18.100B, Fall 2002, Homework 7

Was due by Noon, Tuesday November 5.
This was a bit of a stinker.

Rudin:

(1) Chapter 5, Problem 12

Solution. In $x > 0$, $|x|^3 = x^3$ so is infinitely differentiable, being a polynomial, and has derivative $3x^2$. Similarly in $x < 0$, $|x|^3 = -x^3$ is again a polynomial and has derivative $-3x^2$. The limit

$$\lim_{t \to 0} \frac{f(0) - f(t)}{0 - t} = \lim_{t \to 0} |t|^3 / t = 0$$

so $f$ is differentiable at 0 and $f'(x) = 3x|x|$ everywhere. As already noted this is differentiable in $x \neq 0$ and has derivative $6|x|$. The limit

(1) $$\lim_{0 \neq t \to 0} \frac{f'(0) - f'(t)}{0 - t} = \lim_{0 \neq t \to 0} 3|t| = 0$$

again exists, so $f''(x) = 6|x|$ exists everywhere. Finally the third derivative exists for $x \neq 0$ and is $f^{(3)}(x) = 6 \sgn x$, $\sgn x = \pm 1$ as $x > 0$ or $x < 0$. The limit of

$$\frac{f(0) - f''(t)}{0 - t} = \frac{f(0) - f(t)}{0 - t} 6 \sgn t$$

does not exist as $0 \neq t \to 0$, so $f^{(3)}(0)$ does not exist. \hfill \square

(2) Chapter 5, Problem 14

Solution. By assumption, $f(x)$ is convex and differentiable on $(a,b)$. Thus $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$ for $t \in [0,1]$, $x \leq y \in (a,b)$.

For any three points $x < y < z \in (a,b)$ the difference quotient satisfies

$$\frac{f(x) - f(y)}{x - y} \leq \frac{f(x) - f(z)}{x - z} \leq \frac{f(y) - f(z)}{y - z}$$

as shown last week. Letting $y \downarrow x$ in the first inequality, and using the differentiability of $f$ shows that

$$f'(x) \leq \frac{f(x) - f(z)}{x - z} \leq \frac{f(y) - f(z)}{y - z}$$

where $x, y, z$ are again any points satisfying $x < y < z$. Now setting $y \uparrow z$ we conclude that $f'(x) \leq f''(z)$ if $x < z$.

Conversely, suppose $f'(x)$ is monotonically increasing on $(a,b)$. Using the mean value theorem, if $x < z$ then $f(z) - f(x) = (z-x)f'(y)$ for some $y \in (x,z)$ so $f'(x) \leq \frac{f(z) - f(x)}{z - x} \leq f'(z)$. For three points $x < z < y$ this gives

$$f(z) - f(x) \leq \frac{f(z) - f(y)}{z - y}$$

and setting $t = \frac{y - z}{y - x}$ so $z = tx + (1-t)y$ this is precisely

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \quad \forall \ t \in (0,1)$$

which is convexity.
If $f''(x)$ exists for all $x \in (a, b)$ and $f'' \geq 0$ then $f'(x)$ is increasing and so $f$ is convex. Conversely if $f$ is convex then $f'$ is increasing and hence $f'' \geq 0$. \hfill \square

(3) Chapter 5, Problem 15

I should have said not to do the last part, since I have not talked much about differentiation of vector-valued functions.

**Solution.** The question is quite as clear as should be, you are supposed to assume that $M_0$ and $M_2$ are finite.

Following the hint, recall that Taylor’s theorem shows that

$$f(x + 2h) = f(x) + 2hf'(x) + \frac{(2h)^2}{2}f''(\xi)$$

for some $\xi \in (x, x + 2h)$ which can be written

$$f'(x) = \frac{1}{2h}[f(x + 2h) - f(x)] - hf''(\xi).$$

Thus

$$|f'(x)| \leq \frac{1}{2h}[|f(x + 2h)| + |f(x)|] + h|f''(\xi)|$$

and so with $M_0$ an upper bound for $|f|$ and $M_2$ and upper bound for $f''$, we have

$$|f'(x)| \leq hM_2 + \frac{M_0}{h}, \quad \forall \ h > 0, \ x \in (a, \infty).$$

