Due by Noon, Thursday December 5
Rudin:

(1) Chapter 7, Problem 14

Solution. There is a function $f$ as described, just set

\[
 f(t) = \begin{cases} 
 0 & \frac{1}{2} \leq t \leq \frac{3}{4} \\
 3(t - \frac{1}{2}) & \frac{1}{2} \leq t \leq \frac{3}{4} \\
 1 & \frac{3}{4} < t \leq 1 
\end{cases}
\]

and for instance $f(2 - t) = f(t)$ for $1 \leq t \leq 2$ and then $f(2k + t) = f(t)$ for all $k \in \mathbb{N}$, $k \neq 0$, $t \in [0, 2]$. This gives a continuous function. Consider

\[
 (1) \quad x(t) = \sum_{n=1}^{\infty} 2^{-n} f(3^{2n-1}t), \quad y(t) = \sum_{n=1}^{\infty} 2^{-n} f(3^{2n}t).
\]

The $n$th term in the series for (1) is bounded

\[
 |2^{-n} f(3^{2n-1}t)| \leq 2^{-n}.
\]

By Theorem 7.10, the series converges uniformly. Thus $x(t)$ is continuous by Theorem 7.12. The same argument applies to $y(t)$ so $\Phi(t) = (x(t), y(t))$ is continuous by Theorem 4.10. Now, to see that $\Phi$ is surjective, follow the hint. Each point in $[0, 1]$ has a dyadic decomposition

\[
 x = \sum_{n=1}^{\infty} 2^{-n} b_n, \quad b_n = 0 \text{ or } 1.
\]

Indeed one can compute the successive $b_n$, $n = 1, \ldots, k$ to arrange that

\[
 0 \leq x - \sum_{n=1}^{k} 2^{-n} b_n \leq 2^{-k}.
\]

Then choose $b_{k+1} = 0$ or 1 depending on whether

\[
 0 \leq x - \sum_{n=1}^{k} 2^{-n} b_n < 2^{-k-1} \text{ or } 2^{-k-1} \leq x - \sum_{n=1}^{k} 2^{-n} b_n \leq 2^{-k}.
\]

Now we let the $a_{2n-1}$ be these numbers for $x_0$ and $a_{2n}$ those for $y_0$. It follows that $t_0 = \sum_{i=1}^{\infty} 3^{-i-1}(2a_i)$ converges to a number in $[0, 1]$ and that $3^kt_0 = 2N + \frac{2}{3}a_k + t_k$ where $N$ is an integer, $a_k = 0$ or 1 and $t_k \leq \frac{1}{3}$. The properties of $f$ show that $f(3^kt_0) = f(\frac{2}{3}a_k + t_k) = a_k$ since as $a_k = 0$ or 1 then $\frac{2}{3}a_k + t_k \in [0, \frac{1}{3}]$ of $[\frac{2}{3}, 1]$. From the definition of $x(t)$ and $y(t)$ it follows that $\Phi(t_0) = (x_0, y_0)$ and hence $\Phi$ is surjective.

These points $t_0$ are the ones appearing in the Cantor set as defined in section 2.44.

(2) Chapter 7, Problem 18

Solution. If $f_n$ is a uniformly bounded sequence of functions on $[a, b]$ there exists a constant $M$ such that $|f_n(x)| \leq M$ for all $x \in [a, b]$ and all $n$. Since the $f_n$ are Riemann integrable the functions

\[
 F_n(x) = \int_{a}^{x} f_n(t)dt.
\]
are continuous. We show that the sequence \( F_n \) is equicontinuous. In fact if \( x \leq x' \in [a, b] \) then
\[
|F_n(x) - F_n(x')| \leq \int_x^{x'} |f_n(t)| dt \leq M|x - x'|.
\]
Thus, if \( |x - x'| < \delta = \epsilon/M \) then \( |F_n(x) - F_n(x')| < \epsilon \), showing the equicontinuity. We also have a uniform bound \( |F_n(x)| \leq (b - a)M \). As a sequence of uniformly bounded and equicontinuous functions on a compact metric space we see from Theorem 7.25 that \( \{F_n\} \) has a uniformly convergent subsequence. \( \square \)

(3) Chapter 7, Problem 24

Solution. We define a map \( X \rightarrow C(X) \) into the space of bounded continuous functions on \( X \). Fixing a point \( a \in X \) let
\[
f : X \ni p \mapsto f_p, \quad f_p(x) = d(x, p) - d(x, a).
\]
By the triangle inequality \( d(x, p) \leq d(x, a) + d(a, p) \) and \( d(x, a) \leq d(x, p) + d(a, p) \) shows that \( |f_p(x)| \leq d(a, p) \). Thus \( f_p \) is a bounded function. It is continuous, again by the triangle inequality
\[
|f_p(x) - f_p(q)| \leq |d(x, p) - d(y, p)| + |d(x, a) - d(y, a)| \leq 2d(x, y).
\]
Thus \( f_p \in C(X) \). Now, consider
\[
\|f_p(x) - f_q(x)\| = \sup_{x \in X} |d(x, p) - d(x, a) - d(x, q) + d(x, a)|
\]
\[
= \sup_{x \in X} |d(x, p) - d(x, q)| = |d(p, q)|.
\]

The last inequality follows from the triangle inequality and the fact that \( f_p(q) - f_q(q) = d(p, q) \).

