1 Introduction

In this lecture we are concerned with the existence of equilibrium. Specifically, we discuss:

- Existence of Nash equilibrium in finite games.
- Existence and computation of correlated equilibrium in finite games.
- Continuous strategy spaces.

General Proof Strategy

The general proof strategy for the existence of an equilibrium is based on analyzing the best response correspondence $B$. Let $B : \Sigma \rightarrow \Sigma$ be the best response correspondence of a game, such that

$$B(\sigma) = \{B_i(s_{-i})\}_{i \in I}$$

The existence of equilibrium is then equivalent to the existence of a mixed strategy $\sigma$ such that

$$\sigma \in B(\sigma).$$

This is typically proved with the use of fixpoint theorems. The most commonly used one is Kakutani’s theorem.

**Theorem 1 (Kakutani)** Let $f : A \rightarrow A$ be a correspondence, with $x \in A \rightarrow f(x) \subset A$, satisfying the following conditions:

1. $A$ is a compact, convex, and non-empty subset of a finite dimensional Euclidean space.
2. $f(x)$ is non-empty: $\forall x \in A, f(x)$ is well defined.
3. $f(x)$ is convex: $\forall x \in A, f(x)$ is a convex valued correspondence.
4. $f(x)$ has a closed graph: If $\{x^n, y^n\} \rightarrow \{x, y\}$ with $y^n \in f(x^n)$, and $f$ is an upper semi-continuous correspondence.

Then, $\exists x \in A$, such that $x \in f(x)$.

2 Existence of Nash equilibrium

The following theorem by Weirstrass is used in the proof of Nash’s theorem.

**Theorem 2 (Weirstrass)** Let $f : A \rightarrow \mathbb{R}$ be a continuous function, with $A$ non-empty and compact. Then there exists an optimal solution to the program $\min_{x \in A} f(x)$.

We proceed now to the main result of the section.
Theorem 3 (Nash) Any finite strategic game has a Nash equilibrium

Proof:

The idea is to apply Kakutani's theorem to the best response correspondence $B : \Sigma \Rightarrow \Sigma$. We show that $B(\sigma)$ satisfies the conditions of Kakutani’s theorem.

1. $\Sigma$ is compact, convex, and non-empty.
   
   By definition
   
   $$\Sigma = \prod_{i \in I} \Sigma_i$$
   
   where each $\Sigma_i = \{x \mid \sum x_i = 1\}$ is a simplex of dimension $|S_i| - 1$

2. $B(\sigma)$ is non-empty.
   
   By definition,
   
   $$B_i(\sigma_{-i}) \in \arg \max_{x \in \Sigma_i} u_i(x, \sigma_{-i})$$
   
   where $\Sigma_i$ is non-empty and compact, and $u_i$ is linear in $x$. Hence, $u_i$ is continuous, and by Weirstrass’s theorem $B(\sigma)$ is non-empty.

3. $B(\sigma)$ is convex.

   Equivalently, $B(\sigma) \subset \Sigma$ is convex $\forall \sigma$ iff $B_i(\sigma_{-i})$ is convex $\forall i$.

   Let $\sigma_i', \sigma_i'' \in B_i(\sigma_{-i})$. Then, $\forall \lambda \in [0, 1] \in B_i(\sigma_{-i})$,
   
   $$u_i(\lambda \sigma_i', (1 - \lambda) \sigma_i'') \geq u_i(\tau_i, \sigma_{-i}), \forall \tau_i \in \Sigma_i$$
   
   $$u_i(\sigma_i', \sigma_{-i}) \geq u_i(\tau_i, \sigma_{-i}), \forall \tau_i \in \Sigma_i$$

   Thus,
   
   $$\lambda u_i(\sigma_i', \sigma_{-i}) + (1 - \lambda) u_i(\sigma_i'', \sigma_{-i}) \geq u_i(\tau_i, \sigma_{-i}), \forall \tau_i \in \Sigma_i$$

   By linearity of $u_i$,
   
   $$u_i(\lambda \sigma_i' + (1 - \lambda) \sigma_i'', \sigma_{-i}) \geq u_i(\tau_i, \sigma_{-i}), \forall \tau_i \in \Sigma_i$$

   Therefore, $\lambda \sigma_i' + (1 - \lambda) \sigma_i'' \in B_i(\sigma_{-i})$, and $B$ is convex.

4. $B(\sigma)$ has a closed graph.

   Suppose, for contradiction, that $B(\sigma)$ does not have a closed graph.

   Then, there exists a sequence $(\sigma^n, \hat{\sigma}^n) \rightarrow (\sigma, \hat{\sigma})$ with $\hat{\sigma}^n \in B(\sigma^n)$ but $\hat{\sigma} \notin B(\sigma)$.