Taking the supremum over $x$ for each $h > 0$ we find

$$M_1 \leq hM_2 + \frac{M_0}{h}, \quad \forall \ h > 0.$$

We can assume $M_0$, $M_2 > 0$ since if $M_2 = 0$ then $f$ is linear and $M_0$ is infinite. If $M_0 = 0$ then $f \equiv 0$. The right side is differentiable in $h$ with derivative $M_2 - h^{-2}M_0$. This vanishes when $h = \sqrt{M_0/M_2} > 0$, substituting this gives

$$M_1 \leq 2\sqrt{M_0M_2} \iff M_1^2 \leq 4M_0M_2.$$

For the given

$$f(x) = \begin{cases} 2x^2 - 1 & -1 < x < 0 \\ \frac{x^2 - 1}{x^2 + 1} & 0 \leq x < \infty \end{cases}$$

we see that

$$f'(x) = \begin{cases} 4x & -1 < x < 0 \\ \frac{4x}{(x^2 + 1)^2} & 0 < x < \infty \end{cases}$$

also exists at 0 where it has the value 0. Then $f''(x)$ also exists at 0, taking the value 4 and

$$f''(x) = \begin{cases} 4 & -1 < x < 0 \\ \frac{4(1-x^2)}{(x^2 + 1)^3} & 0 < x < \infty \end{cases}$$

Now, $f' < 0$ in $x < 0$ and $f' > 0$ for $x > 0$ so

$$\sup |f(x)| = M_0 = 1.$$
Similarly $f'' \geq 0$ in $x < 1$ and $f'' < 0$ in $x > 1$ so $f'$ takes its maximum value at $x = 1$ and since it is positive for $x > 0$ its minimum is $-4$ so

$$M_1 = \sup |f'(x)| = 4.$$ 

Finally then $M_2 = \sup |f''| = 4$ since in $x > 0$ it decreases to its zero at $x = 1$ and for $x > 1$, $f'' > -4x^2/(x^2 + 1)^3 \geq -4$. Thus equality can occur.

Yes, the result is true for vector valued functions for the usual Euclidean norms. Let $f = (f_1, f_2, \ldots, f_k)$ be a function with values in $\mathbb{R}^k$. Thus the assumption is that each of the components satisfies the assumptions of the question and we set

$$M_i = \sup_{x \in (a, \infty)} |f^{(i)}(x)|$$

with the Euclidean norm. Now, suppose that $a = (a_1, \ldots, a_k) \in \mathbb{R}^k$ is a (constant) vector. We can apply the result above to $g(x) = a \cdot f(x) = a_1 f_1(x) + \cdots + a_k f_k(x)$. We see then that for any $x \in (a, \infty)$

$$|g'(x)| \leq 4 \sup |g| \sup |g''| \leq 4|a|^2 M_0 M_2.$$ 

Now we can set $a = f'(x)$ for a given $x$ and divide by a factor of $|f'(x)|^2$ and so conclude that

$$|f'(x)|^2 \leq 4M_0 M_2.$$ 

Taking the supremum over $x$ now gives the vector-valued result. 

(4) Chapter 6, Problem 2

Solution. Since $f$ is continuous it is Riemann integrable and $f \geq 0$, either $f = 0$ or there exists an interval of positive length, $l > 0$, in $[a, b]$ on which $f(x) \geq c > 0$. Then there exists a partition, with the end points of this interval as two of its points, such that

$$L(P, f) \geq lc > 0.$$ 

Since $\int_a^b f dx \geq L(P, f)$ for any partition, this implies $\int_a^b f dx > 0$ so $\int_a^b f dx = 0$ must imply $f \equiv 0$.

Or, you could use the fundamental theorem of calculus. 

(5) Chapter 6, Problem 4

Solution. For any partition $P$ we have

$$U(P, f) - L(P, f) = \sum_{i=1, x_{i-1} > x_i}^n (x_i - x_{i-1}) (\sup_{[x_{i-1}, x_i]} f - \inf_{[x_{i-1}, x_i]} f).$$

Now, any interval of non-zero length contains both rational and irrational points, so the difference of sup $f$ and inf $f$ is always one. It follows that

$$U(P, f) - L(P, f) = (b - a)$$

so the function cannot be Riemann integrable.