6.972: Game Theory February 3, 2005

Lecture 2: Static Games

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1 Introduction

The goal of today's lecture is to introduce strategic form games and their solution concepts. A *strategic form* game is a model for a game in which all of the participants act simultaneously, and without knowledge of other players' actions. More formally,

Definition 1 (Strategic Game) *A strategic game is a triplet* $\langle \mathcal{I}, (S_i)_{i\in\mathcal{I}}, (u_i)_{i\in\mathcal{I}} \rangle$ *where*

1. $\mathcal{I}$ *is a finite set of players,* $\mathcal{I} = \{1, \ldots, I\}$.
2. $(S_i)_{i\in\mathcal{I}}$ *is a set of available actions where* S_i *is the non-empty set of actions for player* i*. We denote by* $s_i \in S_i$ *an action for player* i*, and by* $s_{-i} = [s_j]_{j\neq i}$*, a vector of actions for all players* except i*. We will denote by* $S = \prod_i S_i$ *the set of all profiles of actions, and by* $S_{-i} = \prod_{j\neq i} S_j$ *the set of all profiles of actions for all players except* i*. We call the tuple* $(s_i, s_{-i}) \in S$ *an* action profile*, or* outcome.
3. $(u_i)_{i\in\mathcal{I}}$ *is a set of payoff functions where* $u_i : S \to R$ *is a function from the set of all action profiles to the real numbers.*

Remarks:

1. In today's lecture the word *strategy* and the phrase *pure strategy* will denote an action. Later on in the class we will discuss strategies that are 1) randomizations over actions, or 2) in the context of dynamic games, contingency plans over actions.
2. We note that in the game model, we implicitly assume that players have preferences over the outcomes, *and* that these preferences can be captured by assigning utilities to the outcomes. This assumption is a fundamental area of study in economics. Handouts will be posted on the web showing that not all preference relations can be captured by utility functions.
3. The von Neuman-Morgenstern model states that preferences about probability density functions over outcomes can be represented by expected value of a payoff function over deterministic outcomes.

1.1 Finite Strategy Spaces

When the strategy space is finite, and when the number of players and actions is small, a game can be represented in matrix form. The cell indexed by row x and column y contains a pair, (a, b) where $a = u_1(x, y)$ and $b = u_2(x, y)$. For example, consider the following game of "Matching Pennies."

	HEADS	TAILS
HEADS	$-1, 1$	$1, -1$
TAILS	$1, -1$	$-1, 1$

This game represents "pure competition" in the sense that one player's outcome is the opposite of the other's. Another way to view this situation is to note that the sum of the utilities for both players at each outcome is "zero." This class of games, called "zero-sum games" (or equivalently, "constant-sum games") is a special case which has been well-studied.

1.2 Continuous Strategy Spaces

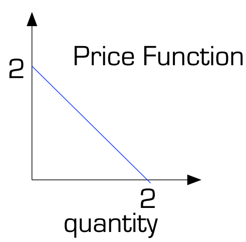
It is also possible for the strategy space for a player to be infinite. Consider the following game of *Cournot competition* which models two firms which produce the same widget and seek to maximize their profits. The formal game $G = \langle \mathcal{I}, (S_i), (u_i) \rangle$ consists of

1. A set of two players, $\mathcal{I} = 1, 2$.
2. The action for player $i \in \mathcal{I}$ is a quantity, $s_i \in [0, \infty]$ which represents the amount of widget that the player manufactures.
3. The utility for each player is its total revenue minus its total cost, which can be written as

$$u_i(s_1, s_2) = s_i p(s_1 + s_2) - c_i s_i$$

   where $p(q)$ is a function which represents the price of the widget, and c_i is the cost-per-unit for firm i.

For simplicity, consider the case where both firms have unit cost, $c_1 = c_2 = 1$. Notice that the price of the widget depends on the *total* amount of widget in the marketplace. Let us assume that $p(q) = \max(0, 2 - q)$ which is depicted below (q must assume a positive value in this context).