   Therefore, $\exists i$ such that $\hat{\sigma}_i \notin B_i(\sigma_{-i})$, which implies that for some $\epsilon > 0$
   
   $$\exists \sigma_i' \in \Sigma_i, \text{ s.t. } u_i(\sigma_i', \sigma_{-i}) > u_i(\hat{\sigma}_i, \sigma_{-i}) + 3\epsilon$$

   For sufficiently large $n$,
   
   $$u_i(\sigma_i', \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i}) - \epsilon$$

   because $\sigma^n_i \rightarrow \sigma_{-i}$ and $u_i$ is continuous.

   Thus
   
   $$u_i(\sigma_i', \sigma_{-i}) > u_i(\hat{\sigma}_i^n, \sigma_{-i}) + 2\epsilon$$

   and
   
   $$u_i(\sigma_i', \sigma_{-i}) \geq u_i(\hat{\sigma}_i^n, \sigma_{-i}) + \epsilon$$

   which is a contradiction, because $\sigma^n$ is a best response.

$\Box$
3 Existence and computation of correlated equilibrium

Every mixed strategy equilibrium is trivially a correlated equilibrium. Therefore, any finite strategic has a correlated equilibrium. In this section we are concerned with correlated equilibria that may lie outside the scope of mixed strategies.

**Proposition 1** Every finite game has a correlated equilibrium

The proposition was first shown by Hart and Schmeidler (1989), using a double mini-max argument. Here we discuss a proof by Papdimitriou (2005), which also yields a polynomial time algorithm for computing a correlated equilibrium. It should be noted that the resulting equilibrium is not necessarily the pareto optimal correlated equilibrium. We sketch Papdimitriou’s proof in the sequel.

Recall that a correlated equilibrium is a probability distribution \( p(\cdot) \) on \( S \) such that \( \forall i \in I, s_i, t_i \in S_i, \)

\[
\sum_{s_{-i}} p(s_i, s_{-i}) \left[ u_i(s_i, s_{-i}) - u_i(t_i, s_{-i}) \right] \geq 0
\]

This leads to an optimization formulation, with \( Is(s-1) \) constraints and a decision vector of dimension \( s^I \). The optimization formulation considers the program

\[
f^* = \max \sum_s x_s
\]

subject to the constraints

\[
Ux \geq 0
\]

\[
x \geq 0
\]

The duality theorem from linear optimization are the basis of Papadimitriou’s proof with this formulation.

**Theorem 4 (LP Duality)** Let \( f^* = \max cx \) subject to the constraints \( Ax \geq b, x \geq 0 \) be the primal problem. The dual problem is defined as \( q^* = \max p^T b \) subject to the constraints \( A^T p \leq 0, p \geq 0 \). Then,

1. **Weak Duality**: \( q^* \leq f^* \)
2. **Strong Duality**: If the primal problem is bounded, then \( q^* = f^* \)

In the primal, either \( f^* = 0 \) or \( f^* = \infty \). If we can show that \( f^* = \infty \), then there exists a correlated equilibrium. More precisely, there exist some \( x \neq 0 \) that can be normalized to yield a correlated equilibrium.

Consider the dual program constraints

\[
U^T p \leq [-1]
\]

\[
p \geq 0
\]

**Claim 1** If the dual is infeasible, then \( f^* = \infty \)
Proof Sketch: Show that if $f^* < \infty$ then the dual is feasible. If $f^* = 0$, then by strong duality $q^* = 0$. If $q^* = 0$, then the dual has a feasible solution. The infeasibility of dual is an immediate consequence of the Papadimitriou’s lemma given below (or else $U^T p \leq [-1]$). □

Lemma 1 (Papadimitriou) For any $p \geq 0$, there exists a probability distribution $x$ such that

$$x^T U^T p = 0$$


4 Continuous strategy spaces

The results we have presented so far concern finite games. As a natural extension, the following theorem states the conditions for the existence of a pure strategy Nash equilibrium in continuous strategy spaces.

Theorem 5 (Debreu, Glicksberg, Fan) Consider a strategic form game $< I, (s_i), (u_i) >$, where $s_i$ is continuous.

Assume:

1. $s_i$ is non-empty, convex, and compact.
2. $u_i(s)$ is continuous in $S$.
3. $u_i(s_i, s_{-i})$ is concave (quasi-concave) in $S_i$.

Then, there exists a pure strategy Nash equilibrium for $< I, (s_i), (u_i) >$.

Example: Unit circle game

Two players pick points $s_1$ and $s_2$ on the unit circle. The payoffs for the two players are

$$u_1(s_1, s_2) = d(s_1, s_2)$$
$$u_2(s_1, s_2) = -d(s_1, s_2)$$

where $d$ is the Euclidean distance metric.

Show that there is no pure strategy Nash equilibrium and find the mixed strategy Nash equilibrium. (Hint: If both players pick the same location, player 1 has incentive to deviate. If they pick different locations, player 2 has incentive to deviate).