# 1 Von Neumann and Morgenstern Preferences

A standard assumption in game theory restricts attention to preferences regarding lotteries (probability distributions) over outcomes that may be represented by the expected value of a payoff function over deterministic outcomes. The first systematic investigation of preferences regarding lotteries represented by the expected value of a payoff function over deterministic outcomes was taken by von Neumann and Morgenstern (1944). Accordingly such preferences are called *vNM preferences*. A payoff function over deterministic outcomes ($u_i$ in our notation) whos expected value represents such preferences is called a *Bernoulli payoff function*.

The following example (known as the Allais paradox) constitutes a challenge to the "expected utility theory". Consider the following two lotteries.

- Lottery 1: You win \$ 2 million with certainty.
- Lottery 2: You win \$ 10 million with probability 0.1, \$ 2 million with probability 0.89, and nothing with probability 0.01.

Which do you prefer? Consider two more lotteries.

- Lottery 3: You win \$ 2 million with probability 0.11, and nothing with probability 0.89.
- Lottery 4: You win \$ 10 million with probability 0.1, and nothing with probability 0.9.

Which do you prefer? Most people express they prefer lottery 1 to 2, and lottery 4 to 3.

These preferences cannot be represented by an expected payoff function. If they could be, there would exist a payoff function $u(\cdot)$ for which the expected payoff of lottery 1 exceeds that of 2:

$$u(2) > 0.1u(10) + 0.89u(2) + 0.01u(0).$$

Subtracting $0.89u(2)$ and adding $0.89u(0)$ to each side, we obtain

$$0.11u(2) + 0.89u(0) > 0.1u(10) + 0.9u(0).$$

But this says the expected payoff to lottery 3 exceeds that of lottery 4!

The preferences need to satisfy some assumptions that allow us to conclude that the decision maker's preferences can be represented by an expected payoff function. The vast majority of models in game theory assume this is the case. We maintain this assumption throughout the course. (We may come back to this issue and examine the assumptions that need to be satisfied by the preferences, if we have time left at the end)

Reference: Chapter 6 of Mas-Colell, Whinston, and Green.