generator** of $CF^*(L_0, L_1)$ to be a path $y : [0,1] \to \mathcal{P}^n$ which is a flowline of the Hamiltonian vector field of H_{01}, together with a pair of points $(\tilde{y}_0, \tilde{y}_1) \in N_0 \times N_1$ such that $L_0(\tilde{y}_0) = y(0)$ and $L_1(\tilde{y}_1) = y(1)$. One defines $CF^*(L_0, L_1)$ to be the $\mathbb{C}$-vector space generated by its generators.

The definition of a perturbation datum on a boundary-punctured disk with Lagrangian labels is the same as in [11, Section 9h].

Definition 1.3.1.6. Given a perturbation datum on a boundary-punctured disk S with Lagrangian boundary labels $(L_0, \ldots, L_k)$, some of which may be immersed, we define an **inhomogeneous pseudo-holomorphic disk** to be a smooth map $u : S \to \mathcal{P}^n$ such that

- $u(C) \in \operatorname{im}(L_C)$ for each boundary component C with label L_C, and
- u satisfies the perturbed holomorphic curve equation [11, Equation (8.9)] with respect to the perturbation datum,

together with a continuous lift $\tilde{u}_C$ of the map $u|_C : C \to \operatorname{im}(L_C)$ to N_C:

$$\begin{tikzcd} & N_C \arrow[d, "L_C"] \\ C \arrow[ur, "\tilde{u}_C"] \arrow[r, "u|_C"'] & \operatorname{im}(L_C) \end{tikzcd}$$

for each boundary component C with label $L_C : N_C \to \mathcal{P}^n$.

Remark 1.3.1.7. Note that the lift $\tilde{u}_C$ exists automatically if L_C is an embedding. When L_C is an immersion, the existence of $\tilde{u}_C$ tells us that the boundary map $u|_C$ does not 'switch sheets' of the immersion along C.

Definition 1.3.1.8. Given generators

$$y_j \in CF^*(L_{j-1}, L_j) \text{ for } j = 1, \ldots, k, \; ,$$

and

$$y_0 \in CF^*(L_0, L_k),$$

we say that an inhomogeneous pseudo-holomorphic disk has **asymptotic conditions** given by $(y_0, \ldots, y_k)$ if, on the strip-like end ϵ_j corresponding to the jth puncture, we have

$$\begin{aligned}
\lim_{s\to+\infty} u(\epsilon_j(s,t)) &= y_j(t), \\
\lim_{s\to+\infty} \tilde{u}(\epsilon_j(s,0)) &= (\tilde{y}_j)_0, \text{and} \\
\lim_{s\to+\infty} \tilde{u}(\epsilon_j(s,1)) &= (\tilde{y}_j)_1.
\end{aligned}$$

(and the analogous condition with $s \to -\infty$ when $j = 0$). We define the moduli space $\mathcal{M}_S(y_0, \ldots, y_k)$ to be the set of inhomogeneous pseudo-holomorphic disks with asymptotic conditions given by the generators $(y_0, \ldots, y_k)$.

To show that $\mathcal{M}_S(y_0, \ldots, y_k)$ is a smooth manifold, we must modify the functional analytic framework of [11, Section 8i] slightly. Namely, we fix $p > 2$, and define a Banach manifold $\mathcal{B}_S(y_0, \ldots, y_k)$ as follows.

A point in $\mathcal{B}_S$ consists of:

- a map $u \in W^{1,p}_{loc}(S, \mathcal{P}^n)$, satisfying $u(C) \in \operatorname{im}(L_C)$;
- continuous lifts $\tilde{u}_C$ of the continuous maps $u|_C : C \to \operatorname{im}(L_C)$ to N_C, for each boundary component C of S,

such that u and $\tilde{u}_C$ are asymptotic to the generators y_j along the strip-like ends, in the sense of Definition 1.3.1.8. Observe that $W^{1,p}$ functions are continuous at the boundary, so the lifting condition makes sense.

Let $\boldsymbol{u} = (u, (\tilde{u}_C)) \in \mathcal{B}_S$ be represented by a smooth map. We define charts for the Banach manifold structure in a neighbourhood of $\boldsymbol{u}$. For each boundary component C of S, we have a continuous Lagrangian embedding of vector bundles,

$$TN_C \xhookrightarrow{(L_C)_*} (L_C)^* T\mathcal{P}^n.$$

Thus, we have a continuous Lagrangian embedding

$$(\tilde{u}_C)^* TN_C \hookrightarrow (\tilde{u}_C)^*(L_C)^* T\mathcal{P}^n \cong (u^* T\mathcal{P}^n)|_C.$$

We define the tangent space to $\mathcal{B}_S(y_0, \dots, y_k)$ at $\boldsymbol{u}$ to be the Banach space

$$T_{\boldsymbol{u}}\mathcal{B}_S(y_0, \dots, y_k) := W^{1,p}(S, u^* T\mathcal{P}^n, \tilde{u}_C^* TN_C)$$

(with the $W^{1,p}$-norm). We choose an exponential map $\exp : T\mathcal{P}^n \to \mathcal{P}^n$ that makes the Lagrangian labels totally geodesic, and denote by $\widetilde{\exp}_N : TN \to N$ the corresponding exponential map on each Lagrangian label. We then define a map

$$\phi_{\boldsymbol{u}} : T_{\boldsymbol{u}}\mathcal{B}_S \to \mathcal{B}_S$$

so that $\phi_{\boldsymbol{u}}(\xi)$ consists of the map $\exp(u, \xi(u))$, together with boundary lifts $\widetilde{\exp}_{N_C}(\tilde{u}_C, \xi(\tilde{u}_C))$. This defines a chart of the Banach manifold structure in a neighbourhood of $\boldsymbol{u}$.

Remark 1.3.1.9. Note that we can **not** define a Banach manifold of locally $W^{1,p}$ maps from S to $\mathcal{P}^n$, sending boundary component C to $\mathrm{im}(L_C)$, then impose the lifting condition separately – this would not define a Banach manifold because the image of L_C may be singular (if $L_C = L^n$).

We now define a Banach bundle $\mathcal{E}_S$ over $\mathcal{B}_S$, and a smooth section given by the perturbed $\bar{\partial}$-operator, as in [11, Section 8i]. The section is Fredholm, because its linearization is a Cauchy-Riemann operator with totally real boundary conditions given by $\tilde{u}_C^* TN_C$. Thus, assuming regularity, the moduli space $\mathcal{M}_S(y_0, \dots, y_k)$ is a smooth manifold with dimension equal to the Fredholm index. We can extend these arguments to show that the moduli space $\mathcal{M}_{\mathcal{S}^{k+1}}(y_0, \dots, y_k)$ of inhomogeneous pseudo-holomorphic disks with arbitrary modulus is also a smooth manifold.

Finally, we must check that Gromov compactness holds. The author is not aware of a proof of Gromov compactness with immersed Lagrangian boundary conditions in the literature, but we can give an ad hoc proof in our special case by passing to a cover of $\mathcal{P}^n$. Namely, by Corollary 1.3.2.3, there is a cover $\widetilde{\mathcal{P}}^n$ of $\mathcal{P}^n$ in which every lift $\widetilde{L}^n$ of L^n is embedded, so all of our lifted boundary conditions are embedded Lagrangians. Any family of inhomogeneous pseudo-holomorphic disks in $\mathcal{P}^n$ lifts to a family in $\widetilde{\mathcal{P}}^n$. Standard Gromov compactness for the family of lifted disks in $\widetilde{\mathcal{P}}^n$, with boundary on the embedded lifts of Lagrangians, implies compactness for the family in $\mathcal{P}^n$.

Everything else works as in [11], so this allows us to define the Fukaya category of $\mathcal{P}^n$ with the extra object L^n, and show that the A_∞ associativity relations hold.

We now consider the A_∞ algebra $\mathcal{A} = CF^*(L^n, L^n)$. We would like to choose the Floer datum for the pair (L^n, L^n) so that the underlying vector space of $\mathcal{A}$ is as small as possible.

Lemma 1.3.1.10. *There exists a Hamiltonian $H \in \mathcal{H}$ such that $(L^n)^*H$ is a Morse function on S^n with exactly two critical points, and $X_H|_{\mathrm{im}(L^n)}$ vanishes only at those critical points.*

Proof. First define H in a neighbourhood of the self-intersections of $\mathrm{im}(L^n)$, in such a way that X_H is transverse to both branches of the image. This defines $(L^n)^*H$ on a neighbourhood of the critical points p_K of f (see Corollary 1.2.2.11). This function can easily be extended to a Morse function on S^n with the desired properties, then extended to a neighbourhood of $\mathrm{im}(L^n)$, then to all of $\mathcal{P}^n$ using a cutoff function. □

Corollary 1.3.1.11. *For an appropriate choice of Floer datum, $CF^*(L^n, L^n)$ has generators p_K indexed by all subsets $K \subset [n+2]$.*

Proof. We scale the H of Lemma 1.3.1.10 so that it is $\ll \epsilon$ (the parameter in the definition of $L^n = L^n_\epsilon$), and use it as the Hamiltonian part of our Floer datum for (L^n, L^n). Let

X_H denote the corresponding Hamiltonian vector field. Now if ϕ^1 is the time-1 flow of X_H, we can arrange that $\phi^1(L^n(p)) = L^n(q)$ if and only if **either**

$$p = q \text{ and } X_H(L^n(p)) = 0,$$

or

$$(p, q) \text{ corresponds to a pair } (p', q') \text{ such that } p' \neq q' \text{ and } L^n(p') = L^n(q')$$

(note that the assumption that $H \ll \epsilon$ ensures that the transverse self-intersections $L^n(p') = L^n(q')$ persist under the flow of one branch of L^n by X_H).

In the first case, we get generators corresponding to the critical points of the Morse function $(L^n)^*H$. We denote the generator corresponding to the minimum, respectively maximum, by p_ϕ, respectively $p_{[n+2]}$. In the second case, we get generators corresponding to pairs $(p', q') = (p_K, p_{\bar{K}})$ where $K \subset [n+2]$ is proper and non-empty, by Corollary 1.2.2.11. We denote the generator corresponding to $(p_K, p_{\bar{K}})$ by p_K, by slight abuse of notation. □

1.3.2 Weights in M

Definition 1.3.2.1. (Compare [9, Section 8b]) Whenever we have an immersed Lagrangian $L : N \to X$ (such that the image of $H_1(N)$ in $H_1(X)$ is trivial), we can assign a weight $w(y) \in H_1(X)$ to each generator y of $CF^*(L, L)$. Namely, choose a path from $\tilde{y}_1$ to $\tilde{y}_0$ in N, and define $w(y)$ be the homology class obtained by composing the image of this path in X with the path y (see Definition 1.3.1.5).

Proposition 1.3.2.2. *In our case, we have*

$$w(p_K) = e_K \in M \cong H_1(\mathcal{P}^n).$$

Proof. By Proposition 1.2.1.6 and Proposition 1.2.2.7, Arg induces a homotopy equivalence between $(\mathcal{P}^n, L^n)$ and $(\mathcal{C}^n, \partial(\pi Z_n))$. Thus, when K is proper and non-empty, $w(p_K)$ is the class of a path from $\pi e_{\bar{K}}$ to πe_K in $H_1(\mathcal{C}^n) \cong M$, which is exactly e_K. When $K = \phi$ or $[n+2]$ it is clear that $w(p_K) = 0$. $\square$

Corollary 1.3.2.3. *There exists a finite cover $\widetilde{\mathcal{P}}^n \to \mathcal{P}^n$ in which every lift $\widetilde{L}^n$ of L^n is embedded.*

Proof. Recall that $\pi_1(\mathcal{P}^n) \cong M$ by Corollary 1.2.1.7. Consider the group homomorphism

$$\begin{aligned} \rho : M &\to \mathbb{Z}_{n+2} \\ \rho(u) &= e_{[n+2]} \cdot u \end{aligned}$$

(this is well defined because $\rho(e_{[n+2]}) \equiv 0 \pmod{(n+2)}$). There is a corresponding $(n+2)$-fold cover of $\mathcal{P}^n$, and we have

$$\rho(w(p_K)) = \rho(e_K) = |K| \neq 0 \pmod{(n+2)}$$

for all proper non-empty $K \subset [n+2]$, so the two lifts of L^n coming together at an intersection point are distinct. $\square$

Proposition 1.3.2.4. *The A_∞ structure maps μ^k are homogeneous with respect to the weight w. In other words, the coefficient of p_{K_0} in $\mu^k(p_{K_1}, \ldots, p_{K_k})$ is non-zero only if*

$$\sum_{j=1}^{k} e_{K_j} = e_{K_0}.$$

Proof. If the coefficient of p_{K_0} in $\mu^k(p_{K_1}, \ldots, p_{K_k})$ is non-zero, then there is a topological disk in $\mathcal{P}^n$ with boundary on the image of L^n,

$$u : (D, \partial D) \to (\mathcal{P}^n, \mathrm{im}(L^n)),$$

whose boundary changes 'sheets' of L^n exactly at the self-intersection points $p_{K_0}, p_{K_1}, \ldots, p_{K_k}$ in that order (ignoring any appearance p_ϕ or $p_{[n+2]}$ on the list). This disk must lift to the universal cover, hence its boundary lifts to a loop in the universal cover.

The boundary always lies on lifts of L^n, which are indexed by the fundamental group M (think of the homotopy-equivalent picture of $M_{\mathbb{R}} \setminus \{\pi Z_n + 2\pi M\}$, with the lifts of L^n being $\partial(\pi Z_n) + 2\pi M$). When the boundary changes sheets at a point p_K, the index of the sheet in M changes by $w(p_K)$ (observe that the points p_ϕ and $p_{[n+2]}$, at which no sheet-changing occurs, have weight 0).

Therefore, if the boundary of our disc changes sheets at $p_{K_0}, p_{K_1}, \ldots, p_{K_k}$, and comes back to the sheet it started on, we must have

$$-w(p_{K_0}) + \sum_{j=1}^{k} w(p_{K_j}) = 0.$$

□

Corollary 1.3.2.5. *The character group of M,*

$$\mathbb{T} := \mathrm{Hom}(M, \mathbb{C}^*),$$

acts on $\mathcal{A}$ via

$$\alpha \cdot p := \alpha(w(p))p.$$

The A_∞ structure on $\mathcal{A}$ is equivariant with respect to this action.

1.3.3 Grading

Recall that, to lift the $\mathbb{Z}_2$-grading on the Fukaya category to a $\mathbb{Z}$-grading, we must equip $\mathcal{P}^n$ with a complex volume form η. We assume that:

- η is compatible with complex conjugation $\tau : \mathcal{P}^n \to \mathcal{P}^n$, in the sense that $\tau^*\eta = \bar{\eta}$;
- η extends to a meromorphic $(n,0)$-form on $\mathbb{CP}^n$, with a pole of order n_j along the divisor D_j (with the usual convention that a zero of order k is a pole of order $-k$).

We set

$$\boldsymbol{n} := \sum_{j=1}^{n+2} n_j e_j \in \widetilde{M}.$$

Observe that

$$\begin{aligned} \boldsymbol{n} \cdot e_{[n+2]} &= \sum_{j=1}^{n+2} n_j \\ &= \deg(K_{\mathbb{CP}^n}) \\ &= n+1. \end{aligned}$$

Observe that there is no canonical choice for η, so our $\mathbb{Z}$-grading will not be canonical.

Proposition 1.3.3.1. *The $\mathbb{Z}$-grading on $\mathcal{A}$ defined by η is*

$$i(p_K) = (2\boldsymbol{n} - e_{[n+2]}) \cdot e_K.$$

In other words, the coefficient of p_{K_0} in $\mu^k(p_{K_1}, \ldots, p_{K_k})$ is non-zero only if

$$i(p_{K_0}) = 2 - k + \sum_{j=1}^{k} i(p_{K_j}).$$

Proof. Recall that the volume form η defines a function

$$\psi : \mathrm{Gr}(T\mathcal{P}^n) \to S^1,$$

where $\mathrm{Gr}(T\mathcal{P}^n)$ is the Lagrangian Grassmannian of $\mathcal{P}^n$ (i.e., the fibre bundle over $\mathcal{P}^n$ whose fibre over a point p is the set of Lagrangian subspaces of $T_p\mathcal{P}^n$). If $V \subset T_p\mathcal{P}^n$ is a Lagrangian subspace, then $\psi(V)$ is defined by choosing a real basis $v_1, \ldots, v_n$ for V and defining

$$\psi(V) := \arg(\eta(v_1, \ldots, v_n)).$$

A grading on L^n is a function $\alpha^\# : S^n \to \mathbb{R}$ such that

$$\pi\alpha^\#(x) = \psi(L^n_*(T_x S^n))$$

(see [33]). Recall from the construction of L^n that, away from the hypersurfaces $D_j^{\mathbb{R}}$, the immersion $L^n : S^n \to \mathbb{CP}^n$ is close to the double cover of the real locus, $\iota : S^n \to \mathbb{CP}^n$. So away from the hypersurfaces $D_j^{\mathbb{R}}$,

$$\psi(L^n_*(T_x S^n)) \approx \psi(\iota_*(T_x S^n)) = 0 \text{ or } \pi,$$

because we assumed η was invariant under complex conjugation, so $\psi(T\mathbb{RP}^n)$ is real. Therefore, away from the hypersurfaces $D_j^{\mathbb{R}}$, $\alpha^\#$ is approximately an integer.

The hypersurfaces $D_j^{\mathbb{R}}$ split S^n into regions S^n_K indexed by proper non-empty subsets $K \subset [n+2]$. Namely, S^n_K is the region where $x_j < 0$ for $j \in K$ and $x_j > 0$ for $j \notin K$, and contains the unique critical point p_K of f. Suppose that $\alpha^\# \approx \alpha^\#_K \in \mathbb{Z}$ in the region S^n_K.

How does $\alpha^\#_K$ change as we cross a hypersurface $D_j^{\mathbb{R}}$? Let p be a point on $D_j^{\mathbb{R}}$, away from the other hypersurfaces $D_k^{\mathbb{R}}$. Let us choose a holomorphic function q in a neighbourhood of $\iota(p)$ in $\mathbb{CP}^n$, compatible with complex conjugation (i.e., $q(\tau(z)) = \overline{q(z)}$), and such that $D_j = \{q = 0\}$. Because η has a pole of order n_j along D_j, we have

$$\eta = q^{-n_j}\eta',$$

where η' is a holomorphic volume form compatible with complex conjugation.

In the same way that η defines the function ψ, η' defines a function

$$\psi' : \mathrm{Gr}(T\mathbb{CP}^n) \to S^1$$

in a neighbourhood of $\iota(p)$. Whereas ψ is not defined on D_j, because η has a pole there, the function ψ' is defined and continuous on D_j, because η' is holomorphic.

We have

$$\psi = \psi' + \arg(q^{-n_j})$$

away from D_j. We can define real functions $\beta^{\#}_{\epsilon}$ on a neighbourhood of p in S^n, for $\epsilon \geq 0$ sufficiently small, so that

$$\pi\beta^{\#}_{\epsilon}(x) \quad = \quad \psi'((L^n_{\epsilon})_*(T_x S^n)).$$

Because $L^n_0 = \iota$, and η' is compatible with complex conjugation, $\beta^{\#}_0$ is a constant integer. Furthermore, away from $D^{\mathbb{R}}_j$, $L^n_{\epsilon} \approx \iota$, so $\beta^{\#}_{\epsilon} \approx \beta^{\#}_0$. It follows that $\beta^{\#}_{\epsilon}$ approximately does not change as we cross $D^{\mathbb{R}}_j$. So the change in $\alpha^{\#}_K$ as we cross the hypersurface $D^{\mathbb{R}}_j$ comes only from the term $\arg(q^{-n_j})$.

We saw in Proposition 1.2.2.7 that $\mathrm{Arg} \circ L^n$ approximates the boundary of the zonotope Z^n. Thus, as we cross $D^{\mathbb{R}}_j$, moving from S^n_K to $S^n_{K \sqcup \{j\}}$, $\mathrm{Arg} \circ L^n$ changes from πe_K to $\pi e_{K \sqcup \{j\}}$, changing by πe_j. It follows that $\arg(q^{-n_j})$ decreases by πn_j. Therefore, $\alpha^{\#}$ approximately decreases by n_j. So we may assume that

$$\alpha^{\#}_K = -\boldsymbol{n} \cdot e_K.$$

To calculate the index of the generator p_K, we observe that the two sheets of L^n that meet at p_K are locally the graphs of the exact 1-forms df and $-df$. It follows by [33, 2d(v)] that the obvious path connecting the tangent spaces of the two sheets in the

Lagrangian Grassmannian has Maslov index equal to the Morse index

$$\mu_{\text{Morse}}(p_K) = n + 1 - |K| \text{ (see Corollary 1.2.2.13).}$$

We also need to take into account the grading shift of $\alpha_K^{\#} - \alpha_{\bar{K}}^{\#}$ between the two sheets. Using [33, 2d(ii)], we have

$$\begin{aligned} i(p_K) &= \mu_{\text{Morse}}(p_K) - \alpha_K^{\#} + \alpha_{\bar{K}}^{\#} \\ &= n + 1 - |K| + \boldsymbol{n} \cdot e_K - \boldsymbol{n} \cdot e_{\bar{K}} \\ &= n + 1 - e_{[n+2]} \cdot e_K + \boldsymbol{n} \cdot (e_K - e_{[n+2]} + e_K) \\ &= (2\boldsymbol{n} - e_{[n+2]}) \cdot e_K \text{ (since } \boldsymbol{n} \cdot e_{[n+2]} = n + 1). \end{aligned}$$

We also note that this equation works for p_ϕ and $p_{[n+2]}$, which have their usual gradings of 0 and n respectively.

The dimension formula for moduli spaces of holomorphic polygons now says that the dimension of the moduli space of $(k+1)$-gons with boundary on L^n, a positive puncture at p_{K_0}, and negative punctures at $p_{K_1}, \ldots, p_{K_k}$ is

$$\dim(\mathcal{M}_{\mathcal{S}}(p_{K_0}, \ldots, p_{K_k})) = k - 2 + i(p_{K_0}) - \sum_{j=1}^{k} i(p_{K_j}).$$

Since we are counting the 0-dimensional component of the moduli space to determine our A_∞ structure coefficients, this dimension should be 0. This proves the stated formula, i.e., that i defines a valid $\mathbb{Z}$-grading on $\mathcal{A}$.

We also observe that i lifts the $\mathbb{Z}_2$-grading: the two sheets of L^n that meet at p_K are locally the graphs of the exact 1-forms df and $-df$, hence the sign of the intersection is

$$n + 1 + \mu_{\text{Morse}}(p_K) \equiv |K| \equiv (2\boldsymbol{n} - e_{[n+2]}) \cdot e_K \pmod 2.$$

□

Corollary 1.3.3.2. *The A_∞ structure on $\mathcal{A}$ admits the fractional grading*

$$|p_K| := \frac{n}{n+2}|K| \in \mathbb{Q},$$

in the sense that the coefficient of p_{K_0} in $\mu^k(p_{K_1}, \ldots, p_{K_k})$ is non-zero only if

$$2 - k + \sum_{j=1}^{k} \frac{n}{n+2}|K_j| = \frac{n}{n+2}|K_0|.$$

Proof. For any such non-zero product, we have

$$-e_{K_0} + \sum_{j=1}^{k} e_{K_j} = q e_{[n+2]}$$

for some $q \in \mathbb{Z}$ (Proposition 1.3.2.4 says that the image of this sum in M is 0, hence it is a multiple of $e_{[n+2]}$ in $\widetilde{M}$). It then follows from Proposition 1.3.3.1 that

$$i(p_{K_0}) \quad = \quad 2 - k + \sum_{j=1}^{k} i(p_{K_j}).$$

Hence, we ought to have

$$\begin{aligned} k - 2 &= (2\boldsymbol{n} - e_{[n+2]}) \cdot \left(-e_{K_0} + \sum_{j=1}^{k} e_{K_j} \right) \\ &= (2\boldsymbol{n} - e_{[n+2]}) \cdot q e_{[n+2]} \\ &= nq \text{ (since } \boldsymbol{n} \cdot e_{[n+2]} = n+1) \\ &= \frac{n}{n+2} e_{[n+2]} \cdot q e_{[n+2]} \\ &= \frac{n}{n+2} e_{[n+2]} \cdot \left(-e_{K_0} + \sum_{j=1}^{k} e_{K_j} \right) \end{aligned}$$

from which the result follows. □

Corollary 1.3.3.3. *The A_∞ products μ^k are non-zero only when $k = 2 + nq$ (where $q \in \mathbb{Z}_{\geq 0}$).*

Proof. This follows from the final set of equations in the proof of Corollary 1.3.3.2. $\square$

Remark 1.3.3.4. We observe that, when $k = 2 + nq$, we must also have

$$\sum_{j=1}^{2+nq} e_{K_j} = e_{K_0} + q e_{[n+2]}$$

(note: this is an equation in $\widetilde{M}$, not M).

Corollary 1.3.3.5. *μ^1 is trivial, and*

$$\mu^2(p_{K_1}, p_{K_2}) = \begin{cases} a(K_1, K_2) p_{K_1 \sqcup K_2} & \text{if } K_1 \cap K_2 = \phi \\ 0 & \text{otherwise,} \end{cases}$$

where $a(K_1, K_2)$ are some integers.

Proof. The fact that $\mu^1 = 0$ follows immediately from Corollary 1.3.3.3.

For the second part of the Proposition, suppose that the coefficient of p_{K_0} in $\mu^2(p_{K_1}, p_{K_2})$ is non-zero. It follows from Proposition 1.3.2.4 that

$$e_{K_1} + e_{K_2} = e_{K_0}$$

in M, and from Corollary 1.3.3.2 that

$$|K_1| + |K_2| = |K_0|.$$

Therefore $K_0 = K_1 \sqcup K_2$, and the result is proven. $\square$

1.3.4 Signs

The main aim of this section is to prove that the cohomology algebra of $\mathcal{A}$ is graded commutative. The basic reason for this is that complex conjugation $\tau : \mathcal{P}^n \to \mathcal{P}^n$ maps L^n to itself. Given a holomorphic disk $u : S \to \mathcal{P}^n$ contributing to the product $a \cdot b$, the corresponding disk $\bar{u} := \tau \circ u : \bar{S} \to \mathcal{P}^n$ (where $\bar{S}$ denotes the disk S with the conjugate complex structure) contributes to the product $b \cdot a$ with the appropriate relative Koszul sign.

Throughout this section, we use the sign conventions of [11].

Definition 1.3.4.1. Given an A_∞ category $\mathcal{C}$, we define its **opposite category** $\mathcal{C}^{op}$ to be the category with the same objects, the 'opposite' morphisms

$$\hom_{\mathcal{C}^{op}}(A, B) := \hom_{\mathcal{C}}(B, A),$$

and compositions defined by

$$\mu^k_{op}(x_1, \ldots, x_k) := (-1)^* \mu^k(x_k, \ldots, x_1),$$

where

$$* = 1 + \frac{k(k-1)}{2} + (k+1)\left(\sum_{j=1}^{k} i(x_j)\right) + \sum_{j<l} i(x_j) i(x_l).$$

It is an exercise to check that $\mathcal{C}^{op}$ is an A_∞-category.

The following proposition is due to [34]:

Proposition 1.3.4.2. *Let $X = (X, \omega, \eta)$ be an exact symplectic manifold with boundary with symplectic form ω, and complex volume form η. Define $X^{op} := (X, -\omega, \bar{\eta})$. Then there is a quasi-isomorphism of A_∞-categories*

$$\mathcal{G} : \mathcal{F}uk(X)^{op} \to \mathcal{F}uk(X^{op}).$$

Proof. See Appendix A. $\square$

Lemma 1.3.4.3. *We denote by $\mathcal{A}^{op}$ the endomorphism algebra of L^n in $\mathcal{F}uk(\mathcal{P}^n)^{op}$. Suppose that $k = 2 + nq$ and $K_0, \dots, K_k$ are subsets of $[n+2]$ such that*

$$\sum_{j=1}^{k} e_{K_j} = e_{K_0} + q e_{[n+2]}$$

in $\widetilde{M}$ (see Remark 1.3.3.4). Then, in $\mathcal{A}^{op}$, we have

$$\mu_{op}^k(p_{K_1}, \dots, p_{K_k}) = (-1)^* \mu^k(p_{K_k}, \dots, p_{K_1})$$

where

$$* = \frac{nq(nq-1)}{2} + (1+nq)|K_0| + \sum_{1 \le j < l} |K_j||K_l|.$$

Proof. Recall from Proposition 1.3.3.1 that the $\mathbb{Z}_2$-grading of $\mathcal{A}$ is

$$i(p_K) \equiv |K| \pmod 2.$$

As noted in Remark 1.3.3.4, the only contributions to $\mu_{op}^k(p_{K_1}, \dots, p_{K_k})$ or $\mu^k(p_{K_k}, \dots, p_{K_1})$ are proportional to p_{K_0}.

By Definition 1.3.4.1, the result holds with the sign

$$* = 1 + \frac{k(k-1)}{2} + (k+1)\left(\sum_{j=1}^{k} i(p_{K_j})\right) + \sum_{j<l} i(p_{K_j}) i(p_{K_l}).$$

The result follows by substituting $k = 2 + nq$, $i(p_K) = |K| \pmod 2$,

$$|K_1| + \ldots + |K_k| = |K_0| + (n+2)q,$$

and simplifying (modulo 2). $\square$

Corollary 1.3.4.4. *There is a quasi-isomorphism of A_∞ algebras, $\mathcal{A} \to \mathcal{A}^{op}$, which is the identity on the level of cohomology.*

Proof. Observe that complex conjugation induces a symplectomorphism $\tau : (\mathcal{P}^n, \omega) \to (\mathcal{P}^n, -\omega)$ and hence a quasi-isomorphism

$$\mathcal{F}uk(\mathcal{P}^n, \omega) \to \mathcal{F}uk(\mathcal{P}^n, -\omega) \to \mathcal{F}uk(\mathcal{P}^n, \omega)^{op},$$

where the second quasi-isomorphism is given by Proposition 1.3.4.2. We observe that this quasi-isomorphism sends our Lagrangian L^n to $\tau \circ L^n$, which is the same as $L^n \circ a$, where $a : S^n \to S^n$ is the antipodal map, by Remark 1.2.2.6. We note that $w_2(S^n) = 0$ and $H^1(S^n) = 0$ for $n \geq 2$, so there is a unique spin structure $P^\#$ on L^n and we must have $a^*(P^\#) \cong P^\#$. Furthermore, an examination of the proof of Proposition 1.3.3.1 quickly shows that

$$a^*(\alpha^\#) = -(n+1) - \alpha^\#$$

from which it follows that our Lagrangian brane $L^\# = (L^n, \alpha^\#, P^\#)$ gets sent, under the above quasi-isomorphism, to the Lagrangian brane $(L^n, n+1+\alpha^\#, P^\#) = L^\#[n+1]$. In particular, the endomorphism algebra gets sent to

$$\mathcal{A} := CF^*(L^\#, L^\#) \to CF^*(L^\#[n+1], L^\#[n+1])^{op} \cong CF^*(L^\#, L^\#)^{op} = \mathcal{A}^{op}.$$

The isomorphism we have defined sends each orientation line o_x to itself without any sign change, so the result follows from Proposition 1.3.4.2. □

Corollary 1.3.4.5. *The cohomology algebra of $\mathcal{A}$, with product*

$$p_{K_1} \cdot p_{K_2} := (-1)^{|K_1|} \mu^2(p_{K_1}, p_{K_2}),$$

is supercommutative:

$$p_{K_1} \cdot p_{K_2} = (-1)^{|K_1||K_2|} p_{K_2} \cdot p_{K_1}.$$

1.4 A Morse-Bott definition of the Fukaya category

The Fukaya A_∞ category was introduced in [23]. There are a number of approaches to transversality issues in its definition – virtual perturbations are used in [24], and explicit perturbations of the holomorphic curve equation are used in [11].

In this section, we describe a 'Morse-Bott' approach which is a modification of the approach in [11], combining it with the approach of [35]. The outline of this approach has appeared in [20, Section 7], and is related to the 'clusters' of [27]. However, the geometric situation we consider is simpler than that of [27], namely we work only in exact symplectic manifolds with convex boundary, which for example rules out disk and sphere bubbling.

Our treatment follows [11, Sections 8 – 12] closely, explaining at each stage how our construction differs. We make use of concepts and terminology from [11] (including abstract Lagrangian branes, strip-like ends and perturbation data) with minimal explanation. We explain, in Section 1.4.8, why our definition of the Fukaya category is quasi-equivalent to that given in [11].

This section deals only with the Fukaya category of embedded Lagrangians. In particular, the Lagrangian immersion $L^n : S^n \to \mathcal{P}^n$ does not fit into this framework. However, the concepts introduced in this section are the basis for the Morse-Bott computation of $\mathcal{A} = CF^*(L^n, L^n)$ that will be explained in Section 1.5.1.

1.4.1 The domain: pearly trees

In this section, we recall the Deligne-Mumford-Stasheff compactification of the moduli space of disks with boundary punctures, and define the analogous moduli space of pearly trees and its compactification.

Suppose that $k \geq 2$, and $\boldsymbol{L} := (L_0, \dots, L_k)$ is a tuple of Lagrangians in X. We denote by $\mathcal{R}(\boldsymbol{L})$ the moduli space of disks with $k+1$ boundary marked points, modulo biholomorphism, with the components of the boundary between marked points labeled $L_0, \dots, L_k$ in order. The marked point between L_k and L_0 is 'positive', and all other marked points are 'negative'. We call $\boldsymbol{L}$ a set of **Lagrangian labels** for our boundary-marked disk (for the purposes of this section, it is not important that the labels correspond to Lagrangians in X – we need only assign certain labels to the boundary components and keep track of which of the labels are identical).

Definition 1.4.1.1. We denote by $\mathcal{S}(\boldsymbol{L}) \to \mathcal{R}(\boldsymbol{L})$ the universal family of boundary-punctured disks with Lagrangian labels $\boldsymbol{L}$, so that the fibre $\mathcal{S}_r$ over a point $r \in \mathcal{R}(\boldsymbol{L})$ is the corresponding disk, with its boundary marked points removed.

We define

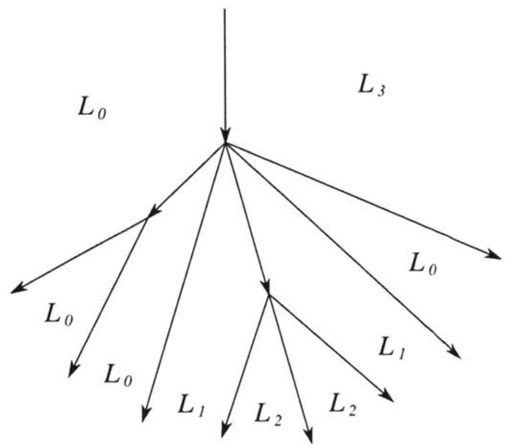
$$Z^{\pm} := \mathbb{R}^{\pm} \times [0,1]$$

with the standard complex structure (where $\mathbb{R}^+, \mathbb{R}^-$ are the positive and negative half-lines respectively). We will use s to denote the $\mathbb{R}^{\pm}$ coordinate and t to denote the $[0,1]$ coordinate. We make a **universal choice of strip-like ends** for the family $\mathcal{S}(\boldsymbol{L}) \to \mathcal{R}(\boldsymbol{L})$, which consists of fibrewise holomorphic embeddings

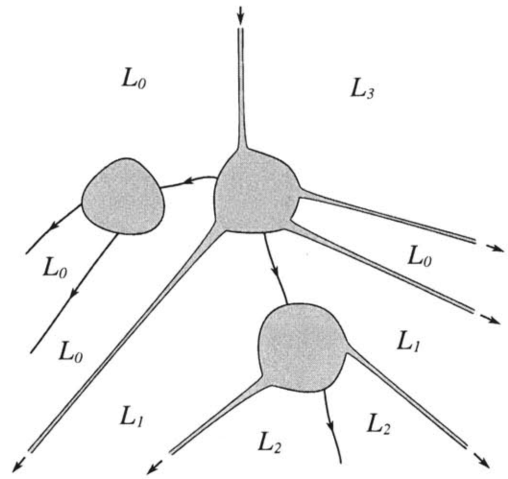
$$\epsilon_j : \mathcal{R}(\boldsymbol{L}) \times Z^{\pm} \to \mathcal{S}(\boldsymbol{L})$$

to a neighbourhood of the jth puncture, for each $j = 0, 1, \dots, k$, where the sign $\pm$ is opposite to the sign of the puncture.

Definition 1.4.1.2. A **directed k-leafed planar tree** T is a directed tree with k semi-infinite 'incoming' edges and one semi-infinite 'outgoing' edge, together with a proper embedding into $\mathbb{R}^2$. Isotopic embeddings are regarded as equivalent. We denote by $V(T)$ the set of vertices of T, by $E(T)$ the set of edges, and by $E_i(T) \subset E(T)$ the set of internal (compact) edges. We say that T has **Lagrangian labels $\boldsymbol{L}$** if the connected components



(a) A k-leafed stable tree T_S is said to have Lagrangian labels $\boldsymbol{L}$ if the connected components of $\mathbb{R}^2 \setminus T$ are labeled by the Lagrangians of $\boldsymbol{L}$, in order. In this figure, $\boldsymbol{L} = (L_0, L_0, L_0, L_1, L_2, L_2, L_1, L_0, L_3)$. A Lagrangian labeling $\boldsymbol{L}$ of T_S induces a labeling $\boldsymbol{L}_v$ of the regions surrounding each vertex v. In this figure, the induced labeling of the regions surrounding the topmost vertex is $\boldsymbol{L}_v = (L_0, L_0, L_1, L_1, L_0, L_3)$.



(b) A pearly tree S, with underlying tree T_S and Lagrangian labels as in Figure 1-1(a). Observe that all edges have the same label on either side, while external strips have different labels on either side.

Figure 1.4.1.1: Pearly trees with Lagrangian labels.

of $\mathbb{R}^2 \setminus T$ are labeled by the Lagrangians of $\boldsymbol{L}$, in order. A Lagrangian labeling $\boldsymbol{L}$ of T induces a labeling $\boldsymbol{L}_v$ of the regions surrounding each vertex $v \in V(T)$ (see Figure 1-1(a)). We call a vertex **stable** if it has valence ≥ 3, and **semi-stable** if it has valence ≥ 2. We call the tree T stable (respectively semi-stable) if all of its vertices are stable (respectively semi-stable).

We define

$$\bar{\mathcal{R}}_T(\boldsymbol{L}) := \left(\prod_{v \in V(T)} \mathcal{R}(\boldsymbol{L}_v) \right) \times (-1, 0]^{E_i(T)}.$$

In other words, $\bar{\mathcal{R}}_T(\boldsymbol{L})$ consists of the data of the planar tree T, a boundary-marked disk $r_v \in \mathcal{R}(\boldsymbol{L}_v)$ for each vertex v, and a **gluing parameter** $\rho_e \in (-1, 0]$ for each internal edge e.

Given an internal edge e of T with gluing parameter $\rho_e \in (-1, 0)$, we can glue the disks r_v at either end of e together along their strip-like ends with gluing parameter ρ_e (corresponding to the 'length' of the gluing region being $l_e := -\log(-\rho_e)$), to obtain an element of $\mathcal{R}_{T/e}(\boldsymbol{L})$ (where T/e denotes the tree obtained from T by contracting the edge e). This defines a **gluing map**

$$\varphi_{T,e} : \{r \in \bar{\mathcal{R}}_T(\boldsymbol{L}) : \rho_e \in (-1, 0)\} \to \bar{\mathcal{R}}_{T/e}(\boldsymbol{L}).$$

Definition 1.4.1.3. We denote by $\bar{\mathcal{R}}(\boldsymbol{L})$ the Deligne-Mumford-Stasheff compactification of $\mathcal{R}(\boldsymbol{L})$ by stable disks:

$$\bar{\mathcal{R}}(\boldsymbol{L}) := \left(\coprod_T \bar{\mathcal{R}}_T(\boldsymbol{L}) \right) / \sim,$$

where

$$r \sim \varphi_{T,e}(r)$$

whenever defined. Given a boundary-punctured disk S with modulus $r \in \mathcal{R}(\boldsymbol{L})$, we call the union of all strip-like ends and gluing regions (under all possible gluing maps) the **thin** part of S, and its complement the **thick** part.

Remark 1.4.1.4. $\bar{\mathcal{R}}$ is the compactification of $\mathcal{R}$ by allowing the gluing parameters ρ_e to take the value 0. This corresponds to allowing the lengths of the gluing regions l_e to be infinite. $\bar{\mathcal{R}}(\boldsymbol{L})$ has the structure of a smooth $(k-2)$-dimensional manifold with corners (where $k := |\boldsymbol{L}| - 1$). The codimension-$d$ boundary strata are indexed by trees T with d internal edges. Namely, T corresponds to the subset of $\bar{\mathcal{R}}_T$ where all d gluing parameters ρ_e are equal to 0.

Definition 1.4.1.5. We denote by $\bar{\mathcal{S}}(\boldsymbol{L}) \to \bar{\mathcal{R}}(\boldsymbol{L})$ the partial compactification of the universal family $\mathcal{S}(\boldsymbol{L}) \to \mathcal{R}(\boldsymbol{L})$ of boundary-punctured disks by stable boundary-punctured disks.

In [11], the coefficients of the A_∞ structure maps

$$\mu^k : CF^*(L_{k-1}, L_k) \otimes \ldots \otimes CF^*(L_0, L_1) \to CF^*(L_0, L_k)$$

are defined by counts of (appropriately perturbed) holomorphic curves $u : \mathcal{S}_r(\boldsymbol{L}) \to X$ for some $r \in \mathcal{R}(\boldsymbol{L})$. The structure of the codimension-1 boundary of $\bar{\mathcal{R}}(\boldsymbol{L})$ leads to the A_∞ associativity equations.

When no two of the Lagrangians in $\boldsymbol{L}$ coincide, we define the A_∞ structure maps in exactly the same way. However, when some of the Lagrangians in $\boldsymbol{L}$ coincide, we alter this definition.

Definition 1.4.1.6. A **pearly tree** S with Lagrangian labels $\boldsymbol{L}$ is specified by the following data:

- A stable directed k-leafed planar tree T_S (the **underlying tree** of S) with Lagrangian labels $\boldsymbol{L}$, such that the labels on either side of an internal edge are identical;
- For each vertex v, a point $r_v \in \mathcal{R}(\boldsymbol{L}_v)$;
- For each internal edge e, a length parameter $l_e \in [0, \infty)$.

We denote by $V(S)$ the set of vertices of the tree T_S, and by $E_L(S)$ the set of edges of T_S with both sides labeled L (internal or external). For each vertex $v \in V(S)$, we define S_v to be the boundary-marked disk with modulus r_v, with all marked points between distinct Lagrangians punctured (but all marked points between identical Lagrangians remain). These are the '**pearls**'. We define

$$S^p := \coprod_{v \in V(S)} S_v.$$

For each internal edge e, we define $S_e := [0, l_e]$. For each external edge e with opposite sides labeled by the same Lagrangian, we define $S_e := \mathbb{R}^{\pm}$, depending on the orientation of the edge. For each Lagrangian $L \in \boldsymbol{L}$, we define

$$S^e(L) := \coprod_{e \in E_L(S)} S_e,$$

and S^e to be the disjoint union of $S^e(L)$ over all L. For each $L \in \boldsymbol{L}$, we define $F_L(S)$ to be the set of flags of T_S with both sides labeled by the same Lagrangian L. We define $F(S)$ to be the union of all $F_L(S)$. For each $f \in F_L(S)$, there is a corresponding marked point on a boundary component of S^p with Lagrangian label L, which we denote by $m(f) \in S^p$. Also corresponding to f, there is a point $b(f) \in S^e(L)$, which is the boundary point of the edge corresponding to the flag f. We finally define

$$S := (S^p \sqcup S^e)/\sim$$

where

$$m(f) \sim b(f) \text{ for all } f \in F(S)$$

(see Figure 1-1(b)).

We now define a topology on the moduli space of pearly trees.

Suppose we are given a stable directed k-leafed planar tree T with Lagrangian labels $\boldsymbol{L}$. If the labels on opposite sides of an edge are distinct, we call the edge a **strip edge**, and if they are identical, we call it a **Morse edge**. We denote by $E_{i,s}(T) \subset E(T)$ the internal strip edges, and $E_{i,M}(T) \subset E(T)$ the internal Morse edges. We define

$$\mathcal{R}_T^{pt}(\boldsymbol{L}) := \left(\prod_{v \in V(T)} \mathcal{R}(\boldsymbol{L}_v) \right) \times (-1,0)^{E_{i,s}(T)} \times (-1,1)^{E_{i,M}(T)}$$

('*pt*' stands for 'pearly tree').

As before, for any internal edge $e \in E_i(T)$, we have a 'gluing map'

$$\varphi_{T,e} : \{r \in \mathcal{R}^{pt}_T(\boldsymbol{L}) : \rho_e \in (-1,0)\} \to \mathcal{R}^{pt}_{T/e}(\boldsymbol{L}).$$

The only difference from the previous construction is that the gluing parameter ρ_e now takes values in $(-1,1)$, rather than $(-1,0)$, for e an internal Morse edge.

Definition 1.4.1.7. We define $\mathcal{R}^{pt}(\boldsymbol{L})$, the moduli space of **pearly trees** with Lagrangian labels $\boldsymbol{L}$:

$$\mathcal{R}^{pt}(\boldsymbol{L}) := \left(\coprod_T \mathcal{R}^{pt}_T(\boldsymbol{L})\right) / \sim,$$

where

$$r \sim \varphi_{T,e}(r)$$

whenever defined. A point $r \in \mathcal{R}^{pt}(\boldsymbol{L})$ corresponds to a pearly tree S_r as follows: we glue along any edge with gluing parameter < 0, so that we get a tree T_S whose only internal edges are Morse edges with gluing parameter $\rho_e \in [0,1)$. We regard these as edges having length parameter

$$l_e := -\log(1-\rho_e)$$

(see Figure 1.4.1.2). This defines a topology on the moduli space $\mathcal{R}^{pt}(\boldsymbol{L})$. Again, we define the **thin** part of S^p to be the union of all strip-like ends and gluing regions (including a strip neighbourhood of each boundary marked point), and the **thick** part of S^p to be its complement.

Remark 1.4.1.8. We could have defined $\mathcal{R}^{pt}(\boldsymbol{L})$ without any reference to strip edges at all, since we can glue along all strip edges. However this would not allow us to define the thick and thin regions, and we will need to consider strip edges soon anyway when we define the compactification of $\mathcal{R}^{pt}(\boldsymbol{L})$.

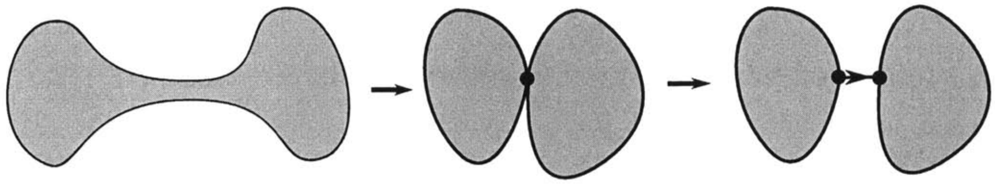


Figure 1.4.1.2: In this figure, we show what happens as the gluing parameter ρ_e for a Morse edge in a pearly tree passes from negative to positive. On the left, $\rho_e < 0$, and we have a 'thin' region in our disk, corresponding to the edge e. As $\rho_e \to 0^-$, the thin region's length becomes infinite, until at $\rho_e = 0$ we have a stable disk (middle picture). On the right, $\rho_e > 0$, and we have two distinct disks connected by an edge of length $l_e = -\log(1 - \rho_e)$. As $\rho_e \to 0^+$, the edge's length goes to 0, until at $\rho_e = 0$ we have the same stable disk.

Definition 1.4.1.9. We denote by

$$\begin{aligned} \mathcal{S}^p(\boldsymbol{L}) &\to \mathcal{R}^{pt}(\boldsymbol{L}), \\ \mathcal{S}^e_L(\boldsymbol{L}) &\to \mathcal{R}^{pt}(\boldsymbol{L}) \text{ (for } L \in \boldsymbol{L}\text{), and} \\ \mathcal{S}^{pt}(\boldsymbol{L}) &\to \mathcal{R}^{pt}(\boldsymbol{L}) \end{aligned}$$

the universal families with fibre $S^p_r, S^e_r(L)$ and S_r respectively, over a point $r \in \mathcal{R}^{pt}(\boldsymbol{L})$.

Definition 1.4.1.10. We define a universal choice of strip-like ends for the family $\mathcal{S}^{pt}(\boldsymbol{L}) \to \mathcal{R}^{pt}(\boldsymbol{L})$ to consist of the embeddings

$$\epsilon_j : \mathcal{R}^{pt}(\boldsymbol{L}) \times Z^{\pm} \to \mathcal{S}^{pt}(\boldsymbol{L})$$

for each external strip edge, coming from our universal choice of strip-like ends for families of boundary-punctured disks, and

$$\epsilon_j : \mathcal{R}^{pt}(\boldsymbol{L}) \times \mathbb{R}^{\pm} \to \mathcal{S}^{pt}(\boldsymbol{L})$$

which are parametrisations of the corresponding external Morse edges (where the sign $\pm$ is determined by the orientation of the edge).

Definition 1.4.1.11. Given a tree T_S as above, and a subset $B \subset E(T_S)$, we define

$\mathcal{R}^{pt}(T_S, B) \subset \mathcal{R}^{pt}(\boldsymbol{L})$ to be the images of pearly trees S with underlying tree T_S, with gluing parameter $\rho_e = 0$ for $e \in B$ and $\rho_e > 0$ for $e \notin B$ (of course this depends on the Lagrangian labels, but we omit $\boldsymbol{L}$ from the notation for readability). Each pearly tree $r \in \mathcal{R}^{pt}(\boldsymbol{L})$ lies in a unique subset $\mathcal{R}^{pt}(T_S, B)$.

Definition 1.4.1.12. Given (T_S, B) as in Definition 1.4.1.11, we define the universal family

$$\mathcal{S}^{pt}(T_S, B) \to \mathcal{R}^{pt}(T_S, B).$$

We now define the compactification of $\mathcal{R}^{pt}(\boldsymbol{L})$. Let

$$\bar{\mathcal{R}}^{pt}_T(\boldsymbol{L}) := \left(\prod_{v \in V(T)} \mathcal{R}(\boldsymbol{L}_v) \right) \times (-1, 0]^{E_{i,s}(T)} \times (-1, 1]^{E_{i,M}(T)}.$$

Note that $\bar{\mathcal{R}}^{pt}_T(\boldsymbol{L})$ contains $\mathcal{R}^{pt}_T(\boldsymbol{L})$ as a dense open subset.

Definition 1.4.1.13. We define the the compactification of $\mathcal{R}^{pt}(\boldsymbol{L})$, the moduli space of **stable pearly trees**,

$$\bar{\mathcal{R}}^{pt}(\boldsymbol{L}) := \left(\coprod_T \bar{\mathcal{R}}^{pt}_T(\boldsymbol{L}) \right) / \sim,$$

where

$$r \sim \varphi_{T,e}(r)$$

whenever defined. We also define the universal family $\bar{\mathcal{S}}^{pt}(\boldsymbol{L}) \to \bar{\mathcal{R}}^{pt}(\boldsymbol{L})$ of stable pearly trees.

Remark 1.4.1.14. In the spaces $\bar{\mathcal{R}}^{pt}_T$, the gluing parameters of strip (respectively Morse) edges can take the value 0 (respectively 1). This corresponds to the length of the gluing region l_e becoming infinite (respectively, the length of the edge l_e becoming infinite). Thus, we are essentially compactifying by allowing the pearls to be stable disks, and the Morse edges to have infinite length. $\bar{\mathcal{R}}^{pt}(\boldsymbol{L})$ has the structure of a smooth $(k-2)$-manifold with corners. The codimension-d boundary strata are indexed by trees T with

Lagrangian labels $\boldsymbol{L}$ and d internal edges. Namely, the boundary stratum corresponding to T is the image of the subset of $\bar{\mathcal{R}}^{pt}_T(\boldsymbol{L})$ where all gluing parameters ρ_e are 0 for strip edges and 1 for Morse edges.

Remark 1.4.1.15. $\bar{\mathcal{R}}^{pt}(\boldsymbol{L})$ is obtained from the usual Deligne-Mumford-Stasheff compactification $\bar{\mathcal{R}}(\boldsymbol{L})$ by adding a 'collar' along each boundary stratum corresponding to a tree with a Morse edge in it.

Naïvely, the structure coefficients of the usual Fukaya category count rigid holomorphic disks $u : \mathcal{S}_r \to X$ for some $r \in \mathcal{R}(\boldsymbol{L})$. In reality, we must perturb the J-holomorphic curve equation to achieve transversality, in particular when two of the Lagrangian boundary conditions coincide. In [11], the equation is perturbed by allowing modulus- and domain-dependent almost-complex structures and Hamiltonian perturbations.

We would like to alter the definition of the Fukaya category so that the structure coefficients are counts of rigid 'holomorphic pearly trees' $u : S_r \to X$ for some $r \in \mathcal{R}^{pt}(\boldsymbol{L})$. Naïvely, a holomorphic pearly tree is a map which is holomorphic on the pearls and given by the Morse flow of some Morse function on the corresponding Lagrangian on each edge. Again, in reality, we have to perturb the holomorphic curve and Morse flow equations by modulus- and domain-dependent perturbations in order to achieve transversality. We describe how to do this in Sections 1.4.2-1.4.4.

1.4.2 Floer data and morphism spaces

Recall, from Section 1.3.1, that we define the Fukaya category of a symplectic manifold (X, ω) with the following properties and structures:

- $\omega = d\theta$ is exact;
- X is equipped with an almost-complex structure J_0, compatible with ω;

- X is convex at infinity, in the sense that there is a bounded below, proper function $h : X \to \mathbb{R}$ such that

$$\theta = -dh \circ J_0;$$

- X is equipped with a complex volume form η (note: we will not take a quadratic complex volume form as in [11], because we will assume our Lagrangians to be oriented).

An object of the Fukaya category of X is a compact, exact, embedded Lagrangian brane $L^{\#}$ (we will neglect the superscript $\#$, denoting the brane structure, for notational convenience).

Definition 1.4.2.1. We define

$$\mathcal{H} := C_c^\infty(X, \mathbb{R}),$$

the space of smooth, compactly supported functions on X (think of this as the space of Hamiltonians), and $\mathcal{J}$, the space of smooth almost-complex structures on X compatible with ω, and equal to the standard complex structure J_0 outside of some compact set. For future use, for each Lagrangian L, we define

$$\mathcal{V}_L := C^\infty(L, TL),$$

the space of smooth vector fields on L.

Definition 1.4.2.2. For each **distinct** pair of objects (L_0, L_1), we choose a **Floer datum** (H_{01}, J_{01}) consisting of

$$H_{01} \in C^\infty([0,1], \mathcal{H}) \text{ and } J_{01} \in C^\infty([0,1], \mathcal{J})$$

satisfying the following property: if ϕ^t denotes the flow of the Hamiltonian vector field of the (time-dependent) Hamiltonian H_{01}, then the time-1 flow $\phi^1(L_0)$ is transverse to

L_1. One then defines a **generator** of $CF^*(L_0, L_1)$ to be a path $y : [0,1] \to X$ which is a flowline of the Hamiltonian vector field of H_{01}, such that $y(0) \in L_0$ and $y(1) \in L_1$ (these correspond to the transverse intersections of $\phi^1(L_0)$ with L_1). One defines $CF^*(L_0, L_1)$ to be the $\mathbb{C}$-vector space generated by its generators. It is $\mathbb{Z}$-graded, as explained in [11, Chapter 11, 12].

In [11], the case $L_0 = L_1$ is treated identically, but we will do something different.

Definition 1.4.2.3. A **Floer datum** for a pair of identical Lagrangians (L, L) is a Morse-Smale pair (h_L, g_L) consisting of a Morse function $h_L : L \to \mathbb{R}$ and a Riemannian metric g_L on L. One then defines $CF^*(L, L) := C^*_M(L)$, the $\mathbb{C}$-vector space generated by critical points of h_L. It is $\mathbb{Z}$-graded by the Morse index.

Remark 1.4.2.4. Intuitively, one should think of this as a limiting case of Definition 1.4.2.2. Namely, we could choose the almost-complex structure part of the perturbation datum to be a time-independent $J \in \mathcal{J}$ which, when combined with ω, induces a Riemannian metric whose restriction to L is g_L. We could then choose the Hamiltonian part of the perturbation datum to be a time-independent function ϵH, where $H|_L = h_L$, and consider the limit $\epsilon \to 0$.

Definition 1.4.2.5. Given a set of Lagrangian labels $\boldsymbol{L} = (L_0, \ldots, L_k)$, an **associated set of generators** is a tuple

$$\boldsymbol{y} = (y_0, \ldots, y_k),$$

where y_j is a generator of $CF^*(L_{j-1}, L_j)$ for each $1 \leq j \leq k$, and y_0 is a generator of $CF^*(L_0, L_k)$. We denote the grading of a generator y by $i(y)$, and define

$$i(\boldsymbol{y}) := i(y_0) - \sum_{j=1}^{k} i(y_j).$$

1.4.3 Perturbation data for fixed moduli

For the purposes of this section, let S be a pearly tree with Lagrangian labels $\boldsymbol{L}$ and fixed modulus $r \in \mathcal{R}^{pt}(\boldsymbol{L})$.

Definition 1.4.3.1. A **perturbation datum** for S consists of the data (K, J, V), where:

- $K \in \Omega^1(S^p, \mathcal{H})$;
- $J \in C^\infty(S^p, \mathcal{J})$;
- V is a tuple of maps $V_L \in C^\infty(S^e(L), \mathcal{V}_L)$ for each $L \in \boldsymbol{L}$,

such that

$$K(\xi)|_{L_C} = 0 \text{ for all } \xi \in TC \subset T(\partial S^p)$$

for each boundary component C of a pearl in S with Lagrangian label L_C.

We also impose a requirement that the perturbation datum be compatible with the Floer data on the strip-like ends, in the following senses:

$$\epsilon_j^* K = H_{j-1,j}(t)dt, \;\; J(\epsilon_j(s,t)) = J_{j-1,j}(t)$$

on each external strip edge;

$$V_{L_j}(\epsilon_j(s)) = \nabla h_{L_j}$$

on each external Morse edge.

Definition 1.4.3.2. Given a pearly tree S with Lagrangian labels $\boldsymbol{L}$ and a perturbation datum (K, J, V), a **holomorphic pearly tree** (or more properly, an inhomogeneous pseudo-holomorphic pearly tree) in X with domain S is a collection $\boldsymbol{u}$ of smooth maps

$$\begin{aligned} u_p : S^p &\to X \text{ and} \\ u_L : S^e(L) &\to L \text{ for all } L \text{ in } \boldsymbol{L}, \end{aligned}$$

satisfying

$$\begin{aligned}
u_p(C) &\in L_C \text{ for each boundary component } C \text{ of } S^p \text{ with label } L_C; \\
u_p(m(f)) &= u_L(b(f)) \text{ for all } f \in F_L(S), \text{ for all } L; \\
(Du_p - Y)^{0,1} &= 0 \text{ on } S^p; \\
Du_L - V &= 0 \text{ on } S^e(L), \text{ for all } L,
\end{aligned}$$

where, for $\xi \in TS$, $Y(\xi)$ is the Hamiltonian vector field of the function $K(\xi)$. Note that the second condition says exactly that $\boldsymbol{u}$ defines a continuous map $S \to X$.

Definition 1.4.3.3. Given $\boldsymbol{y} = (y_-, y_+)$, where $y_\pm$ are generators of $CF^*(L_0, L_1)$, we define the moduli space $\mathcal{M}_Z(\boldsymbol{y})$ of solutions of the holomorphic pearly tree equation with domain $Z = \mathbb{R} \times [0,1]$ (if $L_0 \neq L_1$) or $\mathbb{R}$ (if $L_0 = L_1$), translation-invariant perturbation datum given by the corresponding Floer datum, and asymptotic conditions

$$\lim_{s \to \pm\infty} u(s,t) = y_\pm(t)$$

if $L_0 \neq L_1$, and the same without the t variable if $L_0 = L_1$. We define $\mathcal{M}^*_Z(\boldsymbol{y}) := \mathcal{M}_Z(\boldsymbol{y})/\mathbb{R}$, where $\mathbb{R}$ acts by translation in the s variable.

It is standard (see [36, 37]) that the moduli spaces $\mathcal{M}^*_Z(\boldsymbol{y})$ are smooth manifolds for generic choice of Floer data, and their dimension is $i(\boldsymbol{y}) - 1$.

Definition 1.4.3.4. Suppose that $k \geq 2$. Given a pearly tree S with Lagrangian labels $\boldsymbol{L} = (L_0, \ldots, L_k)$, associated generators $\boldsymbol{y} = (y_0, \ldots, y_k)$, and a perturbation datum, we consider the moduli space $\mathcal{M}_S(\boldsymbol{y})$ of holomorphic pearly trees with domain S, such that

$$\lim_{s \to +\infty} u(\epsilon_j(s,t)) = y_j(t)$$

and

$$\lim_{s \to -\infty} u(\epsilon_0(s,t)) = y_0(t)$$

on external strip edges, and the same (without the t variable) on external Morse edges.

We wish to show that the moduli spaces $\mathcal{M}_S(\boldsymbol{y})$ form smooth, finite-dimensional manifolds for a generic choice of perturbation datum.

Definition 1.4.3.5. Fix $2 < p < \infty$ and define the Banach manifold $\mathcal{B}_S(\boldsymbol{y})$ to consist of collections of maps

$$\boldsymbol{u} = (u_p, \boldsymbol{u_L}) \in W^{1,p}_{loc}(S^p, X) \times \prod_{L \in \boldsymbol{L}} W^{1,p}_{loc}(S^e(L), L)$$

such that

$$u_p(C) \in L_C$$

for each boundary component C of S^p with label L_C, and $\boldsymbol{u}$ converges in $W^{1,p}$-sense to y_j on the jth strip-like end. These boundary and asymptotic conditions make sense because $W^{1,p}$ injects into the space of continuous functions. Henceforth we omit the $\boldsymbol{y}$ from the notation for readability. Note that the tangent space to $\mathcal{B}_S$ is

$$T_{\boldsymbol{u}}\mathcal{B}_S = W^{1,p}(S^p, u_p^*TX, u_p^*TL_C) \oplus \bigoplus_{L \in \boldsymbol{L}} W^{1,p}(S^e(L), u_L^*TL),$$

where for the first component we have used the notation $W^{1,p}(S^p, E, F)$ for the space of $W^{1,p}$ sections of a vector bundle E over S, whose restriction to the boundary lies in the distribution $F \subset E|_{\partial S^p}$.

Definition 1.4.3.6. The maps $\boldsymbol{u} \in \mathcal{B}_S$ are not necessarily continuous at the points where edges join onto pearls. We define

$$\boldsymbol{L}^{F(S)} := \prod_{L \in \boldsymbol{L}} L^{F_L(S)}.$$

Then there are evaluation maps

$$\begin{aligned} \boldsymbol{ev}_m : \mathcal{B}_S &\to \boldsymbol{L}^{F(S)} \\ \boldsymbol{ev}_m(\boldsymbol{u}) &:= (u_p(m(f)))_{f \in F(S)} \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{ev}_b : \mathcal{B}_S &\to \boldsymbol{L}^{F(S)} \\ \boldsymbol{ev}_b(\boldsymbol{u}) &:= (u_L(b(f)))_{f \in F_L(S)}. \end{aligned}$$

We define

$$\begin{aligned} \boldsymbol{ev} : \mathcal{B}_S &\to \boldsymbol{L}^{F(S)} \times \boldsymbol{L}^{F(S)} \\ \boldsymbol{ev} &:= (\boldsymbol{ev}_m, \boldsymbol{ev}_b). \end{aligned}$$

We also define

$$\Delta^S \subset \boldsymbol{L}^{F(S)} \times \boldsymbol{L}^{F(S)}$$

to be the diagonal. An element $\boldsymbol{u} \in \mathcal{B}_S$ is continuous at the points where edges join onto pearls if and only if $\boldsymbol{u} \in \boldsymbol{ev}^{-1}(\Delta^S)$. We define the linearization of $\boldsymbol{ev}$,

$$D(\boldsymbol{ev}) : T_{\boldsymbol{u}}\mathcal{B}_S \to T_{\boldsymbol{ev}(\boldsymbol{u})}\left(\boldsymbol{L}^{F(S)} \times \boldsymbol{L}^{F(S)}\right).$$

Given a point $\boldsymbol{u} \in \boldsymbol{ev}^{-1}(\Delta^S)$, we define the projection of the linearization to the normal bundle of the diagonal,

$$\begin{aligned} D^{ev}_{S,\boldsymbol{u}} : T_{\boldsymbol{u}}\mathcal{B}_S &\to T_{\boldsymbol{ev}_m(\boldsymbol{u})}\boldsymbol{L}^{F(S)}, \\ D^{ev}_{S,\boldsymbol{u}} &:= D(\boldsymbol{ev}_m) - D(\boldsymbol{ev}_b). \end{aligned}$$

Definition 1.4.3.7. Define the Banach vector bundle $\mathcal{E}_S(\boldsymbol{y}) \to \mathcal{B}_S(\boldsymbol{y})$ whose fibre over

$\boldsymbol{u}$ (again omitting the $\boldsymbol{y}$ from the notation) is the space

$$(\mathcal{E}_S)_{\boldsymbol{u}} := L^p(S^p, \Omega_S^{0,1} \otimes u_p^* TX) \oplus \bigoplus_{L \in \boldsymbol{L}} L^p(S^e(L), u_L^* TL).$$

There is a smooth section

$$\begin{aligned} d_S : \mathcal{B}_S &\rightarrow \mathcal{E}_S \\ d_S(\boldsymbol{u}) &= ((Du_p - Y)^{0,1}, (D\boldsymbol{u}_{\boldsymbol{L}} - V)). \end{aligned}$$

We denote the linearization of d_S at $\boldsymbol{u}$ by

$$D^h_{S,\boldsymbol{u}} : T_{\boldsymbol{u}}\mathcal{B}_S \rightarrow (\mathcal{E}_S)_{\boldsymbol{u}}$$

(the 'h' stands for 'holomorphic').

Note that $\mathcal{M}_S(\boldsymbol{y}) = (\boldsymbol{ev}, d_S)^{-1}(\Delta^S, \boldsymbol{0})$ (where $\boldsymbol{0}$ denotes the zero section of the Banach vector bundle $\mathcal{E}_S(\boldsymbol{y})$).

Definition 1.4.3.8. Given $\boldsymbol{u} \in \mathcal{M}_S(\boldsymbol{y})$, we denote by

$$D_{S,\boldsymbol{u}} : T_{\boldsymbol{u}}\mathcal{B}_S \rightarrow T_{\boldsymbol{ev}_m(\boldsymbol{u})}\boldsymbol{L}^{F(S)} \oplus (\mathcal{E}_S)_{\boldsymbol{u}}$$

the projection of the linearization

$$D_{\boldsymbol{u}}(\boldsymbol{ev}, d_S)$$

to the normal bundle of $(\Delta^S, \boldsymbol{0})$. It is given by

$$D_{S,\boldsymbol{u}} = D^{ev}_{S,\boldsymbol{u}} \oplus D^h_{S,\boldsymbol{u}}.$$

We say that $\boldsymbol{u} \in \mathcal{M}_S(\boldsymbol{y})$ is **regular** if $D_{S,\boldsymbol{u}}$ is surjective, and that $\mathcal{M}_S(\boldsymbol{y})$ is regular if every $\boldsymbol{u} \in \mathcal{M}_S(\boldsymbol{y})$ is regular.

It is standard that the operator $D^h_{S,\boldsymbol{u}}$ is Fredholm (compare [11, Section 8i] for the pearls, and [38, Section 2.2] for the edges). Therefore, $D_{\boldsymbol{u}}(\boldsymbol{ev}, d_S)$ is Fredholm also, because the codomain of $\boldsymbol{ev}$ is finite-dimensional. So the map $(\boldsymbol{ev}, d_S)$ is Fredholm. Thus, if $\mathcal{M}_S(\boldsymbol{y})$ is regular, then it is a smooth manifold with dimension given by the Fredholm index of $D_{S,\boldsymbol{u}}$ at each point.

It will follow from our arguments in Section 1.4.6 that, for a generic choice of perturbation datum, $\mathcal{M}_S(\boldsymbol{y})$ is regular.

1.4.4 Perturbation data for families

To define the Fukaya category, we must count moduli spaces of holomorphic pearly trees with varying domain, rather than a fixed domain as in Section 1.4.3. The first step is to define perturbation data for the whole family $\mathcal{S}^{pt}(\boldsymbol{L}) \to \mathcal{R}^{pt}(\boldsymbol{L})$. The following definition is the appropriate notion of a smoothly varying family of perturbation data for each fibre S_r.

Definition 1.4.4.1. A **perturbation datum** for the family $\mathcal{S}^{pt}(\boldsymbol{L}) \to \mathcal{R}^{pt}(\boldsymbol{L})$ consists of the data (K, J, V), where:

- $K \in \Omega^1_{\mathcal{S}^p/\mathcal{R}^{pt}}(\mathcal{S}^p, \mathcal{H})$;
- $J \in C^\infty(\mathcal{S}^p, \mathcal{J})$;
- V is a tuple of maps $V_L \in C^\infty(\mathcal{S}^e_L, \mathcal{V}_L)$ for each $L \in \boldsymbol{L}$,

such that the restriction of (K, J, V) to each fibre S_r is a perturbation datum. We furthermore require some additional, somewhat artificial, conditions to deal with the structure of the moduli space near a point with an edge of length 0 (the situation illustrated in Figure 1.4.1.2). Namely, for any edge e, we require:

- $V|_{S_e} = 0$ whenever $l_e \in [0, 1]$;
- the perturbation data do not change as l_e varies between 0 and 1 (keeping all other parameters fixed);
- $V|_{S_e} = \nabla h_L$ whenever $l_e \geq 2$;
- $K \equiv 0$, and J is constant, on a neighbourhood of each Morse edge of length 0. To see what this means, look at Figure 1.4.1.2: we require that $K \equiv 0$ and J has one fixed value on the long strip on the left, and in a neighbourhood of the boundary marked points at opposite ends of the edge on the right.

We impose the condition $V|_{S_e} = 0$ on edges of length $l_e \leq 1$ because it makes the following Lemma true (a similar trick is used in [35]):

Lemma 1.4.4.2. *Suppose that we have chosen a perturbation datum in accordance with Definition 1.4.4.1, and that $S = S_r$ is a pearly tree with an edge e of length $l_e < 1$. Let $S' = \mathcal{S}^{pt}_{r'}$ denote the pearly tree that is identical to S, except we shrink the edge e to have length $l_e = 0$. Then there is a canonical isomorphism*

$$\mathcal{M}_S(\boldsymbol{y}) \equiv \mathcal{M}_{S'}(\boldsymbol{y})$$

(where both are defined using the restriction of the perturbation datum on $\mathcal{S}^{pt}$ to the fibres S, S').

Proof. The result is clear from the holomorphic pearly tree equation (see Definition 1.4.3.2): because $V|_{S_e} = 0$ for $l_e \in [0, 1]$, the corresponding map $\boldsymbol{u}|_{S_e} : [0, l_e] \to L$ is necessarily constant. Thus the part of the holomorphic pearly tree equation on the edge e reduces to a point constraint, regardless of l_e. Because the perturbation datum does not change as we vary $l_e \in [0, 1]$, the equation on the rest of S does not change, so $\mathcal{M}_S(\boldsymbol{y})$ and $\mathcal{M}_{S'}(\boldsymbol{y})$ can be canonically identified. $\square$

Definition 1.4.4.3. Given a set of Lagrangian labels $\boldsymbol{L} = (L_0, \ldots, L_k)$, associated generators $\boldsymbol{y}$, and a perturbation datum, we consider the moduli space

$$\mathcal{M}_{\mathcal{S}^{pt}}(\boldsymbol{y}) := \{(r, \boldsymbol{u}) : r \in \mathcal{R}^{pt}(\boldsymbol{L}) \text{ and } \boldsymbol{u} \in \mathcal{M}_{S_r}(\boldsymbol{y})\}.$$

We now aim to show that $\mathcal{M}_{\mathcal{S}^{pt}}(\boldsymbol{y})$ is a manifold (whether it is possible to construct a smooth manifold structure is unclear, but this is irrelevant for the purposes of defining the Fukaya category). The complicated part of this is to understand what happens in a neighbourhood of the Morse edges of zero length, because the nature of the domain changes at those points. We start by explaining what happens away from the Morse edges of zero length (i.e., when the modulus $r \in \mathcal{R}^{pt}(T_S, B)$ where $B = \phi$).

Definition 1.4.4.4. Let $U \subset \mathcal{R}^{pt}(\boldsymbol{L})$ be a small connected open subset which makes the strip-like ends constant and avoids a neighbourhood of the pearly trees with some Morse edge of length 0. We define the trivial Banach fibre bundle $\mathcal{B}_{\mathcal{S}^{pt}|_U}(\boldsymbol{y}) \to U$ whose fibre over $r \in U$ is the Banach manifold $\mathcal{B}_{S_r}(\boldsymbol{y})$ defined in Definition 1.4.3.5. There is a Banach vector bundle $\mathcal{E}_{\mathcal{S}^{pt}|_U}(\boldsymbol{y}) \to \mathcal{B}_{\mathcal{S}^{pt}|_U}(\boldsymbol{y})$ whose restriction (omitting the $\boldsymbol{y}$ from the notation) to $\mathcal{B}_{S_r}$ is the Banach vector bundle $\mathcal{E}_{S_r}$ defined in Definition 1.4.3.7. It has a smooth section $d_{\mathcal{S}^{pt}|_U}$ given, over $\mathcal{B}_{S_r}$, by the section d_{S_r} of Definition 1.4.3.7. We have

$$\mathcal{M}_{\mathcal{S}^{pt}|_U}(\boldsymbol{y}) = (\boldsymbol{ev}|_U, d_{\mathcal{S}^{pt}|_U})^{-1}(\Delta^S, \boldsymbol{0})$$

(note that the codomain of $\boldsymbol{ev}$ depends on the underlying tree T_S of S_r; our requirement that U be connected and avoid Morse edges of length 0 ensures that T_S is constant on U). Given $(r, \boldsymbol{u}) \in \mathcal{M}_{\mathcal{S}^{pt}}(\boldsymbol{y})$ with $r \in U$, we denote the linearization of $d_{\mathcal{S}^{pt}|_U}$ at $(r, \boldsymbol{u})$ by

$$D^h_{\mathcal{S}^{pt}|_U, r, \boldsymbol{u}} : T_{(r,\boldsymbol{u})}\left(\mathcal{B}_{\mathcal{S}^{pt}|_U}\right) \to (\mathcal{E}_{S_r})_{\boldsymbol{u}},$$

where we note that

$$T_{(r,\boldsymbol{u})}\left(\mathcal{B}_{\mathcal{S}^{pt}|_U}\right) = T_r\mathcal{R}^{pt} \oplus T_{\boldsymbol{u}}\mathcal{B}_{S_r}.$$

Remark 1.4.4.5. The component

$$T_{\boldsymbol{u}}\mathcal{B}_{S_r} \to (\mathcal{E}_{S_r})_{\boldsymbol{u}}$$

is just the linearized operator $D^h_{S_r,\boldsymbol{u}}$ from Definition 1.4.3.7. The component

$$T_r\mathcal{R}^{pt} \to (\mathcal{E}_{S_r})_{\boldsymbol{u}}$$

corresponds to derivatives of the holomorphic curve equation (Definition 1.4.3.2) with respect to changes of the modulus r.

Definition 1.4.4.6. We denote by

$$D_{\mathcal{S}^{pt}|_U,r,\boldsymbol{u}} : T_{(r,\boldsymbol{u})}\left(\mathcal{B}_{\mathcal{S}^{pt}|_U}\right) \to T_{\boldsymbol{ev}_m(\boldsymbol{u})}\boldsymbol{L}^{F(S)} \oplus (\mathcal{E}_{S_r})_{\boldsymbol{u}}$$

the projection of the linearization

$$D_{r,\boldsymbol{u}}(\boldsymbol{ev}|_U, d_{\mathcal{S}^{pt}}|_U)$$

to the normal bundle of $(\Delta^S, \mathbf{0})$. It is given by

$$D_{\mathcal{S}^{pt}|_U,r,\boldsymbol{u}} = D^{ev}_{S_r,\boldsymbol{u}} \oplus D^h_{\mathcal{S}^{pt}|_U,r,\boldsymbol{u}}.$$

If S_r has no edges of length 0, we say that $(r,\boldsymbol{u})$ is a **regular** point of $\mathcal{M}_{\mathcal{S}^{pt}}(\boldsymbol{y})$ if $D_{\mathcal{S}^{pt}|_U,r,\boldsymbol{u}}$ is surjective (for some open neighbourhood U of r as above). We say that the moduli space $\mathcal{M}_{\mathcal{S}^{pt}|_U}(\boldsymbol{y})$ is regular if every $\boldsymbol{u} \in \mathcal{M}_{\mathcal{S}^{pt}|_U}(\boldsymbol{y})$ is regular.

Proposition 1.4.4.7. *The operator $D_{\mathcal{S}^{pt}|_U,r,\boldsymbol{u}}$ is Fredholm of index*

$$\mathrm{ind}(D_{\mathcal{S}^{pt}|_U,r,\boldsymbol{u}}) = k - 2 + i(\boldsymbol{y})$$

when U avoids a neighbourhood of all pearly trees with edges of length 0.

Proof. See [11, Section 12d] for the pearl component – the inclusion of the Morse flowlines is a trivial addition. □

It follows that, if $\mathcal{M}_{\mathcal{S}^{pt}|_U}(\boldsymbol{y})$ is regular, then it is a smooth manifold with dimension equal to the Fredholm index of $D_{\mathcal{S}^{pt}|_U}$ given above. The transition maps between the spaces $\mathcal{B}_{\mathcal{S}^{pt}|_U}$ are not necessarily smooth, so in general it is not possible to define a Banach manifold '$\mathcal{B}_{\mathcal{S}^{pt}|_U}$' over an arbitrarily large open set U avoiding a neighbourhood of the Morse edges of length 0. However, elliptic regularity ensures that the transition maps between spaces $\mathcal{M}_{\mathcal{S}^{pt}|_U}(\boldsymbol{y})$ are smooth in the regular case, hence they can be patched together to obtain a smooth manifold $\mathcal{M}_{\mathcal{S}^{pt}|_U}(\boldsymbol{y})$ over an arbitrarily large open set U avoiding a neighbourhood of the Morse edges of length 0 (compare [11, Remark 9.4]).

Now we must deal with the Morse edges of length 0, i.e., the case that the modulus $r \in \mathcal{R}^{pt}(T_S, B)$, where $B \neq \phi$ (in the notation of Definition 1.4.1.11).

Definition 1.4.4.8. We define the moduli space

$$\mathcal{M}_{\mathcal{S}^{pt}(T_S,B)}(\boldsymbol{y}) := \{(r, \boldsymbol{u}) \in \mathcal{M}_{\mathcal{S}^{pt}}(\boldsymbol{y}) : r \in \mathcal{R}^{pt}(T_S, B)\}.$$

In order to construct a manifold structure on the moduli space $\mathcal{M}_{\mathcal{S}^{pt}}(\boldsymbol{y})$, we are going to arrange that all of the moduli spaces $\mathcal{M}_{\mathcal{S}^{pt}(T_S,B)}(\boldsymbol{y})$ are regular, then use them to construct charts for the manifold structure on $\mathcal{M}_{\mathcal{S}^{pt}}(\boldsymbol{y})$.