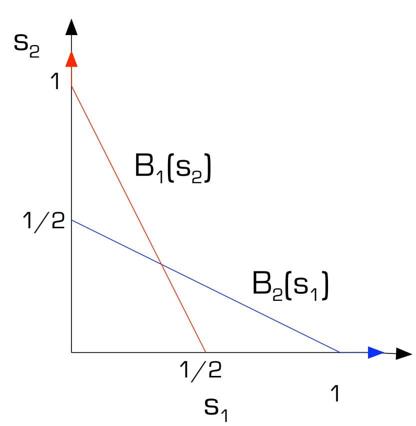
We can analyze this game by considering the *best-response function* for each of the firms. Let us fix the quantity that firm $-i$ produces, and now write the function which maximizes the profit for firm i. Denoting this function as $B_i(s_{-i})$, we have:

$$B_i(s_{-i}) = \arg\max_{s_i \geq 0}(s_i p(s_i + s_{-i}) - s_i)$$

Notice that when $s_{-i} > 2$, then the best response is 0 since the price becomes 0. On the other hand, when $s_i \in [0, 2]$, then by differentiating and solving at zero, we find that

$$\begin{aligned} B_i(s_{-i}) &= \arg\max_{s_i \geq 0}(s_i(2 - s_i - s_{-i}) - s_i) \\ &= \arg\max_{s_i \geq 0}(-s_i^2 + s_i - s_i s_{-i}) \\ &= \frac{1 - s_{-i}}{2} \end{aligned}$$

Graphing both of these functions, the intuitive outcome of this game occurs at the point where both of these functions intersect. In the next section, we shall argue for this outcome using an entirely different approach.



1.3 Dominant/Dominated Strategies

It is easy to predict the outcome of some games, given that all the players are rational, because there is only one *sensible* outcome. Take, for example, the well-studied Prisoner's Dilemma game illustrated below.

	COOPERATE	DON'T COOPERATE
COOPERATE	$-2, -2$	$-10, -1$
DON'T COOPERATE	$-1, -10$	$-5, -5$

Game 2: Prisoner's Dilemma.

Let us consider the row player's situation. Notice that if the row player chooses "Don't Cooperate", then his payoff is $-1 > -2$ if the Column player chooses his first option, and $-5 > -10$ if the Column player chooses his latter option. Thus, no matter what the column player does, the strategy "Don't Cooperate" is always better for the row player than the "Cooperate" strategy. We say that the strategy "Don't Cooperate" is *strictly dominant*, as no matter what action the other player chooses, this strategy is always the best response. By symmetry, the same holds for the Column player. In this case, we can therefore predict, unfortunately, that both players will choose "Don't Cooperate" and spend the next five years in jail!

Lets suppose, however, that we also allow the players to consider the action SUICIDE which has the following payoff structure.

	COOPERATE	DON'T COOPERATE	SUICIDE
COOPERATE	$-2, -2$	$-10, -1$	$0, -20$
DON'T COOPERATE	$-1, -10$	$-5, -5$	$-5, -20$
SUICIDE	$-20, 0$	$-20, -5$	$-20, -20$

Game 2: Prisoner's Dilemma with Suicide.

Indeed, if one player plays SUICIDE, then, due to the lack of witnesses, the other player gets off free. Notice, however, that this strategy of SUICIDE is the *worst* possible option for a player, no matter what the other player does. In this sense, SUICIDE is *strictly dominated* by the other two options. More generally, we say that a strategy is *strictly dominated* if the exists some other strategy that always gives a better outcome.

Definition 2 (Strictly Dominated Strategy) *A strategy $s_i \in S_i$ is strictly dominated for player i if $\exists\ s'_i \in S_i$ such that for all $s_{-i} \in S_{-i}$*

$$u_i(s'_i, s_{-i}) > u_i(s_i, s_{-i}).$$

We can also define a weaker version of dominated strategies.

Definition 3 (Weakly Dominated Strategy) *A strategy $s_i \in S_i$ is weakly dominated for player i if $\exists\ s'_i \in S_i$ such that for all $s_{-i} \in S_{-i}$*

$$u_i(s'_i, s_{-i}) \geq u_i(s_i, s_{-i}),$$

and for some $s_{-i} \in S_{-i}$

$$u_i(s'_i, s_{-i}) > u_i(s_i, s_{-i}).$$

1.4 Iterated Elimination of Strictly Dominated Strategies

Consider the following game.

	LEFT	MIDDLE	RIGHT
UP	$4,3$	$5,1$	$6,2$
MIDDLE	$2,1$	$8,4$	$3,6$
DOWN	$3,0$	$9,6$	$2,8$

Game 3: Example for Iterated Removal of Dominated Strategies.

Note that there do not exist any strategies that are strictly dominated for player 1 (the Row player). On the other hand, note that the strategy MIDDLE is strictly dominated by the strategy RIGHT for player 2 (the Column player). Thus, we conclude that it is never rational for player 2 to play MIDDLE and we can therefore remove this column in the game, resulting in the following game.

	LEFT	RIGHT
UP	$4,3$	$6,2$
MIDDLE	$2,1$	$3,6$
DOWN	$3,0$	$2,8$

Game 3: Game after One Removal of Dominated Strategies.

