6.972: Game Theory	February 17, 2005

Lecture 6: Existence of Equilibrium

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1 Introduction

In this lecture we are concerned with the existence of equilibrium. Specifically, we discuss:

- Existence of Nash equilibrium in finite games.
- Existence and computation of correlated equilibrium in finite games.
- Continuous strategy spaces.

General Proof Strategy

The general proof strategy for the existence of an equilibrium is based on analyzing the best response correspondence B. Let $B : \Sigma \rightrightarrows \Sigma$ be the best response correspondence of a game, such that

$$B(\sigma) = [B_i(s_{-i})]_{i \in \mathcal{I}}$$

The existence of equilibrium is then equivalent to the existence of a mixed strategy σ such that $\sigma \in B(\sigma)$. This is typically proved with the use of fixpoint theorems. The most commonly used one is Kakutani's theorem.

Theorem 1 (Kakutani) *Let $f : A \rightrightarrows A$ be a correspondence, with $x \in A \to \mathbf{f}(x) \subset A$, satisfying the following conditions:*

1. *A is a compact, convex, and non-empty subset of a finite dimensional Euclidean space.*
2. *$f(x)$ is non-empty: $\forall x \in A$, $f(x)$ is well defined.*
3. *$f(x)$ is convex: $\forall x \in A$, $f(x)$ is a convex valued correspondence.*
4. *$f(x)$ has a closed graph: If $\{x^n, y^n\} \to \{x, y\}$ with $y^n \in f(x^n)$, and f is an upper semi-continuous correspondence.*

Then, $\exists x \in A$, such that $x \in f(x)$.

2 Existence of Nash equilibrium

The following theorem by Weirstrass is used in the proof of Nash's theorem.

Theorem 2 (Weirstrass) *Let $f : A \to \Re$ be a continuous function, with A non-empty and compact. Then there exists an optimal solution to the program $\min_{x \in A} f(x)$.*

We proceed now to the main result of the section.

Theorem 3 (Nash) *Any finite strategic game has a Nash equilibrium*

Proof:

The idea is to apply Kakutani's theorem to the best response correspondence $B : \Sigma \rightrightarrows \Sigma$. We show that $B(\sigma)$ satisfies the conditions of Kakutani's theorem.

1. Σ is compact, convex, and non-empty.

By definition

$$\Sigma = \prod_{i \in \mathcal{I}} \Sigma_i$$

where each $\Sigma_i = \{x | \sum x_i = 1\}$ is a *simplex* of dimension $|S_i| - 1$

2. $B(\sigma)$ is non-empty.

By definition,

$$B_i(\sigma_{-i}) \in \arg\max_{x \in \Sigma_i} u_i(x, \sigma_{-i})$$

where Σ_i is non-empty and compact, and u_i is linear in x. Hence, u_i is continuous, and by Weirstrass's theorem $B(\sigma)$ is non-empty.

3. $B(\sigma)$ is convex.

Equivalently, $B(\sigma) \subset \Sigma$ is convex $\forall \sigma$ iff $B_i(\sigma_{-i})$ is convex $\forall i$.

Let $\sigma_i', \sigma_i'' \in B_i(\sigma_{-i})$. Then, $\forall \lambda \in [0, 1] \in B_i(\sigma_{-i})$,

$$u_i(\sigma_i', \sigma_{-i}) \geq u_i(\tau_i, \sigma_{-i}), \forall \tau_i \in \Sigma_i$$

$$u_i(\sigma_i'', \sigma_{-i}) \geq u_i(\tau_i, \sigma_{-i}), \forall \tau_i \in \Sigma_i$$

Thus,

$$\lambda u_i(\sigma_i', \sigma_{-i}) + (1 - \lambda) u_i(\sigma_i'', \sigma_{-i}) \geq u_i(\tau_i, \sigma_{-i}), \forall \tau_i \in \Sigma_i$$

By linearity of u_i,

$$u_i(\lambda \sigma_i' + (1 - \lambda)\sigma_i'', \sigma_{-i}) \geq u_i(\tau_i, \sigma_{-i}), \forall \tau_i \in \Sigma_i$$

Therefore, $\lambda \sigma_i' + (1 - \lambda)\sigma_i'' \in B_i(\sigma_{-i})$, and B is convex.

4. $B(\sigma)$ has a closed graph.

Suppose, for contradiction, that $B(\sigma)$ does not have a closed graph.

Then, there exists a sequence $(\sigma^n, \hat{\sigma}^n) \to (\sigma, \hat{\sigma})$ with $\hat{\sigma}^n \in B(\sigma^n)$ but $\hat{\sigma} \notin B(\sigma)$.