Definition 1.4.4.9. Let $U \subset \mathcal{R}^{pt}(T_S, B)$ be a small connected open subset which makes the strip-like ends constant and avoids a neighbourhood of the pearly trees with some Morse edge not in B having length 0. We define $\mathcal{B}_{\mathcal{S}^{pt}(T_S,B)|_U}, \mathcal{E}_{\mathcal{S}^{pt}(T_S,B)|_U}, d_{\mathcal{S}^{pt}(T_S,B)|_U}$ by restricting $\mathcal{B}_{\mathcal{S}^{pt}|_U}, \mathcal{E}_{\mathcal{S}^{pt}|_U}, d_{\mathcal{S}^{pt}|_U}$ to $\mathcal{R}^{pt}(T_S, B)$. We have

$$\mathcal{M}_{\mathcal{S}^{pt}(T_S,B)|_U}(\boldsymbol{y}) = (\boldsymbol{ev}|_U, d_{\mathcal{S}^{pt}(T_S,B)|_U})^{-1}(\Delta^S, \boldsymbol{0}).$$

The projection of the linearization

$$D_{r,\boldsymbol{u}}(\boldsymbol{ev}|_U, d_{\mathcal{S}^{pt}(T_S,B)|_U})$$

to the normal bundle of $(\Delta^S, \mathbf{0})$ is the restriction of $D_{\mathcal{S}^{pt}|_U,r,\boldsymbol{u}}$ to the codimension-$|B|$ subspace

$$T_r\mathcal{R}^{pt}(T_S,B) \oplus T_{\boldsymbol{u}}\mathcal{B}_{S_r} \subset T_r\mathcal{R}^{pt} \oplus T_{\boldsymbol{u}}\mathcal{B}_{S_r}.$$

We denote it by $D_{\mathcal{S}^{pt}(T_S,B)|_U,r,\boldsymbol{u}}$. By Proposition 1.4.4.7, it is Fredholm of index

$$\operatorname{ind}(D_{\mathcal{S}^{pt}(T_S,B)|_U,r,\boldsymbol{u}}) = k - 2 + i(\boldsymbol{y}) - |B|.$$

Definition 1.4.4.10. We say that $(r, \boldsymbol{u})$ is a regular point of $\mathcal{M}_{\mathcal{S}^{pt}}(\boldsymbol{y})$ if $r \in \mathcal{R}^{pt}(T_S, B)$ and the operator $D_{\mathcal{S}^{pt}(T_S,B)|_U,r,\boldsymbol{u}}$ is surjective (for some open neighbourhood $U \subset \mathcal{R}^{pt}(T_S, B)$ of r as above). We say that the moduli space $\mathcal{M}_{\mathcal{S}^{pt}}(\boldsymbol{y})$ is regular if every $(r, \boldsymbol{u}) \in \mathcal{M}_{\mathcal{S}^{pt}}(\boldsymbol{y})$ is regular.

It follows that, if $\mathcal{M}_{\mathcal{S}^{pt}}(\boldsymbol{y})$ is regular, then each moduli space $\mathcal{M}_{\mathcal{S}^{pt}(T_S,B)|_U}$ is a smooth manifold with dimension equal to the Fredholm index of $D_{\mathcal{S}^{pt}(T_S,B)}$ given above.

Assuming regularity, we now construct charts for a manifold structure on $\mathcal{M}_{\mathcal{S}^{pt}}(\boldsymbol{y})$.

Definition 1.4.4.11. Let $U \subset \mathcal{R}^{pt}(T_S, B)$ be a small connected open subset which makes the strip-like ends constant and avoids a neighbourhood of the pearly trees with some Morse edge not in B having length 0. Given $\epsilon > 0$, denote by $U_\epsilon \subset \mathcal{R}^{pt}$ the image of the map

$$U \times (-\epsilon, \epsilon)^B \to \mathcal{R}^{pt}$$

obtained by interpreting the parameter in $(-\epsilon, \epsilon)$ corresponding to the edge $e \in B$ as a gluing parameter ρ_e for e. Note that U_ϵ is open in $\mathcal{R}^{pt}$.

Proposition 1.4.4.12. *Suppose that $\mathcal{M}_{\mathcal{S}^{pt}}$ is regular. Then for some $\epsilon > 0$ sufficiently* *small, there is a homeomorphism*

$$\mathcal{M}_{\mathcal{S}^{pt}(T_S,B)|_U} \times (-\epsilon,\epsilon)^B \to \mathcal{M}_{\mathcal{S}^{pt}|_{U_\epsilon}}$$

which makes the following diagram commute:

$$\begin{array}{ccc} \mathcal{M}_{\mathcal{S}^{pt}(T_S,B)|_U} \times (-\epsilon,\epsilon)^B & \longrightarrow & \mathcal{M}_{\mathcal{S}^{pt}|_{U_\epsilon}} \\ \downarrow & & \downarrow \\ U \times (-\epsilon,\epsilon)^B & \longrightarrow & U_\epsilon. \end{array}$$

Proof. If two pearls are joined by a Morse edge e of length zero, then they form a nodal disk. In a neighbourhood of the node, the Hamiltonian perturbation is identically 0 and the almost-complex structure is constant, by the conditions we placed on our perturbation datum. A standard gluing argument shows that there is a family of pearls with gluing parameter $\rho_e \in (-\epsilon, 0]$, converging to this nodal disk. A standard compactness argument shows that any sequence of pearls with gluing parameter $\rho_e \to 0^-$ converges to such a nodal disk. More generally, allowing for multiple Morse edges of length 0, one can show that there is a homeomorphism

$$\mathcal{M}_{\mathcal{S}^{pt}(T_S,B)|_U} \times (-\epsilon,0]^B \to \mathcal{M}_{\mathcal{S}^{pt}|_{\mathrm{im}(U\times(-\epsilon,0]^B)}}$$

for some $\epsilon > 0$ sufficiently small.

It then follows from Lemma 1.4.4.2 that this map extends to a homeomorphism

$$\mathcal{M}_{\mathcal{S}^{pt}(T_S,B)|_U} \times (-\epsilon,\epsilon)^B \to \mathcal{M}_{\mathcal{S}^{pt}|_{U_\epsilon}}$$

with the desired properties. □

We have an open cover of $\mathcal{R}^{pt}$ by the sets of the form U_ϵ, for some $U \subset \mathcal{R}^{pt}(T_S, B)$ and some T_S, B. Therefore, we have an open cover of $\mathcal{M}_{\mathcal{S}^{pt}}(\boldsymbol{y})$ by sets $\mathcal{M}_{\mathcal{S}^{pt}|_{U_\epsilon}}(\boldsymbol{y})$ which are homeomorphic to smooth manifolds of dimension $k-2+i(\boldsymbol{y})$. So they are the charts of a topological manifold structure on $\mathcal{M}_{\mathcal{S}^{pt}}$. We have proven:

Proposition 1.4.4.13. *If $\mathcal{M}_{\mathcal{S}^{pt}}(\boldsymbol{y})$ is regular, then it has the structure of a topological manifold of dimension*

$$\dim(\mathcal{M}_{\mathcal{S}^{pt}}(\boldsymbol{y})) = k - 2 + i(\boldsymbol{y}).$$

Remark 1.4.4.14. One can show that the embeddings of Proposition 1.4.4.12 respect orientations, and hence that the manifold $\mathcal{M}_{\mathcal{S}^{pt}}(\boldsymbol{y})$ is oriented.

1.4.5 Consistency and compactness

Definition 1.4.5.1. A universal choice of perturbation data is a choice of perturbation datum for each family $\mathcal{S}^{pt}(\boldsymbol{L})$ (for all choices of Lagrangian labels $\boldsymbol{L}$).

Definition 1.4.5.2. (Compare [11, Section 9i]) Given a tree T with Lagrangian labels $\boldsymbol{L}$, the gluing construction defines a map to a collar neighbourhood of the boundary stratum corresponding to T:

$$\left\{r \in \bar{\mathcal{R}}^{pt}_T : \rho_e \in (-\epsilon, 0] \text{ for } e \text{ a strip edge, and } \rho_e \in (1-\epsilon, 1] \text{ for } e \text{ a Morse edge}\right\} \to \bar{\mathcal{R}}^{pt}.$$

Because the perturbation data are standard along the strip-like ends (given by the Floer data), we can glue the perturbation data on the families $\mathcal{S}^{pt}(\boldsymbol{L}_v)$, for each vertex v of T, together to obtain a perturbation datum (K_T, J_T, V_T) on this collar neighbourhood. Furthermore, this perturbation datum extends smoothly to the boundary stratum corresponding to T. We say that a universal choice of perturbation data is **consistent** if the perturbation datum (K, J, V) on $\mathcal{S}^{pt}(\boldsymbol{L})$ also extends smoothly to the compactification $\bar{\mathcal{S}}^{pt}(\boldsymbol{L})$, and agrees with the perturbation datum (K_T, J_T, V_T) on the boundary stratum corresponding to T, for all such $\boldsymbol{L}$ and T.

Proposition 1.4.5.3. *Consistent universal choices of perturbation data exist.*

Proof. The proof is essentially the same as [11, Lemma 9.5]. ☐

Definition 1.4.5.4. Suppose we have made a consistent universal choice of perturbation data, and all moduli spaces are regular. Let $\boldsymbol{L}$ be a set of Lagrangian labels and $\boldsymbol{y}$ an associated set of generators. A **stable holomorphic pearly tree** consists of the following data:

- A semi-stable directed planar tree T with Lagrangian labels $\boldsymbol{L}$;
- For each edge e of T, a generator $y_e \in CF^*(L_{r(e)}, L_{l(e)})$, where $L_{r(e)}, L_{l(e)}$ are the Lagrangian labels to the right and left of e respectively, such that the generators are given by $\boldsymbol{y}$ for the external edges;
- For each stable vertex v (i.e., v has valence ≥ 3), an element

$$(r_v, \boldsymbol{u}_v) \in \mathcal{M}_{\mathcal{S}^{pt}(\boldsymbol{L}_v)}(\boldsymbol{y}_v),$$

  where $\boldsymbol{y}_v$ denotes the set of chosen generators for the edges adjacent to v;
- for each vertex v of valence 2, an element

$$u_v \in \mathcal{M}_Z^*(\boldsymbol{y}_v).$$

We define $\mathcal{M}^T_{\bar{\mathcal{S}}^{pt}}(\boldsymbol{y})$ to be the set of all equivalence classes of stable holomorphic pearly trees modeled on the tree T.

Definition 1.4.5.5. We define the moduli space

$$\bar{\mathcal{M}}_{\bar{\mathcal{S}}^{pt}}(\boldsymbol{y}) := \coprod_T \mathcal{M}^T_{\bar{\mathcal{S}}^{pt}}(\boldsymbol{y})$$

of stable holomorphic pearly trees, as a set.

Proposition 1.4.5.6. *$\bar{\mathcal{M}}_{\bar{\mathcal{S}}^{pt}}(\boldsymbol{y})$ has the structure of a compact topological manifold with corners. Its codimension-d strata are the sets $\mathcal{M}^T_{\bar{\mathcal{S}}^{pt}}$ where T has d internal edges. In* *particular, the open stratum (corresponding to the one-vertex tree) is the moduli space* $\mathcal{M}_{\mathcal{S}^{pt}}(\boldsymbol{y})$.

Proof. Observe that each stratum

$$\mathcal{M}^T_{\bar{\mathcal{S}}^{pt}}(\boldsymbol{y})$$

has the structure of a smooth manifold, since it is a product of smooth manifolds. By standard gluing arguments, there are maps

$$\mathcal{M}^T_{\bar{\mathcal{S}}^{pt}}(\boldsymbol{y}) \times (-\epsilon, 0]^{E(T)} \to \bar{\mathcal{M}}_{\bar{\mathcal{S}}^{pt}}(\boldsymbol{y}).$$

We define the topology on $\bar{\mathcal{M}}_{\bar{\mathcal{S}}^{pt}}(\boldsymbol{y})$ so that all of these maps are continuous. This defines a manifold-with-corners structure on the moduli space of stable holomorphic pearly trees.

We prove compactness by considering each underlying tree type T_S for a pearly tree separately. Given T_S, consider the moduli space of stable holomorphic pearly trees such that, if we contract all edges of length 0, we get a tree of type T_S. The space of possible stable pearls corresponding to vertices of T_S is compact, by standard Gromov compactness as in [39]. Similarly, the space of possible broken Morse flowlines corresponding to edges of T_S is compact, by standard compactness results in Morse theory as in [38, Section 2.4]. Thus, the full moduli space is a closed subset (defined by the incidence conditions of marked boundary points on pearls and ends of edges) of the compact set of all possible pearl and edge maps. By considering all possible tree types T_S, we obtain a covering of $\bar{\mathcal{M}}_{\bar{\mathcal{S}}^{pt}}(\boldsymbol{y})$ by a finite number of compact sets, hence the moduli space is compact. □

1.4.6 Transversality

Proposition 1.4.6.1. *The moduli spaces $\mathcal{M}_{\mathcal{S}^{pt}}(\boldsymbol{y})$ are regular for generic consistent universal choices of perturbation data.*

Proof. Make a consistent universal choice of perturbation data. For each set of Lagrangian labels $\boldsymbol{L}$, we show that it is possible to modify the perturbation data (K, J, V) slightly to make our moduli spaces regular. In fact it is sufficient only to perturb (K, J), assuming we have already chosen the Floer data (h_L, g_L) to be Morse-Smale for each L. Our situation is very similar to that considered in [11, Section 9k].

A deformation of (K, J) is given by a choice of:

- $\delta K \in \Omega^1_{\mathcal{S}^p/\mathcal{R}^{pt}}(\mathcal{S}^p, \mathcal{H})$;
- $\delta J \in C^\infty(\mathcal{S}^p, T_J\mathcal{J})$,

such that $(\delta K, \delta J)$ vanish on the strip-like ends and $\delta K(\xi)|_{L_C} = 0$ for each $\xi \in TC$, where C is a boundary component of a pearl and L_C its Lagrangian label.

We choose an open set $\Omega \subset \mathcal{S}^p$ such that, for each $r \in \mathcal{R}^{pt}$, $\Omega \cap \mathcal{S}^p_r$ lies within the 'thick' region of Definition 1.4.1.7, and intersects each connected component of the thick region in a non-empty, connected set that intersects each boundary component (see Figure 1.4.6.1). To retain consistency of our perturbation datum, we require that $(\delta K, \delta J)$ are zero outside Ω, and extend smoothly to a pair $(\overline{\delta K}, \overline{\delta J})$ defined on $\bar{\mathcal{S}}^p$ which vanish to infinite order along the boundary.

Let $\mathcal{T}$ denote the space of all such $(\delta K, \delta J)$. Given $t \in \mathcal{T}$, we can exponentiate it to an actual perturbation datum, and we define

$$\mathcal{M}^t_{\mathcal{S}^{pt}}(\boldsymbol{y})$$

to be the moduli space of holomorphic pearly trees with respect to this perturbation datum. We define the universal moduli space

$$\mathcal{M}^{univ}_{\mathcal{S}^{pt}}(\boldsymbol{y}) := \{(t, r, \boldsymbol{u}) : t \in \mathcal{T}, (r, \boldsymbol{u}) \in \mathcal{M}^{t}_{\mathcal{S}^{pt}}(\boldsymbol{y})\}.$$

We have the associated universal linearized operators

$$D^{univ}_{\mathcal{S}^{pt}, r, \boldsymbol{u}} : \mathcal{T} \oplus T_r \mathcal{R}^{pt} \oplus T_{\boldsymbol{u}} \mathcal{B}_{S_r} \to T_{\boldsymbol{ev}_m(\boldsymbol{u})} \boldsymbol{L}^{F(S)} \oplus (\mathcal{E}_{S_r})_{\boldsymbol{u}},$$

given by

$$D^{univ}_{\mathcal{S}^{pt}, r, \boldsymbol{u}} = D^{def}_{\mathcal{S}^{pt}, r, \boldsymbol{u}} \oplus D_{\mathcal{S}^{pt}, r, \boldsymbol{u}},$$

where $D_{\mathcal{S}^{pt}, r, \boldsymbol{u}}$ is as defined in Definition 1.4.4.4 and

$$D^{def}_{\mathcal{S}^{pt}, r, \boldsymbol{u}} : \mathcal{T} \to (\mathcal{E}_{S_r})_{\boldsymbol{u}}$$

takes the derivative of the holomorphic pearly tree equation with respect to changes in the perturbation datum. We should really work in a local trivialization of $\mathcal{S}^{pt}$ over a small set U, as we did in Section 1.4.4, but we gloss over this point to make things readable.

We claim that the universal operator $D^{univ}_{\mathcal{S}^{pt}, r, \boldsymbol{u}}$ is surjective. Let S denote the pearly tree with modulus r. The codomain of $D^{univ}_{\mathcal{S}^{pt}, r, \boldsymbol{u}}$ is a direct sum

$$T_{\boldsymbol{ev}_m(\boldsymbol{u})} \boldsymbol{L}^{F(S)} \oplus L^p(S^p, \Omega^{0,1}_S \otimes u_p^* TX) \oplus \bigoplus_{L \in \boldsymbol{L}} L^p(S^e(L), u_L^* TL).$$

The operator $D_{\mathcal{S}^{pt}, r, \boldsymbol{u}}$ always maps

$$W^{1,p}(S^e(L), u_L^* TL) \to L^p(S^e(L), u_L^* TL)$$

surjectively, for each $L \in \boldsymbol{L}$ (the moduli spaces of Morse flowlines are always regular –

we are not imposing any boundary conditions here).

The space of deformations $\mathcal{T}$ maps surjectively to

$$L^p(S^p, \Omega_S^{0,1} \otimes u_p^* TX)$$

(see [11, Section 9k]). To complete the proof of surjectivity, we show that the tangent space to the zero set of the universal section

$$d_S^{univ}|_{S^p} : \mathcal{T} \times W^{1,p}_{loc}(S^p, X) \to L^p(S^p, \Omega_S^{0,1} \otimes u_p^* TX)$$

maps surjectively to

$$T_{\boldsymbol{ev}_m(\boldsymbol{u})} \boldsymbol{L}^{F(S)},$$

using a modification of an argument given in [40, Section 3.4]. The essential observation is that the group of Hamiltonian diffeomorphisms fixing the Lagrangians in $\boldsymbol{L}$ acts on the space of perturbation data and associated holomorphic pearly trees with labels $\boldsymbol{L}$.

Let $h : S^p \to \mathcal{H}$ be a smooth function which is locally (in the z coordinates on S^p) equal to a constant $H \in \mathcal{H}$ outside of $\Omega \cap S^p$, and such that $h|_C$ vanishes on the Lagrangian L_C, for any boundary component C of S^p with label L_C. Denote by $\phi_z : X \to X$ the time-1 flow of the Hamiltonian $h(z)$, for $z \in S^p$. Then we can define a map from

$$\mathcal{T} \times W^{1,p}_{loc}(S^p, X)$$

to itself by

$$\begin{aligned} u_p(z) &\mapsto \phi_z(u_p(z)), \\ K(z) &\mapsto \phi_z^* K(z) - dh(z), \\ J(z) &\mapsto J(z) \circ \phi_z \end{aligned}$$

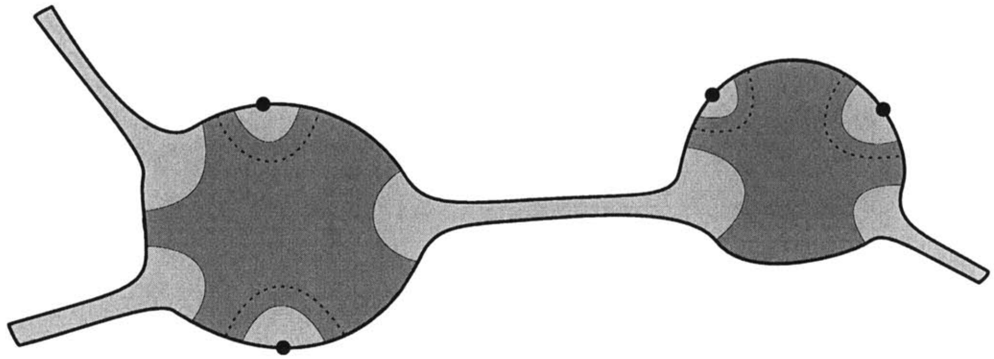


Figure 1.4.6.1: The region $\Omega \cap \mathcal{S}_r^p$ (shaded dark grey) inside $\mathcal{S}_r^p$ (shaded light grey), for some $r \in \mathcal{R}^{pt}$. Note that Ω avoids all thin regions. The solid circles denote marked points. For each marked point $m(f)$, there is a curve inside Ω (drawn as a dotted line) which separates $m(f)$ from all other marked points and punctures.

where $dh(z)$ denotes the differential of $h(z)$ with respect to the coordinates z on S^p. In particular, $dh(z)$ is supported in Ω, so the new perturbation datum still lies in $\mathcal{T}$. One can show that this action preserves the section $d_S^{univ}|_{S^p}$ and in particular preserves its zero set.

By our definition of Ω, for each flag $f \in F(S)$ we can choose a curve in Ω that cuts the pearl containing $m(f)$ into two regions, one of which contains the marked point $m(f)$ and no other punctures or marked points. We can make these curves disjoint for different f (see Figure 1.4.6.1). Then we can define $h_f : S^p \to \mathcal{H}$ which is supported in the region containing $m(f)$, and constant equal to some Hamiltonian H_f in the portion of that region that lies outside of Ω. By making different choices of the functions H_f, we can independently move the points $\phi_{m(f)}(u_p(m(f)))$ in any direction we please, so the linearization of the evaluation map is surjective from the tangent space to the zero set of $d_S^{univ}|_{S^p}$ onto $T_{\boldsymbol{ev}_m(\boldsymbol{u})}\boldsymbol{L}^{F(S)}$. This completes the proof of surjectivity of the universal linearized operator.

Therefore, the universal moduli spaces $\mathcal{M}_{S^{pt}}^{univ}$ are Banach manifolds. Similarly, one can show that the universal moduli spaces

$$\mathcal{M}^{univ}_{\mathcal{S}^{pt}(T_S,B)}(\boldsymbol{y}) := \{(t, r, \boldsymbol{u}) : t \in \mathcal{T}, (r, \boldsymbol{u}) \in \mathcal{M}^t_{\mathcal{S}^{pt}(T_S,B)}(\boldsymbol{y})\}$$

are Banach manifolds for each (T_S, B) (see Definitions 1.4.1.11, 1.4.4.8). The regular values of the projections of each of these universal moduli spaces to $\mathcal{T}$ are of the second category, by the Sard-Smale theorem (see Remark 1.4.6.2). Taking the intersection of regular values of the projection, over all (T_S, B), shows that for a generic choice of deformed perturbation datum in $\mathcal{T}$, the moduli spaces

$$\mathcal{M}_{\mathcal{S}^{pt}(T_S,B)}(\boldsymbol{y})$$

are all simultaneously regular. This was our definition of regularity of the moduli space $\mathcal{M}_{\mathcal{S}^{pt}}(\boldsymbol{y})$ (see Definition 1.4.4.10). □

Remark 1.4.6.2. We have glossed over one technical issue: the space of admissible deformed perturbation data is not a Banach space as we have defined it, but rather a Fréchet space, and hence the Sard-Smale theorem does not apply. To fix this, we should work with the Banach spaces of C^l perturbation data, then take the intersection over all l (see [40, Section 3.1] for details).

1.4.7 A_∞ structure maps

In this section we give our definition of the Fukaya category. We do not discuss signs, but they work in essentially the same way as in [11] (using Remark 1.4.4.14).

We make a choice of Floer data and a consistent universal choice of perturbation data, and assume that all moduli spaces $\mathcal{M}_{\mathcal{S}^{pt}}(\boldsymbol{y})$ (as well as those used in the definition of the Floer differential) are regular.

We define the differential

$$\mu^1 : CF^*(L_0, L_1) \to CF^*(L_0, L_1)$$

to be the standard Floer differential if L_0, L_1 are distinct, and to be the Morse differential (for Morse **co**homology) for (h_L, g_L) if $L_0 = L_1 = L$.

Given Lagrangian labels $\boldsymbol{L} = (L_0, \ldots, L_k)$, we define the higher products

$$\mu^k : CF^*(L_{k-1}, L_k) \otimes \ldots \otimes CF^*(L_0, L_1) \to CF^*(L_0, L_k)[2-k]$$

as follows: given an associated set of generators $\boldsymbol{y} = (y_0, \ldots, y_k)$, such that

$$i(y_0) = i(y_1) + \ldots + i(y_k) + 2 - k,$$

we define the coefficient of y_0 in

$$\mu^k(y_k, \ldots, y_1)$$

to be the count of points in the moduli space

$$\mathcal{M}_{\mathcal{S}^{pt}}(\boldsymbol{y})$$

(which is 0-dimensional by Proposition 1.4.4.13), with appropriate signs. Note that the condition on degrees of the y_j mean that the maps μ^k respect the $\mathbb{Z}$-grading in the appropriate sense for an A_∞ category.

Proposition 1.4.7.1. *The operations μ^k satisfy the A_∞ associativity equations, with signs and $\mathbb{Z}$-gradings.*

Proof. The proof follows familiar lines: given a set of generators $\boldsymbol{y}$ associated to Lagrangian labels $\boldsymbol{L}$, we consider the 1-dimensional component of the moduli space $\bar{\mathcal{M}}_{\bar{\mathcal{S}}^{pt}}(\boldsymbol{y})$. The signed count of its boundary components is 0. By the results outlined in Section

1.4.5, the codimension-1 boundary strata of $\bar{\mathcal{M}}_{\mathcal{S}^{pt}}(\boldsymbol{y})$ consist of those stable holomorphic pearly trees modeled on trees T with one internal edge. The fact that their signed count is 0 means that the coefficient of y_0 in

$$\sum_{a,b}(-1)^{\star}\mu^{k-a+1}(y_k,\ldots,y_{a+b+1},\mu^a(y_{a+b},\ldots,y_{b+1}),y_b,\ldots,y_1)$$

is 0, where

$$\star = i(y_1)+\ldots+i(y_b)-b.$$

This means exactly that the A_∞ associativity equations hold. □

Proposition 1.4.7.2. *The Fukaya category is independent, up to quasi-isomorphism, of the choices of strip-like ends, Floer data and perturbation data made in its definition.*

Proof. Compare [11, Chapter 10]. □

Proposition 1.4.7.3. *The A_∞ algebra $CF^*(L,L)$ is quasi-isomorphic to the differential graded cohomology algebra $C^*(L)$.*

Proof. We can choose the Hamiltonian perturbations of the moduli spaces used to define $CF^*(L,L)$ to be zero, so that all pearls are constant by exactness of L. It is not difficult to show that transversality can be achieved with this class of perturbation data, by perturbing V. The definition of $CF^*(L,L)$ then coincides with the definition of the A_∞ algebra $CM^*(L)$ given in [35, Section 2.2] (by counting Morse flow trees on L). The result now follows from [35, Section 3]. □

1.4.8 Compatibility with other definitions

In this section, we explain why our definition of the Fukaya category (which we denote, for the purposes of this section, by $\mathcal{F}uk^1(X)$) is quasi-equivalent to that in [11] (which we denote by $\mathcal{F}uk^2(X)$). We define an auxiliary A_∞ category, $\mathcal{F}uk^{12}(X)$, which contains two objects, L^1 and L^2, for each object L of the usual Fukaya category. We define Floer data for each pair (L^1, L^1) to consist of a Morse-Smale pair on L, but for all other pairs of objects (L_0^i, L_1^j), including the case $L_0 = L_1$, we define the Floer data as if the objects were distinct in Definition 1.4.2.2 (i.e., the Floer datum consists of a Hamiltonian component whose time-1 flow makes L_0 and L_1 transverse, and an almost-complex structure component). We define the A_∞ structure coefficients by counting holomorphic pearly trees as before, but we only allow Morse flowlines if an edge has labels L^1 on opposite sides for some L.

There are A_∞ embeddings

$$\mathcal{F}uk^1(X) \hookrightarrow \mathcal{F}uk^{12}(X) \hookleftarrow \mathcal{F}uk^2(X)$$

defined by $L \mapsto L^1$, $L \mapsto L^2$ respectively.

Proposition 1.4.8.1. *The objects L^1, L^2 are quasi-isomorphic, for any L.*

Proof. The Piunikhin-Salamon-Schwarz isomorphism [41] gives isomorphisms on the level of cohomology,

$$HF^*(L^1, L^2) \cong HF^*(L^2, L^1) \cong HF^*(L^2, L^2) \cong H^*(L),$$

and says that the product

$$HF^*(L^1, L^2) \otimes HF^*(L^2, L^1) \to HF^*(L^2, L^2)$$

agrees with the cup product on cohomology (note that the moduli spaces defining this product involve no holomorphic pearly trees, only disks).

In particular, if we choose morphisms

$$f_{12} \in CF^*(L^1, L^2)\ ,\ \ f_{21} \in CF^*(L^2, L^1)$$

corresponding to the identity in cohomology, then the PSS isomorphism tells us that the product

$$\mu^2(f_{21}, f_{12}) \in CF^*(L^2, L^2)$$

corresponds to the identity in cohomology. Thus, because $HF^*(L^1, L^1)$ and $HF^*(L^2, L^2)$ have the same rank (both are isomorphic to $H^*(L)$), the morphisms f_{12} and f_{21} induce isomorphisms on cohomology.

Thus, L^1 and L^2 are quasi-isomorphic, as required. □

Corollary 1.4.8.2. *The embeddings*

$$\mathcal{F}uk^1(X) \hookrightarrow \mathcal{F}uk^{12}(X) \hookleftarrow \mathcal{F}uk^2(X)$$

are quasi-equivalences, and in particular, the A_∞ categories $\mathcal{F}uk^1(X)$ and $\mathcal{F}uk^2(X)$ are quasi-equivalent.

Proof. See [11, Section 10a]. □

1.5 Computation of $\mathcal{A}$

The aim of this section is to prove Theorem 3, which identifies the cohomology algebra of $\mathcal{A}$ as an exterior algebra, and Proposition 1.5.4.3, which gives a description of $\mathcal{A}$ up to quasi-isomorphism.

The outline of the section is as follows: Section 1.5.1 gives a Morse-Bott description of $CF^*(L^n, L^n)$. We define an A_∞ category $\mathcal{C}$ with two objects: one is the Lagrangian immersion $L^n : S^n \to \mathcal{P}^n$, and the other is the Lagrangian immersion $L' : S^n \to \mathbb{CP}^n$ which is the double cover of the real locus $\mathbb{RP}^n$. The situation is analogous to that in Section 1.4.8, in which we explained why our Morse-Bott description of the Fukaya category using pearly trees was equivalent to the standard one using disks. Namely, we will define the A_∞ structure maps so that the A_∞ endomorphism algebra of L^n counts holomorphic disks as in Section 1.3.1, and in particular is the same as $\mathcal{A}$, while the A_∞ endomorphism algebra of L' counts Morse-Bott objects which we call 'admissible flipping holomorphic pearly trees'.

Recall that one can think of a pearly tree as a degeneration of holomorphic disks, as the Hamiltonian part of the Floer datum for the pair (L, L) converges to 0 (see Remark 1.4.2.4). Similarly, one should think of L' as the limit of L^n_ϵ as $\epsilon \to 0$, i.e., the double cover of $\mathbb{RP}^n \subset \mathbb{CP}^n$ by S^n (recall that L^n_ϵ is constructed as the graph of an exact 2-valued 1-form ϵdf in the cotangent disk bundle $D^*_\eta \mathbb{RP}^n$ embedded in $\mathbb{CP}^n$). One should think of a flipping holomorphic pearly tree as a degeneration of a holomorphic pearly tree with boundary on L^n_ϵ, in the limit $\epsilon \to 0$.

Because we wish to consider only holomorphic pearly trees which lie inside $\mathcal{P}^n$ (i.e., do not intersect the boundary divisors), we must impose an additional condition ('admissibility') on our flipping holomorphic pearly trees. Thus, although the admissible flipping holomorphic pearly trees themselves may intersect the boundary divisors, they should be thought of as degenerations of holomorphic pearly trees which avoid the boundary divisors.

We show that, for sufficiently small $\epsilon > 0$, the objects L' and L^n_ϵ of this A_∞ category are quasi-isomorphic, and hence that we can compute $\mathcal{A} := CF^*(L^n_\epsilon, L^n_\epsilon)$ up to quasi-isomorphism by computing $\mathcal{A}' := CF^*(L', L')$.

In Section 1.5.2 we describe some features of pearly trees, which help us to explicitly identify the moduli spaces of flipping holomorphic pearly trees that give the structure coefficients of $\mathcal{A}'$. This is possible because the pearls involved are just holomorphic disks in $\mathbb{CP}^n$ with boundary on $\mathbb{RP}^n$ (with some additional restrictions), hence well-understood.

In Section 1.5.3, we carry this out. In particular, we prove Theorem 3, which identifies the cohomology algebra of $\mathcal{A}'$ (and hence $\mathcal{A}$) as an exterior algebra. We also identify certain higher A_∞ structure maps of $\mathcal{A}'$.

Finally, in Section 1.5.4, we show that $\mathcal{A}'$ is versal in the class of A_∞ algebras with cohomology algebra the exterior algebra, and the equivariance and grading properties established in Section 1.3. This identifies $\mathcal{A}'$ (and hence $\mathcal{A}$) up to quasi-isomorphism, in the sense that any A_∞ algebra in the same class must be quasi-isomorphic to $\mathcal{A}$.

1.5.1 Flipping pearly trees

For the purposes of this section, we think of $\mathbb{CP}^n$ as the hyperplane

$$\left\{\sum_j z_j = 0\right\} \subset \mathbb{CP}^{n+1},$$

$\mathbb{RP}^n$ as its real locus, $L' : S^n \to \mathbb{CP}^n$ the composition of the double cover of $\mathbb{RP}^n$ with the inclusion $\mathbb{RP}^n \hookrightarrow \mathbb{CP}^n$, and $\{x_j\}$ the real coordinates on S^n. We define an A_∞ category $\mathcal{C}$ with two objects: one is the Lagrangian immersion $L^n : S^n \to \mathcal{P}^n$, and the other is the Lagrangian immersion $L' : S^n \to \mathbb{CP}^n$ just defined.

Definition 1.5.1.1. We define Floer data and morphism spaces for the pairs of objects $(L_0, L_1) = (L^n, L^n), (L^n, L')$ or (L', L^n) as in Definition 1.3.1.5.

Definition 1.5.1.2. The Floer datum for the pair (L', L') consists of **two** Morse functions on S^n: one is h, a function whose only critical points are a maximum $p_{[n+2]}$ and minimum p_ϕ. The other is f, the function constructed in Definition 1.2.2.3, which has critical points p_K for each proper, non-empty subset $K \subset [n+2]$, as shown in Corollary 1.2.2.11. Both, when paired with the standard round metric g on S^n, form a Morse-Smale pair. One then defines

$$CF^*(L', L') := CM^*(h) \oplus CM^*(f) \cong \bigoplus_{K \subset [n+2]} \mathbb{C}\langle p_K \rangle.$$

We equip it with the $\mathbb{Q}$-grading

$$i(p_K) := \frac{n}{n+2}|K|$$

(compare Corollary 1.3.3.2).

Remark 1.5.1.3. Given a complex volume form η on $\mathcal{P}^n$, we can define a $\mathbb{Z}$-grading on the morphism spaces $CF^*(L_0, L_1)$ as usual.

Definition 1.5.1.4. We call generators of $CF^*(L', L')$ corresponding to critical points of f **flipping generators**, and those corresponding to critical points of h **non-flipping generators**.

Definition 1.5.1.5. Suppose we are given a set of Lagrangian labels $\boldsymbol{L}$, consisting only of the objects L' and L^n of $\mathcal{C}$. We define a pearly tree with labels $\boldsymbol{L}$ to be a pearly tree as in Definition 1.4.1.6, except that we only allow edges labeled L' (not L^n).

Definition 1.5.1.6. We define a perturbation datum (K, J, V) for the family of pearly trees as in Sections 1.4.3, 1.4.4, with one difference. Namely, the part of the perturbation datum V (associated to the edges, which all have label L') now consists of two components: the 'flipping component'

$$V^f \in C^\infty(\mathcal{S}^e, \mathcal{V}_{S^n})$$

and the 'non-flipping component'

$$V^{nf} \in C^\infty(\mathcal{S}^e, \mathcal{V}_{S^n}).$$

We require that

$$V^f = V^{nf} = 0$$

on an internal edge e of length $l_e \leq 1$, and

$$V^f = \nabla f \text{ and } V^{nf} = \nabla h$$

on an external edge or an edge e of length $l_e \geq 2$.

Definition 1.5.1.7. (Compare Definition 1.4.3.2) Given a set of Lagrangian labels $\boldsymbol{L}$ and associated generators $\boldsymbol{y}$, we define a **flipping holomorphic pearly tree** with labels $\boldsymbol{y}$ to consist of the following data:

- A designation of certain edges as **flipping** and the remaining edges as **non-flipping**, such that external flipping edges are labeled by flipping generators and external non-flipping edges are labeled by non-flipping generators. We call the marked points attached to flipping edges **flipping marked points** and those attached to non-flipping edges **non-flipping marked points**;

- A smooth map

  $$u_e : S^e \to S^n$$

  satisfying

  $$\begin{aligned} Du_e - V^f &= 0 \text{ on flipping edges, and} \\ Du_e - V^{nf} &= 0 \text{ on non-flipping edges;} \end{aligned}$$

- A smooth map

  $$u_p : S^p \to \mathbb{CP}^n$$

  satisfying

  $$(Du_p - Y)^{0,1} = 0,$$

such that

$$u_p(C) \in \mathrm{im}(L_C) \text{ for each boundary component } C \text{ of } S^p \text{ with label } L_C;$$

- A lift $\tilde{u}_C$ of the map $u_p|_C : C \to \mathrm{im}(L_C)$ to S^n,

$$\begin{tikzcd} & S^n \arrow[d, "L_C"] \\ C \arrow[ur, "\tilde{u}_C"] \arrow[r, "u_p|_C"'] & \mathrm{im}(L_C), \end{tikzcd}$$

  for each boundary component C with label L_C,

satisfying the following conditions:

- $\tilde{u}_C$ is continuous except at flipping marked points, where it changes sheets of the covering;
- We have
$$\tilde{u}^{\pm}(m(f)) = u_e(b(f)) \text{ for all } f \in F^{\pm}(S),$$
  where we denote by $\tilde{u}^{+}$, respectively $\tilde{u}^{-}$, the right, respectively left, limit of $\tilde{u}$ (this is necessary because $\tilde{u}$ is discontinuous exactly at the flipping marked points), and where $F^{+}(S)$, respectively $F^{-}(S)$, denotes the subset of flags whose orientation agrees, respectively disagrees, with the orientation of the tree;
- The external edges are asymptotic to the generators $\boldsymbol{y}$, in the same sense as in Definition 1.3.1.8.

Recall that $\mathcal{P}^n$ is obtained from $\mathbb{CP}^n$ by removing the divisor D which is the union of the divisors $D_j = \{z_j = 0\}$ for $j = 1, \ldots, n+2$. We wish to count only flipping holomorphic pearly trees that do not 'intersect' the divisors D_j. We now explain how to do this in a well-defined way.

Definition 1.5.1.8. Given a flipping holomorphic pearly tree $\boldsymbol{u}$ as defined above, one obtains a well-defined homology class $[\boldsymbol{u}] \in H_2(\mathbb{CP}^n, L^n)$ as follows:

- Start with the continuous map $\boldsymbol{u} : S \to \mathbb{CP}^n$ associated with the flipping holomorphic pearly tree.
- Glue a thin strip along the boundary of the flipping pearly tree (see Figure 1-1(a));
- If the boundary component or edge has label L^n, then it already gets mapped to L^n, so we map the strip into $\mathbb{CP}^n$ by making it constant along its width.
- If the boundary component or edge has label L', then by construction, there is a continuous lift of the boundary of the strip to S^n. Namely, it is given by the lift $\tilde{u}_C$ along a boundary component C of a pearl with label L'; by a flowline of ∇f and its antipode along the boundary of a strip coming from a flipping edge; and by a flowline of ∇h on both sides of the boundary of a strip coming from a non-flipping edge.
- Thus, we can map the strip into $\mathbb{CP}^n$ by letting it interpolate between the zero section and the graph of ϵdf in the Weinstein neighbourhood $D^*_\eta S^n$ used in the construction of L^n_ϵ. Thus, boundary components of the strip with label L' now lie on L^n_ϵ.

We now define the intersection number $\boldsymbol{u} \cdot D_j$ to be the topological intersection number of this class $[\boldsymbol{u}] \in H_2(\mathbb{CP}^n, L^n)$ with $D_j \in H_{2n-2}(\mathbb{CP}^n)$. We say that a flipping holomorphic pearly tree $\boldsymbol{u}$ is **admissible** if $\boldsymbol{u} \cdot D_j = 0$ for all j.

Proposition 1.5.1.9. *Let $\boldsymbol{u}$ be a flipping holomorphic pearly tree. Then the intersection numbers $\boldsymbol{u} \cdot D_j$ are non-negative. Furthermore, in nice situations they can be calculated: Suppose that the boundary lifts $\tilde{u}_C$ of each boundary component C with label L' are transverse to the real hypersurface $D_j^{\mathbb{R}} \subset S^n$, and no flipping marked points lie on $D_j^{\mathbb{R}}$. Then one can calculate $\boldsymbol{u} \cdot D_j$ by counting the usual intersection number for internal* *intersections of each pearl u_v with D_j (this is positive by positivity of intersections), +1 for each time a flipping edge of $\boldsymbol{u}$ crosses $D_j^{\mathbb{R}}$, and +1 for each time a boundary lift $\tilde{u}_C$ crosses $D_j^{\mathbb{R}}$ in the negative direction.*

Proof. We observe that the first statement follows from the second: in the transverse situation the intersection number is non-negative because the only contributions are positive. We can put ourselves in the transverse situation by making a small perturbation of the divisor D_j. Namely, define a 1-parameter family of divisors

$$D_j^t := \left\{ z_j + t \sum_k \alpha_k z_k = 0 \right\}$$

for $t \in [0, \delta]$, where $\alpha_j \in \mathbb{R}$ and $\delta > 0$ is real and sufficiently small that the real part $(D_j^t)^{\mathbb{R}}$ remains transverse to the gradient vector field ∇f, and hence D_j^t avoids the Lagrangian L^n (by Lemma 1.2.2.2). We also make δ small enough that D_j^t avoids all other Lagrangian labels of the flipping holomorphic pearly tree. Therefore the intersection number $\boldsymbol{u} \cdot D_j^t$ remains constant, so we can compute $\boldsymbol{u} \cdot D_j$ by computing $\boldsymbol{u} \cdot D_j^{\delta}$. That D_j^{δ} can be made transverse to the boundary lifts $\tilde{u}_C$ is an easy application of Sard's theorem. Furthermore, one can easily make D_j^{δ} avoid all marked points and critical points of pearls (since these are isolated).

Now we prove the second statement. Internal intersections of $\boldsymbol{u}$ with D_j contribute the usual intersection number (which is positive by positivity of intersections, recalling that the almost-complex structure is standard near the divisors D_j). The other intersections happen near boundary components of $\boldsymbol{u}$ with label L':

- If a flipping edge crosses $D_j^{\mathbb{R}}$, one can see that the image of the surrounding strip under projection to the z_j plane looks like Figure 1-1(b), hence contributes +1 to the intersection number;
- If a non-flipping edge crosses $D_j^{\mathbb{R}}$, the image of the strip under projection to the z_j

plane looks like Figure 1-1(b) except that the strip gets folded in two, so that both edges get sent to the same sheet of L^n, and the contribution to the topological intersection number is 0;

- If a boundary lift $\tilde{u}_C$ crosses $D_j^{\mathbb{R}}$ positively, the projection of the strip and nearby disk to the z_j plane looks like Figure 1-1(c) (the projection is a holomorphic map, which by assumption has no singularities near the divisor D_j, and its boundary crosses $D_j^{\mathbb{R}}$ positively, hence maps to the upper half plane in a neighbourhood of this point). There is a 'fold' along the real axis, and one can see that the contribution to the topological intersection number with D_j is 0;

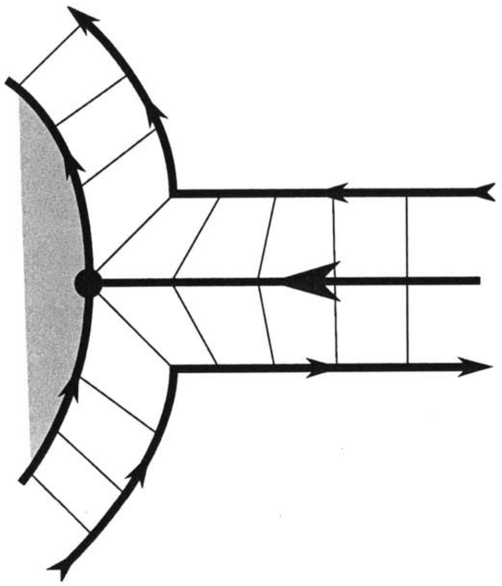
- If a boundary lift $\tilde{u}_C$ crosses $D_j^{\mathbb{R}}$ negatively, the projection of the strip and nearby disk to the z_j plane looks like Figure 1-1(d) (as before, because the disk is holomorphic, non-singular, and its boundary crosses $D_j^{\mathbb{R}}$ negatively, it must get sent to the lower half plane in a neighbourhood of this point). Thus the contribution to the topological intersection number with D_j is $+1$.

This completes the proof. $\square$

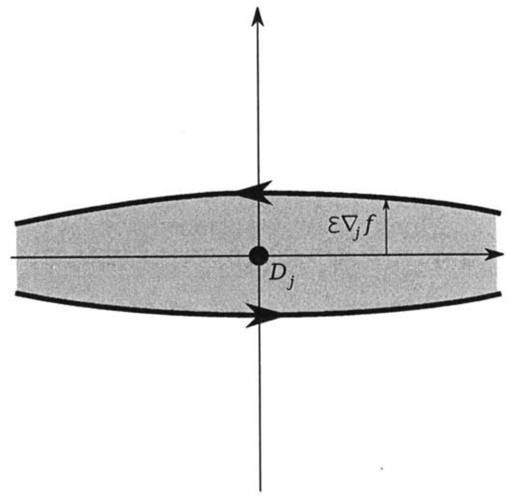
Corollary 1.5.1.10. *In an admissible flipping holomorphic pearly tree, the flipping edges can not cross the hypersurfaces $D_j^{\mathbb{R}}$ and the boundary lifts can only cross $D_j^{\mathbb{R}}$ in the positive direction.*

Definition 1.5.1.11. We define the moduli space $\mathcal{M}_{\mathcal{S}^{fpt}}(\boldsymbol{y})$ of admissible flipping holomorphic pearly trees with asymptotic conditions $\boldsymbol{y}$, by analogy with Definition 1.3.1.8.

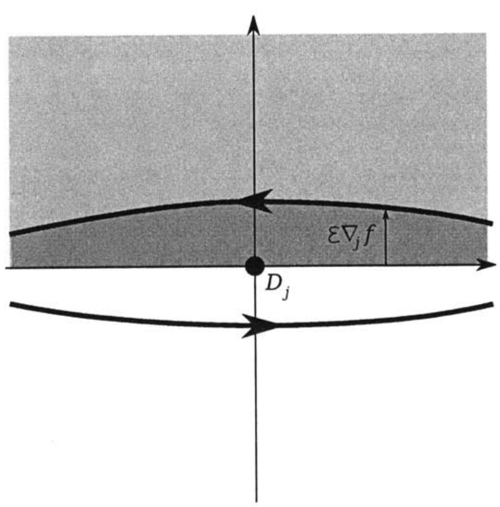
Remark 1.5.1.12. We remark that it follows from the proof of Proposition 1.5.1.9 that, if $\boldsymbol{u}$ is an admissible flipping holomorphic pearly tree, then its homology class $[\boldsymbol{u}]$ can be represented by a smooth disk in $\mathcal{P}^n$ with boundary on L^n. Namely, we perturb the divisors D_j to put ourselves in the transverse situation as described. The disk defining $[\boldsymbol{u}]$ can only intersect the divisors D_j when a boundary lift $\tilde{u}_C$ crosses $D_j^{\mathbb{R}}$ in the positive



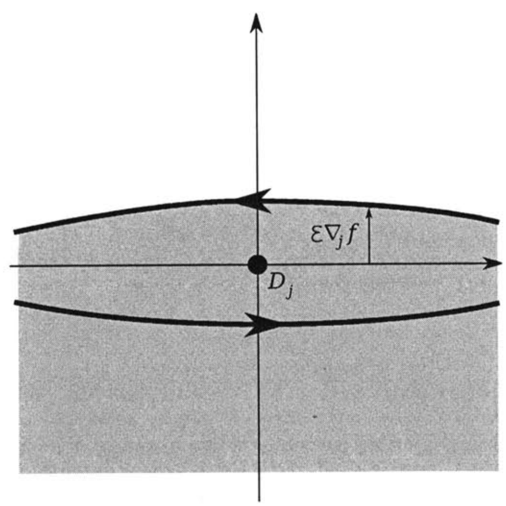
(a) Adding a strip to a flipping pearly tree, to define its homology class in $H_2(\mathbb{CP}^n, L^n)$.



(b) Projection of the strip surrounding a flipping edge crossing the hypersurface $D_j^{\mathbb{R}}$ transversely, to the z_j plane. The topological intersection number with D_j (which corresponds to the point 0 in this projection, drawn as a solid circle) is $+1$.



(c) Projection of part of the disk and strip near a positive crossing of a boundary lift $\tilde{u}_C$ with $D_j^{\mathbb{R}}$, to the z_j plane. There is a 'fold' along the real axis, so the topological intersection number with D_j is 0.



(d) Projection of part of the disk and strip near a negative crossing of a boundary lift $\tilde{u}_C$ with $D_j^{\mathbb{R}}$, to the z_j plane. The topological intersection with D_j is $+1$.

Figure 1.5.1.1: Defining and calculating $\boldsymbol{u} \cdot D_j$.

direction. It is obvious from Figure 1-1(c) that the disk can be perturbed to avoid the divisor in this case.

It follows that admissible flipping pearly trees inherit any properties of holomorphic disks in $\mathcal{P}^n$ with boundary on L^n that depend only on the topology. For example, the energy of an admissible flipping holomorphic pearly tree is given by the differences of symplectic action functionals of input and output generators, and in particular is constant in the moduli space $\mathcal{M}_{\mathcal{S}^{fpt}}(\boldsymbol{y})$. Furthermore, we can prove the following:

Proposition 1.5.1.13. *Suppose that $\boldsymbol{L}$ is a set of Lagrangian labels and $\boldsymbol{y}$ an associated set of generators. Then, for generic choice of perturbation data, $\mathcal{M}_{\mathcal{S}^{fpt}}(\boldsymbol{y})$ is a manifold of dimension*

$$\dim(\mathcal{M}_{\mathcal{S}^{fpt}}(\boldsymbol{y})) = i(\boldsymbol{y}) + k - 2.$$

Proof. The proof follows that of Proposition 1.4.4.13 – we must construct charts from the moduli spaces $\mathcal{M}_{\mathcal{S}^{fpt}(T_S,B)}(\boldsymbol{y})$ for each (T_S, B) as in Definition 1.4.4.8, and glue the pieces $\mathcal{M}_{\mathcal{S}^{fpt}(T_S,B)}(\boldsymbol{y}) \times (-\epsilon, \epsilon)^B$ together to obtain a manifold, using an analogue of Proposition 1.4.4.12.

The dimension is given by the index of the Fredholm operator used to cut out the moduli space. One might worry that the index theory of Cauchy-Riemann operators depends on a choice of holomorphic volume form η on $\mathcal{P}^n$, and our holomorphic pearls can intersect the boundary divisors D_j, where η is not defined. However, this is dealt with by Remark 1.5.1.12, which shows how to construct a smooth disk in $\mathcal{P}^n$ with boundary on L^n, near any given admissible holomorphic flipping pearly tree. One can show that the Fredholm index of the operator cutting out the moduli space of flipping pearly trees is equal to the index of the pseudo-holomorphic curve equation on the nearby disk, which depends only on the homology class of the disk in $\mathcal{P}^n$ relative to its Lagrangian boundary conditions. This is sufficient to prove the dimension formula.

Now observe that, when a new Morse edge with label L' is created as in Figure

1.4.1.2, there are two possibilities: either the lifts $\tilde{u}$ of the two boundary components of the strip on the left are antipodes, in which case a flipping edge is created, or they coincide, in which case a non-flipping edge is created. With this convention, the gluing maps of Proposition 1.4.4.12 define boundary lifts $\tilde{u}_C$ as well as the map u_p. They also preserve the homology class of Definition 1.5.1.8, and hence admissibility. □

Definition 1.5.1.14. We define a **stable flipping holomorphic pearly tree** by analogy with the definition of stable pearly trees (Definition 1.4.5.4). The only difference is for edges of trees T with both sides labeled L': these can be broken Morse flowlines of f (for flipping edges) or h (for non-flipping edges). We define a **stable admissible flipping holomorphic pearly tree** to be a stable flipping holomorphic pearly tree, each component of which is admissible.

Remark 1.5.1.15. We observe that the admissibility condition rules out sphere bubbling in families of admissible flipping holomorphic pearly trees: any sphere bubble must have intersection number 0 with the divisors D_j by admissibility, and hence have trivial homology class. But then its symplectic area is 0, so it must be constant.

Proposition 1.5.1.16. *The moduli space of stable admissible flipping holomorphic pearly trees has the structure of a compact manifold with corners.*

Proof. As in Section 1.3.1, we run into the problem that we can not appeal to a Gromov compactness theorem for immersed Lagrangians. Furthermore, we can not bypass this problem by passing to the cover $\widetilde{\mathcal{P}}^n$ of $\mathcal{P}^n$ defined in Corollary 1.3.2.3, as we did in Section 1.3.1, because the image of the Lagrangian immersion L' does not lie in $\mathcal{P}^n$. Even if we considered the corresponding branched cover of $\mathbb{CP}^n$ (branched around the divisors D_j), the Lagrangian immersion L' would only lift to a piecewise smooth embedded Lagrangian, with 'edges' along the branching divisors D_j. Again, there is no Gromov compactness theorem that deals with piecewise smooth Lagrangians.

Instead, consider the quadric

$$Q^n := \left\{ \sum_{j=0}^{n+2} z_j^2 = 0, \sum_{j=1}^{n+2} z_j = 0 \right\} \subset \mathbb{CP}^{n+2},$$

and the branched double cover

$$\begin{aligned} \rho : Q^n &\to \mathbb{CP}^n \\ \rho([z_0 : \ldots : z_{n+2}]) &= [z_1 : \ldots : z_{n+2}]. \end{aligned}$$

The cover is branched along the divisor

$$\widetilde{Q}^n := \left\{ \sum_j z_j^2 = 0 \right\} \subset \mathbb{CP}^n.$$

The real locus of Q^n in the affine chart $z_0 = i$ is the unit sphere S^n, and $\rho|_{S^n}$ is the double cover of the real locus $\mathbb{RP}^n$ of $\mathbb{CP}^n$. It is well-known that there is a symplectomorphism

$$T^*S^n \to Q^n \setminus \{z_0 = 0\},$$

sending the zero section to the real locus. This sends the radius-η disk bundle $D^*_\eta S^n$ to a neighbourhood of $\mathbb{RP}^n$, as in the construction of L^n (Section 1.2.2). Thus, the lifts of L^n and L' to $T^*S^n \subset Q^n$ are embedded. L^n lifts as the graphs of the exact one-forms $\pm\epsilon df$, and L' lifts to the zero section via the identity and via the antipodal map.

For any flipping holomorphic pearly tree $\boldsymbol{u} \in \mathcal{M}_{\mathcal{S}^{fpt}}(\boldsymbol{y})$, the topological intersection number $[\boldsymbol{u}] \cdot \widetilde{Q}^n$ depends only on the generators $\boldsymbol{y}$ (compare Proposition 1.5.2.3). We can arrange that positivity of intersection with $\widetilde{Q}^n$ holds in our moduli space, for appropriate choice of perturbation datum, and then each flipping holomorphic pearly tree in the moduli space intersects $\widetilde{Q}^n$ some finite number of times, which is bounded above by the topological intersection number. Then the lifts of flipping holomorphic pearly trees

$\boldsymbol{u} \in \mathcal{M}_{\mathcal{S}^{fpt}}(\boldsymbol{y})$ to the branched cover Q^n are branched over some finite number of points, hence have bounded genus. Gromov compactness for curves with bounded genus and boundary (see, for example, [42, 43]) then implies that the lifted family has a convergent subsequence, which corresponds to a convergent subsequence downstairs.

This shows that a sequence of admissible flipping holomorphic pearly trees has a subsequence converging to a stable flipping holomorphic pearly tree whose intersection number with each divisor D_j is 0. The intersection number of the stable flipping holomorphic pearly tree with D_j is the sum of intersection numbers of each component flipping holomorphic pearly tree with D_j. Since these are all non-negative by Proposition 1.5.1.9, they must all be 0. Thus the limit stable flipping holomorphic pearly tree is also admissible, and we have proven compactness. $\square$

We define A_∞ structure maps μ^k as in Section 1.4.7, by counting rigid flipping holomorphic pearly trees. The proof that they satisfy the A_∞ associativity equations essentially follows that of Proposition 1.4.7.1. The proof that the A_∞ product is $\mathbb{Q}$-graded relies on Proposition 1.5.1.13.

Proposition 1.5.1.17. *For sufficiently small $\epsilon > 0$, the objects L' and L^n_ϵ are quasi-isomorphic.*

Proof. We observe that $\mathbb{RP}^n$ and L^n_ϵ intersect transversely in the points p_K. Therefore we can choose the Hamiltonian component of the Floer datum for the pairs (L', L^n) and (L^n, L') to be 0. The morphism space $CF^*(L', L^n)$ is generated by pairs of points $(p, q) \in S^n \times S^n$ that get sent to the same point by the respective Lagrangian immersions defining L', L^n. Thus p is a critical point of f, and q is either equal to p or its antipode. As we saw in Corollary 1.2.2.11, there is a critical point p_K of f for each proper non-empty subset $K \subset [n+2]$. Therefore, we can label the generators of $CF^*(L', L^n)$ as $p^M_K := (p_K, p_K)$ and $p^S_K := (p_K, a(p_K))$ (M stands for 'Morse' because the generators p^M_K correspond to the Morse cohomology of L^n, and S stands for 'self-intersection' because the generators p^S_K correspond to the self-intersections of L^n). So, additively,

$$CF^*(L', L^n) \cong CM^*_M(f) \oplus CM^*_S(f)$$

and similarly for $CF^*(L^n, L')$. One can check that the gradings of these generators are

$$i(p^S_K) = \frac{n}{n+2}|K|, \quad i(p^M_K) = n - \mu_M(p_K) = n + 1 - |K|.$$

Now observe that we have natural inclusions

$$\begin{aligned} CM^*_M(f) &\overset{\varphi_1}{\hookrightarrow} CF^*(L', L^n), \\ CM^*_M(f) &\overset{\varphi_2}{\hookrightarrow} CF^*(L^n, L') \end{aligned}$$

as graded vector spaces.

Lemma 1.5.1.18. *For sufficiently small $\epsilon > 0$, the inclusions φ_j are chain maps.*

Proof. We first observe that, for sufficiently small $\epsilon > 0$, the holomorphic strips

$$u : Z \to \mathbb{CP}^n$$

used to define the differential

$$\mu^1 : CF^*(L', L^n) \to CF^*(L', L^n)$$

must remain entirely within the Weinstein neighbourhood $D^*_\eta\mathbb{RP}^n$ used in the construction of L^n_ϵ. To see why, suppose that u passes through some point p of distance $> \eta$ from $\mathbb{RP}^n$. Then for sufficiently small $\epsilon > 0$, the ball $B(p; \eta/2)$ is disjoint from L^n_ϵ and L'. Therefore, by the monotonicity lemma (see [44, 3.15]), the symplectic area of the intersection of u with the ball $B(p; \eta/2)$ is at least $c(\eta/2)^2$ for some constant c. However, the symplectic area of u is given by the difference in symplectic actions of the generators (see Remark 1.5.1.12 and its sequel), which is proportional to ϵ and hence can be made arbitrarily small. Thus, for sufficiently small $\epsilon > 0$, the strips never leave the Weinstein neighbourhood $D^*_\eta\mathbb{RP}^n$.

Now we observe that any strip u contributing to the differential on $CF^*(L', L^n)$ lifts to the double cover $D^*_\eta S^n \to D^*_\eta\mathbb{RP}^n$, because it comes equipped with a lift of one boundary component to S^n by definition. This lifted strip contributes to the differential

$$\mu^1 : CF^*(S^n, \Gamma(\epsilon df)) \to CF^*(S^n, \Gamma(\epsilon df))$$

in the Fukaya category of T^*S^n. Conversely, any strip u contributing to the differential on $CF^*(S^n, \Gamma(\epsilon df))$ projects to a strip contributing to the differential on $CF^*(L', L^n_\epsilon)$. The only thing to check is that these projected strips are all admissible – for this one needs a certain amount of control on the topology of u. It was proven in [45, Proposition 9.8] that, given $\delta > 0$, there exists $\epsilon_0 > 0$ such that for any strip contributing to the differential on $CF^*(S^n, \Gamma(\epsilon df))$, with $\epsilon < \epsilon_0$, there is a Morse flowline of f,

$$\gamma : \mathbb{R} \to S^n$$

such that

$$d(u(s,t), \gamma(\epsilon s)) < \delta \text{ for all } s, t.$$

Because Morse flowlines of f cross the hypersurfaces $D^{\mathbb{R}}_j$ positively, it follows from Proposition 1.5.1.9 that all such strips are admissible.

It follows that the inclusion

$$CF^*(S^n, \Gamma(\epsilon df)) \hookrightarrow CF^*(L', L^n)$$

(where the left hand side is a morphism space in the Fukaya category of T^*S^n and the right hand side is a morphism space in the Fukaya category of $\mathcal{P}^n$ as we have defined it) is a chain map. Now the Lagrangians $S^n, \Gamma(\epsilon df)$ in T^*S^n are Hamiltonian isotopic, hence quasi-isomorphic in the Fukaya category of T^*S^n. So there is a quasi-isomorphism

$$CF^*(S^n, \Gamma(\epsilon df)) \cong CF^*(S^n, S^n) \cong CM^*(f)$$

(the second quasi-isomorphism comes from Proposition 1.4.7.3). Thus, there is a chain map

$$CM^*_M(f) \cong CF^*(S^n, S^n) \cong CF^*(S^n, \Gamma(\epsilon df)) \hookrightarrow CF^*(L', L^n)$$

as required. □

Now consider the elements

$$f_1 \in CF^*(L^n, L'), \quad f_2 \in CF^*(L', L^n)$$

that correspond to the identity in $CM^*_M(S^n)$. Explicitly,

$$f_1 = \sum_{j=1}^{n+2} p^M_{\{j\}}$$

(and the same for f_2).

Lemma 1.5.1.19. *For sufficiently small $\epsilon > 0$, we have*

$$\begin{aligned} \mu^1(f_j) &= 0 \textit{ for } j = 1, 2, \textit{ and} \\ \mu^2(f_1, f_2) &= p_\phi \in CF^*(L', L') \end{aligned}$$

Proof. The fact that $\mu^1(f_j) = 0$ follows from Lemma 1.5.1.18. We now prove that $\mu^2(f_1, f_2) = p_\phi$.

Observe that $i(f_1) = i(f_2) = 0$, so $i(\mu^2(f_1, f_2)) = 0$. Therefore, p_ϕ is the only term that can appear in the product $\mu^2(f_1, f_2)$. Its coefficient is the signed count of points

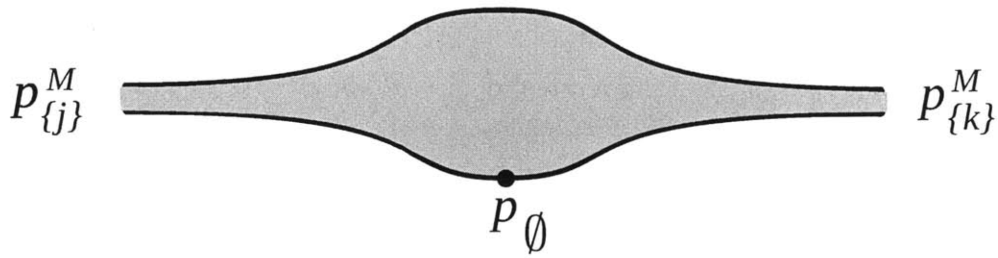


Figure 1.5.1.2: The flipping holomorphic pearly trees whose count gives the coefficient of p_ϕ in $\mu^2(f_1, f_2)$. The solid circle denotes a non-flipping point. The upper half of the boundary gets sent to L^n, and the lower half to L'.

in the moduli space of flipping holomorphic pearly trees which are holomorphic strips running between some intersections $p^M_{\{j\}}$ and $p^M_{\{k\}}$ of L^n and L', with one marked point on the boundary labeled L' which gets sent to p_ϕ (see Figure 1.5.1.2). As we saw in the proof of Lemma 1.5.1.18, such strips must lie inside the Weinstein neighbourhood $D^*_\eta \mathbb{RP}^n$, and lift canonically to the double cover $D^*_\eta S^n$. The lift is a holomorphic pearly tree contributing to the product

$$\mu^2 : CF^*(\Gamma(\epsilon df), S^n) \otimes CF^*(S^n, \Gamma(\epsilon df)) \to CF^*(S^n, S^n).$$

Conversely, by the same argument as in the proof of Lemma 1.5.1.18, any holomorphic pearly tree contributing to this product projects to an admissible flipping holomorphic pearly tree contributing to the product $\mu^2(f_1, f_2)$.

It now follows from the quasi-isomorphisms (in the Fukaya category of T^*S^n)

$$CF^*(S^n, \Gamma(\epsilon df)) \cong CF^*(S^n, S^n) \cong CF^*(\Gamma(\epsilon df), S^n)$$

and

$$CF^*(S^n, S^n) \cong CM^*(S^n)$$

that, on the level of cohomology,

$$[\mu^2(f_1, f_2)] = [p_\phi]$$

(product of identity with identity is identity in $CM^*(S^n)$). But $CF^0(L', L')$ has only the single generator p_ϕ, so we have

$$\mu^2(f_1, f_2) = p_\phi$$

as required. ☐

Because $CF^*(L', L')$ and $CF^*(L^n, L^n)$ have the same rank (by Corollary 1.3.1.11), it follows that f_1 and f_2 induce mutually inverse isomorphisms on the level of cohomology, and therefore are mutually inverse quasi-isomorphisms in the category $\mathcal{C}$. This completes the proof that L' and L^n are quasi-isomorphic, for sufficiently small $\epsilon > 0$. ☐

1.5.2 Properties of the A_∞ algebra $\mathcal{A}' := CF^*(L', L')$

We define the A_∞ algebra $\mathcal{A}' := CF^*(L', L')$. It follows from Proposition 1.5.1.17 that $\mathcal{A}$ and $\mathcal{A}'$ are quasi-isomorphic A_∞ algebras. Henceforth we will only be concerned with computing the A_∞ structure of $\mathcal{A}'$. In particular, we will assume that our flipping holomorphic pearly trees have all boundary components labeled L'.

Lemma 1.5.2.1. *If $\boldsymbol{u}$ is an admissible flipping holomorphic pearly tree with associated morphisms $\boldsymbol{y} = (p_{K_0}, \ldots, p_{K_k})$, then*

$$\sum_{j=1}^{k} e_{K_j} = e_{K_0}$$

in M.

Proof. The proof is identical to that of Proposition 1.3.2.4, since the proof relies only on the homology class $[\boldsymbol{u}] \in H_2(\mathbb{CP}^n, L^n)$, which is determined by the admissibility condition. $\square$

Lemma 1.5.2.2. *$\mathcal{A}'$ inherits the following properties of $\mathcal{A}$:*

- *It is $\mathbb{T}$-equivariant in the same sense as in Corollary 1.3.2.5, by Lemma 1.5.2.1;*
- *It has the $\mathbb{Q}$-grading given by $n/(n+2)$ times the normal $\mathbb{Z}$-grading, as in Corollary 1.3.3.2;*
- *As a consequence of these two properties, it satisfies the analogue of Corollary 1.3.3.3, namely the only non-zero A_∞ products are μ^{2+nq} for $q \in \mathbb{Z}_{\geq 0}$;*
- *It satisfies the analogues of Corollaries 1.3.4.4 and 1.3.4.5 (regarding signs and supercommutativity).*

We now establish some results about flipping holomorphic pearly trees which will be used in Section 1.5.3 to identify the moduli spaces that give rise to the A_∞ structure coefficients of $\mathcal{A}'$.

Proposition 1.5.2.3. *For $K \subset [n+2]$, define*

$$|K|' = \begin{cases} \frac{n+2}{2} & K = \phi, [n+2] \\ |K| & \textit{otherwise.} \end{cases}$$

If $\boldsymbol{u}$ is an admissible flipping holomorphic pearly tree with labels $\boldsymbol{y} = (p_{K_0}, \ldots, p_{K_k})$, then the homology class of $\boldsymbol{u}$ in $H_2(\mathbb{CP}^n, \mathbb{RP}^n) \cong \mathbb{Z}$ is given by the formula

$$d_{\boldsymbol{u}} = 2\frac{|K_0|' - \sum_{j=1}^k |K_j|'}{n+2} + k - 1.$$

Proof. Note that the Fubini-Study symplectic form ω acts on $H_2(\mathbb{CP}^n, \mathbb{RP}^n)$, with value 2π on the generator. It follows that

$$\omega(\boldsymbol{u}) = 2\pi d_{\boldsymbol{u}},$$

so we can compute $d_{\boldsymbol{u}}$ by computing $\omega(u)$.

Recall that we add a strip to $\boldsymbol{u}$ to obtain a disk $\tilde{u} : (D, \partial D) \to (\mathbb{CP}^n, L^n)$. Note that the symplectic area of the strip we add is $\mathcal{O}(\epsilon)$. So we can compute $\omega(\boldsymbol{u})$ by evaluating $\omega(\tilde{u})$ in the limit $\epsilon \to 0$.

The Fubini-Study form is given by the Kähler potential

$$\rho = \log\left(\sum_{j=1}^{n+2}\left|\frac{z_j}{z_1}\right|^2\right) = \log\left(\sum_{j=1}^{n+2} e^{2r_j}\right) - 2r_1$$

on $\mathbb{CP}^{n+1} \setminus D_1$, where $z_j = \exp(r_j + i\theta_j)$. Thus

$$\omega = dd^c\rho,$$

(recall that $d^c\rho = d\rho \circ J$), so we define

$$\begin{aligned}
\alpha &= d^c\rho \\
&= \frac{\sum_{j=1}^{n+2} 2e^{2r_j} d^c r_j}{\sum_{j=1}^{n+2} e^{2r_j}} - 2d^c r_1 \\
&= -2\frac{\sum_{j=1}^{n+2} e^{2r_j} d\theta_j}{\sum_{j=1}^{n+2} e^{2r_j}} + 2d\theta_1.
\end{aligned}$$

Then $\omega = d\alpha$. Of course this is really $\pi^*\alpha$, where $\pi : \mathbb{C}^{n+2} - \{0\} \to \mathbb{CP}^{n+1}$ is the projection.

Because $\tilde{u} \cdot D_1 = 0$ by admissibility, we can deform $\tilde{u}$ to avoid D_1 then apply Stokes' theorem to obtain

$$\int_{\partial D} \tilde{u}^*\alpha = \int_D \tilde{u}^*\omega.$$

Now recall the lift of L^n to $\mathbb{C}^{n+2}$ that arose in the construction of L^n, namely

$$\begin{array}{rcl} \left\{\sum_{j=1}^{n+2} x_j^2 = 1, \sum_{j=1}^{n+2} x_j = 0\right\} & \to & \mathbb{C}^{n+2} \\ (x_1, \ldots, x_{n+2}) & \mapsto & (x_1 + i\epsilon f_1, \ldots, x_{n+2} + i\epsilon f_{n+2}) + \mathcal{O}(\epsilon^2). \end{array}$$

We can lift ∂D to $\mathbb{C}^{n+2}$ (the result will not be a cycle, because when ∂D changes sheets of L^n the lift stops and reappears at the antipode). Call the lift l. Then

$$\int_{\partial D} \alpha = \int_{\pi_* l} \alpha = \int_l \pi^* \alpha.$$

Observe that on the lift of L^n, $d\theta_k$ is small everywhere except for when r_k is small, and when r_k is small then

$$\frac{e^{2r_k}}{\sum_{j=1}^{n+2} e^{2r_j}}$$

is small. Thus the first term in $\pi^*\alpha$ is negligible. So

$$\int_l \pi^* \alpha = \int_l 2d\theta_1 + \mathcal{O}(\epsilon).$$

The projection of the lift of the point p_K to the angular variables is πe_K (now thought of as living in $\widetilde{M_{\mathbb{R}}}$ rather than $M_{\mathbb{R}}$). Thus, as the lift of ∂D travels from p_{K_i} to $p_{\bar{K}_{i+1}}$, the contribution to the integral is (to order ϵ)

$$\int_{p_{K_j}}^{p_{\bar{K}_{j+1}}} 2d\theta_1 = 2\pi e_1 \cdot \left(e_{\bar{K}_{j+1}} - e_{K_j}\right).$$

An exception occurs when K_j (respectively $\bar{K}_{j+1}$) $= \phi$ or $[n+2]$, in which case p_{K_j} (respectively $p_{\bar{K}_{j+1}}$) represents the bottom or top cohomology class of L^n, so ∂D does not change sheets of L^n as it passes through p_{K_j} (respectively $p_{\bar{K}_{j+1}}$). In this case we should simply replace $e_1 \cdot (e_{K_j})$ (respectively $e_1 \cdot (e_{\bar{K}_{j+1}})$) in the expression above by 0.

For the moment, assume that $K_j \neq \phi$ or $[n+2]$. Adding up and regrouping the contributions of each part of ∂D, and recalling that p_{K_0} is the 'outgoing' point, we obtain:

$$\begin{aligned}
\int_{\partial D} \alpha &= 2\pi e_1 \cdot \left(e_{K_0} - e_{\bar{K}_0} + \sum_{j=1}^{k} e_{\bar{K}_j} - e_{K_j} \right) + \mathcal{O}(\epsilon) \\
&= 2\pi \left(2e_1 \cdot \left(e_{K_0} - \sum_{j=1}^{k} e_{K_j} \right) + k - 1 \right) + \mathcal{O}(\epsilon) \\
&= 2\pi \left(2\frac{|K_0| - \sum_{j=1}^{k} |K_j|}{n+2} + k - 1 \right) + \mathcal{O}(\epsilon)
\end{aligned}$$

(in the last step we used the fact that the vector is a multiple of $e_{[n+2]}$ by Proposition 1.3.2.4).

Now if $K_j = \phi$ or $[n+2]$, recall that we must replace $e_1 \cdot (e_{\bar{K}_j} - e_{K_j})$ by 0 in the first two lines above. This is equivalent to replacing $|K_j|$ by $|K_j|'$ in the final line. This completes our proof. $\square$

Definition 1.5.2.4. Given an admissible flipping holomorphic pearly tree, it is useful to label certain points on its boundary with proper, non-empty subsets of $[n+2]$, as follows: At each flipping marked point, the boundary immediately before and after the point get sent (by the lift $\tilde{u}$ of the boundary) to antipodal points of $S^n \setminus D^{\mathbb{R}}$. Thus they lie in the antipodal regions $S^n_K, S^n_{\bar{K}}$ respectively, for some $K \subset [n+2]$ (recall that S^n_K is defined to be the region where $x_j < 0$ for $j \in K$ and $x_j > 0$ for $j \notin K$). We will ignore the case where a flipping marked point lies on some $D^{\mathbb{R}}_j$, but it presents no real additional problem in our subsequent arguments. We label the point immediately before our flipping marked point with K, and the point immediately after with $\bar{K}$. Non-flipping marked points do not get labels.

Remark 1.5.2.5. We observe that, because of the condition that Morse flowlines do not cross the hypersurfaces $D^{\mathbb{R}}_j$ (by Corollary 1.5.1.10), the labels at opposite ends of an internal flipping Morse flowline are identical. Furthermore, at a flipping marked point connected by an incoming edge to the flipping generator p_K, the label immediately before is K and the label immediately after is $\bar{K}$. Also, by Corollary 1.5.1.10, the boundary lifts can only cross the hypersurfaces positively. So as we follow the boundary around anti-clockwise between two adjacent flipping marked points, the label at the beginning of the segment contains (not necessarily strictly) the label at the end of the segment. Suppose the pearl corresponding to vertex v of the underlying tree has degree $d_v \in H_2(\mathbb{CP}^n, \mathbb{RP}^n) \cong \mathbb{Z}$. Then it must intersect D_j d_v times, and none of the intersections can be internal by admissibility, so the boundary lift must intersect $D_j^{\mathbb{R}}$ d_v times. It follows that

$$\sum_{j \bmod k_v} e_{\bar{K}_{j-1}} - e_{K_j} = d_v e_{[n+2]}$$

in $\widetilde{M}$, where $K_1, \ldots, K_{k_v}$ are the labels given to the points immediately before the flipping points (traversing the boundary of the pearl in positive direction) on the pearl corresponding to v. It follows quickly that

$$\sum_{j=1}^{k_v} e_{K_j} = \frac{k_v - d_v}{2} e_{[n+2]}$$

for each pearl. Figure 1.5.2.1 shows a possible labeling of a flipping holomorphic pearly tree.

Remark 1.5.2.6. We will choose the almost-complex structure component of our perturbation data to be equal to the standard integrable complex structure J_0, and the Hamiltonian perturbation to be identically 0. Then the pearls in a flipping holomorphic pearly tree with labels L' are holomorphic disks with boundary on $\mathbb{RP}^n$, hence they can be 'doubled' to a holomorphic sphere by the Schwarz reflection principle. It follows from [40, Proposition 7.4.3] that the moduli space of holomorphic spheres in $\mathbb{CP}^n$, in a given homology class, is automatically regular. The moduli space of pearls is the real part of the moduli space of spheres, hence also regular. It follows that for every

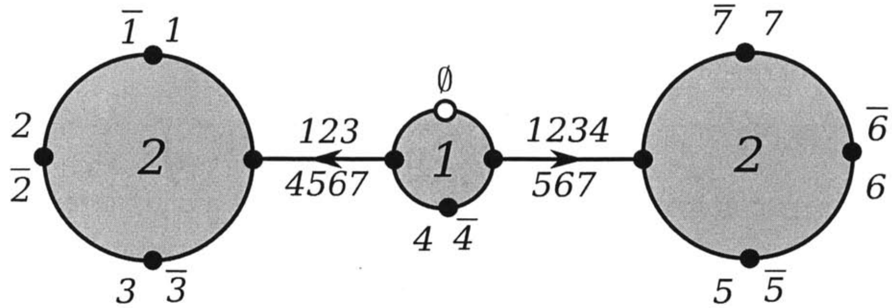


Figure 1.5.2.1: An example of a legal labeling of a flipping holomorphic pearly tree, which might contribute to the coefficient of p_{ϕ} in the A_∞ product $\mu^7(p_{\{1\}}, \ldots, p_{\{7\}})$. We have illustrated a simple case, in which all external flowlines are constant because the points $p_{\{j\}}$ are maxima of the Morse function f. The external label '1' means the set $\{1\}$, while '$\overline{1}$' means the complement $\{2,3,4,5,6,7\}$. The big label '1' in the middle of a pearl means that the pearl has degree 1.

$(r, \boldsymbol{u}) \in \mathcal{M}_{\mathcal{S}^{fpt}}(\boldsymbol{y})$, the linearized operator

$$D^h_{\mathcal{S}^{fpt},r,\boldsymbol{u}} : T_{(r,\boldsymbol{u})}\left(\mathcal{B}_{\mathcal{S}^{fpt}}\right) \to (\mathcal{E}_{S_r})_{\boldsymbol{u}}$$

of Definition 1.4.4.4 is automatically surjective. Thus, to show that a moduli space $\mathcal{M}_{\mathcal{S}^{fpt}}(\boldsymbol{y})$ of flipping holomorphic pearly trees is regular, we need only check that the evaluation map

$$\boldsymbol{ev} : \ker(d_{\mathcal{S}^{fpt}}) \to T_{\boldsymbol{u}}\left((S^n)^{F(S)}\right)$$

is surjective at each $(r, \boldsymbol{u}) \in \mathcal{M}_{\mathcal{S}^{fpt}}(\boldsymbol{y})$. Note that $\ker(d_{\mathcal{S}^{fpt}})$ is the space of holomorphic pearls and Morse flowlines, without the constraint $\boldsymbol{ev}(\boldsymbol{u}) \in \Delta^S$.

Definition 1.5.2.7. The following notation will be useful. If $K_1, \ldots, K_k$ are disjoint subsets of $[n+2]$, we define

$$F_{K_1,K_2,\ldots,K_k} := \{\boldsymbol{x} \in S^n : x_l = x_m \text{ for all } l, m \in K_i, \text{ for all } i\}.$$

Remark 1.5.2.8. Observe that

$$F_{K,\bar{K}} = \{p_K, p_{\bar{K}}\}.$$

As we saw in Lemma 1.2.2.12, the unstable manifold $\mathcal{U}(K)$ of p_K is an open subset of $F_{\bar{K}}$, and the stable manifold $\mathcal{S}(K)$ is an open subset of F_K.

1.5.3 Computation of $\mathcal{A}'$

In this section we compute the A_∞ structure of $\mathcal{A}'$.