It follows that \( f \) is a continuous map. It is an isometry, namely the distance in \( C(X) \) is \( \|f - g\| \). In particular this shows that \( f \) is 1-1. Let \( Y \) be the closure of \( f(X) \subset C(X) \). Then we may regard \( X \subset Y \) using \( f \). As a closed subset of \( C(X) \), \( Y \) is complete as a metric space. Thus \( X \) is isometric to a subset of the complete space \( Y \) in which it is dense. \( \square \)

(4) Chapter 5, Problem 26

Solution. We assume that \( f \) is differentiable on \( [a, b] \), \( f(a) = 0 \) and there is a real number \( A \) such that \( |f'(x)| \leq A|f(x)| \) for all \( x \in [a, b] \). Following the hint, for a chosen \( x_0 \in (a, b] \), set \( M_0 = \sup |f(x)| \) and \( M_1 = \sup |f'(x)| \) with the suprema over \( [a, x_0] \); where the second exists because of the assumption. It follows that \( M_1 \leq AM_0 \). By the mean value theorem \( f(x) - f(a) = f'(y)(x - a) \) for some \( y \in (a, x_0) \) so
\[
|f(x) - f(a)| = |f(x)| = |f'(y)||x - a| \leq M_1(x_a) \leq AM_0(x_0 - a).
\]

Taking the sup over \( x \in [a, x_0] \) we see that \( M_0 \leq AM_0(x_0 - a) \). Taking \( x_0 - a \) so small that \( A(x_0 - a) < 1 \) we see \( M_0 \leq 0 \) and hence \( M_0 = 0 \). Thus \( f(x) = 0 \) on \( [a, x_0] \). Set \( z = \sup \{x_0; f(x) = 0 \text{ on } [a, x_0]\} \). If \( z = b \) we are finished, since \( f \) is continuous so \( f(z) = 0 \). If \( z < b \) then we may apply the argument again to find a contradiction. \( \square \)

(5) Chapter 5, Problem 27
Solution. I did this in class in a slightly different way, so I am just asking you to write down the proof. Namely if \( f_i(x) \) are two solutions, for \( i = 1, 2 \), then we may integrate the equation to see that
\[
f_i(x) = c + \int_a^x \phi(t, f_i(t))dt.
\]
It follows that the difference \( f(x) = f_2(x) - f_1(x) \) satisfies
\[
f(a) = 0, \quad f'(x) = \phi(x, f_1(x)) - \phi(x, f_2(x)).
\]
We see that \( |f'(x)| \leq A|f(x)| \) where \( A \) is the constant in the Lipschitz condition. Applying the previous problem, we conclude that \( f = 0 \).

Differentiating \( y = \frac{1}{4}x^2, \ y' = \frac{1}{2}x = y^\frac{1}{2} \) shows that it is a solution in \([0, 1]\). Thus both \( y \equiv 0 \) and \( y = \frac{1}{4}x^2 \) are both solutions.

These are not the only solutions, since if \( x_0 > 0 \) and we define
\[
y = \begin{cases} 
0 & x < x_0 \\
\frac{1}{4}(x - x_0)^2 & x \geq x_0 
\end{cases}
\]
we get a continuously differentiable solution by the same argument. Conversely every solution, \( y \geq 0 \), of \( y' = y^\frac{1}{2} \) is of this form. Indeed, if \( y(t) > 0 \) for some \( t \in (0, 1) \) then, being continuous, it is positive nearby. Thus we can divide by \( y^\frac{1}{2} \) and conclude that \( \frac{d}{dx} \left( 2y^{-\frac{1}{2}} \right) = 1 \) on any interval where \( y > 0 \). This implies that \( 2y^{-\frac{1}{2}} = x - x_0 \) for some constant \( x_0 \) and hence that (2) holds in any interval where \( y > 0 \). In principle there could be different values of \( x_0 \) on different intervals. However, \( y \) in (2) is increasing, so the set on which it is strictly positive must be of the form \((x_0, 1]\) for some \( x_0 \in [0, 1] \), since \( y(0) = 0 \). Thus the general solution is (2) for some \( x_0 \in [0, 1] \).

PS. I don’t have the book with me, perhaps the upper limit of the interval here is \( \infty \), but the argument is the same. \( \square \)