Modeling and Analysis of Two-Part Type Manufacturing Systems

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***Abstract*— This paper presents a model and analysis of a synchronous tandem flow line that produces two different part types on unreliable machines. The machines operate according to a static priority rule, operating on the highest priority part whenever possible, and operating on lower priority parts only when unable to produce those with higher priorities. We develop a new *decomposition method* to analyze the behavior of the manufacturing system by decomposing the long production line into small analytically tractable components. As a first step in modeling a production line with more than one part type, we restrict ourselves to the case where there are two part types. Detailed modeling and derivations are presented with a small two-part type production line that consists of two processing machines and two demand machines. Estimates for performance measures, such as average buffer levels and production rates, are presented and compared to extensive discrete event simulation.**

***Index Terms*— manufacturing system, flexible, flow line, finite buffers, unreliable machines, markov chain model, decomposition.**

I. Introduction

This paper presents a model and analysis of a synchronous tandem production line that produces two different part types on unreliable machines. Inventory is stored between machines in finite buffers. We assume that machines in the processing line are flexible in that they can operate on different part types, and there are no set-up penalties incurred when machines switch production from one part type to another. The machines operate according to a static priority rule, operating on the highest priority part whenever possible, and operate on lower priority parts only when unable to produce those with higher priorities due to either blockage or starvation.

Gershwin [1] introduced a decomposition method that analyzes the behavior of the manufacturing system with a stochastic queuing model. This method models a manufacturing system as a flow line with unreliable machines and finite buffers. This decomposition method was limited to a single part type case. Nemec [2] formulated a deterministic single failure multi-part type transfer line. However, this formulation worked only for small two-part type lines, and there is no clear way of generalizing his equations for longer lines. Syrowicz [3] proposed a way of analyzing two-part type line with multiple-failure modes. This approach made the decomposition of the multi-part type line easier than the decomposition introduced by Nemec [2]. However, the Markov model of the two-machine line, the basic building block of the decomposition, for this model was complex. Moreover, there were too many variables and equations to solve with this.

As a first step in modeling a transfer line with more than one part type, we restrict ourselves to the case where there are two part types. We verify our results by comparing them with simulation. The qualitative behavior of the multiple-part-type processing line under different supply and demand scenarios is also investigated.

This paper is organized as follows. Section II introduces a Markov model of a processing line with two different part types. The decomposition of the long line into smaller, tractable two-machine lines is also discussed. Section III presents the analysis of the Markov chain for the two-machine lines. Compared to the single-part-type line, the two-part-type line behavior is very complicated. However, all of the fundamental concepts of the decomposition of a two-part-type production line can be described in terms of the small production line system, composed of two processing machines and two demand machines, without becoming burdened with the algebraic difficulties of the longer production line system. Section IV introduces the modeling process of the small multiple-stage production line. Then Type 1 and Type 2 part decomposition methods are introduced for the small production line in Section V and in Section VI.An algorithm to solve the decomposition is presented in Section VII, as are numerical results concerning the accuracy of the decomposition, and the qualitative behavior of the system.

II. Two-Part-Type Processing Line

A. Notation

Figure 1 represents a production line processing two different part types. The line consists of two kinds of components: processing machines M_i, denoted by the squares and finite-capacity storage buffers $B_{i,j}$ for work in process inventory, denoted by the circles. Let us define K to be the number of machines that are processing two different part types in the line, not including the supply and demand machines. At the beginning and end of the line, there are supply machines $M_{0,1}$, and $M_{0,2}$, and demand machines: $M_{K+1,1}$, and $M_{K+1,2}$.

Machines $M_{0,1}$ and $M_{K+1,1}$ process only Type 1 parts, while machines $M_{0,2}$ and $M_{K+1,2}$ process only Type 2 parts. Each machine, other than the supply and demand machines, process both part types. We assume that there is no set-up time incurred when the machines switch production from one part type to another. When M_i completes work on a part, it sends the part to a buffer downstream of the machine. Each part type has a distinct buffer after each machine. Therefore, a Type 1 part processed at M_i would be sent to $B_{i,1}$. A Type 2 part processed at the same machine would be sent to $B_{i,2}$.