First, we observe that the analogue of Corollary 1.3.3.5 holds for $\mathcal{A}'$. I.e., $\mu^1 = 0$ and the only possibly non-zero μ^2 products are

$$\mu^2_{\mathcal{A}'}(p_{K_1}, p_{K_2}) = a'(K_1, K_2) p_{K_1 \sqcup K_2}$$

for disjoint K_1, K_2. The proof is exactly the same, using the corresponding properties of $\mathcal{A}'$ given in Lemma 1.5.2.2.

Proposition 1.5.3.1. *We have*

$$a'(K_1, K_2) = \pm 1.$$

Proof. Let $K_3 := \overline{K_1 \sqcup K_2}$, so $K_1 \sqcup K_2 \sqcup K_3 = [n+2]$. If any of K_1, K_2, K_3 are ϕ or $[n+2]$, the result is easy as the corresponding holomorphic disks are constant. If that is not the case, then $a'(K_1, K_2)$ is given by a count of flipping holomorphic pearly trees. The homology class of such a flipping holomorphic pearly tree is

$$\left(2\frac{|K_1 \sqcup K_2|' - |K_1|' - |K_2|'}{n+2} + 2 - 1 \right) = 1$$

by Proposition 1.5.2.3. Therefore the corresponding flipping holomorphic pearly tree has two incoming and one outgoing legs, and a single pearl with the homology class of half of a line in $\mathbb{CP}^n$ with boundary on $\mathbb{RP}^n$.

The real part of such a pearl is a line. Thus, $a'(K_1, K_2)$ counts lines passing through the unstable manifolds $\mathcal{U}(K_1)$, $\mathcal{U}(K_2)$, $\mathcal{U}(K_3)$. Recall from Lemma 1.2.2.12 that the unstable manifolds $\mathcal{U}(K_i)$ are contained in the linear spaces $F_{\bar{K}_i}$ (see Definition 1.5.2.7).

Given points $p_1 \in F_{\bar{K}_1}$ and $p_2 \in F_{\bar{K}_2}$, the line through p_1 and p_2 is contained in the linear space $F_{\bar{K}_1 \cap \bar{K}_2} = F_{K_3}$. This space intersects $F_{\bar{K}_3}$ transversely at p_{K_3}. Therefore there is a unique line (namely F_{K_1,K_2,K_3}) that intersects $\mathcal{U}(K_1)$, $\mathcal{U}(K_2)$, $\mathcal{U}(K_3)$ (at p_{K_1}, p_{K_2}, p_{K_3} respectively), and the intersections are transverse so the flipping holomorphic pearly tree is regular.

We check that it is admissible, using Proposition 1.5.1.9. Firstly, the Morse flowlines are constant at the p_{K_i}, hence do not cross the hypersurfaces $D_j^{\mathbb{R}}$. Secondly, the boundary lifts as

$$p_{K_1} \rightsquigarrow p_{K_2 \sqcup K_3} \to p_{K_2} \rightsquigarrow p_{K_1 \sqcup K_3} \to p_{K_3} \rightsquigarrow p_{K_1 \sqcup K_2} \to p_{K_1}$$

where $\to$ denotes a straight line connecting two points and $\rightsquigarrow$ denotes changing sheet. This lift clearly crosses all hypersurfaces $D_j^{\mathbb{R}}$ positively (since the label at the beginning of a straight line always contains the label at the end), so the flipping holomorphic pearly tree is admissible and regular.

Thus $a'(K_1, K_2) = \pm 1$ as required. $\square$

We are now in a position to prove Theorem 1. It is implied by the following:

Theorem 3. *The cohomology algebra of $\mathcal{A}$ is*

$$H^*(\mathcal{A}) \cong \Lambda^* \widetilde{M}_{\mathbb{C}}$$

as $\mathbb{Z}_2$-graded associative $\mathbb{C}$-algebras. The isomorphism is given by

$$p_K \mapsto \sigma_K \bigwedge_{j \in K} e_j,$$

for some sign $\sigma_K = \pm 1$.

Proof. We define a homomorphism of $\mathbb{C}$-algebras from the tensor algebra of $\widetilde{M}_{\mathbb{C}}$ to the cohomology algebra of $\mathcal{A}$, by

$$\begin{aligned} \bigoplus_{k=1}^{\infty}(\widetilde{M}_{\mathbb{C}})^{\otimes k} &\to H^*(\mathcal{A}), \\ e_j &\mapsto p_{\{j\}} \text{ for all } j \in [n+2]. \end{aligned}$$

By Corollary 1.3.4.5, this descends to a homomorphism

$$\Lambda^* \widetilde{M}_{\mathbb{C}} \to H^*(\mathcal{A}).$$

It follows from Proposition 1.5.3.1 that the elements $p_{\{j\}}$ generate the algebra $H^*(\mathcal{A}')$, and hence the corresponding elements generate $H^*(\mathcal{A})$, by Proposition 1.5.1.17. Therefore this homomorphism is surjective, so because both sides have the same rank it must be an isomorphism. $\square$

Now we consider the next non-trivial A_∞ product in $\mathcal{A}'$, μ^{n+2}. We aim to compute

$$\mu^{n+2}(p_{\{\sigma(1)\}}, \ldots, p_{\{\sigma(n+2)\}}),$$

where σ is a permutation of $[n+2]$ (these are the important products to compute in order to apply deformation theory, because they determine the deformation class of the A_∞ structure (see Section 1.5.4).

Proposition 1.5.3.2. *In $\mathcal{A}'$, we have*

$$\mu^{n+2}(p_{\{\sigma(1)\}}, \ldots, p_{\{\sigma(n+2)\}}) = \pm p_{\phi},$$

for exactly one permutation σ of $[n+2]$. For all other permutations, the result is 0. A different choice of the point p_ϕ (the minimum of the Morse function h) will lead to a different permutation σ.

Proof. First, note that p_ϕ is the only term that can appear in this product, for grading reasons (Corollary 1.3.3.2).

Note also that $\mathcal{U}(p_{\{j\}}) = \{p_{\{j\}}\}$ and $\mathcal{S}(p_\phi) = \{p_\phi\}$, so the external gradient flowlines of the flipping holomorphic pearly trees contributing to the coefficient of p_ϕ in this product are constant. We split the proof into two parts: counting the flipping holomorphic pearly trees with a single 'pearl' (we show that these give the desired answer) and proving that there are no 'multiple-pearl trees' contributing to the product.

For the first part, Proposition 1.5.2.3 shows that a disk contributing to this product must have degree n. By pairing such a disk with its conjugate we obtain a degree-n curve through the $n+3$ points $p_{\{1\}}, \ldots, p_{\{n+2\}}, p_\phi$. It is a classical theorem of Veronese that there is a unique rational normal curve through $n+3$ generic points in $\mathbb{CP}^n$. A constructive proof is given in [46, p. 10]. We just need to check that this curve satisfies the conditions required for the definition of an admissible flipping holomorphic pearly tree – namely, the curve should be real, and its real part should admit a lift to S^n which changes sheet at each point $p_{\{j\}}$ and crosses the hypersurfaces $D_k^{\mathbb{R}}$ positively.

By the construction in [46], we can parametrize our curve as $u : \mathbb{CP}^1 \to \mathbb{CP}^n$,

$$\begin{aligned} u(z) &:= \begin{pmatrix} n+1 & -1 & \ldots & -1 \\ -1 & n+1 & \ldots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \ldots & n+1 \end{pmatrix} \begin{pmatrix} (z-\nu_1)^{-1} \\ (z-\nu_2)^{-1} \\ \vdots \\ (z-\nu_{n+2})^{-1} \end{pmatrix} \\ &= \left[\frac{n+1}{z-\nu_1} - \sum_{j\neq 1} \frac{1}{z-\nu_j} : \frac{n+1}{z-\nu_2} - \sum_{j\neq 2} \frac{1}{z-\nu_j} : \ldots : \frac{n+1}{z-\nu_{n+2}} - \sum_{j\neq n+2} \frac{1}{z-\nu_j} \right]. \end{aligned}$$

Observe that this curve has degree n: if we clear denominators, the leading coefficients z^{n+1} in all factors cancel, leaving polynomials of degree n. Furthermore, we have

$$u(\nu_j) = [-1 : -1 : \ldots : n+1 : \ldots : -1] = p_{\{j\}}.$$

We choose the ν_j so that $u(0) = p_\phi$, i.e.,

$$\begin{pmatrix} n+1 & -1 & \dots & -1 \\ -1 & n+1 & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \dots & n+1 \end{pmatrix} \begin{pmatrix} \nu_1^{-1} \\ \nu_2^{-1} \\ \vdots \\ \nu_{n+2}^{-1} \end{pmatrix} = p_\phi.$$

Note that this parametrization automatically gives a lift of the boundary $\mathbb{RP}^1$ to $\mathbb{R}^{n+1} \setminus \{0\}$ and hence to S^n. Furthermore, the parametrization changes sheets exactly at the flipping points ν_j, because the sign of the dominant term $(z-\nu_j)^{-1}$ changes there. We just have to check that it crosses all of the real hypersurfaces $D_k^{\mathbb{R}}$ positively. This is true because if

$$\frac{n+1}{z-\nu_k} - \sum_{j \neq k} \frac{1}{z-\nu_j} = 0,$$

then the derivative

$$-\frac{n+1}{(z-\nu_k)^2} + \sum_{j \neq k} \frac{1}{(z-\nu_j)^2} > 0$$

by the quadratic-arithmetic mean inequality (alternatively one can graph the function).

Thus, the two halves of this curve are the only disks that can contribute to such a product, and only one passes through p_ϕ (the other has the opposite lift of the boundary, hence passes through the antipode of p_ϕ). The permutation σ is determined by the ordering of the coordinates of the chosen point p_ϕ.

It is clear from our construction that this pearl is regular. Namely, because we have exhibited a construction of a degree-n curve through $n+3$ arbitrary generic points in $\mathbb{RP}^n$, if we fix all boundary points $p_{\{j\}}, p_\phi$ except for one, then the evaluation map at the remaining point is transverse to the point.

Now we proceed with the second part of the proof, namely showing that multiple-pearl trees do not contribute. Suppose we have a contribution from a multiple-pearl tree.

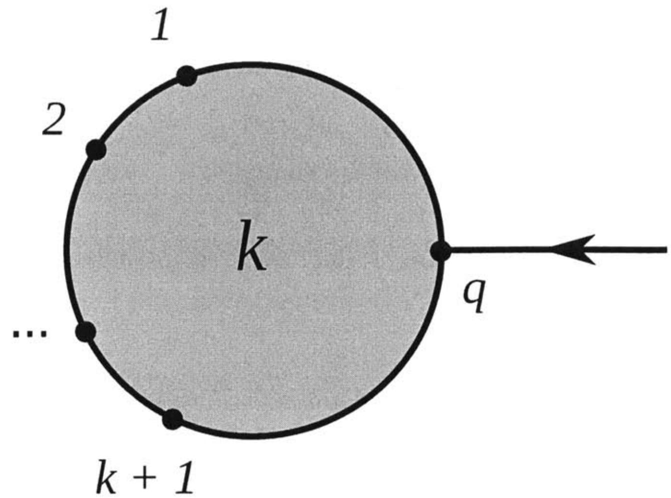


Figure 1.5.3.1: Part of a multiple-pearl tree that may contribute to μ^{n+2}. The label 'j' on a marked point means that marked point gets mapped to $p_{\{j\}}$, while the big label 'k' in the middle of the pearl means that pearl has degree k.

The tree must contain a pearl with exactly one internal edge attached. Without loss of generality it has input flipping generators $p_{\{1\}}, \ldots, p_{\{k+1\}}$ and a single Morse flowline attached at point q, as shown in Figure 1.5.3.1 (it may also have the 'output' point p_ϕ on its boundary, but whether it does or not is irrelevant to the following argument). If q is non-flipping then it follows from Remark 1.5.2.5 that $k = n + 1$, so this is not a multiple-pearl tree. If q is flipping, then it follows by Remark 1.5.2.5 that it has degree k, where we assume $k < n$.

Any degree-k curve in $\mathbb{CP}^n$ is contained in a linear subspace of dimension k (this can be proved by induction on n: choose any $k + 1$ points on the curve and a hyperplane through those points, then the hyperplane intersects the degree-k curve in more than k points so the curve is contained in the hyperplane by Bezout's Theorem). In our case, there is a unique dimension-k linear subspace through the points $p_{\{1\}}, \ldots, p_{\{k+1\}}$, namely $F_{\overline{[k+1]}}$ (to clarify: $\overline{[k+1]} = \{k+2, \ldots, n+2\}$).

Therefore our pearl is a degree-k curve in a k-dimensional projective space, so by the first half of the argument, the evaluation map at q runs over an open subset of $F_{\overline{[k+1]}}$.

But this subspace is preserved by the Morse flow of f, by the equivariance of f with respect to the S_{n+2} action. Hence the Morse flow at q is parallel to the evaluation map, so the evaluation map at p_ϕ has dimension (at least) 1 less than expected. Thus, for a generic choice of p_ϕ, the moduli space will be empty.

Thus the only contributions to the product come from the single-pearl tree, which gives the advertised result. $\square$

Remark 1.5.3.3. We observe that the final argument, in which we showed that multiple-pearl flipping holomorphic pearly trees do not contribute to the product, remains true even if we make a small change in our perturbation data: observe that, by Remark 1.5.2.5, q lies in the region $S^n_{[k+1]}$. If we perturb the holomorphic curve equation by a small amount, the perturbed evaluation map at q can be made arbitrarily C^0-close to the unperturbed one. Thus, the image of the perturbed evaluation map at q is contained in an arbitrarily small open neighbourhood of $F_{\overline{[k+1]}} \cap S^n_{[k+1]}$.

Now the Morse flowline emanating from q remains inside the region $S^n_{[k+1]}$, since flipping flowlines cannot cross the hypersurfaces by Corollary 1.5.1.10. But $F_{\overline{[k+1]}} \cap S^n_{[k+1]}$ is exactly the intersection of the unstable manifold of $p_{[k+1]}$ with $S^n_{[k+1]}$, so the flowline remains inside an arbitrarily small open neighbourhood of $F_{\overline{[k+1]}} \cap S^n_{[k+1]}$. Given that, for generic p_ϕ, the evaluation map at the other end of the Morse flowline misses $F_{\overline{[k+1]}}$, it also misses a sufficiently small neighbourhood of it. Therefore, for a sufficiently small perturbation, the moduli space remains empty.

1.5.4 Versality of $\mathcal{A}'$

We aim to prove Theorem 2 by applying the techniques of [9, Section 3], in the equivariant setting. All our conventions on signs and gradings are taken from that paper. We review some necessary definitions and results.

Definition 1.5.4.1. Consider the $\mathbb{Q}$-graded algebra

$$A := \Lambda^* \left(\widetilde{M_{\mathbb{C}}} \right),$$

where the grading is given by $n/(n+2)$ times the normal ($\mathbb{Z}$-)grading. Define an action of the character group of M,

$$\mathbb{T} := \mathrm{Hom}(M, \mathbb{C}^*),$$

on A by

$$\alpha \cdot e := \alpha(e)e.$$

Let $\mathfrak{A}(A)$ denote the set of $\mathbb{Q}$-graded, $\mathbb{T}$-equivariant A_∞-algebras with underlying graded vector space A, $\mu^1 = 0$ and

$$\mu^2(a_2, a_1) = (-1)^{|a_1|} a_2 \wedge a_1.$$

Proposition 1.5.4.2. *Recall that the ($\mathbb{T}$-equivariant) Hochschild cohomology of A is given by the Hochschild-Kostant-Rosenberg isomorphism [47]:*

$$HH^{s+t}(A, A)^{t,\mathbb{T}} \cong \bigoplus_{\frac{2}{n+2}s + \frac{n}{n+2}j = s+t} \left(Sym^s(\widetilde{M_{\mathbb{C}}^{\vee}}) \otimes \Lambda^j \left(\widetilde{M_{\mathbb{C}}} \right) \right)^{\mathbb{T}}.$$

For $d > 2$, we have

$$HH^2(A, A)^{2-d,\mathbb{T}} = \begin{cases} \mathbb{C} \cdot W & \text{for } d = n+2 \\ 0 & \text{otherwise,} \end{cases}$$

where $W = z_1 \ldots z_{n+2} = z^{e_{[n+2]}}$ is the superpotential of the mirror, viewed as an element of the symmetric tensor product $Sym^{n+2}(\widetilde{M_{\mathbb{C}}^{\vee}})$.

Proof. Suppose we have a generator

$$z^a \bigwedge_{k \in K} e_k \in HH^2(A, A)^{2-d, \mathbb{T}}.$$

Here $a \in \widetilde{M}^{\vee}_{\geq 0}$, $K \subset [n+2]$ and $d = \deg(z^a) > 2$. $\mathbb{T}$-equivariance simply says that

$$a = e_K + q e_{[n+2]}$$

for some $q \in \mathbb{Z}_{\geq 0}$ (here we identify $\widetilde{M}^{\vee}$ with $\widetilde{M}$ in the natural way). To lie in HH^2 we must have

$$\begin{aligned} 2 &= \frac{2}{n+2}\deg(z^a) + \frac{n}{n+2}|K| \\ &= \frac{2}{n+2}(|K| + q(n+2)) + \frac{n}{n+2}|K| \\ &= |K| + 2q. \end{aligned}$$

Now we have

$$2 < d = \deg(z^a) = |K| + (n+2)q = 2 + nq,$$

hence $q > 0$. Therefore, we must have $K = \phi, q = 1$ and $a = e_{[n+2]}$. Thus the generator is $z^a = W$. □

Proposition 1.5.4.3. *$\mathcal{A}'$ is a versal element of $\mathfrak{A}(A)$, in the sense of a $\mathbb{T}$-equivariant version of [9, Lemma 3.2], with deformation class $\pm W \in HH^2(A, A)^{-n, \mathbb{T}}$. In particular, any element of $\mathfrak{A}(A)$ with the same deformation class is quasi-isomorphic to $\mathcal{A}'$.*

Proof. The fact that $\mathcal{A}'$ lies in $\mathfrak{A}(A)$ follows from our previous results, namely Lemma 1.5.2.2:

- $\mu^1 = 0$ as the only non-zero A_∞ products are μ^{2+nq} for $q \in \mathbb{Z}_{\geq 0}$;

- the underlying algebra is A (Theorem 3);

- the grading on A is $n/(n+2)$ times the usual grading;
- it is equivariant with respect to the action of $\mathbb{T}$.

The fact that $\mathcal{A}'$ is versal follows from the results:

- $\mu^k = 0$ for $2 < k < n+2$ (by the analogue of Corollary 1.3.3.3);
- The first non-trivial higher product μ^{n+2} satisfies

$$\mu^{n+2}(e_1, \ldots, e_{n+2}) = \pm 1$$

(without loss of generality) but is 0 on all other permutations of the generators e_i (Proposition 1.5.3.2). Therefore the deformation class of $\mathcal{A}'$ in $HH^2(A,A)^{-n}$ is given (by the HKR isomorphism) by

$$\mu^{n+2}(\boldsymbol{z}, \ldots, \boldsymbol{z}) = \pm z_1 \ldots z_{n+2} = \pm W(z),$$

where $\boldsymbol{z} = \sum_j z_j e_j$. Combining this with Proposition 1.5.4.2 gives the result.

$\square$

1.6 Matrix factorizations

We now consider the other side of mirror symmetry. Recall (from the Introduction) that the putative mirror to $\mathcal{P}^n$ is the Landau-Ginzburg model $(\mathrm{Spec}(R), W)$, where

$$\begin{aligned} R &:= \mathbb{C}\left[\widetilde{M}\right] \\ W &= z^{e_{[n+2]}}. \end{aligned}$$

Observe that there is a natural action of $\mathbb{T}$ on R that preserves W (recall $\mathbb{T} := \mathrm{Hom}(M, \mathbb{C}^*)$).

Also recall (from the Introduction) that the B-model on $(\mathrm{Spec}(R), W)$ is given by the triangulated category of singularities of $W^{-1}(0)$, which is quasi-equivalent (by [12, Theorem 3.9]) to the category $MF(R, W)$ of matrix factorizations of W. The object corresponding to our Lagrangian L^n is the skyscraper sheaf at the origin,

$$\mathcal{O}_0 \in D^b_{\mathsf{Sing}}(W^{-1}(0)).$$

Henceforth, we work entirely in the category $MF(R, W)$. We abuse notation, and denote also by $\mathcal{O}_0$ the matrix factorization corresponding to $\mathcal{O}_0$ under the above quasi-equivalence.

To prove Theorem 2, we must show that the differential $\mathbb{Z}_2$-graded algebra of endomorphisms of $\mathcal{O}_0$,

$$\mathcal{B} := \mathrm{Hom}^*_{MF(R,W)}(\mathcal{O}_0, \mathcal{O}_0),$$

is quasi-isomorphic to $\mathcal{A}$.

It is explained in [25] how to compute a minimal A_∞ model for the endomorphism algebra of $\mathcal{O}_0$. That paper focuses on the case where W has an isolated singularity at 0, which is certainly not true in our case, but the computation of the minimal A_∞ model does not rely on this assumption. We briefly review the construction, explaining how the $\mathbb{T}$-action enters the picture.

The matrix factorisation corresponding to $\mathcal{O}_0$ is the Koszul resolution of $\mathcal{O}_0$

$$R \otimes \Lambda^* \widetilde{M}$$

with the deformed differential

$$\delta := \iota_u + v \wedge \cdot$$

where

$$\begin{aligned} u &= \sum_{j=1}^{n+2} z_j \theta_j^\vee \in R \otimes \Lambda^* \widetilde{M}^\vee \\ v &= \sum_{j=1}^{n+2} a_j \frac{W}{z_j} \theta_j \in R \otimes \Lambda^* \widetilde{M} \end{aligned}$$

where $\{\theta_j\}$ is a relabeling of the canonical basis for $\widetilde{M}$, $\{\theta_j^\vee\}$ is the dual basis of $\widetilde{M}^\vee$, and a_j are numbers adding up to 1. Alternatively, we can write this matrix factorisation as

$$\left(R\left\langle \theta_1, \ldots, \theta_{n+2} \right\rangle, \delta\right),$$

where

$$\delta = \sum_j z_j \frac{\partial}{\partial \theta_j} + a_j \frac{W}{z_j} \theta_j.$$

The endomorphism algebra of $\mathcal{O}_0$ is the algebra

$$R \otimes \Lambda^* \widetilde{M}^\vee \otimes \Lambda^* \widetilde{M}.$$

This can be thought of as the commutative algebra of differential operators

$$\mathcal{B} := R\left\langle \theta_1, \ldots, \theta_{n+2}, \frac{\partial}{\partial \theta_1}, \ldots, \frac{\partial}{\partial \theta_{n+2}} \right\rangle$$

with the differential given by $d = [\delta, -]$. One can check that

$$\begin{aligned} d(\theta_j) &= z_j \\ d\left(\frac{\partial}{\partial \theta_j}\right) &= a_j \frac{W}{z_j}. \end{aligned}$$

Thus the cohomology algebra $H^*(\mathcal{B}, d)$ is generated by the elements

$$\bar{\partial}_j := \frac{\partial}{\partial \theta_j} - a_j \frac{W}{z_j z_k} \theta_k$$

for some $k \neq j$ (this is proven in [25] by constructing an explicit homotopy contracting $\mathcal{B}$ onto the subcomplex generated by the $\bar{\partial}_j$). The generators $\bar{\partial}_j$ supercommute, so the cohomology algebra can be naturally identified with

$$A = \Lambda^*(\widetilde{M}_{\mathbb{C}})$$

via

$$\bar{\partial}_j \mapsto e_j.$$

This proves that

$$H^*\left(\mathrm{Hom}^*_{MF(R,W)}(\mathcal{O}_0, \mathcal{O}_0)\right) \cong \Lambda^*\mathbb{C}^{n+2}$$

as $\mathbb{Z}_2$-graded associative $\mathbb{C}$-algebras.

We observe that the action of $\mathbb{T}$ extends in the natural way to $\mathcal{B}$, and that δ is invariant under the action of $\mathbb{T}$, so the differential algebra structure of $\mathcal{B}$ is $\mathbb{T}$-equivariant.

Furthermore, observe that if we assign $\mathbb{Q}$-gradings

$$|z_j| = \frac{2}{n+2}, \; |\theta_j| = -\frac{n}{n+2}, \; \left|\frac{\partial}{\partial\theta^j}\right| = \frac{n}{n+2},$$

then the product structure on $\mathcal{B}$ respects the grading (because $|\theta_j| + |\partial/\partial\theta_j| = 0$), and the differential on $\mathcal{B}$ has degree $|\delta| = +1$. Therefore $(\mathcal{B}, d)$ is a $\mathbb{T}$-equivariant differential ($\mathbb{Q}$-)graded algebra. Observe that the grading on the cohomology algebra A is $n/(n+2)$ times the usual one, as

$$|\bar{\partial}_j| = \frac{n}{n+2}.$$

In [25, Section 4], it is shown how to construct a homotopy contracting $\mathcal{B}$ onto its cohomology, and hence (via the homological perturbation lemma) a minimal A_∞ model for $\mathcal{B}$. The homotopy used is manifestly $\mathbb{T}$-equivariant in our setting (see [25] to check this), so the resulting minimal model is also $\mathbb{T}$-equivariant. Furthermore, the homotopy has degree 0 with respect to the grading introduced above, so the $\mathbb{Q}$-grading is preserved under the perturbation lemma construction (in the sense that the A_∞ product μ^k has degree $2-k$ with respect to this grading). Thus we obtain a $\mathbb{T}$-equivariant, $\mathbb{Q}$-graded minimal A_∞ model for $\mathcal{B}$, which we shall denote by $\mathcal{B}'$. It is clear from our discussion that $\mathcal{B}'$ satisfies the necessary conditions to lie in $\mathfrak{A}(A)$.

Proposition 1.6.0.4. *$\mathcal{B}'$ is a versal element of $\mathfrak{A}(A)$, in the same $\mathbb{T}$-equivariant sense as in Proposition 1.5.4.3. It has the same deformation class as $\mathcal{A}'$.*

Proof. The fact that $\mathcal{B}'$ lies in $\mathfrak{A}(A)$ follows from the preceding discussion. The fact that $\mathcal{B}'$ is versal with the same deformation class as $\mathcal{A}'$ follows from the results:

- $\mu^k = 0$ for $2 < k < n+2$ because of the grading and $\mathbb{T}$-equivariance (exactly as in Corollary 1.3.3.3);

- The first non-trivial higher product μ^{n+2} satisfies

$$\mu^{n+2}(e_1, \dots, e_{n+2}) = \pm 1$$

  for an appropriate choice of contracting homotopy h (see [25, Theorem 4.8]) but is 0 on all other permutations of the generators e_j (by similar computations – one can show that only one tree gives a non-zero contribution to such a product). Therefore the deformation class of $\mathcal{B}'$ in $HH^2(A, A)^{-n}$ is given (by the HKR isomorphism) by

$$\mu^{n+2}(\boldsymbol{z}, \dots, \boldsymbol{z}) = \pm z_1 \dots z_{n+2} = \pm W(z),$$

  where $\boldsymbol{z} = \sum_j z_j e_j$.

Combining this with Propositions 1.5.4.2 and 1.5.4.3 gives the result. □

Corollary 1.6.0.5. *There are quasi-isomorphisms*

$$\mathcal{A} \cong \mathcal{A}' \cong \mathcal{B}' \cong \mathcal{B}.$$

In particular, Theorem 2 is proved.

Proof. That $\mathcal{A} \cong \mathcal{A}'$ follows from Proposition 1.5.1.17. That $\mathcal{A}' \cong \mathcal{B}'$ follows from Propositions 1.5.4.3 and 1.6.0.4, by a $\mathbb{T}$-equivariant version of [9, Lemma 3.2]. That $\mathcal{B}' \cong \mathcal{B}$ follows by construction. $\square$

Chapter 2

Homological mirror symmetry for Calabi-Yau hypersurfaces in projective space

2.1 Introduction

2.1.1 Mirror symmetry context

The mirror symmetry phenomenon was discovered by string theorists. In its original form, it dealt with Calabi-Yau Kähler manifolds. On such a manifold, one can define symplectic invariants (the 'A-model') and complex invariants (the 'B-model'). Broadly, mirror symmetry says that there exist pairs of manifolds (M, N) such that the A-model on M is equivalent to the B-model on N, and vice versa.

The closed-string version of the A-model is encoded in Gromov-Witten invariants (the quantum cohomology ring), and the closed-string version of the B-model is encoded in period integrals of the holomorphic volume form. Mirror symmetry first came to the attention of mathematicians in 1991, when Candelas, de la Ossa, Green and Parkes [48] applied it to make predictions about rational curve counts on the quintic three-fold M. They constructed a manifold N which ought to be mirror to M, and computed the closed-string B-model on N. Assuming mirror symmetry to hold, this allowed them to predict the closed-string A-model on M, from which they extracted rational curve counts. This closed-string version of mirror symmetry was proven by Givental for Calabi-Yau (and Fano) complete intersections in toric varieties in 1996 [49, 50].

In the meantime, Kontsevich had introduced the open-string version of mirror symmetry, called the Homological Mirror Symmetry conjecture [3] in 1994. He proposed that the A-model should be encoded in the Fukaya category, and the B-model should be encoded in the category of coherent sheaves. Then if M and N are mirror Calabi-Yaus, there should be an equivalence of the Fukaya category of M with the category of coherent sheaves on Y (on the derived level). Complete or partial proofs of homological mirror symmetry for Calabi-Yau varieties are known for elliptic curves [4, 5], abelian varieties [6] (see [7] for the case of the four-torus), Strominger-Yau-Zaslow dual torus fibrations

[51], and the quartic K3 surface [9].

In this paper we consider a smooth Calabi-Yau hypersurface $M^n \subset \mathbb{CP}^{n-1}$, and its (predicted) mirror N^n. Our main result is to prove one direction of homological mirror symmetry: we prove that there is an equivalence of triangulated categories between the split-closed derived Fukaya category of M^n and the bounded derived category of coherent sheaves on N^n (see Theorem 4 for the precise statement). M^3 is the elliptic curve considered by Polishchuk and Zaslow, M^4 is the quartic surface considered by Seidel, and M^5 is the quintic three-fold. We remark that Nohara and Ueda have also considered the case of the quintic three-fold [52], using the results of Seidel [9] and our earlier paper [1].

In future work [53], we will extend this result to the case of Fano hypersurfaces in projective space, and we plan also to consider the case of hypersurfaces of general type.

2.1.2 Statement of the main result

First let us introduce the coefficient rings of the categories we will be considering.

Definition 2.1.2.1. We define the **universal Novikov ring** Λ_0, whose elements are formal sums

$$\psi(r) = \sum_{j=1}^{\infty} c_j r^{\lambda_j},$$

where $c_j \in \mathbb{C}$, and $\lambda_j \in \mathbb{R}_{\geq 0}$ is an increasing sequence of non-negative reals such that

$$\lim_{j \to \infty} \lambda_j = \infty.$$

It is a valuation ring whose value group is $\mathbb{R}$. Its field of fractions is the **universal Novikov field**

$$\Lambda := \Lambda_0[r^{-1}].$$

We denote the maximal ideal by $\mathfrak{m} \subset \Lambda_0$.

Definition 2.1.2.2. Given an element $\psi \in \mathbb{C}[[r]]$ such that $\psi(0) \neq 0$, we obtain an automorphism

$$\begin{aligned} f_\psi : \Lambda &\to \Lambda, \text{ such that} \\ f_\psi(r) &= r\psi(r). \end{aligned}$$

If $\mathcal{C}$ is a Λ-linear category, and ψ such an element, we define a new Λ-linear category $\psi \cdot \mathcal{C}$: it is the same category as $\mathcal{C}$, but the Λ-vector space structure of the morphism spaces is changed via f_ψ. We say that $\psi \cdot \mathcal{C}$ is obtained from $\mathcal{C}$ via a **change of variables** in the Novikov field.

Now we introduce the relevant categories.

Definition 2.1.2.3. On the symplectic side, let $M^n \subset \mathbb{CP}^{n-1}$ be a smooth hypersurface of degree n. M^n is an $(n-2)$-dimensional Calabi-Yau. The Fukaya category $\mathcal{F}(M^n)$ (as defined in [54]) is a $\mathbb{Z}$-graded Λ-linear A_∞ category. The split-closed derived Fukaya category $D^\pi\mathcal{F}(M^n)$ (see [11, Section I.4]) is a Λ-linear triangulated category. We remark that the Fukaya category is a symplectic invariant (up to A_∞ quasi-isomorphism), so it does not matter which smooth hypersurface M^n we choose.

Definition 2.1.2.4. On the algebraic side, we define

$$w_{nov} := u_1 \dots u_n + r \sum_{j=1}^{n} u_j^n \in \Lambda[u_1, \dots, u_n].$$

We set $\widetilde{N}^n_{nov} := \{w_{nov} = 0\} \subset \mathbb{P}^{n-1}_\Lambda$. We equip $\widetilde{N}^n_{nov}$ with the action of a finite group. Observe that the group

$$\tilde{\Gamma}^*_n := (\mathbb{Z}_n)^n/(1, 1, \dots, 1)$$

acts on $\mathbb{P}^{n-1}_\Lambda$ by multiplying the homogeneous coordinates u_j by nth roots of unity (we include the '$*$' for consistency with later notation). We have a homomorphism

$$\tilde{\Gamma}_n^* \to \mathbb{Z}_n,$$

given by summing the coordinates. We denote the kernel of this homomorphism by Γ_n^*. Note that the action of Γ_n^* preserves $\widetilde{N}_{nov}^n$, so Γ_n^* acts on $\widetilde{N}_{nov}^n$. We define

$$N_{nov}^n := \widetilde{N}_{nov}^n / \Gamma_n^*.$$

We consider the bounded derived category of coherent sheaves on N_{nov}^n, which is

$$D^b Coh(N_{nov}^n) := D^b \left(Coh^{\Gamma_n^*} \left(\widetilde{N}_{nov}^n \right) \right).$$

It is a triangulated category over Λ.

Theorem 4. *If $n \geq 5$, then there exists a power series $\psi \in \mathbb{C}[[r]]$, $\psi(0) = \pm 1$, and an equivalence of Λ-linear triangulated categories*

$$D^\pi \mathcal{F}(M^n) \cong \psi \cdot D^b Coh(N_{nov}^n).$$

In fact, we will see that $\psi \in \mathbb{C}[[r^n]] \subset \mathbb{C}[[r]]$.

Remark 2.1.2.5. The requirement that $n \geq 5$ can be removed without difficulty, but would require some ad-hoc arguments in the cases $n = 3$ and $n = 4$. We prefer to leave these to the reader, as they distract from the main flow of the argument.

We remark that the reference [54], in which the full Fukaya category $\mathcal{F}(M^n)$ is defined, and one of the results we use in the proof of Theorem 4 is proven (we have stated it as Theorem 8), is still in preparation. This is not an ideal situation, but we reassure the reader that this paper is written with minimal reliance on [54]. Let us explain what we mean.

We rigorously define a version of the Fukaya category called the 'relative' Fukaya category $\mathcal{F}(M^n, \boldsymbol{D})$. It is an A_∞ category defined over some coefficient ring R, and there are natural homomorphisms of $\mathbb{C}$-algebras

$$R \to \mathbb{C}[[r]] \hookrightarrow \Lambda.$$

We also define a certain category of matrix factorizations, denoted by $\boldsymbol{p}_1^* MF^{\boldsymbol{G}}(S, w)$, which is a differential graded category over the same coefficient ring R. We introduce full subcategories

$$\widetilde{\mathcal{A}} \subset \mathcal{F}(M^n, \boldsymbol{D}), \quad \widetilde{\mathcal{B}} \subset \boldsymbol{p}_1^* MF^{\boldsymbol{G}}(S, w),$$

and prove (without reference to [54]) that there exists $\psi \in R$ and an A_∞ equivalence

$$\psi \cdot \widetilde{\mathcal{B}} \cong \widetilde{\mathcal{A}}.$$

For a more precise statement, see Theorem 5. Using a theorem of Orlov relating matrix factorizations to coherent sheaves, we show that there is an equivalence

$$D^b Coh(N^n_{nov}) \cong \mathrm{Ho}\left(\boldsymbol{p}_1^* MF^{\boldsymbol{G}}(S, w) \otimes_R \Lambda\right)$$

(the 'Ho' on the right-hand side denotes the homotopy category of a differential graded category), and the full subcategory $\widetilde{\mathcal{B}} \otimes_R \Lambda$ generates. It follows that we have a fully faithful embedding

$$\psi \cdot D^b Coh(N^n_{nov}) \hookrightarrow D^b\left(\mathcal{F}(M^n, \boldsymbol{D}) \otimes_R \Lambda\right).$$

We can come to this point without reference to [54], but no further.

In Section 2.8, we make the assumption (Assumption 2.8.1.1) that there is a fully faithful embedding

$$\mathcal{F}(M^n, \boldsymbol{D}) \otimes_R \Lambda \hookrightarrow \mathcal{F}(M^n),$$

where $\mathcal{F}(M^n)$ is defined as in [54]. We justify this assumption 'heuristically', and show that the full subcategory

$$\widetilde{\mathcal{A}} \otimes_R \Lambda \subset \mathcal{F}(M^n)$$

split-generates, by applying the split-generation criterion Theorem 8 (also due to [54]). This completes the proof of Theorem 4.

In the rest of this section, we give an overview of the main techniques introduced in the rest of the paper, and how they are used to prove Theorem 4. We will make a few imprecise statements and definitions, in the interest of giving the reader the correct intuitive picture.

2.1.3 Affine and relative Fukaya categories

Let M be a Kähler manifold, with a tuple $\boldsymbol{D} = (D_1, \ldots, D_k)$ of smooth, ample, irreducible divisors with normal crossings. In this paper, we consider three versions of the Fukaya category: the **affine Fukaya category** $\mathcal{F}(M \setminus \boldsymbol{D})$, the **relative Fukaya category** $\mathcal{F}(M, \boldsymbol{D})$, and the **full Fukaya category** $\mathcal{F}(M)$.

First let us describe the affine Fukaya category. It is closely related to the exact Fukaya category of the exact symplectic manifold $M \setminus \boldsymbol{D}$, as defined by Seidel in [11], but it has a more interesting grading structure.

First we explain the objects of the affine Fukaya category. If P is a symplectic manifold, we denote by $\mathcal{G}P \to P$ the Lagrangian Grassmannian, whose fibre over $p \in P$ is the space of Lagrangian subspaces of T_pP. Any smooth Lagrangian immersion $i : L \to P$ comes with an associated lift $i_* : L \to \mathcal{G}P$. We define an **anchored Lagrangian brane**

$L^{\#}$ in P to be a Lagrangian immersion $i : L \to P$, together with a lift

$$\begin{tikzcd} & \widetilde{\mathcal{G}}P \arrow[d] \\ L \arrow[ur, "i^{\#}"] \arrow[r, "i_*"] & \mathcal{G}P, \end{tikzcd}$$

where $\widetilde{\mathcal{G}}P$ is the universal cover of the total space of $\mathcal{G}P$, together with a choice of Pin structure on L. The terminology comes from [55], where a closely-related concept is studied.

We define the objects of the affine Fukaya category $\mathcal{F}(M \setminus \boldsymbol{D})$ to be compact, exact, embedded, anchored Lagrangian branes in $M \setminus \boldsymbol{D}$. The category is $\mathbb{C}$-linear, and morphism spaces and A_∞ structure maps are defined in exactly the same way as in the exact Fukaya category of the exact symplectic manifold $M \setminus \boldsymbol{D}$. Namely, the morphism spaces are

$$CF^*_{\mathcal{F}(M\setminus \boldsymbol{D})}(L_0, L_1) := \bigoplus_{p \in L_0 \cap L_1} \mathbb{C} \cdot p$$

(with appropriate modifications to allow for non-transverse intersection of L_0 with L_1). The A_∞ structure map

$$\mu^s : CF^*(L_{s-1}, L_s) \otimes \ldots \otimes CF^*(L_0, L_1) \to CF^*(L_0, L_s)$$

is defined by counting rigid holomorphic disks with $s+1$ boundary punctures in $M \setminus \boldsymbol{D}$. Namely, the coefficient of p_0 in $\mu^s(p_s, \ldots, p_1)$ is given by the signed count of rigid holomorphic disks

$$u : \mathbb{D} \setminus \{\zeta_0, \ldots, \zeta_s\} \to M \setminus \boldsymbol{D},$$

sending the jth boundary component to Lagrangian L_j, and asymptotic at puncture ζ_j to intersection point p_j. However, we treat the grading differently.

To start with, we equip each morphism space with a grading in the abelian group $H_1(M \setminus \boldsymbol{D})$. The A_∞ structure maps respect this grading, essentially because if there is a holomorphic disk contributing to some A_∞ product in $M \setminus \boldsymbol{D}$, then its boundary is nullhomologous in $M \setminus \boldsymbol{D}$. Gradings of this type have appeared in, for example, [20, 1].

Remark 2.1.3.1. Another way of defining this grading (following [55]) would be to equip M with a basepoint $q \in M \setminus \boldsymbol{D}$, and define an object of the Fukaya category to be a Lagrangian $L \subset M \setminus \boldsymbol{D}$, equipped with a path from q to a point $q_L \in L$, inside $M \setminus \boldsymbol{D}$. Then, given an intersection point $p \in L_0 \cap L_1$, which by definition is a generator of $CF^*(L_0, L_1)$, we can define a class in $H_1(M \setminus \boldsymbol{D})$: start from q and follow the path from q to q_{L_0}, then follow a path in L_0 to the intersection point p, then follow a path in L_1 to q_{L_1}, then follow the path back to q. The class of this path in $H_1(M \setminus \boldsymbol{D})$ defines the grading of p.

Secondly, if we were given a quadratic complex volume form on $M \setminus \boldsymbol{D}$, we could define a $\mathbb{Z}$-grading of the morphism spaces (as in [11]). This defines a $\mathbb{Z} \oplus H_1(M \setminus \boldsymbol{D})$-graded category. However, this formulation is unsatisfactory: the $\mathbb{Z}$-grading and H_1-grading are related. For example, changing the quadratic volume form has the effect of changing the $\mathbb{Z} \oplus H_1$ grading by an automorphism preserving the H_1 factor. We really want a new notion of grading.

In Section 2.2, we define a **grading datum $\boldsymbol{G}$** to be an abelian group Y together with a morphism $f : \mathbb{Z} \to Y$. We say that an A_∞ category is **$\boldsymbol{G}$-graded** if its morphism spaces are Y-graded, and the A_∞ structure map μ^s has degree $f(2-s)$. We also study the deformation theory of $\boldsymbol{G}$-graded A_∞ algebras and categories, and prove various classification results about them.

In Section 2.3.1, we introduce a grading datum $\boldsymbol{G}(M, \boldsymbol{D})$ associated to $M \setminus \boldsymbol{D}$, as follows: we consider the fibre bundle $\mathcal{G}(M \setminus \boldsymbol{D})$, with the associated fibration

$$\mathcal{G}_p(M \setminus \boldsymbol{D}) \hookrightarrow \mathcal{G}(M \setminus \boldsymbol{D}) \to M \setminus \boldsymbol{D}.$$

Taking the abelianization of the associated exact sequence of homotopy groups, we obtain an exact sequence

$$H_1\left(\mathcal{G}_p(M \setminus \boldsymbol{D})\right) \to H_1\left(\mathcal{G}(M \setminus \boldsymbol{D})\right) \to H_1(M \setminus \boldsymbol{D}).$$

We observe that $\mathcal{G}_p(M \setminus \boldsymbol{D})$ is the Lagrangian Grassmannian of the symplectic vector space T_pM, so $H_1(\mathcal{G}_p(M \setminus \boldsymbol{D})) \cong \mathbb{Z}$ (see [56]). Thus we can define the grading datum $\boldsymbol{G}(M, \boldsymbol{D})$ to be given by the first morphism in this exact sequence. We show that the affine Fukaya category is naturally $\boldsymbol{G}(M, \boldsymbol{D})$-graded. Observe that the second map in the exact sequence gives the $H_1(M \setminus \boldsymbol{D})$-grading mentioned earlier.

Next, we introduce the relative Fukaya category, denoted $\mathcal{F}(M, \boldsymbol{D})$ (following [57, 9]). Its objects are exactly the same as those of $\mathcal{F}(M \setminus \boldsymbol{D})$: compact, exact, embedded, anchored Lagrangian branes in $M \setminus \boldsymbol{D}$. It is defined over the coefficient ring

$$R(M, \boldsymbol{D}) := \mathbb{C}[[r_1, \ldots, r_k]],$$

a power series ring with one generator for each divisor. We will often write R instead of $R(M, \boldsymbol{D})$ when no confusion is possible. We define morphism spaces by

$$CF^*_{\mathcal{F}(M,\boldsymbol{D})}(L_0, L_1) := \bigoplus_{p \in L_0 \cap L_1} R \cdot p.$$

The A_∞ structure maps μ^s count rigid boundary-punctured holomorphic disks in M. Namely, the coefficient of p_0 in $\mu^s(p_s, \ldots, p_1)$ is given by a signed count of rigid holomorphic disks

$$u : \mathbb{D} \setminus \{\zeta_0, \ldots, \zeta_s\} \to M$$

with boundary and asymptotic conditions as before. Each such disk u contributes a term

$$r^{u \cdot \boldsymbol{D}} := r_1^{u \cdot D_1} \ldots r_k^{u \cdot D_k} \in R,$$

where $u \cdot D_j$ denotes the topological intersection number of u with D_j. We observe that

$u \cdot D_j \geq 0$ by positivity of intersection, so indeed the coefficients lie in R.

$\mathcal{F}(M, \boldsymbol{D})$ is still a $\boldsymbol{G}(M, \boldsymbol{D})$-graded category, but we remark that the coefficient ring R must have a non-trivial grading for this to be true. For example, consider the $H_1(M \setminus \boldsymbol{D})$-grading that we mentioned earlier. It is no longer true that a holomorphic disk u contributing to an A_∞ product μ^s has nullhomologous boundary in $M \setminus \boldsymbol{D}$, because the disk now maps into M, not $M \setminus \boldsymbol{D}$ as it did for the affine Fukaya category. However, if we remove small balls surrounding each intersection point of u with a divisor D_j, then the resulting surface defines a homology in $M \setminus \boldsymbol{D}$ between the boundary of u and a collection of meridian loops around the divisors. Thus, if we define the $H_1(M \setminus \boldsymbol{D})$-grading of the generator $r_j \in R$ to be the class of a meridian loop around divisor D_j, then the A_∞ structure respects the H_1-grading. Of course there remains more to check to show that $\mathcal{F}(M, \boldsymbol{D})$ is $\boldsymbol{G}(M, \boldsymbol{D})$-graded, but this is the basic idea – the details can be found in Section 2.5.

We observe that the zeroth-order part of $\mathcal{F}(M, \boldsymbol{D})$ defines an A_∞ category over $\mathbb{C}$, in which a holomorphic disk $u : \mathbb{D} \to M$ contributes to μ^s only if $u \cdot D_j = 0$ for all j. This corresponds to counting only holomorphic disks which avoid the divisors $\boldsymbol{D}$, i.e., which lie in $M \setminus \boldsymbol{D}$. By definition, this corresponds to the affine Fukaya category $\mathcal{F}(M \setminus \boldsymbol{D})$. We therefore say that $\mathcal{F}(M, \boldsymbol{D})$ is a **$\boldsymbol{G}(M, \boldsymbol{D})$-graded deformation** of $\mathcal{F}(M \setminus \boldsymbol{D})$ over $R(M, \boldsymbol{D})$.

Finally, we recall the definition of the full Fukaya category $\mathcal{F}(M)$, as in [24, 54], where M is a Calabi-Yau symplectic manifold. Its objects are graded spin Lagrangians $L \subset M$. It is linear over the Novikov field Λ. Its morphism spaces are $\mathbb{Z}$-graded Λ-vector spaces (where Λ has degree $0 \in \mathbb{Z}$), defined by

$$CF^*_{\mathcal{F}(M)}(L_0, L_1) := \bigoplus_{p \in L_0 \cap L_1} \Lambda \cdot p.$$

The A_∞ structure maps are defined by signed counts of rigid boundary-punctured holomorphic disks $u : \mathbb{D} \to M$ as before. Each such disk u contributes a term $r^{\omega(u)} \in \Lambda$ to the corresponding coefficient of μ^s.

If we define the ring homomorphism

$$\begin{aligned} R &\to \Lambda \\ r_j &\mapsto r \text{ for all } j, \end{aligned}$$

then Λ becomes an R-algebra. We expect that there is a fully faithful embedding of $\mathbb{Z}$-graded Λ-linear A_∞ categories,

$$\mathcal{F}(M, \boldsymbol{D}) \otimes_R \Lambda \to \mathcal{F}(M).$$

We do not rigorously establish that such an embedding exists, so we state this as Assumption 2.8.1.1, and give some justification in Remark 2.8.1.2.

We summarize the properties of the three versions of the Fukaya category in the following table:

Notation	Affine: $\mathcal{F}(M \setminus \boldsymbol{D})$	Relative: $\mathcal{F}(M, \boldsymbol{D})$	Full: $\mathcal{F}(M)$
Objects	Eq. branes $L^\#$ in $M \setminus \boldsymbol{D}$	Same as for $\mathcal{F}(M \setminus \boldsymbol{D})$	Graded, Spin $L \subset M$
Coefficients	$\mathbb{C}$	$R(M, \boldsymbol{D})$	Λ
Morphisms	$\mathbb{C}\langle L_0 \cap L_1 \rangle$	$R\langle L_0 \cap L_1 \rangle$	$\Lambda\langle L_0 \cap L_1 \rangle$
Grading	$\boldsymbol{G}(M, \boldsymbol{D})$-graded	$\boldsymbol{G}(M, \boldsymbol{D})$-graded	$\mathbb{Z}$-graded (if M is C-Y)
A_∞ maps μ^s	$\#\{u : \mathbb{D} \to M \setminus \boldsymbol{D} \text{ hol.}\}$	$\#\{u : \mathbb{D} \to M \text{ hol.}\}$, with $r^{u \cdot \boldsymbol{D}} \in R$	$\#\{u : \mathbb{D} \to M \text{ hol.}\}$, with $r^{\omega(u)} \in \Lambda$

$$\mathcal{F}(M \setminus \boldsymbol{D}) \xrightarrow{\boldsymbol{G}(M, \boldsymbol{D})\text{-graded deformation}} \mathcal{F}(M, \boldsymbol{D}) \xrightarrow{\otimes_R \Lambda} \mathcal{F}(M).$$

2.1.4 The B-model mirror to the affine, relative and full Fukaya categories

We define the smooth Calabi-Yau Fermat hypersurface

$$M^n := \left\{ \sum_{j=1}^{n} z_j^n = 0 \right\} \subset \mathbb{CP}^{n-1},$$

with ample divisors $D_j := \{z_j = 0\}$ for $j = 1, \ldots, n$. In this section, we will introduce the B-models which ought to be mirror to $\mathcal{F}(M^n \setminus \boldsymbol{D})$, $\mathcal{F}(M^n, \boldsymbol{D})$, and $\mathcal{F}(M^n)$.

We define the power series ring

$$R := \mathbb{C}[[r_1, \ldots, r_n]],$$

which is the coefficient ring of $\mathcal{F}(M^n, \boldsymbol{D})$. We define the R-algebra

$$S := R[u_1, \ldots, u_n],$$

and equip it with the $\mathbb{Z}$-grading so that R is concentrated in degree 0, and each u_j has degree 1. We define the element

$$w = u_1 \ldots u_n + \sum_{j=1}^{n} r_j u_j^n \in S,$$

of degree n.

Now note that $\mathrm{Proj}(S) = \mathbb{P}_R^{n-1}$. We consider the variety

$$\widetilde{N}^n := \{w = 0\} \subset \mathbb{P}_R^{n-1},$$

and equip it with the action of Γ_n^*, exactly as we did for $\widetilde{N}_{nov}^n$ in Definition 2.1.2.4, then define

$$N^n := \widetilde{N}^n / \Gamma_n^*.$$

We note that the algebraic torus

$$\mathbb{T} := \{(\lambda_1, \ldots, \lambda_n) \in (\mathbb{C}^*)^n : \lambda_1 \ldots \lambda_n = 1\}$$

acts on S, by sending

$$\begin{aligned} u_j &\mapsto \lambda_j u_j, \\ r_j &\mapsto \lambda_j^{-n} r_j. \end{aligned}$$

This action preserves w, and commutes with the action of Γ_n^*, and therefore defines an action of $\mathbb{T}$ on N^n.

We note that there are homomorphisms

$$\mathbb{C} \leftarrow R \rightarrow \Lambda,$$

given by

$$0 \leftarrow\!\shortmid r_j \mapsto r$$

for all j. Hence, by base change, we obtain

$$\begin{tikzcd} N_0^n \arrow[r] \arrow[d] & N^n \arrow[d] & N_{nov}^n \arrow[l] \arrow[d] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r] & \mathrm{Spec}(R) & \mathrm{Spec}(\Lambda). \arrow[l] \end{tikzcd}$$

We call N^n the **total space** of the family over $\mathrm{Spec}(R)$, we call N_0^n the **special fibre**, and N_{nov}^n the **generic fibre**. We observe that this N_{nov}^n coincides with the definition given in Definition 2.1.2.4.

We expect mirror relationships as follows:

Coefficients	$\mathbb{C}$	R	Λ
B-model	$\mathrm{Perf}(N_0^n)$	$\mathrm{Perf}(N^n)$	$Coh(N^n_{nov})$
A-model	$\mathcal{F}(M^n \setminus \boldsymbol{D})$	$\mathcal{F}(M^n, \boldsymbol{D})$	$\mathcal{F}(M^n)$

(however note that we do **not** necessarily claim to prove all of these equivalences – this table is included to assist the reader in seeing the 'big picture').

Note that the map from R to $\mathbb{C}$ is naturally $\mathbb{T}$-equivariant. Thus, we can define $\mathbb{T}$-equivariant sheaves on N_0^n and N^n. However it is impossible to equip Λ with a $\mathbb{T}$-action so that the map $R \to \Lambda$ is $\mathbb{T}$-equivariant, so we can not talk about $\mathbb{T}$-equivariant sheaves on N^n_{nov}.

We expect that $\mathbb{T}$-equivariant sheaves correspond to anchored Lagrangian branes. Namely, we have seen that the morphism spaces between anchored Lagrangian branes admit a grading in the group $H_1(M^n \setminus \boldsymbol{D})$, and hence an action of its character group. In this case, there is a natural isomorphism

$$\mathrm{Hom}\,(H_1(M^n \setminus \boldsymbol{D}), \mathbb{C}^*) \cong \mathbb{T},$$

and we expect the mirror correspondences in the above table to be $\mathbb{T}$-equivariant (excluding the last column). In fact, we expect something stronger: they should be equivalences of $\boldsymbol{G}$-graded categories.

In [1, Theorem 7.4], we proved that there is a fully faithful, $\mathbb{T}$-equivariant embedding

$$\mathrm{Perf}(N_0^n) \hookrightarrow D^b\mathcal{F}(M^n \setminus \boldsymbol{D}).$$

In this paper, we extend this to prove results about the other columns.

Our ultimate aim is to understand the final column, and give a proof of Theorem 4. Thus we need a method of making computations in $D^bCoh(N^n_{nov})$. For this purpose, we use the category of **graded matrix factorizations**. Let us denote

$$S_{nov} := S \otimes_R \Lambda = \Lambda[u_1, \ldots, u_n],$$

with

$$w_{nov} := w \otimes 1 \in S_{nov}.$$

Then S_{nov} is $\mathbb{Z}$-graded, and w_{nov} is homogeneous of degree n. In [58], Orlov introduced the differential graded category of graded matrix factorizations of a homogeneous superpotential, $\mathrm{GrMF}(S_{nov}, w_{nov})$, and proved that there is an equivalence

$$\mathrm{Ho}\left(\mathrm{GrMF}(S_{nov}, w_{nov})\right) \cong D^bCoh\left(\widetilde{N}^n_{nov}\right),$$

where

$$\widetilde{N}^n_{nov} := \{w_{nov} = 0\} \subset \mathbb{P}^{n-1}_{\Lambda}$$

(see [58, Theorem 3.11]). Similarly, there is an equivalence of Γ^*_n-equivariant categories,

$$\mathrm{Ho}\left(\mathrm{GrMF}(S_{nov}, w_{nov})^{\Gamma^*_n}\right) \cong D^bCoh\left(N^n_{nov}\right).$$

Orlov's theorem applies because we work over the field Λ, and $\widetilde{N}^n_{nov} = \{w_{nov} = 0\}$ is smooth and Calabi-Yau. However, recall that by passing from the variety N^n, defined over R, to the variety N^n_{nov}, defined over Λ, we lose the $\mathbb{T}$-action. This is a disadvantage, because $\mathbb{T}$-equivariance constrains the algebraic structures we consider significantly, and makes our classification problems tractable.

Therefore, we introduce (in Section 2.7) the category $\mathrm{GrMF}(S, w)$ of graded matrix factorizations of $w \in S$, over the coefficient ring R (Orlov mainly considered graded matrix factorizations over a field in [58], but the definition works over any ring). The coefficient ring R and the graded R-algebra S admit an action of $\mathbb{T}$, which preserves w. Therefore we can talk about $\mathbb{T}$-equivariant objects in $\mathrm{GrMF}(S, w)$, and furthermore there is a fully faithful embedding

$$\mathrm{GrMF}(S, w) \otimes_R \Lambda \hookrightarrow \mathrm{GrMF}(S_{nov}, w_{nov}).$$

However, because R is not a field, Orlov's theorem does **not** apply to give a relationship between the categories $\mathrm{GrMF}(S, w)$ and $\mathrm{Perf}(\widetilde{N}^n)$ (although it seems reasonable to hope that some sort of relationship might hold).

In fact, we first introduce a differential $\boldsymbol{G}$-graded category $MF^{\boldsymbol{G}}(S, w)$ of $\boldsymbol{G}$-graded matrix factorizations of $w \in S$ (these combine the $\mathbb{T}$-action with the $\mathbb{Z}$-grading, in the same way that anchored Lagrangian branes combine the $H_1(M^n \setminus \boldsymbol{D})$-grading with the $\mathbb{Z}$-grading in the Fukaya category). We show that $\mathrm{GrMF}(S, w)^{\Gamma_n^*}$ is some 'orbifolding' of it.

The starting point for this is the observation that $\mathrm{GrMF}(S, w)$ is a $\mathbb{Z}_n$-equivariant version of $MF^{\boldsymbol{G}}(S, w)$ (compare [59, 60]). In fact there is an action of $\tilde{\Gamma}_n^*$ on $MF^{\boldsymbol{G}}(S, w)$, and we show that there is a fully faithful embedding

$$MF^{\boldsymbol{G}}(S, w)^{\tilde{\Gamma}_n^*} \hookrightarrow \mathrm{GrMF}(S, w)^{\Gamma_n^*}$$

(recall that $\tilde{\Gamma}_n^*$ is an extension of $\mathbb{Z}_n$ by Γ_n^*; the $\mathbb{Z}_n$ got eaten up turning MF into GrMF). In fact, our notation for the $\tilde{\Gamma}_n^*$-equivariant category is different – we will write it as

$$\boldsymbol{p}_1^* MF^{\boldsymbol{G}}(S, w) \equiv MF^{\boldsymbol{G}}(S, w)^{\tilde{\Gamma}_n^*}.$$

We will not give the precise meaning of '$\boldsymbol{p}_1^*$' in this introduction, but will continue to use it for consistency with our later notation.

We consider the object $\mathcal{O}_0$ of $MF^{\boldsymbol{G}}(S, w)$, corresponding to the ideal $(u_1, \ldots, u_n) \subset S$.

We denote its endomorphism algebra by $\mathcal{B}$. It is a deformation of the exterior algebra

$$\mathrm{Ext}^*_{Coh(\mathbb{C}^n)}(\mathcal{O}_0, \mathcal{O}_0) \cong \Lambda^*\mathbb{C}^n$$

over the power series ring R. Deformations of $\Lambda^*\mathbb{C}^n$ are governed by the Hochschild cohomology, which is given by polyvector fields, by the Hochschild-Kostant-Rosenberg isomorphism:

$$HH^*(\Lambda^*\mathbb{C}^n) \cong \mathbb{C}[[u_1, \dots, u_n]][\theta_1, \dots, \theta_n],$$

where the variables u_j commute and the variables θ_j anti-commute. We construct a minimal A_∞ model for this A_∞ deformation of $\Lambda^*\mathbb{C}^n$, and prove that its deformation classes are given exactly by the coefficients of w (following [21]). We prove a classification theorem (Theorem 6), which shows that these deformation classes, together with the $\boldsymbol{G}$-grading, are enough to determine the deformation up to A_∞ quasi-isomorphism and formal change of variables.

We then consider the full subcategory of $\boldsymbol{p}_1^* MF^{\boldsymbol{G}}(S, w)$ whose objects are the equivariant twists of $\mathcal{O}_0$. We denote it by $\widetilde{\mathcal{B}}$. It can be determined completely from $\mathcal{B}$. We also introduce a full subcategory

$$\widetilde{\mathcal{A}} \subset \mathcal{F}(M^n, \boldsymbol{D})$$

(the remaining sections of this introduction will consist of an explanation of how to compute $\widetilde{\mathcal{A}}$). We prove the following generalization of [1, Theorem 7.4]:

Theorem 5. *There exists a $\psi \in \mathbb{C}[[T]] \subset R$, where $T = r_1 \dots r_n$, with $\psi(0) = \pm 1$, and a quasi-isomorphism of $\boldsymbol{G}$-graded R-linear A_∞ categories*

$$\psi \cdot \widetilde{\mathcal{B}} \cong \widetilde{\mathcal{A}}.$$

By our previous discussion, there is a fully faithful embedding

$$\mathrm{Ho}\left(\boldsymbol{p}_1^* MF^{\boldsymbol{G}}(S,w)\otimes_R \Lambda\right) \hookrightarrow D^bCoh(N^n_{nov}) \cong D^bCoh^{\Gamma^*_n}\left(\widetilde{N}^n_{nov}\right).$$

The images of the equivariant twists of $\mathcal{O}_0$ correspond to equivariant twists of the restrictions of the Beilinson exceptional collection $\Omega^j(j)$ restricted to $\widetilde{N}^n_{nov}$ (for $j = 0, 1, \ldots, n-1$) by characters of Γ^*_n.

The equivariant twists of the restrictions of the Beilinson exceptional collection generate $D^bCoh(N^n_{nov})$, so it follows immediately from Theorem 5, by tensoring with Λ, that there is an equivalence of triangulated categories

$$\psi \cdot D^bCoh(N^n_{nov}) \cong \mathrm{Ho}\left(Tw\left(\widetilde{\mathcal{A}}\otimes_R \Lambda\right)\right) \subset D^\pi \mathcal{F}(M^n)$$

(under our assumption (Assumption 2.8.1.1) that the second embedding above exists).

Finally, to complete the proof, we wish to show that $\widetilde{\mathcal{A}} \otimes_R \Lambda$ split-generates the Fukaya category. We do this by applying the split-generation result of [54], which says that, if the closed-open string map

$$\mathcal{CO}: QH^*(M^n) \to HH^*\left(\widetilde{\mathcal{A}}\otimes_R \Lambda\right)$$

is non-zero in the top degree $2(n-2)$, then $\widetilde{\mathcal{A}} \otimes_R \Lambda$ split-generates $D^\pi\mathcal{F}(M^n)$.

We observe that

$$\mathcal{CO}([\omega]) = r\frac{\partial \mu^*}{\partial r} \in HH^2\left(\widetilde{\mathcal{A}}\otimes_R \Lambda\right)$$

(in words, the image of the class of the symplectic form under the closed-open string map is the class in HH^2 corresponding to deforming the Fukaya category by scaling the symplectic form). We now observe that $\mathcal{CO}$ is a Λ-algebra homomorphism, so

$$\mathcal{CO}([\omega]^{n-2}) = \left(r\frac{\partial \mu^*}{\partial r}\right)^{n-2}.$$

We then compute that this class is non-zero in the Hochschild cohomology. It follows from the split-generation criterion that $\widetilde{\mathcal{A}}\otimes_R\Lambda$ split-generates $D^\pi\mathcal{F}(M^n)$. This completes the proof of Theorem 4.

In the rest of this introduction, we explain how we make computations in the relative Fukaya category $\mathcal{F}(M^n, \boldsymbol{D})$, which are sufficient to prove Theorem 5.

2.1.5 Behaviour of the Fukaya category under branched covers

Suppose that N and M are compact Kähler manifolds with ample normal-crossings divisors $\boldsymbol{E} \subset N$ and $\boldsymbol{D} \subset M$ as before, and that

$$\phi : (N, \boldsymbol{E}) \to (M, \boldsymbol{D})$$

is a branched cover ramified about the divisors $\boldsymbol{E}$, sending divisor E_j to divisor D_j, and with ramification of degree a_j about divisor E_j. We aim to understand how the affine and relative Fukaya categories of $(N, \boldsymbol{E})$ and $(M, \boldsymbol{D})$ are related.

First, we observe that the map

$$\phi : N \setminus \boldsymbol{E} \to M \setminus \boldsymbol{D}$$

is an unbranched cover. Therefore, any holomorphic disk in $M \setminus \boldsymbol{D}$ lifts to $N \setminus \boldsymbol{E}$, because it is contractible. It follows that the problem of relating $\mathcal{F}(N \setminus \boldsymbol{E})$ to $\mathcal{F}(M \setminus \boldsymbol{D})$ is essentially one of algebraic bookkeeping: we need to keep track of how the holomorphic disks lift, but do not need to compute any new moduli spaces of disks. This leads one to the statement that $\mathcal{F}(N \setminus \boldsymbol{E})$ is a 'semi-direct product of $\mathcal{F}(M \setminus \boldsymbol{D})$ with the character group of the covering group of ϕ' (see [9, Section 8b] and [20, Section 9]). We rephrase this in Section 2.3.4 using the language of $\boldsymbol{G}$-graded categories, in which we write

$$\mathcal{F}(N \setminus \boldsymbol{D}) \cong \boldsymbol{p}^* \mathcal{F}(M \setminus \boldsymbol{D})$$

(we won't explain this notation in the introduction).

Now we try to understand the behaviour of the relative Fukaya category with respect to branched covers. This is not as simple as the unramified cover case, because holomorphic disks may pass through the branching locus, and then they do not lift to the cover. In order to relate $\mathcal{F}(N, \boldsymbol{E})$ to $\mathcal{F}(M, \boldsymbol{D})$, we introduce a 'smooth orbifold relative Fukaya category' $\mathcal{F}(M, \boldsymbol{D}, \boldsymbol{a})$, where $\boldsymbol{a} = (a_1, \ldots, a_k)$ denotes the degrees of ramification of the cover about the divisors (but we could define the orbifold Fukaya category for any tuple $\boldsymbol{a}$ of k positive integers).

The objects, morphism spaces and coefficient ring of $\mathcal{F}(M, \boldsymbol{D}, \boldsymbol{a})$ are the same as for $\mathcal{F}(M, \boldsymbol{D})$. The A_∞ structure maps μ^s, however, count holomorphic disks $u : \mathbb{D} \to M$ that have ramification of degree a_j about divisor D_j wherever they intersect it. Each such disk contributes

$$r_1^{\#(\text{intersection points with } D_1)} \ldots r_k^{\#(\text{intersection points with } D_k)} \in R(M, \boldsymbol{D}).$$

In particular, if $\boldsymbol{a} = (1, 1, \ldots, 1)$ then we recover the relative Fukaya category. The category is still $\boldsymbol{G}(M, \boldsymbol{D})$-graded, but the coefficient ring $R(M, \boldsymbol{D})$ is equipped with a different $\boldsymbol{G}(M, \boldsymbol{D})$-grading depending on $\boldsymbol{a}$.

The holomorphic disks $u : \mathbb{D} \to M$ contributing to the orbifold relative Fukaya category, now do lift to holomorphic disks $u : \mathbb{D} \to N$ (by the homotopy lifting criterion). Thus, the relative Fukaya category $\mathcal{F}(N, \boldsymbol{E})$ is related to $\mathcal{F}(M, \boldsymbol{D}, \boldsymbol{a})$ in exactly the same way that the affine Fukaya category $\mathcal{F}(N \setminus \boldsymbol{E})$ is related to $\mathcal{F}(M \setminus \boldsymbol{D})$: in the language of $\boldsymbol{G}$-graded categories,

$$\mathcal{F}(N, \boldsymbol{D}) \cong \mathbf{p}^* \mathcal{F}(M, \boldsymbol{D}, \boldsymbol{a}).$$

It now remains to relate $\mathcal{F}(M, \boldsymbol{D}, \boldsymbol{a})$ to $\mathcal{F}(M, \boldsymbol{D})$. In fact, we are only able to relate the 'first-order' parts of the categories (but this turns out to be enough for our purposes). The first-order relative Fukaya category is defined to be

$$\mathcal{F}(M, \boldsymbol{D})/\mathfrak{m}^2 := \mathcal{F}(M, \boldsymbol{D}) \otimes_R R/\mathfrak{m}^2,$$

where $\mathfrak{m} \subset R$ is the maximal ideal. It is linear over $R/\mathfrak{m}^2$. It retains only the information about rigid holomorphic disks $u : \mathbb{D} \to M$ passing through a **single** divisor D_j (with multiplicity 1).

Let us write

$$\mu^* = \mu_0^* + \mu_1^*$$

for the A_∞ structure maps μ^* in $\mathcal{F}(M, \boldsymbol{D})/\mathfrak{m}^2$, where μ_0^* gives the affine Fukaya category and μ_1^* gives the first-order terms. Then the A_∞ relations tell us that μ_1^* is a Hochschild cocycle, hence defines an element

$$\sum_{j=1}^{k} r_j \alpha_j \in HH^*(\mathcal{F}(M \setminus \boldsymbol{D})) \otimes \mathfrak{m}/\mathfrak{m}^2.$$

We call α_j the **first-order deformation classes** of $\mathcal{F}(M, \boldsymbol{D})$.

We prove (Theorem 7) that, if $\mathcal{F}(M, \boldsymbol{D})$ has first-order deformation classes α_j, then $\mathcal{F}(M, \boldsymbol{D}, \boldsymbol{a})$ has first-order deformation classes $\alpha_j^{a_j}$, where the power is taken with respect to the Yoneda product on Hochschild cohomology. The proof looks very similar to the proof that the map $QH^*(M) \to HH^*(\mathcal{F}(M))$ is a ring homomorphism.

Remark 2.1.5.1. At first sight, this may seem a strange result: first-order deformation classes of a category live in HH^2, and the Yoneda product respects the $\mathbb{Z}$-grading, so one would expect the class $\alpha_j^{a_j}$ to no longer live in HH^2 and therefore not be an appropriate first-order deformation class. The solution lies in the fact that the coefficient rings R have non-trivial gradings, and in fact the coefficient rings for $\mathcal{F}(M, \boldsymbol{D})$ and $\mathcal{F}(M, \boldsymbol{D}, \boldsymbol{a})$ have **different** gradings: thus, both $r_j \alpha_j$ and $r_j \alpha_j^{a_j}$ have degree 2 in the respective Hochschild cohomology groups in which they live.

Combining this result with our previous observations, if we have a branched cover $\phi : (N, \boldsymbol{E}) \to (M, \boldsymbol{D})$, then we can compute $\mathcal{F}(N, \boldsymbol{E})$ to first order if we know $\mathcal{F}(M, \boldsymbol{D})$ to first order.

2.1.6 The Fukaya category of M^n

We will now explain how to compute the Fukaya category of M^n. We will keep the one-dimensional case ($n = 3$) as a running example throughout, despite the fact that we do not prove this case of Theorem 4 completely in this paper. We do this because one can see all of the holomorphic disks in the Fukaya category in this case, and gain intuition for the various versions of the Fukaya category that we introduce, and results that we prove about it.

We consider the Fermat hypersurfaces

$$M_a^n := \left\{ \sum_{j=1}^n z_j^a = 0 \right\} \subset \mathbb{CP}^{n-1}$$

with the smooth ample normal-crossings divisors $D_j = \{z_j = 0\}$ for $j = 1, \ldots, n$. There is a branched cover

$$\begin{aligned} \phi : (M_a^n, \boldsymbol{D}) &\to (M_1^n, \boldsymbol{D}), \\ [z_1 : \ldots : z_n] &\mapsto [z_1^a : \ldots : z_n^a]. \end{aligned}$$

Thus, applying the results described in the previous section, if we can compute $\mathcal{F}(M_1^n, \boldsymbol{D})$ to first order, then we can compute $\mathcal{F}(M_a^n, \boldsymbol{D})$ to first order.

We observe that $M_1^n \cong \mathbb{CP}^{n-2}$, and $\boldsymbol{D}$ consists of n hyperplanes in general position. $M_1^n \setminus \boldsymbol{D}$ is called the $(n-2)$-dimensional **pair of pants**. In Section 2.6, we construct a Lagrangian immersion

$$L : S^{n-2} \to M_1^n \setminus \boldsymbol{D}$$

which has an anchored Lagrangian brane structure. This Lagrangian was introduced in [1].

Example 2.1.6.1. $M_1^3 = \mathbb{CP}^1$, and $\boldsymbol{D}$ consists of three divisors (points). The Lagrangian immersion $L : S^1 \to \mathbb{CP}^1 \setminus \boldsymbol{D}$ is shown in Figure 2.1.6.1. The A_∞ algebra $CF^*(L, L)$ was described in [20]. It was introduced as a $\mathbb{Z} \oplus H_1(M_1^3 \setminus \boldsymbol{D})$-graded category, but it is not hard to see the underlying $\boldsymbol{G}(M_1^3, \boldsymbol{D})$-graded structure.

The reason this Lagrangian is important is because it can be regarded as a 'fibre' in a Strominger-Yau-Zaslow fibration. See Figure 2.1.6.2 for the picture in the one-dimensional case. More generally, as shown in [22], the pair of pants $\mathbb{CP}^{n-2} \setminus \boldsymbol{D}$ is a singular torus fibration over the 'tropical amoeba of the pair of pants', which is some space stratified by affine manifolds. The torus fibration is non-singular over the top-dimensional faces of the tropical pair of pants.

We suggest that one should think, not of an SYZ fibration of the pair of pants over the tropical pair of pants, with some singular fibres, but rather of an SYZ **family** of objects of the Fukaya category, parametrized by the tropical pair of pants. The immersed Lagrangian sphere L is the object corresponding to the central point in the tropical pair of pants in this picture. The objects corresponding to points of the top-dimensional strata are Lagrangian torus fibres (recall that the fibration is non-singular there). The objects corresponding to points on the in-between strata are lower-dimensional incarnations of L, crossed with tori. We provided some evidence for this philosophy in [1], where we

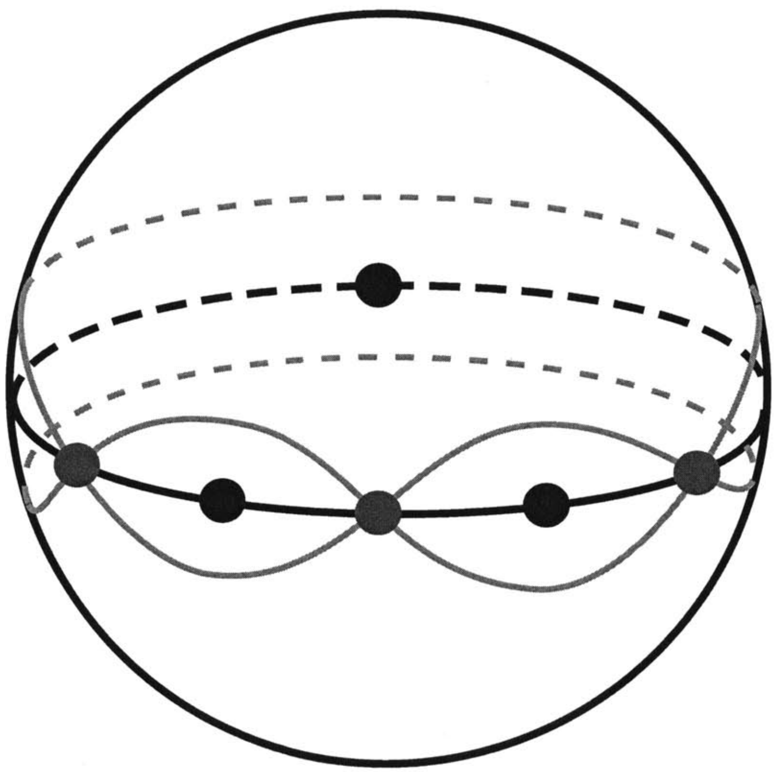


Figure 2.1.6.1: $\mathbb{CP}^1$ with its real locus $\mathbb{RP}^1$ is shown in black, with the divisors $\boldsymbol{D}$ indicated by black dots. The Lagrangian immersion $L : S^1 \to \mathbb{CP}^1 \setminus \boldsymbol{D}$ is shown in red, and its self-intersection points are marked in blue.

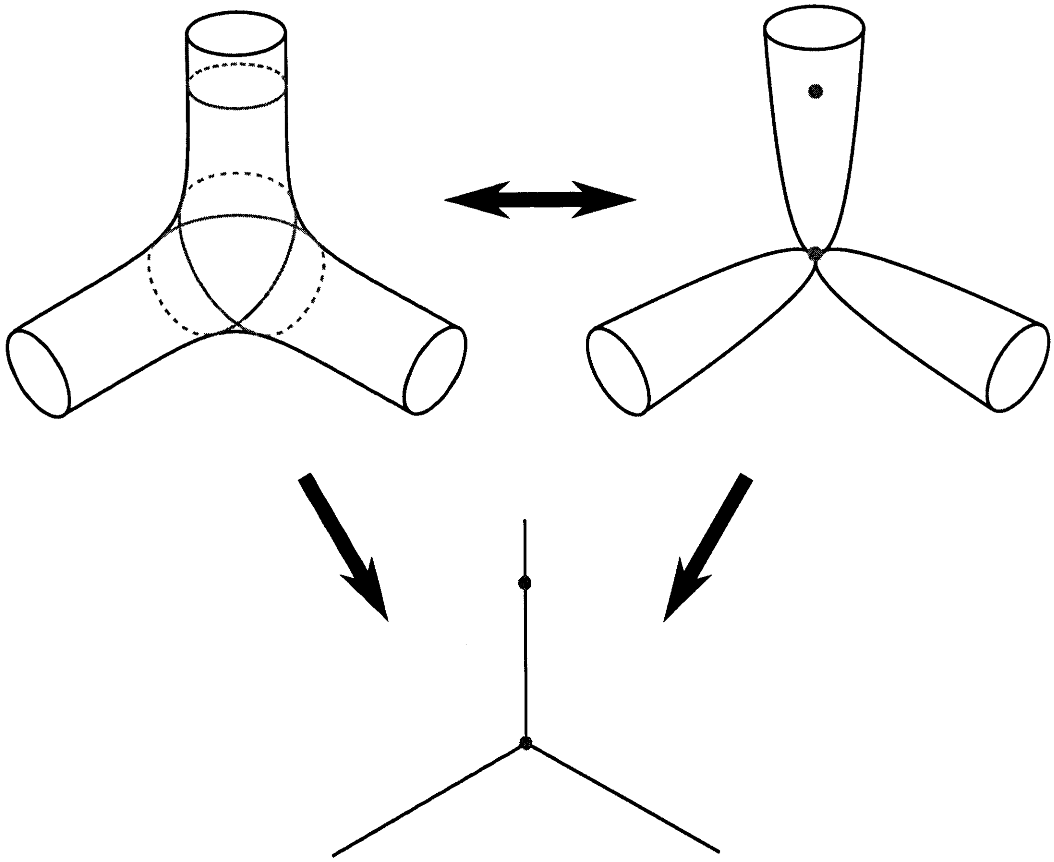


Figure 2.1.6.2: The Lagrangian $L : S^1 \to \mathbb{CP}^1 \setminus \boldsymbol{D}$ can also be drawn as a 'trefoil', shown here in red on the pair of pants (upper left). It lies over the central point (shown in red) in the tropical pair of pants (bottom). The mirror is the union of coordinate axes in $\mathbb{C}^3$ (upper right), and the object corresponding to L is the structure sheaf of the origin (shown in red).

showed that the endomorphism algebra of L in $\mathcal{F}(\mathbb{CP}^{n-2} \setminus \boldsymbol{D})$ is quasi-isomorphic to the endomorphism algebra of the structure sheaf of the origin in the mirror category of matrix factorizations. We plan to make this picture more precise in future work.