Therefore, $\exists i$ such that $\hat{\sigma}_i \notin B_i(\sigma_{-i})$, which implies that for some $\epsilon > 0$

$$\exists \sigma_i' \in \Sigma_i, \text{s.t. } u_i(\sigma_i', \sigma_{-i}) > u_i(\hat{\sigma}_i, \sigma_{-i}) + 3\epsilon$$

For sufficiently large n,

$$u_i(\sigma_i', \sigma_{-i}^n) \geq u_i(\sigma_i', \sigma_{-i}) - \epsilon$$

because $\sigma_{-i}^n \to \sigma_{-i}$ and u_i is continuous.

Thus

$$u_i(\sigma_i', \sigma_{-i}^n) > u_i(\hat{\sigma}_i^n, \sigma_{-i}) + 2\epsilon$$

and

$$u_i(\sigma_i', \sigma_{-i}^n) \geq u_i(\hat{\sigma}_i^n, \sigma_{-i}^n) + \epsilon$$

which is a contradiction, because σ^n is a best response.

$\square$

3 Existence and computation of correlated equilibrium

Every mixed strategy equilibrium is trivially a correlated equilibrium. Therefore, any finite strategic has a correlated equilibrium. In this section we are concerned with correlated equilibria that may lie outside the scope of mixed strategies.

Proposition 1 *Every finite game has a correlated equilibrium*

The proposition was first shown by Hart and Schmeidler (1989), using a double mini-max argument. Here we discuss a proof by Papdimitriou (2005), which also yields a polynomial time algorithm for computing a correlated equilibrium. It should be noted that the resulting equilibrium is not necessarily the pareto optimal correlated equilibrium. We sketch Papdimitriou's proof in the sequel.

Recall that a correlated equilibrium is a probability distribution $p(\cdot)$ on S such that $\forall i \in \mathcal{I}, s_i, t_i \in S_i$,

$$\sum_{s_{-i}} p(s_i, s_{-i}) \left[u_i(s_i, s_{-i}) - u_i(t_i, s_{-i})\right] \geq 0$$

This leads to an optimization formulation, with $Is(s-1)$ constraints and a decision vector of dimension s^I. The optimization formulation considers the program

$$f^* = \max \sum_s x_s$$

subject to the constraints

$$Ux \geq 0$$

$$x \geq 0$$

The duality theorem from linear optimization are the basis of Papadimitriou's proof with this formulation.

Theorem 4 (LP Duality) *Let $f^* = \max cx$ subject to the constraints $Ax \geq b, x \geq 0$ be the primal problem. The dual problem is defined as $q^* = \max p^T b$ subject to the constraints $A^T p \leq 0, p \geq 0$. Then,*

1. *Weak Duality: $q^* \leq f^*$*
2. *Strong Duality: If the primal problem is bounded, then $q^* = f^*$*

In the primal, either $f^* = 0$ or $f^* = \infty$. If we can show that $f^* = \infty$, then there exists a correlated equilibrium. More precisely, there exist some $x \neq 0$ that can be normalized to yield a correlated equilibrium.

Consider the dual program constraints

$$U^T p \leq [-1]$$

$$p \geq 0$$

Claim 1 *If the dual is infeasible, then $f^* = \infty$*

Proof Sketch: Show that if $f^* < \infty$ then the dual is feasible. If $f^* = 0$, then by strong duality $q^* = 0$. If $q^* = 0$, then the dual has a feasible solution. The infeasibility of dual is an immediate consequence of the Papadimitriou's lemma given below (or else $U^T p \leq [-1]$). □

Lemma 1 (Papadimitriou) *For any $p \geq 0$, there exists a probability distribution x such that*

$$x^T U^T p = 0$$

Further details are available in C. Papadimitriou, "Computing Correlated Equilibria in Multiplayer Games", STOC 2005.

4 Continuous strategy spaces

The results we have presented so far concern finite games. As a natural extension, the following theorem states the conditions for the existence of a pure strategy Nash equilibrium in continuous strategy spaces.

Theorem 5 (Debreu, Glicksberg, Fan) *Consider a strategic form game $< \mathcal{I}, (s_i), (u_i) >$, where s_i is continuous.*

Assume:

1. *s_i is non-empty, convex, and compact.*
2. *$u_i(s)$ is continuous in S.*
3. *$u_(s_i, s_{-i})$ is concave (quasi-concave) in S_i.*

Then, there exists a pure strategy Nash equilibrium for $< \mathcal{I}, (s_i), (u_i) >$.

Example: Unit circle game

Two players pick points s_1 and s_2 on the unit circle. The payoffs for the two players are

$$u_1(s_1, s_2) = d(s_1, s_2)$$

$$u_2(s_1, s_2) = -d(s_1, s_2)$$

where d is the Euclidean distance metric.

Show that there is no pure strategy Nash equilibrium and find the mixed strategy Nash equilibrium. (Hint: If both players pick the same location, player 1 has incentive to deviate. If they pick different locations, player 2 has incentive to deviate).