We assume that all the machines in the line, including supply and demand machines, are unreliable. Let α denote the state of a machine. If $\alpha = 1$, the machine is said to be *up* or *working*. If $\alpha = 0$, the machine is said to be *down* or *failed*. We let $\alpha_{0,1}(t)$ denote the state of supply machine $M_{0,1}$ at the end of time t. We define $\alpha_{0,2}(t)$ similarly for $M_{0,2}$. For the demand machine, $M_{K+1,1}$ and $M_{K+1,2}$, we let the corresponding state variables be $\alpha_{K+1,1}(t)$ and $\alpha_{K+1,2}(t)$. For processing machine M_i, the state variable representing the state of the machine at the end of time t is written $\alpha_i(t)$. We make the assumption that all the machines in the line, including the supply and demand machines, have *homogeneous processing times*. That is, the lengths of time that parts spend in machines are fixed, known in advance, and the same for all the machines. For convenience, the processing times are assumed to

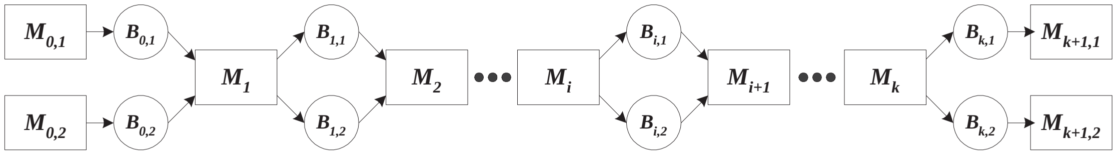

Fig. 1. A two-part type production line

be scaled to unity. Furthermore, we assume that the yield of all machines is 100%. That is, we do not allow the scrapping or rework of parts.

We assume that all buffers, including the supply and demand buffers, have finite size. The size of buffer $B_{i,j}$ is denoted $N_{i,j}$, where i indicates the production stage, and $j = 1$ or 2, represents the part type. We let buffers $B_{0,1}$ and $B_{0,2}$ denote the supply buffers for Type 1 and Type 2, respectively. Likewise, buffers $B_{K,1}$ and $B_{K,2}$ denote the demand buffers for Type 1 and Type 2, respectively. We denote the current level of $B_{i,j}$ at the end of time t by $n_{i,j}(t)$. Therefore, $0 \leq n_{i,j}(t) \leq N_{i,j}$, for all (i, j), and for all $t \geq 0$. A machine is said to be *starved* for a given part type if the upstream buffer corresponding to that part type is empty. It is *blocked* for a given part type if the corresponding downstream buffer is full. We make the assumptions that the supply machines are never starved and the demand machines are never blocked.

B. Machine Parameters and Dynamics

As mentioned earlier, all machines in the line are assumed to be unreliable. We further assume that machines cannot fail if they are idle. This is called *operation dependent failures*. It means that the supply machines cannot fail if they are blocked and the demand machines cannot fail if they are starved. A processing machine cannot fail if it is either starved or blocked for Type 1 parts, and at the same time starved or blocked for Type 2 parts.

All machines are assumed to have geometrically distributed up and down times. We assume that the probability that M_i fails is the same, regardless of the part type the processing machine is working on. We let r_i represent the probability that M_i is up in time $t+1$, given it was down in time t. Likewise, p_i represents the probability that M_i is down in time $t+1$, given it was up and not blocked or starved in time t. For the supply machines, we let $r_{0,1}$ and $r_{0,2}$ represent the probability that $M_{0,1}$ and $M_{0,2}$ are up in time $t+1$, given they were down in time t. Also, $p_{0,1}$ and $p_{0,2}$ represent the probability that $M_{0,1}$ and $M_{0,2}$ are down in time $t+1$, given they were up *and* not blocked in time t. For the demand machines $M_{K+1,1}$ and $M_{K+1,2}$, the corresponding parameters are written $r_{K+1,1}$, $p_{K+1,1}$, $r_{K+1,2}$, and $p_{K+1,2}$. For M_i, the machine parameters can be written as:

$$\begin{aligned} r_i &= Pr\left[\alpha_i(t+1) = 1 | \alpha_i(t) = 0\right] \quad (1)\\ p_i &= Pr\left[\alpha_{i,1}(t+1) = 0 \right| \\ &\quad \{\alpha_{i,1}(t) = 1 \cap n_{i-1,1}(t) > 0 \cap n_{i,1}(t) < N_{i,1}\} \cup \\ &\quad \{\alpha_{i,1}(t) = 1 \cap (n_{i-1,1}(t) = 0 \cup n_{i,1}(t) = N_{i,1}) \\ &\quad \cap n_{i-1,2}(t) > 0 \cap n_{i,2}(t) < N_{i,2}\}] \\ &\text{for } i = 1, \ldots, K \end{aligned}$$