In Section 2.6, we compute $CF^*(L, L)$ to first order in $\mathcal{F}(\mathbb{CP}^{n-2}, \boldsymbol{D})$, using a Morse-Bott model for the relative Fukaya category, based on the 'cluster homology' of [27]. We compute that the underlying vector space is an exterior algebra:

$$CF^*(L, L) \cong \Lambda^*\mathbb{C}^n \cong \mathbb{C}[\theta_1, \dots, \theta_n],$$

where the variables θ_j anti-commute. For example, when $n = 3$, $CF^*(L, L)$ is generated by $H^*(S^1)$ (whose two generators we identify as the bottom and top classes 1 and $\theta_1 \wedge \theta_2 \wedge \theta_3$), together with two generators for each self-intersection point, which we label as in Figure 2.1.6.3.

We next show that the zeroth-order algebra structure, μ_0^2, coincides with the exterior algebra. In the case $n = 3$, the corresponding holomorphic triangles are shown in Figure 2-3(a). The shaded triangle can be viewed as having inputs θ_1 and θ_2 and output $\theta_1 \wedge \theta_2$, while the corresponding triangle on the back of the figure can be viewed as having inputs θ_2 and θ_1 and output $-\theta_1 \wedge \theta_2$. The other products follow similarly.

A_∞ structures with underlying cohomology algebra $\Lambda^*\mathbb{C}^n$ are classified by the Hochschild cohomology, which is given by polyvector fields, by the Hochschild-Kostant-Rosenberg isomorphism:

$$HH(\Lambda^*\mathbb{C}^n) \cong \mathbb{C}[[u_1, \dots, u_n]][\theta_1, \dots, \theta_n],$$

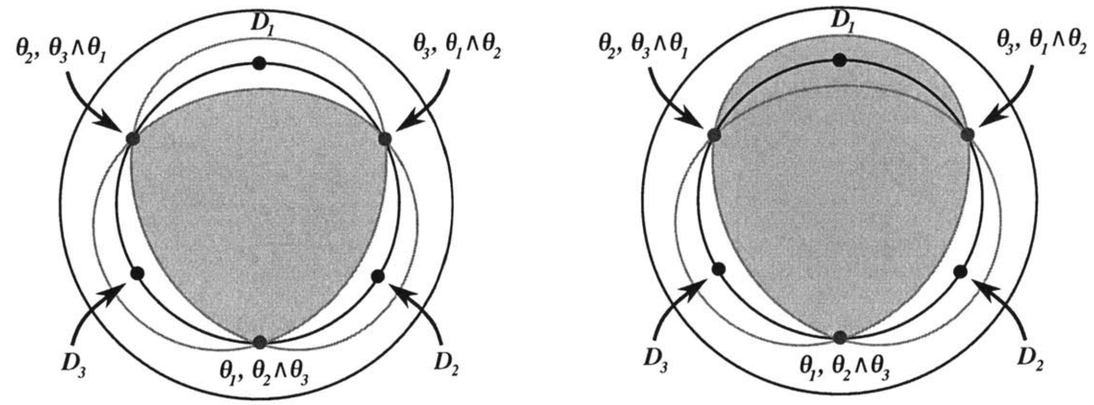
where variables u_i commute and θ_i anti-commute. We show that the endomorphism algebra $CF^*(L, L)$ in the affine Fukaya category $\mathcal{F}(\mathbb{CP}^{n-2} \setminus \boldsymbol{D})$ is completely determined, up to A_∞ quasi-isomorphism, by a single higher-order product, having the form

$$\mu^n(\theta_1, \dots, \theta_n) = 1,$$

corresponding to the Hochschild cohomology class

$$u_1 \dots u_n \in \mathbb{C}[[u_1, \dots, u_n]][\theta_1, \dots, \theta_n].$$

In the case $n = 3$, we can see the corresponding holomorphic disk in Figure 2-3(a). It is



(a) Disks contributing to $CF^*(L, L)$ in the affine Fukaya category, $\mathcal{F}(\mathbb{CP}^1 \setminus \boldsymbol{D})$.

(b) Disks contributing to the first-order deformation class of $CF^*(L, L)$ in the relative Fukaya category, $\mathcal{F}(\mathbb{CP}^1, \boldsymbol{D})$.

Figure 2.1.6.3: Holomorphic disks contributing to $CF^*(L, L)$.

the shaded triangle, which we view as a degenerate 4-gon having inputs θ_1, θ_2, θ_3, and output a degenerate vertex on one of the sides of the triangle, corresponding to 1.

We next compute that the endomorphism algebra of $CF^*(L, L)$ in the first-order relative Fukaya category $\mathcal{F}(\mathbb{CP}^{n-2}, \boldsymbol{D})/\mathfrak{m}^2)$ is determined by structure maps of the form

$$\mu^1(\theta_j) = r_j \cdot 1,$$

corresponding to first-order deformation classes

$$r_j u_j \in \mathbb{C}[[u_1, \ldots, u_n]][\theta_1, \ldots, \theta_n] \otimes \mathfrak{m}/\mathfrak{m}^2.$$

When $n = 3$, we can see the corresponding holomorphic disks in Figure 2-3(b). The shaded 'teardrop' shape has one input θ_1, and a degenerate output vertex corresponding to 1. It intersects divisor D_1 exactly once, and does not intersect the other divisors, hence contributes with a coefficient r_1. Thus it gives rise to the term $\mu^1(\theta_1) = r_1 \cdot 1$.

It follows from the result described in Section 2.1.5 that the first-order deformation classes of the A_∞ algebra

$$\mathcal{A} := CF^*_{\mathcal{F}(\mathbb{CP}^{n-2},\boldsymbol{D},(n,\ldots,n))}(L, L)$$

in the orbifold Fukaya category, are $r_j u_j^n$. Thus, the full deformation class of $\mathcal{A}$ is

$$u_1 \ldots u_n + \sum_{j=1}^{n} r_j u_j^n + \mathcal{O}(r^2),$$

which we observe coincides with the defining polynomial w of N^n, to first order.

We prove a classification theorem (Theorem 6) which shows that this is enough information to determine the full $\boldsymbol{G}$-graded deformation, up to A_∞ quasi-isomorphism and formal change of variables.

We show that the A_∞ algebra

$$\mathcal{B} := \mathrm{Hom}_{MF^{\boldsymbol{G}}(S,w)}(\mathcal{O}_0, \mathcal{O}_0)$$

also has the same underlying algebra, $\boldsymbol{G} \cong \boldsymbol{G}(\mathbb{CP}^{n-2}, \boldsymbol{D})$-grading, and deformation classes (recall the deformation classes were given exactly by w itself). Therefore, by the above-mentioned classification theorem, we have a formal change of variables ψ, and an A_∞ quasi-isomorphism

$$\mathcal{B} \cong \psi \cdot \mathcal{A}.$$

Now the algebraic procedures of passing from $\mathcal{F}(\mathbb{CP}^{n-2}, \boldsymbol{D}, (n, \ldots, n))$ to the branched cover $\mathcal{F}(M^n, \boldsymbol{D})$, and of passing from $MF^{\boldsymbol{G}}(S, w)$ to $\boldsymbol{p}_1^* MF^{\boldsymbol{G}}(S, w)$, are equivalent: we have

$$\widetilde{\mathcal{B}} \cong \boldsymbol{p}_1^*\mathcal{B} \cong \psi \cdot \boldsymbol{p}_1^*\mathcal{A} \cong \psi \cdot \widetilde{\mathcal{A}}.$$

This completes the proof of Theorem 5. See Figure 2.1.6.4 for a picture in the case $n = 3$.

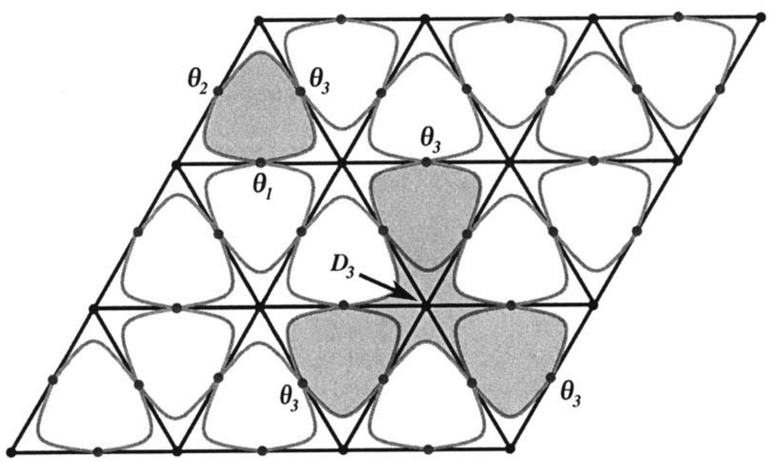


Figure 2.1.6.4: A fundamental domain of the elliptic curve M^3, with the divisor $\boldsymbol{D}$ consisting of nine points, indicated by black dots. The 9-fold cover $(M^3, \boldsymbol{D}) \to (\mathbb{CP}^1, \boldsymbol{D})$ has ramification of order 3 about each divisor. The black lines are the pullback of $\mathbb{RP}^1$. The lifts of L are the curves shown in red, with their intersection points in blue. We have shaded two of the holomorphic disks contributing to the A_∞ products between lifts of L. Note that the triangle of Figure 2-3(a), which contributes to the affine Fukaya category, lifts directly to M^3, whereas the 'teardrop' illustrated in Figure 2-3(b), which contributes the term $r_1 u_1$ to the deformation class of the relative Fukaya category, does not lift, but rather gives rise to the 'clover-leaf' shape, which contributes the deformation class $r_3 u_3^3$.

2.2 Graded and equivariant categories

The main purpose of this section is to introduce the relevant notions of graded and equivariant algebraic objects, and modify the results in [9, Section 3] to classify such objects.

2.2.1 Grading data

For the purposes of this section, we fix an integer $n \geq 3$.

Definition 2.2.1.1. An **unsigned grading datum** $\boldsymbol{G}$ is an abelian group Y together with a morphism $f : \mathbb{Z} \to Y$. We will use the shorthand $\boldsymbol{G} = \{\mathbb{Z} \xrightarrow{f} Y\}$. We will often write $\boldsymbol{G}$ as

$$\mathbb{Z} \xrightarrow{f} Y \xrightarrow{g} X \longrightarrow 0,$$

where X is the cokernel of f. We say that $\boldsymbol{G}$ is **exact** if the map $\mathbb{Z} \to Y$ is injective.

Definition 2.2.1.2. A **morphism** of unsigned grading data, $\boldsymbol{p} : \boldsymbol{G}_1 \to \boldsymbol{G}_2$, is a morphism $p : Y_1 \to Y_2$ that makes the following diagram commute:

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "f_1"] \arrow[d, equal] & Y_1 \arrow[d, "p"] \\ \mathbb{Z} \arrow[r, "f_2"] & Y_2, \end{tikzcd}$$

Composition of morphisms is defined in the obvious way, and this defines a category of unsigned grading data. We say that a morphism of unsigned grading data is **injective** (respectively **surjective**) if the map p is injective (respectively surjective). We will sometimes write $\boldsymbol{p}$ as

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "f_1"] \arrow[d, equal] & Y_1 \arrow[r, "g_1"] \arrow[d, "p"] & X_1 \arrow[r] \arrow[d, "p_X"] & 0 \\ \mathbb{Z} \arrow[r, "f_2"] & Y_2 \arrow[r, "g_2"] & X_2 \arrow[r] & 0, \end{tikzcd}$$

where p_X is the map induced by p.

Definition 2.2.1.3. We define the **sign grading datum**, $\boldsymbol{G}_\sigma := \{\mathbb{Z} \to \mathbb{Z}_2\}$.

Definition 2.2.1.4. We define a **grading datum** $(\boldsymbol{G}, \boldsymbol{\sigma})$ to be an unsigned grading datum $\boldsymbol{G}$, together with a **sign morphism**, which is a morphism of unsigned grading data,

$$\boldsymbol{\sigma} : \boldsymbol{G} \to \boldsymbol{G}_\sigma.$$

We define a morphism of grading data to be a morphism of unsigned grading data that is compatible with sign morphisms. Henceforth, we will often omit the sign morphism $\boldsymbol{\sigma}$ from the notation to avoid clutter.

The sign morphism is important because it allows us to define certain signs in our algebraic objects, which allows us to work over fields of characteristic not equal to 2.

Example 2.2.1.5. We define the grading datum $\boldsymbol{G}_\mathbb{Z} := \{\mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z}\}$, with the obvious morphism $\boldsymbol{\sigma}$. It is an initial object in the category of grading data.

In practice, it is often simpler to work with objects called pseudo-grading data.

Definition 2.2.1.6. A **pseudo-grading datum** $\boldsymbol{H}$ is a morphism of abelian groups $f : Z \to Y$, together with an element $c \in \mathrm{Hom}(Z, \mathbb{Z})$, whose image lies inside $2\mathbb{Z} \subset \mathbb{Z}$.

Definition 2.2.1.7. A **morphism** of pseudo-grading data,

$$\boldsymbol{p} : \boldsymbol{H}_1 \to \boldsymbol{H}_2,$$

consists of maps p_Z and p_Y that make the following diagram commute:

$$\begin{tikzcd} Z_1 \arrow[r, "f_1"] \arrow[d, "p_Z"'] & Y_1 \arrow[d, "p_Y"'] \\ Z_2 \arrow[r, "f_2"] & Y_2, \end{tikzcd}$$

together with an element $d \in \mathrm{Hom}(Y_1, \mathbb{Z})$, whose image lies inside $2\mathbb{Z} \subset \mathbb{Z}$, such that

$$c_1 = p_Z^*(c_2) + f_1^*(d).$$

Definition 2.2.1.8. Given a pseudo-grading datum $\boldsymbol{H}$:

$$Z \xrightarrow{f} Y,$$

together with c, we define a grading datum $\boldsymbol{G}(\boldsymbol{H})$ by

$$\mathbb{Z} \longrightarrow (\mathbb{Z} \oplus Y)/Z,$$

where we define the map $Z \to \mathbb{Z} \oplus Y$ by

$$z \mapsto c(-z) \oplus f(z)$$

and the other maps in the obvious way. We define the sign morphism $\boldsymbol{\sigma} : \boldsymbol{G}(\boldsymbol{H}) \to \boldsymbol{G}_\sigma$ by

$$\begin{aligned} \sigma : (\mathbb{Z} \oplus Y)/Z &\to \mathbb{Z}_2, \\ \sigma(j \oplus y) &:= j. \end{aligned}$$

Observe that the condition that the image of c lies in $2\mathbb{Z}$ ensures that σ is well-defined.

Definition 2.2.1.9. Given a morphism of pseudo-grading data $\boldsymbol{p} : \boldsymbol{H}_1 \to \boldsymbol{H}_2$ as in Definition 2.2.1.7, we define a corresponding morphism of grading data

$$\boldsymbol{G}(\boldsymbol{p}) : \boldsymbol{G}(\boldsymbol{H}_1) \to \boldsymbol{G}(\boldsymbol{H}_2),$$

where

$$\boldsymbol{G}(\boldsymbol{p})(j \oplus y_1) := (j + d(y_1)) \oplus p(y_1).$$

It is not hard to check that this defines a functor from the category of pseudo-grading data to the category of grading data.

Example 2.2.1.10. If we denote by $\mathbf{0} := \{0 \to 0\}$ the zero morphism, then $\boldsymbol{G}(\mathbf{0}) = \boldsymbol{G}_{\mathbb{Z}}$.

Example 2.2.1.11. Given $n \geq 0$, we define the pseudo-grading datum $\boldsymbol{H}_{MF(n)} := \{\mathbb{Z} \xrightarrow{\times n} \mathbb{Z}\}$, with $c = 2$. We denote the corresponding grading datum by $\boldsymbol{G}_{MF(n)}$. It is exact, and has corresponding short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{f} (\mathbb{Z} \oplus \mathbb{Z})/(2, -n) \xrightarrow{g} \mathbb{Z}_n \longrightarrow 0,$$

This grading datum is important because it controls Orlov's 'category of graded matrix factorizations' of a superpotential of degree n (hence our terminology).

Now we will introduce another important grading datum, although first we introduce a bit of convenient notation.

Definition 2.2.1.12. We denote $[k] := \{1, \ldots, k\}$ for any positive integer k and, for any $K \subset [k]$,

$$y_K := \sum_{j \in K} y_j \in \mathbb{Z}\langle y_1, \ldots, y_k \rangle \cong \mathbb{Z}^k.$$

Example 2.2.1.13. Given $n \geq 1$, we denote by $\boldsymbol{H}_a^n$ the pseudo-grading datum

$$\mathbb{Z} \xrightarrow{y_{[n]}} \mathbb{Z}\langle y_1, \ldots, y_n \rangle,$$

together with $c = 2(n - a)$. We denote by $\boldsymbol{G}_a^n$ the corresponding grading datum. This grading datum is important because it controls both the Fukaya category and the category of equivariant matrix factorizations that we will consider.

Now we prove a Lemma relating some of the grading data that we have introduced. We will use it to relate the category of matrix factorizations to the category of coherent sheaves.

Lemma 2.2.1.14. *For any $n \geq 1$, there is a commutative square of grading data:*

$$\begin{tikzcd} \boldsymbol{G}_n^n \arrow[r, two heads, "\boldsymbol{q}_1"] \arrow[d, hook, "\boldsymbol{p}_1"'] & \boldsymbol{G}_{\mathbb{Z}} \arrow[d, hook, "\boldsymbol{p}_2"'] \\ \boldsymbol{G}_1^n \arrow[r, two heads, "\boldsymbol{q}_2"] & \boldsymbol{G}_{MF(n)}, \end{tikzcd}$$

such that $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ are injective.

Proof. The morphism $\boldsymbol{p}_2$ is uniquely defined because $\boldsymbol{G}_{\mathbb{Z}}$ is an initial object. It is clearly injective. The morphism $\boldsymbol{q}_1$ comes from the zero morphism of pseudo-grading data, with $d = 0$ (note this is a morphism of pseudo-grading data because $c = 0$ in $\boldsymbol{H}_n^n$).

The morphism $\boldsymbol{p}_1$ comes from the morphism of pseudo-grading data,

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "y_{[n]}"] \arrow[d, "\times n"'] & \mathbb{Z}^n \arrow[d, "\times n"'] \\ \mathbb{Z} \arrow[r, "y_{[n]}"] & \mathbb{Z}^n, \end{tikzcd}$$

with $d = 2(1-n)y_{[n]}$ (where here we denote by $y_{[n]}$ the element of the dual space $(\mathbb{Z}^n)^\vee \cong \mathbb{Z}^n$). It is clearly injective.

The morphism $\boldsymbol{q}_2$ comes from the morphism of pseudo-grading data,

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "y_{[n]}"] \arrow[d, "\times(-1)"'] & \mathbb{Z}^n \arrow[d, "-y_{[n]}"'] \\ \mathbb{Z} \arrow[r, "\times n"] & \mathbb{Z}, \end{tikzcd}$$

with $d = 2y_{[n]}$.

It is a simple exercise, applying Definition 2.2.1.9, to check that the diagram commutes. $\square$

2.2.2 Graded vector spaces

We recall that, if Y is an abelian group, then a Y**-graded vector space** is a vector space V, together with a collection of vector spaces V_y, indexed by $y \in Y$, and an isomorphism

$$V \cong \bigoplus_{y \in Y} V_y.$$

Definition 2.2.2.1. Let $p : Y_1 \to Y_2$ be morphism of abelian groups. Given a Y_1-graded vector space V, we define the Y_2-graded vector space $p_* V$, where

$$(p_* V)_b := \bigoplus_{p(a)=b} V_a.$$

In particular, the underlying vector space V does not change.

Definition 2.2.2.2. Let $p : Y_1 \to Y_2$ be morphism of abelian groups. Given a Y_2-graded vector space V, we define the Y_1-graded vector space $p^* V$, where

$$(p^* V)_a := V_{p(a)}.$$

Remark 2.2.2.3. If $p : Y_1 \to Y_2$ is injective, then $p^* V$ is just the part of V whose Y_2-degree lies in $\mathrm{im}(p)$.

Definition 2.2.2.4. Let $\boldsymbol{G} = \{\mathbb{Z} \to Y\}$ be a grading datum. A $\boldsymbol{G}$-graded vector space V is the same thing as a Y-graded vector space V.

In fact, for the purposes of this section, '$\boldsymbol{G}$-graded' is virtually identical to 'Y-graded'. Things will get more complicated in the subsequent sections.

Definition 2.2.2.5. Given an element $y \in Y$, and a $\boldsymbol{G}$-graded vector space V, we define $V[y]$ to be V with grading shifted by y.

Definition 2.2.2.6. Given a morphism $\boldsymbol{p}$ of grading data, we define operations $\boldsymbol{p}_*$ and $\boldsymbol{p}^*$ on $\boldsymbol{G}$-graded vector spaces to be identical to p_* and p^*.

Definition 2.2.2.7. If V is a $\boldsymbol{G}$-graded vector space, then it automatically becomes a $\mathbb{Z}_2$-graded vector space:

$$V \cong \boldsymbol{\sigma}_* V,$$

where $\boldsymbol{\sigma}$ is the sign morphism of $\boldsymbol{G}$. Given $v \in V$ of pure degree with respect to this $\mathbb{Z}_2$-grading, we denote its $\mathbb{Z}_2$-degree by $\sigma(v)$.

Definition 2.2.2.8. A $\boldsymbol{G}$-graded algebra is a Y-graded algebra, i.e., one whose multiplication respects the Y-grading.

Example 2.2.2.9. Recall the grading datum $\boldsymbol{G}_1^n$ of Example 2.2.1.13. We define the n-dimensional $\boldsymbol{G}_1^n$-graded vector space

$$U_n := \mathbb{C}\langle u_1, \dots, u_n \rangle,$$

where we equip u_j with degree $(-1, y_j) \in (\mathbb{Z} \oplus Y_n)/(2(1-n) \oplus y_{[n]})$.

Example 2.2.2.10. Let a be an integer. We define the $\boldsymbol{G}_1^n$-graded vector space

$$V_a^n := \mathbb{C}\langle r_1, \dots, r_n \rangle,$$

where we equip r_j with degree $(2-2a, ay_j) \in (\mathbb{Z} \oplus Y_n)/(2(1-n) \oplus y_{[n]})$.

Remark 2.2.2.11. We observe that, if $\boldsymbol{p}_1$ and $\boldsymbol{q}_1$ are the morphisms of grading data defined in Lemma 2.2.1.14, then

$$\boldsymbol{q}_{1*} \boldsymbol{p}_1^* V_n^n \cong V_n^n$$

is a $\mathbb{Z}$-graded vector space concentrated in degree 0.

We remark that, if V is a $\boldsymbol{G}$-graded vector space, then the exterior algebra $\Lambda(V)$ and symmetric algebra $\mathrm{Sym}(V)$ have natural $\boldsymbol{G}$-graded algebra structures.

Definition 2.2.2.12. We define the G_1^n-graded exterior algebra

$$A_n := \Lambda(U_n) \cong \mathbb{C}[\theta_1, \ldots, \theta_n],$$

where the variable θ_j anti-commute.

Definition 2.2.2.13. We define the G_1^n-graded power series ring

$$R_a^n := \mathbb{C}[[V_a^\vee]] \cong \mathbb{C}[[r_1, \ldots, r_n]].$$

The next Lemma looks abstruse, but will be used in Section 2.7.5 to relate equivariant matrix factorizations to equivariant coherent sheaves:

Lemma 2.2.2.14. *Suppose we are given a commutative diagram of exact morphisms of grading data:*

$$\begin{tikzcd} \boldsymbol{G}_1 \arrow[r, "q_1"] \arrow[d, hook, "p_1"'] & \boldsymbol{G}_\mathbb{Z} \arrow[d, hook, "p_2"'] \\ \boldsymbol{G} \arrow[r, "q_2"] & \boldsymbol{G}_2, \end{tikzcd}$$

where $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ are injective, and a $\boldsymbol{G}$-graded vector space V. Taking the X-part of the grading data and morphisms, we find there are morphisms

$$X_1 \xrightarrow{p_{1,X}} X \xrightarrow{q_{2,X}} X_2$$

whose composition is 0 *(by commutativity of the diagram). We define the group Γ to be the homology of this sequence:*

$$\Gamma := \ker(q_{2,X})/\mathrm{im}(p_{1,X}).$$

Then $\boldsymbol{p}_2^*\boldsymbol{q}_{2*}V$ admits a Γ-grading, and hence an action of the character group Γ^*, and there is an isomorphism

$$(\boldsymbol{p}_2^*\boldsymbol{q}_{2*}V)^{\Gamma^*} \cong \boldsymbol{q}_{1*}\boldsymbol{p}_1^*V$$

as $\mathbb{Z}$-graded vector spaces, where the superscript Γ^* denotes the Γ^*-invariant part (or equivalently the part of degree $0 \in \Gamma$).

Proof. We have, by definition, the degree-j parts

$$(\boldsymbol{p}_2^*\boldsymbol{q}_{2*}V)^j := \bigoplus_{q_2(y)=p_2(j)} A_y,$$

and

$$(\boldsymbol{q}_1^*\boldsymbol{p}_{1*}V)^j := \bigoplus_{q_1(y)=j} A_{p_1(y)}.$$

Note that the first of these is equal to the part of A whose Y-grading lies in $q_2^{-1}(\mathrm{im}(p_2))$, while the second is equal to the part of A whose Y-grading lies in $\mathrm{im}(p_1)$ (using the exactness of $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$). By commutativity of the diagram,

$$\begin{aligned} y &= p_1(y_1) \\ \Rightarrow q_2(y) &= p_2(q_1(y_1)) \\ \Rightarrow y &\in q_2^{-1}(\mathrm{im}(q_1)), \end{aligned}$$

so $\mathrm{im}(p_1) \subset q_2^{-1}(\mathrm{im}(p_2))$. Therefore, we have

$$\boldsymbol{q}_1^*\boldsymbol{p}_{1*}V \subset \boldsymbol{p}_2^*\boldsymbol{q}_{2*}V.$$

Furthermore, the left-hand side is exactly equal to the part of the right-hand side whose Y-grading lies in

$$\mathrm{im}(p_1) \subset q_2^{-1}(\mathrm{im}(p_2)),$$

so we can equip the right-hand side with a grading in

$$q_2^{-1}(\mathrm{im}(p_2))/\mathrm{im}(p_1) \cong \Gamma,$$

and the left-hand side is equal to the part of degree $0 \in \Gamma$.

The fact that the $\mathbb{Z}$-gradings match up also follows from commutativity of the diagram:

$$q_1(y) = j \Rightarrow p_2(j) = q_2(p_1(y)).$$

This completes the proof. $\square$

2.2.3 $\boldsymbol{G}$-graded A_∞ algebras and Hochschild cohomology

We now define appropriate notions of $\boldsymbol{G}$-graded A_∞ algebras and Hochschild cohomology. For the purposes of this section, let $\boldsymbol{G}$ be an exact grading datum.

Definition 2.2.3.1. Let R be a $\boldsymbol{G}$-graded algebra, and let A, B be $\boldsymbol{G}$-graded R-bimodules. For each $s \geq 0$, we define a $Y \oplus \mathbb{Z}$-graded R-bimodule, whose degree-(y, s) part is

$$CC_c^y(A, B|R)^s := \operatorname{Hom}_{R-bimod}(A^{\otimes_R s}, B)_{y-f(s)},$$

called **compactly supported Hochschild cochains of length s and degree y**. Note that the Y-grading is not quite the obvious one: if ϕ^s changes Y-degree by y', then we define the Y-grading of ϕ^s to be $y := f(s) + y'$. We will omit the '$|R$' from the notation unless it is necessary to avoid confusion. There is a natural filtration of $CC_c^*(A, B)$, called the **length filtration**, given by

$$(CC_c^*(A, B))^s := \bigoplus_{s' \geq s} (CC_c^*(A, B)^s).$$

We also define a filtered $\boldsymbol{G}$-graded R-bimodule, whose degree-y part is

$$CC^y(A, B) := \prod_{s \geq 0} CC^y(A, B)^s,$$

which is the $\boldsymbol{G}$-graded completion of $CC_c^*(A, B)$ with respect to the length filtration. Note that we call it a '$\boldsymbol{G}$-graded' completion because it is the completion in the category of $\boldsymbol{G}$-graded R-bimodules – this is different from taking the completion of $CC_c^*(A,B)$ in the category of R-bimodules, which would no longer be Y-graded (elements could be sums of non-zero elements of infinitely many different Y-degrees). If $B = A$, we denote

$$CC^*(A) := CC^*(A,A).$$

Given $\phi \in CC^*(A,B)$, we write ϕ^s for the length-s component of ϕ.

Definition 2.2.3.2. We consider the morphism $f : \mathbb{Z} \to Y$ coming from the grading datum $\boldsymbol{G}$, and define

$$\begin{aligned} p : \mathbb{Z} \oplus \mathbb{Z} &\to Y \oplus \mathbb{Z} \\ (j,k) &\mapsto (f(j), j-k). \end{aligned}$$

Then we define the $\mathbb{Z} \oplus \mathbb{Z}$-graded R-bimodule

$$CC^*_{c,\boldsymbol{G}}(A,B) := p^* CC_c^*(A,B).$$

We denote the degree-$(s+t,t)$ part of $CC^*_{\boldsymbol{G}}(A,B)$ by

$$CC^{s+t}_{c,\boldsymbol{G}}(A,B)^t := \text{ the degree-}f(t)\text{ part of } CC^s(A,B).$$

We also define the $\mathbb{Z}$-graded R-bimodule

$$CC^*_{\boldsymbol{G}}(A,B) := f^* CC^*(A,B),$$

which is the completion of $CC^*_{c,\boldsymbol{G}}(A,B)$ with respect to the length filtration.

Remark 2.2.3.3. Note that $CC^*_{\boldsymbol{G}}(A,B)$ is the part of $CC^*(A,B)$ whose Y-grading lies in the image of $\mathbb{Z} \to Y$. In our applications, $CC^*_{\boldsymbol{G}}(A,B)$ will be easier to compute (being smaller), and most of our deformation theory problems take place inside it. It is this observation that makes many of our deformation theory problems tractable.

Definition 2.2.3.4. We also define a 'truncated' version of $CC^*_{\boldsymbol{G}}(A,B)$, by restricting to elements of non-positive degree t. Namely,

$$TCC^*_{\boldsymbol{G}}(A,B) := \prod_{s\geq 0}\bigoplus_{t\leq 0} CC^{s+t}_{\boldsymbol{G}}(A,B)^t.$$

Definition 2.2.3.5. Suppose that R is a $\boldsymbol{G}$-graded algebra, and A and B are $\boldsymbol{G}$-graded R-bimodules. The **Gerstenhaber product** is a map of degree $(f(-1),-1) \in Y \oplus \mathbb{Z}$,

$$\begin{aligned} CC^*_c(A,B) \otimes_R CC^*_c(A) &\to CC^*_c(A,B), \text{ which we denote by} \\ \phi \otimes \psi &\mapsto \phi \circ \psi, \text{ and which is defined by} \\ \phi \circ \psi(a_n, \ldots, a_1) &:= \end{aligned}$$

$$\sum_{i+j+k=n} (-1)^{\dagger} \phi^{i+k+1}(a_{i+j+k}, \ldots, a_{i+j+1}, \psi^j(a_{i+j}, \ldots, a_{i+1}), a_i, \ldots, a_1),$$

where

$$\dagger = (\sigma(\psi)+1)(\sigma(a_1) + \ldots + \sigma(a_i) - i)$$

(recalling Definition 2.2.2.7). If the left and right actions of R on the R-bimodules A and B coincide, then the Gerstenhaber product is R-bilinear; otherwise it is only R-linear in ϕ. Because the Gerstenhaber product respects the length filtration, it defines a product

$$CC^*(A,B) \otimes_R CC^*(A) \to CC^*(A,B)$$

of degree $f(-1)$, also called the Gerstenhaber product.

Definition 2.2.3.6. If R is a $\boldsymbol{G}$-graded algebra and A is a $\boldsymbol{G}$-graded R-bimodule, then we define the **Gerstenhaber bracket**, which is a Lie bracket of degree $f(-1)$ on $CC^*(A)$, by

$$[\phi, \psi] := \phi \circ \psi - (-1)^{(\sigma(\phi)+1)(\sigma(\psi)+1)} \psi \circ \phi.$$

Definition 2.2.3.7. If R is a $\boldsymbol{G}$-graded algebra, then a **$\boldsymbol{G}$-graded associative** R-

algebra is a $\boldsymbol{G}$-graded R-bimodule A, together with an element

$$\mu^2 \in CC^2_{c,\boldsymbol{G}}(A)^0,$$

satisfying the associativity relation,

$$\mu^2 \circ \mu^2 = 0.$$

Remark 2.2.3.8. If A is a $\boldsymbol{G}$-graded associative R-algebra, then the product

$$a_2 \cdot a_1 := (-1)^{\sigma(a_1)} \mu^2(a_1, a_2)$$

is associative and respects the $\boldsymbol{G}$-grading, and makes A into a $\boldsymbol{G}$-graded associative R-algebra in the usual sense.

Remark 2.2.3.9. If R is semisimple:

$$R := \bigoplus_{i \in I} \mathbb{C} u_i, \quad u_i u_j = \delta_{ij} u_i$$

(where I is finite) and A is unital, then this is equivalent to a category with objects indexed by I.

Definition 2.2.3.10. If (A, μ^2) is a $\boldsymbol{G}$-graded associative R-algebra, then we define the **Hochschild differential**

$$\begin{aligned} \delta : CC^*(A) &\rightarrow CC^*(A) \\ \delta(\tau) &:= \left[\mu^2, \tau\right]. \end{aligned}$$

It has degree $f(1) \in Y$ and increases length by 1 (i.e., it is induced by a similar differential of degree $(f(1), 1)$ on $CC^*_c(A)$). It follows from the fact that $\mu^2 \circ \mu^2 = 0$ that δ is a differential, i.e., $\delta^2 = 0$. We define the **Hochschild cohomology** of A to be its cohomology,

$$HH^*(A) := H^*(CC^*(A), \delta),$$

which is a $\boldsymbol{G}$-graded R-bimodule (as δ has pure degree in Y), and similarly

$$HH^*_{\boldsymbol{G}}(A) := H^*(CC^*_{\boldsymbol{G}}(A), \delta),$$

which is $\mathbb{Z}$-graded. Furthermore, because δ is pure of degree $(1, f(1))$ on $CC^*_c(A)$, we can also define compactly-supported versions, $HH^*_c(A)$ which is $Y \oplus \mathbb{Z}$-graded and $HH^*_{c,\boldsymbol{G}}(A)$ which is $\mathbb{Z} \oplus \mathbb{Z}$-graded. As before, we denote the degree-(j,k) part by $HH^j_{c,\boldsymbol{G}}(A)^k$.

Definition 2.2.3.11. If R is a $\boldsymbol{G}$-graded algebra, then a **$\boldsymbol{G}$-graded A_∞ algebra** over R is a $\boldsymbol{G}$-graded R-bimodule A, together with an element

$$\mu \in CC^2_{\boldsymbol{G}}(A),$$

satisfying $\mu^0 = 0$, and such that the A_∞ **associativity relation**

$$\mu \circ \mu = 0$$

is satisfied. We denote A_∞ algebras by $\mathcal{A} := (A, \mu^*)$. If $\mu^1 = 0$ (or equivalently, if μ sits inside $TCC^2_{\boldsymbol{G}}(A)$), we say that $\mathcal{A}$ is **minimal**. If we have μ^* such that $\mu^0 \neq 0$, but still $\mu \circ \mu = 0$, then (A, μ^*) is called a **curved** A_∞ algebra.

Remark 2.2.3.12. In other words, A is a Y-graded R-bimodule, equipped with R-multilinear maps

$$\mu^s : A^{\otimes_R s} \to A$$

of degree $f(2-s) \in Y$, for all $s \geq 1$, satisfying the A_∞ associativity relations.

Definition 2.2.3.13. If A and B are $\boldsymbol{G}$-graded R-bimodules, we define a new 'product'

$$\begin{aligned} CC^*(B) \otimes_R CC^*(A,B) &\to CC^*(A,B), \text{ denoted by} \\ \phi \otimes \psi &\mapsto \phi \diamond \psi, \text{ and which we define by} \\ (\phi \diamond \psi)^n(a_n, \ldots, a_1) &:= \end{aligned}$$

$$\sum_{i_1+\ldots+i_j=n} \phi^j(\psi^{i_1}(a_{i_1+\ldots+i_j}, \ldots, a_{1+i_2+\ldots+i_j}), \psi^{i_2}(a_{i_2+\ldots+i_j}, \ldots), \ldots, \psi^{i_j}(a_{i_j}, \ldots, a_1)).$$

It is R-linear only in the first variable ϕ. Note that $\diamond$ is clearly associative: $(F \diamond G) \diamond H = F \diamond (G \diamond H)$.

Remark 2.2.3.14. We remark that $\diamond$ descends to a product

$$CC^*_{\boldsymbol{G}}(B) \otimes CC^*_{\boldsymbol{G}}(A,B) \to CC^*_{\boldsymbol{G}}(A,B),$$

because the Y-degree of a summand of $\phi \diamond \psi$ clearly lies in the image of $f : \mathbb{Z} \to Y$. Although $\diamond$ does not have a pure grading, it is an easy exercise to check that it maps

$$CC^s_{\boldsymbol{G}}(B) \otimes CC^1_{\boldsymbol{G}}(A,B) \to CC^s_{\boldsymbol{G}}(A,B).$$

Definition 2.2.3.15. If $\mathcal{A} = (A, \mu)$ and $\mathcal{B} = (B, \eta)$ are $\boldsymbol{G}$-graded A_∞ algebras over R, then an A_∞ morphism from $\mathcal{A}$ to $\mathcal{B}$ is an element $F \in TCC^1_{\boldsymbol{G}}(A,B)$ such that

$$F \circ \mu - \eta \diamond F = 0 \in CC^2_{\boldsymbol{G}}(A,B)$$

(note that the class lives in degree 2 by Remark 2.2.3.14). Composition of two A_∞ morphisms is defined using the product $\diamond$. F is called **strict** if $F^j = 0$ for all $j \geq 2$.

If an A_∞ morphism induces an isomorphism on the level of cohomology, then it is said to be a **quasi-isomorphism**. In fact, when our A_∞ algebras are minimal, there is an easier notion, that of **formal diffeomorphism** (see [11, Section 1c]).

Definition 2.2.3.16. If R is a $\boldsymbol{G}$-graded $\mathbb{C}$-algebra, and A and B are $\boldsymbol{G}$-graded R-modules, then a **$\boldsymbol{G}$-graded formal diffeomorphism** from A to B is an element

$$F \in TCC^1_{\boldsymbol{G}}(A, B)$$

such that

$$F^1 : A \to B$$

is an isomorphism of R-modules.

Lemma 2.2.3.17. *If (A, μ) is a minimal $\boldsymbol{G}$-graded A_∞ algebra over R, and $F \in TCC^1_{\boldsymbol{G}}(A, B)$ is a formal diffeomorphism from A to B, then there exists a unique $\boldsymbol{G}$-graded minimal A_∞ structure $F_*\mu$ on B, such that F defines an A_∞ morphism from (A, μ) to $(B, F_*\mu)$.*

Proof. Briefly, $F_*\mu$ is determined inductively in the length filtration: if $F_*\mu$ is determined to length $\leq s-1$, then at order s we have (schematically):

$$(F_*\mu)^s(F^1(a_s), F^1(a_{s-1}), \ldots, F^1(a_1)) = \sum_{s' \leq s-1} (F_*\mu)^{s'}(F(\ldots), F(\ldots), \ldots, F(\ldots)) + \sum_j \mu^j(\ldots, F(\ldots), \ldots),$$

and since F^1 is an isomorphism, this determines F^s uniquely. One can quickly check that $F_*\mu$ satisfies the A_∞ equations. $\square$

Lemma 2.2.3.18. *Formal diffeomorphisms can be composed using $\diamond$: If F is a $\boldsymbol{G}$-graded formal diffeomorphism from A to B and G is a $\boldsymbol{G}$-graded formal diffeomorphism from B to C, then $G \diamond F$ is a $\boldsymbol{G}$-graded formal diffeomorphism from A to C. Furthermore, if μ is a $\boldsymbol{G}$-graded A_∞ structure on A, then*

$$(G \diamond F)_*\mu = G_*(F_*\mu).$$

Furthermore, $\diamond$ is associative.

Lemma 2.2.3.19. *$\boldsymbol{G}$-graded formal diffeomorphisms can be strictly inverted: if F is a $\boldsymbol{G}$-graded formal diffeomorphism from A to B, then there exists a unique $\boldsymbol{G}$-graded formal diffeomorphism G from B to A such that $G \diamond F = \mathrm{id}$ (where* id *denotes the formal diffeomorphism whose length-*1 *component is the identity, and all other components vanish).*

Proof. The construction of G is by induction on length, as in the proof of Lemma 2.2.3.17. □

Definition 2.2.3.20. If A is a $\boldsymbol{G}$-graded vector space over $\mathbb{C}$, then there is a group

$$\mathfrak{G}(A) := \{F \in TCC^1_{\boldsymbol{G}}(A) : F^1 = \mathrm{id}\},$$

called $\boldsymbol{G}$-graded formal diffeomorphisms from A to itself. It follows from Lemmata 2.2.3.17, 2.2.3.18 and 2.2.3.19 that $\mathfrak{G}(A)$ is a group, and that it acts on the space of minimal $\boldsymbol{G}$-graded A_∞ structures on A. Note that this action preserves the underlying algebra (A, μ^2), and that F defines an A_∞ quasi-isomorphism from μ to $F_*\mu$.

Definition 2.2.3.21. If we are given a $\boldsymbol{G}$-graded associative algebra A over $\mathbb{C}$, we define $\mathfrak{A}(A)$, the set of $\boldsymbol{G}$-graded minimal A_∞ algebras $\mathcal{A} = (A, \mu)$ over $\mathbb{C}$, with μ^2 coinciding with the product on A.

Definition 2.2.3.22. Suppose that $\mathcal{A} = (A, \mu) \in \mathfrak{A}(A)$, and $\mu^s = 0$ for $2 < s < d$. Then the A_∞ associativity relations $\mu \circ \mu = 0$ imply that μ^d is a Hochschild cocycle for A. The class

$$[\mu^d] \in HH^2_{c,\boldsymbol{G}}(A)^{2-d}$$

is called the **order-d deforming class** of $\mathcal{A}$.

Remark 2.2.3.23. In [9] and [1], the class $[\mu^d]$ is called an order-d **deformation class**. However a large part of this paper is devoted to studying different objects (see Definition 2.2.4.8), also elements of Hochschild cohomology, and also called deformation classes in

[9]. That is why, to avoid confusing the reader, and only for the purposes of the current paper, we use the terminology 'deforming class' to distinguish this object.

We recall a versality result from [11], appropriately modified to take into account $\boldsymbol{G}$-grading:

Proposition 2.2.3.24. *Suppose that A is a $\boldsymbol{G}$-graded associative algebra over $\mathbb{C}$, and there exists $d > 2$ such that*

$$HH^2_{c,\boldsymbol{G}}(A)^{2-s} \cong \begin{cases} \mathbb{C} & \textit{for } s = d \\ 0 & \textit{for all } s > 0,\ s \neq d. \end{cases}$$

Suppose that $\mathcal{A}_1 = (A, \mu_1)$ and $\mathcal{A}_2 = (A, \mu_2)$ both lie in $\mathfrak{A}(A)$, satisfy $\mu_1^s = \mu_2^s = 0$ for all $2 < s < d$, and have non-trivial order-d deforming class in $HH^2_{c,\boldsymbol{G}}(A)^{2-d}$. Then $\mathcal{A}_1$ and $\mathcal{A}_2$ are quasi-isomorphic.

Proof. The proof is by a straightforward order-by-order construction of a formal diffeomorphism F such that $F_*\mu_1 = \mu_2$, showing that all obstructions to the existence of F vanish (see also [9, Lemma 3.2]). $\square$

Now we define Hochschild cohomology of an A_∞ algebra:

Definition 2.2.3.25. Suppose that $\mathcal{A} = (A, \mu)$ is a $\boldsymbol{G}$-graded A_∞ algebra. We define the Hochschild differential

$$\begin{aligned} \delta : CC^*(A) &\rightarrow CC^*(A), \\ \delta(\tau) &:= [\mu, \tau]. \end{aligned}$$

It follows from the fact that $\mu \circ \mu = 0$ that δ is a differential, i.e., $\delta^2 = 0$. We define the Hochschild cohomology of $\mathcal{A}$ to be the cohomology,

$$HH^*(\mathcal{A}) := H^*(CC^*(A), \delta).$$

The Hochschild differential has pure degree $f(1)$, so $HH^*(\mathcal{A})$ is $\boldsymbol{G}$-graded. However, note that the Hochschild differential is no longer pure with respect to the length, so we can not define the $Y \oplus \mathbb{Z}$-graded compactly-supported version, as we could for an associative algebra. However, δ does always increase (or preserve) the length; therefore, we do still have the length filtration on the Hochschild cochain complex.

Definition 2.2.3.26. Similarly, we can define the $\boldsymbol{G}$-graded Hochschild cohomology,

$$HH^*_{\boldsymbol{G}}(\mathcal{A}) := H^*(CC^*_{\boldsymbol{G}}(A), \delta).$$

It is $\mathbb{Z}$-graded and admits a length filtration.

Definition 2.2.3.27. If $\mathcal{A} = (A, \mu)$ is a $\boldsymbol{G}$-graded **minimal** A_∞ algebra (i.e., $\mu \in TCC^2_{\boldsymbol{G}}(A)$), then the Hochschild differential preserves the truncated Hochschild cochains. Thus it makes sense to define the **truncated Hochschild complex** $(TCC^*_{\boldsymbol{G}}(\mathcal{A}), \delta)$, and call its cohomology the **truncated Hochschild cohomology** $THH^*_{\boldsymbol{G}}(\mathcal{A})$.

Remark 2.2.3.28. Suppose that A is a $\boldsymbol{G}$-graded associative algebra and $\mathcal{A} \in \mathfrak{A}(A)$. The length filtration on the Hochschild cochain complex $CC^*(\mathcal{A})$ yields a '$\boldsymbol{G}$-graded spectral sequence' $(E^{*,*}_d, \delta^{*,*}_d)$. The E_d page is $Y \oplus \mathbb{Z}$ graded, and δ_d has degree $(f(1), d)$. We have

$$E^*_2 = HH^*_c(A).$$

When we need to prove that it converges to $HH^*(\mathcal{A})$, we will apply the 'complete convergence theorem' [61, Theorem 5.5.10]. The length filtration is clearly bounded above, because all Hochschild cochains have length $s \geq 0$, and hence it is also exhaustive. It is also complete in the category of $\boldsymbol{G}$-graded R-bimodules (because the Hochschild cochain complex is defined to be a direct product). Therefore, to prove that the spectral sequence converges to $HH^*(\mathcal{A})$, we must show that it is **regular**: for each $(y, s) \in Y \oplus \mathbb{Z}$, the differentials

$$\delta_d : E^{(y,s)}_d \to E^{(y+f(1),s+d)}_d$$

vanish for sufficiently large d. When we need to prove that the spectral sequence converges, we will in fact show that the differentials δ_d vanish whenever d is sufficiently large (independent of y, s).

Similarly, the length filtration on $CC^*_{\boldsymbol{G}}(\mathcal{A})$ induces a cohomological spectral sequence $(E_d^{*,*}, \delta_d^{*,*})$ with the standard grading convention, and

$$E_2^{s,t} = HH^{s+t}_{c,\boldsymbol{G}}(A)^t.$$

Remark 2.2.3.29. Suppose that A is a $\boldsymbol{G}$-graded associative algebra over $\mathbb{C}$, and $\mathcal{A} \in \mathfrak{A}(A)$ has $\mu^s = 0$ for $2 < s < d$, and order-d deforming class $[\mu^d]$. Then the first non-zero differential in the spectral sequence is δ_{d-1}, and it is given by

$$\begin{aligned} \delta_{d-1} : HH_c^{(y,s)}(A) &\rightarrow HH_c^{(y+f(1),s+d-1)}(A), \\ \delta_{d-1}(\phi) &= \left[[\mu^d], \phi\right], \end{aligned}$$

where we observe that the Gerstenhaber bracket $[-,-]$ descends to the cohomology.

Now we will define the appropriate notions of $\boldsymbol{G}$-graded A_∞ categories and their Hochschild cohomology.

Definition 2.2.3.30. Let $\boldsymbol{G} = \{\mathbb{Z} \rightarrow Y\}$ be a grading datum, and R a $\boldsymbol{G}$-graded algebra. A **$\boldsymbol{G}$-graded pre-category** $\mathcal{C}$ over R is a set of objects $Ob(\mathcal{C})$, together with morphism spaces

$$\mathrm{Hom}_{\mathcal{C}}(K, L)$$

which are $\boldsymbol{G}$-graded R-bimodules, and an action of Y on $Ob(\mathcal{C})$ by 'shifts' $[y]$, together with (compatible) isomorphisms

$$\mathrm{Hom}_{\mathcal{C}}(K[y_1], L[y_2]) \cong \mathrm{Hom}_{\mathcal{C}}(K, L)[y_2 - y_1].$$

We can think of this as equipping $\mathcal{C}$ with an $R[Y]$-bimodule structure. We define the group $CC^*(\mathcal{C})$, and the $\boldsymbol{G}$-graded version $CC^*_{\boldsymbol{G}}(\mathcal{C})$, by analogy with Definitions 2.2.3.1 and 2.2.3.2, restricting to the parts that respect the above isomorphisms. We can think of this strict equivariance requirement as taking $CC^*(\mathcal{C}|R[Y])$, where $\mathcal{C}$ acquires an $R[Y]$-linear structure from the Y-action. Explicitly, it means that for $\phi \in CC^s(\mathcal{C})$, and any $y_0, \ldots, y_s \in Y$, the following diagram commutes:

$$\begin{tikzcd}
\operatorname{Hom}(L_{s-1}, L_s) \otimes \ldots \otimes \operatorname{Hom}(L_0, L_1) \arrow[r, "\phi^s"] \arrow[d] & \operatorname{Hom}(L_0, L_s) \arrow[d] \\
\operatorname{Hom}(L_{s-1}[y_{s-1}], L_s[y_s]) \otimes \ldots \otimes \operatorname{Hom}(L_0[y_0], L_1[y_1])[y_s - y_0] \arrow[r, "\phi^s"] & \operatorname{Hom}(L_0[y_0], L_s[y_s])[y_s - y_0].
\end{tikzcd}$$

Definition 2.2.3.31. We define a **$\boldsymbol{G}$-graded A_∞ category** $\mathcal{C}$ over R to be a $\boldsymbol{G}$-graded pre-category together with

$$\mu \in CC^2_{\boldsymbol{G}}(\mathcal{C})$$

satisfying $\mu^0 = 0$ and $\mu \circ \mu = 0$. An A_∞ category is said to be **cohomologically unital** if its cohomological category is unital. We define the Hochschild cohomology $HH^*(\mathcal{C})$ and $HH^*_{\boldsymbol{G}}(\mathcal{C})$ by analogy with Definition 2.2.3.25. We define $\boldsymbol{G}$-graded A_∞ functors by analogy with $\boldsymbol{G}$-graded A_∞ morphisms. We also consider the case where $\mu^0 \neq 0$; in this case we say that $\mathcal{C}$ is a **curved A_∞ category**.

Definition 2.2.3.32. A $\boldsymbol{G}$-graded A_∞ category is said to be **minimal** if μ lies in $TCC^2_{\boldsymbol{G}}(\mathcal{C})$.

Remark 2.2.3.33. The notions of unitality and equivalence for minimal A_∞ categories are simpler than for non-minimal categories. Because there is no differential μ^1 on the morphisms spaces, μ^2 is strictly associative and therefore defines a category. Thus, if $(\mathcal{C}, \mu)$ is minimal and cohomologically unital, then $(\mathcal{C}, \mu^2)$ is a category, in particular is unital. We say that two objects of $\mathcal{C}$ are **quasi-isomorphic** if they are quasi-isomorphic as objects of $(\mathcal{C}, \mu^2)$. We say that an A_∞ functor $\mathcal{F} : (\mathcal{C}, \mu) \to (\mathcal{D}, \eta)$ between minimal A_∞ categories is a **quasi-equivalence** if the functor $\mathcal{F}^1 : (\mathcal{C}, \mu^2) \to (\mathcal{D}, \eta^2)$ is a quasi-equivalence.

Lemma 2.2.3.34. *If $\mathcal{F} : (\mathcal{C}, \mu) \to (\mathcal{D}, \eta)$ is an A_∞ quasi-equivalence between minimal*

A_∞ categories, then there exists an A_∞ functor $\mathcal{G} : (\mathcal{D}, \eta) \to (\mathcal{C}, \mu)$ such that $\mathcal{F}^1$ and $\mathcal{G}^1$ are mutually inverse quasi-equivalences.

Proof. Follows from Lemma 2.2.3.19. $\square$

Definition 2.2.3.35. If $\mathcal{A}$ is a $\boldsymbol{G}$-graded A_∞ algebra, then we denote by $\underline{\mathcal{A}}$ the smallest $\boldsymbol{G}$-graded A_∞ category with an object whose endomorphism algebra is $\mathcal{A}$. Namely, $\underline{\mathcal{A}}$ has objects $K[y]$ indexed by Y, and morphism spaces

$$\mathrm{Hom}(K[y_1], K[y_2]) := \mathcal{A}[y_2 - y_1].$$

By definition, there are isomorphisms

$$CC^*(\mathcal{A}) \cong CC^*(\underline{\mathcal{A}})$$

and

$$CC^*_{\boldsymbol{G}}(\mathcal{A}) \cong CC^*_{\boldsymbol{G}}(\underline{\mathcal{A}}).$$

We define the A_∞ structure maps μ^* on $\underline{\mathcal{A}}$ to be the image of those on $\mathcal{A}$ under the latter isomorphism.

Now we explain how $\boldsymbol{G}$-graded A_∞ categories can be 'pulled back' along injective morphisms of grading data, and how this operation affects the Hochschild cohomology.

Definition 2.2.3.36. Let $\boldsymbol{p} : \boldsymbol{G}_1 \hookrightarrow \boldsymbol{G}_2$ be an injective morphism of grading data, R a $\boldsymbol{G}_1$-graded algebra, and $\mathcal{C}$ a $\boldsymbol{G}_2$-graded pre-category over $\boldsymbol{p}_* R$. We define $\boldsymbol{p}^*\mathcal{C}$, a $\boldsymbol{G}_1$-graded pre-category over R, to have the same objects as $\mathcal{C}$, but $\boldsymbol{G}_1$-graded morphism spaces

$$\mathrm{Hom}_{\boldsymbol{p}^*\mathcal{C}}(K, L) := \boldsymbol{p}^*\mathrm{Hom}_{\mathcal{C}}(K, L)$$

(thus it is a faithful but not full sub-pre-category). We note that $\boldsymbol{p}^*\mathcal{C}$ still has an action of Y_2 by shifts, so the subgroup $Y_1 \subset Y_2$ acts, equipping $\boldsymbol{p}^*\mathcal{C}$ with the structure of a

$\boldsymbol{G}_1$-graded pre-category over R.

Because Y_2 acts on $\boldsymbol{p}^*\mathcal{C}$ by shifts (shifting all objects simultaneously by the same $y \in Y_2$), it acts on $CC^*(\boldsymbol{p}^*\mathcal{C})$. However the action of the subgroup Y_1 is trivial by definition, since we restrict to Hochschild cochains that respect the shifts by Y_1. So there is an action of the group $\Gamma := Y_2/Y_1$ on $CC^*(\boldsymbol{p}^*\mathcal{C})$. It is not hard to see that the Γ-fixed part is isomorphic to $\boldsymbol{p}^*CC^*(\mathcal{C})$. Thus, we have

$$\boldsymbol{p}^*CC^*(\mathcal{C}) \cong CC^*(\boldsymbol{p}^*\mathcal{C})^\Gamma,$$

and it follows that

$$CC^*_{\boldsymbol{G}_2}(\mathcal{C}) \cong CC^*_{\boldsymbol{G}_1}(\boldsymbol{p}^*\mathcal{C})^\Gamma \subset CC^*_{\boldsymbol{G}_1}(\boldsymbol{p}^*\mathcal{C}).$$

Thus we can make the following:

Definition 2.2.3.37. If $\mathcal{C}$ is a $\boldsymbol{G}_2$-graded A_∞ category over R, with structure maps μ^*, then we define $\boldsymbol{p}^*\mathcal{C}$, a $\boldsymbol{G}_1$-graded A_∞ category over R, whose A_∞ structure maps are given by the image of μ^* under the inclusion

$$CC^2_{\boldsymbol{G}_2}(\mathcal{C}) \cong CC^2_{\boldsymbol{G}_1}(\boldsymbol{p}^*\mathcal{C})^\Gamma \subset CC^2_{\boldsymbol{G}_1}(\boldsymbol{p}^*\mathcal{C}).$$

Remark 2.2.3.38. It follows that there are isomorphisms

$$\boldsymbol{p}^*HH^*(\mathcal{C}) \cong HH^*(\boldsymbol{p}^*\mathcal{C})^\Gamma$$

and

$$HH^*_{\boldsymbol{G}_2}(\mathcal{C}) \cong HH^*_{\boldsymbol{G}_1}(\boldsymbol{p}^*\mathcal{C})^\Gamma \subset HH^*_{\boldsymbol{G}_1}(\boldsymbol{p}^*\mathcal{C}).$$

Now we explain how a $\boldsymbol{G}$-graded A_∞ category can be 'pushed forward' along a surjective morphism of grading data.

Definition 2.2.3.39. Let $\boldsymbol{p} : \boldsymbol{G}_1 \to \boldsymbol{G}_2$ be a surjective morphism of grading data, and $\mathcal{C}$

a $\boldsymbol{G}_1$-graded A_∞ category over a $\boldsymbol{G}_1$-graded algebra R. We now define a $\boldsymbol{G}_2$-graded pre-category $\boldsymbol{p}_*\mathcal{C}$ over $\boldsymbol{p}_*R$, as follows: First, observe that there are canonical isomorphisms of $\boldsymbol{G}_2$-graded vector spaces

$$\boldsymbol{p}_*\mathrm{Hom}_{\mathcal{C}}(K[y], L) \cong \boldsymbol{p}_*\mathrm{Hom}_{\mathcal{C}}(K, L)$$

for any $y \in \Gamma$. We define the set of objects of $\boldsymbol{p}_*\mathcal{C}$ to be the quotient of the set of objects of $\mathcal{C}$ by the action of $\Gamma \subset Y_1$. We define the $\boldsymbol{G}_2$-graded morphism spaces to be

$$\mathrm{Hom}_{\boldsymbol{p}_*\mathcal{C}}(K, L) := \boldsymbol{p}_*\mathrm{Hom}_{\mathcal{C}}(K, L).$$

This is well-defined by our previous remarks. Furthermore, because $\boldsymbol{p}$ is surjective, there is an obvious action of Y_2 on $\boldsymbol{p}_*\mathcal{C}$ by shifts, so $\boldsymbol{p}_*\mathcal{C}$ is a $\boldsymbol{G}_2$-graded pre-category. In some sense we have

$$\boldsymbol{p}_*\mathcal{C} = \mathcal{C} \otimes_{R[Y_1]} R[Y_2].$$

Now it is not hard to see that $CC^y(\boldsymbol{p}_*\mathcal{C})$ is just the completion of $(\boldsymbol{p}_*CC^*(\mathcal{C}))^y$ with respect to the length filtration, for all $y \in Y_2$ (observe that the completion is only needed when $\boldsymbol{p}$ has an infinite kernel). It follows that there is an inclusion

$$CC^*_{\boldsymbol{G}_1}(\mathcal{C}) \hookrightarrow CC^*_{\boldsymbol{G}_2}(\boldsymbol{p}_*\mathcal{C}).$$

Definition 2.2.3.40. If $\mathcal{C}$ is a $\boldsymbol{G}_1$-graded A_∞ category over R, with structure maps μ^*, and $\boldsymbol{p} : \boldsymbol{G}_1 \to \boldsymbol{G}_2$ a surjective morphism of grading data, then we define $\boldsymbol{p}_*\mathcal{C}$, a $\boldsymbol{G}_2$-graded A_∞ category over $\boldsymbol{p}_*R$, whose A_∞ structure maps are given by the image of μ^* under the inclusion

$$CC^2_{\boldsymbol{G}_1}(\mathcal{C}) \hookrightarrow CC^2_{\boldsymbol{G}_2}(\boldsymbol{p}_*\mathcal{C}).$$

Remark 2.2.3.41. It follows that $HH^y(\boldsymbol{p}_*\mathcal{C})$ is the completion of $(\boldsymbol{p}_*HH^*(\mathcal{C}))^y$ with respect to the length filtration, for all $y \in Y_2$. Observe that the completion is only needed when $\boldsymbol{p}$ has infinite kernel and $HH^*(\mathcal{C})$ has infinite rank.

2.2.4 Deformations of A_∞ algebras

For the purposes of this section, let us fix a grading datum $\boldsymbol{G}$, and a power series ring

$$R \cong \mathbb{C}[[r_1, \ldots, r_k]]$$

with a $\boldsymbol{G}$-grading.

We also equip R with its natural filtration by order. We denote the order-j part of R by R^j. There is a natural projection $R \to R^0 \cong \mathbb{C}$, given by setting all $r_j = 0$, and we denote the kernel of this projection by $\mathfrak{m} \subset R$.

We also denote by R_0 the part of R of degree $0 \in Y$.

Definition 2.2.4.1. Given $\psi \in R_0$, there is a $\boldsymbol{G}$-graded algebra homomorphism

$$\begin{aligned} f_\psi : R &\to R, \\ f_\psi(r^j) &:= \psi^j \cdot r^j, \text{ where } r^j \in R^j \text{ has order } j. \end{aligned}$$

We now define a group, which by abuse of notation we call $\mathrm{Aut}(R)$, by setting

$$\begin{aligned} \mathrm{Aut}(R) &:= \{\psi \in R_0 : \psi(0) \neq 0\}, \\ \phi \cdot \psi &:= f_\phi(\psi)\phi, \end{aligned}$$

and an action of $\mathrm{Aut}(R)$ on R by $\boldsymbol{G}$-graded algebra isomorphisms. Note that the condition $\psi(0) \neq 0$ ensures that ψ has a unique inverse in $\mathrm{Aut}(R)$, which can be constructed order-by-order.

Definition 2.2.4.2. If A is a $\boldsymbol{G}$-graded vector space, then $A \otimes R$ is a $\boldsymbol{G}$-graded R-bimodule, and we have an isomorphism

$$CC^*(A \otimes R, A \otimes R|R) \cong CC^*(A, A \otimes R|\mathbb{C}).$$

If we have

$$\begin{aligned}
\phi &\in CC^*(A, A \otimes R), \\
\phi &= \sum_{j \geq 0} \phi_j, \text{ where} \\
\phi_j &\in CC^*(A, A \otimes R^j),
\end{aligned}$$

then we call ϕ_j the **order-j component** of ϕ.

Definition 2.2.4.3. Let $\mathcal{A} = (A, \mu_0)$ be a $\boldsymbol{G}$-graded minimal A_∞ algebra over $\mathbb{C}$. A **$\boldsymbol{G}$-graded deformation of $\mathcal{A}$ over** R is an element

$$\mu \in CC^2_{\boldsymbol{G}}(A \otimes R | R) \cong CC^2_{\boldsymbol{G}}(A, A \otimes R | \mathbb{C})$$

that makes $A \otimes R$ into an A_∞ algebra over R (i.e., $\mu \circ \mu = 0$), and whose order-0 component is μ_0. If $\mu \in TCC^2_{\boldsymbol{G}}(A, A \otimes R)$, then the deformation is said to be **minimal**.

Remark 2.2.4.4. Observe that $CC^2_{\boldsymbol{G}}(A, A \otimes R)$ and $TCC^2_{\boldsymbol{G}}(A, A \otimes R)$ are R_0-modules in the obvious way.

Definition 2.2.4.5. The group $\mathrm{Aut}(R)$ acts on the set of $\boldsymbol{G}$-graded deformations of $\mathcal{A}$ over R, via its action on R. We write $\psi \cdot \mu$ for the action of $\psi \in \mathrm{Aut}(R)$ on a deformation μ.

Now suppose that (A, μ_0) and (B, η_0) are $\boldsymbol{G}$-graded A_∞ algebras over $\mathbb{C}$, and (A, μ) and (B, η) are $\boldsymbol{G}$-graded A_∞ deformations of these over R. We recall (from Definition 2.2.3.15) that a $\boldsymbol{G}$-graded A_∞ morphism from (A, μ) to (B, η) over R is an element

$$F \in TCC^1_{\boldsymbol{G}}(A \otimes R, B \otimes R | R) \cong TCC^1_{\boldsymbol{G}}(A, B \otimes R | \mathbb{C})$$

such that

$$F \circ \mu - \eta \diamond F = 0.$$

Once again, we write

$$F = \sum_{j \geq 0} F_j,$$

where F_j is the order-j component of F. Observe that F_0 is a $\boldsymbol{G}$-graded A_∞ morphism from (A, μ_0) to (B, η_0).

The notion of A_∞ morphisms over R is not very well-behaved (for example, it is not clear that A_∞ quasi-isomorphisms can be inverted over R). It turns out that it will be sufficient for us to work with **minimal** A_∞ algebras, and formal diffeomorphisms. We recall the notion of $\boldsymbol{G}$-graded formal diffeomorphisms from Definition 2.2.3.16, and make appropriate modifications for the case of A_∞ algebras defined over R:

Definition 2.2.4.6. If A and B are $\boldsymbol{G}$-graded vector spaces over $\mathbb{C}$, and R a $\boldsymbol{G}$-graded power series ring, then a **$\boldsymbol{G}$-graded formal diffeomorphism** from $A \otimes R$ to $B \otimes R$ is an element

$$F \in TCC^1_{\boldsymbol{G}}(A, B \otimes R)$$

such that

$$F^1_0 : A \to B$$

is an isomorphism of vector spaces (to clarify: this is the order-0 component of the length-1 part of F).

As before (see Lemmata 2.2.3.17, 2.2.3.18, 2.2.3.19), formal diffeomorphisms can be composed, used to push forward minimal $\boldsymbol{G}$-graded A_∞ structures over R, and they can be strictly inverted. This last point is particularly important, because (as we stated above), there is no reason for an arbitrary A_∞ quasi-isomorphism over R to be invertible.

Now we introduce an analogue of Definition 2.2.3.20 for minimal deformations of A_∞ algebras.

Definition 2.2.4.7. If $\mathcal{A} = (A, \mu_0)$ is a $\boldsymbol{G}$-graded minimal A_∞ algebra over $\mathbb{C}$, we consider the group of $\boldsymbol{G}$-graded formal diffeomorphisms from A to itself, whose leading-order term is the identity:

$$\mathfrak{G}_R(A) := \{F \in TCC^1_{\boldsymbol{G}}(A, A \otimes R) : F^1 = \mathrm{id} + \mathfrak{m}\}.$$

By the analogues of Lemmata 2.2.3.17, 2.2.3.18 and 2.2.3.19, $\mathfrak{G}_R(A)$ forms a group, and this group acts on the set of $\boldsymbol{G}$-graded minimal A_∞ structures on $A \otimes R$. The action of F on $\mu \in TCC^2_{\boldsymbol{G}}(A, A \otimes R)$ is denoted by $F_*\mu \in TCC^2_{\boldsymbol{G}}(A, A \otimes R)$, and it is the unique element such that F defines an A_∞ morphism from μ to $F_*\mu$.

Definition 2.2.4.8. Suppose that μ is a $\boldsymbol{G}$-graded deformation of the A_∞ algebra $\mathcal{A} = (A, \mu_0)$ over R. The first-order component of the A_∞ relation $\mu \circ \mu = 0$ tells us that

$$\mu_1 \in CC^2_{\boldsymbol{G}}\left(A, A \otimes R^1\right)$$

is a Hochschild cochain. Thus, we obtain an element

$$[\mu_1] \in HH^2_{\boldsymbol{G}}(\mathcal{A}, \mathcal{A} \otimes R^1),$$

which we call the **first-order deformation class** of μ.

Definition 2.2.4.9. If μ is a $\boldsymbol{G}$-graded **minimal** deformation of the **minimal** A_∞ algebra $\mathcal{A}$ over R, then the first-order component of μ defines an element in the **truncated** Hochschild cohomology,

$$[\mu_1] \in THH^2_{\boldsymbol{G}}(\mathcal{A}, \mathcal{A} \otimes R^1),$$

which we also call the **first-order deformation class** of the minimal deformation μ.

We are now almost ready to prove our main classification result for deformations of A_∞ algebras. It turns out that in our particular situation, we need to incorporate a finite group action into the picture, so we now briefly explain how to do that.

Definition 2.2.4.10. Let H be a finite group. An **action** of H on a grading datum $\boldsymbol{G}$

is an action α of H on Y by group homomorphisms,

$$\alpha : H \to \mathrm{Hom}_{Ab}(Y, Y),$$

which preserves $\mathbb{Z}$. We will denote $\alpha(h)$ by α_h.

Example 2.2.4.11. In the case of Example 2.2.1.13, there is an action of the symmetric group $H = S_n$ on $\boldsymbol{G}_n$, by permuting the generators of $\mathbb{Z}^n$.

Definition 2.2.4.12. Suppose that H acts on the grading datum $\boldsymbol{G}$, and V is a $\boldsymbol{G}$-graded vector space, and we have an action

$$\rho : H \to \mathrm{Hom}_{Vect}(V, V).$$

We will denote $\rho(h)$ by ρ_h. We say that the action ρ is **$\boldsymbol{G}$-graded** if $\rho(h)$ maps V_y to $V_{\alpha_h(y)}$.

Definition 2.2.4.13. Suppose that we have compatible $\boldsymbol{G}$-graded actions of H on the $\boldsymbol{G}$-graded algebra R, and on the $\boldsymbol{G}$-graded R-bimodules A and B. Then H acts on the $\mathbb{Z}$-graded vector space $CC^*_{\boldsymbol{G}}(A, B)$ (preserving the grading), via

$$(h \cdot \phi)^s(a_s, \ldots, a_1) := h^{-1} \cdot \phi^s(h \cdot a_s, \ldots, h \cdot a_1).$$

We denote the H-invariant part of $CC^*_{\boldsymbol{G}}(A, B)$ by

$$CC^*_{\boldsymbol{G}}(A, B)^H := \{\phi \in CC^*_{\boldsymbol{G}}(A, B) : h \cdot \phi = \phi \text{ for all } h \in H\}.$$

Definition 2.2.4.14. We say that a $\boldsymbol{G}$-graded A_∞ algebra $\mathcal{A} = (A, \mu)$ over R is **strictly H-equivariant** if μ lies in $CC^2_{\boldsymbol{G}}(A)^H \subset CC^2_{\boldsymbol{G}}(A)$. Equivalently, we have

$$\mu^k(h \cdot a_k, \ldots, h \cdot a_1) = h \cdot \mu^k(a_k, \ldots, a_1)$$

for all k and all $h \in H$.

Remark 2.2.4.15. We remark that $TCC^2_{\boldsymbol{G}}(A, A \otimes R)^H$ is naturally a R_0^H-module (compare Remark 2.2.4.4).

Now we prove our main classification result for $\boldsymbol{G}$-graded deformations of A_∞ algebras over R:

Proposition 2.2.4.16. *Suppose that $\mathcal{A} = (A, \mu_0)$ is a $\boldsymbol{G}$-graded minimal A_∞ algebra over $\mathbb{C}$, and furthermore is strictly H-equivariant with respect to the action of some finite group H on A. Suppose that*

$$THH^2_{\boldsymbol{G}}(\mathcal{A}, \mathcal{A} \otimes R)^H$$

is generated, as an R_0^H-module, by its first-order part

$$THH^2_{\boldsymbol{G}}(\mathcal{A}, \mathcal{A} \otimes R^1)^H,$$

and this first-order part is one-dimensional as a $\mathbb{C}$-vector space. Then any two strictly H-equivariant $\boldsymbol{G}$-graded minimal deformations of $\mathcal{A}$, whose first-order deformation classes are non-zero in

$$THH^2_{\boldsymbol{G}}(\mathcal{A}, \mathcal{A} \otimes R^1)^H,$$

are related by an element of $\operatorname{Aut}(R)^H$ composed with a $\boldsymbol{G}$-graded formal diffeomorphism.

Proof. Suppose that (A, μ) and (A, η) are two such deformations. We will construct, order-by-order, elements $\psi \in \operatorname{Aut}(R)^H$ and $F \in TCC^1_{\boldsymbol{G}}(A, A \otimes R)^H$ so that $\psi \cdot \mu = F_* \eta$. The equation that ψ and F must satisfy is

$$\psi \cdot \mu = F_* \eta.$$

We call this the A_∞ relation for the purposes of this proof.

We denote

$$\begin{aligned}\psi &= \sum_{j\geq 0} \psi_j, \text{ where } \psi_j \in (R^j)_0^H, \text{ and} \\ F &= \sum_{j\geq 0} F_j, \text{ where } F_j \in TCC^1_{\boldsymbol{G}}(A, A\otimes R^j)^H.\end{aligned}$$

We start with $F_0 = \mathrm{id}$. The order-zero component of the A_∞ equation says that $\mu_0 = \eta_0$, which is true by assumption.

Now suppose, inductively, that we have determined F_j and ψ_{j-1} for all $j \leq k-1$, H-invariant and $\boldsymbol{G}$-graded, and that

$$(\psi\cdot\mu - F_*\eta)_j = 0 \text{ for all } j \leq k-1.$$

We show that it is possible to choose F_k and ψ_{k-1} so that

$$(\psi\cdot\mu - F_*\eta)_k = 0.$$

The left hand side lies in $TCC^2_{\boldsymbol{G}}(A, A\otimes R^k)^H$.

First, we observe that

$$[\psi\cdot\mu - F_*\eta, \psi\cdot\mu + F_*\eta] = 0,$$

by expanding out the brackets: the cross-terms vanish by symmetry ($\psi\cdot\mu$ and $F_*\eta$ both have degree 2), and the other terms vanish because $\psi\cdot\mu$ and $F_*\eta$ are A_∞ structures. Now note that

$$(\psi\cdot\mu + F_*\eta)_0 = \mu_0 + \eta_0 = 2\mu_0,$$

so taking the order-k component of the previous equation gives us

$$\begin{aligned}[\psi \cdot \mu - F_*\eta, \mu_0]_k &= 0 \\ \Rightarrow \delta\left((\psi \cdot \mu - F_*\eta)_k\right) = 0, &\end{aligned}$$

regardless of what F_k and ψ_{k-1} are.

Now we pick out the terms in $(\psi \cdot \mu - F_*\eta)_k$ that involve F_k and ψ_{k-1}. After a little calculation along the lines of the proof of Lemma 2.2.3.17, we see that it has the form

$$(\psi \cdot \mu - F_*\eta)_k = \psi_{k-1} \cdot \mu_1 + F_k \circ \mu_0 - \eta_0 \circ F_k + D_k,$$

where D_k contains all the terms that do not involve ψ_{k-1} or F_k. We recall that $\eta_0 = \mu_0$, and identify the Hochschild differential:

$$(\psi \cdot \mu - F_*\eta)_k = \psi_{k-1} \cdot \mu_1 + \delta(F_k) + D_k.$$

Note that if we set $F_k = 0$ and $\psi_{k-1} = 0$, our previous argument shows that $\delta(D_k) = 0$, so D_k defines a class

$$[D_k] \in THH^2_G(\mathcal{A}, \mathcal{A} \otimes R^k)^H.$$

Thus, we need to choose ψ_{k-1} so that $\psi_{k-1} \cdot [\mu_1] = -[D_k]$ in the truncated Hochschild cohomology $THH^2_G(\mathcal{A}, \mathcal{A} \otimes R^k)^H$. We can do this by our assumption that $[\mu_1]$ is non-zero, hence generates the one-dimensional first-order component $THH^2_G(\mathcal{A}, \mathcal{A} \otimes R^1)^H$, which generates $THH^2_G(\mathcal{A}, \mathcal{A} \otimes R)^H$ as a R_0^H-module. We then choose F_k to effect the Hochschild coboundary between $\psi_{k-1} \cdot \mu_1$ and $-D_k$. We can make F_k H-invariant by averaging over H.

Finally, note that at first order, we have

$$\psi_0 \cdot [\mu_1] = [\eta_1],$$

from which it follows that $\psi_0 \neq 0$, because $[\mu_1]$ and $[\eta_1]$ are both non-zero, so indeed $\psi \in \mathrm{Aut}(R)^H$. $\square$

2.2.5 Computations

In this section, we introduce specific $\boldsymbol{G}$-graded A_∞ algebras and deformations, and use the results of the previous sections to prove classification theorems for them.

Let us introduce some notation. We fix an integer $n \geq 4$ (we will be considering hypersurfaces in $\mathbb{CP}^{n-1}$) and $a \geq 1$ (this will be the degree of the hypersurface in $\mathbb{CP}^{n-1}$ that we will consider). In our intended applications in the current paper, a will be either 1 or n.

Throughout this section, we will be using the grading datum $\boldsymbol{G} := \boldsymbol{G}_1^n$ from Example 2.2.1.13. For an element $y \in Y$, we will denote

$$|y| := y_{[n]} \cdot y.$$

We will denote by H the symmetric group on n elements, and recall that it acts on $\boldsymbol{G}$ (see Example 2.2.4.11).

We recall the $\boldsymbol{G}$-graded exterior algebra

$$A := A_n := \Lambda(U_n)$$

of Definition 2.2.2.12. For each subset $K \subset [n]$, we denote the corresponding element of A by

$$\theta^K := \bigwedge_{j \in K} \theta^j.$$

We equip the vector space U_n with an H-action, which up to sign is the obvious action by permuting basis elements. In other words,

$$h(u_j) = \pm u_{h(j)}.$$

We will not need to specify the actual signs. There is an induced action of H on A.

We recall the $\boldsymbol{G}$-graded ring

$$R_a := R_a^n := \mathbb{C}[[r_1, \dots, r_n]]$$

from Definition 2.2.2.13. We give a name to one important element of R_a: we denote

$$T := r^{y_{[n]}} = r_1 \dots r_n.$$

We equip R_a with an H action, which up to sign is the obvious action by permuting basis elements. We furthermore denote

$$R := R_n$$

because $a = n$ is the most important case we will consider. Finally, we will denote by R_a^j the order-j part of R_a. Note the change of notation from Definition 2.2.2.13, where the superscript denoted the number of generators. We hope this does not cause confusion.

Definition 2.2.5.1. Suppose that $A \cong \Lambda(U)$ is an exterior algebra over $\mathbb{C}$, and R is an arbitrary commutative ring. We define the R-algebra $A \otimes R$, and the **Hochschild-Kostant-Rosenberg map**

$$\begin{aligned} \Phi : CC^*(A \otimes R|R) &\rightarrow R[[U]] \otimes A, \\ \phi &\mapsto \sum_{s=0}^{\infty} \phi^s(\boldsymbol{u}, \dots, \boldsymbol{u}), \end{aligned}$$

where

$$\boldsymbol{u} := \sum_{j=1}^{n} u_j \theta^j.$$

It has the following properties:

- If R is a field, then Φ induces an isomorphism on the level of cohomology (the Hochschild-Kostant-Rosenberg isomorphism [47]);
- If S is an R-algebra, then there is a commutative diagram

$$\begin{tikzcd} CC^*(A \otimes R|R) \otimes_R S \arrow[r, "\Phi_R \otimes \mathbf{1}"] \arrow[d] & (R[[U]] \otimes A) \otimes_R S \arrow[d] \\ CC^*(A \otimes S|S) \arrow[r, "\Phi_S"] & S[[U]] \otimes A. \end{tikzcd}$$

We call the latter property 'naturality' of the HKR map.

Definition 2.2.5.2. Let us fix $a = n$. We say that a $\boldsymbol{G}$-graded A_∞ algebra $\mathcal{A}$ over R has **type A** if it satisfies the following properties:

- Its underlying R-module and order-0 cohomology algebra is

$$(\mathcal{A}, \mu_0^2) \cong A \otimes R;$$

- It is strictly H-equivariant;
- It satisfies

$$\Phi(\mu^{\geq 3}) = u_1 \dots u_n + \sum_{j=1}^{n} r_j u_j^n + \mathcal{O}(r^2).$$

Now we state the main result we will prove in this section:

Theorem 6. *Suppose that $\mathcal{A}_1 = (A \otimes R, \mu)$ and $\mathcal{A}_2 = (A \otimes R, \eta)$ are two $\boldsymbol{G}$-graded A_∞*

algebras over R of type A. Then there exists $\psi \in R_0^H$, of the form

$$\psi(r_1, \ldots, r_n) = 1 + \sum_{j=1}^{\infty} c_j T^j,$$

and a $\boldsymbol{G}$-graded formal diffeomorphism $F \in \mathfrak{G}_R(A)$, such that

$$\psi \cdot \mathcal{A}_1 = F_* \mathcal{A}_2.$$

We will now give a brief outline of the proof. The first step is to prove (in Corollary 2.2.5.7, using Proposition 2.2.3.24) that the order-0 parts (A, μ_0) and (A, η_0) are quasi-isomorphic. The classification of the order-0 part is governed by the Hochschild cohomology $HH^*_{\boldsymbol{G}}(A)$, which is determined via the Hochschild-Kostant-Rosenberg (HKR) isomorphism. We thereafter denote by $\mathcal{A} \cong (A, \mu_0)$ this (unique up to formal diffeomorphism) order-0 part.

