10.13 Notes on error estimates.

Each of the standard convergence tools for series brings with it a method for estimating the error made in approximating the limit by taking only finitely many terms of the series. We treat this method for both the integral test and the ratio test.

Theorem 1. (Integral estimate.) *Let f be a positive decreasing function defined for $x \geqslant 1$. Let $a_k = f(k)$. If the integral $\int_1^\infty f$ exists, then the series $\Sigma\, a_k$ converges, and*

$$\int_{N+1}^{\infty} f(x)\,dx \leqslant \left(\sum_{k=N+1}^{\infty} a_k\right) \leqslant \int_N^{\infty} f(x)\,dx.$$

The expression in the middle is of course the error made in using the finite sum $a_1 + \ldots + a_N$ as an approximation to the sum of the series.

Theorem 2. (Ratio estimate.) *Suppose that $\Sigma\, a_k$ is a series of non-zero terms. Suppose α is a number less than 1, and that*

$$|a_{k+1}/a_k| \leqslant \alpha \quad \text{for all} \quad k > N.$$

Then

$$\left|\sum_{k=N+1}^{\infty} a_k\right| \leqslant \left|a_{N+1}\right| \left(\frac{1}{1-\alpha}\right).$$

The expression on the left is again the error made in approximating the series $\Sigma\, a_k$ by the finite sum $a_1 + \ldots + a_N$.

Exercises

1. Prove Theorem 1.

2. (a) Use Theorem 1 to estimate the error made in approximating the number

$$a = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

by the finite sum $1 + 1/4 + 1/9 + 1/16 + 1/25$. (In fact, $a = \pi^2/6$.)

(b) Given N, estimate the error made by using

$$\sum_{n=1}^{N} \frac{1}{n^2}$$

as an approximation to a.

(c) Estimate the error made by using

$$\sum_{n=1}^{N} \frac{1}{n^2} + \frac{1}{N+1}$$

as an approximation to a.

(d) Estimate the error made in approximating a by the sum $1 + 1/4 + 1/9 + 1/16 + 1/25 + 1/6$.

3. Prove Theorem 2.

4. Estimate the error made in approximating the number $\Sigma_{n=1}^{\infty} n/2^n$ by the finite sum $1/2 + 2/4 + 3/8 + 4/16 + 5/32$.