Likewise, for the supply and demand machines, the machine parameters are defined as:

$$\begin{aligned} r_{0,i} &= Pr\left[\alpha_{0,i}(t+1) = 1 | \alpha_{0,i}(t) = 0\right] \\ p_{0,i} &= Pr\left[\alpha_{0,i}(t+1) = 0 | \alpha_{0,i}(t) = 1 \cap n_{0,i}(t) < N_i\right] \\ &\text{for } i = 1, 2 \end{aligned}$$

$$\begin{aligned} r_{K+1,j} &= Pr\left[\alpha_{K+1,j}(t+1) = 1 | \alpha_{K+1,j}(t) = 0\right] \\ p_{K+1,j} &= Pr\left[\alpha_{K+1,j}(t+1) = 0 \right| \\ &\quad \alpha_{K+1,j}(t) = 1 \cap n_{K,j}(t) > 0] \\ &\text{for } j = 1, 2 \end{aligned}$$

C. Part Type Priority Policy

Since each machine in the production line must choose which part to work on when it has a choice, we are required to state a policy by which that choice is made. Our assumption is that each machine will work on Type 1 parts whenever the machine is up, the upstream buffer for Type 1 parts is not empty, and the downstream buffer for Type 1 parts is not full. Each machine will only work on Type 2 parts if it is up, and either blocked or starved for Type 2 parts, and not starved or blocked for Type 2 parts.

D. Production Rate

Let us denote the production rate of Type 1 parts at M_i by $E_{i,1}$. This is the fraction of time that M_i is working on Type 1 parts. We know that M_i will make a Type 1 part at the end of time $t+1$ if M_i is not starved for Type 1 parts at time t, M_i is not blocked for Type 1 parts at time t, and M_i is up at the end of time $t+1$. This probability is expressed as follows:

$$\begin{aligned} E_{i,1} &= Pr\left[\alpha_i(t+1) = 1 \cap \right. \\ &\quad \left. n_{i-1,1}(t) > 0 \cap n_{i,1}(t) < N_{i,1}\right] \end{aligned} \quad (2)$$

Let the quantity $E_{i,2}$ denote the production rate of Type 2 parts. This is the fraction of time that M_i is working on Type 1 parts. From our assumptions, we know that M_i will make a Type 2 part at time $t+1$, if M_i is either blocked or starved for Type 1 at time t; M_i is not starved or blocked for Type 2; and M_i is up at the end of time $t+1$. This is:

$$\begin{aligned} E_{i,2} &= Pr\left[\alpha_i(t+1) = 1 \cap (n_{i-1,1}(t) = 0 \cup \right. \\ &\quad \left. n_{i,1}(t) = N_{i,1}) \cap n_{i-1,2}(t) > 0 \cap n_{i,2}(t) < N_{i,2}\right] \end{aligned} \quad (3)$$

In steady state, because of conservation of flow, we require that each machine in the line makes the same number of Type 1 and Type 2 parts. If we denote the throughput for the demand machine for Type j parts by $E_{K+1,j}$, and the supply machine for Type j parts by $E_{0,j}$, then we must have

$$E_{0,j} = E_{1,j} = E_{2,j} = \ldots = E_{i,k} = E_{K+1,j}, \text{ for } j = 1, 2$$

E. Basic Idea of Decomposition

We intend to break down the larger system into analytically tractable two-machine lines, and capture the local behavior of the long line, as seen by an observer in a buffer, by choosing appropriate parameters of the two-machine lines. This decomposition procedure is represented in Figure 2. As discussed earlier, the idea is to fool an observer in a buffer in the long, multi-part type processing line into thinking he is in a two-machine line. In the figure, the inflow and outflow behavior of material an observer in buffer $B_{i,1}$ could see is modeled by the two-machine, one-part line $L(i, 1)$.

Close observation of the dynamics of the long line, however, shows the necessity for a new two-machine line model. The reason is as follows. Suppose that we take the point of view of an observer in $B_{i,1}$. We misinform this observer: we lead him to believe that he is watching the flow in the only buffer in a two-machine, one-buffer, one-part type system. Let us assume that the observer sees that the outflow from his buffer has ceased, but the inflow has not. Eventually, unless the outflow resumes or the inflow ceases, $B_{i,1}$ will fill up. According to our scheduling rule, M_i will immediately begin making Type 2 parts, if it is able to. Suppose it does, and that M_i fails while making a Type 2 part. Now suppose that while M_i is down, the outflow from $B_{i,1}$ begins again. Then the sequence of events that the observer will see are that the outflow ceased, the buffer filled up, but when the outflow began again, the inflow did not. As far as the observer in the buffer is concerned, the machine upstream of him failed while it was blocked.