We then study deformations of $\mathcal{A}$ over R. The classification of such deformations is governed by the Hochschild cohomology $HH^*_{\boldsymbol{G}}(\mathcal{A}, \mathcal{A} \otimes R)$, which is also determined from the HKR isomorphism, via a spectral sequence. We apply Proposition 2.2.4.16 to show that such deformations are unique up to A_∞ quasi-isomorphism and the action of $\mathrm{Aut}(R)^H$.

Now let us begin. We first explain how to use the HKR isomorphism [47] to calculate $HH^*_{\boldsymbol{G}}(A)$, and more generally $HH^*_{\boldsymbol{G}}(A, A \otimes R_a)$. The HKR isomorphism show that, if $A = \Lambda(U)$ is a $\boldsymbol{G}$-graded exterior algebra over $\mathbb{C}$, then the map

$$\Phi : (CC^*_c(A), \delta) \to (\mathbb{C}[U] \otimes \Lambda(U), 0),$$

is a quasi-isomorphism of $Y \oplus \mathbb{Z}$-graded chain complexes. This allows us to determine

$$HH^*(A) \cong \mathbb{C}[[U]] \otimes \Lambda(U)$$

by taking the completion with respect to the length filtration, and also $HH^*_{\boldsymbol{G}}(A)$, where now we equip A with its $\boldsymbol{G}$-grading. We will make the $\boldsymbol{G}$-grading of the right-hand side explicit in Lemma 2.2.5.4.

More generally, it follows that there is an isomorphism of $Y \oplus \mathbb{Z}$-graded vector spaces,

$$\Phi : HH^*_c(A, A \otimes R_a) \to \mathbb{C}[U] \otimes \Lambda(U) \otimes R_a.$$

$\mathbb{C}[U] \otimes \Lambda(U) \otimes R_a$ is generated by terms $u^{\boldsymbol{b}}\theta^K r^{\boldsymbol{c}}$, where $\boldsymbol{b}, \boldsymbol{c} \in Y_{\geq 0}$ (recall $Y \cong \mathbb{Z}^n$, so we define $Y_{\geq 0} := \mathbb{Z}^n_{\geq 0}$) and $K \subset [n]$. If $\boldsymbol{b} = \sum_j b_j e_j$, then this is the image under Φ of a Hochschild cochain which sends

$$\bigotimes_j (\theta^j)^{\otimes b_j} \mapsto r^{\boldsymbol{c}}\theta^K.$$

We start by examining what the various gradings on a Hochschild cochain tell us.

Lemma 2.2.5.3. *If a generator $\tau \in CC^{s+t}_{c,\boldsymbol{G}}(A, A \otimes R^j_a)^t$ sends*

$$\bigotimes_{i=1}^{s} \theta^{K_i} \mapsto r^{\boldsymbol{c}}\theta^{K_0},$$

then we have

$$\begin{aligned} j &= |\boldsymbol{c}| && (2.2.5.1) \\ a\boldsymbol{c} + y_{K_0} - \sum_{i=1}^{s} y_{K_i} &= q y_{[n]} \text{ in } Y, \text{ for some } q && (2.2.5.2) \\ t &= (n-2)q + (2-a)j. && (2.2.5.3) \end{aligned}$$

Proof. Equation (2.2.5.1) follows by definition. To prove Equations (2.2.5.2) and (2.2.5.3), we recall that the grading of θ_j is $(-1, y_j)$ (Example 2.2.2.9) and the grading of r_j is $(2-2a, ay_j)$ (Example 2.2.2.10). We alter the grading datum $\boldsymbol{G}$ by an automorphism sending

$$(j, y) \mapsto (j + y_{[n]} \cdot y, y).$$

This is equivalent to considering the pseudo-grading datum with the same exact sequence as $\boldsymbol{H}_n$, but

$$c = 2(n-1) - y_{[n]} \cdot y_{[n]} = n - 2.$$

Then θ_j has grading $(0, y_j)$ and r_j has grading $(2-a, ay_j)$.

If the grading of τ is $f(t) = (t, 0) \in (\mathbb{Z} \oplus Y)/Z$, then

$$(0, y_{K_0}) + ((2-a)|\boldsymbol{c}|, a\boldsymbol{c}) - \sum_{j=1}^{s}(0, y_{K_j}) \cong (t, 0) \text{ modulo } Z.$$

Recalling that the image of Z in $\mathbb{Z} \oplus Y$ is given by

$$((2-n)q, qy_{[n]})$$

(in the altered grading datum), we have

$$(0, y_{K_0}) + ((2-a)|\boldsymbol{c}|, a\boldsymbol{c}) - \sum_{j=1}^{s}(0, y_{K_j}) = (t, 0) + q(2-n, y_{[n]}),$$

from which the result follows. □

Now recalling the Hochschild-Kostant-Rosenberg isomorphism, a generator of $HH^{s+t}_{c,\boldsymbol{G}}(A, A \otimes R_a)^t$ has the form $u^{\boldsymbol{b}}\theta^K r^{\boldsymbol{c}}$, where $\boldsymbol{b}$ and $\boldsymbol{c}$ are elements of $Y_{\geq 0}$, and $K \subset [n]$. We examine what the gradings tell us about such a generator of the Hochschild cohomology.

Lemma 2.2.5.4. *If $u^{\boldsymbol{b}}\theta^K r^{\boldsymbol{c}}$ is a generator of $HH^{s+t}_{c,\boldsymbol{G}}(A, A \otimes R_a^j)^t$, then the following* *equations hold:*

$$\begin{aligned} j &= |\boldsymbol{c}| && (2.2.5.4) \\ s &= |\boldsymbol{b}| && (2.2.5.5) \\ y_K + a\boldsymbol{c} - \boldsymbol{b} &= q y_{[n]} \textit{ in } Y\textit{, for some } q && (2.2.5.6) \\ t &= (n-2)q + (2-a)j && (2.2.5.7) \\ |K| &= s + t + 2(q-j). && (2.2.5.8) \\ (n-2)|K| &= (n-2)s + nt + 2(a-n)j && (2.2.5.9) \end{aligned}$$

Proof. Equations (2.2.5.4) and (2.2.5.5) hold by definition. Equation (2.2.5.6) follows from Equation (2.2.5.2). Equation (2.2.5.7) follows from Equation (2.2.5.3). The first step in proving Equations (2.2.5.8) and (2.2.5.9) is to take the dot product of Equation (2.2.5.6) with $y_{[n]}$:

$$\begin{aligned} y_{[n]} \cdot (y_K + a\boldsymbol{c} - \boldsymbol{b}) &= q y_{[n]} \cdot y_{[n]} \\ \Rightarrow |K| &= |\boldsymbol{b}| - a|\boldsymbol{c}| + nq \\ &= s - aj + nq. \end{aligned}$$

To prove equation (2.2.5.8), we use equation (2.2.5.7) to substitute for nq:

$$\begin{aligned} |K| &= s - aj + t + (a-2)j + 2q \\ &= s + t + 2(q-j). \end{aligned}$$

We use the same equation to prove equation (2.2.5.9), but this time we first multiply by n then substitute in equation (2.2.5.7):

$$\begin{aligned} (n-2)|K| &= (n-2)(s - aj + nq) \\ &= (n-2)s - (n-2)aj + n(t + (a-2)j) \\ &= (n-2)s + nt + 2(a-n)j. \end{aligned}$$

$\square$

We now set about determining the order-0 part of an A_∞ algebra of type A. We identify the possible $\boldsymbol{G}$-graded A_∞ algebras $\mathcal{A} = (A, \mu_0)$ over $\mathbb{C}$ with underlying vector space and μ_0^2 given by A.

Lemma 2.2.5.5. *We have*

$$CC^*_{c,\boldsymbol{G}}(A)^t \cong 0$$

unless t is divisible by $(n-2)$.

Proof. Follows from Equation (2.2.5.7) with $j = 0$. $\square$

Lemma 2.2.5.6. *The Hochschild cohomology of A satisfies, for $d > 0$,*

$$HH^2_{c,\boldsymbol{G}}(A)^{2-d} \cong \begin{cases} \mathbb{C} \cdot u^{y_{[n]}} & \text{if } d = n, \\ 0 & \text{otherwise,} \end{cases}$$

where $u^{y_{[n]}} = u_1 \dots u_n$.

Proof. Let $u^{\boldsymbol{b}}\theta^K$ be a generator of $HH^2_{c,\boldsymbol{G}}(A)^{2-d}$. Equation (2.2.5.7), with $j = 0$, yields

$$d = 2 - (n-2)q.$$

We want $d > 0$, so we have $q < 0$. Now equation (2.2.5.8), with $j = 0$, yields

$$2 + 2q = |K| \geq 0.$$

Thus, $-1 \leq q < 0$, so $q = -1$ and $|K| = 0$, so $K = \phi$.

Equation (2.2.5.6) now yields $\boldsymbol{b} = y_{[n]}$. Therefore $u^{\boldsymbol{b}}\theta^K = u^{y_{[n]}}$, as required. $\square$

Corollary 2.2.5.7. *There is an A_∞ quasi-isomorphism between the 0th-order parts of $\mathcal{A}_1$ and $\mathcal{A}_2$:*

$$(A, \mu_0) \cong (A, \eta_0).$$

Proof. First, we observe that $CC^2_{c,\boldsymbol{G}}(A)^1 \cong 0$ by Equation (2.2.5.3) (with $j = 0$, and assuming $n \geq 4$), so the A_∞ algebras are minimal. Furthermore, $\mu_0^d = 0$ for $2 < d < n$ because $CC^2_{c,\boldsymbol{G}}(A)^{2-d} \cong 0$ by Lemma 2.2.5.5. The result now follows from Proposition 2.2.3.24, because the deforming class of an algebra of type A is

$$\Phi_0(\mu_0) = u_1 \ldots u_n = u^{y_{[n]}}$$

by definition. $\square$

Henceforth, we will denote by $\mathcal{A}$ any $\boldsymbol{G}$-graded minimal A_∞ algebra over $\mathbb{C}$ with cohomology algebra given by A, and whose order-n deforming class is a non-zero multiple of $u^{y_{[n]}}$. The previous lemma says that $\mathcal{A}$ is well-defined up to quasi-isomorphism.

We will consider deformations of $\mathcal{A}$ over R_a, which are controlled by the Hochschild cohomology with coefficients in R_a,

$$HH^*_{\boldsymbol{G}}(\mathcal{A}, \mathcal{A} \otimes R_a).$$

Recall from Remark 2.2.3.28 that the filtration by length on the Hochschild complex $CC(A)$ yields a spectral sequence for the Hochschild cohomology, with

$$E_2^{*,*} = HH^*_c(A, A).$$

Lemma 2.2.5.8. *The spectral sequence induced by the length filtration on the Hochschild cochain complex $CC^*(\mathcal{A})$ converges to the Hochschild cohomology $HH^*(\mathcal{A})$.*

Proof. By Remark 2.2.3.28, it suffices to prove that the spectral sequence is regular.

A generator $u^{\boldsymbol{b}}\theta^K$ of $HH_c^*(A)$ changes the Y-degree by

$$(-|K| + |\boldsymbol{b}|, -\boldsymbol{b} + y_K) \in \mathbb{Z} \oplus Y/(2, y_{[n]}),$$

and its length is $s = |\boldsymbol{b}|$. Therefore, its grading is

$$(-|K| + 2|\boldsymbol{b}|, -\boldsymbol{b} + y_K)$$

(recalling the conventions for grading Hochschild cochains). Using a suitably altered pseudo-grading datum as in the proof of Lemma 2.2.5.3, this is equivalent to a grading

$$(|K|, -\boldsymbol{b} + y_K)$$

(note: $c = 2$ in this altered pseudo-grading datum).

Now recall that the differential on page d of the spectral sequence maps

$$\delta_d : E_d^{(y,s)} \to E_d^{(y+f(1),s+d)}.$$

If both domain and codomain of the differential are to be non-zero, then we must have generators $u^{\boldsymbol{b}_1}\theta^{K_1}$ and $u^{\boldsymbol{b}_2}\theta^{K_2}$ such that

$$(|K_1| + 1, -\boldsymbol{b}_1 + y_{K_1}) = (|K_2|, -\boldsymbol{b}_2 + y_{K_2}) + q(2, y_{[n]})$$

and

$$|\boldsymbol{b}_2| = |\boldsymbol{b}_1| + d.$$

Because $0 \leq |K_1|, |K_2| \leq n$, and $|K_1| + 1 = |K_2| + 2q$, we must have $q \leq (n+1)/2$. We also have

$$\boldsymbol{b}_2 = \boldsymbol{b}_1 + y_{K_2} - y_{K_1} + q y_{[n]},$$

and hence that

$$|\boldsymbol{b}_2| = |\boldsymbol{b}_1| + |K_2| - |K_1| + nq = |\boldsymbol{b}_1| + 1 + (n-2)q.$$

Therefore, for the differential δ_d to be non-zero, we must have

$$d = |\boldsymbol{b}_2| - |\boldsymbol{b}_1| = (n-2)q + 1 \leq \frac{(n+1)(n-2)}{2} + 1.$$

Hence, for d sufficiently large, the differential vanishes, so the spectral sequence is regular.

□

Now, if we are to apply the deformation theory of Section 2.2.4, we need to know that our deformations are minimal.

Lemma 2.2.5.9. *If $a = n$, and $n \geq 5$, then any $\boldsymbol{G}$-graded deformation of $\mathcal{A}$ over R is minimal.*

Proof. Equation (2.2.5.3) with $a = n$ shows that

$$CC^2_{c,\boldsymbol{G}}(A, A \otimes R)^t \cong 0$$

unless t is divisible by $(n-2)$. The μ^0 and μ^1 terms live in the spaces with $t = 2$ and $t = 1$, respectively. So when $n \geq 5$, any deformation of $\mathcal{A}$ over R satisfies $\mu^0 = 0$ and $\mu^1 = 0$, hence is minimal. □

Remark 2.2.5.10. Lemma 2.2.5.9 is the only place where we need the assumption $n \geq 5$, as opposed to $n \geq 4$. If $n = 4$, then $CC^2_{c,\boldsymbol{G}}(A, A \otimes R)^0$ is one-dimensional, generated by the element $T\theta^{[4]}$. These deformations can be ruled out in our situation with some additional input (namely, by building a strictly unital model for our category), but we prefer to avoid this additional complication. See [9].

The next step is to determine the first-order deformation space.

Lemma 2.2.5.11. *If $n \geq 4$, then the vector space*

$$HH^2_{\boldsymbol{G}}(A, A \otimes R^1_a)$$

is generated by the elements

$$r_j u_j^a,$$

for $j = 1, \ldots, n$.

Proof. We apply Lemma 2.2.5.4 with $s + t = 2$ and $j = 1$. Equation (2.2.5.8) yields

$$|K| = 2q \geq 0.$$

Furthermore, Equation (2.2.5.7) gives

$$\begin{aligned} 2 - s &= (n-2)q + 2 - a \\ \Rightarrow a - s &= (n-2)q. \end{aligned}$$

Furthermore, we have $\boldsymbol{c} = y_k$ for some k. Taking the dot product of y_k with Equation (2.2.5.6) gives us

$$\begin{aligned} y_k \cdot (y_K + a\boldsymbol{c} - \boldsymbol{b}) &= q y_k \cdot y_{[n]} \\ \Rightarrow (0 \text{ or } 1) &= -a + y_k \cdot \boldsymbol{b} + q \\ &\leq -a + s + q \\ &= (3-n)q. \end{aligned}$$

If $n \geq 4$, then $3 - n < 0$ and $q \geq 0$, so we must have $q = 0$. Thus, we have $|K| = 2q = 0$, so $K = \phi$. From Equation (2.2.5.6), we obtain $\boldsymbol{b} = a y_j$, so the generator has the form

$$r_j u_j^a.$$

$\square$

Having determined the first-order deformation space, we now seek to understand the higher-order parts of the deformation. The following lemma will be useful:

Lemma 2.2.5.12. *If $r^{\boldsymbol{c}}u^{\boldsymbol{b}}\theta^K$ is a generator of $THH^2_G(A, A\otimes R_a)$, and $j = |\boldsymbol{c}| \geq 2$, then $\boldsymbol{c} \geq y_{[n]}$.*

Proof. We apply Lemma 2.2.5.4, with $s + t = 2$. Equation (2.2.5.8) yields

$$0 \leq |K| = 2(1 + q - j).$$

It follows that $q \geq j - 1$. Now we split into two cases, depending on whether $q = j - 1$ or $q > j - 1$.

Case 1: $q = j - 1$. In this case, we have $|K| = 0$ so $K = \phi$. Thus Equation (2.2.5.6) gives

$$a\boldsymbol{c} = qy_{[n]} + \boldsymbol{b}.$$

Taking the dot product with y_k yields

$$ac_k = q + b_k \geq q = j - 1 > 0,$$

since $j \geq 2$. Hence $c_k \geq 1$ for all k, and the result is proved.

Case 2: $q > j - 1$. In this case, we have $q > j - 1 \geq 1$, so $q \geq 2$. Thus, taking the dot product of Equation (2.2.5.6) with y_k gives

$$ac_k = q + b_k - y_k \cdot y_K \geq q - 1 > 0.$$

Hence $c_k \geq 1$ for all k, and the result is proved. $\square$

Now if $a = n$, then the grading of $T = r^{y_{[n]}} \in R_0$ is 0 in $\mathbb{Z}$, and furthermore T is clearly H-invariant. Thus $\mathbb{C}[[T]] \subset R_0^H$ (in fact one can show they are equal). It follows that $THH^2_G(\mathcal{A}, \mathcal{A} \otimes R)^H$ is a $\mathbb{C}[[T]]$-module.

Lemma 2.2.5.13. *Suppose that $a = n$, and $n \geq 4$. Then the H-invariant part of the truncated Hochschild cohomology group*

$$THH^2_G(\mathcal{A}, \mathcal{A} \otimes R)^H$$

is generated, as a $\mathbb{C}[[T]]$-module, by its first-order part

$$THH^2_G(\mathcal{A}, \mathcal{A} \otimes R^1)^H.$$

Proof. We prove the result by induction on the order j, and using the spectral sequence induced by the length filtration.

When $j = 0$, we must show that $THH^2_G(\mathcal{A}) \cong 0$. Let $u^b\theta^K$ be a generator of $THH^2_G(A)$. By equation (2.2.5.7) with $j = 0$, we have

$$t = (n-2)q \leq 0$$

(since we are considering the truncated Hochschild cohomology). By equation (2.2.5.8) with $j = 0$, we have

$$|K| = 2(1+q) \geq 0.$$

Thus we have $-1 \leq q \leq 0$. We have two cases:

Case 1: $q = 0$. In this case, $|K| = 2$, $t = 0$, so $s = 2$. Equation (2.2.5.6) yields

$$y_K = \boldsymbol{a},$$

so our generator has the form $u_i u_k \theta^i \wedge \theta^k$. But now, if $\sigma_{ik} \in H$ denotes the transposition of elements i and k, we have

$$\sigma_{ik}(u_i u_k \theta^i \wedge \theta^k) = u_k u_i \theta^k \wedge \theta^i = -u_i u_k \theta^i \wedge \theta^k.$$

Hence, the coefficient of this term in any H-invariant element of the truncated Hochschild cohomology must be 0.

Remark 2.2.5.14. Recall that H may differ from the obvious action by permutation of basis elements, by some signs. However, each sign appears twice (once for u_i, once for θ^i), so they do not affect this computation.

Case 2: $q = -1$. In this case, $|K| = 0$, so $K = \phi$. Equation (2.2.5.6) yields

$$y_{[n]} = \boldsymbol{b}.$$

Thus, the generator is $u^{y_{[n]}}$, the deforming class identified in Lemma 2.2.5.6. However, this generator gets killed by the first non-trivial differential of the spectral sequence. To see this, observe that, because $\mu^d = 0$ for $2 < d < n$ by Lemma 2.2.5.5, the first non-trivial differential is δ_{n-1}, and is given by

$$\delta_{n-1}(\phi) = [[\mu^n], \phi] = [u^{y_{[n]}}, \phi]$$

(see Remark 2.2.3.29). The Gerstenhaber bracket on $CC^*(A)$ gets carried to the Schouten bracket on polyvector fields $\mathbb{C}[[U]] \otimes \Lambda(U)$. It follows quickly from the explicit form of the Schouten bracket (see [20, Equation (3.7)]) that for any $W \in \mathbb{C}[[U]]$,

$$[W, \eta] = \iota_{dW}(\eta),$$

and therefore that the cohomology of $[W, -]$ is the cohomology of the Koszul complex associated to dW. In particular, the class W itself gets killed by taking the cohomology of this differential. In our case, this means that $u^{y_{[n]}}$ is killed by δ_{n-1}.

This completes the proof that $THH^2_G(\mathcal{A}) \cong 0$.

When $j = 1$, the statement is simply that the first-order part is generated by the first-order part. So it remains to prove that, for $j \geq 2$, the order-j part is generated by the first-order part.

Suppose, inductively, that

$$THH^2_G(\mathcal{A}, \mathcal{A} \otimes R^k) \subset \mathbb{C}[[T]] \cdot THH^2_G(\mathcal{A}, \mathcal{A} \otimes R^1)$$

for all $k \leq j-1$, for some $j \geq 2$. We prove that the same holds for $k = j$. Let

$$r^{\boldsymbol{c}} u^{\boldsymbol{b}} \theta^K \in THH^2_G(A, A \otimes R^j)$$

be a generator. Because $j \geq 2$, it follows from Lemma 2.2.5.12 that $r^{\boldsymbol{c}} = T \cdot r^{\boldsymbol{c}'}$, where

$$\boldsymbol{c}' = \boldsymbol{c} - y_{[n]} \geq 0.$$

Because $T \in R_0$, we have

$$r^{\boldsymbol{c}'} u^{\boldsymbol{b}} \theta^K \in THH^2_G(A, A \otimes R^{j-n}).$$

It follows that

$$THH^2_G(A, A \otimes R^j) = T \cdot THH^2_G(A, A \otimes R^{j-n}),$$

and hence, from the spectral sequence induced by the length filtration, that

$$THH^2_G(\mathcal{A}, \mathcal{A} \otimes R^j) = T \cdot THH^2_G(\mathcal{A}, \mathcal{A} \otimes R^{j-n}).$$

The result now follows by taking H-invariant parts of this equation.

This completes the inductive step, and hence the proof. □

Corollary 2.2.5.15. *There exists a* $\psi \in \mathbb{C}[[T]]$*, with* $\psi(0) = 1$*, and an* $\boldsymbol{G}$*-graded formal diffeomorphism* F*, such that*

$$\psi \cdot \mathcal{A}_1 = F_* \mathcal{A}_2.$$

Proof. By Corollary 2.2.5.7, there is a formal diffeomorphism F_0 from (A, η_0) to (A, μ_0). Therefore we can push forward η by F_0, and reduce to the case where $\mu_0 = \eta_0$. The result then follows from Proposition 2.2.4.16 and Lemma 2.2.5.13. □

This completes the proof of Theorem 6.

We now make one final computation of Hochschild cohomology. For this computation, we will be interested in an extra structure on Hochschild cohomology: the **Yoneda product**, which makes it into an associative algebra. In fact, together with the Gerstenhaber bracket, this makes the Hochschild cohomology into a **Gerstenhaber algebra** (see [62]).

Definition 2.2.5.16. The Hochschild cohomology $HH^*(\mathcal{A})$ of a $\boldsymbol{G}$-graded A_∞ algebra $\mathcal{A}$ carries the $\boldsymbol{G}$-graded, associative **Yoneda product**, which has the form

$$(\phi \bullet \psi)^n(x_n, \ldots, x_1) :=$$

$$\sum_{i+j+k+l+m=n} (-1)^{\dagger} \mu^{m+k+i+2}(x_{i+j+k+l+m}, \ldots, \phi^l(x_{i+j+k+l}, \ldots), x_{i+j+k}, \ldots, \psi^j(x_{i+j}, \ldots), x_i, \ldots),$$

where $\dagger$ is some sign.

Remark 2.2.5.17. We record the following useful information about the Yoneda product:

- If $\boldsymbol{k}$ is a field, and U a $\boldsymbol{k}$-vector space, then the HKR isomorphism

$$\Phi : HH^*(\Lambda(U)) \to \boldsymbol{k}[[U]] \otimes \Lambda(U),$$

is an isomorphism of $\boldsymbol{k}$-algebras, where the product on $HH^*(\Lambda(U))$ is the Yoneda product, and the product on polyvector fields is the wedge product (see [63, Section 8]);

- The spectral sequence $E_d^{s,t} \Rightarrow HH^*(\mathcal{A})$ induced by the length filtration (Remark 2.2.3.28) respects the multiplicative structure of the Yoneda product.

We now carry out our final Hochschild cohomology computation. Suppose that $\mathcal{A}$ is an A_∞ algebra of type A, and $\underline{\mathcal{A}}$ its extension to a $\boldsymbol{G}_1^n$-graded category. We denote

$$\widetilde{\mathcal{A}} := \boldsymbol{p}_1^* \underline{\mathcal{A}},$$

where $\boldsymbol{q}_1$ and $\boldsymbol{p}_1$ are the morphisms of grading data defined in Lemma 2.2.1.14. Note that $\boldsymbol{q}_{1*}\widetilde{\mathcal{A}}$ is a $\boldsymbol{G}_{\mathbb{Z}}$-graded (or equivalently, $\mathbb{Z}$-graded) A_∞ category, over the $\mathbb{Z}$-graded coefficient ring $\boldsymbol{q}_{1*}\boldsymbol{p}_1^* R \cong R$, which has degree $0 \in \mathbb{Z}$, by Remark 2.2.2.11. Therefore, the homomorphism

$$\begin{aligned} R &\rightarrow \Lambda, \\ r_j &\mapsto r \text{ for all } j \end{aligned}$$

respects the $\mathbb{Z}$-grading, and we can form the $\boldsymbol{G}_{\mathbb{Z}}$-graded Λ-linear A_∞ category

$$\widetilde{\mathcal{A}}_{nov} := \boldsymbol{q}_{1*}\widetilde{\mathcal{A}} \otimes_R \Lambda.$$

Our aim now is to compute a certain part of

$$HH^*\left(\widetilde{\mathcal{A}}_{nov}\middle| \Lambda\right).$$

We observe that Y_1^n acts on $\underline{\mathcal{A}}$ by shifts, hence also acts on $\widetilde{\mathcal{A}}_{nov}$, hence also acts on $CC^*(\widetilde{\mathcal{A}}_{nov})$. The action of $\ker(Y_n^n \to \mathbb{Z}) \subset Y_1^n$ is trivial, because shifts in this subgroup become trivial when we take $\boldsymbol{q}_{1*}$ of our category. The action of $\mathbb{Z}$ is also trivial, because we consider only $\mathbb{Z}$-equivariant cochains, by definition. It follows that the action of the image of Y_n^n in Y_1^n is trivial, so there is an action of

$$\tilde{\Gamma}_n^* \cong Y_1^n / Y_n^n$$

on $CC^*(\widetilde{\mathcal{A}}_{nov})$. Note that this is the same $\tilde{\Gamma}_n^*$ as in Definition 2.1.2.4.

Lemma 2.2.5.18. *There is an isomorphism of $\mathbb{Z}$-graded Λ-algebras,*

$$HH^*\left(\widetilde{\mathcal{A}}_{nov}\right)^{\tilde{\Gamma}_n^*} \cong \Lambda[\alpha]/\alpha^{n-1},$$

where α has degree 2.

Proof. We consider the spectral sequence induced by the length filtration on

$$CC^*\left(\widetilde{\mathcal{A}}_{nov}\right)^{\tilde{\Gamma}_n^*}.$$

As we saw in Remark 2.2.3.28, the filtration is bounded above, hence exhaustive, and is also complete. So if we can prove that it is regular, then it must converge to the Hochschild cohomology.

We observe that there is an obvious morphism of chain complexes,

$$CC^*\left(\boldsymbol{q}_{1*}\boldsymbol{p}_1^*\underline{\mathcal{A}}\right) \otimes_R \Lambda \to CC^*\left(\widetilde{\mathcal{A}}_{nov}\right),$$

and indeed the A_∞ structure maps on $\widetilde{\mathcal{A}}_{nov}$ are the image of the A_∞ structure maps of $\mathcal{A}$ under this morphism. However, this is **not** necessarily a quasi-isomorphism, because the Hochschild cochain complex is defined as a direct product over cochains of all lengths $s \geq 0$, and arbitrary direct products do not commute with $\otimes_R \Lambda$.

On the other hand, **finite** direct products **do** commute with $\otimes_R \Lambda$, so the above morphism of chain complexes does induce an isomorphism after we quotient by chains of length greater than some fixed s. It follows that there is an isomorphism between the spectral sequences induced by the length filtrations on

$$CC^*\left(\boldsymbol{q}_{1*}\boldsymbol{p}_1^*\underline{\mathcal{A}}\right) \otimes_R \Lambda$$

and on

$$CC^*\left(\widetilde{\mathcal{A}}_{nov}\right)$$

(it is just that the filtration on the former chain complex is not complete, so the associated spectral sequence may not converge to its cohomology).

Now, applying Remarks 2.2.3.41 and 2.2.3.38, we have isomorphisms

$$\begin{aligned} CC^*\left(\boldsymbol{q}_{1*}\boldsymbol{p}_1^*\underline{\mathcal{A}}\right)^{\tilde{\Gamma}_n^*} &\cong \boldsymbol{q}_{1*}CC^*\left(\boldsymbol{p}_1^*\underline{\mathcal{A}}\right)^{\tilde{\Gamma}_n^*} \text{ (up to completing with respect to the length filtration)} \\ &\cong \boldsymbol{q}_{1*}\boldsymbol{p}_1^*CC^*\left(\underline{\mathcal{A}}\right) \\ &\cong \boldsymbol{q}_{1*}\boldsymbol{p}_1^*CC^*\left(\mathcal{A}\right). \end{aligned}$$

It follows from the preceding discussion that, if it is regular, then the spectral sequence induced by the length filtration on

$$\boldsymbol{q}_{1*}\boldsymbol{p}_1^*CC^*\left(\mathcal{A}\right) \otimes_R \Lambda$$

converges to

$$HH^*\left(\widetilde{\mathcal{A}}_{nov}\right)^{\tilde{\Gamma}_n^*}.$$

Now, note that $\mu^s = 0$ in $\mathcal{A}$ unless $s-2$ is divisible by $(n-2)$ (by Equation (2.2.5.3) with $a = n$ and $s + t = 2$). In particular, $\mathcal{A}$ is minimal. Next, we show that there is an algebra isomorphism $(\mathcal{A}, \mu_0^2) \cong (\mathcal{A}, \mu^2)$. I.e., the higher-order terms in the product μ^2 can be absorbed into the order-0 product μ_0^2, which we know to be the exterior product.

This follows immediately from the calculation

$$THH^2_G(A, A \otimes R)^{0,H} \cong 0,$$

which was carried out in the proof of Lemma 2.2.5.13 (as 'Case 1'), together with a deformation theory argument. Thus, we can replace $\mathcal{A}$ by a quasi-isomorphic A_∞ algebra $\mathcal{A}'$, whose underlying algebra is the exterior algebra

$$(\mathcal{A}', \mu^2) \cong A \otimes R.$$

Furthermore, $\mathcal{A}'$ and $\mathcal{A}$ coincide to order n, because the higher-order terms in μ^2 were functions of $T = r_1 \ldots r_n$. So $\mathcal{A}'$ is still of type A. Henceforth we'll simply write $\mathcal{A}$ for this replacement.

Now the E_2 page of the spectral sequence induced by the length filtration on

$$CC^*(\mathcal{A}) \otimes_R \Lambda$$

is the Hochschild cohomology of the associative algebra $(\mathcal{A} \otimes_R \Lambda, \mu^2 \otimes 1)$. By the preceding arguments, this is exactly $A \otimes \Lambda$, the exterior algebra over the field Λ. Its Hochschild cohomology is given by the HKR isomorphism (because Λ is a field), so we have

$$E_2 = \boldsymbol{q}_{1*}\boldsymbol{p}_1^* \left(\Lambda[[U]] \otimes A\right).$$

Because $\mu^s = 0$ for $2 < s < n$, the first non-trivial differential is δ_{n-1}, which is given by Gerstenhaber bracket with the order-n deformation class $[\mu^n]$. The Gerstenhaber bracket gets carried to the Schouten bracket under the HKR isomorphism, and μ^n gets carried to

$$w_{nov} := \Phi([\mu^n]) \in \Lambda[[U]] \otimes A,$$

where Φ is the HKR isomorphism. By the 'naturality' property of the HKR map (see

Definition 2.2.5.1), w_{nov} is the image of the element

$$w \otimes 1 \in (R[[U]] \otimes A) \otimes_R \Lambda$$

under the natural map

$$(R[[U]] \otimes A) \otimes_R \Lambda \to \Lambda[[U]] \otimes A,$$

where w is the image of $[\mu^n]$ in $R[[U]] \otimes A$ under the HKR map.

Now, by the grading computations in the proof of Lemma 2.2.5.13, w has the form

$$w = f_1(T)u_1 \dots u_n + f_2(T) \sum_{j=1}^{n} r_j u_j^n \in R[[u_1, \dots, u_n]],$$

where $f_1, f_2 \in \mathbb{C}[[T]]$, and we recall that $T := r_1 \dots r_n$. By the definition of 'type A', we have $f_1(0) = 1$ and $f_2(0) = 1$. It follows that

$$w_{nov} = f_1(r^n)u_1 \dots u_n + f_2(r^n) \sum_{j=1}^{n} r u_j^n \in \Lambda[[u_1, \dots, u_n]].$$

In particular, w_{nov} lies in $\Lambda[[U]]$, i.e., there are no non-trivial polyvector field terms appearing. Therefore, as we saw in the proof of Lemma 2.2.5.13,

$$\delta_{n-1}(-) = [\Phi(\mu^n), -] = \iota_{dw_{nov}}(-)$$

gives the Koszul complex for the sequence

$$\frac{\partial w_{nov}}{\partial u_1}, \dots, \frac{\partial w_{nov}}{\partial u_n} \in \Lambda[[U]].$$

Now we show that this sequence is regular. This follows because w_{nov} has an isolated singularity at the origin. To see this, observe that we have relations

$$\begin{aligned} f_1(r^n)\frac{u_1 \ldots u_n}{u_j} &\equiv -nrf_2(r^n)u_j^{n-1} \\ \Rightarrow \frac{u_1 \ldots u_n}{u_j} &\equiv rf(r)u_j^{n-1}, \end{aligned}$$

in the ring

$$\Lambda[[u_1, \ldots, u_n]]/(\partial_1 w_{nov}, \ldots, \partial_n w_{nov}),$$

where

$$f(r) = -n\frac{f_2(r^n)}{f_1(r^n)} \in \mathbb{C}[[r^n]], \ f(0) \neq 0.$$

Taking the product of these relations gives

$$(u_1 \ldots u_n)^{n-1} \equiv r^n f(r)^n (u_1 \ldots u_n)^{n-1},$$

and hence that

$$(u_1 \ldots u_n)^{n-1} \equiv 0,$$

because $1 + \mathcal{O}(r)$ is invertible in Λ. Returning to the original relation, we have

$$(rf(r)u_j^n)^{n-1} = (u_1 \ldots u_n)^{n-1} = 0,$$

and hence sufficiently high powers of each generator u_j vanish (recalling $f \neq 0$, so $f \in \Lambda^*$ is invertible). Therefore w_{nov} has an isolated singularity at 0, so the sequence is regular, so the cohomology of the Koszul complex is the Jacobian ring.

Now we observe that

$$\boldsymbol{q}_{1*}\boldsymbol{p}_1^*\Lambda[[U]]$$

is generated, as a Λ-algebra, by the elements

$$u_1 \ldots u_n, u_1^n, \ldots, u_n^n,$$

and each of these has degree $2 \in \mathbb{Z}$. To see this, note that for $u^{\boldsymbol{b}}$ to be in $\boldsymbol{p}_1^*\Lambda[[U]]$, its degree must be in the image of Y_n^n in Y_1^n. From the definition, this means that

$$(|\boldsymbol{b}|, -\boldsymbol{b}) = (t + 2(n-1)|\boldsymbol{m}|, -n\boldsymbol{m}) - q(2(1-n), y_{[n]})$$

for some $t, q \in \mathbb{Z}$ and $\boldsymbol{m} \in \mathbb{Z}^n$, or in other words, setting $s = |\boldsymbol{b}|$ for the length,

$$\begin{aligned} s &= t + 2(n-1)|\boldsymbol{m}| + 2(n-1)q, \\ \boldsymbol{b} &= n\boldsymbol{m} + qy_{[n]}. \end{aligned}$$

It is not hard to show that, since $\boldsymbol{b} \geq \boldsymbol{0}$, we can arrange that $\boldsymbol{m} \geq \boldsymbol{0}$ and $q \geq 0$, so $u^{\boldsymbol{b}}$ is a product of the generators $u_1 \dots u_n, u_1^n, \dots, u_n^n$, as claimed. Furthermore, one can show that $s + t = 2q + 2|\boldsymbol{m}|$, where $s = |\boldsymbol{b}|$ is the length and t is the $\mathbb{Z}$-grading. It follows that the degree of each generator is 2, when regarded as an element of Hochschild cohomology.

It follows that the E_n page of our spectral sequence is given by

$$\Lambda[[u_1, \dots, u_n]]/(\partial_1 w_{nov}, \dots, \partial_n w_{nov}) \cap \Lambda[[u_1 \dots u_n, u_1^n, \dots, u_n^n]].$$

From the relations in the Jacobian ring, we have

$$u_1 \dots u_n = rf(r)u_j^n \text{ where } f(r) \neq 0,$$

so we only need a single generator $\alpha := u_1 \dots u_n$, and furthermore this generator satisfies

$$\alpha^{n-1} = 0,$$

as we showed above. It is easy to check (for example using Gröbner bases) that α^{n-2} does not lie in the ideal generated by the partial derivatives $\partial_j w_{nov}$. It follows that the

E_n page of the spectral sequence is given by

$$\Lambda[\alpha]/\alpha^{n-1}.$$

Because α has degree $2 \in \mathbb{Z}$, E_n is graded in even degrees so all subsequent differentials in the spectral sequence vanish, so the spectral sequence is regular. Since the length filtration is bounded above and complete, it follows by [61, Theorem 5.5.10] that the spectral sequence converges to the Hochschild cohomology. Thus the E_n page is isomorphic to the associated graded algebra of the Hochschild cohomology; but since it is one-dimensional in each degree, it is in fact isomorphic to the Hochschild cohomology. This completes the proof. $\square$

Now we observe that the element

$$r_j \frac{\partial \mu^*}{\partial r_j} \in CC^2_{G^n_n}\left(\widetilde{\mathcal{A}}\right)$$

is a Hochschild cochain (this follows by applying $r_j \partial/\partial r_j$ to the A_∞ relation $\mu \circ \mu = 0$). Hence it defines an element in $HH^*(\widetilde{\mathcal{A}})$. We denote by β the element of $HH^*(\widetilde{\mathcal{A}}_{nov})$ that is the image of

$$\left(r_j \frac{\partial \mu^*}{\partial r_j}\right) \otimes 1 \in HH^*\left(\widetilde{\mathcal{A}}\right) \otimes_R \Lambda$$

under the obvious map.

Lemma 2.2.5.19. *The element β lies in the $\tilde{\Gamma}^*_n$-equivariant part of $HH^*(\widetilde{\mathcal{A}}_{nov})$, and corresponds to $g \cdot \alpha$ for some invertible $g \in \Lambda^*$ under the isomorphism of Lemma 2.2.5.18.*

Proof. The fact that β is $\tilde{\Gamma}^*_n$-equivariant follows immediately from the fact that μ^* is. We recall from the proof of Lemma 2.2.5.18 that, firstly, we arrange that μ^2 is independent of r_j and, secondly, the image of $\mu^{\geq 3}$ under the HKR map to $R[[U]] \otimes A$ has the form

$$w = f_1(T) u_1 \dots u_n + f_2(T) \sum_{j=1}^{n} r_j u_j^n.$$

It follows that the image of $r_j\partial\mu^*/\partial r_j$ under the HKR map has the form

$$r_j\frac{\partial w}{\partial r_j} = Tf_1'(T)u_1\ldots u_n + f_2(T)r_ju_j^n + Tf_2'(T)\sum_{j=1}^{n} r_ju_j^n.$$

It follows from naturality of the HKR map that the image of $\beta = (r_j\partial\mu^*/\partial r_j)\otimes 1$ on the E_2 page of the spectral sequence has the form

$$r^nf_1'(r^n)u_1\ldots u_n + f_2(r^n)ru_j^n + r^nf_2'(r^n)\sum_{j=1}^{n} ru_j^n.$$

We now recall from the proof of Lemma 2.2.5.18 that the E_n page of the spectral sequence is the Jacobian ring $\Lambda[[u_1,\ldots,u_n]]/(\partial_1 w_{nov},\ldots,\partial_n w_{nov})$. We recall (from the proof of Lemma 2.2.5.18) that, in the Jacobian ring, we have relations

$$f_1(r^n)u_1\ldots u_n \equiv -nf_2(r^n)ru_j^n,$$

and we set $\alpha := u_1\ldots u_n$, so the image of β on the E_n page of the spectral sequence is equivalent to

$$\left(r^nf_1'(r^n) - \frac{1}{n}f_1(r^n) - r^n\frac{f_2'(r^n)f_1(r^n)}{f_2(r^n)}\right)\alpha = g(r^n)\alpha,$$

where $g(r^n)\in\mathbb{C}[[r^n]]$ and $g(0)\neq 0$.

Recalling that the spectral sequence degenerates at the E_n page, this completes the proof. $\square$

2.2.6 First-order deformations

In this section, we will consider a very specific situation. Let $\mathcal{A} = (A,\mu_0)$ be a $\boldsymbol{G}$-graded minimal A_∞ algebra over $\mathbb{C}$, and V a $\boldsymbol{G}$-graded vector space. We consider the $\boldsymbol{G}$-graded

$\mathbb{C}$-algebra

$$R = \mathbb{C}[V]/\mathfrak{m}^2,$$

where $\mathfrak{m} \subset \mathbb{C}[V]$ is the maximal ideal corresponding to 0. A $\boldsymbol{G}$-graded **first-order deformation** of $\mathcal{A}$ over V is an element

$$\mu \in CC^2_{\boldsymbol{G}}(A \otimes R, A \otimes R),$$

whose order-0 component agrees with μ_0, and such that $\mu \circ \mu = 0$. Equivalently, we have

$$\mu = \mu_0 + \mu_1,$$

where

$$\mu_1 \in CC^2_{\boldsymbol{G}}(\mathcal{A}, \mathcal{A} \otimes V^\vee)$$

is a Hochschild cocycle. Its class in $HH^2_{\boldsymbol{G}}(\mathcal{A}, \mathcal{A} \otimes V^\vee)$ is called the **deformation class** of the deformation. Two first-order deformations are A_∞ quasi-isomorphic if and only if their deformation classes coincide.

Let $\mathcal{C}$ be an A_∞ category over $\mathbb{C}$ with two quasi-isomorphic objects, which we call L_0 and L_1. We denote the A_∞ endomorphism algebras of the two objects by $\mathcal{A}_j := \mathrm{Hom}(L_j, L_j)$ for $j = 0, 1$. It is standard that $\mathcal{A}_0$ and $\mathcal{A}_1$ are A_∞ quasi-isomorphic.

Let $\mathcal{C}_j$ be the full subcategory with a single object L_j, for $j = 0, 1$. Then the inclusions

$$\mathcal{C}_0 \hookrightarrow \mathcal{C} \hookleftarrow \mathcal{C}_1$$

are quasi-equivalences. It follows by Morita invariance (see, for example, [11, Lemma 2.6]) that the restriction maps

$$HH^*_{\boldsymbol{G}}(\mathcal{C}_0, \mathcal{C}_0) \leftarrow HH^*_{\boldsymbol{G}}(\mathcal{C}, \mathcal{C}) \rightarrow HH^*_{\boldsymbol{G}}(\mathcal{C}_1, \mathcal{C}_1)$$

are isomorphisms.

Now suppose that we have a $\boldsymbol{G}$-graded first-order deformation of $\mathcal{C}$ over V. As we observed above, this is equivalent (up to quasi-isomorphism) to a choice of deformation class

$$[\mu_1] \in HH^2_{\boldsymbol{G}}(\mathcal{C}, \mathcal{C} \otimes V^\vee).$$

The deformation classes of the induced first-order deformations of the endomorphism algebras $\mathcal{A}_j$ are given by the images of $[\mu_1]$ under the restriction maps

$$HH^2_{\boldsymbol{G}}(\mathcal{C}, \mathcal{C} \otimes V^\vee) \to HH^2_{\boldsymbol{G}}(\mathcal{A}_j, \mathcal{A}_j \otimes V^\vee),$$

and hence they correspond under the isomorphism

$$HH^2_{\boldsymbol{G}}(\mathcal{A}_0, \mathcal{A}_0 \otimes V^\vee) \cong HH^2_{\boldsymbol{G}}(\mathcal{A}_1, \mathcal{A}_1 \otimes V^\vee)$$

induced by the quasi-isomorphism of $\mathcal{A}_0$ with $\mathcal{A}_1$.

2.3 The affine Fukaya category

In this section, we introduce the affine Fukaya category $\mathcal{F}(M)$ of an exact symplectic manifold, and explain its relationship with the exact Fukaya category as defined in [11], which we will denote by $\mathcal{F}'(M)$. The difference is essentially that the affine Fukaya category has less objects, but a richer grading structure.

2.3.1 Grading data from the Lagrangian Grassmannian

We recall some notions from [11, Chapters 11, 12]. Let M be a symplectic manifold. We denote by $\mathcal{G}M$ the bundle of Lagrangian subspaces of TM. Observe that, because we have a fibration

$$\begin{array}{ccc} \mathcal{G}_p M & \hookrightarrow & \mathcal{G} M \\ & & \downarrow \\ & & M, \end{array}$$

there is an exact sequence

$$\ldots \to \pi_1(\mathcal{G}_p M) \to \pi_1(\mathcal{G} M) \to \pi_1(M) \to *.$$

We recall that

$$\pi_1(\mathcal{G}_p M) \cong \mathbb{Z}.$$

Assumption 2.3.1.1. For the purposes of this paper, we will only consider manifolds M such that $\pi_1(M)$ is abelian. The reason for this assumption is that we will use $\pi_1(M)$ to define a grading datum $\boldsymbol{G}(M)$, and the affine Fukaya category will be $\boldsymbol{G}(M)$-graded. However we have only set up the theory of grading data (Section 2.2) for abelian groups. One should be able to define a sensible theory of non-abelian grading data, and apply it to study Fukaya categories of exact symplectic manifolds M with non-abelian fundamental group. On the other hand, when one studies the the relative Fukaya category (Section 2.5), it becomes absolutely necessary to consider abelian grading data. In this case, one should consider the universal **abelian** cover of $\mathcal{G}M$, rather than the universal cover. Since these issues do not concern us in this paper, we ignore them by making this assumption.

In particular, Assumption 2.3.1.1 implies that that $\pi_1(M) \cong H_1(M)$ and $\pi_1(\mathcal{G}M) \cong H_1(\mathcal{G}M)$.

Definition 2.3.1.2. We define a grading datum $\boldsymbol{G}(M)$:

$$\begin{array}{ccccccc} \mathbb{Z} & \xrightarrow{f} & Y(M) & \xrightarrow{g} & X(M) & \longrightarrow & 0 \\ \| & & \| & & \| & & \\ H_1(\mathcal{G}_p M) & \to & H_1(\mathcal{G} M) & \to & H_1(M) & \longrightarrow & 0. \end{array}$$

To define the sign morphism $\boldsymbol{\sigma}$, we must specify a map

$$\sigma : H_1(\mathcal{G}M) \to \mathbb{Z}_2.$$

To do this, we consider the real vector bundle $\mathcal{L} \to \mathcal{G}M$, whose fibre over a point is identified with the Lagrangian subspace at that point. The first Stiefel-Whitney class defines an element

$$w_1(\mathcal{L}) \in H^1(\mathcal{G}M; \mathbb{Z}_2),$$

and σ is defined by pairing with this element.

We will see that it is natural to define the Fukaya category as a $\boldsymbol{G}(M)$-graded category. That is because of the relationship between $\boldsymbol{G}(M)$ and index theory of Cauchy-Riemann operators in M, which we now explain.

We recall (from [40, Appendix C.3]) the definition of a **bundle pair** (Σ, E, F) over a Riemann surface with boundary Σ. It is a complex vector bundle $E \to \Sigma$ together with a totally real subbundle $F \subset E|_{\partial\Sigma}$. We recall that a bundle pair defines a Cauchy-Riemann operator

$$D : \Omega^0_F(\Sigma, E) \to \Omega^{0,1}_F(\Sigma, E),$$

whose zeroes are holomorphic sections of E whose boundary values lie in F. We recall also the **boundary Maslov index** of a bundle pair, which is an integer $\mu(\Sigma, E, F)$ such that the index of (an appropriate Sobolev-space version of) the Cauchy-Riemann operator D associated to the bundle pair (Σ, E, F) is

$$\mathrm{ind}(D) = n\chi(\Sigma) + \mu(\Sigma, E, F),$$

where n is the complex dimension of a fibre of E.

Now suppose we are given a map $u : \Sigma \to M$, together with a Lagrangian subbundle

$F \subset u^*TM|_{\partial\Sigma}$. F defines a lift

$$\begin{tikzcd} & \mathcal{G}M \arrow[d] \\ \partial\Sigma \arrow[ur, "\rho"] \arrow[r, "u|_{\partial\Sigma}"'] & M, \end{tikzcd}$$

which defines a class $[\rho] \in H_1(\mathcal{G}M) = Y(M)$, which is a lift of

$$u|_{\partial\Sigma} = \partial u = 0 \in H_1(M) = X(M).$$

So by exactness of the sequence $\boldsymbol{G}(M)$, $[\rho]$ lies in the image of $\mathbb{Z}$.

Lemma 2.3.1.3. *We have*

$$[\rho] = f(\mu(\Sigma, u^*TM, F)),$$

in $Y(M)$, where $f : \mathbb{Z} \to Y(M)$ comes from the grading datum.

Proof. Define a decomposition of bundle pairs,

$$(\Sigma, u^*TM, F) = (\Sigma_1, u^*TM, F) \cup (\Sigma_2, u^*TM, F),$$

where Σ_1 is a small ball in the interior of Σ, Σ_2 is its complement, and the totally real subbundle of $u^*TM|_{\partial\Sigma_1}$ is defined by a lift

$$\rho' : \partial\Sigma_1 \to \mathcal{G}M,$$

chosen in such a way that there is a trivialization

$$(\Sigma_2, u^*TM, F) \cong (\Sigma_2, \Sigma_2 \times \mathbb{C}^n, \Sigma_2 \times \mathbb{R}^n).$$

It follows quickly from the properties of the boundary Maslov index (see [40, Theorem

C.3.5]) that

$$\mu(\Sigma_2, u^*TM, F) = 0.$$

Then, by the composition property of the boundary Maslov index,

$$\mu(\Sigma, u^*TM, F) = \mu(\Sigma_1, u^*TM, F) + \mu(\Sigma_2, u^*TM, F) = \mu(\Sigma_1, u^*TM, F).$$

By our definition of (Σ_2, u^*TM, F), there is a lift

$$\begin{tikzcd} & \mathcal{G}M \arrow[d] \\ \Sigma_2 \arrow[ur, "P"] \arrow[r, "u|_{\Sigma_2}"'] & M, \end{tikzcd}$$

and P defines a homology between the class $[\rho]$ and the class $[\rho']$. If the ball Σ_1 is centred on a point p, then there is an obvious isomorphism

$$u^*TM|_{\Sigma_1} \cong \Sigma_1 \times T_pM,$$

so ρ' defines a class in $H_1(\mathcal{G}_pM) \cong \mathbb{Z}$, which is exactly equal to $\mu(\Sigma_1, u^*TM, F)$ (essentially from the definition of the Maslov class). The result follows. $\square$

2.3.2 Anchored branes

In this section, we will define the notion of an **anchored Lagrangian brane** in M. These will be the objects of the affine Fukaya category $\mathcal{F}(M)$ (when M is exact). First we recall the notion of a (non-anchored) graded Lagrangian brane from [11, Chapter 11], which is an object of $\mathcal{F}'(M)$.

If $2c_1(M) = 0$, and M is equipped with a quadratic complex volume form η, then we can construct a **phase map**

$$\alpha_M : \mathcal{G}M \to S^1.$$

Now if $i : L \to M$ is a Lagrangian immersion, there is a canonical lift of

$$\begin{tikzcd} & \mathcal{G}M \arrow[d] \\ L \arrow[ur, "i_*"] \arrow[r, "i"'] & M. \end{tikzcd}$$

Hence, given η, there is a map

$$\begin{aligned} \alpha_L : L &\to S^1, \\ \alpha_L &:= \alpha_M \circ i_*. \end{aligned}$$

A (non-anchored) **graded Lagrangian brane** in M is a compact embedded Lagrangian $L \subset M$, together with a lift $\alpha_L^\#$ of α_L to $\mathbb{R}$, and a Pin structure on L.

Now we introduce a new notion. Let

$$\pi : \widetilde{\mathcal{G}M} \to \mathcal{G}M$$

denote the universal cover of the manifold $\mathcal{G}M$.

Definition 2.3.2.1. An **anchored Lagrangian brane** $L^\#$ in M is a Lagrangian immersion $i : L \to M$ of a compact manifold L into M, together with a lift $i^\#$ as follows:

$$\begin{tikzcd} & \widetilde{\mathcal{G}M} \arrow[d, "\pi"'] \\ L \arrow[ur, "i^\#"] \arrow[r, "i_*"'] & \mathcal{G}M, \end{tikzcd}$$

and a Pin structure. Note that we do **not** need a quadratic complex volume form to define the notion of an anchored Lagrangian brane.

We observe that there is a natural action of the covering group $\pi_1(\mathcal{G}M) \cong Y(M)$ on $\widetilde{\mathcal{G}M}$, and hence on anchored Lagrangian branes. We denote the action of $y \in Y(M)$ on

$L^{\#}$ by $y \cdot L^{\#}$.

Now we explain the relationship between anchored Lagrangian branes and (non-anchored) graded Lagrangian branes. We observe that, given a quadratic complex volume form η_M, there exists a lift of the squared phase map

$$\begin{tikzcd} \widetilde{\mathcal{G}}M \arrow[r, "\tilde{\alpha}_M"] \arrow[d] & \mathbb{R} \arrow[d] \\ \mathcal{G}M \arrow[r, "\alpha_M"] & S^1. \end{tikzcd}$$

Definition 2.3.2.2. Given a quadratic volume form η_M, a lift $\tilde{\alpha}_M$ as above, and an embedded anchored Lagrangian brane $L^{\#} = (L, i^{\#})$, we define a Lagrangian brane $\mathfrak{f}(L^{\#})$ in M ($\mathfrak{f}$ for 'forgetting' the anchored structure), with the same underlying Lagrangian L and Pin structure, and

$$\alpha_L^{\#} = \tilde{\alpha}_M \circ i^{\#}.$$

2.3.3 The affine Fukaya category

Let M be an exact symplectic manifold with convex boundary, with $\pi_1(M)$ abelian (see Assumption 2.3.1.1). In this section, we define the **affine Fukaya category**, $\mathcal{F}(M)$. It will be a $\boldsymbol{G}(M)$-graded A_∞ category. The definition is very closely related to the definition of the affine Fukaya category $\mathcal{F}'(M)$, given in [11, Section 12], to which the reader is referred for all technical details. We will see that $\mathcal{F}(M)$ is essentially a full subcategory of $\mathcal{F}'(M)$ with a richer grading structure – in particular, the analytic details of defining moduli spaces of pseudoholomorphic disks to define the A_∞ structure maps are completely analogous.

Objects of $\mathcal{F}(M)$ are embedded anchored Lagrangian branes. For each pair of objects, we choose a Floer datum on M, and for all moduli spaces of boundary-punctured holomorphic disks with boundary components labelled by anchored Lagrangian branes, we make a consistent universal choice of perturbation data on M. We assume that the action of $Y(M)$ on anchored Lagrangian branes lifts to an action on Floer and perturbation data (i.e., if we change some of the boundary labels of a boundary-punctured holomorphic disk by the action of $Y(M)$, then the perturbation datum does not change).

We now define the $\boldsymbol{G}(M)$-graded morphism spaces in $\mathcal{F}(M)$. Now, given anchored Lagrangian branes $L_0^\#, L_1^\#$, we define the morphism space $CF^*(L_0^\#, L_1^\#)$ to be generated by paths $p : [0,1] \to M$ satisfying $p(0) \in L_0$, $p(1) \in L(1)$, which are flowlines of the Hamiltonian vector field associated with the corresponding Floer datum.

Given such a p, we define its grading $y \in Y(M)$ to be the unique element such that p lifts to a path from $L_0^\#$ to $y \cdot L_1^\#$ in $\widetilde{\mathcal{G}}M$, which has Maslov index $0 \in \mathbb{Z}$. To explain what this means, we observe that there is a commutative diagram

$$\begin{array}{ccc} \widetilde{\mathcal{G}}M & \longrightarrow & \widetilde{M} \\ \downarrow & & \downarrow \\ \mathcal{G}M & \longrightarrow & M, \end{array}$$

where $\widetilde{M} \to M$ is the universal cover of M. Thus, associated with any anchored Lagrangian brane $L^\#$ is a lift, $\widetilde{L}$, of L to $\widetilde{M}$. The fact that p must lift to $\widetilde{\mathcal{G}}M$ implies that it must lift to $\widetilde{M}$; this already defines $y \in Y(M)$ up to addition of an element in the image of $\mathbb{Z} \to Y(M)$. Now we observe that the fibres of the bundle

$$\widetilde{\mathcal{G}}M \to \widetilde{M}$$

are the universal covers of the fibres of the Lagrangian Grassmannian $\mathcal{G}\widetilde{M}$. Thus, the anchored brane structures $L_0^\#, L_1^\#$ equip $\widetilde{L}_0, \widetilde{L}_1$ with the structure of 'abstract Lagrangian branes' (see [11, Section 12a]). Therefore, if the path p lifts to a path $\tilde{p}$ from $\widetilde{L}_0$ to $y \cdot \widetilde{L}_1$ in $\widetilde{M}$, then we can define the Maslov index i of any lift of $\tilde{p}$ to $\widetilde{\mathcal{G}}M$, and it is equal to the relative Maslov index of the abstract linear Lagrangian branes at either end of $\tilde{p}$. It is this index that we require to be 0. Given p, it is clear that we can find $y' \in Y(M)$

such that the path p lifts to a path $\tilde{p}$ from $\widetilde{L}_0$ to $y' \cdot \widetilde{L}_1$, but the Maslov index i may not be zero. However, we then necessarily have

$$y = y' - f(i),$$

so the $Y(M)$-grading of p is well-defined.

We define the A_∞ structure maps in $\mathcal{F}(M)$ by counting rigid pseudo-holomorphic disks in M. That is, given objects

$$L_0^\#, \ldots, L_s^\#$$

and morphisms

$$p_j \in CF^*(L_{j-1}^\#, L_j^\#) \text{ for } j = 1, \ldots, s$$

and

$$p_0 \in CF^*(L_0^\#, L_s^\#),$$

we define the coefficient of p_0 in $\mu^s(p_s, \ldots, p_1)$ to be the count of rigid pseudolomorphic disks in M with boundary conditions on L_j, asymptotic to the generators p_j.

We now explain why these structure maps are $\boldsymbol{G}(M)$-graded. Firstly, observe that the structure maps respect the action of $Y(M)$ on objects, because we chose the perturbation data to do so. From [11, Section 11l], we recall the definition of an orientation operator D_p corresponding to a generator p of $CF^*(L_0^\#, L_1^\#)$. We lift p to a path

$$\rho : [0,1] \to \widetilde{\mathcal{G}}M$$

connecting $L_0^\#(p(0))$ to $y \cdot L_1^\#(p(1))$, where $y \in Y(M)$ is the degree of p. Now define a smooth, non-decreasing function $\psi : \mathbb{R} \to [0,1]$ such that $\psi(s) = 0$ for $s \ll 0$ and $\psi(s) = 1$ for $s \gg 0$. We consider the Hermitian vector bundle over the upper half plane $\mathbb{R} \times \mathbb{R}_{\geq 0}$, with fibre over (s,t) given by $T_{p(\psi(s))}M$. We introduce Lagrangian boundary conditions along the real axis, given by $\rho(\psi(s))$. These Lagrangian boundary conditions define a Cauchy-Riemann operator, which we denote by D_p. D_p is Fredholm, its index is equal to the relative Maslov index of the abstract Lagrangian branes at either end of p, which is 0 by the definition of y, and its determinant line is canonically isomorphic to the orientation line o_p of p.

Given a holomorphic disk u contributing to an A_∞ product μ^s, we denote the linearized operator of the perturbed holomorphic curve equation at u (with fixed domain S) by $D_{S,u}$. It is a Cauchy-Riemann operator on the trivial Hermitian vector bundle u^*TM over S. We can glue the orientation operators $D_{p_1}, \ldots, D_{p_s}$ and $D_{p_0}^\vee$ to $D_{S,u}$ along the strip-like ends to obtain a new Cauchy-Riemann operator over the closed disk. We denote this operator by $\overline{D}$. The gluing formula then implies that

$$\begin{aligned} i(\overline{D}) &= i(D_{S,u}) + i(p_1) + \ldots + i(p_s) + (n - i(p_0)) \\ &= i(D_{S,u}) + n, \end{aligned}$$

and there is a canonical isomorphism

$$\det(\overline{D}) \cong \det(D_u) \otimes o_{p_1} \otimes \ldots \otimes o_{p_s} \otimes o_{p_0}^\vee.$$

Now the Cauchy-Riemann operator $\overline{D}$ is given by a bundle pair (D^2, E, F), which is equivalent to a bundle pair (D^2, u^*TM, F), where $u : D^2 \to M$ is obtained from the original disk u (which had strip-like ends converging to the generators p_j) by gluing the orientation operators onto the ends. The boundary conditions for $\overline{D}$ define a map $\rho : \partial D^2 \to \mathcal{G}M$ which lifts the boundary map ∂u.

If we think of ρ as a map

$$\rho : [0, 1] \to \mathcal{G}M$$

such that $\rho(0) = \rho(1)$ lies on L_0, then we have a lift

$$\begin{tikzcd} & \widetilde{\mathcal{G}}M \arrow[d] \\ {[0,1]} \arrow[ur, "\tilde{\rho}"] \arrow[r, "\rho"] & \mathcal{G}M. \end{tikzcd}$$

The boundary conditions L_j lift to $(y_1 + \ldots + y_j) \cdot L_j^{\#}$, and finally $\rho(1)$ lands on $(y_1 + \ldots + y_s - y_0) \cdot L_0^{\#}$. It follows that

$$[\rho] = -y_0 + \sum_{j=1}^{s} y_j.$$

Lemma 2.3.1.3 now implies that

$$\begin{aligned} f\left(i(\overline{D}) - n\chi\left(D^2\right)\right) &= -y_0 + \sum_{j=1}^{s} y_j \\ \Rightarrow f(i(D_{S,u})) &= -y_0 + \sum_{j=1}^{s} y_j \text{ in } Y(M). \end{aligned}$$

We now recall that, for the disk to be rigid, the **extended** linearized operator D_u (in which the modulus of the domain is allowed to vary, as well as the map) should have index zero. The dimension of the moduli space of disks with $s+1$ marked boundary points is $s-2$, so this means that

$$y_0 = f(2-s) + \sum_{j=1}^{s} y_j$$

in $Y(M)$. It follows that the affine Fukaya category is a $\boldsymbol{G}(M)$-graded A_∞ category (see Remark 2.2.3.12). We observe that the A_∞ associativity equations are satisfied, by the same argument as for $\mathcal{F}'(M)$ ([11, Proposition 12.3]).

We will now explain how $\mathcal{F}(M)$ is related to the $\mathbb{Z}$-graded exact Fukaya category $\mathcal{F}'(M)$, as defined in [11]. We recall that, to define $\mathbb{Z}$-gradings on $\mathcal{F}'(M)$, we require that $2c_1(M) = 0$ and equip η with a quadratic volume form η_M.

Recall from Definition 2.3.2.2 that, if we equip M with a quadratic volume form η_M, then we obtain a squared phase map

$$\alpha_M : \mathcal{G}M \to S^1,$$

and if we define a lift

$$\tilde{\alpha}_M : \widetilde{\mathcal{G}}M \to \mathbb{R},$$

then we obtain a forgetful map $\mathfrak{f}$ from anchored Lagrangian branes to (non-anchored) Lagrangian branes.

Now on the level of H_1, α_M induces a map

$$Y(M) \to \mathbb{Z}.$$

This defines a morphism of grading data, $\boldsymbol{p}^\eta : \boldsymbol{G}(M) \to \boldsymbol{G}_\mathbb{Z}$. It follows that $\boldsymbol{p}^\eta_* \mathcal{F}(M)$ is a $\mathbb{Z}$-graded A_∞ category. We have:

Lemma 2.3.3.1. *The forgetful map $\mathfrak{f}$ on objects extends to a fully faithful embedding of $\mathbb{Z}$-graded A_∞ categories,*

$$\mathfrak{f} : \boldsymbol{p}^\eta_* \mathcal{F}(M) \to \mathcal{F}'(M).$$

Remark 2.3.3.2. The image of this embedding consists of all (non-anchored) Lagrangian branes L such that the image of $H_1(L)$ in $H_1(M)$ is zero.

2.3.4 Covers

We explain how the affine Fukaya category behaves with respect to finite covers (essentially following [9, Section 8b]). Suppose that M, N are exact symplectic manifolds with convex boundary, with assumptions as in Section 2.3.3, and $\phi : N \to M$ is an exact symplectic covering (i.e., a covering such that the Liouville one-form on N is pulled back from that on M via ϕ). Then we have an induced covering

$$\begin{tikzcd} \mathcal{G}N \arrow[r] \arrow[d, "\phi_*"'] & N \arrow[d, "\phi"] \\ \mathcal{G}M \arrow[r] & M, \end{tikzcd}$$

and hence an injective morphism of grading data, $\boldsymbol{p} : \boldsymbol{G}(N) \to \boldsymbol{G}(M)$, given by

$$\begin{tikzcd} H_1(\mathcal{G}_pN) \arrow[r] \arrow[d, equal, "\phi_*"'] & H_1(\mathcal{G}N) \arrow[r] \arrow[d, "\phi_*"'] & H_1(N) \arrow[r] \arrow[d, "\phi_*"'] & 0 \\ H_1(\mathcal{G}_{\phi(p)}M) \arrow[r] & H_1(\mathcal{G}M) \arrow[r] & H_1(M) \arrow[r] & 0. \end{tikzcd}$$

Proposition 2.3.4.1. *There is a fully faithful embedding of $\boldsymbol{G}(N)$-graded categories,*

$$\boldsymbol{p}^*(\mathcal{F}(M)) \hookrightarrow \mathcal{F}(N).$$

Proof. We remark that the universal covers of $\mathcal{G}N$ and $\mathcal{G}M$ are isomorphic, so there is an obvious correspondence between anchored Lagrangian branes on N and on M. One can similarly set up a correspondence between morphism spaces and moduli spaces of pseudoholomorphic disks defining the A_∞ structure maps, and show that the gradings correspond, so the categories are strictly equivalent. □

2.3.5 The relative case

Now we specialize to a particular type of exact symplectic manifold with corners.

Definition 2.3.5.1. A **Kähler pair** $(M, \boldsymbol{D})$ consists of:

- A smooth complex projective variety M, equipped with a positive holomorphic line bundle $\mathcal{L}$ with a Hermitian metric, so that the curvature of $\mathcal{L}$ defines a Kähler form ω, with $[\omega] = c_1(\mathcal{L})$;

- A tuple $\boldsymbol{D} = (D_1, D_2, \ldots, D_k)$ of smooth irreducible divisors $D_j \subset M$ with normal crossings, each corresponding to some positive power d_j of the bundle $\mathcal{L}$.

That is, for each $j = 1, 2, \ldots, k$, we have a section

$$h_j \in \Gamma(M, \mathcal{L}^{\otimes d_j})$$

such that

$$D_j = \{h_j = 0\}.$$

We furthermore assume that

- $\pi_1(M) = 0$;
- $\pi_1(M \setminus \boldsymbol{D})$ is abelian;
- $k \geq n+1$, where k is the number of divisors and n is the complex dimension of M.

We define the **affine part**,

$$M \setminus \boldsymbol{D} := M \setminus \bigcup_j D_j.$$

We equip $M \setminus \boldsymbol{D}$ with a Kähler potential

$$h := \frac{\sum_{j=1}^k \log\left(\|h_j\|^2\right)}{\sum_{j=1}^k d_j},$$

so that $\alpha = -dh \circ J_0$ is a Liouville one-form (i.e., $\omega = d\alpha$), and α is convex at infinity (h is exhausting and bounded below on $M \setminus \boldsymbol{D}$).

Example 2.3.5.2. We consider the Fermat hypersurfaces,

$$M_a^n := \left\{ \sum_{j=1}^n z_j^a = 0 \right\} \subset \mathbb{CP}^{n-1},$$

with the ample divisors

$$D_j := \{z_j = 0\}$$

for $j = 1, \ldots, n$.

Now recall that there is a grading datum $\boldsymbol{G}(M \setminus \boldsymbol{D})$ associated to $M \setminus \boldsymbol{D}$, with exact sequence

$$\mathbb{Z} \to H_1(\mathcal{G}(M \setminus \boldsymbol{D})) \to H_1(M \setminus \boldsymbol{D}) \to 0.$$

We will denote

$$\boldsymbol{G}(M, \boldsymbol{D}) := \boldsymbol{G}(M \setminus \boldsymbol{D})$$

in the relative case, and write the exact sequence as

$$\mathbb{Z} \to \widetilde{Y}(M, \boldsymbol{D}) \to \widetilde{X}(M, \boldsymbol{D}) \to 0.$$

We now introduce a pseudo-grading datum $\boldsymbol{H}(M, \boldsymbol{D})$, whose associated grading datum is $\boldsymbol{G}(M, \boldsymbol{D})$. Compare [64, Section 5].

If we set U to be a neighbourhood of the union of the divisors $\boldsymbol{D}$ in M, and V to be $M \setminus \boldsymbol{D}$, then part of the Mayer-Vietoris long exact sequence for reduced homology gives

$$H_2(M) \to H_1(U \cap V) \to H_1(U) \oplus H_1(V) \to H_1(M) \to 0,$$

which becomes

$$H_2(M) \to \mathbb{Z}\langle y_1, \ldots, y_k \rangle \to H_1(M \setminus \boldsymbol{D}) \to 0,$$

since we are assuming $H_1(M) = 0$. Here $y_j \in H_1(U \cap V)$ is the class of a meridian loop around divisor D_j.

Definition 2.3.5.3. We define a pseudo-grading datum $\boldsymbol{H}(M, \boldsymbol{D})$, as follows: the exact sequence is

$$\begin{tikzcd}
Z(M,\boldsymbol{D}) \arrow[r, "f"] \arrow[d, equal] & Y(M,\boldsymbol{D}) \arrow[r, "g"] \arrow[d, equal] & X(M,\boldsymbol{D}) \arrow[r] \arrow[d, equal] & 0 \\
H_2(M;\mathbb{Z}) \arrow[r, "f"] & \mathbb{Z}\langle y_1,\ldots,y_k\rangle \arrow[r, "g"] & H_1(M\setminus\boldsymbol{D};\mathbb{Z}) \arrow[r] & 0,
\end{tikzcd}$$

where we define

$$\begin{aligned} f(u) &= \sum_{i=1}^{k}(u\cdot D_i)y_i, \text{ and} \\ g(y_i) &= \text{the class of a loop around } D_i. \end{aligned}$$

We define the element $c \in \mathrm{Hom}(Z,\mathbb{Z})$ to be given by $2c_1(TM) \in H^2(M)$.

We now define a morphism of pseudo-grading data, $\boldsymbol{p} : \boldsymbol{H}(M,\boldsymbol{D}) \to \boldsymbol{G}(M,\boldsymbol{D})$. We define $p_Z : H_2(M) \to \mathbb{Z}$ to be given by $2c_1(TM)$, and p_X to be the identity. We define the map d to be 0. To define the map

$$p_Y : \mathbb{Z}\langle y_1,\ldots,y_k\rangle \to H_1(\mathcal{G}(M\setminus\boldsymbol{D})),$$

it is sufficient to define the action on the generators y_i. We denote the image of y_i by $\bar{y}_i$. To define $\bar{y}_i$, we consider a disk

$$u_i : (D^2,\partial D^2) \to (M, M\setminus\boldsymbol{D})$$

such that $u_i \cdot D_j = \delta_{ij}$. We trivialize the symplectic vector bundle u_i^*TM, and choose a lift of ∂u_i:

$$\begin{tikzcd}
 & \mathcal{G}M \arrow[d] \\
S^1 \arrow[ur, "\bar{y}_i"] \arrow[r, "\partial u_i"'] & M,
\end{tikzcd}$$

so that the boundary Maslov index

$$\mu(D^2, u_i^*TM, \bar{y}_i) = 0.$$

This defines the element $p_Y(y_i) := \bar{y}_i \in H_1(\mathcal{G}(M \setminus \boldsymbol{D}))$.

Lemma 2.3.5.4. *The diagram*

$$\begin{tikzcd} H_2(M) \arrow[r] \arrow[d, "2c_1"'] & \mathbb{Z}\langle y_1, \ldots, y_k\rangle \arrow[r] \arrow[d, "p_Y"'] & H_1(M \setminus \boldsymbol{D}) \arrow[r] \arrow[d, equal] & 0 \\ \mathbb{Z} \arrow[r] & H_1(\mathcal{G}(M \setminus \boldsymbol{D})) \arrow[r] & H_1(M \setminus \boldsymbol{D}) \arrow[r] & 0 \end{tikzcd}$$

commutes.

Proof. The only non-trivial thing to check is the commutativity of the left square. Suppose that $u : \Sigma \to M$ is a surface representing a homology class in $H_2(M)$, and intersecting the divisors D_j transversely. One side of the square maps

$$u \mapsto \sum_{i=1}^{k} (u \cdot D_i)\bar{y}_i,$$

while the other maps

$$u \mapsto f(2c_1(u)).$$

We consider the bundle pair (Σ, E, ϕ) with empty boundary, simply given by the complex vector bundle $E := u^*(TM)$. Its Maslov index is $\mu(\Sigma, E, \phi) = 2c_1(u)$. We now define a decomposition of this bundle pair: $\Sigma = \Sigma_1 \cup \Sigma_2$, where Σ_1 is a union of small balls around each of the intersection points of u with divisors D_i, and Σ_2 is the rest of Σ. We define the Lagrangian boundary conditions along $\partial\Sigma_1$ by requiring that the bundle pair over each ball around a single intersection point has boundary Maslov index zero. Then the composition property for bundle pairs (see [40, Appendix C.3]) says that

$$2c_1(u) = \mu(\Sigma, E, \phi) = \mu(\Sigma_1, E, F) + \mu(\Sigma_2, E, F) = \mu(\Sigma_2, E, F).$$

We note that the boundary conditions we have associated to a small ball around an intersection point of u with the divisor D_i define a map

$$S^1 \to \mathcal{G}(M \setminus \boldsymbol{D})$$

representing the class $\bar{y}_i$, by definition. Because u maps Σ_2 into $M \setminus \boldsymbol{D}$, it follows from Lemma 2.3.1.3 and the previous argument that

$$f(2c_1(u)) = f(\mu(\Sigma_2, E, F)) = \sum_{i=1}^{k} (u \cdot D_i)\bar{y}_i,$$

which proves that the left square commutes. □

Corollary 2.3.5.5. *The grading datum corresponding to the pseudo-grading datum $\boldsymbol{H}(M, \boldsymbol{D})$ is*

$$\boldsymbol{G}(\boldsymbol{H}(M, \boldsymbol{D})) \cong \boldsymbol{G}(M, \boldsymbol{D}).$$

Now suppose that we equip M with a meromorphic n-form η (i.e., an $(n, 0)$-form), whose zeroes and poles lie along the divisors D_j. Then we obtain a quadratic complex volume form η^2 on $M \setminus \boldsymbol{D}$, and recall that this defines a morphism $\boldsymbol{p}^\eta : \boldsymbol{G}(M, \boldsymbol{D}) \to \boldsymbol{G}_{\mathbb{Z}}$, allowing us to equip our category with a $\mathbb{Z}$-grading.

Lemma 2.3.5.6. *The morphism of grading data $\boldsymbol{p}^\eta$ is induced by the morphism of pseudo-grading data defined by*

$$\begin{aligned} p_Y : \mathbb{Z}\langle y_1, \ldots, y_k \rangle &\to \mathbb{Z}, \\ p_Y(y_j) &= 2p_j, \end{aligned}$$

where p_j is the order of the pole of η along divisor D_j.

Proof. Follows essentially from the definition of the boundary Maslov index, see [40, Theorem C.3.5, 'Normalization' property]. □

Remark 2.3.5.7. If M is Calabi-Yau, then it admits a nowhere-vanishing holomorphic volume form η, so there is a canonical morphism of grading data

$$\boldsymbol{G}(M, \boldsymbol{D}) \to \boldsymbol{G}_{\mathbb{Z}},$$

which is induced by the zero morphism of pseudo-grading data, in accordance with Lemma 2.3.5.6.

Lemma 2.3.5.8. *The pseudo-grading datum associated to the Fermat hypersurfaces with coordinate divisors, $(M_a^n, \boldsymbol{D})$ (see Example 2.3.5.2) is*

$$\boldsymbol{H}(M_a^n, \boldsymbol{D}) \cong \boldsymbol{H}_a^n,$$

where $\boldsymbol{H}_a^n$ is the pseudo-grading datum of Example 2.2.1.13.

Proof. Follows from the fact that $H_2(M_a^n) \cong \mathbb{Z}$, generated by the class of a line $[P]$, that

$$[P] \cdot D_j = 1$$

for all j, and that

$$c_1([P]) = n - a.$$

□

Definition 2.3.5.9. Suppose that $(N, \boldsymbol{E})$ and $(M, \boldsymbol{D})$ are Kähler pairs (each with k divisors), and $\boldsymbol{a} = (a_1, \ldots, a_k)$ is a tuple of positive integers. An ***a*-branched cover** of Kähler pairs,

$$\phi : (N, \boldsymbol{E}) \to (M, \boldsymbol{D}),$$

is a branched cover $\phi : N \to M$ which maps divisor E_j to D_j, and has branching of order a_j along divisor E_j (and no branching anywhere else).

Example 2.3.5.10. There is an $(a, \ldots, a)$-branched cover of Fermat hypersurfaces

$$\begin{aligned} \phi_a : (M_a^n, \boldsymbol{D}) &\to (M_1^n, \boldsymbol{D}), \\ \phi_a([z_1 : \ldots : z_n]) &= [z_1^a : \ldots : z_n^a]. \end{aligned}$$

Lemma 2.3.5.11. *Let $\phi : (N, \boldsymbol{E}) \to (M, \boldsymbol{D})$ be an $\boldsymbol{a}$-branched cover. The unbranched cover*

$$\phi : N \setminus \boldsymbol{E} \to M \setminus \boldsymbol{D}$$

induces an injective morphism of grading data

$$\boldsymbol{p} : \boldsymbol{G}(N, \boldsymbol{E}) \to \boldsymbol{G}(M, \boldsymbol{D}),$$

as in Section 2.3.4. This morphism of grading data is induced by a morphism of pseudo-grading data

$$\tilde{\boldsymbol{p}} : \boldsymbol{H}(N, \boldsymbol{E}) \to \boldsymbol{H}(M, \boldsymbol{D}),$$

where

$$\begin{aligned} \tilde{p}_Y(y_j) &= a_j y_j, \text{ and} \\ d(y_j) &= 2(1 - a_j) \end{aligned}$$

Proof. It suffices to prove that

$$p_Y(0 \oplus y_j) = (2(1 - a_j) \oplus a_j y_j).$$

This follows easily from the definition, and the local form

$$(z_1, z_2, \ldots, z_n) \mapsto (z_1^{a_j}, z_2, \ldots, z_n)$$

of ϕ near divisor E_j. $\square$

Corollary 2.3.5.12. *If $\phi_n : (M_n^n, \boldsymbol{D}) \to (M_1^n, \boldsymbol{D})$ is the $(n, \ldots, n)$-branched cover of Fermat hypersurfaces introduced in Example 2.3.5.10, then the induced morphism of grading data*

$$\boldsymbol{p} : \boldsymbol{G}(M_n^n, \boldsymbol{D}) \to \boldsymbol{G}(M_1^n, \boldsymbol{D})$$

Coincides with the morphism

$$\boldsymbol{p}_1 : \boldsymbol{G}_n^n \to \boldsymbol{G}_1^n$$

of Lemma 2.2.1.14 (recall also Lemma 2.3.5.8).