Now, note that both the actions MIDDLE and DOWN are strictly dominated by the action UP for player 1, which means that both these rows can be removed, resulting in the following game.

	LEFT	RIGHT
UP	$4,3$	$6,2$

Game 3: Game after Three Iterated Removals of Dominated Strategies.

We are left with a game where player 1 does not have any choice in his actions, while player 2 can choose between LEFT and RIGHT. Since LEFT will maximize the utility of player 2, we conclude that the only rational action profile in the game is (UP,LEFT).

Remark: One might worry that different orderings of the removals of dominated strategies can yield different results. One of the problems in the first homework will show that this can not happen. That is, *the ordering in which strategies are eliminated does not affect the set of strategies that survive iterated strict dominance.*

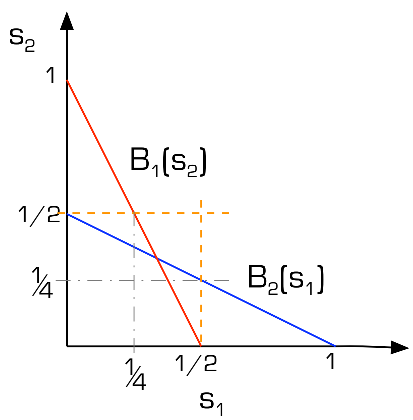
1.5 Back to the Cournot Example

We now apply the technique of iterated elimination of strictly dominated strategies to the Cournot Competition. (Our terminology in the following example is somewhat loose and informal.) In the first step, we note that both firms must choose a quantity between $[0, \infty]$. We denote this:

$$A_1^1 = [0, \infty]$$
$$A_2^1 = [0, \infty]$$

Notice, however, that it is not rational for player 1 to choose any quantity that is outside of the range $[0, 1/2]$ since player 1's best response function is 0 outside of that range. Therefore, playing *any* strategy which is defined over $[0, \infty]$ is dominated by playing one over $[0, 1/2]$. The same reasoning holds for player 2. Thus, at the second iteration, we can argue that the pair of best responses must be

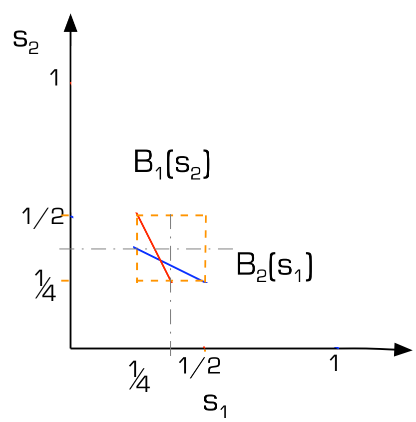
$$A_1^2 = [0, 1/2]$$
$$A_2^2 = [0, 1/2]$$



Given that player 2 only plays in the range $[0, 1/2]$, then player 1 can *restrict* his best response function to only these values. In the graph below, this is depicted with the orange dashed lines. Consider the point where the horizontal orange line intersects $B_1(s_2)$. Since player 2 will only play strategies below the dashed-orange line, then player 1 need only consider strategies between $[1/4, 1/2]$. The same situation holds for player 2.

$$A_1^3 = [1/4, 1/2]$$
$$A_2^3 = [1/4, 1/2]$$

The graph below depicts the same analysis. A formal argument can be made to show that the limit of this process will converge on the point where the two best response functions intersect.



2 Nash Equilibrium

We end this lecture by introducing the famous Nash Equilibrium. We will discuss this Equilibrium in more detail during the next lecture.

At a high-level, a Nash Equilibrium is a profile of actions, which has the property that no *single* player can profit by deviating from the action profile, assuming that all other players act according to it. More formally,

Definition 4 (Nash Equilibrium) *A pure strategy Nash Equilibrium of a strategic game* $\langle \mathcal{I}, (S_i), (u_i)_{i \in \mathcal{I}} \rangle$ *is an action profile* $s^* \in S$ *s.t.* $\forall i \in \mathcal{I}$ *the following condition holds*

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) \quad \forall s_i \in S_i.$$

We note that the definition can be restated in terms of a best-response function:

Definition 5 (Nash Equilibrium - Restated) *Let* $\langle \mathcal{I}, (S_i), (u_i)_{i \in \mathcal{I}} \rangle$ *be a strategic game. For any* $s_{-i} \in S_{-i}$*, define the best-response function* $B_i(s_{-i})$*,*

$$B_i(s_{-i}) = \{s_i \in S_i | u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}') \quad \forall s_i' \in S_i\}$$

We say that an action profile s^* *is a Nash Equilibrium iff*

$$s_i^* \in B_i(s_{-i}^*) \quad \forall i.$$