2.4 Moduli Spaces of Disks

In this section we introduce the various moduli spaces of pseudo-holomorphic disks that we will need to define the versions of the Fukaya category that we will consider.

2.4.1 Moduli spaces of holomorphic spheres and disks

Definition 2.4.1.1. Given an unordered set $\boldsymbol{E}$, with $|\boldsymbol{E}| \geq 3$, we define $\mathcal{R}_0(\boldsymbol{E})$, the moduli space of holomorphic spheres with distinct marked points q_e indexed by $e \in \boldsymbol{E}$, up to biholomorphism preserving marked points.

Definition 2.4.1.2. Given an ordered tuple $\boldsymbol{L} = (L_0, \ldots, L_d)$, a **disk with boundary labels $\boldsymbol{L}$** is a disk with $d+1$ distinct boundary marked points, $\zeta_0, \zeta_1, \ldots, \zeta_d$, with the boundary component between ζ_i and ζ_{i+1} labelled L_i (understood modulo $d+1$).

We define three types of moduli spaces of disks:

Definition 2.4.1.3. Given a tuple $\boldsymbol{L}$, and a set $\boldsymbol{E}$, with $|\boldsymbol{L}| + 2|\boldsymbol{E}| \geq 3$, we define $\mathcal{R}(\boldsymbol{L}, \boldsymbol{E})$ to be the moduli space of holomorphic disks S with boundary labels $\boldsymbol{L}$, together with distinct internal marked points q_e indexed by $e \in \boldsymbol{E}$. We consider these objects up to biholomorphism preserving all marked points.

Definition 2.4.1.4. Given a tuple $\boldsymbol{L}$, we define $\mathcal{R}_1(\boldsymbol{L}) := \mathcal{R}(\boldsymbol{L}, \{1\})$, the moduli space of holomorphic disks S with boundary labels $\boldsymbol{L}$ and a single interior marked point q.

Definition 2.4.1.5. Given a tuple $\boldsymbol{L}$, we define $\mathcal{R}_2(\boldsymbol{L}) \subset \mathcal{R}(\boldsymbol{L}, \{1, 2\})$ to be the moduli space of holomorphic disks S with Lagrangian labels $\boldsymbol{L}$, together with interior marked points q_1, q_2, such that there is a biholomorphism of S with the unit disk $\{|w| \leq 1\} \subset \mathbb{C}$ sending

$$\begin{array}{rcl} \zeta_0 & \mapsto & -i \\ q_1 & \mapsto & -t \\ q_2 & \mapsto & t \end{array}$$

for some $t \in (0, 1) \subset \mathbb{R}$ (see Figure 2.4.1.1).

Given a point r in one of these moduli spaces, we denote by S_r the corresponding (marked) disk, with all boundary marked points removed.

2.4.2 Deligne-Mumford compactifications

We make a universal choice of strip-like and cylindrical ends for each of these moduli spaces. We denote by $\overline{\mathcal{R}}_0(\boldsymbol{E})$, $\overline{\mathcal{R}}(\boldsymbol{L}, \boldsymbol{E})$, $\overline{\mathcal{R}}_1(\boldsymbol{L})$, $\overline{\mathcal{R}}_2(\boldsymbol{L})$ the Deligne-Mumford compactifications of these moduli spaces by stable spheres and disks. We now describe these compactifications.

The Deligne-Mumford compactification of $\mathcal{R}_0(\boldsymbol{E})$ consists of stable trees of spheres. Boundary strata are indexed by stable trees T, with semi-infinite edges indexed by $\boldsymbol{E}$. We denote by $V(T)$ the set of vertices of T, and $E(T)$ the set of edges of T. A tree is

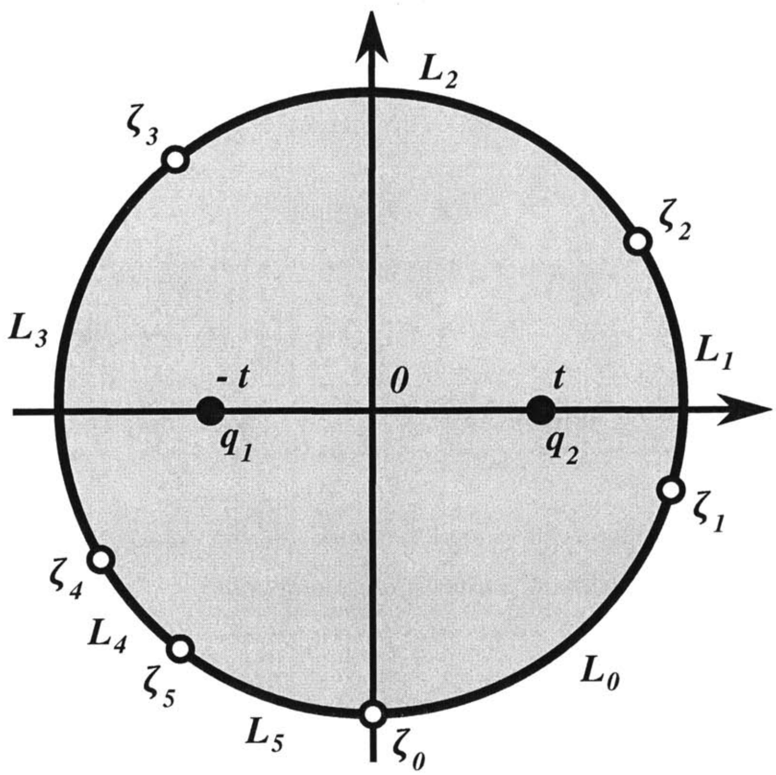


Figure 2.4.1.1: The moduli space $\mathcal{R}_2(\boldsymbol{L})$, where $|\boldsymbol{L}| = 6$.

called **stable** if each vertex has valence ≥ 3. For each vertex v of T, we denote by $\boldsymbol{E}_v$ the set of edges of T incident to v. The boundary stratum indexed by T is

$$\mathcal{R}_0^T(\boldsymbol{E}) := \prod_v \mathcal{R}_0(\boldsymbol{E}_v).$$

Points in this stratum correspond to trees of sphere bubbles, with semi-infinite edges corresponding to marked points, and finite edges corresponding to nodes (see [40, Section D.3, Figure 4]). The Deligne-Mumford (or Grothendieck-Knudsen) compactification, as a set, is the union of all such strata. It is a smooth manifold (see [40, Section D.5]). The codimension of the stratum indexed by T is $2(|V(T)|-1)$.

Definition 2.4.2.1. A **directed d-leafed planar tree** is a directed d-leafed tree T embedded in $\mathbb{R}^2$. It consists of the following data:

- a finite set of vertices $V(T)$;
- a set of d semi-infinite **outgoing edges**;
- a single semi-infinite **incoming edge**, connected to a vertex $v \in V(T)$ called the **root** of T;
- a set $E(T)$ of internal edges.

A vertex is allowed to have zero outgoing edges, but must always have exactly one incoming edge. We say that a vertex $v \in V(T)$ is **stable** if it has ≥ 2 outgoing edges, and **semi-stable** if it has ≥ 1 outgoing edges. Given a tuple $\boldsymbol{L}$, we say that T **has labels $\boldsymbol{L}$** if the connected components of $\mathbb{R}^2 \setminus T$ are labeled by the elements of $\boldsymbol{L}$, in order. A labeling of T induces a labeling $\boldsymbol{L}_v$ of the regions surrounding each vertex $v \in V(T)$ (see Figure 2.4.2.1).

The Deligne-Mumford compactification of $\overline{\mathcal{R}}(\boldsymbol{L}, \boldsymbol{E})$ consists of stable trees of disk and sphere bubbles with appropriate markings (see [39, Section 2.3]). It is a smooth

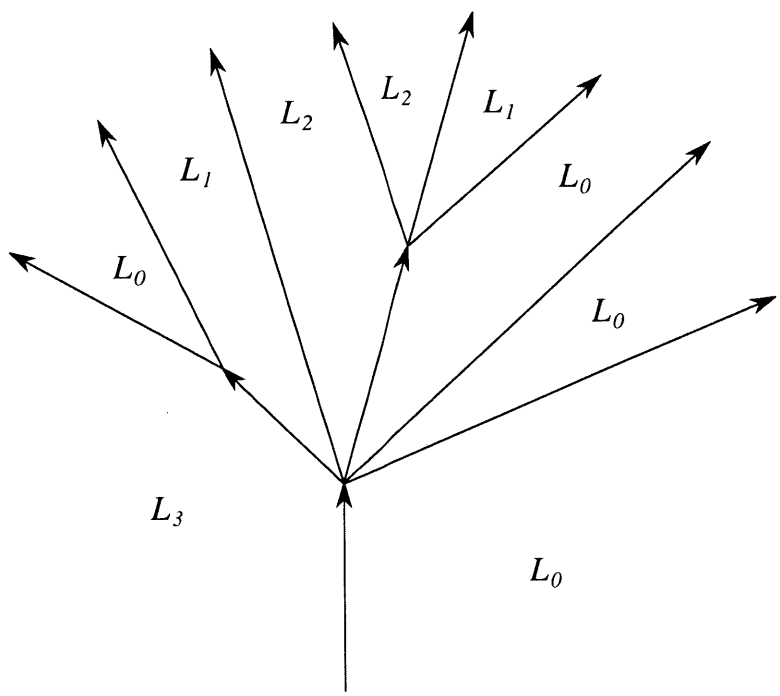


Figure 2.4.2.1: If $\boldsymbol{L}$ is some tuple, then a k-leafed stable tree T is said to have labels $\boldsymbol{L}$ if the connected components of $\mathbb{R}^2 \setminus T$ are labeled by the elements of $\boldsymbol{L}$, in order. In this figure, $\boldsymbol{L} = (L_0, L_0, L_0, L_1, L_2, L_2, L_1, L_0, L_3)$. A labeling $\boldsymbol{L}$ of T induces a labeling $\boldsymbol{L}_v$ of the regions surrounding each vertex v. In this figure, the induced labeling of the regions surrounding the uppermost vertex v is $\boldsymbol{L}_v = (L_0, L_1, L_2, L_2)$.

manifold with corners. Each boundary stratum is indexed by a directed tree T, together with a directed planar subtree T_L with labels $\boldsymbol{L}$. We denote $T_E := T \setminus T_L$. We require that the semi-infinite edges of T_E are indexed by $\boldsymbol{E}$. For each vertex $v \in T_L$, we have a labeling $\boldsymbol{L}_v$ as above, and denote by $\boldsymbol{E}_v$ the set of edges incident to v in T_E. For each vertex $v \in T_E$, we denote by $\boldsymbol{E}_v$ the set of edges incident to v in T_E. We require that the tree is stable, in the sense that for each vertex $v \in T_L$,

$$|\boldsymbol{L}_v| + 2|\boldsymbol{E}_v| \geq 3,$$

while for each vertex $v \in T_E$,

$$|\boldsymbol{E}_v| \geq 3.$$

The tree T corresponds to the stratum

$$\mathcal{R}^T(\boldsymbol{L}, \boldsymbol{E}) \cong \prod_{v \in V(T_L)} \mathcal{R}(\boldsymbol{L}_v, \boldsymbol{E}_v) \times \prod_{v \in V(T_E)} \mathcal{R}_0(\boldsymbol{E}_v).$$

Points in this stratum correspond to nodal disks, with semi-infinite edges of T_L corresponding to boundary marked points, finite edges of T_L corresponding to boundary nodes, semi-infinite edges of T_E corresponding to internal marked points, and finite edges of T_E corresponding to internal nodes. The codimension of this stratum is

$$|V(T_L)| + 2|V(T_E)| - 1.$$

The boundary strata of $\overline{\mathcal{R}}_2(\boldsymbol{L})$ fall into three types (we have illustrated the codimension-1 part of each stratum in Figure 2.4.2.2):

- strata indexed by directed planar trees T with boundary labels $\boldsymbol{L}$, together with a distinguished vertex v_1, so that all vertices other than possibly v_1 have valence

≥ 3; these correspond to codimension-$(|V(T)|-1)$ strata

$$\mathcal{R}_2^{1,T}(\boldsymbol{L}) \cong \mathcal{R}_2(\boldsymbol{L}_{v_1}) \times \prod_{v \in V(T) \setminus \{v_1\}} \mathcal{R}(\boldsymbol{L}_v)$$

which consist of nodal disks glued together in the obvious way (see Figure 2-2(a));

- another set of strata indexed by directed planar trees T with boundary labels $\boldsymbol{L}$, together with a distinguished vertex v_1, so that all vertices other than possibly v_1 have valence ≥ 3; these correspond to codimension-$|V(T)|$ strata

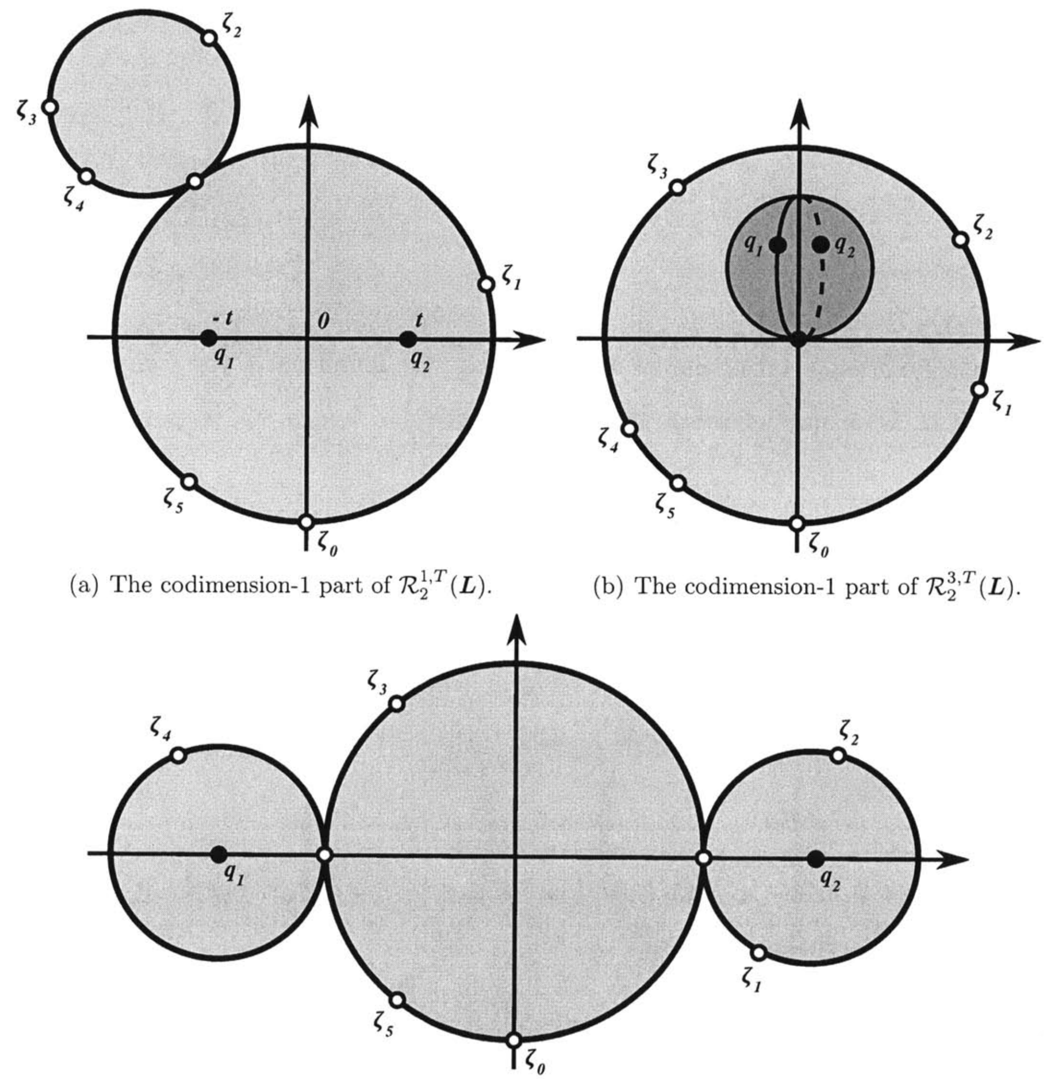
$$\mathcal{R}_2^{2,T}(\boldsymbol{L}) \cong \mathcal{R}_1(\boldsymbol{L}_{v_1}) \times \prod_{v \in V(T) \setminus \{v_1\}} \mathcal{R}(\boldsymbol{L}_v)$$

  which consist of nodal disks glued in the obvious way, together with a sphere with three marked points, two of which are the marked points q_1, q_2 and one of which is a node, identified to the internal marked point q in the disk coming from the factor $\mathcal{R}_1(\boldsymbol{L}_{v_1})$ (see Figure 2-2(b));

- strata indexed by directed planar trees T with boundary labels $\boldsymbol{L}$ and two (different) distinguished vertices v_1, v_2, so that all vertices other than possibly v_1 and v_2 have valence ≥ 3, and the branch of T containing v_1 lies strictly to the left of the branch containing v_2; these correspond to codimension-$(|V(T)|-2)$ strata

$$\mathcal{R}_2^{3,T}(\boldsymbol{L}) \cong \mathcal{R}_1(\boldsymbol{L}_{v_1}) \times \mathcal{R}_1(\boldsymbol{L}_{v_2}) \times \prod_{v \in V(T) \setminus \{v_1, v_2\}} \mathcal{R}(\boldsymbol{L}_v)$$

  consisting of nodal disks glued together in the obvious way, where the internal marked point in the disk coming from the factor $\mathcal{R}_1(\boldsymbol{L}_{v_j})$ corresponds to the marked point q_j, for $j = 1, 2$ (see Figure 2-2(c)).



(a) The codimension-1 part of $\mathcal{R}_2^{1,T}(\boldsymbol{L})$.

(b) The codimension-1 part of $\mathcal{R}_2^{3,T}(\boldsymbol{L})$.

(c) The codimension-1 part of $\mathcal{R}_2^{2,T}(\boldsymbol{L})$.

Figure 2.4.2.2: The codimension-1 boundary components of $\mathcal{R}_2(\boldsymbol{L})$, where $|\boldsymbol{L}| = 6$.

2.4.3 Moduli spaces of pseudoholomorphic disks

Let $(M, \boldsymbol{D})$ be a Kähler pair (see Definition 2.3.5.1). Thus, M is a Kähler manifold, and $\boldsymbol{D} = D_1 \cup \ldots \cup D_k$ is a union of smooth ample divisors with normal crossings. In this section we will define moduli spaces of pseudoholomorphic disks mapping into M.

Definition 2.4.3.1. Let $\boldsymbol{F}$ be a finite set, and let

$$\ell : \boldsymbol{F} \to [k]$$

be a function from $\boldsymbol{F}$ to the set $[k] := \{1, \ldots, k\}$ indexing the divisors $D_1, \ldots, D_k$. We call such a function a **labelling** of $\boldsymbol{F}$. Recalling the definition of the pseudo-grading datum $\boldsymbol{H}(M, \boldsymbol{D})$ from Definition 2.3.5.3, we denote

$$\boldsymbol{d}(\ell) := \sum_{j=1}^{k} |\ell^{-1}(j)| y_j \in Y_{\geq 0}$$

(where $Y_{\geq 0} := \mathbb{Z}_{\geq 0}\langle y_1, \ldots, y_k \rangle$). If $\boldsymbol{a} = (a_1, \ldots, a_k)$ is a tuple of positive integers, then we define

$$\boldsymbol{d}_{\boldsymbol{a}}(\ell) := \sum_{j=1}^{k} a_j |\ell^{-1}(j)| y_j \in Y_{\geq 0}.$$

We denote $\mathbf{1} = (1, \ldots, 1)$ (k copies), so that $\boldsymbol{d} = \boldsymbol{d}_{\mathbf{1}}$.

Definition 2.4.3.2. Let $\boldsymbol{E}$ and $\boldsymbol{F}$ be finite sets, $|\boldsymbol{E}| + |\boldsymbol{F}| \geq 3$. Let $\ell : \boldsymbol{F} \to [k]$ be a labelling of $\boldsymbol{F}$. We define

$$\mathcal{R}_0(\boldsymbol{E}, \ell) := \mathcal{R}_0(\boldsymbol{E} \sqcup \boldsymbol{F}).$$

Definition 2.4.3.3. Given a tuple of objects $\boldsymbol{L}$, finite sets $\boldsymbol{E}, \boldsymbol{F}$ such that $|\boldsymbol{L}| + 2|\boldsymbol{E}| + 2|\boldsymbol{F}| \geq 3$, and a labelling $\ell : \boldsymbol{F} \to [k]$, we define the moduli space

$$\mathcal{R}(\boldsymbol{L}, \boldsymbol{E}, \ell) := \mathcal{R}(\boldsymbol{L}, \boldsymbol{E} \sqcup \boldsymbol{F}).$$

For each pair of objects in the affine Fukaya category $\mathcal{F}(M \setminus \boldsymbol{D})$, we choose a Floer datum (see [11, Section 8e]). We make a universal choice of perturbation data (in the sense of [11, Section 9h]) on each of the moduli spaces $\mathcal{R}_0(\boldsymbol{E}, \ell)$, $\mathcal{R}(\boldsymbol{L}, \boldsymbol{E}, \ell)$, $\mathcal{R}_1(\boldsymbol{L})$ and $\mathcal{R}_2(\boldsymbol{L})$. Note that the choice of perturbation data on the moduli spaces $\mathcal{R}_0$, $\mathcal{R}$ may be different for different labellings ℓ, even though they have the same number of boundary components and internal marked points. Also note that we are choosing Floer and perturbation data which are defined on all of M, not just on $M \setminus \boldsymbol{D}$.

We require that

- the Hamiltonian part of each perturbation datum is 0 on the moduli spaces $\mathcal{R}_0(\boldsymbol{E})$;
- the Hamiltonian part of each perturbation datum vanishes, with its first derivatives, along each divisor D_j;
- the almost-complex structure part of each perturbation datum makes each divisor D_j an almost-complex manifold;
- on the strip-like ends, the perturbation datum agrees with the associated Floer datum;
- the choices of perturbation data are consistent with respect to the Deligne-Mumford compactifications outlined in Section 2.4.1, in the sense of [11, Section 9i];
- the choices of perturbation data are invariant under shifts of the anchored Lagrangian branes by the covering group action

  $$\pi_1(\mathcal{G}(M \setminus \boldsymbol{D})) \cong \widetilde{Y}(M, \boldsymbol{D}),$$

  as was the case for the affine Fukaya category (see Section 2.3.3).

We will use the shorthand

$$u \cdot \boldsymbol{D} := \sum_{j=1}^{k} (u \cdot D_j) y_j \in Y(M, \boldsymbol{D})$$

(see Definition 2.3.5.3 for the definition of $Y(M, \boldsymbol{D})$), where $u \cdot D_j$ denotes the topological intersection number, for any class $u \in H_2(M)$ or $H_2(M, M \setminus \boldsymbol{D})$. Note that with our choice of perturbation data, any pseudoholomorphic curve u intersects the divisors D_j positively, so

$$u \cdot \boldsymbol{D} \in Y(M, \boldsymbol{D})_{\geq 0}$$

if u is a pseudo-holomorphic disk or sphere that is not contained inside any of the divisors D_j.

Definition 2.4.3.4. We define an element of the moduli space $\mathcal{M}_0(\boldsymbol{E}, \ell)$ to be a pair (r, u), where r is an element of $\mathcal{R}_0(\boldsymbol{E}, \ell)$ and $u : S_r \to M$ is a smooth map, such that:

- u satisfies the perturbed holomorphic curve equation;
- $u(q_f) \in D_{\ell(f)}$ for each $f \in \boldsymbol{F}$;
- $u \cdot \boldsymbol{D} = \boldsymbol{d}(\ell)$

(see [11, Equations (8.9) and (9.17)] for the perturbed holomorphic curve equation). There is an evaluation map

$$\boldsymbol{ev} : \mathcal{M}_0(\boldsymbol{E}, \ell) \to M^{\boldsymbol{E}}.$$

If $\boldsymbol{E}$ is empty we may omit it from the notation, and we will also write $\mathcal{M}_0(\boldsymbol{E}, \boldsymbol{d})$ for $\mathcal{M}_0(\boldsymbol{E}, \ell)$ where $\boldsymbol{d} = \boldsymbol{d}(\ell)$.

Remark 2.4.3.5. Suppose that $u \in \mathcal{C}_0(\boldsymbol{E}, \ell)$ is not contained in divisor D_j. Then our assumptions on the perturbation data ensure that u intersects D_j in isolated points with positive multiplicity. Since each marked point q_f with $\ell(f) = j$ contributes at least 1 to $u \cdot D_j$, our requirement that $u \cdot \boldsymbol{D} = \boldsymbol{d}(\ell)$ ensures that u intersects D_j only at the marked points q_f with $\ell(f)$, and each intersection has multiplicity 1.

Definition 2.4.3.6. Given a tuple $\boldsymbol{L} = (L_0^{\#}, \ldots, L_s^{\#})$ of anchored Lagrangian branes, an **associated set of generators** is a tuple $\boldsymbol{p} = (p_0, \ldots, p_s)$ where p_j is a generator of $CF^*(L_j^{\#}, L_{j-1}^{\#})$ for $j \geq 1$, and p_0 is a generator of $CF^*(L_0^{\#}, L_s^{\#})$.

Definition 2.4.3.7. Given a tuple of objects $\boldsymbol{L}$ with associated generators $\boldsymbol{p}$, finite sets $\boldsymbol{E}, \boldsymbol{F}$ such that $|\boldsymbol{L}|+2|\boldsymbol{E}|+2|\boldsymbol{F}| \geq 3$, and a labelling $\ell : \boldsymbol{F} \to [k]$ as in Definition 2.4.3.1, we define an element of the moduli space $\mathcal{M}(\boldsymbol{p}, \boldsymbol{E}, \ell)$ to be a pair (r, u), where r is an element of $\mathcal{R}(\boldsymbol{L}, \boldsymbol{E}, \ell)$ and $u : S_r \to M$ is a smooth map, such that:

- u satisfies the perturbed holomorphic curve equation, with Lagrangian boundary conditions given by the labels $\boldsymbol{L}$;
- u is asymptotic to the generators $\boldsymbol{p}$ along the corresponding strip-like ends;
- $u(q_f) \in D_{\ell(f)}$ for each $f \in \boldsymbol{F}$;
- $u \cdot \boldsymbol{D} = \boldsymbol{d}(\ell)$.

(see [11, Equation (8.10)] for the definition of 'asymptotic to the generators $\boldsymbol{p}$'). There is an evaluation map

$$\boldsymbol{ev} : \mathcal{M}(\boldsymbol{p}, \boldsymbol{E}, \ell) \to M^{\boldsymbol{E}}.$$

If $\boldsymbol{E}$ is empty, we may omit it from the notation, and we will also write $\mathcal{M}(\boldsymbol{p}, \boldsymbol{E}, \boldsymbol{d})$ instead of $\mathcal{M}(\boldsymbol{p}, \boldsymbol{E}, \ell)$, where $\boldsymbol{d} = \boldsymbol{d}(\ell)$.

By the same reasoning as in Remark 2.4.3.5, every intersection point of an element $u \in \mathcal{M}(\boldsymbol{p}, \boldsymbol{E}, \ell)$ with a divisor D_j is a marked point (and the intersection has multiplicity 1).

Definition 2.4.3.8. For a tuple of 2 Lagrangian labels $\boldsymbol{L}$ with associated generators $\boldsymbol{p}$, we define $\mathcal{M}(\boldsymbol{p}, 0)$, the set of holomorphic strips with boundary conditions on $\boldsymbol{L}$, intersection number 0 with the divisors $\boldsymbol{D}$, translation-invariant perturbation coming from the corresponding Floer datum, asymptotic to the generators $\boldsymbol{p}$, modulo translation by $\mathbb{R}$ (see [11, Equation (8.8)]).

Given a holomorphic curve with an internal marked point, we define the notion of 'tangency to a divisor to order k' at the marked point, in accordance with [65]:

Definition 2.4.3.9. Suppose we are given:

- a Riemann surface S with an internal marked point $q \in S$;
- a perturbed holomorphic curve $u : S \to M$;
- a choice of divisor $D_j \subset M$;
- an integer $k \geq 1$.

We say that u is **tangent to D_j at q to order k** if

- $u(q) \in D_j$;
- all partial derivatives of u at q of order $\leq k$ lie inside the tangent space TD_j.

We remark that this does not depend on the choice of coordinates.

When $k = 0$, this is the same thing as a point constraint $u(q) \in D_j$. For $k \geq 1$, one should think of the curve u having 'ramification of order $k+1$' about the divisor D_j. We can find local holomorphic coordinates $(z_1, \ldots, z_n)$ for M near $u(q)$, such that $u(q)$ corresponds to the origin, and D_j corresponds to $\{z_1 = 0\}$, and a local holomorphic coordinate z for S near q such that q corresponds to the origin. If we assume that the almost-complex structure part of the perturbation datum is equal to the standard complex structure along the cylindrical end associated to the marked point q, and the Hamiltonian part of the perturbation datum vanishes, then u takes the form $(u_1(z), \ldots, u_n(z))$ in these coordinates, where $u_j(z)$ are holomorphic functions, and u_1 has a zero of order $\geq k+1$ at the origin. It follows that the point q contributes at least $k+1$ to the intersection number of u with D_j.

Definition 2.4.3.10. Suppose that $\boldsymbol{a} = (a_1, \ldots, a_k)$ is a tuple of k positive integers. We define the moduli spaces $\mathcal{M}_0(\boldsymbol{E}, \ell, \boldsymbol{a})$ and $\mathcal{M}(\boldsymbol{p}, \boldsymbol{E}, \ell, \boldsymbol{a})$ in exactly the same way as we did $\mathcal{M}_0(\boldsymbol{E}, \ell)$ and $\mathcal{M}(\boldsymbol{p}, \boldsymbol{E}, \ell)$, with the following exceptions:

- For each $f \in \boldsymbol{F}$, we require u to be tangent to $D_{\ell(f)}$ at q_f to order $a_{\ell(f)} - 1$;

- We require that $u \cdot \boldsymbol{D} = d_{\boldsymbol{a}}(\ell)$.

In particular, we have isomorphisms

$$\mathcal{M}_0(\boldsymbol{E}, \ell) \cong \mathcal{M}_0(\boldsymbol{E}, \ell, \mathbf{1})$$

and

$$\mathcal{M}(\boldsymbol{p}, \boldsymbol{E}, \ell) \cong \mathcal{M}(\boldsymbol{p}, \boldsymbol{E}, \ell, \mathbf{1}).$$

Note that, because each marked point q_f contributes $\geq a_{\ell(f)}$ to $u \cdot D_{\ell(f)}$, elements $u \in \mathcal{M}_0(\boldsymbol{E}, \ell, \boldsymbol{a})$ do not intersect any of the divisors D_j anywhere other than at marked points q_f, where they intersect with multiplicity $a_{\ell(f)}$, and similarly for $\mathcal{M}(\boldsymbol{p}, \boldsymbol{E}, \ell, \boldsymbol{a})$.

Definition 2.4.3.11. Let $\boldsymbol{L}$ be a tuple of Lagrangians with associated generators $\boldsymbol{p}$, a a positive integer, and $j \in [k]$. We define $\boldsymbol{F}$ to be a set with a single element, and ℓ a labelling which assigns j to this element. We let $\boldsymbol{a}$ be any tuple such that $a_j = a$. Then we define

$$\mathcal{M}_1(\boldsymbol{p}, j, a) := \mathcal{M}(\boldsymbol{p}, \ell, \boldsymbol{a}),$$

the moduli space of pseudoholomorphic disks with a single internal marked point, which is tangent to divisor D_j to order $a - 1$. Note that we're not defining anything new; this is just convenient notation for us to have.

Definition 2.4.3.12. Given a tuple of Lagrangians $\boldsymbol{L}$ with associated generators $\boldsymbol{p}$, a positive integer a, together with a choice of divisor D_j, we define an element of the moduli space $\mathcal{M}_2(\boldsymbol{p}, j, a)$ to consist of the following data:

- a point $r \in \mathcal{R}_2(\boldsymbol{L})$;

- a smooth map $u : S_r \to M$,

such that:

- u satisfies the perturbed holomorphic curve equation;
- u is asymptotic to the generators $\boldsymbol{p}$ along the strip-like ends;
- $u \cdot \boldsymbol{D} = (a+1)e_j$;
- u is tangent to D_j at q_1 to order $a-1$, and to D_j at q_2 to order 0.

Note that, as before, elements $u \in \mathcal{M}_2(\boldsymbol{p}, j, a)$ do not intersect the divisors D_i for $i \neq j$, and intersect D_j only at q_1 (with multiplicity a) and q_2 (with multiplicity 1).

2.4.4 Indices and orientations

Each of the moduli spaces of pseudoholomorphic curves we have defined can be defined as the set of zeroes of a smooth section of a Banach bundle over a Banach manifold of maps (more precisely, the moduli space can be covered by such). We follow [11, Chapters 8 and 9] in defining the functional analytic framework, with modifications following [65, Section 6] to take into account the 'orders of tangency' restrictions. This just means we have to use $W^{k+1,p}$ maps rather than $W^{1,p}$, so that we can make sense of 'derivatives of order $\leq k$'.

The linearization of this smooth section defines a Fredholm operator. The moduli spaces are said to be regular when the linearization is surjective everywhere. When they are regular, the moduli spaces are smooth manifolds, with dimension given by the index of the Fredholm operator. In this section, we will outline the calculation of this dimension.

Lemma 2.4.4.1. *Let $\boldsymbol{L}$ be a tuple of anchored Lagrangian branes with associated generators $\boldsymbol{p}$, and u an element of $\mathcal{M}(\boldsymbol{p}, \boldsymbol{E}, \ell)$. Let $\tilde{y}_j \in \widetilde{Y}(M, \boldsymbol{D})$ be the degree of p_j. Then* *the index of the extended linearized operator D_u at u satisfies*

$$f(i(D_u) - 2|\boldsymbol{E}| + 2 - s) = -\tilde{y}_0 + \sum_{j=1}^{s} \tilde{y}_j + p_Y(\boldsymbol{d}(\ell)),$$

and there is a canonical identification of orientation lines

$$o_{p_0} \cong \det(D_u) \otimes o_{p_1} \otimes \ldots \otimes o_{p_s}.$$

Proof. Each path p_j lifts to a path from $L^{\#}_{j-1}$ to $\tilde{y}_j \cdot L^{\#}_j$ in $\widetilde{\mathcal{G}}(M \setminus \boldsymbol{D})$, which we use to define orientation operator D_{p_j} whose index is 0.

We denote the linearized operator of the perturbed holomorphic curve equation at u (with fixed domain S) by $D_{S,u}$. We glue the orientation operators D_{p_j} to $D_{S,u}$ along the strip-like ends to obtain a bundle pair over the closed disk, and hence a Cauchy-Riemann operator $\overline{D}$. The gluing formula implies that

$$\begin{aligned} i(\overline{D}) &= i(D_{S,u}) + i(D_{p_1}) + \ldots + i(D_{p_s}) + (n - i(D_{p_0})) \\ &= i(D_{S,u}) + n, \end{aligned}$$

and there is a canonical isomorphism

$$\det(\overline{D}) \cong \det(D_{S,u}) \otimes o_{p_1} \otimes \ldots \otimes o_{p_s} \otimes o_{p_0}^{\vee}.$$

As in Section 2.3.3, this bundle pair defining $\overline{D}$ is equivalent to a bundle pair (D^2, u^*TM, ρ), where $\rho : \partial D^2 \to \mathcal{G}(M \setminus \boldsymbol{D})$ which lifts the boundary map ∂u, and hence has index

$$i(\overline{D}) = n + \mu(D^2, u^*TM, \rho).$$

As in Section 2.3.3, we can compute the homology class

$$[\rho] = -\tilde{y}_0 + \sum_{j=1}^{s} \tilde{y}_j.$$

As in the proof of Lemma 2.3.5.4, we define a decomposition of the bundle pair (D^2, u^*TM, ρ) into two bundle pairs: (Σ_1, u^*TM, ρ'), consisting of a union of small balls surrounding each of the points q_f for $f \in \boldsymbol{F}$, with boundary conditions given by $\bar{y}_{\ell(f)}$, and $(\Sigma_2, u^*TM, \rho \cup \rho')$ the complement of Σ_1. Then the decomposition property of the boundary Maslov index, together with Lemma 2.3.1.3, say that

$$\begin{aligned} \mu(D^2, u^*TM, \rho) &= \mu(\Sigma_1, u^*TM, \rho) + \mu(\Sigma_2, u^*TM, \rho \cup \rho') \\ &= \mu(\Sigma_2, u^*TM, \rho \cup \rho') \\ \Rightarrow f(\mu(D^2, u^*TM, \rho)) &= [\rho] + [\rho'] \\ &= -\tilde{y}_0 + \sum_{j=1}^{s} \tilde{y}_j + p_Y(\boldsymbol{d}(\ell)). \end{aligned}$$

The result now follows, as

$$\begin{aligned} i(D_u) &= i(D_{S,u}) + \dim(\mathcal{R}(\boldsymbol{L}, \boldsymbol{E})) \\ &= i(\overline{D}) - n + s - 2 + 2|\boldsymbol{E}| \\ &= \mu(D^2, u^*TM, \rho) + s - 2 + 2|\boldsymbol{E}|. \end{aligned}$$

The isomorphism of orientation lines follows after we fix an orientation for $\mathcal{R}(\boldsymbol{L}, \boldsymbol{E})$. □

Lemma 2.4.4.2. *Suppose we have objects* $\boldsymbol{L}$ *with associated generators* $\mathbf{p}$*, a* k*-tuple* $\boldsymbol{a} = (a_1, \ldots, a_k)$ *of positive integers, and an element* $u \in \mathcal{M}(\mathbf{p}, \boldsymbol{E}, \ell, \boldsymbol{a})$*. Let* $\tilde{y}_j \in \widetilde{Y}(M, \boldsymbol{D})$ *be the degree of* p_j*. Then the index of the extended linearized operator* D_u *at* u *satisfies*

$$f\left(i(D_u) + 2 - s - 2|\boldsymbol{E}| - 2\left|\boldsymbol{d}(\ell) - \boldsymbol{d}_{\boldsymbol{a}}(\ell)\right|\right) = -\tilde{y}_0 + \sum_{j=1}^{s} \tilde{y}_j + p_Y(\boldsymbol{d}_{\boldsymbol{a}}(\ell)).$$

Proof. Follows from Lemma 2.4.4.1, where we observe that being tangent to $D_{\ell(f)}$ at q_f to order $a_{\ell(f)}$ imposes an additional $2(a_{\ell(f)} - 1)$-dimensional constraint on the disk, leading to the final term on the left-hand side, which is equal to

$$2 \sum_{f \in \boldsymbol{F}} 1 - a_{\ell(f)}.$$

□

We can perform similar calculations for $\mathcal{M}_0$, $\mathcal{M}_1$ and $\mathcal{M}_2$. We obtain:

Lemma 2.4.4.3. *If D_u is the extended linearized operator at $u \in \mathcal{M}_0(\boldsymbol{E}, \ell, \boldsymbol{a})$, then*

$$f(i(D_u) - 2n + 6 - 2|\boldsymbol{E}| - 2\left|\boldsymbol{d}(\ell) - \boldsymbol{d_a}(\ell)\right|) = p_Y(\boldsymbol{d_a}(\ell))).$$

Lemma 2.4.4.4. *If D_u is the extended linearized operator at $u \in \mathcal{M}_2(\boldsymbol{p}, j, a)$, then*

$$f(i(D_u) + 2a + 1 - s) = -\tilde{y}_0 + \sum_{l=1}^{s} \tilde{y}_l + p_Y((a+1)y_j),$$

and there is a canonical isomorphism of orientation lines as before. To clarify, recall that $\tilde{y}_l \in \widetilde{Y}(M, \boldsymbol{D})$ is the degree of p_l, but y_j is the jth generator of $Y(M, \boldsymbol{D})$.

One can prove that these moduli spaces are regular for generic choices of perturbation data, by essentially the same arguments as in [65] and [11, Section 9k]. Namely, for each map u in the Banach manifold of maps, one can choose the perturbation datum essentially arbitrarily on an open subset of the domain, and this is enough to achieve transversality. There are two exceptions: firstly, the moduli space of holomorphic strips that do not intersect the boundary divisors is defined using a translation-invariant perturbation data (see Definition 2.4.3.8). It is shown in [36, 37] that these moduli spaces are regular for generic choice of Floer data. The second exception is for moduli spaces of holomorphic spheres contained entirely within one of the divisors D_j: our assumptions on the perturbation data along the divisors make it impossible to guarantee regularity in this situation. We will explain in Section 2.4.5 why this is not a problem for our purposes.

2.4.5 Gromov compactness

We now describe Gromov compactifications of the moduli spaces we have defined. We observe that each moduli space has bounded energy, and standard Gromov compactness shows that any sequence in one of these moduli spaces has a subsequence which converges, up to sphere and disk bubbling (see [40] for spheres and [39] for disks). We would like to show that any sphere or disk that bubbles off must be stable, so that the Gromov compactification of our moduli spaces is regular. First we discuss the case of sphere bubbles.

Any non-constant sphere bubble u must have positive intersection with each divisor D_j. So unless it is contained inside one of the divisors, it has $\geq k \geq 3$ marked points (where k is the number of divisors). In particular, all such sphere bubbles have stable domain, so they are regular for generic choice of perturbation data.

The main difficulty occurs when there are sphere bubbles lying entirely inside one of the divisors D_j, because we restricted our perturbation data so that the Hamiltonian part vanishes along D_j, and the almost-complex structure component makes D_j an almost-complex submanifold. So we cannot guarantee regularity of these sphere bubbles, and we need separate arguments to deal with this case.

Definition 2.4.5.1. Let $\boldsymbol{E}, \boldsymbol{F}$ be finite sets, and $\ell : \boldsymbol{F} \to [k]$ a labelling. A stratum of $\overline{\mathcal{M}}_0(\boldsymbol{E}, \ell)$ is indexed by a tree T whose semi-infinite edges are indexed by $\boldsymbol{E} \sqcup \boldsymbol{F}$. For each vertex v, we denote by $\boldsymbol{F}_v$ the set of semi-infinite edges incident to v with index in $\boldsymbol{F}$, and by $\ell_v : \boldsymbol{F}_v \to [k]$ the induced labelling of $\boldsymbol{F}_v$. We denote by $\boldsymbol{E}_v$ the set of the remaining edges (finite or semi-infinite) incident to v. The tree is required to be **stable**, in the sense that for each vertex $v \in V(T)$ we have $|\boldsymbol{E}_v| + |\boldsymbol{F}_v| \geq 3$.

Definition 2.4.5.2. Given such a tree T, we define the corresponding stratum of the Gromov compactification. For each vertex $v \in V(T)$, we define

$$\mathcal{M}_0^T(v) := \mathcal{M}_0(\boldsymbol{E}_v, \ell_v).$$

We then define

$$(\Pi\mathcal{M})_0^T(\boldsymbol{E}, \ell) := \prod_{v \in V(T)} \mathcal{M}_0^T(v).$$

If $\boldsymbol{E}_{\mathrm{int}}$ denotes the set of internal (finite) edges of T, then there is an evaluation map

$$\boldsymbol{ev}^T : (\Pi\mathcal{M})_0^T(\boldsymbol{E}, \ell) \to M^{\boldsymbol{E}_{\mathrm{int}}} \times M^{\boldsymbol{E}_{\mathrm{int}}}.$$

Note that each edge appears twice on the right-hand side, once for each of its endpoints. We define

$$\mathcal{M}_0^T(\boldsymbol{E}, \ell) := \left(\boldsymbol{ev}^T\right)^{-1}\left(\triangle^{\boldsymbol{T}}\right),$$

where

$$\triangle^{\boldsymbol{T}} \subset M^{\boldsymbol{E}_{\mathrm{int}}} \times M^{\boldsymbol{E}_{\mathrm{int}}}$$

denotes the diagonal.

Definition 2.4.5.3. As a set, we define

$$\overline{\mathcal{M}}_0(\boldsymbol{E}, \ell) := \coprod_T \mathcal{M}_0^T(\boldsymbol{E}, \ell).$$

We equip it with the Gromov topology.

Proposition 2.4.5.4. *If $|\boldsymbol{E}| \geq 1$, then the space $\overline{\mathcal{M}}_0(\boldsymbol{E}, \ell)$ is compact.*

Proof. Standard Gromov compactness (see [40]) says that any sequence in $\overline{\mathcal{M}}_0(\boldsymbol{E}, \ell)$ has a subsequence which Gromov-converges to a tree of nodal spheres. The space $\overline{\mathcal{M}}_0(\boldsymbol{E}, \ell)$, by definition, contains all **stable** nodal spheres. These are the nodal spheres such that each sphere bubble has ≥ 3 marked points. Thus, to prove the result, we must show that, if a non-constant sphere bubbles off in our moduli space, it must be stable.

Suppose that u is a non-constant pseudoholomorphic sphere bubble appearing in some nodal sphere which is the Gromov limit of a sequence in $\overline{\mathcal{M}}_0(\boldsymbol{E}, \ell)$. If u intersects a divisor D_j in an isolated point, then the point must be a marked point. This is because the intersection point persists in a neighbourhood of the nodal sphere in the Gromov topology, and any isolated intersection point of a sphere in $\mathcal{M}_0(\boldsymbol{E}, \ell)$ with a divisor D_j is a marked point, by Remark 2.4.3.5. Now u has positive intersection with each of the divisors D_j. Hence, if u has isolated intersection points with the divisors, it has ≥ 3 marked points where it intersects the divisors, and its domain is therefore stable.

If u does not have isolated intersection points with some divisor D_j, it must be contained in D_j by analytic continuation (and our assumptions on the perturbation data near the divisors).

Definition 2.4.5.5. If $K \subset [k]$, we denote

$$D_K := \bigcap_{j \in K} D_j.$$

Suppose that $u \subset D_K$, but has transverse intersections with all other divisors D_j. The dimension of D_K is $2n - 2|K|$ (by the normal crossings condition), so for the sphere to be non-constant we require that $|K| \leq n-1$. Then, because there are $\geq n+1$ divisors (by definition of a Kähler pair, see Definition 2.3.5.1), there remain ≥ 2 divisors D_j with which u has isolated intersections. The sphere u must intersect these divisors positively, and the intersections must be marked points. So u has ≥ 2 marked points coming from the corresponding intersections, as well as the marked point corresponding to the node (or to the marked point $q \in \boldsymbol{E}$), hence its domain is stable. $\square$

The virtual codimension of the stratum $\mathcal{M}_0^T(\boldsymbol{E}, \ell)$ in $\overline{\mathcal{M}}(\boldsymbol{E}, \ell)$ is $2(\#\text{ vertices of } T_E)-1$. If every stratum is regular, then standard gluing theorems show that $\overline{\mathcal{M}}_0(\boldsymbol{E}, \ell)$ has the structure of a compact topological manifold with corners, and the actual codimension of each stratum is equal to the virtual codimension. Regularity of the stratum indexed by the tree T means that the moduli spaces $\mathcal{M}_0^T(v)$ are regular, and the evaluation map $\boldsymbol{ev}^T$ is transverse to the diagonal $\triangle^T$.

If all nodal spheres in the moduli spaces $\mathcal{M}_0^T(v)$ are transverse to the divisors $\boldsymbol{D}$, then the moduli space is regular for generic choice of perturbation data, as we can change the perturbation arbitrarily away from $\boldsymbol{D}$ (compare [40, Section 6.3]). However, it is possible that one of the holomorphic spheres appearing in this moduli space is contained in some divisor D_j. In that case, our assumptions on the perturbation data along the divisor D_j make it impossible to guarantee regularity of this moduli space in M.

However, with some additional assumptions on the perturbation data, one can show that, if the virtual dimension of this stratum of $\mathcal{M}_0^T(\boldsymbol{E}, \ell)$ is negative, then it is actually empty, even if there are sphere bubbles contained inside some of the divisors D_j. We can also show that transversality can still be achieved with these additional assumptions on the perturbation data. This suffices for our purposes of defining algebraic structures out of these moduli spaces.

Remark 2.4.5.6. We remark that this is the point at which we need the assumption (see Definition 2.3.5.1) that each divisor D_j is ample.

We now describe these assumptions:

Let $\boldsymbol{E}, \boldsymbol{F}$ be finite sets, and $\ell : \boldsymbol{F} \to [k]$ a labelling of $\boldsymbol{F}$. Given a subset $K \subset [k]$, let

$$\boldsymbol{F}^K := \{f \in \boldsymbol{F} : \ell(f) \notin K\},$$

and

$$\ell^K : \boldsymbol{F}^K \to \overline{K}$$

the restriction of ℓ. There is a forgetful map

$$\mathfrak{f}^K : \mathcal{R}_0(\boldsymbol{E}, \ell) \to \mathcal{R}_0(\boldsymbol{E}, \ell^K).$$

Condition 2.4.5.7. For each $K \subset [k]$, the perturbation data on $\mathcal{R}_0(\boldsymbol{E}, \ell)$ coincides with the pullback of the perturbation data on $\mathcal{R}_0(\boldsymbol{E}, \ell^K)$ by $\mathfrak{f}^K$, when restricted to D_K.

Observe that even in the presence of Condition 2.4.5.7, we can still perturb the almost-complex structure arbitrarily away from the divisors D_j, so our previous transversality arguments for pseudoholomorphic spheres which are not contained in a divisor D_j are unaffected by this additional assumption.

Definition 2.4.5.8. Given a moduli space $\mathcal{M}$ cut out locally by a Fredholm section of a Banach vector bundle, we denote by

$$\text{v.d.}(\mathcal{M})$$

the Fredholm index of the section, where 'v.d.' stands for 'virtual dimension': this is the expected dimension of the moduli space, and the actual dimension if it is regular.

Proposition 2.4.5.9. *Suppose $|\boldsymbol{E}| \geq 1$. For a generic choice of perturbation data satisfying Condition 2.4.5.7, any stratum $\mathcal{M}_0^T(\boldsymbol{E}, \ell)$ of $\overline{\mathcal{M}}_0(\boldsymbol{E}, \ell)$ whose virtual dimension is negative, is actually empty.*

Proof. For clarity, we will omit the $(\boldsymbol{E}, \ell)$ from the notation throughout the proof. Let us consider a stratum indexed by a fixed tree T. For the purposes of this proof, we will make T a directed tree (arbitrarily), and denote by $h(e) \in V(T)$ the head of the directed edge $e \in \boldsymbol{E}_{int}$, and by $t(e) \in V(T)$ the tail.

Recall that any holomorphic disk either intersects a divisor transversely, or is contained in it. For each vertex $v \in V(T)$, we define a stratification

$$\mathcal{M}_0^T(v) = \bigcup_{K \subset [k]} \mathcal{M}_0^{T,K}(v),$$

where

$$\mathcal{M}_0^{T,K}(v) := \{u_v \in \mathcal{M}_0^T(v) : u_v \subset D_K, u_v \not\subset D_j \text{ for } j \notin K\}$$

(recall Definition 2.4.5.5).

We now consider the manifold D_K, with divisors $D_j \cap D_K$ for $j \notin K$ (these are divisors are normal crossings in D_K, because of the normal crossings condition in M). We define the moduli space

$$\widetilde{\mathcal{M}}_0^{T,K}(v) := \left\{u_v \in \mathcal{M}_0\left(\boldsymbol{E}_v, \ell_v^K\right) : u_v \subset D_K\right\}.$$

Assuming Condition 2.4.5.7 is satisfied, there is a forgetful map

$$\mathfrak{f}_v^{T,K} : \mathcal{M}_0^{T,K}(v) \to \widetilde{\mathcal{M}}_0^{T,K}(v),$$

obtained by forgetting the marked points with label $j \in K$. Furthermore, the evaluation map

$$\boldsymbol{ev}_v : \mathcal{M}_0^{T,K}(v) \to M^{\boldsymbol{E}_v}$$

factors as follows:

$$\begin{tikzcd} \mathcal{M}_0^{T,K}(v) \arrow[r, "\boldsymbol{ev}_v"] \arrow[d, "\mathfrak{f}_v^{T,K}"'] & M^{\boldsymbol{E}_v} \\ \widetilde{\mathcal{M}}_0^{T,K}(v) \arrow[r, "\widetilde{\boldsymbol{ev}}_v^{T,K}"] & D_K^{\boldsymbol{E}_v}. \arrow[u, hook] \end{tikzcd}$$

Lemma 2.4.5.10. *For generic choice of perturbation data satisfying Condition 2.4.5.7,*

$\widetilde{\mathcal{M}}_0^{T,K}(v)$ *is a smooth manifold of dimension*

$$\dim\left(\widetilde{\mathcal{M}}_0^{T,K}(v)\right) \leq \text{v.d.}_M\left(\mathcal{M}_0^T(v)\right) - 2|K|.$$

Proof. As we saw in the proof of Proposition 2.4.5.4, $|\boldsymbol{F}_v^K| \geq 2$ and $|\boldsymbol{E}_v| \geq 1$, so the domain of u_v is stable. Because we can perturb the restriction of the almost-complex structure part of the perturbation data to D_K essentially arbitrarily away from the divisors $D_j \cap D_K$, the moduli space $\widetilde{\mathcal{M}}_0^{T,K}(v)$ is regular in D_K for generic choice of perturbation data (observe that this is **not** necessarily the same as being regular in M). Its virtual dimension at u_v (hence its actual dimension, when regular) in D_K is

$$\text{v.d.}_{D_K}\left(\widetilde{\mathcal{M}}_0^{T,K}(v)\right) = 2(n-|K|) + 2|\boldsymbol{E}_v| + 2c_1(TD_K)(u_v) - 6.$$

Denoting the normal bundle of the divisor D_j by ND_j, we have

$$\begin{aligned} c_1(TM)(u_v) &= c_1\left(TD_K \oplus \bigoplus_{j\in K} ND_j\right)(u_v) \\ &= c_1(TD_K)(u_v) + \sum_{j\in K} c_1(ND_j)(u_v) \\ &= c_1(TD_K)(u_v) + \sum_{j\in K} c_1\left(\mathcal{L}^{\otimes d_j}|_{D_j}\right)(u_v) \\ &= c_1(TD_K)(u_v) + \sum_{j\in K} d_j\omega(u_v) \\ &\geq c_1(TD_K)(u_v), \end{aligned}$$

because the symplectic area of a pseudoholomorphic sphere is non-negative. It follows that

$$\begin{aligned} \text{v.d.}_M\left(\mathcal{M}_0^T(v)\right) - 2|K| &= 2n + 2|\boldsymbol{E}_v| + 2c_1(TM)(u_v) - 6 - 2|K| \\ &\geq 2(n-|K|) + 2|\boldsymbol{E}_v| + 2c_1(TD_K)(u_v) - 6 \\ &= \text{v.d.}_{D_K}\left(\widetilde{\mathcal{M}}_0^{T,K}(v)\right). \end{aligned}$$

The actual dimension of $\widetilde{\mathcal{M}}_0^{T,K}(v)$ near u_v is generically the same as the virtual dimension in D_K, since it is generically regular in D_K. □

Thus, we have a stratification of

$$(\Pi\mathcal{M})_0^T(\boldsymbol{E},\ell) = \prod_v \mathcal{M}_0^T(v)$$

by smooth manifolds

$$(\Pi\mathcal{M})_0^{T,\boldsymbol{K}} := \prod_v \mathcal{M}_0^{T,K_v}(v),$$

where $\boldsymbol{K}$ denotes the choice of a subset $K_v \subset [k]$ for each $v \in V(T)$. We define

$$\left(\Pi\widetilde{\mathcal{M}}\right)_0^{T,\boldsymbol{K}} := \prod_{v\in V(T)} \widetilde{\mathcal{M}}_0^{T,K_v}(v)$$

and

$$D_{\boldsymbol{K}}^{\boldsymbol{E}_{int}} := \prod_{e\in\boldsymbol{E}_{int}} D_{K_{h(e)}} \times D_{K_{t(e)}},$$

and observe that the evaluation map $\boldsymbol{ev}^{T,\boldsymbol{K}}$ factors as follows:

$$\begin{tikzcd}
\Pi\mathcal{M}_0^{T,\boldsymbol{K}} \arrow[r, "\boldsymbol{ev}^T"] \arrow[d, "\mathfrak{f}^{T,\boldsymbol{K}}"'] & M^{\boldsymbol{E}_{int}} \times M^{\boldsymbol{E}_{int}} \\
\Pi\widetilde{\mathcal{M}}_0^{T,\boldsymbol{K}} \arrow[r, "\widetilde{\boldsymbol{ev}}^{T,\boldsymbol{K}}"] & D_{\boldsymbol{K}}^{\boldsymbol{E}_{int}}, \arrow[u, hook]
\end{tikzcd}$$

where $\mathfrak{f}^{T,\boldsymbol{K}}$ is a forgetful map.

Now, observe that the map

$$D_{\boldsymbol{K}}^{\boldsymbol{E}_{int}} \to M^{\boldsymbol{E}_{int}} \times M^{\boldsymbol{E}_{int}}$$

is **not** necessarily transverse to Δ^T, and in particular we cannot hope for the map

$$\Pi\widetilde{\mathcal{M}}_0^{T,\boldsymbol{K}} \to M^{\boldsymbol{E}_{int}} \times M^{\boldsymbol{E}_{int}}$$

to be transverse to Δ^T. However, we will explain that

$$\widetilde{\boldsymbol{ev}}^{T,\boldsymbol{K}} : \Pi\widetilde{\mathcal{M}}_0^{T,\boldsymbol{K}} \to D_{\boldsymbol{K}}^{\boldsymbol{E}_{int}}$$

is generically transverse to the diagonal

$$\Delta^{T,\boldsymbol{K}} := \Delta^T \cap D_{\boldsymbol{K}}^{\boldsymbol{E}_{int}}.$$

We have

$$\Delta^{T,\boldsymbol{K}} \cong \prod_{e\in\boldsymbol{E}_{int}} D_{K_{h(e)}\cup K_{t(e)}} \hookrightarrow \prod_{e\in\boldsymbol{E}_{int}} D_{K_{h(e)}} \times D_{K_{t(e)}},$$

where the inclusion is via the 'diagonal map' $p \mapsto (p,p)$. The map $\widetilde{\boldsymbol{ev}}^{T,\boldsymbol{K}}$ is then transverse to $\Delta^{T,\boldsymbol{K}}$ because, for any $e \in \boldsymbol{E}_{int}$, sphere bubbles inside $D_{K_{h(e)}}$ are generically regular, so we can perturb them to make the marked point move in any direction within $TD_{K_{h(e)}}$. Similarly for the sphere bubble in $D_{K_{t(e)}}$.

It follows that the manifold

$$\left(\widetilde{\boldsymbol{ev}}^{T,\boldsymbol{K}}\right)^{-1}(\Delta^{T,\boldsymbol{K}})$$

is generically smooth of dimension

$$\begin{aligned}
\dim\left(\left(\widetilde{\boldsymbol{ev}}^{T,\boldsymbol{K}}\right)^{-1}(\Delta^{T,\boldsymbol{K}})\right) &= \dim\left(\Pi\widetilde{\mathcal{M}}_0^{T,\boldsymbol{K}}\right) - \operatorname{codim}\left(\Delta^{T,\boldsymbol{K}}\right) \\
&= \sum_v \dim\left(\widetilde{\mathcal{M}}_0^{T,K_v}(v)\right) - \sum_{e\in\boldsymbol{E}_{int}} \left(2n - 2|K_{h(e)} \cap K_{t(e)}|\right) \\
&\leq \sum_{v\in V(T)} \left(\text{v.d.}\left(\mathcal{M}_0^T(v)\right) - 2|K_v|\right) - \sum_{e\in\boldsymbol{E}_{int}} 2n - 2|K_{h(e)} \cap K_{t(e)}| \\
&= \text{v.d.}\left(\Pi\mathcal{M}_0^T\right) - 2n|\boldsymbol{E}_{int}| - 2\sum_{v\in V(T)} |K_v| + 2\sum_{e\in\boldsymbol{E}_{int}} |K_{h(e)} \cap K_{t(e)}| \\
&\leq \text{v.d.}\left(\mathcal{M}_0^T\right) - 2\sum_{e\in\boldsymbol{E}_{int}} |K_{h(e)}| - |K_{h(e)} \cap K_{t(e)}| \\
&\leq \text{v.d.}\left(\mathcal{M}_0^T\right),
\end{aligned}$$

where the second-last line follows as each $v \in V(T)$ is the head of at most one edge $e \in \boldsymbol{E}_{int}(T)$.

In particular, if the expected dimension of $\mathcal{M}_0^T$ is negative, then the manifold is empty. It follows that the intersection of the subspace

$$\boldsymbol{ev}^{-1}\left(\Delta^T\right) \cap \Pi \mathcal{M}_0^{T,\boldsymbol{K}} = \left(\mathfrak{f}^{T,\boldsymbol{K}}\right)^{-1}\left(\left(\widetilde{\boldsymbol{ev}}^{T,\boldsymbol{K}}\right)^{-1}\left(\Delta^{T,\boldsymbol{K}}\right)\right)$$

is empty, for each $\boldsymbol{K}$.

This completes the proof. $\square$

We have an analogous result for the moduli spaces $\overline{\mathcal{M}}_0(\boldsymbol{E}, \ell, \boldsymbol{a})$. Now we describe the Gromov compactification of the moduli spaces $\mathcal{M}(\boldsymbol{p}, \boldsymbol{E}, \ell, \boldsymbol{a})$.

Definition 2.4.5.11. Let $\boldsymbol{L}$ be a tuple of elements of the affine Fukaya category, $\boldsymbol{p}$ an associated set of generators, $\boldsymbol{E}, \boldsymbol{F}$ finite sets, $\ell : \boldsymbol{F} \to [k]$ a labelling, such that $|\boldsymbol{L}| + 2|\boldsymbol{E}| + 2|\boldsymbol{F}| \geq 2$, and $\boldsymbol{a}$ a k-tuple of positive integers. A stratum of $\overline{\mathcal{M}}(\boldsymbol{p}, \boldsymbol{E}, \ell, \boldsymbol{a})$ is indexed by an object $\boldsymbol{T}$, where $\boldsymbol{T}$ consists of the following data:

- A tree T, together with a directed planar subtree T_L with labels $\boldsymbol{L}$;
- An indexing of the semi-infinite edges of $T_E := T \setminus T_L$ by $\boldsymbol{E} \sqcup \boldsymbol{F}$;
- For each edge e of T_L, a choice of generator $p_e \in CF^*(L_{r(e)}, L_{l(e)})$, where $L_{r(e)}, L_{l(e)}$ are the Lagrangian labels to the right and left of e respectively, such that the generators are given by $\boldsymbol{p}$ for the external edges.

For each vertex $v \in V(T)$, we denote by $\boldsymbol{F}_v$ the set of semi-infinite edges in T_E that are incident to v and have index in $\boldsymbol{F}$, and by $\ell_v : \boldsymbol{F}_v \to [k]$ the labelling induced by ℓ. We denote by $\boldsymbol{E}_v$ the remaining edges (finite or semi-infinite) in T_E that are incident to v. For each vertex $v \in T_L$, we denote by $\boldsymbol{L}_v$ the tuple of Lagrangians labelling the regions surrounding v, and by $\boldsymbol{p}_v$ the set of chosen generators for the edges adjacent to v. The tree T is required to be **semi-stable**, in the sense that for each vertex $v \in V(T_L)$ we have

$$|\boldsymbol{L}_v| + 2|\boldsymbol{E}_v| + 2|\boldsymbol{F}_v| \geq 2,$$

while for each vertex $v \in V(T_E)$,

$$|\boldsymbol{E}_v| + |\boldsymbol{F}_v| \geq 3.$$

Definition 2.4.5.12. Given such an object $\boldsymbol{T}$, we define the corresponding stratum of the Gromov compactification. For vertices $v \in T_E$, we define

$$\mathcal{M}^{\boldsymbol{T}}(v) := \mathcal{M}_0(\boldsymbol{E}_v, \ell_v, \boldsymbol{a}).$$

For $v \in T_L$, we define

$$\mathcal{M}^{\boldsymbol{T}}(v) := \mathcal{M}(\boldsymbol{p}_v, \boldsymbol{E}_v, \ell_v, \boldsymbol{a}).$$

Now, letting $\boldsymbol{E}_{\text{int}}$ denote the internal (finite) edges of T_E, we have an obvious evaluation map

$$\boldsymbol{ev}^{\boldsymbol{T}} : \prod_{v \in V(T)} \mathcal{M}^{\boldsymbol{T}}(v) \to M^{\boldsymbol{E}_{\text{int}}} \times M^{\boldsymbol{E}_{\text{int}}}.$$

We define

$$\mathcal{M}^{\boldsymbol{T}}(\boldsymbol{p}, \boldsymbol{E}, \boldsymbol{\ell}, \boldsymbol{a}) := \left(\boldsymbol{ev}^{\boldsymbol{T}}\right)^{-1} \left(\triangle^{\boldsymbol{T}}\right),$$

where

$$\triangle^{\boldsymbol{T}} \subset M^{\boldsymbol{E}_{\text{int}}} \times M^{\boldsymbol{E}_{\text{int}}}$$

denotes the diagonal.

Definition 2.4.5.13. As a set, we define

$$\overline{\mathcal{M}}(\boldsymbol{p}, \boldsymbol{E}, \ell, \boldsymbol{a}) := \coprod_{\boldsymbol{T}} \mathcal{M}^{\boldsymbol{T}}(\boldsymbol{p}, \boldsymbol{E}, \ell, \boldsymbol{a}).$$

We equip it with the Gromov topology.

Proposition 2.4.5.14. *The space $\overline{\mathcal{M}}(\boldsymbol{p}, \boldsymbol{E}, \ell, \boldsymbol{a})$, is compact.*

Proof. Standard Gromov compactness (see [39]) says that any sequence in $\overline{\mathcal{M}}(\boldsymbol{p}, \boldsymbol{E}, \ell, \boldsymbol{a})$ has a subsequence which Gromov-converges to a nodal disk. The space $\overline{\mathcal{M}}(\boldsymbol{p}, \boldsymbol{E}, \ell, \boldsymbol{a})$, by definition, contains all **semi-stable** nodal disks. These are the nodal disks such that each sphere bubble has ≥ 3 marked points, and each disk bubble has

$$\#\text{ boundary punctures} + 2(\#\text{ marked points}) \geq 2.$$

Thus, to prove the result, we must show that, if a non-constant sphere or disk bubbles off in our moduli space, it must be semi-stable. Sphere bubbling was dealt with in Proposition 2.4.5.4. Any unstable disk has one boundary puncture and no internal marked points; these are constant by exactness of the Lagrangians in $M \setminus \boldsymbol{D}$. Thus there can be no unstable sphere or disk bubbling. $\square$

Lemma 2.4.5.15. *Suppose that all strata $\mathcal{M}^{\boldsymbol{T}}(\boldsymbol{p}, \boldsymbol{E}, \ell, \boldsymbol{a})$ of $\overline{\mathcal{M}}(\boldsymbol{p}, \boldsymbol{E}, \ell, \boldsymbol{a})$ which contain a sphere bubble have negative virtual dimension. Then, for a generic choice of perturbation data satisfying Condition 2.4.5.7, $\overline{\mathcal{M}}(\boldsymbol{p}, \boldsymbol{E}, \ell, \boldsymbol{a})$ is a compact topological manifold with corners, and each stratum has the expected dimension.*

Proof. It follows as in Proposition 2.4.5.9 that any stratum whose virtual dimension is negative is actually empty. Therefore, there are no sphere bubbles. Holomorphic disks intersect the divisors transversely, and are therefore regular. Standard gluing theorems then show that the moduli space is a topological manifold with corners. $\square$

We are particularly interested in moduli spaces of dimension 0 and 1, since we use those to define the Fukaya category. In particular, since sphere bubbles always have codimension ≥ 2, we have the following result:

Corollary 2.4.5.16. *Suppose we are given a set of Lagrangian boundary conditions $\boldsymbol{L}$ with associated generators $\boldsymbol{p}$, a labelling $\ell : \boldsymbol{F} \to [k]$, and a tuple $\boldsymbol{a}$. Then, for a generic choice of perturbation data,*

- *If the virtual dimension of $\overline{\mathcal{M}}(\boldsymbol{p}, \ell, \boldsymbol{a})$ is 0, then it is a compact zero-dimensional manifold;*
- *If the virtual dimension of $\overline{\mathcal{M}}(\boldsymbol{p}, \ell, \boldsymbol{a})$ is 1, then it is a compact one-dimensional manifold, with boundary consisting of the 0-dimensional moduli spaces $\mathcal{M}^{\boldsymbol{T}}(\boldsymbol{p}, \ell, \boldsymbol{a})$, where $\boldsymbol{T}$ has two vertices, both contained in T_L.*

Now let $\boldsymbol{L}$ be a set of Lagrangian labels, $\boldsymbol{p}$ an associated set of generators, a be a positive integer, and D_j be one of the divisors. We define the three types of boundary strata in the Gromov compactification of $\mathcal{M}_2(\boldsymbol{p}, j, a)$ (compare Figure 2.4.2.2). We observe that any spheres bubbling off from a sequence in $\mathcal{M}_2(\boldsymbol{p}, j, a)$ are necessarily constant, because they do not intersect the divisors D_i for $i \neq j$; thus we need only consider strata consisting of disk bubbles.

Definition 2.4.5.17. Let $\boldsymbol{T}$ consist of the following data:

- A directed planar tree T with Lagrangian labels $\boldsymbol{L}$, and a distinguished vertex v_1;
- For each edge e of T, a generator $p_e \in CF^*(L_{r(e)}, L_{l(e)})$,

such that all vertices are semi-stable with the possible exception of v_1. We define

$$\mathcal{M}_2^{1,\boldsymbol{T}}(\boldsymbol{p}, j, k) := \mathcal{M}_2(\boldsymbol{p}_{v_1}, j, a) \times \prod_{v \neq v_1} \mathcal{M}(\boldsymbol{p}_v, \ell_v)$$

(note that for $v \neq v_1$, $\boldsymbol{F}_v = \phi$, so the ℓ_v is irrelevant but we include it in the notation for consistency).

The second stratum corresponds to $t \to 0$, so the marked points z_1 and z_2 come together and bubble off a pseudo-holomorphic sphere. This sphere has intersection number 0 with all the divisors other than D_j, hence it must be constant. Thus, the holomorphic disk attached to the sphere has intersection number $(a+1)$ with the divisor D_j, and 0 with the other divisors, and only intersects D_j at the nodal point z where it is attached to the constant sphere. It follow that the disk is tangent to D_j at z to order $a+1$. In other words, it is an element of $\mathcal{M}_1(\boldsymbol{p}, j, a+1)$ (i.e., we can choose our perturbation data to make this so).

Definition 2.4.5.18. Let $\boldsymbol{T}$ consist of the following data:

- A directed planar tree T with Lagrangian labels $\boldsymbol{L}$, and a distinguished vertex v_1;
- For each edge e of T, a generator $p_e \in CF^*(L_{r(e)}, L_{l(e)})$;

such that all vertices are semi-stable with the possible exception of v_1. We define

$$\mathcal{M}_2^{2,\boldsymbol{T}}(\boldsymbol{p}, j, a) := \mathcal{M}(\boldsymbol{p}_{v_1}, j, a+1) \times \prod_{v \neq v_1} \mathcal{M}(\boldsymbol{p}_v, \ell_v).$$

The third stratum corresponds to $t \to 1$, so the marked points z_1 and z_2 move to the boundary and bubble off disks at the boundary.

Definition 2.4.5.19. Let $\boldsymbol{T}$ consist of the following data:

- A directed planar tree T with Lagrangian labels $\boldsymbol{L}$, and two distinguished vertices v_1 and v_2;
- For each edge e of T, a generator $p_e \in CF^*(L_{r(e)}, L_{l(e)})$,

such that all vertices are semi-stable with the possible exception of v_1 and v_2, and the branch of T containing v_1 lies strictly to the left of the branch containing v_2. We define

$$\mathcal{M}_2^{3,\boldsymbol{T}}(\boldsymbol{p}, j, a) := \mathcal{M}_1(\boldsymbol{p}_{v_1}, j, a) \times \mathcal{M}_1(\boldsymbol{p}_{v_2}, j, 1) \times \prod_{v \neq v_1, v_2} \mathcal{M}(\boldsymbol{p}_v, \ell_v).$$

Definition 2.4.5.20. We define the moduli space

$$\overline{\mathcal{M}}_2(\boldsymbol{p}, j, a) := \left(\coprod_{\boldsymbol{T}} \mathcal{M}_2^{1,\boldsymbol{T}}(\boldsymbol{p}, j, a)\right) \coprod \left(\coprod_{\boldsymbol{T}} \mathcal{M}_2^{2,\boldsymbol{T}}(\boldsymbol{p}, j, a)\right) \coprod \left(\coprod_{\boldsymbol{T}} \mathcal{M}_2^{3,\boldsymbol{T}}(\boldsymbol{p}, j, a)\right)$$

as a set.

Lemma 2.4.5.21. *For generic choice of perturbation data, $\overline{\mathcal{M}}_2(\boldsymbol{p}, j, a)$ has the structure of a compact manifold with corners, of the expected dimension (see Lemma 2.4.4.4). Furthermore,*

- *The stratum $\mathcal{M}_2^{1,\boldsymbol{T}}(\boldsymbol{p}, j, k)$ has codimension $|V(T)| - 1$,*
- *The stratum $\mathcal{M}_2^{2,\boldsymbol{T}}(\boldsymbol{p}, j, k)$ has codimension $|V(T)|$, and*
- *The stratum $\mathcal{M}_2^{3,\boldsymbol{T}}(\boldsymbol{p}, j, k)$ has codimension $|V(T)| - 2$.*

2.4.6 Branched covers

Let $\phi : (N, \boldsymbol{D}') \to (M, \boldsymbol{D})$ be an $\boldsymbol{a}$-branched cover of Kähler pairs (see Definition 2.3.5.9).

Let $\boldsymbol{L}$ be a tuple of anchored Lagrangian branes in $N \setminus \boldsymbol{D}'$, $\boldsymbol{p}$ an associated set of generators, $\boldsymbol{E}$ a finite set, and ℓ a labelling. Denote by $\phi(\boldsymbol{L})$ the image of these branes in $M \setminus \boldsymbol{D}$, and by $\phi(\boldsymbol{p})$ the associated set of generators. We would like to related the moduli space $\mathcal{M}(\boldsymbol{p}, \boldsymbol{E}, \ell)$ of disks in N and the moduli space $\mathcal{M}(\phi(\boldsymbol{p}), \boldsymbol{E}, \ell, \boldsymbol{a})$ of disks in M.

Let us choose perturbation data for the moduli space $\mathcal{M}(\phi(\boldsymbol{p}), \boldsymbol{E}, \ell, \boldsymbol{a})$ in M.

Condition 2.4.6.1. In a neighbourhood of the divisors $\boldsymbol{D} \subset M$, the almost-complex structure part of the perturbation datum is equal to the standard (integrable) complex structure.

Under Condition 2.4.6.1, the pullback of the perturbation data in M by ϕ to N is a valid choice of perturbation data in N. Note that this is not true for generic perturbation data: the pullback of a generic almost-complex structure by ϕ may be singular along the divisors $\boldsymbol{D}$.

Lemma 2.4.6.2. *If our perturbation data in M satisfy Condition 2.4.6.1, and we use the pulled-back perturbation data to define the moduli space in N, then there is an isomorphism of moduli spaces:*

$$\begin{aligned} \mathcal{M}(\boldsymbol{p}, \boldsymbol{E}, \ell) &\xrightarrow{\cong} \mathcal{M}(\phi(\boldsymbol{p}), \boldsymbol{E}, \ell, \boldsymbol{a}), \\ u &\mapsto \phi \circ u. \end{aligned}$$

Proof. It is clear that this map is well-defined and injective. It is also surjective: suppose we are given $u \in \mathcal{M}(\phi(\boldsymbol{p}), \boldsymbol{E}, \ell, \boldsymbol{a})$. It is clear that, locally, u lifts to a pseudoholomorphic curve in $N \setminus \boldsymbol{D}'$, away from the marked points q_f. At a marked point q_f, u is tangent to the divisor $D_{\ell(f)}$ to order $a_{\ell(f)} - 1$, and it follows that a loop around q_f gets mapped to a loop going $a_{\ell(f)}$ times around divisor $D_{\ell(f)}$. Therefore, a punctured neighbourhood of q_f lifts to $N \setminus \boldsymbol{D}'$. By the removable singularity theorem, the point q_f also lifts, so u lifts locally on a neighbourhood of the marked points q_f. Therefore, since the disk is contractible, u lifts to N, and the lift is clearly an element of $\mathcal{M}(\boldsymbol{p}, \boldsymbol{E}, \ell, \boldsymbol{a})$. □

We observe that it is possible to achieve regularity of the moduli spaces $\mathcal{M}(\phi(\boldsymbol{p}), \boldsymbol{E}, \ell)$ if we require our perturbation data to satisfy Condition 2.4.6.1, because we can still perturb the Hamiltonian part of the perturbation data essentially arbitrarily away from the strip-like ends and marked points (the transversality argument follows [11, Section 9k]).

However, it is **not** possible to achieve regularity of the moduli spaces $\mathcal{M}_0(\boldsymbol{E}, \ell)$: in particular, on a sphere bubble contained inside one of the divisors, the perturbation datum is required to be equal to the standard complex structure, which may not be regular. Therefore we can in general **not** guarantee regularity of the Gromov compactification

$$\overline{\mathcal{M}}(\phi(\boldsymbol{p}), \boldsymbol{E}, \ell),$$

since its strata involve sphere bubbles.

However, we recall that spheres can not bubble off from the moduli spaces $\mathcal{M}_1(\boldsymbol{p}, j, a)$ and $\mathcal{M}_2(\boldsymbol{p}, j, a)$, because a non-constant sphere bubble must intersect all of the divisors. Therefore, we **can** achieve regularity of the Gromov compactifications

$$\overline{\mathcal{M}}_1(\phi(\boldsymbol{p}), j, a) \text{ and } \overline{\mathcal{M}}_2(\phi(\boldsymbol{p}), j, a)$$

with perturbation data satisfying Condition 2.4.6.1. This allows us to prove:

Lemma 2.4.6.3. *There exist choices of perturbation data in M and N such that Lemma 2.4.5.15 remains true in both M and N, and Lemma 2.4.5.21 remains true in M, and such that there are furthermore isomorphisms of moduli spaces*

$$\begin{aligned} \mathcal{M}_1(\boldsymbol{p}, j, 1) &\stackrel{\cong}{\to} \mathcal{M}_1(\phi(\boldsymbol{p}), j, a_j) \\ u &\mapsto \phi \circ u \end{aligned}$$

for all $\mathbf{p}, j$.

Proof. First, we choose perturbation data on the moduli spaces $\mathcal{M}_1(\phi(\boldsymbol{p}), j, a_j)$ satisfying Condition 2.4.6.1. We define the perturbation data on $\mathcal{M}_1(\boldsymbol{p}, j, 1)$ to be the pullback of this perturbation data under ϕ. We can then extend these choices to consistent choices of perturbation data for **all** of the moduli spaces $\mathcal{M}_0$, $\mathcal{M}$, $\mathcal{M}_1$, $\mathcal{M}_2$, separately in M and N, such that the moduli spaces and their Gromov compactifications are regular. $\square$

2.5 The relative Fukaya category

In this section, we give our definition of the **relative Fukaya category** of a Kähler pair $(M, \boldsymbol{D})$, which we denote $\mathcal{F}(M, \boldsymbol{D})$. It is a (possibly curved) $\boldsymbol{G}(M, \boldsymbol{D})$-graded deformation of the affine Fukaya category, in the sense of Definition 2.2.4.3. We also define an orbifold version of the relative Fukaya category. In Section 2.5.2, we describe the behaviour of the relative Fukaya category with respect to branched covers.

2.5.1 The definition

Suppose that $(M, \boldsymbol{D})$ is a Kähler pair, and $\boldsymbol{a} = (a_1, \ldots, a_k)$ a tuple of k positive integers (where k is the number of divisors in $\boldsymbol{D}$).

Definition 2.5.1.1. We define the ring

$$R_{\boldsymbol{a}} := \mathbb{C}[[r_1, \ldots, r_k]],$$

and equip it with the $\boldsymbol{G}(M, \boldsymbol{D})$-grading, where r_j has grading

$$(2(1 - a_j), a_j y_j) \in (\mathbb{Z} \oplus Y)/Z$$

(see Corollary 2.3.5.5).

Example 2.5.1.2. Suppose that $(M, \boldsymbol{D}) = (M_1^n, \boldsymbol{D})$ as in Example 2.3.5.2, and $\boldsymbol{a} = (a, \ldots, a)$. Then we have

$$R_{\boldsymbol{a}} \cong R_a^n,$$

where R_a^n is the $\boldsymbol{G}(M, \boldsymbol{D})$-graded ring introduced in Definition 2.2.2.13.

We give a definition of the **smooth orbifold relative Fukaya category** $\mathcal{F}(M, \boldsymbol{D}, \boldsymbol{a})$, based on the definition of the relative Fukaya category given in [57]. It is a (possibly curved) $\boldsymbol{G}(M, \boldsymbol{D})$-graded deformation of the affine Fukaya category over $R_{\boldsymbol{a}}$. We denote

$$\mathcal{F}(M, \boldsymbol{D}) := \mathcal{F}(M, \boldsymbol{D}, \mathbf{1}),$$

and call it the **relative Fukaya category**.

The objects of $\mathcal{F}(M, \boldsymbol{D}, \boldsymbol{a})$ are the same as in the affine Fukaya category: anchored Lagrangian branes in $M \setminus \boldsymbol{D}$. Given a tuple of anchored Lagrangian branes $\boldsymbol{L}$ with associated generators $\boldsymbol{p}$, and an element $\boldsymbol{d} \in \mathbb{Z}^k_{\geq 0}$, we choose a labelling ℓ such that $\boldsymbol{d}(\ell) = \boldsymbol{d}$, and define the coefficient of $r^{\boldsymbol{d}} p_0$ in $\mu^s(p_s, \ldots, p_1)$ to be

$$\frac{\#\left(\mathcal{M}(\boldsymbol{p}, \ell, \boldsymbol{a})\right)}{\boldsymbol{d}!},$$

where we denote

$$\boldsymbol{d}! := d_1! d_2! \ldots d_k!,$$

and $\#$ denotes a signed count of the zero-dimensional part of the moduli space, with signs defined according to the canonical isomorphism of orientation spaces given by Lemma 2.4.4.2. We observe that this is a finite sum, by Corollary 2.4.5.16. Note that, while the affine Fukaya category is not curved (any pseudoholomorphic disk with boundary on an exact Lagrangian is constant, since it has zero energy), the relative Fukaya category may be curved.

It follows from the index computation in Lemma 2.4.4.2 that the structure maps μ^s define a $\boldsymbol{G}(M, \boldsymbol{D})$-graded A_∞ deformation of the affine Fukaya category over $R_{\boldsymbol{a}}$. Observe that the order-0 component of μ^s counts disks that completely avoid the divisors $\boldsymbol{D}$, and therefore coincides with the definition of the structure maps in the affine Fukaya category.

The fact that $\mu \circ \mu = 0$ follows also from Corollary 2.4.5.16, since the signed count of boundary points of a compact 1-dimensional manifold with boundary is 0. The sign computation follows directly from that of [11, Section 12g], essentially because the fibres of the forgetful maps $\mathcal{R}(\boldsymbol{L}, \boldsymbol{E}) \to \mathcal{R}(\boldsymbol{L}, \phi)$ have a complex structure, hence are canonically oriented.

Observe that we needed the factor $(\boldsymbol{d}!)^{-1}$ in the definition of the structure coefficients μ^s. This is because, if $\boldsymbol{d} = \boldsymbol{d}_1 + \boldsymbol{d}_2$, then given a labelling $\ell : \boldsymbol{F} \to [k]$ with $\boldsymbol{d}(\ell) = \boldsymbol{d}$, there are $\boldsymbol{d}!/(\boldsymbol{d}_1!\boldsymbol{d}_2!)$ ways of choosing a partition $\boldsymbol{F} = \boldsymbol{F}_1 \sqcup \boldsymbol{F}_2$ such that the restricted labellings ℓ_1, ℓ_2 on $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ satisfy $\boldsymbol{d}(\ell_1) = \boldsymbol{d}_1$ and $\boldsymbol{d}(\ell_2) = \boldsymbol{d}_2$. So, in the boundary of the one-dimensional component of the moduli space $\mathcal{M}(\boldsymbol{p}, \ell)$, which consists of nodal disks with two components, there are $\boldsymbol{d}!/(\boldsymbol{d}_1!\boldsymbol{d}_2!)$ ways for the marked points q_f to be distributed between the two components.

It is important to consider in what sense the smooth orbifold relative Fukaya category is dependent on the choices (of Floer and perturbation data) involved in its construction. We recall the argument of [11, Section 10a]: Let I denote a set of possible choices of Floer and perturbation data. For each $i \in I$, we denote by $\mathcal{F}(M, \boldsymbol{D}, \boldsymbol{a})^i$ the smooth orbifold relative Fukaya category defined using those choices. We define a new A_∞ category, the **total** category $\mathcal{F}(M, \boldsymbol{D}, \boldsymbol{a})^{\mathrm{tot}}$, as follows:

- Objects are pairs (L, i) where L is an object of $\mathcal{F}(M, \boldsymbol{D}, \boldsymbol{a})$ and $i \in I$;
- For each pair of objects we choose a Floer datum, and for each set of labels of objects we choose a perturbation datum;
- We require that, for a pair $(L_0, i), (L_1, i)$, the Floer datum is that given by i;
- We require that, for a set of labels $(L_0, i), \ldots, (L_k, i)$, the perturbation data are those given by the index i;
- The rest of the Floer and perturbation data we choose arbitrarily.

The rest of the construction (of morphism spaces and composition maps) follows that of the smooth orbifold relative Fukaya category. It follows that for each $i \in I$, there is a full embedding

$$\mathcal{F}(M, \boldsymbol{D}, \boldsymbol{a})^i \hookrightarrow \mathcal{F}(M, \boldsymbol{D}, \boldsymbol{a})^{\mathrm{tot}}$$

which is given, on the level of objects, by

$$L \mapsto (L, i).$$

When restricting to the affine Fukaya category ($r_j = 0$), it follows from the PSS isomorphism that these embeddings are quasi-equivalences, and hence invertible, and therefore that the affine Fukaya category does not depend on the choice of data $i \in I$, up to quasi-equivalence (see [11, Section 10a].

This need no longer be the case for the relative Fukaya category (these embeddings need not be quasi-equivalences nor invertible), and in general the relative Fukaya may depend on the data used to define it (compare [9, Section 8f]). However, in this paper we are only interested in a certain full subcategory of the relative Fukaya category which is necessarily **minimal** (for grading reasons). Therefore, rather than quasi-equivalence (which we recall is not necessarily a well-behaved notion over the power series ring R), we can use the simpler notion of formal diffeomorphism.

Let $\mathcal{L}$ be a set of objects of $\mathcal{F}(M, \boldsymbol{D}, \boldsymbol{a})$, and I be some set of possible choices of Floer and perturbation data for the full subcategory $\mathcal{C} \subset \mathcal{F}(M, \boldsymbol{D}, \boldsymbol{a})$ with objects $\mathcal{L}$. Let us form the total category $\mathcal{C}^{\mathrm{tot}}$, as above.

Lemma 2.5.1.3. *Suppose that we can choose Floer data and perturbation data for $\mathcal{C}^{\mathrm{tot}}$ so that it is minimal. Then, for any $i, j \in I$, there is a $\boldsymbol{G}$-graded quasi-equivalence of minimal A_∞ categories over R,*

$$\mathcal{C}^i \cong \mathcal{C}^j.$$

Proof. First, note that we have A_∞ embeddings of minimal A_∞ categories

$$\mathcal{C}^i \hookrightarrow \mathcal{C}^{\mathrm{tot}} \hookleftarrow \mathcal{C}^j,$$

as above. Now note that, because the objects (L, i) and (L, j) are quasi-isomorphic when restricted to the affine Fukaya category, and $\mathcal{C}^{\mathrm{tot}}$ is minimal, these objects are quasi-isomorphic in $\mathcal{C}^{\mathrm{tot}}$. It follows that the above embeddings are quasi-equivalences. By Lemma 2.2.3.34, quasi-equivalences of minimal A_∞ categories can be inverted over R. It follows that there is a quasi-equivalence $\mathcal{C}^i \cong \mathcal{C}^j$, as required. $\square$

In other words, if we can choose the category $\mathcal{C}^{\mathrm{tot}}$ to be minimal (e.g., for grading reasons), then $\mathcal{C}$ is independent of the choice of perturbation data $i \in I$ made in its construction, up to formal diffeomorphism.

Remark 2.5.1.4. If M is Calabi-Yau, then by Remark 2.3.5.7, there is a canonical morphism of grading data

$$\boldsymbol{p} : \boldsymbol{G}(M, \boldsymbol{D}) \to \boldsymbol{G}_{\mathbb{Z}}.$$

Thus, we obtain a canonical $\mathbb{Z}$-grading on the category

$$\mathcal{F}(M, \boldsymbol{D}) \cong \boldsymbol{p}_* \mathcal{F}(M, \boldsymbol{D}),$$

such that the $\mathbb{Z}$-grading of R is zero.

2.5.2 Behaviour with respect to ramified covers

Suppose that $\phi : (N, \boldsymbol{D}') \to (M, \boldsymbol{D})$ is an $\boldsymbol{a}$-branched cover of Kähler pairs, where $\boldsymbol{a} = (a_1, \ldots, a_k)$. In this section, we will examine the relationship between the following three categories:

$$\mathcal{F}(M, \boldsymbol{D}),\ \mathcal{F}(M, \boldsymbol{D}, \boldsymbol{a}), \text{ and } \mathcal{F}(N, \boldsymbol{D}').$$

We first recall the behaviour of the affine Fukaya category under covers. Note that $\phi : N \backslash \boldsymbol{D}' \to M \backslash \boldsymbol{D}$ is a finite (unramified) cover. Thus, it induces an injective morphism of grading data,

$$\boldsymbol{p} : \boldsymbol{G}(N, \boldsymbol{D}') \to \boldsymbol{G}(M, \boldsymbol{D}).$$

We recall from Proposition 2.3.4.1 that there is a fully faithful embedding

$$\boldsymbol{p}^* \mathcal{F}(M \setminus \boldsymbol{D}) \to \mathcal{F}(N \setminus \boldsymbol{D}').$$

We observe that, if it were possible to choose perturbation data in M such that Condition 2.4.6.1 were satisfied, then Lemma 2.4.6.2 would imply, by a similar argument, that there is a fully faithful embedding

$$\boldsymbol{p}^* \mathcal{F}(M, \boldsymbol{D}, \boldsymbol{a}) \to \mathcal{F}(N, \boldsymbol{D}').$$

Remark 2.5.2.1. We observe that $\boldsymbol{p}^* R_{\boldsymbol{a}}$ is exactly the $\boldsymbol{G}(N, \boldsymbol{D}')$-graded coefficient ring over which $\mathcal{F}(N, \boldsymbol{D}')$ is defined; this follows immediately from the definition of $R_{\boldsymbol{a}}$ (Definition 2.5.1.1) and Lemma 2.3.5.11.

However, we recall (see discussion in Section 2.4.6) that it is **not** possible to guarantee sufficient regularity of all of our moduli spaces under Condition 2.4.6.1, so we can not quite make this statement. Ideally, we would find a better version of Condition 2.4.6.1 that would allow us to guarantee lifting and regularity. However, we have been unable to do this, and instead circumvent this problem by a rather ugly and ad-hoc method, which we now describe.

We recall, from the discussion at the end of Section 2.4.6, that it **is** possible to obtain regularity for moduli spaces with only a single marked point, under Condition 2.4.6.1. Therefore, we can make sure that the result is true 'to first order' (see Section 2.2.6). Recall (Section 2.2.6) that we denote by $\mathfrak{m} \subset R$ the maximal ideal. If $\mathcal{F}$ is an R-linear

A_∞ category, then $\mathcal{F}/\mathfrak{m}^2$ is an $R/\mathfrak{m}^2$-linear category, which retains only the information about the first-order part of $\mathcal{F}$. Now we prove the result:

Proposition 2.5.2.2. *Given an $\boldsymbol{a}$-branched cover of Kähler pairs $\phi : (N, \boldsymbol{D}') \to (M, \boldsymbol{D})$, there exists a $\boldsymbol{G}(M, \boldsymbol{D})$-graded A_∞ category $\mathcal{F}(\phi)$ over $R_{\boldsymbol{a}}$, such that there exist A_∞ functors*

$$\mathcal{G}_1 : \boldsymbol{p}^*\mathcal{F}(\phi) \to \mathcal{F}(N, \boldsymbol{D}'),$$

and

$$\mathcal{G}_2 : \mathcal{F}(\phi)/\mathfrak{m}^2 \to \mathcal{F}(M, \boldsymbol{D}, \boldsymbol{a})/\mathfrak{m}^2,$$

and both $\mathcal{G}_1/\mathfrak{m}$ and $\mathcal{G}_2/\mathfrak{m}$ are quasi-equivalences of the zeroth-order categories.

Proof. We would like to choose perturbation data in N which are invariant with respect to the action of $Y(M, \boldsymbol{D})$, but this will be problematic, because the geometric action of $Y(M, \boldsymbol{D})$ on N is non-trivial, so we need to employ a trick to bypass this problem (compare [9, Section 8b]).

The covering map ϕ induces an injective homomorphism

$$p : Y(N, \boldsymbol{D}') \to Y(M, \boldsymbol{D}).$$

We denote

$$\Gamma := Y(M, \boldsymbol{D})/Y(N, \boldsymbol{D}').$$

As part of the construction of $\mathcal{F}(\phi)$, we choose a function

$$q : \Gamma \to Y(M, \boldsymbol{D})$$

(**not** necessarily a group homomorphism) that splits the map $Y(M, \boldsymbol{D}) \to \Gamma$.

Objects of $\mathcal{F}(\phi)$ are pairs (L, γ), where L is an anchored Lagrangian brane in $M \setminus \boldsymbol{D}$

and $\gamma \in \Gamma$. We define $Y(M, \boldsymbol{D})$ to act on our objects by

$$y \cdot (L, \gamma) := ((y - q(y)) \cdot L, \gamma + y),$$

and Γ to act by

$$\gamma \cdot (L, \gamma') := (L, \gamma + \gamma').$$

On the level of objects, the A_∞ morphisms are given by

$$(L, \gamma) \mapsto q(\gamma) \cdot L$$

(recall that there is a correspondence between anchored Lagrangian branes in $M \setminus \boldsymbol{D}$ and anchored Lagrangian branes in $N \setminus \boldsymbol{D}'$).

We now choose Floer data for our anchored Lagrangian branes in M, satisfying Condition 2.4.6.1, invariant under the action of $Y(M, \boldsymbol{D}) \oplus \Gamma$. This is possible because the action of this group on the underlying geometric Lagrangians is trivial. We define the morphism spaces as usual, with grading defined so that

$$CF^*_{\mathcal{F}(\phi)}((L_0, \gamma_0), (L_1, \gamma_1)) \cong CF^*_{\mathcal{F}(M \setminus \boldsymbol{D})}(q(\gamma_0) \cdot L_0, q(\gamma_1) \cdot L_1)$$

as $\boldsymbol{G}$-graded vector spaces.

This makes $\mathcal{F}(\phi)$ into a $\boldsymbol{G}(M, \boldsymbol{D})$-graded pre-category. For labellings ℓ with $|\boldsymbol{d}(\ell)| \leq 1$, we choose perturbation data in M for the moduli spaces $\mathcal{M}(\boldsymbol{p}, \ell, \boldsymbol{a})$, also satisfying Condition 2.4.6.1 and invariant under the action of $Y(M, \boldsymbol{D}) \oplus \Gamma$. This allows us to define $\mathcal{F}(\phi)$ to first order, by counting holomorphic disks in M intersecting only a single divisor, exactly by analogy with $\mathcal{F}(M, \boldsymbol{D}, \boldsymbol{a})/\mathfrak{m}^2$. Thus, we in fact have an isomorphism of $\boldsymbol{G}(M, \boldsymbol{D})$-graded categories

$$\mathcal{F}(\phi)/\mathfrak{m}^2 \cong \mathcal{F}(M, \boldsymbol{D}, \boldsymbol{a})/\mathfrak{m}^2 \otimes \mathbb{C}[\Gamma],$$

where $\gamma \in \Gamma$ has degree $q(\gamma)$. In particular, the categories are quasi-equivalent.

Furthermore, we observe that the above-defined map on the level of objects,

$$\begin{aligned} \boldsymbol{p}^*\mathcal{F}(\phi) &\rightarrow \mathcal{F}(N, \boldsymbol{D}') \\ (L, \gamma) &\mapsto q(\gamma) \cdot L, \end{aligned}$$

defines an equivalence of $\boldsymbol{G}(N, \boldsymbol{D}')$-graded pre-categories. I.e., a generator p of $CF^*((L_0, \gamma_0), (L_1, \gamma_1))$ only lifts to a generator of $CF^*(q(\gamma_0) \cdot L_0, q(\gamma_1) \cdot L_1)$ if its degree $y + q(\gamma_0 - \gamma_1) \in Y(M, \boldsymbol{D})$ lies in the image of $Y(N, \boldsymbol{D}')$.

Now we extend the definition of $\mathcal{F}(\phi)$ to higher-order terms. Given objects $\boldsymbol{L} := ((L_0, \gamma_0), \ldots, (L_s, \gamma_s))$, with associated generators $\boldsymbol{p} = (p_0, \ldots, p_s)$ of degrees $(y_0, \ldots, y_s)$ in $Y(M, \boldsymbol{D})$, and ℓ a labelling, such that

$$y_0 = f(2 - s - 2|\boldsymbol{d}(\ell) - \boldsymbol{d}_{\boldsymbol{a}}(\ell)|) + p_Y(\boldsymbol{d}_{\boldsymbol{a}}(\ell)) + \sum_{j=1}^{s} y_j$$

(recall that this condition must be satisfied if the coefficient of $r^{\boldsymbol{d}(\ell)} p_0$ in $\mu^s(p_s, \ldots, p_1)$ is to be non-zero), we define $\phi^*(\boldsymbol{L}, \boldsymbol{p})$ to be the tuple of anchored Lagrangian branes in $N \setminus \boldsymbol{D}'$,

$$(q(\gamma_0) \cdot L_0, (y_1 + q(\gamma_1)) \cdot L_1, \ldots, (y_1 + \ldots + y_s + q(\gamma_s)) \cdot L_s),$$

with the associated generators which are the lifts of the p_j (note that p_0 does lift, by the equation we imposed on the y_j). We define perturbation data for the moduli spaces $\mathcal{M}(\phi^*(\boldsymbol{L}, \boldsymbol{p}), \ell)$ in N, such that:

- They are given by the pullback under ϕ of the perturbation data on $\mathcal{M}(\boldsymbol{p}, \ell, \boldsymbol{a})$ in M, for $|\boldsymbol{d}(\ell)| \leq 1$ (recalling that these perturbation data satisfy Condition 2.4.6.1, and hence can be pulled back under ϕ);
- They are invariant with respect to the action of $Y(M, \boldsymbol{D})$ on objects.

Note that, we can still achieve transversality in the presence of this final assumption. To see why, observe that the action of $Y(M, \boldsymbol{D})$ may involve a non-trivial geometric action of the covering group Γ. However, under the action

$$y \cdot (L, \gamma) = ((y - q(y)) \cdot L, \gamma + y),$$

the element in the first factor $y - q(y)$ acts trivially on N. Thus, we can choose perturbation data by considering only the first factors of our objects, on which the action of $Y(M, \boldsymbol{D})$ is trivial, then push these perturbation data forward using the geometric action of the second factors via q.

We now explain why the Gromov compactifications $\overline{\mathcal{M}}(\phi^*(\boldsymbol{L}, \boldsymbol{p}), \ell)$ of zero- and one-dimensional moduli spaces are generically regular. We recall the discussion of Section 2.4.6: the only obstruction to regularity is the appearance of sphere bubbles contained entirely within one of the divisors $\boldsymbol{D}$. If we imposed Condition 2.4.6.1 on all of our moduli spaces, we could not guarantee this, because the almost-complex structure on each divisor would be required to be the standard (integrable) complex structure, and hence not necessarily regular. However, we have only imposed this condition on moduli spaces $\mathcal{M}(\phi^*(\boldsymbol{L}, \boldsymbol{p}), \ell)$ where $|\boldsymbol{d}(\ell)| \leq 1$, and any sphere bubbling off from such a moduli space is necessarily constant, because it does not intersect some divisor D_i. Therefore, these moduli spaces are generically regular. For the remaining moduli spaces with $|\boldsymbol{d}(\ell)| \geq 2$, we can perturb the almost-complex structure arbitrarily on the divisors, and therefore we can apply the argument of Proposition 2.4.5.9 to prove regularity. Thus, our moduli spaces satisfy the analogue of Corollary 2.4.5.16.

We now define the coefficient of $r^{\boldsymbol{d}} \mu^s(p_s, \ldots, p_1)$ to be the signed count of points in the zero-dimensional part of the moduli space $\mathcal{M}(\phi^*(\boldsymbol{L}, \boldsymbol{p}), \ell)$, where $\boldsymbol{d}(\ell) = \boldsymbol{d}$. It follows as in the definition of the relative Fukaya category that μ^s define a $\boldsymbol{G}(M, \boldsymbol{D})$-graded A_∞ deformation of $\mathcal{F}(M \setminus \boldsymbol{D})$ over $R_{\boldsymbol{a}}$. It follows immediately from the definition that there is an A_∞ quasi-equivalence

$$\boldsymbol{p}^*\mathcal{F}(\phi) \to \mathcal{F}(N, \boldsymbol{D}'),$$

sending $(L, \gamma) \mapsto \gamma \cdot L$ on the level of objects. This completes the proof. $\square$

Remark 2.5.2.3. We recall that the notion of quasi-equivalence of A_∞ categories over R is not necessarily well-behaved; however, in the situation in which we will apply Proposition 2.5.2.2, the parts of the categories $\mathcal{F}(\phi)$, $\mathcal{F}(M, \boldsymbol{D}, \boldsymbol{a})$ and $\mathcal{F}(N, \boldsymbol{D}')$ will necessarily be minimal, so quasi-equivalences are well-behaved by Lemma 2.2.3.34.

Now we would like to relate $\mathcal{F}(M, \boldsymbol{D})$ to $\mathcal{F}(M, \boldsymbol{D}, \boldsymbol{a})$. Again, we will only relate the first-order part of the deformations. We recall from Section 2.2.6 that a first-order deformation of the A_∞ category $\mathcal{F} := \mathcal{F}(M \setminus \boldsymbol{D})$ over R consists of A_∞ structure maps

$$\mu^* = \mu_0^* + \mu_1^*,$$

where μ_0^* gives the structure maps of $\mathcal{F}$, and the A_∞ relations say that μ_1^* defines a class

$$[\mu_1] \in HH^2(\mathcal{F}, \mathcal{F} \otimes R^1_{\boldsymbol{1}}).$$

Theorem 7. *Let $\mathcal{F} := \mathcal{F}(M \setminus \boldsymbol{D})$, $\boldsymbol{G} := \boldsymbol{G}(M, \boldsymbol{D})$, and let*

$$[\mu_{1,\boldsymbol{1}}] := \sum_{j=1}^{k} r_j \alpha_j \in HH^2_{\boldsymbol{G}}(\mathcal{F}, \mathcal{F} \otimes R^1_{\boldsymbol{1}})$$

be the first-order deformation class of $\mathcal{F}(M, \boldsymbol{D}, \boldsymbol{1}) \cong \mathcal{F}(M, \boldsymbol{D})$. Then the first-order deformation class of $\mathcal{F}(M, \boldsymbol{D}, \boldsymbol{a})$ is given by

$$[\mu_{1,\boldsymbol{a}}] = \sum_{j=1}^{k} r_j \alpha_j^{a_j} \in HH^2_{\boldsymbol{G}}(\mathcal{F}, \mathcal{F} \otimes R^1_{\boldsymbol{a}}),$$

where the power is taken with respect to the Yoneda product on $HH^*(\mathcal{F})$.

Proof. We define elements

$$\beta_j(b) \in CC^*(\mathcal{F}),$$

for $b \geq 1$, as follows: Let $\boldsymbol{L}$ be a tuple of anchored Lagrangian branes in $M \setminus \boldsymbol{D}$, with associated generators $\boldsymbol{p}$. Then the coefficient of p_0 in $\beta_j(b)^s(p_s, \ldots, p_1)$ is given by the count of rigid elements in the moduli space $\mathcal{M}_1(\boldsymbol{p}, j, b)$. It follows from the fact that the signed count of points in the boundary of the one-dimensional component of the moduli space $\overline{\mathcal{M}}_1(\boldsymbol{p}, j, b)$ is 0, that each $\beta_j(b)$ is a Hochschild cocycle. Furthermore, by definition we have

$$[\beta_j(1)] = \alpha_j$$

and

$$\mu_{1,\boldsymbol{a}} = \sum_{j=1}^{k} r_j \beta_j(a_j).$$

We also define elements

$$H_j(b) \in CC^*(\mathcal{F})$$

by counting rigid elements in $\mathcal{M}_2(\boldsymbol{p}, j, b)$.

Lemma 2.5.2.4. *We have*

$$\beta_j(b+1) = \beta_j(b) \bullet \beta_j(1) \pm \partial(H_j(b))$$

in $CC^*(\mathcal{F})$, where $\bullet$ denotes the Yoneda product and ∂ denotes the Hochschild differential.

Proof. The result follows from the fact that the signed count of points in the boundary of the one-dimensional component of the moduli space $\overline{\mathcal{M}}_2(\boldsymbol{p}, j, b)$ is 0. See Lemma 2.4.5.21 for the description of the boundary components. The boundary points $\mathcal{M}^{1,\boldsymbol{T}}(\boldsymbol{p}, j, b)$ contribute the term $\partial H_j(b)$ to the sum, the boundary points $\mathcal{M}^{2,\boldsymbol{T}}(\boldsymbol{p}, j, b)$ contribute the term $\beta_j(b+1)$, and the boundary points $\mathcal{M}^{3,\boldsymbol{T}}(\boldsymbol{p}, j, b)$ contribute the term $\beta_j(b) \bullet \beta_j(1)$. $\square$

It follows that, on the level of Hochschild cohomology,

$$[\beta_j(b+1)] = [\beta_j(b)] \bullet \alpha_j,$$

and hence, by induction, that

$$[\beta_j(b)] = \alpha_j^b.$$

The result follows immediately. $\square$

2.6 Morse-Bott computations in the Fukaya category

In this section, we consider the Kähler pair $(M, \boldsymbol{D}) = (M_1^n, \boldsymbol{D})$ of Example 2.3.5.2, and the tuple

$$\boldsymbol{a} := (n, \ldots, n)$$

(n copies) associated to the branched cover of Kähler pairs

$$\phi : (M_n^n, \boldsymbol{D}) \to (M_1^n, \boldsymbol{D})$$

(see Example 2.3.5.10). We consider the grading datum $\boldsymbol{G} := \boldsymbol{G}(M, \boldsymbol{D})$. We define the $\boldsymbol{G}$-graded rings

$$R := R_1^n \cong \mathbb{C}[[r_1, \ldots, r_n]]$$

and

$$R_n := R_n^n \cong \mathbb{C}[[r_1, \ldots, r_n]]$$

from Definition 2.2.2.13, and recall that they are the coefficient rings of $\mathcal{F}(M, \boldsymbol{D})$ and $\mathcal{F}(M, \boldsymbol{D}, \boldsymbol{a})$ respectively (see Example 2.5.1.2). Throughout this section, we denote by

$$Y := \mathbb{Z}\langle y_1, \ldots, y_n \rangle$$

the abelian group appearing in the pseudo-grading datum $\boldsymbol{H}(M, \boldsymbol{D})$.

We recall that

$$M := \left\{ \sum_j z_j = 0 \right\} \subset \mathbb{CP}^{n-1},$$

with the divisors $D_j := \{z_j = 0\}$. Thus $M \cong \mathbb{CP}^{n-2}$, and $\boldsymbol{D}$ consists of n hyperplanes with normal crossings. $M \setminus \boldsymbol{D}$ is called the (generalized) **pair of pants**.

We construct an immersed Lagrangian sphere $L^n : S^{n-2} \to M \setminus \boldsymbol{D}$ in the pair of pants (summarising the construction in [1]). The main result of this section (Corollary 2.6.5.6) is that the endomorphism algebra $CF^*(L^n, L^n)$, computed in $\mathcal{F}(M, \boldsymbol{D}, \boldsymbol{a})$, is of type A (see Definition 2.2.5.2).

To do this, we first give a Morse-Bott description of the endomorphism algebra $CF^*(L^n, L^n)$ in the relative Fukaya category, $\mathcal{F}(M, \boldsymbol{D})$, to first order. Structure coefficients in this description are given by counts of 'holomorphic flipping pearly trees' rather than holomorphic disks. A holomorphic flipping pearly tree is a Morse-Bott version of a holomorphic disk, made out of holomorphic disks and Morse flowlines. We introduce them because it is often possible to explicitly identify moduli spaces of flipping pearly trees, and therefore to make explicit computations of the structure coefficients in the Fukaya category.

The construction is based on [1] (note that the original idea comes from [27]). The extra content here is that, whereas [1] describes the endomorphism algebra in the affine Fukaya category only, we will describe the endomorphism algebra in the first-order relative Fukaya category.

There are some transversality issues in the definition of moduli spaces of flipping pearly trees, involving the possibility of unstable disk and sphere bubbles. In the moduli spaces of holomorphic disks that we used to define the relative Fukaya category, we avoided this problem by introducing extra internal marked points where they intersected the divisors $\boldsymbol{D}$. This approach is no longer possible for flipping pearly trees: they can intersect the divisors on their boundary. For this reason, we can not guarantee transversality and therefore can not give a complete Morse-Bott description of $CF^*(L^n, L^n)$ in the relative Fukaya category.

However, for moduli spaces with very few intersections with the divisors (in particular, those which intersect only a single divisor), we can rule out any unstable disk or sphere bubbles in an ad hoc way. Thus, we are able to give a Morse-Bott description of $CF^*(L^n, L^n)$ to first order, and in particular to identify the first-order deformation class in $\mathcal{F}(M, \boldsymbol{D})$. This allows us, via Theorem 7, to determine the first-order deformation class in $\mathcal{F}(M, \boldsymbol{D}, \boldsymbol{a})$. The first-order deformation class is all we need to determine that the algebra is of type A, so the failure of transversality in our Morse-Bott model at higher order does not concern us (to clarify: $CF^*(L^n, L^n)$ is perfectly well-defined at all orders, but our Morse-Bott model for it is well-defined only to first order).

After describing the construction of the Lagrangian L^n in Section 2.6.1, the structure of this section follows that of Section 2.4: first we introduce the moduli space of pearly trees (possible domains for a flipping pearly tree), then we describe our choice of perturbation data, then we describe the moduli space of flipping pearly trees (pseudoholomorphic maps into M), explain why transversality holds, then describe Gromov compactness.

2.6.1 The Lagrangian immersion $L^n : S^{n-2} \to M \setminus \boldsymbol{D}$

In [1], we introduced a one-parameter family of Lagrangian immersions

$$L^n_\epsilon : S^{n-2} \to M \setminus \boldsymbol{D},$$

for $\epsilon > 0$ sufficiently small (actually we called these L^{n-2}; apologies for the change in notation, but it makes many formulae cleaner). We briefly recall the construction of L^n.

We consider the Lagrangian immersion $L' : S^{n-2} \to M$ which is the double cover of the real locus $\mathbb{RP}^{n-2}$ of M. If we think of

$$S^{n-2} := \left\{ \sum_{j=1}^n x_j = 0, \sum_{j=1}^n x_j^2 = 1 \right\} \subset \mathbb{R}^n,$$

then the immersion is given by

$$(x_1, \ldots, x_n) \mapsto [x_1 : \ldots : x_n].$$

We construct the immersion L^n by perturbing the immersion L'.

Namely, by the Weinstein Lagrangian neighbourhood theorem, L' can be extended to an immersion of the radius-η cotangent disk bundle

$$D^*_\epsilon S^{n-2} \to M,$$

which is J_0-holomorphic along the zero section, and such that complex conjugation acts by -1 on the covector.

We construct a function $f : S^{n-2} \to \mathbb{R}$ by setting

$$f(x_1, \ldots, x_n) = \sum_{j=1}^n g(x_j),$$

where $g : \mathbb{R} \to \mathbb{R}$ has the properties

1. $g'(x) > 0$;
2. $g(-x) = -g(x)$;
3. $g(x) = x$ for $|x| < \delta$;
4. $g'(x)$ is a strictly decreasing function of $|x|$ for $|x| > \delta$;
5. $g'(x) < \delta$ for $|x| > 2\delta$,

for some small $\delta > 0$.

We then define $L^n_\epsilon : S^{n-2} \to M$ to be the image of the graph of the exact one-form ϵdf in $D^*_\eta S^{n-2}$, under the immersion into M, so that $L' = L^n_0$. The fact that ∇f is transverse to the hypersurfaces $\{x_j = 0\}$ implies that the image L^n_ϵ avoids the divisors $\{z_j = 0\}$ for $\epsilon > 0$, so we obtain a Lagrangian immersion $L^n_\epsilon : S^{n-2} \to M \setminus \boldsymbol{D}$. It has self-intersections at the critical points of f (where it intersects the other branch of the double cover). We observe that, for $n \geq 4$, L^n automatically lifts to $\widetilde{\mathcal{G}}(M \setminus \boldsymbol{D})$, because $\pi_1(S^{n-2}) = 0$. We choose such a lift, and hence define an anchored brane structure on the Lagrangian L^n.

The flowlines of ∇f are illustrated in Figure 2.6.1.1, in the case $n = 4$. The hypersurfaces $\{x_j = 0\}$ split S^{n-2} into $2^n - 2$ regions, indexed by the proper non-empty sets $K \subset [n]$. Namely, K corresponds to the region where coordinates x_j are negative for $j \in K$ and positive for $x_j \notin K$. Each region contains a unique critical point p_K of f.

The Floer endomorphism algebra $CF^*(L^n, L^n)$ can be defined, despite L^n being immersed (see [1, Section 3.1]), and is generated by the self-intersection points of L^n (which are the points p_K indexed by proper non-empty sets $K \subset [n]$), together with the Morse cohomology of S^n (which we choose to have generators p_ϕ and $p_{[n]}$, corresponding to the

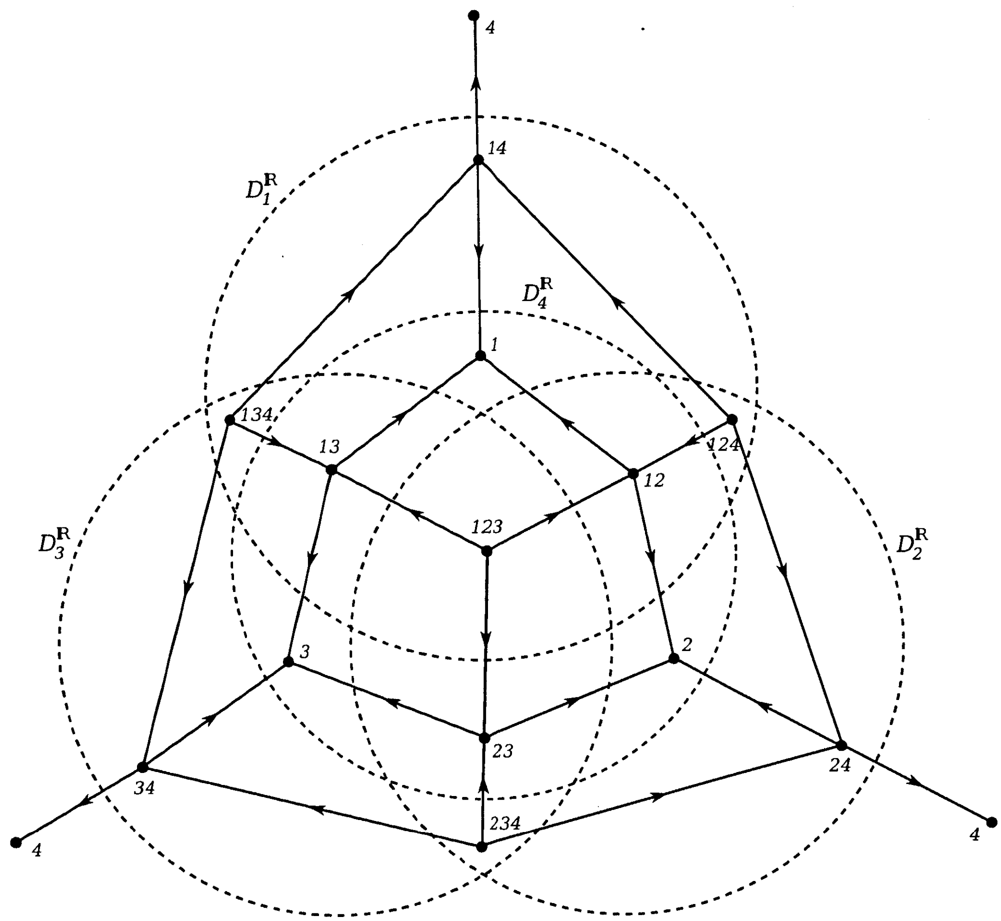


Figure 2.6.1.1: The case $n = 4$. The dashed circles represent the hypersurfaces $D_j^{\mathbb{R}} = D_j \cap \mathbb{RP}^2$ as labeled. Each region is labeled with the list of coordinates that are negative in that region (e.g., the label '124' means that $x_1 < 0, x_2 < 0, x_3 > 0, x_4 < 0$ in that region). The arrows represent the index-1 Morse flow lines of ∇f. The dots represent critical points of f. The picture really lives on a sphere, and the three points labeled '4' should be identified (at infinity).

identity and top class respectively). Thus, $CF^*(L^n, L^n)$ has generators p_K indexed by subsets $K \subset [n]$.

It follows from [1, Proposition 3.3] and [1, Proposition 3.7] that there is an isomorphism

$$CF^*(L^n, L^n) \cong A$$

as $\boldsymbol{G}$-graded vector spaces, where $A \cong A_n$ is the $\boldsymbol{G}$-graded exterior algebra of Definition 2.2.2.12.

We now observe that we can define the endomorphism algebra $CF^*(L^n, L^n)$ in $\mathcal{F}(M, \boldsymbol{D})/\mathfrak{m}^2$. Most of the rest of this section is concerned with computing $CF^*(L^n, L^n)$ in $\mathcal{F}(M, \boldsymbol{D})/\mathfrak{m}^2$, using 'flipping pearly trees'.

Remark 2.6.1.1. Note that, in [1, Section 3.1], well-definedness of $CF^*(L^n, L^n)$ in the affine Fukaya category was proved by passing to the cover $M^n \setminus \boldsymbol{D}$ of $M \setminus \boldsymbol{D}$, to bypass proving Gromov compactness for immersed Lagrangians. One may worry that this will no longer be valid when we consider disks passing through the divisors $\boldsymbol{D}$, about which the cover $M^n \to M$ has some branching. However, as we saw in the proof of Proposition 2.5.2.2, for the first-order relative Fukaya category it is possible to choose perturbation data that lift to the branched cover; and therefore we can apply the same trick to rigorously define $CF^*(L^n, L^n)$ in $\mathcal{F}(M, \boldsymbol{D})/\mathfrak{m}^2$.

The main result we will prove is:

Proposition 2.6.1.2. *There exists a $\boldsymbol{G}$-graded A_∞ category $\mathcal{F}'$ over $R/\mathfrak{m}^2$, such that;*

- *$\mathcal{F}'$ has two objects: L and L';*
- *L and L' are quasi-isomorphic in the zeroth-order category $\mathcal{F}'/\mathfrak{m}$;*
- *The endomorphism algebra of L in $\mathcal{F}'$ coincides with the endomorphism algebra of L^n in $\mathcal{F}(M, \boldsymbol{D})/\mathfrak{m}^2$;*

Furthermore, the endomorphism algebra of L' in $\mathcal{F}'$ satisfies:

$$CF^*_{\mathcal{F}'}(L', L') \cong (A, \mu^*),$$

where:

- *A is the $\boldsymbol{G}$-graded vector space of Definition 2.2.2.12;*
- *(A, μ_0^2) coincides with the exterior algebra multiplication on A;*
- *We have*

$$\Phi(\mu^*) = u_1 \dots u_n + \sum_{j=1}^{n} r_j u_j \in HH^2_{\boldsymbol{G}}(A, A \otimes R/\mathfrak{m}^2)$$

  where Φ is the HKR map (see Definition 2.2.5.1).

Remark 2.6.1.3. The main result of [1] was the zeroth-order part of Proposition 2.6.1.2. It remains to prove that flipping pearly trees can be made to work to first order, and that the first-order deformation classes are as claimed.

2.6.2 Flipping Pearly trees

For the purposes of this section, $\boldsymbol{L}$ will denote a tuple of objects of $\mathcal{F}'$ (i.e., it consists only of two types of entries: L (representing the Lagrangian immersion $L^n_\epsilon : S^{n-2} \to M$ for some $\epsilon > 0$) or L' (representing the Lagrangian immersion $L' : S^{n-2} \to M$).

Definition 2.6.2.1. If T is a semi-stable directed planar tree with labels $\boldsymbol{L}$, we introduce the following notation:

- $V(T)$ is the set of vertices of T;
- $E(T)$ is the set of edges of T;
- $E'(T) \subset E(T)$ is the subset of edges with both sides labeled L';

- $F'(T)$ is the set of flags (v, e) of T such that $e \in E'(T)$;
- $C(T)$ is the set of 'segments' between consecutive edges around a vertex (these are indexed by pairs of consecutive flags around a vertex);
- If $C \in C(T)$, then $L_C \in \boldsymbol{L}$ is the label associated to C.

Definition 2.6.2.2. Let $\boldsymbol{L}$ be a tuple. We denote by $\mathcal{R}_3(\boldsymbol{L})$ the moduli space of **flipping pearly trees**, where a flipping pearly tree $r \in \mathcal{R}_3(\boldsymbol{L})$ consists of the following data:

- A semi-stable directed planar tree T_r with labels $\boldsymbol{L}$, such that all internal edges have both sides labeled L' (i.e., all internal edges are contained in $E'(T_r)$);
- A designation of each edge $e \in E'(T_r)$ as either **flipping** or **non-flipping**;
- For each stable vertex $v \in V(T_r)$, a point $r_v \in \mathcal{R}(\boldsymbol{L}_v, \phi)$;
- For each internal edge e, a length parameter $l_e \in [0, \infty)$.

See Figure 2.6.2.1 for a picture of a flipping pearly tree. We also allow one special case: if $\boldsymbol{L} = (L', L')$, then we permit T_r to have no vertices, just a single edge with both sides labeled L'.

Now, given a flipping pearly tree $r \in \mathcal{R}_3(\boldsymbol{L})$, we define an associated topological space S_r. There are a few special cases first:

Definition 2.6.2.3. If $\boldsymbol{L} = (L', L')$, and T_r is the tree with a single edge, then we define $S_r := \mathbb{R}$. If $|\boldsymbol{L}| = 2$ but $\boldsymbol{L} \neq (L', L')$, then $S_r := \mathbb{R} \times [0, 1]$.

Now we define S_r in the remaining cases:

Definition 2.6.2.4. Given a flipping pearly tree $r \in \mathcal{R}_3(\boldsymbol{L})$, we define a topological space S_r as follows:

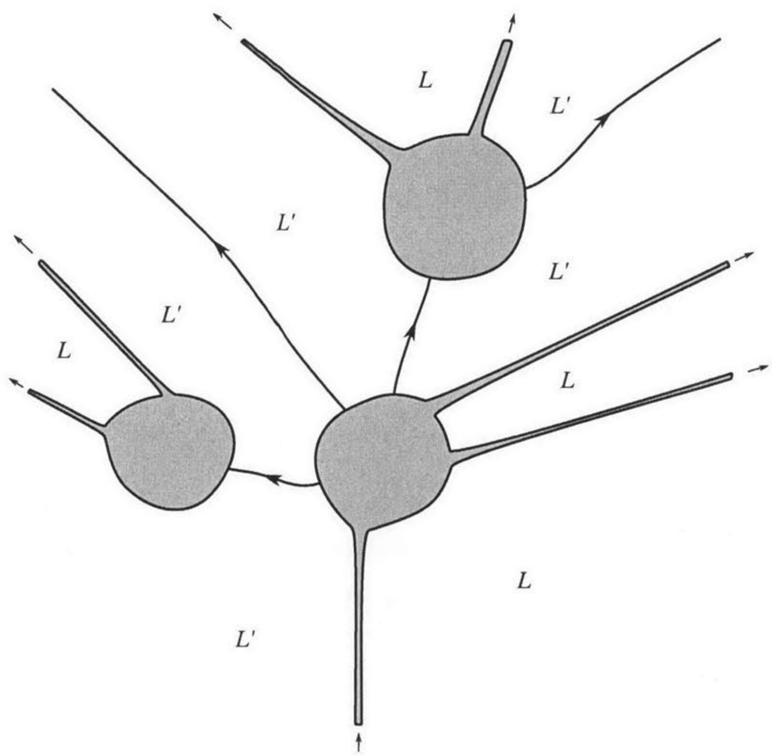


Figure 2.6.2.1: A flipping pearly tree S. Observe that all edges have label L' on either side.

- For each semi-stable vertex $v \in V(T_r)$ with both sides labeled L', we define S_v to be a disk with two boundary marked points, corresponding to the edges incident to v.
- For each stable vertex $v \in V(T_r)$, we define S_v to be the boundary-marked disk with modulus r_v, with all marked points punctured except for those corresponding to edges in $E'(T_r)$ (they remain as marked points). These are the '**pearls**'.
- We define

$$S^p := \coprod_{v \in V(T_r)} S_v.$$

- For each internal edge e, we define $S_e := [0, l_e]$. For each external edge e in $E'(T_r)$, we define $S_e := \mathbb{R}^{\pm}$, with the $+$ or $-$ depending on the orientation of the edge.
- We define

$$S^e := \coprod_{e \in E(r)} S_e.$$

- For each flag $f = (v, e) \in F'(T_r)$, there is a corresponding marked boundary point $m(f) \in S_v$ and boundary point $b(f) \in S_e$.
- We define

$$S_r := (S^p \sqcup S^e)/\sim$$

  where

$$m(f) \sim b(f) \text{ for all } f \in F'(T_r).$$

Definition 2.6.2.5. Given $r \in \mathcal{R}_3(\boldsymbol{L})$, we also define a 'boundary' $(\partial S)_r$ and a continuous map

$$(\partial S)_r \to S_r,$$

as follows:

- For each segment $C \in C(T_r)$ adjacent to vertex $v \in V(T_r)$, we define $(\partial S)_C$ to be the corresponding component of the boundary of S_v. Thus, $(\partial S)_C$ is an interval, and its two ends correspond to consecutive marked points on the disk with modulus r_v. If the marked point is punctured in S_v (i.e., if its sides are not both labeled L'), then that end of the interval $(\partial S)_C$ is open, and if the marked point remains in S_v (i.e., if its sides are both labeled L'), then that end of the interval $(\partial S)_C$ is closed.

- We define

$$\begin{aligned}(\partial S)^p &:= \coprod_{C \in C(T_r)} (\partial S)_C, \text{ with the obvious map} \\ (\partial S)^p &\to S^p.\end{aligned}$$

- For each edge e, we define $(\partial S)_e := S_e \times \{0, 1\}$ (two copies of S_e).

- We define

$$\begin{aligned}(\partial S)^e &:= \coprod_{e \in E(T_r)} (\partial S)^e, \text{ with the obvious map} \\ (\partial S)^e &\to S^e.\end{aligned}$$

- For each flag $f = (v, e) \in F'(T_r)$, there are points $\tilde{m}_j(f) \in (\partial S)_v$ for $j = 0, 1$, from the boundary components to the right and left of $m(f)$ respectively, and points $\tilde{b}_j(f) = (b(f), j) \in (\partial S)_e$, for $j = 0, 1$.

- We define

$$(\partial S)_r := ((\partial S)^p \sqcup (\partial S)^e) / \sim,$$

where

$$\tilde{m}_j(f) \sim \tilde{b}_j(f) \text{ for all } f \in F'(r), \text{ and } j = 0, 1$$

(see Figure 2.6.2.2).

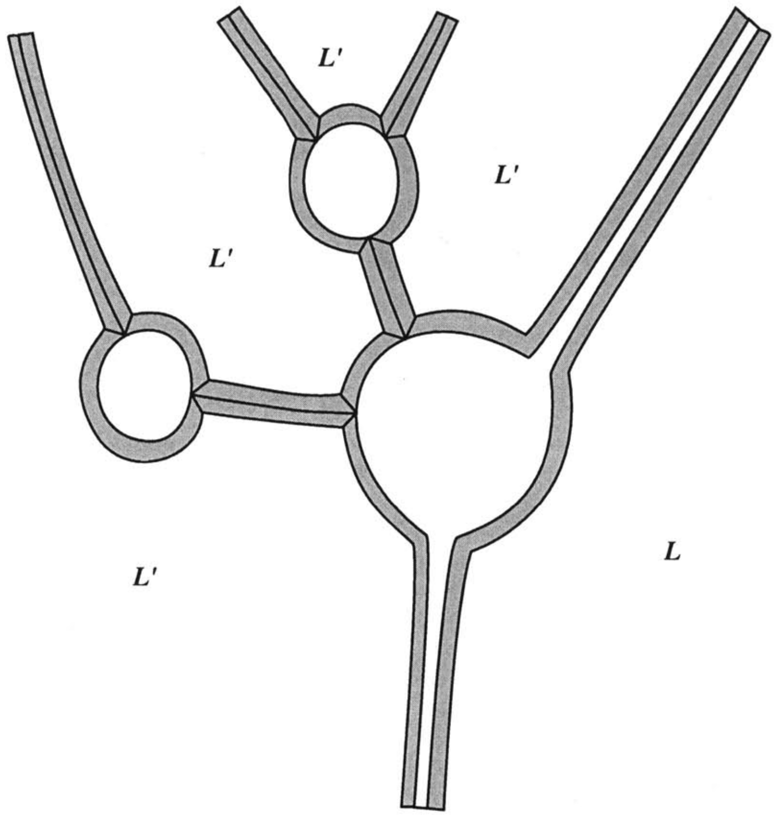


Figure 2.6.2.2: Defining the boundary $\partial S \to S$ of a flipping pearly tree S, and attaching a strip (shaded in grey) along it.

- It is clear that there is a continuous map

$$(\partial S)_r \to S_r.$$

Definition 2.6.2.6. An **automorphism** of a flipping pearly tree is a map $S_r \to S_r$ such that each pearl gets sent to itself by a biholomorphism which preserves the marked points, and each edge gets sent to itself by a translation preserving marked points (in particular, edges are fixed by any automorphism, unless they are infinite).

In particular, the possible non-trivial automorphisms of a flipping pearly tree are:

- If $\boldsymbol{L} = (L', L')$ and $S_r = \mathbb{R}$, then automorphisms are translations of $\mathbb{R}$;

- If $|\boldsymbol{L}| = 2$ but $\boldsymbol{L} \neq (L', L')$, then $S_r = \mathbb{R} \times [0,1]$ and automorphisms are translations in the $\mathbb{R}$-direction;
- If v is a semi-stable vertex of $T(r)$, then S_v is a disk with two marked boundary points, and there is an $\mathbb{R}$ family of automorphisms (translations) of S_v preserving the marked boundary points.

Definition 2.6.2.7. From our universal choice of strip-like ends for the moduli spaces $\mathcal{R}(\boldsymbol{L}, \phi)$, we can define a subset

$$S^p_{\text{thin}} \subset S^p,$$

called the **thin** region, which consists of the images of strip-like ends under gluing maps (see [11, Remark 9.1]). To clarify: the thin region includes a neighbourhood of each boundary marked point of a pearl, and also **all** of any semi-stable pearl S_v (see Figure 2.6.2.3). We define the corresponding **thick** region

$$S^p_{\text{thick}} := S^p \setminus S^p_{\text{thin}}.$$

Definition 2.6.2.8. We define the region

$$S^e_{\text{thin}} \subset S^e$$

to be the set of points on edges which are distance > 1 from the boundary of the edge, and

$$S^e_{\text{thick}} := S^e \setminus S^e_{\text{thin}}$$

(see Figure 2.6.2.3).

As in [1, Section 4.1], we can define a topology on the moduli space $\mathcal{R}_3(\boldsymbol{L})$. The important point is that thin regions with opposite sides labelled L' can stretch until they 'break', then become an internal edge (see Figure 2.6.2.4). The difference from [1] is that we now allow semi-stable vertices, and this means that $\mathcal{R}_3(\boldsymbol{L})$ no longer has the

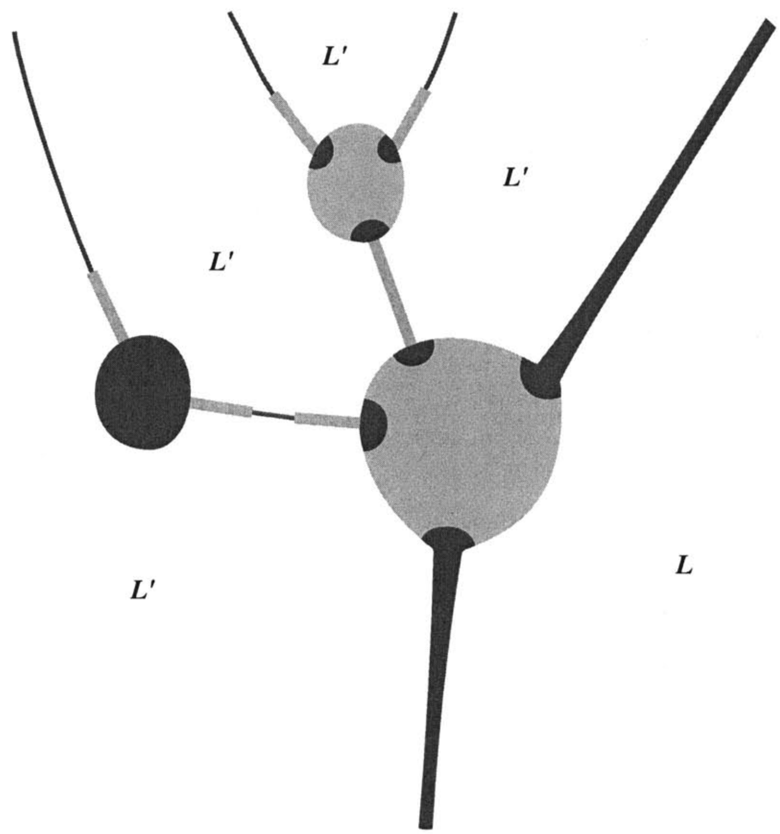


Figure 2.6.2.3: The thick and thin regions of the same flipping pearly tree S illustrated in Figure 2.6.2.2. The thick regions are shown in light grey, and the thin regions in dark grey. Note that the unstable disk (with two marked points) is entirely thin, and also that it is possible for an internal edge to be entirely thick (when it has length ≤ 2).

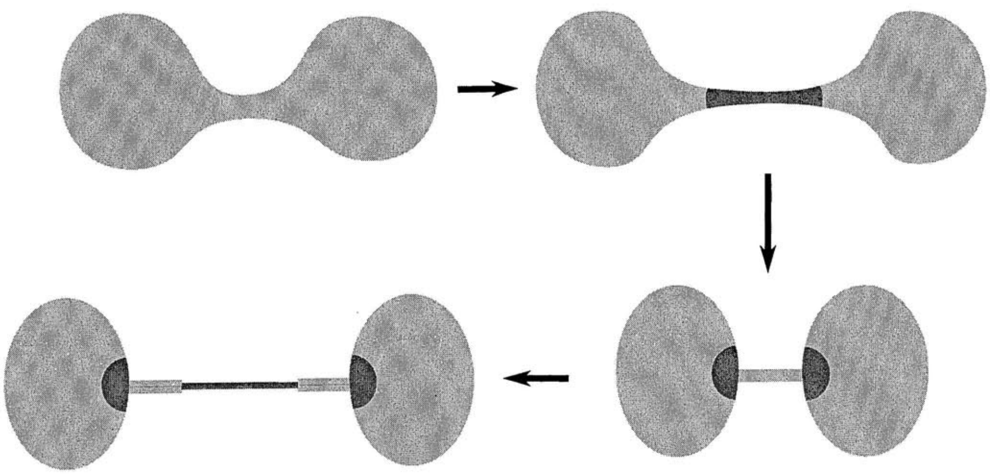


Figure 2.6.2.4: A thin region with both sides labelled L' (upper left) can stretch until it becomes a thin region (upper right), then break, becoming a thick internal edge (lower right), which then stretches until it has a thin region in its interior (lower left).

structure of a manifold with boundary. All we can say is that it is stratified by manifolds – however we will see later that the space of holomorphic maps of pearly trees into our manifold **is** a manifold with boundary, which is what we need to define our algebraic structures.

The strata of $\overline{\mathcal{R}}_3(\boldsymbol{L})$ are indexed by semi-stable directed planar trees T with labels $\boldsymbol{L}$. Note there is no requirement that internal edges have opposite sides labeled L' here. The tree T corresponds to the codimension-$(|V(T)|-1)$ stratum

$$\mathcal{R}_3^T(\boldsymbol{L}) \cong \prod_{v \in V(T)} \mathcal{R}_3(\boldsymbol{L}_v).$$

Points in this stratum correspond to flipping pearly trees, where the pearls are allowed to be nodal and the edges are allowed to have infinite length.

Remark 2.6.2.9. The moduli space $\overline{\mathcal{R}}_3(\boldsymbol{L})$ is not compact, because our flipping pearly trees can have arbitrarily many semi-stable vertices. In practice (see the proof of Proposition 2.6.4.1), we have an a priori upper bound N on the number of semi-stable vertices that we need consider. We consider the subspace

$$\overline{\mathcal{R}}_3(\boldsymbol{L}, N) \subset \overline{\mathcal{R}}_3(\boldsymbol{L}),$$

consisting of stable flipping pearly trees with $\leq N$ semi-stable vertices. This subspace is compact.

2.6.3 Floer and perturbation data

Definition 2.6.3.1. We define

$$\mathcal{H}^e := C^\infty(S^n, \mathbb{R})$$

(think of this as the space of Morse functions on S^n), and

$$\mathcal{H}^p := \{H \in C^\infty(M, \mathbb{R}) : H \text{ vanishes, with its first derivatives, along each divisor } D_j\}$$

(think of this as the space of Hamiltonians on M), and $\mathcal{J}$, the space of smooth almost-complex structures on M which are compatible with ω, and make the divisors D_j almost-complex submanifolds.

Definition 2.6.3.2. Let $\boldsymbol{L} = (L_0, L_1)$ be a 2-element tuple. For each such tuple, we choose a Floer datum $(H_{\boldsymbol{L}}, J_{\boldsymbol{L}})$ consisting of

$$H_{\boldsymbol{L}} \in C^\infty([0,1], \mathcal{H}^p) \text{ and } J_{\boldsymbol{L}} \in C^\infty([0,1], \mathcal{J})$$

such that:

- $H_{\boldsymbol{L}} = 0$ unless $\boldsymbol{L} = (L, L)$;
- the time-1 Hamiltonian flow of $H_{(L,L)}$ makes L^n transverse to itself;

- $J_{(L',L')} = J_0$ is constant, equal to the standard integrable complex structure.

Now, if $(L_0, L_1) \neq (L', L')$, then we define a **generator** of $CF^*_{\mathcal{F}'}(L_0, L_1)$ to be a path $p : [0,1] \to M$ which is a flowline of the Hamiltonian vector field of $H_{\boldsymbol{L}}$, such that $p(0) \in L_0$ and $p(1) \in L_1$. One defines $CF^*_{\mathcal{F}'}(L_0, L_1)$ to be the $R/\mathfrak{m}^2$-module generated by its generators. It is $\boldsymbol{G}$-graded.

Remark 2.6.3.3. The $\boldsymbol{G}$-grading is defined exactly as in the affine Fukaya category (Section 2.3.3). One might worry that L' intersects the divisors D_j, hence doesn't lie in $M \setminus \boldsymbol{D}$. However, we simply push L' off itself using ∇f (as in the definition of L^n_ϵ), and use the pushed-off version of L' for all grading (and index) computations – see [1, Proof of Proposition 5.3]. The point is that the grading of the relative Fukaya category arises from index computations of the relevant Fredholm operators, which are computed purely topologically.

Definition 2.6.3.4. If $(L_0, L_1) = (L', L')$, then we define the Floer datum to contain additional information, namely:

- the Morse function $f : S^n \to \mathbb{R}$;
- another Morse function $h : S^n \to \mathbb{R}$, with exactly two critical points.

One defines a **generator** of $CF^*_{\mathcal{F}'}(L', L')$ to be a critical point of one of the Morse functions f or h, and $CF^*_{\mathcal{F}'}(L', L')$ to be the $R/\mathfrak{m}^2$-module generated by these critical points. We identify the critical points of f as p_K for $K \subset [n]$ proper and non-empty, and the critical points of h as p_ϕ and $p_{[n]}$. Then $CF^*_{\mathcal{F}'}(L', L')$ is $\boldsymbol{G}$-graded, where p_K has the same grading as the corresponding generator θ^K of A, where A is the $\boldsymbol{G}$-graded algebra of Definition 2.2.2.12.

Lemma 2.6.3.5. *The $\boldsymbol{G}$-graded morphism spaces in $\mathcal{F}'$ are as follows:*

$$\begin{aligned} CF^*_{\mathcal{F}'}(L,L) &\cong CF^*_{\mathcal{F}'}(L',L') \\ &\cong A \otimes R/\mathfrak{m}^2 \\ &\cong \bigoplus_{K \subset [n]} R/\mathfrak{m}^2 \cdot p_K, \end{aligned}$$

and

$$\begin{aligned} CF^*_{\mathcal{F}'}(L,L') &\cong CF^*_{\mathcal{F}'}(L',L) \\ &\cong A \otimes R/\mathfrak{m}^2 \oplus CM^*(f) \otimes R/\mathfrak{m}^2 \\ &\cong \bigoplus_{K \subset [n]} R/\mathfrak{m}^2 \cdot p_K \oplus \bigoplus_{K \subset [n], K \neq \phi, [n]} R/\mathfrak{m}^2 \cdot q_K, \end{aligned}$$

where $CM^*(f)$ is the Morse complex of the function f. The $\boldsymbol{G}$-grading of generators labelled by $p_K \in A$ is as in Definition 2.2.2.12. The $\boldsymbol{G}$-grading of generators labelled by q_K is $(|K|-1, 0)$ (where $|K|-1$ is the Morse index of q_K).

Proof. See [1, Proof of Proposition 5.5]. □

Definition 2.6.3.6. In each morphism space, we call the generators labelled p_K, where $K \subset [n]$ is not equal to ϕ or $[n]$, the **flipping generators**, and the others (labelled q_K, p_ϕ or $p_{[n]}$) the **non-flipping generators**. The terminology comes from the definition of L^n as a perturbation of the double cover $S^n \to \mathbb{RP}^n \hookrightarrow \mathbb{CP}^n$. Flipping generators correspond to paths p from one sheet of the cover to the opposite sheet; non-flipping generators correspond to paths from one sheet to the same sheet.

Definition 2.6.3.7. A **perturbation datum** for a fixed flipping pearly tree $r \in \mathcal{R}_3(\boldsymbol{L})$ consists of the data (K^e, K^p, J), where:

- $K^e \in C^\infty(S^e, \mathcal{H}^e)$;
- $K^p \in \Omega^1(S^p, \mathcal{H}^p)$;

- $J \in C^\infty(S^p, \mathcal{J})$,

such that, for each boundary component C of a pearl in S with Lagrangian label L_C,

$$K^p(\xi)|_{L_C} = 0 \text{ for all } \xi \in TC \subset T(\partial S).$$

Definition 2.6.3.8. We say that a perturbation datum is **compatible with the Floer data** if, on each component of the thin regions of S^p and S^e, the perturbation datum agrees with the corresponding (translation-invariant) Floer datum. Explicitly, this means that:

- On each strip-like end of a pearl, (K^p, J) is given by the translation-invariant extension of the Floer datum $(H_{\boldsymbol{L}}, J_{\boldsymbol{L}})$;
- In a neighbourhood of each boundary marked point of a pearl, and also on all of each semi-stable pearl with both sides labeled L', we have $(K^p, J) = (0, J_0)$;
- On each thin region of a flipping edge, $K^e = f$;
- On each thin region of a non-flipping edge, $K^e = h$.

Remark 2.6.3.9. Note that, if our perturbation datum is compatible with the Floer data, then it is preserved by any automorphism of the flipping pearly tree.

Definition 2.6.3.10. We define the notion of a **compatible universal choice of perturbation data** for the moduli spaces $\mathcal{R}(\boldsymbol{L})$, by analogy with [11, Section 9i].

Definition 2.6.3.11. Let $\boldsymbol{L}$ be a tuple of objects of $\mathcal{F}'$, and $\boldsymbol{p}$ an associated set of generators. A **holomorphic flipping pearly tree $\boldsymbol{u}$** with ends on $\boldsymbol{p}$ consists of the following data:

- A flipping pearly tree $r \in \mathcal{R}_3(\boldsymbol{L})$;

- For each vertex $v \in V(T(r))$, a smooth map $u_v : S_v \to M$;
- For each edge $e \in E(r)$ with both sides labeled L', a smooth map $u_e : S_e \to S^n$;
- A continuous map $\tilde{u} : \partial S_r \to S^n$.

We impose the following requirements on these maps:

- $\boldsymbol{u}$ is asymptotic to the generators $\boldsymbol{p}$ along the strip-like ends and external edges;
- For each semi-stable vertex v, the map u_v satisfies the perturbed holomorphic curve equation
$$(Du_v - Y)^{0,1} = 0,$$
where, for $\xi \in TS$, $Y(\xi)$ is the Hamiltonian vector field of the function $K^p(\xi)$;
- The maps u_e satisfy the Morse flow equation
$$Du_e - \nabla K^e \quad = \quad 0;$$
- For each boundary component C of a pearl S_v,
$$L_C \circ \tilde{u}|_C = u_v|_C;$$
- For each edge e with both sides labeled L',
$$\tilde{u}|_{S_e \times \{0\}} = u_e,$$
and
$$\tilde{u}|_{S_e \times \{1\}} = \begin{cases} u_e & \text{if } e \text{ is non-flipping} \\ a \circ u_e & \text{if } e \text{ is flipping} \end{cases}$$
where we recall that $a : S^n \to S^n$ is the antipodal map.

- If $v \in V(T(r))$ is semi-stable, then the map u_v is non-constant.

Definition 2.6.3.12. Two holomorphic flipping pearly trees are **equivalent** if they are related by an automorphism of the domain (recall from Remark 2.6.3.9 that any automorphism of the domain preserves the perturbation datum and hence acts on the space of holomorphic flipping pearly trees).

Definition 2.6.3.13. Given a flipping holomorphic pearly tree $\boldsymbol{u}$ as defined above, one obtains a well-defined homology class $[\boldsymbol{u}] \in H_2(M, L^n)$ as follows (see Figure 2.6.2.2):

- Start with the continuous map $\boldsymbol{u} : S \to \mathbb{CP}^n$ associated with the flipping holomorphic pearly tree.
- Glue a thin strip along the boundary ∂S of the flipping pearly tree;
- If the boundary component or edge has label L, then it already gets mapped to L^n, so we map the strip into $\mathbb{CP}^n$ by making it constant along its width.
- If the boundary component or edge has label L', then by construction, there is a continuous lift $\tilde{u}$ of the boundary of the strip to S^n.
- Thus, we can map the strip into $\mathbb{CP}^n$ by letting it interpolate between the zero section and the graph of ϵdf in the Weinstein neighbourhood $D^*_\eta S^n$ used in the construction of L^n. Thus, boundary components of the strip with label L' now lie on L^n.

We now define the intersection number $\boldsymbol{u} \cdot D_j$ to be the topological intersection number of this class $[\boldsymbol{u}] \in H_2(\mathbb{CP}^n, L^n)$ with $D_j \in H_{2n-2}(\mathbb{CP}^n)$, and

$$\boldsymbol{u} \cdot \boldsymbol{D} := \sum_j ([\boldsymbol{u}] \cdot D_j)\, y_j \in Y.$$

Definition 2.6.3.14. Let $\boldsymbol{u}$ be a holomorphic flipping pearly tree. For each $v \in V(T(r))$, the map u_v defines a homology class in $H_2(\mathbb{CP}^n, \mathbb{RP}^n) \cong \mathbb{Z}$, because its boundary gets mapped to a Weinstein neighbourhood of $\mathbb{RP}^n$. We denote this homology class by $d_v \in \mathbb{Z}_{\geq 0}$ (it is non-negative because holomorphic disks have non-negative area). We denote the sum of all homology classes d_v by $d_{\boldsymbol{u}} \in \mathbb{Z}_{\geq 0}$.

Now we explain how to compute these intersection numbers in a simple way. It helps if the holomorphic flipping pearly trees are in general position, in the following sense:

Definition 2.6.3.15. We say that a holomorphic flipping pearly tree $\boldsymbol{u}$ is **in general position** if:

- Each boundary component C with label L' is transverse to the real hypersurfaces $D_j^{\mathbb{R}} \subset S^n$;
- No flipping marked points lie on the hypersurfaces $D_j^{\mathbb{R}}$.

Lemma 2.6.3.16. *Given a holomorphic flipping pearly tree $\boldsymbol{u}$, we can perturb the defining equations of the divisors D_j so that*

- *The intersection numbers $\boldsymbol{u} \cdot D_j$ do not change;*
- *$\boldsymbol{u}$ is in general position with respect to the perturbed divisors.*

Proof. See the proof of [1, Proposition 5.1]. □

If $\boldsymbol{u}$ is in general position, then we can split the surface defining our homology class $[\boldsymbol{u}]$ into regions $[u_v]$ corresponding to the pearls v, and $[u_e]$ corresponding to the edges e of the pearly tree, in such a way that the boundary of each such region does not intersect the divisors D_j. We cut the pearls off from the strips in the obvious way – since they are joined at boundary marked points, which don't lie on the hypersurfaces $D_j^{\mathbb{R}}$, the cuts we introduce do not intersect the divisors D_j.

Thus, each region defines a class in $H_2(M, M \setminus \boldsymbol{D})$, and $[\boldsymbol{u}] \cdot \boldsymbol{D}$ is equal to the sum of $[u_v] \cdot \boldsymbol{D}$ and $[u_e] \cdot \boldsymbol{D}$ over all pearls v and edges e of the pearly tree.

Lemma 2.6.3.17. *Let $\boldsymbol{u}$ be a holomorphic flipping pearly tree in general position. Then we have:*

- *For each non-flipping edge e, $[u_e] \cdot D_j = 0$;*
- *For each flipping edge e, $[u_e] \cdot D_j$ is equal to the topological intersection number of the edge $u_e : [0, l_e] \to S^n$ with the hypersurface $D_j^{\mathbb{R}}$ (this is non-negative because the gradient of the function f crosses all hypersurfaces $D_j^{\mathbb{R}}$ positively);*
- *For each pearl v, $[u_v] \cdot D_j$ is equal to the sum of the number of internal intersections of u_v with D_j (these are counted positively by positivity of intersections), together with $+1$ for each time a boundary lift $\tilde{u}|_C$ with label L' crosses $D_j^{\mathbb{R}}$ in the negative direction (and 0 if the lift crosses in the positive direction).*

Proof. See [1, Proposition 5.1]. ☐

Corollary 2.6.3.18. *If $\boldsymbol{u}$ is a flipping holomorphic pearly tree, then the intersection numbers $\boldsymbol{u} \cdot D_j$ are non-negative.*

Definition 2.6.3.19. Let $\boldsymbol{L}$ be a tuple, $\boldsymbol{p}$ an associated set of generators, and $\boldsymbol{d} \in Y_{\geq 0}$. We define $\mathcal{M}_3(\boldsymbol{p}, \boldsymbol{d})$ to be the moduli space of holomorphic flipping pearly trees $\boldsymbol{u}$ with labels $\boldsymbol{L}$ and ends on $\boldsymbol{p}$, and such that

$$\boldsymbol{u} \cdot \boldsymbol{D} = \boldsymbol{d},$$

modulo equivalence.

Proposition 2.6.3.20. *Let $\boldsymbol{L}$ be a tuple, $\boldsymbol{p}$ an associated set of generators. Suppose that $(p_{K_1}, \ldots, p_{K_k})$ is the tuple obtained from $\boldsymbol{p}$ by keeping only the flipping generators* *(see Definition 2.6.3.6). Then we have*

$$[\boldsymbol{u}] \cdot \boldsymbol{D} = q y_{[n]} + \sum_{j=1}^{k} y_{K_j}$$

for some $q \in \mathbb{Z}$, and the homology class of $[\boldsymbol{u}] \in H_2(\mathbb{CP}^n, \mathbb{RP}^n) \cong \mathbb{Z}$ is given by the formula

$$[\boldsymbol{u}] = 2q + k.$$

Proof. Follows from [1, Lemma 5.8], and a slight modification of [1, Proposition 5.10]. □

Definition 2.6.3.21. Let $\boldsymbol{u}$ be a holomorphic flipping pearly tree in general position, all of whose boundary components are labelled L'. For each flag f corresponding to a flipping edge of the tree, we have corresponding marked points in the boundary $(\partial S)_r$, namely

$$\tilde{m}_0(f) \sim \tilde{b}_0(f) \text{ and } \tilde{m}_1(f) \sim \tilde{b}_1(f).$$

The boundary map $\tilde{u}$ sends these points to antipodal regions $S^n_K, S^n_{\bar{K}}$ respectively, for some $K \subset [n]$ (recall that S^n_K is defined to be the region where $x_j < 0$ for $j \in K$ and $x_j > 0$ for $j \notin K$). We attach the labels K and $\bar{K}$ to the marked points $\tilde{m}_0(f)$ and $\tilde{m}_1(f)$, respectively. Figure 2.6.3.1 shows a possible labeling of a flipping holomorphic pearly tree.

Lemma 2.6.3.22. *Let $\boldsymbol{u}$ be a holomorphic flipping pearly tree with all sides labelled L', in general position, and equipped with labels as above. We decompose $[\boldsymbol{u}]$ into regions $[u_e]$, corresponding to edges, and $[u_v]$, corresponding to pearls, as before. Then we have:*

- *For each flipping edge e, the label K_0 at the start of the edge contains the label K_1 at the end of the edge, and*

$$[u_e] \cdot \boldsymbol{D} = y_{K_1} - y_{K_0};$$

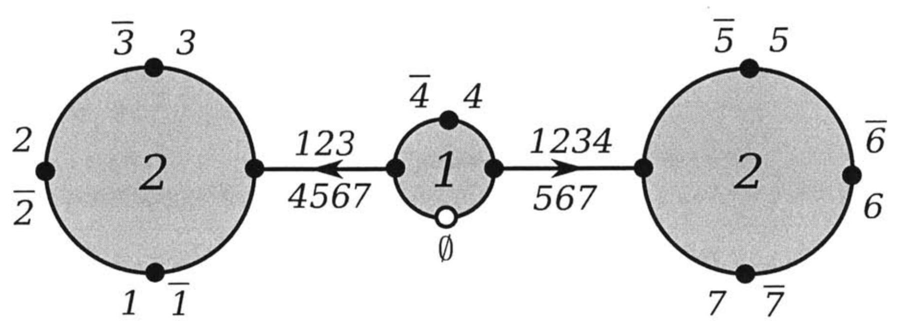


Figure 2.6.3.1: An example of a legal labeling of a flipping holomorphic pearly tree, which might contribute to the coefficient of p_ϕ in the A_∞ product $\mu^7(p_{\{1\}}, \ldots, p_{\{7\}})$. We have illustrated a simple case, in which all external flowlines are constant because the points $p_{\{j\}}$ are maxima of the Morse function f. The external label '1' means the set $\{1\}$, while '$\overline{1}$' means the complement $\{2, 3, 4, 5, 6, 7\}$. The big label '1' in the middle of a pearl means that the pearl has degree 1.

- *For each pearl v such that:*
  - *The homology class is $d_v \in H_2(\mathbb{CP}^n, \mathbb{RP}^n) \cong \mathbb{Z}$;*
  - *The points immediately after the flipping marked points of u_v have labels $K_1, \ldots, K_{k_v}$, if we traverse the boundary in positive direction (in other words, the points $\tilde{m}_1(f)$ for all outgoing flags, and $\tilde{m}_0(f)$ for the incoming flag);*

  then we have:
  - *For some $q_v \in \mathbb{Z}$,*

$$[\boldsymbol{u}] \cdot \boldsymbol{D} = q_v y_{[n]} + \sum_{j=1}^{k_v} y_{K_j};$$

  - *The homology class d_v satisfies*

$$d_v = 2q_v + k_v.$$

We recall that non-flipping edges u_e do not contribute to the intersection numbers.

Proof. For edges, the statement follows from the fact that ∇f only crosses divisors positively, and $[u_e]$ picks up an intersection point with D_j each time u_e crosses D_j

positively (compare [1, Figure 9(b)]). For pearls, the statement follows from Proposition 2.6.3.20. □

2.6.4 Transversality and compactness

Proposition 2.6.4.1. *For generic choice of Floer and perturbation data, the components of $\mathcal{M}_3(\boldsymbol{p}, \boldsymbol{d})$ with $|\boldsymbol{d}| \leq 1$ are regular, and have the structure of topological manifolds of the expected dimension.*

Proof. The moduli spaces $\mathcal{M}_3(\boldsymbol{p}, \boldsymbol{d})$ are constructed by gluing together pieces corresponding to the different possible underlying trees (see [1, Section 4.4]). If we are to obtain a topological manifold, we need each piece to be cut out transversely, and also for the 'seams' along which the pieces are glued (corresponding to holomorphic flipping pearly trees with nodal pearls, where a Morse flowline is about to form as in Figure 2.6.2.4) to be regular.

This amounts to checking the following:

- The Cauchy-Riemann operator $(Du^p - Y)^{0,1}$ on each stable pearl is surjective;
- For the semi-stable pearls, we require that the moduli space of J_0-holomorphic disks with two boundary marked points, modulo translation, is regular;
- The Morse flow operator $(Du^e - \nabla K^e)$ on each edge is surjective;
- For moduli spaces consisting of a single Morse edge, we require that the moduli space of Morse flowlines of f or h is regular;
- The restriction that marked points on pearls coincide with ends of edges is cut out transversely;

- The restriction that marked points on pearls coincide at a node (with both sides labelled L') is cut out transversely (this is the requirement that the 'seams' are regular).

The Cauchy-Riemann operators on pearls corresponding to stable vertices are generically surjective, by perturbing K^p (as in [11, Section 9k]). The moduli spaces of semi-stable pearls with sides labelled anything other than (L', L') are regular for generic choice of Floer data, by the arguments of [36, 37]. The semi-stable pearls with opposite sides labelled L' are different – we have required that $K^p = 0$ and $J = J_0$ on these pearls, which is not a generic condition. However, the moduli space of such pearls is regular by the 'automatic regularity' result of [40, Proposition 7.4.3] (the automatic regularity result deals with holomorphic spheres in $\mathbb{CP}^n = M$, and our moduli space of holomorphic disks is the real locus of this moduli space, hence also regular).

The Morse flow operators are surjective for generic choice of Floer and perturbation data.

The intersections of marked points on pearls with the endpoints of edges are generically transverse, by perturbing K^e near the end of the edge.

Now we deal with intersections of marked points at a node connecting two pearls, which are necessary to make the 'seams' along which we glue different parts of our moduli space. If one of the pearls involved is stable, then the intersection is transverse, because we can perturb K^p on the stable pearl (see [1, Section 4.6]) to move the marked point in any direction we please. A problem arises if both pearls are semi-stable (with both sides labelled L'). However, this situation does not arise in the moduli spaces $\mathcal{M}_3(\boldsymbol{p}, \boldsymbol{d})$ with $|\boldsymbol{d}| \leq 1$: any semi-stable pearl must contribute at least 1 to one of the intersection numbers $\boldsymbol{u} \cdot D_j$.

This follows from Lemma 2.6.3.22: if $[\boldsymbol{u}] \cdot \boldsymbol{D} = 0$ for a pearl with two boundary marked points with labels K_0 and K_1, then

$$qy_{[n]} + y_{K_0} + y_{K_1} = 0.$$

If $K_0 = \phi$ or $[n]$, then Lemma 2.6.3.22 shows that $d_v = 0$ so the unstable pearl u_v has zero energy and must be constant, which is not allowed. If $K_0 \neq \phi$ or $[n]$, then necessarily $q = -1$, and Lemma 2.6.3.22 shows that

$$d_v = 2q_v + k_v = 0,$$

so again u_v is constant. □

Proposition 2.6.4.2. *For generic choice of Floer and perturbation data, the components of $\mathcal{M}_3(\boldsymbol{p}, \boldsymbol{d})$ of virtual dimension ≤ 1, and with $|\boldsymbol{d}| \leq 1$, have the structure of compact topological manifolds with boundary, of the expected dimension. The boundary strata of the one-dimensional moduli spaces correspond to nodal holomorphic flipping pearly trees (see [1, Definition 4.33, 4.34]).*

Proof. This essentially follows from Gromov compactness, as outlined in [1, Proposition 4.6]. However, we must do a bit more work in this case: we must check that the number of semi-stable pearls is bounded, so that we are only gluing together finitely many pieces to make our moduli space (see Remark 2.6.2.9). This is true because the homology class $d_{\boldsymbol{u}}$ of any $\boldsymbol{u} \in \mathcal{M}_3(\boldsymbol{p}, \boldsymbol{d})$ is fixed, by Proposition 2.6.3.20, and any semi-stable pearl v contributes at least 1 to $d_{\boldsymbol{u}}$, because it is required to be non-constant.

Furthermore, we must rule out the possibility of sphere and disk bubbling in our moduli spaces. Sphere bubbling is easy to rule out: any non-constant sphere bubble must intersect each divisor D_j at least once, hence contribute at least n to $\boldsymbol{d}$. So sphere bubbling does not happen in moduli spaces with $|\boldsymbol{d}| \leq 1$.

Disk bubbling needs a little more work. Suppose a holomorphic disk bubbles off some pearl in a holomorphic flipping pearly tree. Let us denote by u_1 the disk, and by u_2 the rest of the holomorphic flipping pearly tree. We will assume that u_1 has boundary on L' (the case with boundary on L^n is not very different). We can regard this configuration as a holomorphic flipping pearly tree with a vertex v of valence 1, connected by a Morse edge of length 0 to the rest of the holomorphic flipping pearly tree. We will show that, if u_1 is non-constant, then its virtual dimension (by which we mean the virtual dimension of the moduli space in which it lies) is $\geq n+1$. This is the same as showing that its Maslov index is at least 3. This will show that disk bubbling generically does not happen, because the rest of the holomorphic flipping pearly tree has virtual dimension < 0.

Firstly, by Lemma 2.6.3.22 (which works exactly the same if there are pearls of valence 1), if the edge is non-flipping then the disk has zero energy, and hence is constant.

So let us assume that the edge is flipping, and the label attached to it is $K \subset [n]$ (as in Definition 2.6.3.21). By Lemma 2.6.3.22, we have

$$[u_1] \cdot \boldsymbol{D} = q y_{[n]} + y_K.$$

Because $|\boldsymbol{d}| \leq 1$, it must be that $[u_1] \cdot \boldsymbol{D} = y_j$, $q = 0$, and $K = \{j\}$ for some $j \in [n]$. It follows, by Lemma 2.6.3.22, that the homology class of u_1 is $d_1 = 1$. Thus, the virtual dimension of u_1 is (by the real analogue of Lemma 2.4.4.3)

$$\text{v.d.}(u_1) = n + d_1(n+1) - 3 + 1 = 2n - 1 \geq n + 1$$

(because $n \geq 2$).

The connect sum formula now says that the virtual dimension of $\boldsymbol{u}$ is

$$\text{v.d.}(\boldsymbol{u}) = \text{v.d.}(u_1) + \text{v.d.}(u_2) + 1 - n \leq 1$$

(since we are considering moduli spaces of virtual dimension ≤ 1), and hence

$$\text{v.d.}(u_2) \le n - \text{v.d.}(u_1) \le -1.$$

We saw in Proposition 2.6.4.1 that moduli spaces of holomorphic flipping pearly trees are generically regular, hence a moduli space of virtual dimension < 0 is generically empty. So we may conclude that no disk bubbling occurs for generic choices of Floer and perturbation data. $\square$

2.6.5 Morse-Bott model for the first-order Fukaya category

This section contains the proof of Proposition 2.6.1.2.

We define a $\boldsymbol{G}$-graded A_∞ category $\mathcal{F}'$ over $R/\mathfrak{m}^2$, with two objects, L and L'. The $\boldsymbol{G}$-graded morphism spaces $CF^*(L_0, L_1)$ are the $R/\mathfrak{m}^2$-module freely generated by their generators, and are described explicitly in Lemma 2.6.3.5. Given a tuple $\boldsymbol{L}$ with associated generators $\boldsymbol{p} = (p_0, p_1, \ldots, p_s)$, the coefficient of $r^{\boldsymbol{d}} p_0$ in the A_∞ product $\mu^s(p_s, \ldots, p_1)$ is given by the signed count of holomorphic flipping pearly trees in the zero-dimensional component of the moduli space $\mathcal{M}_3(\boldsymbol{p}, \boldsymbol{d})$ (by analogy with the usual definition of the Fukaya category). By the standard argument, the composition maps μ^s satisfy the A_∞ relations.

Furthermore, they are $\boldsymbol{G}$-graded, by the same argument as for the relative Fukaya category (keeping in mind Remark 2.6.3.3). To see why, recall that the grading depended on the index theory of a Cauchy-Riemann operator coming from a bundle pair (D^2, u^*TM, F), where $u : D^2 \to M \setminus \boldsymbol{D}$ was some smooth map, and F a lift of ∂u to $\mathcal{G}(M \setminus \boldsymbol{D})$. Given a holomorphic flipping pearly tree $\boldsymbol{u}$, we can construct (by a modification of the construction of the smooth surface representing the homology class $[\boldsymbol{u}]$) a smooth boundary-punctured disk $\tilde{\boldsymbol{u}}$ mapping to $M \setminus \boldsymbol{D}$, with boundary on L^n, such that the index of the associated Cauchy-Riemann operator is equal to the index of the Fredholm operator defining the moduli space of holomorphic flipping pearly trees near $\boldsymbol{u}$. Note that $\tilde{\boldsymbol{u}}$ will **not** be holomorphic itself: we are simply using it to compute the index topologically.

Lemma 2.6.5.1. *For small enough $\epsilon > 0$, the objects L^n_ϵ and L' are quasi-isomorphic in $\mathcal{F}'/\mathfrak{m}$.*

Proof. See [1, Proposition 5.5]. $\square$

Lemma 2.6.5.2. *For appropriate choices of perturbation data, there is a strict isomorphism*

$$CF^*_{\mathcal{F}(M,\boldsymbol{D})/\mathfrak{m}^2}(L^n, L^n) \cong CF^*_{\mathcal{F}'}(L, L).$$

Proof. Suppose that $\boldsymbol{L}$ is a tuple, all of whose entries are L^n, $|\boldsymbol{L}| \geq 3$, and ℓ is a labelling such that $\boldsymbol{d}(\ell) = y_j$ for some j. Then there is a forgetful map $\mathcal{R}(\boldsymbol{L}, \ell) \to \mathcal{R}_3(\boldsymbol{L})$, obtained by simply forgetting the internal marked point. Suppose that we choose Floer and perturbation data on the moduli space $\mathcal{R}(\boldsymbol{L}, \ell)$ by pulling back the corresponding data from $\mathcal{R}_3(\boldsymbol{L})$.

With these assumptions, if $\boldsymbol{p}$ is a set of generators associated with the tuple $\boldsymbol{L}$, then there is a forgetful map $\mathcal{M}(\boldsymbol{p}, \ell) \to \mathcal{M}_3(\boldsymbol{p}, e_j)$. This map is clearly bijective: given a holomorphic disk with topological intersection number 1 with the divisor D_j, there exists a unique internal point which gets mapped to D_j (uniqueness follows from positivity of intersections).

If $|\boldsymbol{L}| = 2$, we choose the perturbation data on the moduli space $\mathcal{R}(\boldsymbol{L}, \ell)$ to be translation-invariant, like for $\mathcal{R}_3(\boldsymbol{L}, \ell)$. The definitions of the two moduli spaces now are different: $\mathcal{M}_3(\boldsymbol{p}, e_j)$ is a quotient of a moduli space of holomorphic strips by translation, whereas $\mathcal{M}(\boldsymbol{p}, \ell)$ is a moduli space of holomorphic strips with an internal marked point getting mapped to D_j. However, this map is still bijective: given a strip with topological intersection number 1 with divisor D_j, there is a unique internal point which gets mapped to D_j. Introducing this as a marked point defines a class in $\mathcal{M}(\boldsymbol{p}, \ell)$, and the result is independent of translation of the domain in the original strip.

Observe that, because the moduli spaces $\mathcal{M}_3(\boldsymbol{p}, e_j)$ are regular, the moduli spaces $\mathcal{M}(\boldsymbol{p}, \ell)$ are regular, even though we have made a non-generic choice of perturbation data. So it is legitimate to compute coefficients in the relative Fukaya category using the moduli spaces $\mathcal{M}(\boldsymbol{p}, \ell)$ defined using these Floer and perturbation data. $\square$

Now we recall (Lemma 2.6.3.5) that the underlying $\boldsymbol{G}$-graded vector space of $CF^*_{\mathcal{F}'}(L', L')$ is A. So $CF^*_{\mathcal{F}'}(L', L')$ has the form (A, μ^*).

Lemma 2.6.5.3. *As a $\boldsymbol{G}$-graded algebra, we have*

$$(A, \mu_0^2) \cong (A, \wedge),$$

where '$\wedge$' denotes the usual product on the exterior algebra.

Proof. See [1, Theorem 5.12]. $\square$

Now recall from Section 2.2.5 that there is a map

$$\Phi : CC^*_{\boldsymbol{G}}(A, A \otimes R) \to p^* \left(\mathbb{C}[[U]] \otimes A \otimes R \right).$$

Lemma 2.6.5.4. *We have*

$$\Phi\left(\mu^*\right) = u_1 \dots u_n + \sum_{j=1}^{n} r_j u_j + \mathfrak{m}^2$$

(up to some signs).

Proof. See [1, Proposition 5.15] for the proof that

$$\Phi(\mu_0^*) = \pm u_1 \dots u_n.$$

Our aim now is to calculate the first-order terms in $\Phi(\mu^*)$. By Lemma 2.2.5.11, we know that the degree-2 part of

$$HH_G^*(A, A \otimes R^1) \cong p^* \left(\mathbb{C}[[U]] \otimes A \otimes R^1\right)$$

is generated by the elements $r_j u_j$. Thus, the first-order part of $\Phi(\mu^*)$ can be written as

$$\sum_{j=1}^{n} c_j r_j u_j$$

for some numbers $c_j \in \mathbb{C}$.

The number c_j is given by the count of holomorphic flipping pearly trees $\boldsymbol{u}$ in the moduli space $\mathcal{M}_3((p_\phi, p_{\{j\}}), e_j)$. Such a holomorphic flipping pearly tree must be a chain of semi-stable J_0-holomorphic pearls (see Definition 2.6.3.8).

By Proposition 2.6.3.20, the homology class of such $\boldsymbol{u}$ is 1. Because semi-stable pearls are not allowed to be constant, this means there can only be a single semi-stable pearl. This pearl is a J_0-holomorphic disk with boundary on $\mathbb{RP}^n$,

$$u_v : (D, \partial D) \to (\mathbb{CP}^n, \mathbb{RP}^n),$$

together with two marked boundary points, considered up to translation. The homology class $[u_v] \in H_2(\mathbb{CP}^n, \mathbb{RP}^n)$ is 1. Thus, u_v is one half of a J_0-holomorphic sphere of degree 1 in $\mathbb{CP}^n$. That is, it is one half of a complex line in $\mathbb{CP}^n$, and its boundary is a real line in $\mathbb{RP}^n$.

The Morse index of the input $p_{\{j\}}$ is n (see [1, Corollary 2.11]), so the corresponding

Morse edge must be constant: this means we just have a point constraint that our real line must pass through the point $L'(p_{\{j\}})$. Similarly, the Morse index of p_ϕ is 0, so the corresponding Morse edge must be constant: this means we have a point constraint that our real line must pass through the point $L'(p_\phi)$. There is a unique real line through the points $L'(p_{\{j\}})$ and $L'(p_\phi)$, so the pearl must be one half of the corresponding complex line.

Furthermore, the pearl must admit a lift $\tilde{u}_v$ of the boundary to S^n, which changes sheets at $p_{\{j\}}$ but not at p_ϕ. Since u_v is one half of a complex line, this lift $\tilde{u}_v$ must be one half of a great circle, from $p_{\{j\}}$ to its antipode $p_{\overline{\{j\}}}$, and passing through the point p_ϕ. Clearly, there is exactly one such half-great circle. Furthermore, the orientation of this half-great circle determines uniquely which half of the complex line we must take. Thus, we have uniquely determined our holomorphic flipping pearly tree $\boldsymbol{u}$.

It follows from Proposition 2.6.3.20 that $[\boldsymbol{u}] \cdot \boldsymbol{D} = y_j$.

We have shown that the moduli space $\mathcal{M}((p_{\{j\}}, p_\phi), e_j)$ contains a unique element $\boldsymbol{u}$. Thus, each coefficient c_j must be ± 1. $\square$

Remark 2.6.5.5. We could also have calculated $[\boldsymbol{u}] \cdot \boldsymbol{D}$ using Lemma 2.6.3.17, and the exercise helps us to get a picture of $\boldsymbol{u}$: the edges u_e are constant, hence do not contribute to $[\boldsymbol{u}] \cdot \boldsymbol{D}$. The pearl u_v does not intersect any divisor D_i in its interior (it is half of a complex line, hence intersects the divisor D_i exactly once, and that intersection is on the real locus $\mathbb{RP}^n$). So we get no contributions to $[\boldsymbol{u}] \cdot \boldsymbol{D}$ from interior intersection points. The boundary lift $\tilde{u}_v$ moves along a great circle from $p_{\{j\}}$ to $p_{\overline{\{j\}}}$, hence crosses the divisors D_i positively for $i \neq j$, and the divisor D_j once negatively. Therefore, we have $[\boldsymbol{u}] \cdot \boldsymbol{D} = y_j$. See Figure 2-3(b) for the picture in the one-dimensional case.

This completes the proof of Proposition 2.6.1.2.

Corollary 2.6.5.6. *Let us denote*

$$\mathcal{A} := CF^*_{\mathcal{F}(\phi)}(L^n, L^n),$$

where $\mathcal{F}(\phi)$ is the category of Proposition 2.5.2.2 and $\phi : (M^n, \boldsymbol{D}) \to (M, \boldsymbol{D})$ is the branched cover of Example 2.3.5.10. Then $\mathcal{A}$ satisfies all of the conditions required to be of type A, in the sense of Definition 2.2.5.2, except it may not be strictly H-equivariant.

Remark 2.6.5.7. If one is willing to accept that $CF^*_{\mathcal{F}(M,\boldsymbol{D},\boldsymbol{a})}(L^n, L^n)$ can be defined to all orders, despite L^n being immersed (compare Remark 2.6.1.1), then we could substitute it for $CF^*_{\mathcal{F}(\phi)}(L^n, L^n)$ in the statement of Corollary 2.6.5.6. We prefer to work with $\mathcal{F}(\phi)$ because it does not require us to assume a general statement of Gromov compactness for immersed Lagrangians.

Proof. It follows from Proposition 2.6.1.2 and Proposition 2.5.2.2 that there is a quasi-isomorphism

$$\mathcal{A}/\mathfrak{m} \cong CF^*_{\mathcal{F}'/\mathfrak{m}}(L', L'),$$

because

$$\mathcal{F}(M, \boldsymbol{D}, \boldsymbol{a})/\mathfrak{m} \cong \mathcal{F}(M, \boldsymbol{D})/\mathfrak{m}.$$

It follows that the underlying $\boldsymbol{G}$-graded vector space of $\mathcal{A}$ is A, and the product μ_0^2 is the exterior product. Furthermore, if $\mathcal{A} = (A, \mu^*)$, it follows that

$$\Phi(\mu^*) = u_1 \ldots u_n + \mathfrak{m}$$

(see Definition 2.2.5.1).

Now let $\mathcal{A} := \mathcal{A}/\mathfrak{m}$. We recall from Lemma 2.2.5.8 that the spectral sequence induced by the length filtration on $CC^*(\mathcal{A})$ has E_2 page

$$E_2^{*,*} \cong HH_c^*(A),$$

and converges to $HH^*(\mathcal{A})$. It follows from Proposition 2.6.1.2 that the first-order deformation class of

$$CF^*_{\mathcal{F}(M,\boldsymbol{D})}(L^n, L^n)$$

is given by

$$[\mu_1] = \sum_{j=1}^{n} r_j u_j + (\text{higher-order in length filtration}).$$

It follows from Theorem 7, and the fact that the spectral sequence induced by the length filtration respects the multiplication given by the Yoneda product, that the first-order deformation class of

$$CF^*_{\mathcal{F}(M,\boldsymbol{D},\boldsymbol{a})}(L^n, L^n)$$

is given by

$$[\mu_{1,\boldsymbol{a}}] = \sum_{j=1}^{n} r_j u_j^n + (\text{higher-order in length filtration}).$$

It follows that

$$\Phi(\mu^*) = u_1 \ldots u_n + \sum_{j=1}^{n} r_j u_j^n + \mathfrak{m}^2,$$

from which the result follows, by Proposition 2.5.2.2. □

We now have to deal with the fact that $\mathcal{A}$ may not be strictly H-equivariant. It is clear that H acts on $(M, \boldsymbol{D})$, preserving the anchored Lagrangian brane L^n, because all of our constructions of $(M, \boldsymbol{D})$ and L^n have been symmetric with respect to permuting the coordinates. However, it may not be possible to choose our perturbation data H-equivariantly and still achieve transversality, so we may only have H-equivariance 'up to homotopy'. We can fix this using the arguments of Appendix B, which says that we can replace $\mathcal{A}$ by a quasi-equivalent algebra which is strictly H-equivariant: essentially, we just apply the proof of Theorem 6 to this strictly H-equivariant replacement.

Corollary 2.6.5.8. *Suppose that $\mathcal{B}$ is a $\boldsymbol{G}$-graded A_∞ algebra over R of type A. Then* *there exists $\psi \in \mathrm{Aut}(R)$, and an A_∞ quasi-isomorphism*

$$\mathcal{B} \cong \psi \cdot \mathcal{A}.$$

Proof. First, observe that Corollary 2.2.5.7 does not require strict H-equivariance, so there is a quasi-isomorphism

$$\mathcal{A}/\mathfrak{m} \cong \mathcal{B}/\mathfrak{m} =: \mathcal{A}.$$

Now, by a version of Proposition B.0.2.7 the subcategory of $\mathcal{F}(\phi)$ with object L^n embeds, H-equivariantly, into a strictly H-equivariant A_∞ category, in such a way that the order-0 component of the embedding is a quasi-equivalence. Now recall that $\mathcal{A}$ is necessarily minimal (by Lemma 2.2.5.9), and it follows easily from the proof of Proposition B.0.2.7 that this strictly H-equivariant A_∞ category can be chosen to be minimal too. Any A_∞ functor between minimal A_∞ categories, whose order-0 component is a quasi-equivalence, is necessarily a quasi-equivalence. Using the fact that quasi-equivalences of minimal A_∞ categories can be inverted (Lemma 2.2.3.34), we can apply [9, Lemma 4.3] to prove that $\mathcal{A}$ is quasi-equivalent to a strictly H-equivariant A_∞ algebra of type A. The result now follows from Theorem 6. □

Definition 2.6.5.9. We denote

$$\widetilde{\mathcal{A}} \subset \mathcal{F}(M^n, \boldsymbol{D})$$

the full $\boldsymbol{G}^n_n$-graded subcategory whose objects are the lifts of L^n.

Corollary 2.6.5.10. *If $\mathcal{B}$ is any A_∞ algebra of type A, then there exists $\psi[[T]] \in \mathrm{Aut}(R)$ and an A_∞ quasi-isomorphism of $\boldsymbol{G}^n_n$-graded R-linear A_∞ categories,*

$$\widetilde{\mathcal{A}} \cong \psi \cdot \boldsymbol{p}_1^* \underline{\mathcal{B}}.$$

Proof. Consider the branched cover

$$\phi : (M^n_n, \boldsymbol{D}) \to (M^n_1, \boldsymbol{D})$$

of Example 2.3.5.10. By Proposition 2.5.2.2, there is a fully faithful embedding

$$\boldsymbol{p}_1^* \mathcal{F}(\phi) \to \mathcal{F}(M^n, \boldsymbol{D}),$$

and in particular we have a quasi-equivalence

$$\boldsymbol{p}_1^* \underline{\mathcal{A}} \cong \widetilde{\mathcal{A}}.$$

The result now follows from Corollary 2.6.5.8. □

2.7 The B-model

The aim of this section is to prove Theorem 5.

2.7.1 Homological perturbation lemma

We will use a version of the homological perturbation lemma which is not quite the usual one (for which see, for example, [66], [67]), but rather the slightly modified version used in [20], so we feel it is as well to state it.

Suppose we are given:

- An A_∞ algebra (B, μ^*) (over a $\mathbb{C}$-algebra R);
- A map $\partial : B \to B$ that is a Maurer-Cartan element for (B, μ^*), in the sense that

$\tilde{\mu}^* := (\mu^1 + \partial, \mu^2, \ldots)$ is an A_∞ structure on B;

- A chain complex (C, d_C);
- Chain maps

$$(C, d_C) \underset{p}{\overset{i}{\rightleftarrows}} (B, \mu^1);$$

- A map

$$h : B \to B,$$

such that

- $pi = \mathrm{id}$;
- h defines a homotopy between ip and the identity, which just means that

$$ip = \mathrm{id} - [\mu^1, h];$$

- the **side conditions**

$$\begin{aligned} h^2 &= 0, \\ hi &= 0, \text{ and} \\ ph &= 0. \end{aligned}$$

  are satisfied;
- there exists some integer N such that $(\partial h)^N = 0$.

Then we construct:

- An A_∞ structure ν^* on C;

- An A_∞ morphism I^* from $(B, \tilde{\mu}^*)$ to (C, ν^*);
- An A_∞ morphism P^* from (C, ν^*) to $(B, \tilde{\mu}^*)$,

such that I^* and P^* are mutually inverse A_∞ quasi-isomorphisms. In fact, we can show that $P^1 \circ I^1 = \mathrm{id}$, and we can construct an A_∞ homotopy H^* such that

$$I^* \circ P^* = \mathrm{id} - [\tilde{\mu}^*, H^*].$$

This result is proved in the case $\partial = 0$ in [66]. The operations ν^*, I^*, P^*, H^* are defined by certain counts over stable directed planar trees (we use the opposite orientation convention from Definition 2.4.2.1, so trees have s incoming edges and a single outgoing edge). We attach the operation μ^k to each vertex of arity k (arity = number of incoming edges). If $\partial \neq 0$, then we make exactly the same construction, but sum instead over **semistable** directed planar trees, and attach the operation ∂ to each vertex of arity 1. The assumption that $(\partial h)^N = 0$ ensures that we need only sum over a finite number of trees, because a tree with a sufficiently long chain of vertices of arity 1 does not contribute to the sum.

For example, to define ν^s, we sum over semistable directed planar trees with s incoming edges. We attach operations to each vertex and edge of such a tree, as follows (omitting signs):

- to each vertex of arity 1, attach ∂;
- to each vertex of arity $k \geq 2$, attach μ^k;
- to each internal edge, attach h;
- to each incoming edge, attach i;
- to each outgoing edge, attach p.

Composing the operations as prescribed by the tree determines a map $C^{\otimes s} \to C$. Summing these maps, over all such trees, defines ν^s.

The modifications in the definitions of I^*, P^*, H^*, and the proofs that ν^* is an A_∞ structure, that I^* and P^* are A_∞ morphisms, and that H^* defines an A_∞ homotopy from $I^* \circ P^*$ to id, should all be clear from [66]. The fact that $P^1 \circ I^1 = \mathrm{id}$ follows easily from the side conditions.

2.7.2 Matrix factorization computations

Let k be a field of characteristic 0, and R a commutative k-algebra. Consider the polynomial R-algebra $S := R[x_1, \ldots, x_n]$. Suppose we are given $w \in S$. We consider the differential $\mathbb{Z}_2$-graded category of matrix factorizations $MF(S, w)$. Objects are finitely-generated free $\mathbb{Z}_2$-graded S-modules K, equipped with a 'differential' $\delta_K : K \to K$ of odd degree such that $\delta_K^2 = w \cdot \mathrm{id}$. Morphisms are S-module homomorphisms, with the standard differential and compositions.

By [68, Theorem 3.9], there is an exact equivalence between $\mathrm{Ho}(MF(S, w))$ (where 'Ho' denotes the homotopy category) and Orlov's 'derived category of singularities' $D^b_{\mathrm{Sing}}(w^{-1}(0))$. We consider a matrix factorization (B, δ_B) which corresponds to the ideal

$$\mathcal{O}_0 := (x_1, \ldots x_n)$$

under this equivalence. Following the method described in [25, Section 2.3], we take B to be the free finitely-generated $\mathbb{Z}_2$-graded algebra generated by odd supercommuting variables $\theta_1, \ldots, \theta_n$ (with S in even degree). That is,

$$B := S[\theta_1, \ldots, \theta_n].$$

We define the differential on B to be

$$\delta := \delta_0 + \delta_1,$$

where

$$\begin{aligned} \delta_0 &= \sum_{j=1}^{n} x_j \frac{\partial}{\partial \theta_j}, \\ \delta_1 &= \sum_{j=1}^{n} w_j \theta_j, \end{aligned}$$

where $w_j \in S$ are elements chosen such that

$$w = \sum_{j=1}^{n} w_j x_j.$$

Observe that

$$\begin{aligned} \delta_0^2 &= 0, \\ \delta_1^2 &= 0, \text{ and} \\ [\delta_0, \delta_1] &= w. \end{aligned}$$

It follows that $\delta^2 = w \cdot \mathrm{id}$ as required.

Now consider the differential $\mathbb{Z}_2$-graded algebra

$$\mathcal{B} := \mathrm{Hom}^*_{MF(S,w)}(B, B).$$

Again following [25, Section 2.3], we take the underlying vector space to be the algebra of differential operators

$$\mathcal{B} := S\left[\theta_1, \ldots, \theta_n, \frac{\partial}{\partial \theta_1}, \ldots, \frac{\partial}{\partial \theta_n}\right],$$

acting on B in the obvious way, with the natural multiplication and the differential $d := d_0 + d_1$, where $d_j := [\delta_j, -]$. $\mathcal{B}$ is freely generated, as an R-module, by generators $x^{\boldsymbol{b}}\theta^J\partial^K$, where $\boldsymbol{b} = (b_1, \ldots, b_n)$ is a multi-index, $J \subset [n]$, $K \subset [n]$. We will use the shorthand ∂_j for $\partial/\partial\theta_j$.

Now we use the homological perturbation lemma to construct a minimal A_∞ model for $\mathcal{B}$.

To put ourselves in the situation of Section 2.7.1, let us consider $\mathcal{B}$ with the A_∞ (in fact, differential graded) structure given by the differential $\mu^1 = d_0$ and standard multiplication μ^2, and let $\partial := d_1$. Then $(\mathcal{B}, \tilde{\mu}^*) = \mathcal{B}$. Furthermore, let

$$\begin{aligned}
C &:= R[\partial_1, \ldots, \partial_n],\\
d_C &= 0,\\
i : (C, d_C) &\to (\mathcal{B}, d_0) \text{ the obvious inclusion},\\
p : (\mathcal{B}, d_0) &\to (C, d_C) \text{ the projection defined by}\\
p(x^{\boldsymbol{b}}\theta^J\partial^K) &= \begin{cases} \partial^K & \text{if } \boldsymbol{b} = 0 \text{ and } J = \phi, \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

so that $pi = \mathrm{id}$. We define

$$\begin{aligned}
\tilde{h} : B &\to B,\\
\tilde{h}(f\theta^J\partial^K) &:= \left(\sum_{j=1}^{n} \frac{\partial f}{\partial x_j}\theta_j\right)\theta^J\partial^K.
\end{aligned}$$

One can check that

$$[d_0, \tilde{h}]\left(x^{\boldsymbol{b}}\theta^J\partial^K\right) = (|\boldsymbol{b}| + |J|)\left(x^{\boldsymbol{b}}\theta^J\partial^K\right).$$

Therefore, if we define

$$h\left(x^{\boldsymbol{b}}\theta^J\partial^K\right) := \begin{cases} 0 & \text{if } \boldsymbol{b}=0 \text{ and } J=\phi, \\ \frac{1}{|\boldsymbol{b}|+|J|}\tilde{h}\left(x^{\boldsymbol{b}}\theta^J\partial^K\right) & \text{otherwise,} \end{cases}$$

then we have

$$ip = \mathrm{id} - [d_0, h].$$

Furthermore, we can check that the side conditions are satisfied:

$$\begin{aligned} h^2 &= 0 \text{ (for the same reason the exterior derivative squares to 0)}, \\ hi &= 0, \text{ and} \\ ph &= 0. \end{aligned}$$

Finally, observe that $h\partial$ decreases the grading $|x^{\boldsymbol{b}}\theta^J\partial^K| := |K|$ by 1, so $(h\partial)^{n+1} = 0$.

Thus, we can apply Section 2.7.1 to construct an A_∞ structure ν^* on C, which is quasi-isomorphic to $\mathcal{B}$. If w has degree ≥ 3, then the differential $\nu^1 = 0$ and the product ν^2 is the standard (exterior algebra) product on C. Thus, $\nu^{\geq 3}$ defines a Maurer-Cartan element in the Hochschild cochain complex $CC^*(C)$.

Now we recall, from Definition 2.2.5.1, the Hochschild-Kostant-Rosenberg map from the Hochschild cohain complex to the space of polyvector fields,

$$\Phi : CC^*(C) \to S[\partial_1, \ldots, \partial_n],$$

given by

$$\Phi(\alpha) := \sum_{s\geq 0} \alpha^s(\boldsymbol{x}, \ldots, \boldsymbol{x}),$$

where we denote

$$\boldsymbol{x} := \sum_{j=1}^n x_j \partial_j.$$

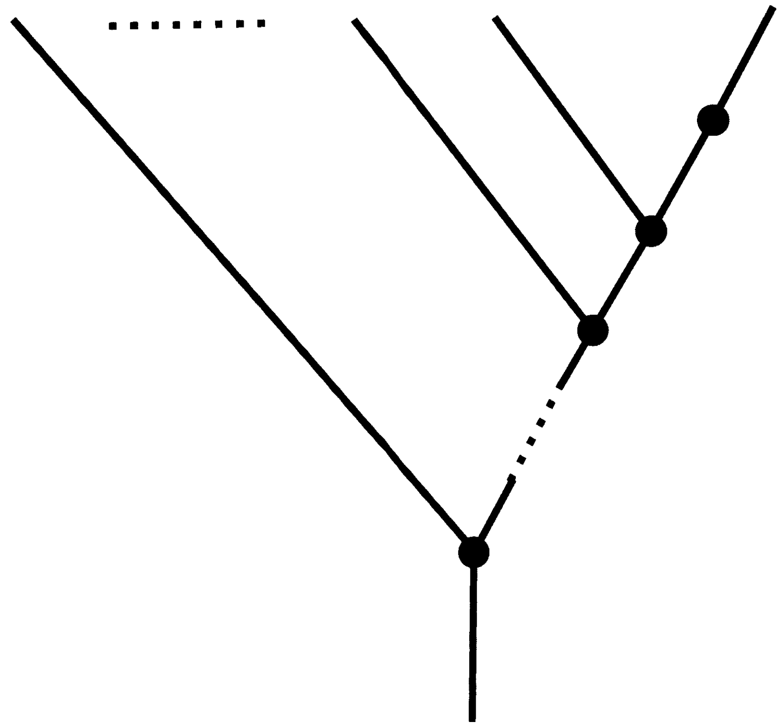


Figure 2.7.2.1: The only trees contributing to the homological perturbation lemma computation.

Proposition 2.7.2.1. *(see [21, Proposition 7.1]) The image of the Maurer-Cartan element $\nu^{\geq 3}$ under Φ is exactly the superpotential w.*

Proof. We recall the construction of the maps ν^k from Section 2.7.1, by summing over trees. The only trees that give a non-zero contribution to a product $\nu^k(\partial_{i_1}, \ldots, \partial_{i_k})$ are those depicted in Figure 2.7.2.1.

Given such a tree, with inputs $\partial_{i_1}, \ldots, \partial_{i_k}$, one easily determines the output: it is the constant term of

$$\partial_{i_k} \ldots \partial_{i_2} w_{i_1},$$

divided by $(k-1)!$ (coming from the terms in the denominator of h). Given a monomial $x^{\boldsymbol{b}}$ in w_j, there are $|\boldsymbol{b}|!/\boldsymbol{b}!$ ways of choosing the order of the inputs ∂_{i_m} we take from the term

$$\nu^{|\boldsymbol{b}|+1}(\boldsymbol{x}, \dots, \boldsymbol{x})$$

of $\Phi(\nu^{\geq 3})$, in such a way that $j = i_1$ comes first, and the constant term of

$$\partial_{i_k} \dots \partial_{i_2} w_j$$

is non-zero. Thus, each monomial $x^{\boldsymbol{b}}$ of w_j contributes a term

$$x_j \frac{1}{|\boldsymbol{b}|!} \frac{|\boldsymbol{b}|!}{\boldsymbol{b}!} \left(\partial^{\boldsymbol{b}} x^{\boldsymbol{b}}\right) x^{\boldsymbol{b}} = x_j x^{\boldsymbol{b}}$$

to $\Phi(\nu^{\geq 3})$, and the result follows. □

2.7.3 G-Graded matrix factorizations

For the purposes of this section, let $\boldsymbol{G}$ be a grading datum:

$$\mathbb{Z} \xrightarrow{f} Y \to X \to 0,$$

with sign morphism $\boldsymbol{\sigma}$.

Let S be a $\boldsymbol{G}$-graded algebra such that $\boldsymbol{\sigma}_* S$ is concentrated in degree $0 \in \mathbb{Z}_2$, and let $w \in S$ be an element of degree $f(2) \in Y$.

Definition 2.7.3.1. A **$\boldsymbol{G}$-graded matrix factorization** of $w \in S$ is a $\boldsymbol{G}$-graded finitely-generated free S-module K, together with a homomorphism

$$\delta_K \in \mathrm{Hom}_S(K, K)$$

of degree $f(1) \in Y$, such that

$$\delta_K^2 = w \cdot \mathrm{id}.$$

Definition 2.7.3.2. We define the **differential $\boldsymbol{G}$-graded category of matrix factorizations**, $MF^{\boldsymbol{G}}(S,w)$:

- Objects are $\boldsymbol{G}$-graded matrix factorizations of w;
- Morphisms are S-module homomorphisms:

$$\mathrm{Hom}((K,\delta_K),(L,\delta_L)) := \mathrm{Hom}_S(K,L);$$

- Differential on morphism spaces is as usual:

$$\partial(F) := \delta_L \circ F - (-1)^{\sigma(F)} F \circ \delta_K;$$

- Composition is composition of S-module homomorphisms.

We note that the morphism spaces are naturally $\boldsymbol{G}$-graded S-modules, the differential and composition maps have degrees $f(1)$ and $f(0) \in Y$ respectively, and they satisfy the Leibniz rule. It follows that $MF^{\boldsymbol{G}}(S,w)$ is a $\boldsymbol{G}$-graded A_∞ category over S (see Remark 2.2.3.12). In fact it is a differential $\boldsymbol{G}$-graded category, since all $\mu^{\geq 3}$ are zero.

We observe that ordinary matrix factorizations are nothing more than $\boldsymbol{G}_\sigma$-graded matrix factorizations. It follows that there is a fully faithful embedding

$$\boldsymbol{\sigma}_* MF^{\boldsymbol{G}}(S,w) \to MF(S,w).$$

Now let us introduce our main example. We will introduce graded matrix factorizations mirror to the smooth orbifold relative Fukaya category (compare Section 2.6.1). Let $\boldsymbol{G} := \boldsymbol{G}_1^n$ be the grading datum introduced in Example 2.2.1.13. Let

$$R := R_n^n \cong \mathbb{C}[[r_1,\dots,r_n]]$$

be the $\boldsymbol{G}$-graded power series ring introduced in Definition 2.2.2.13.

Let U be the $\boldsymbol{G}$-graded vector space of Example 2.2.2.9. We consider the $\boldsymbol{G}$-graded algebra

$$S := R[U] \cong \mathbb{C}[[r_1, \ldots, r_n]][u_1, \ldots, u_n].$$

We consider the element

$$w = u_1 \ldots u_n + \sum_{j=1}^{n} r_j u_j^n \in S,$$

which has degree $f(2) \in Y$, hence we can define the $\boldsymbol{G}$-graded category of matrix factorizations $MF^{\boldsymbol{G}}(S, w)$.

We consider the $\boldsymbol{G}$-graded S-module

$$K := R[U] \otimes \Lambda\left(U^{\vee}\right) \cong \mathbb{C}[[r_1, \ldots, r_n]][u_1, \ldots, u_n][\theta_1, \ldots, \theta_n],$$

where the variables θ_j anti-commute. We introduce the differential

$$\begin{aligned} \delta_K : K &\to K \\ \delta_K &= \sum_{j=1}^{n} u_j \frac{\partial}{\partial \theta_j} + w_j \theta_j, \end{aligned}$$

where

$$w_j = \frac{u_1 \ldots u_n}{n u_j} + r_j u_j^{n-1}.$$

We observe that δ_K has degree $f(1) \in Y$, and that

$$\delta_K^2 = w \cdot \mathrm{id}.$$

Thus, (K, δ_K) is a $\boldsymbol{G}$-graded matrix factorization of w. We denote it by $\mathcal{O}_0$. Finally, we observe that δ_K is H-invariant, where H is the symmetric group acting in the obvious way.

Corollary 2.7.3.3. *Let us define the $\boldsymbol{G}$-graded A_∞ algebra over R:*

$$\mathcal{B} := \mathrm{Hom}_{MF^{\boldsymbol{G}}(S,w)}(\mathcal{O}_0, \mathcal{O}_0).$$

Then $\mathcal{B}$ is of type A, in the sense of Definition 2.2.5.2.

Proof. Follows from Proposition 2.7.2.1. We observe that it is necessary to check that the $\boldsymbol{G}$-grading and H-equivariance interact appropriately with the homological perturbation lemma construction, but this is clear. □

Corollary 2.7.3.4. *Let*

$$\mathcal{A} := CF^*_{\mathcal{F}(\phi)}(L^n, L^n),$$

where $\mathcal{F}(\phi)$ is the category of Proposition 2.5.2.2 and $\phi : (M^n, \boldsymbol{D}) \to (M, \boldsymbol{D})$ is the branched cover of Example 2.3.5.10. Let

$$\mathcal{B} := \mathrm{Hom}_{MF^{\boldsymbol{G}}(S,w)}(\mathcal{O}_0, \mathcal{O}_0)$$

as above. Both $\mathcal{A}$ and $\mathcal{B}$ are $\boldsymbol{G}$-graded A_∞ algebras over R, and there exists a power series $\psi \in \mathbb{C}[[T]]$, with $\psi(0) = 1$ (recalling $T = r_1 \ldots r_n$), and an A_∞ quasi-isomorphism

$$\mathcal{A} \cong \psi \cdot \mathcal{B}.$$

Proof. Follows from Corollary 2.7.3.3 and Corollary 2.6.5.8. □

2.7.4 Equivariant matrix factorizations

Suppose that $\boldsymbol{p} : \boldsymbol{G}' \to \boldsymbol{G}$ is an injective morphism of grading data. Then we consider the $\boldsymbol{G}'$-graded category

$$\boldsymbol{p}^* MF^{\boldsymbol{G}}(S, w).$$

Remark 2.7.4.1. Objects of $\boldsymbol{p}^* MF^{\boldsymbol{G}}(S, w)$ are again $\boldsymbol{G}$-graded matrix factorizations, but the morphism spaces are just the parts whose Y-grading lies in the image of $p : Y' \to Y$. Thus $\boldsymbol{p}^* MF^{\boldsymbol{G}}(S, w)$ embeds, fully faithfully, in the category of coker$(p)^*$-equivariant matrix factorizations.

For example, let $\boldsymbol{p}_1 : \boldsymbol{G}_n^n \to \boldsymbol{G}_1^n$ be the morphism defined in Lemma 2.2.1.14.

Definition 2.7.4.2. We denote

$$\widetilde{\mathcal{B}} := \boldsymbol{p}_1^* \underline{\mathcal{B}} \subset \boldsymbol{p}_1^* MF^{\boldsymbol{G}}(S, w).$$

It is the full subcategory whose objects are $\mathcal{O}_0$ and its shifts by elements $y \in Y$.

We obtain the following:

Corollary 2.7.4.3. *There exists a power series $\psi \in \mathbb{C}[[T]]$, $\psi(0) = 1$, and an A_∞ quasi-isomorphism of $\boldsymbol{G}_n^n$-graded R-linear A_∞ categories,*

$$\widetilde{\mathcal{A}} \cong \psi \cdot \widetilde{\mathcal{B}}.$$

Proof. Follows from Corollary 2.7.3.3 and Corollary 2.6.5.10. □

2.7.5 Coherent sheaves

In this section, we will explain how to relate equivariant categories of coherent sheaves on a projective variety to equivariant categories of graded matrix factorizations. We state a result closely related to [59, Proposition 1.2.2] (see also [60, Section 2]), in the language of $\boldsymbol{G}$-graded matrix factorizations. First, we recall Orlov's **category of graded matrix factorizations** (see [58, Section 3.1]).

Let S be a $\mathbb{Z}$-graded ring, and $w \in S$ homogeneous of degree d. Recall from Example 2.2.1.11 the grading datum $\boldsymbol{G}_{MF(d)}$. We define

$$\begin{aligned} q : \mathbb{Z} &\to \mathbb{Z} \oplus \mathbb{Z}/(2,-d), \\ q(j) &= (0,j), \end{aligned}$$

so we can equip $S \cong q_* S$ with a $\boldsymbol{G}_{MF(d)}$-grading. Then w has degree $(0,d) \sim (2,0) = f(2) \in Y$, so we can define the category of $\boldsymbol{G}_{MF(d)}$-graded matrix factorizations. Now there is a unique injective morphism of grading data, $\boldsymbol{p} : \boldsymbol{G}_{\mathbb{Z}} \to \boldsymbol{G}_{MF(d)}$. We define

$$\mathrm{GrMF}(S,w) := \boldsymbol{p}^* MF^{\boldsymbol{G}_{MF(d)}}(S,w).$$

It is not hard to see that this definition coincides with Orlov's category of graded matrix factorizations of w (actually Orlov defines graded matrix factorizations to be the homotopy category of this differential graded category). Namely, if we denote

$$\pi : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}/(2,-d),$$

then given a $\boldsymbol{G}_{MF(d)}$-graded S-module K, $\pi^* K$ is a quasi-periodic complex of graded S-modules, as in the usual definition of the category of graded matrix factorizations.

We recall the relationship of GrMF to coherent sheaves. Suppose that k is a field, $S = k[u_1, \ldots, u_n]$ is the $\mathbb{Z}$-graded polynomial ring, and $w \in S$ is homogeneous of degree n. Suppose that the variety

$$X := \{w = 0\} \subset \mathbb{P}^{n-1}_k$$

is smooth. Then, because X is Calabi-Yau, [58, Theorem 3.11] says that there is an equivalence of triangulated categories,

$$D^b Coh(X) \cong \mathrm{Ho}(\mathrm{GrMF}(S,w)).$$

Remark 2.7.5.1. We observe that the matrix factorization $\mathcal{O}_0$ is always $\boldsymbol{G}_{MF(n)}$-graded. If $\mathcal{O}_0[j]$ denotes the shift of $\mathcal{O}_0$ by the integer j (i.e., by $f(j)$), then we can arrange that the object

$$\bigoplus_{j=0}^{n-1} \mathcal{O}_0[j] \in Ob(\mathrm{GrMF}(S,w))$$

corresponds, under Orlov's equivalence, to the restriction of the Beilinson exceptional collection

$$\bigoplus_{j=0}^{n-1} i^*\Omega^j(j),$$

where $i : X \hookrightarrow \mathbb{P}_k^{n-1}$ denotes the inclusion. See [69, Remark 5.20], also [70, Section IV.A].

Now let us consider the situation of Section 2.7.4. Recall the commutative square of grading data of Lemma 2.2.1.14. We have

$$R := \mathbb{C}[[r_1, \ldots, r_n]]$$

and

$$S := R[u_1, \ldots, u_n]$$

are $\boldsymbol{G}_1^n$-graded rings, and $w \in S$ has degree $f(2) \in Y$. Recall that the Novikov field Λ is an R-algebra, via the map

$$\begin{aligned} R &\to \Lambda, \\ r_j &\mapsto r. \end{aligned}$$

We define

$$S_{nov} := S \otimes_R \Lambda \cong \Lambda[u_1, \ldots, u_n],$$

with the object

$$w_{nov} := w \otimes 1 \in S_{nov}.$$

Equip S_{nov} with the standard $\mathbb{Z}$-grading (where Λ is concentrated in degree 0, and each u_j has degree 1). Consider the variety

$$\widetilde{N}^n_{nov} := \{w_{nov} = 0\} \subset \mathrm{Proj}(S_{nov}) \cong \mathbb{P}^{n-1}_{\Lambda}.$$

Define the group Γ_n to be the kernel of the map

$$(\mathbb{Z}_n)^n / y_{[n]} \to \mathbb{Z}_n,$$

and equip $\widetilde{N}^n_{nov}$ with the action of the character group Γ_n^*, acting by multiplying the coordinates u_j by nth roots of unity. We denote

$$N^n_{nov} := \widetilde{N}^n_{nov} / \Gamma^*_n.$$

We observe that these definitions coincide with those given in the Introduction.

Lemma 2.7.5.2. *There is a fully faithful embedding of triangulated categories,*

$$\mathrm{Ho}\left(\boldsymbol{q}_{1*}\boldsymbol{p}_1^* MF^{\boldsymbol{G}}(S,w)\right) \otimes_R \Lambda \to D^b Coh\left(N^n_{nov}\right).$$

Proof. First, observe that $\boldsymbol{q}_{2*}S$ is a $\boldsymbol{G}_{MF(n)}$-graded ring. Furthermore, it is easy to check that $\boldsymbol{q}_{2*}R$ is concentrated in degree 0, and u_i has degree $(0,1)$. This coincides with the $\boldsymbol{G}_{MF(n)}$-grading of S induced by the standard $\mathbb{Z}$-grading.

There is a fully faithful embedding

$$\boldsymbol{q}_{2*} MF^{\boldsymbol{G}_1^n}(S,w) \to MF^{\boldsymbol{G}_{MF(n)}}(\boldsymbol{q}_{2*}S, w),$$

which sends

$$(K, \delta_K) \mapsto (\boldsymbol{q}_{2*}K, \delta_K).$$

Therefore, there is a fully faithful embedding

$$\boldsymbol{p}_2^* \boldsymbol{q}_{2*} MF^{\boldsymbol{G}}(S, w) \to \mathrm{GrMF}(S, w).$$

Now note that $\boldsymbol{p}_2^* \boldsymbol{q}_{2*} R \cong R$ is concentrated in degree $0 \in \mathbb{Z}$, so the morphism $R \to \Lambda$ respects the $\mathbb{Z}$-grading, and we obtain a fully faithful embedding of Λ-linear differential graded categories

$$\left(\boldsymbol{p}_2^* \boldsymbol{q}_{2*} MF^{\boldsymbol{G}}(S, w)\right) \otimes_R \Lambda \to \mathrm{GrMF}(S, w) \otimes_R \Lambda.$$

There is a fully faithful embedding of Λ-linear differential graded categories

$$\mathrm{GrMF}(S, w) \otimes_R \Lambda \to \mathrm{GrMF}(S_{nov}, w_{nov}),$$

which sends

$$K \mapsto K \otimes_R \Lambda$$

on the level of objects (recall that K is by definition a **free** S-module). By Orlov's theorem [58, Theorem 3.11], there is an equivalence of Λ-linear triangulated categories,

$$\mathrm{Ho}\left(\mathrm{GrMF}(S_{nov}, w_{nov})\right) \cong D^b Coh\left(\widetilde{N}^n_{nov}\right).$$

It follows that there is an equivalence of triangulated categories

$$\mathrm{Ho}\left(\mathrm{GrMF}(S_{nov}, w_{nov})^{\Gamma^*_n}\right) \cong D^b Coh^{\Gamma^*_n}\left(\widetilde{N}^n_{nov}\right).$$

Hence, by the argument above, there is a fully faithful embedding

$$\mathrm{Ho}\left(\boldsymbol{p}_2^* \boldsymbol{q}_{2*} MF^{\boldsymbol{G}}(S, w)^{\Gamma^*_n} \otimes_R \Lambda\right) \to D^b Coh^{\Gamma^*_n}\left(\widetilde{N}^n_{nov}\right) \cong D^b Coh(N^n_{nov}).$$

Now we recall Lemma 2.2.2.14. It shows that there is an isomorphism

$$\boldsymbol{p}_2^* \boldsymbol{q}_{2*} MF^{\boldsymbol{G}}(S,w)^{\Gamma_n^*} \cong \boldsymbol{q}_{1*} \boldsymbol{p}_1^* MF^{\boldsymbol{G}}(S,w),$$

where

$$\Gamma_n := \ker(q_{2,X})/\mathrm{im}(p_{1,X}).$$

In this case, an examination of Lemma 2.2.1.14 shows that Γ_n is indeed the kernel of the map

$$(\mathbb{Z}_n)^n / y_{[n]} \to \mathbb{Z}_n$$

given by summing the coordinates, and the Γ_n-gradings of morphism spaces correspond under Orlov's equivalence. The result follows. □

Corollary 2.7.5.3. *There is an equivalence of Λ-linear triangulated categories,*

$$\left(\boldsymbol{q}_{1*}\widetilde{\mathbb{B}}\right) \otimes_R \Lambda \cong D^b Coh\left(N^n_{nov}\right).$$

Proof. It follows from Lemma 2.7.5.2 that there is a fully faithful embedding of the left-hand side into the right-hand side. In fact, the embedding is essentially surjective. This follows from Remark 2.7.5.1, together with (the Γ_n^*-equivariant version of) [9, Lemma 5.4], together with (the Γ_n^*-equivariant version of) Beilinson's generation result [71]. □

Definition 2.7.5.4. We denote

$$\widetilde{\mathcal{A}}_{nov} := \boldsymbol{q}_{1*}\widetilde{\mathcal{A}} \otimes_R \Lambda.$$

It is a full subcategory of $\boldsymbol{q}_{1*}\mathcal{F}(M^n, \boldsymbol{D}) \otimes_R \Lambda$.

Corollary 2.7.5.5. *There exists $\psi \in \mathbb{C}[[r^n]]$, and an equivalence of Λ-linear triangulated categories,*

$$\psi \cdot D^b Coh(N^n_{nov}) \cong \mathrm{Ho}\left(Tw\left(\widetilde{\mathcal{A}}_{nov}\right)\right),$$

where 'Tw' denotes forming the category of twisted complexes.

Proof. Follows from Corollary 2.7.5.3 and Corollary 2.7.4.3. □

2.8 The full Fukaya category

In this Section, we consider the full Fukaya category $\mathcal{F}(M^n)$. We explain why there is an embedding

$$\mathcal{F}(M^n, \boldsymbol{D}) \otimes_R \Lambda \to \mathcal{F}(M^n),$$

and prove that the full subcategory

$$\widetilde{\mathcal{A}} \otimes_R \Lambda \subset \mathcal{F}(M^n)$$

(see Definition 2.6.5.9) split-generates, using the criterion of [54]. This allows us to complete the proof of Theorem 4.

2.8.1 Relating the full and relative Fukaya categories

Let (M, ω) be a compact symplectic manifold, satisfying $c_1(M) = 0$. The Fukaya category $\mathcal{F}(M)$ is defined in [54]. It depends on the choice of a bulk class $\mathfrak{b} \in H^{even}(M; \Lambda_0)$, and a background class $st \in H^2(M; \mathbb{Z}_2)$. We choose both of these to be 0. The objects of $\mathcal{F}(M)$ are (L, b), where $L \subset M$ is a graded spin Lagrangian submanifold, and $b \in H^1(L; \Lambda_0)$ is a weak bounding cochain. Composition maps are defined by counting holomorphic disks. $\mathcal{F}(M)$ is a $\mathbb{Z}$-graded A_∞ category.

We would like to relate $\mathcal{F}(M, \boldsymbol{D})$ to $\mathcal{F}(M)$. First, we observe that there is a canonical morphism of grading data,

$$\boldsymbol{q} : \boldsymbol{G}(M, \boldsymbol{D}) \to \boldsymbol{G}_{\mathbb{Z}},$$

by Remark 2.3.5.7. Thus we can define a $\boldsymbol{G}_{\mathbb{Z}}$-graded (i.e., $\mathbb{Z}$-graded) A_∞ category

$$\boldsymbol{q}_*\mathcal{F}(M, \boldsymbol{D}).$$

$\mathcal{F}(M, \boldsymbol{D})$ is defined over the coefficient ring $R \cong \mathbb{C}[[r_1, \ldots, r_k]]$. It is a simple task to show that the coefficient ring $\boldsymbol{q}_* R$ has degree $0 \in \mathbb{Z}$. Therefore, the ring homomorphism

$$\begin{aligned} R &\to \Lambda, \\ r_j &\mapsto r^{d_j}, \end{aligned}$$

respects the $\mathbb{Z}$-grading, because both have degree $0 \in \mathbb{Z}$. This makes Λ into an R-module, and means that we can define the $\mathbb{Z}$-graded, Λ-linear A_∞ category

$$\boldsymbol{q}_*\mathcal{F}(M, \boldsymbol{D}) \otimes_R \Lambda.$$

We remark that, up until this point, we have given a complete definition of the relative Fukaya category $\mathcal{F}(M, \boldsymbol{D})$ of a Kähler pair (satisfying assumptions as in Definition 2.3.5.1), using explicit domain-dependent perturbations of the holomorphic curve equation. To define the full Fukaya category $\mathcal{F}(M)$ however, we need virtual perturbations as in [24, 54]. These two categories should be related as follows:

Assumption 2.8.1.1. There is a fully faithful embedding

$$\boldsymbol{q}_*\mathcal{F}(M, \boldsymbol{D}) \otimes_R \Lambda \to \mathcal{F}(M)$$

of $\mathbb{Z}$-graded, Λ-linear A_∞ categories.

To prove this, we would have to relate the two different perturbation schemes (explicit domain-dependent perturbations versus perturbations of Kuranishi structures), which would take us beyond the scope of this paper. So we leave it as an assumption. We do, however, provide the following justification:

Remark 2.8.1.2. We observe that there is an obvious map on the level of unobstructed objects (Lagrangians with $\mu^0 = 0$):

$$\begin{aligned} Ob(\boldsymbol{q}_*\mathcal{F}(M,\boldsymbol{D}))_{unob} &\to Ob(\mathcal{F}(M)), \\ L &\mapsto (L,0), \end{aligned}$$

and anchored Lagrangian branes automatically come with a grading (recall that taking $\boldsymbol{q}_*$ of a category involves identifying certain objects; in this case, this exactly means that we identify all anchored Lagrangian branes which have the same grading). However, there is no map in the other direction: objects of $\mathcal{F}(M)$ may intersect the divisors $\boldsymbol{D}$. Suppose now that:

- $\boldsymbol{L}$ is a tuple of exact, transversely-intersecting anchored Lagrangian branes in $M \setminus \boldsymbol{D}$;
- $\boldsymbol{p}$ is an associated set of generators (intersection points) of $\boldsymbol{L}$;
- the moduli space of rigid, boundary-punctured holomorphic disks in M with boundary on $\boldsymbol{L}$, asymptotic to $\boldsymbol{p}$, is regular.

We show that a rigid holomorphic disk in this moduli space contributes the same term to an A_∞ structure map μ^s in $\mathcal{F}(M,\boldsymbol{D}) \otimes_R \Lambda$ and in $\mathcal{F}(M)$. We have

$$\omega(u) = -\alpha(p_0) + \sum_{j=1}^{s} \alpha(p_j) + |u \cdot \boldsymbol{D}|$$

by Stokes' theorem, where $\alpha(p)$ denotes the symplectic action of generator p. In particular, if we define the map

$$\begin{aligned} CF^*_{\mathcal{F}(M,\boldsymbol{D})\otimes_R\Lambda}(L_0,L_1) &\to CF^*_{\mathcal{F}(M)}(L_0,L_1), \\ p &\mapsto r^{\alpha(p)}p, \end{aligned}$$

then the holomorphic disk u contributes the same term $\pm r^{\omega(u)}$ to μ^s in both categories. Note that, in $\mathcal{F}(M, \boldsymbol{D})$, we have $(u \cdot \boldsymbol{D})!$ choices for the labelling of the marked points mapping to the divisors $\boldsymbol{D}$, so this disk in fact contributes $(u \cdot \boldsymbol{D})!$ identical terms to μ^s, each of which is $\pm r^{\omega(u)}/(u \cdot \boldsymbol{D})!$ (see the definition in Section 2.5.1). Thus its total contribution is exactly $\pm r^{\omega(u)}$.

2.8.2 Split-generation

We recall the subcategory

$$\widetilde{\mathcal{A}}_{nov} \subset \boldsymbol{q}_{1*}\mathcal{F}(M^n, \boldsymbol{D}) \otimes_R \Lambda$$

from Definition 2.7.5.4. By abuse of notation, we will identify this category with its image under the embedding of Assumption 2.8.1.1. Our aim in this section is to prove that $\widetilde{\mathcal{A}}_{nov}$ split-generates $D^\pi \mathcal{F}(M^n)$.

Definition 2.8.2.1. We define the **closed-open string map** from the (small) quantum cohomology ring of M (with coefficients in Λ) to the Hochschild cohomology of $\mathcal{F}(M)$,

$$\mathcal{CO} : QH^*(M) \to HH^*(\mathcal{F}(M)),$$

as follows: let $\alpha \in H^j(M; \mathbb{C})$ be Poincaré dual to a smooth cycle $A \subset M$. Let $\boldsymbol{L}$ be a tuple of objects with associated generators $\boldsymbol{p}$. We consider the moduli space $\mathcal{M}_4(\boldsymbol{p}, A)$, whose objects consist of pairs (r, u), where $r \in \mathcal{R}_1(\boldsymbol{L})$ and $u : S_r \to M$ is a smooth map, such that

- u satisfies the (perturbed) holomorphic curve equation, with Lagrangian boundary conditions given by the labels $\boldsymbol{L}$;
- u is asymptotic to the generators $\boldsymbol{p}$ at the boundary punctures;

- $u(q) \in A$, where $q \in S_r$ is the internal marked point.

Then each rigid disk $u \in \mathcal{M}_4(\boldsymbol{p}, A)$ contributes a term $\pm r^{\omega(u)} p_0$ to

$$\mathcal{CO}(\alpha)(p_s, \ldots, p_1).$$

We remark that

- When we say 'perturbed' holomorphic curve equation, it really means we must define a Kuranishi structure on $\mathcal{M}_4(\boldsymbol{p}, A)$ and introduce virtual perturbations thereof (see [54]);

- The map $\mathcal{CO}$ is a homomorphism of $\mathbb{Z}$-graded Λ-algebras, where the product on $QH^*(M)$ is quantum cup product $*$, and the product on $HH^*(\mathcal{F}(M))$ is the Yoneda product, and the $\mathbb{Z}$-gradings are the standard ones.

We now aim to apply the following result, which is due to [54]:

Theorem 8. *If (M, ω) is a compact $2d$-dimensional Calabi-Yau symplectic manifold, $\mathcal{L}$ a full subcategory of $\mathcal{F}(M)$ with some finite set of objects, and the map*

$$\mathcal{CO}^{2d} : QH^{2d}(M) \to HH^{2d}(\mathcal{L})$$

is non-zero, then $\mathcal{L}$ split-generates $\mathcal{F}(M)$.

We consider the subcategory $\widetilde{\mathcal{A}}_{nov} \subset \mathcal{F}(M^n)$ (actually this is not a finite collection of Lagrangians, because we include all shifts, but it will suffice to choose one representative of each geometric lift of L^n to M^n). We aim to understand the map

$$\mathcal{CO}^{2(n-2)} : QH^{2(n-2)}(M^n) \to HH^{2(n-2)}\left(\widetilde{\mathcal{A}}_{nov}\right)$$

by first understanding the degree-2 part of the map, $\mathcal{CO}^2$, then using the fact that $\mathcal{CO}$ is a Λ-algebra homomorphism. It is expected that the image of the class of the symplectic form,

$$\mathcal{CO}([\omega]) \in HH^2(\mathcal{F}(M)),$$

should be the class corresponding to the deformation of $\mathcal{F}(M)$ given by scaling the Novikov parameter r. In fact, in our relative setting, we can make a statement with a cleaner proof:

Lemma 2.8.2.2. *Let $(M, \boldsymbol{D})$ be a Kähler pair. Consider the full subcategory*

$$\boldsymbol{q}_* \mathcal{F}(M, \boldsymbol{D}) \otimes_R \Lambda \subset \mathcal{F}(M)$$

of Assumption 2.8.1.1. Then $\mathcal{CO}([\omega])$ is the image of the class

$$\frac{1}{d_j}\left(r_j \frac{\partial \mu^*}{\partial r_j}\right) \otimes 1 \in HH^*(\mathcal{F}(M, \boldsymbol{D})) \otimes_R \Lambda$$

in $HH^*(\mathcal{F}(M, \boldsymbol{D}) \otimes_R \Lambda)$, for any j.

To clarify: $\mu^* \in CC^*(\mathcal{F}(M, \boldsymbol{D}))$ is the A_∞ structure map, and

$$r_j \frac{\partial \mu^*}{\partial r_j} \in CC^*(\mathcal{F}(M, \boldsymbol{D}))$$

is a Hochschild cochain, as can be seen by applying $r_j \partial / \partial r_j$ to the A_∞ associativity equation $\mu^* \circ \mu^* = 0$. Thus, it defines a class in $HH^*(\mathcal{F}(M, \boldsymbol{D})$, and we consider the image of this class under the map

$$HH^*(\mathcal{F}(M, \boldsymbol{D})) \otimes_R \Lambda \to HH^*(\mathcal{F}(M, \boldsymbol{D}) \otimes_R \Lambda).$$

Proof. Given Lagrangian branes $\boldsymbol{L}$ with associated generators $\boldsymbol{p}$, each rigid holomorphic disk u with boundary on $\boldsymbol{L}$, asymptotic to $\boldsymbol{p}$, contributes a term

$$\pm r_1^{u \cdot D_1} \dots r_k^{u \cdot D_k}$$

to the coefficient of p_0 in $\mu^s(p_s, \dots, p_1)$, and hence a term

$$\pm(u \cdot D_j) r_1^{u \cdot D_1} \dots r_k^{u \cdot D_k}$$

to the corresponding coefficient of $r_j \partial \mu^s / \partial r_j$, and hence a term

$$\pm(u \cdot D_j) r^{\omega(u)}$$

to the corresponding coefficient of $(r_j \partial \mu^s / \partial r_j) \otimes_R 1$ (see Remark 2.8.1.2).

On the other hand, recall that D_j is Poincaré dual to $d_j \omega$ by definition of a Kähler pair. So, by definition of the map $\mathcal{CO}$, each such holomorphic disk u together with an internal marked point q mapping to D_j, contributes a term $\pm r^{\omega(u)}$ to the corresponding coefficient of $\mathcal{CO}(d_j \omega)$. There are $u \cdot D_j$ choices for the internal marked point q, so the total contribution of each such holomorphic disk u is $\pm(u \cdot D_j) r^{\omega(u)}$.

Therefore,

$$\mathcal{CO}(d_j \omega) = \left(r_j \frac{\partial \mu^*}{\partial r_j} \right) \otimes_R 1$$

as required. □

Remark 2.8.2.3. Again, Lemma 2.8.2.2 should perhaps be thought of as an assumption, for the same reason that we make Assumption 2.8.1.1.

Proposition 2.8.2.4. *Let M^n be the Calabi-Yau Fermat hypersurface of Example 2.3.5.2, and recall the full subcategory $\widetilde{\mathcal{A}}_{nov} \subset \mathcal{F}(M^n)$. The map*

$$\mathcal{CO}^{2(n-2)} : QH^{2(n-2)}(M^n) \to HH^{2(n-2)} \left(\widetilde{\mathcal{A}}_{nov} \right)$$

is non-zero.

Proof. We recall from Lemma 2.2.5.18 that there is an action of $\tilde{\Gamma}_n^*$ on $HH^*(\widetilde{\mathcal{A}}_{nov})$, and the $\tilde{\Gamma}_n^*$-invariant part is

$$HH^*_{G_{\mathbb{Z}}}\left(\widetilde{\mathcal{A}}_{nov}\right)^{\tilde{\Gamma}_n^*} \cong \Lambda[\alpha]/\alpha^{n-1}$$

as a $\mathbb{Z}$-graded Λ-algebra, where α has degree 2.

It follows from Lemma 2.8.2.2 and Lemma 2.2.5.19 that the image of $\mathcal{CO}([\omega])$ under this isomorphism is $g \cdot \alpha$, for some invertible $g \in \Lambda^*$. Therefore, because $\mathcal{CO}$ is a Λ-algebra homomorphism, the image of $\mathcal{CO}([\omega]^{n-2})$ under this isomorphism is $g^{n-2}\alpha^{n-2}$, which does not vanish in $\Lambda[\alpha]/\alpha^{n-1}$. Because it has degree $2(n-2)$, this completes the proof. □

Corollary 2.8.2.5. *The full subcategory*

$$\widetilde{\mathcal{A}}_{nov} \subset \mathcal{F}(M^n)$$

split-generates the Fukaya category.

Proof. Follows from Proposition 2.8.2.4 and Theorem 8. □

Theorem 4 now follows from Corollary 2.7.5.5, Assumption 2.8.1.1, and Corollary 2.8.2.5.

Appendix A

Signs

This Appendix contains the proof of the following, due to [34]:

Proposition A.0.2.6. *(Proposition 1.3.4.2) Let* $X = (X, \omega, \eta)$ *be an exact symplectic manifold with boundary with symplectic form* ω*, and complex volume form* η*. Define* $X^{op} := (X, -\omega, \bar{\eta})$*. Then there is a quasi-isomorphism of* A_∞*-categories*

$$\mathcal{G} : \mathcal{F}uk(X)^{op} \to \mathcal{F}uk(X^{op})$$

(where the opposite category of an A_∞ *category was defined in Definition 1.3.4.1).*

Proof. We assume the conventions and notation of [11, Sections 8-11] – in particular, the concepts of Lagrangian branes, determinant lines and perturbation data are used with minimal explanation.

Recall that, to define the Fukaya category $\mathcal{F}uk(X)$ as in [11], one must make a choice of universal perturbation data (essentially, a consistent choice of domain-dependent Hamiltonian perturbations and almost-complex structures). Having made such a choice for X, it is clear that we obtain a 'conjugate' choice of universal perturbation data for X^{op} by reversing the sign of all almost-complex structures (because if ω and J are compatible then $-\omega$ and $-J$ are compatible). We will show that there is a strict isomorphism of A_∞ categories from $\mathcal{F}uk(X)^{op}$, defined with the given perturbation data, to $\mathcal{F}uk(X^{op})$, defined using the conjugate perturbation data. The result then follows from the independence of the Fukaya category of the choice of perturbation data, up to quasi-isomorphism.

On the level of objects, a Lagrangian brane $L^\# = (L, \alpha^\#, P^\#)$ (where L is a Lagrangian in X, $\alpha^\#$ a grading of L, and $P^\#$ a Pin structure on L) gets sent to the brane $\mathcal{G}(L^\#) = (L, -\alpha^\#, P^\#)$. Suppose we have a morphism

$$x \in \hom_{\mathcal{F}uk(X,\omega)^{op}}(L_1^\#, L_2^\#) = \hom_{\mathcal{F}uk(X,\omega)}(L_2^\#, L_1^\#).$$

We send it to the morphism $\tilde{x} = \mathcal{G}(x)$ corresponding to the same intersection point as x in

$$\hom_{\mathcal{F}uk(X,-\omega)}(\mathcal{G}(L_1^\#), \mathcal{G}(L_2^\#)).$$

We must also define an isomorphism of the orientation lines $o_x \cong o_{\tilde{x}}$.

Recall the notion of an orientation operator for a morphism $y \in L_1 \cap L_2$ in the Fukaya category: choose a path in the space $\mathrm{Gr}^\#(T_yX)$ of abstract Lagrangian branes at y, $\lambda : [0,1] \to \mathrm{Gr}^\#(T_yX)$ from $(L_1^\#)_y$ to $(L_2^\#)_y$ (i.e., a path in the ordinary Lagrangian Grassmannian $\mathrm{Gr}(T_yX)$ that is compatible with the grading and Pin structures). Define a Cauchy-Riemann operator D_y on the complex vector bundle $H \times (T_yX, J)$ over the upper half plane H, with boundary values specified by $\lambda(s)$ along the real axis. D_y is called an orientation operator for y, and there is a canonical isomorphism

$$o_y \cong \det(D_y).$$

In our case, we choose the path $\lambda(s)$ from $(L_2^\#)_x$ to $(L_1^\#)_x$ and the orientation operator

D_x is on $H \times (T_xX, J)$ with boundary conditions given by $\lambda(s)$. It is not hard to see that the reverse path $\lambda(1-s)$ runs from $(\mathcal{G}(L_1^{\#}))_{\bar{x}}$ to $(\mathcal{G}(L_2^{\#}))_{\bar{x}}$ and gives boundary conditions for the orientation operator $D_{\bar{x}}$ on the complex vector bundle $H \times (T_xX, -J)$. Our two orientation operators are isomorphic, via reflection about the imaginary axis in H. We define our isomorphism of orientation lines to be the composition of isomorphisms

$$o_x \cong \det(D_x) \cong \det(D_{\bar{x}}) \cong o_{\bar{x}}.$$

Now we must check that the composition maps μ^k agree. This amounts to proving that

$$\mathcal{G}\left(\mu^k_{(X,\omega)}(x_k, \ldots, x_1)\right) = (-1)^* \mu^k_{(X,-\omega)}\left(\mathcal{G}(x_1), \ldots, \mathcal{G}(x_k)\right).$$

where $*$ is the sign given in the statement of the Proposition. We prove this equality by showing that the holomorphic disks contributing to each product are in bijective correspondence.

Suppose we are given a disk $S = D^2 \setminus \{k+1 \text{ boundary points}\}$, equipped with a choice of strip-like ends, a map $u : S \to X$ satisfying the perturbed J-holomorphic curve equation (according to our perturbation data for X), sending the jth boundary component to the Lagrangian L^j, contributing a term x_0 to the product on the left hand side. Then the disk $\bar{S}$ (S with the conjugate complex structure), equipped with the same map $u : \bar{S} \to X$, satisfies the perturbed $(-J)$-holomorphic curve equation (according to the conjugate perturbation data for X^{op}), and contributes to the product on the right hand side. We define the conjugate boundary lift of any boundary component C with label L^n (see Section 1.3.1) by $\bar{\tilde{u}}_C = a \circ \tilde{u}_C$, where $a : S^n \to S^n$ is the antipodal map (recall that $\tau \circ L^n = L^n \circ a$ so this is a valid boundary lift).

We just need to show that these disks contribute with the appropriate relative sign. We recall, briefly, how signs are calculated: The linearized J-holomorphic curve equation along u yields a linearized operator

$$D_{S,u} : W^{1,p}(S, u^*TX, u^*TL^j) \to L^p(S, \Omega_S^{0,1} \otimes u^*TX)$$

whose kernel is isomorphic to the tangent space at u of the space of J-holomorphic curves $v : S \to X$ with the same boundary conditions as u.

When defining the Fukaya category, we are concerned with families of J-holomorphic curves whose modulus can vary. Let $\mathcal{S}^{k+1} \to \mathcal{R}^{k+1}$ denote the universal family of disks with $k+1$ boundary marked points, and suppose that the modulus of S is $r \in \mathcal{R}$. Then we can also define an extended linearized operator (again by linearizing the J-holomorphic curve equation)

$$D_{\mathcal{S},r,u} : T_r\mathcal{R}^{k+1} \times W^{1,p}(S, u^*TX, u^*TL^j) \to L^p(S, \Omega_S^{0,1} \otimes u^*TX).$$

The kernel of $D_{\mathcal{S},r,u}$ is isomorphic to the tangent space at u of the space of J-holomorphic curves with the same boundary conditions as u, and possibly varying modulus of the domain. One says that u is **regular** if this operator is surjective, and **rigid** if it is regular and has index 0. Observe that there is a canonical isomorphism

$$\det(D_{\mathcal{S},r,u}) \cong \Lambda^{top}(T_r\mathcal{R}^{k+1}) \otimes \det(D_{S,u})$$

(obtained by deforming $D_{\mathcal{S},r,u}$ to $0 \oplus D_{S,u}$ through Fredholm operators).

The structure coefficients of the A_∞ maps μ^k are defined to be counts of rigid curves, so we assume that u is rigid and therefore there is a canonical isomorphism

$$\det(D_{\mathcal{S},r,u}) \cong \mathbb{R}.$$

We choose an orientation of the moduli space $\mathcal{R}^{k+1}$ by fixing the first three boundary points and taking the induced orientation from the coordinates of the remaining ones.

This defines an isomorphism

$$\Lambda^{top}(T_r\mathcal{R}^{k+1}) \cong \mathbb{R},$$

and hence an isomorphism

$$\det(D_{S,u}) \cong \mathbb{R}.$$

One now chooses orientation operators D_{x_j} for each $1 \leq j \leq k$. One then glues the orientation operators $D_{x_k}, \ldots, D_{x_1}$, in that order, to the operator $D_{S,u}$ to obtain an orientation operator D_{x_0} for x_0. This gives a canonical isomorphism

$$o_{x_0} \cong \det(D_{x_0}) \cong \det(D_{S,u}) \otimes \det(D_{x_k}) \otimes \ldots \otimes \det(D_{x_1}) \cong o_{x_k} \otimes \ldots \otimes o_{x_1}.$$

This, together with an auxiliary sign

$$i(x_1) + 2i(x_2) + \ldots + ki(x_k)$$

(necessary to realize the correct signs in the A_∞ associativity equation, see [11, Equation (12.24)]) defines the sign with which u contributes a term x_0 to the product $\mu^k(x_k, \ldots, x_1)$.

We need to explain how this sign changes under $\mathcal{G}$. The determination of the sign with which the conjugate disk contributes is almost completely isomorphic, except for the following three changes:

- Complex conjugation of the domain of u acts on our chosen orientation of the space $\mathcal{R}^{k+1}$ with a sign
$$1 + \frac{k(k-1)}{2};$$
- We glue the orientation operators $D_{x_k}, \ldots, D_{x_1}$ to $D_{S,u}$ in that order, whereas for the complex conjugate our convention demands that we glue the orientation operators $D_{\mathcal{G}(x_1)}, \ldots, D_{\mathcal{G}(x_k)}$ to $D_{\bar{S},u}$ in that order. This difference in ordering

results in a Koszul sign difference

$$\sum_{1 \le j < l} i(x_j)i(x_l)$$

between the corresponding isomorphisms

$$\begin{array}{ccc} o_{x_0} & \longrightarrow & \det(D_{S,u}) \otimes o_{x_k} \otimes \dots o_{x_1} \\ \downarrow & & \downarrow \\ o_{\mathcal{G}(x_0)} & \longrightarrow & \det(D_{\bar{S},u}) \otimes o_{\mathcal{G}(x_1)} \otimes \dots o_{\mathcal{G}(x_k)}; \end{array}$$

- The auxiliary signs differ by

$$i(x_1)+2i(x_2)+\dots+ki(x_k)-i(x_k)-2i(x_{k-1})-\dots-ki(x_1) = (k+1)(i(x_1)+\dots+i(x_k)).$$

Combining these three sign differences gives the desired result. $\square$

Appendix B

Strict group actions on Fukaya categories

This section is based on the argument of [9, Section 8b]. Suppose that we have a finite group Γ, which acts on a Kähler pair $(M, \boldsymbol{D})$, permuting the divisors $\boldsymbol{D}$, and preserving the Liouville one-form α. We consider the relative Fukaya category $\mathcal{F}(M, \boldsymbol{D})$ defined in Section 2.5.1. There is an obvious action of Γ on the objects of the relative Fukaya category. We would like to say that this action extends to an action on the relative Fukaya category, in a suitable sense.

Recall the notion of a strictly Γ-equivariant A_∞ structure from Definition 2.2.4.14. Naïvely, one might try to argue that Γ acts on the moduli spaces of pseudoholomorphic disks used to define the structure maps of the Fukaya category, and hence the structure maps are strictly Γ-equivariant. However, this does not work: for Γ to act on the moduli spaces of pseudoholomorphic disks, we would have to make a Γ-equivariant choice of perturbation data, which would destroy our chances of achieving transversality. Instead, we have the following:

Proposition B.0.2.7. *In the situation described above, there is a fully faithful embedding* *of $\mathcal{F}(M, \boldsymbol{D})$ into a strictly Γ-equivariant A_∞ category. The order-0 part of this embedding is a quasi-equivalence, and respects the action of Γ on objects, in the sense that $F(\gamma \cdot L)$ is quasi-isomorphic to $\gamma \cdot F(L)$.*

Proof. We consider a category $\widetilde{\mathcal{F}}(M, \boldsymbol{D})$ with objects (γ, L), where L is an object of $\mathcal{F}(M, \boldsymbol{D})$ and $\gamma \in \Gamma$. Think of (γ, L) as representing the object $\gamma \cdot L$ of $\mathcal{F}(M, \boldsymbol{D})$, but we now have $|\Gamma|$ copies of each object.

We define an action of Γ on these objects, via

$$\gamma_1 \cdot (\gamma_2, L) := (\gamma_1 \cdot \gamma_2, L).$$

Now for each pair of objects $((1, L_0), (\gamma, L_1))$ of $\widetilde{\mathcal{F}}(M, \boldsymbol{D})$, we choose a regular Floer datum for the objects $(L_0, \gamma \cdot L_1)$ of $\mathcal{F}(M, \boldsymbol{D})$. We then define Floer data for pairs of objects $((\gamma_0, L_0), (\gamma_1, L_1))$ by acting with γ_0 on the Floer data for $((1, L_0), (\gamma_0^{-1} \cdot \gamma_1, L_1))$. We thus define morphism spaces

$$CF^*((\gamma_0, L_0), (\gamma_1, L_1))$$

for all pairs of objects. We define the Floer differential μ_0^1 as before, and note that it is now strictly Γ-equivariant.

Now for each tuple of objects $\boldsymbol{L} := ((1, L_0), (\gamma_1, L_1), \ldots, (\gamma_k, L_k))$, with associated generators $\boldsymbol{y}$, we choose regular, consistent perturbation data on the moduli spaces $\mathcal{R}(\boldsymbol{p}, \ell)$. We then define perturbation data for tuples $\boldsymbol{L} = ((\gamma_0, L_0), \ldots, (\gamma_k, L_k))$ by acting with γ_0 on the perturbation data chosen for $((1, L_0), (\gamma_0^{-1} \cdot \gamma_1, L_1), \ldots, (\gamma_0^{-1} \cdot \gamma_k, L_k))$. This allows us to define the rest of the Floer products μ^k. Note that they are strictly Γ-equivariant.

Observe now that the full subcategory with objects $(1, L)$ is equivalent to $\mathcal{F}(M, \boldsymbol{D})$

(making the corresponding choice of perturbation data). Thus, we have an inclusion of $\mathcal{F}(M, \boldsymbol{D})$ as a full subcategory of $\widetilde{\mathcal{F}}(M, \boldsymbol{D})$. Furthermore, if we restrict to the affine Fukaya category, then this inclusion is a quasi-equivalence, because each object (γ, L) is quasi-isomorphic to the element $(1, \gamma \cdot L)$ of the subcategory. This concludes the proof. $\square$

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