There is a subtlety here that must be paid close attention to. While this apparent idleness failure is behavior that an observer in a buffer sees, it is important to remember that the real machines do not fail when they are idle. It only appears to the observer that the machine has failed during an idle period, because the observer believes that he is in a two-machine, one-part type line. Therefore, while in our previous model we assumed that both the real machines and the pseudo-machines in the two-machine sub-lines had operation dependent failures, we must relax that assumption for the two-machine sub-lines in the two-part type case. Thus, a new two-machine line model is in order. We present a discrete-time, discrete-state Markov model of precisely such a line in Section III.

III. Two-Machine Line with Idleness Failure

A. Idleness Failure and Failure-Mode Change

As discussed in the previous section, in order to decompose the Markov chain model of the two-part-type processing line, we need a new two-machine line. The two-machine line presented here is similar to the deterministic processing time with multi-failure-mode model described by Tolio [4].

1) Idleness Failure: As in the Tolio decomposition, the upstream machine or downstream machine in the two-machine-line can fail into *local failure modes* and *remote failure modes*. The local failure mode is the failure of the real machine as represented by the upstream machines or downstream machines in the two-machine line. The remote failure mode is the failure introduced to account for the effect of a local failure caused by a machine outside of the two-machine-line. We follow the concept of multi-failure mode in constructing the two-machine-line. However, in our model, as discussed earlier, the machines in the two-machine-line are no longer restricted to failing only if they are not blocked or starved. Since a machine in the two-machine line can fail while it is idle – starved or blocked – we call the line, *a two-machine line with idleness failure*.

2) Failure Mode Change: When an upstream or downstream machine in the two-machine line for type-one parts is in a remote failure mode, the real machine represented by the upstream or downstream machine could work on type-two parts. If this real machine fails while working on a type-two; the upstream or downstream machine will realize that the failure mode which it is in has been shifted from the remote failure mode, which is an initial failure, to the local failure mode. We call this shifting mode change *a failure mode change*. There are two important observations about failure-mode changes. The first is that a failure mode can only change to a mode corresponding to a machine which is closer to the observer. The reason for this is that the initiating failure corresponds to a real failure of some machines, which has propagated by means of starvation or blockage to the observer's location.

B. Two-Machine-Line Notations and Parameters

The two-machine lines are illustrated in Figure 2. As is our convention, the machines are denoted by squares, and the buffer by circles. We denote the upstream machine by M^u, and the downstream machine by M^d. We denote the size of the intermediate buffer by N, and the current level of the intermediate buffer by n. It follows that $0 \leq n \leq N$. We define the state of the two-machine line to be $s = (n, \alpha^u, \alpha^d)$. α^u is Δ_i if M^u is down at mode i, and Υ^u if M^u is up. α^d is Δ_j^d if M^d is down at mode j, and Υ^d if M^d is up.

Material flows into the upstream machine from an infinite supply, is processed by the machine, and when processing is complete, the material is placed into the buffer until it is processed by the downstream machine. Upon finishing being processed by the downstream machine, the part leaves the line. We assume that there is always room for the downstream machine to unload a part it has just completed processing. We make the assumption that there is only one class of parts produced by the line, and that the production time at each of the machines is identical, and equal to one.

The machines are unreliable and can fail in multiple failure modes. We assume that the machines can fail while they are either operating on a part or idle, but we do not assume that the probabilities of failure are identical. In particular, we assume that the probability that M^u fails into mode i while it is working on a part, given it is not blocked, is p_i^u, and the probability that it fails into mode j while it is blocked is q_j^u. Note that we do *not* assume that there is a new failure mode, but only that there is a new way of reaching the failure mode. We define the quantities p_i^d and q_j^d for M^d similarly. Finally, we denote the probabilities that M^u and M^d are repaired while they are down at failure mode j by r_j^u and r_j^d, respectively.

A probability expressed as $z_{i,i'}^u$ represents the probability of the upstream machine having a change from down mode i to down mode i'. The expression $z_{j,j'}^d$ represents the probability that the downstream machine has a change from down mode j to down mode j'. Defining $\alpha^{\dagger}(t)$ as the state (up state or down state) of a machine † at time t (where † is either u or d for upstream or downstream), then we can define r, p, q, and z as

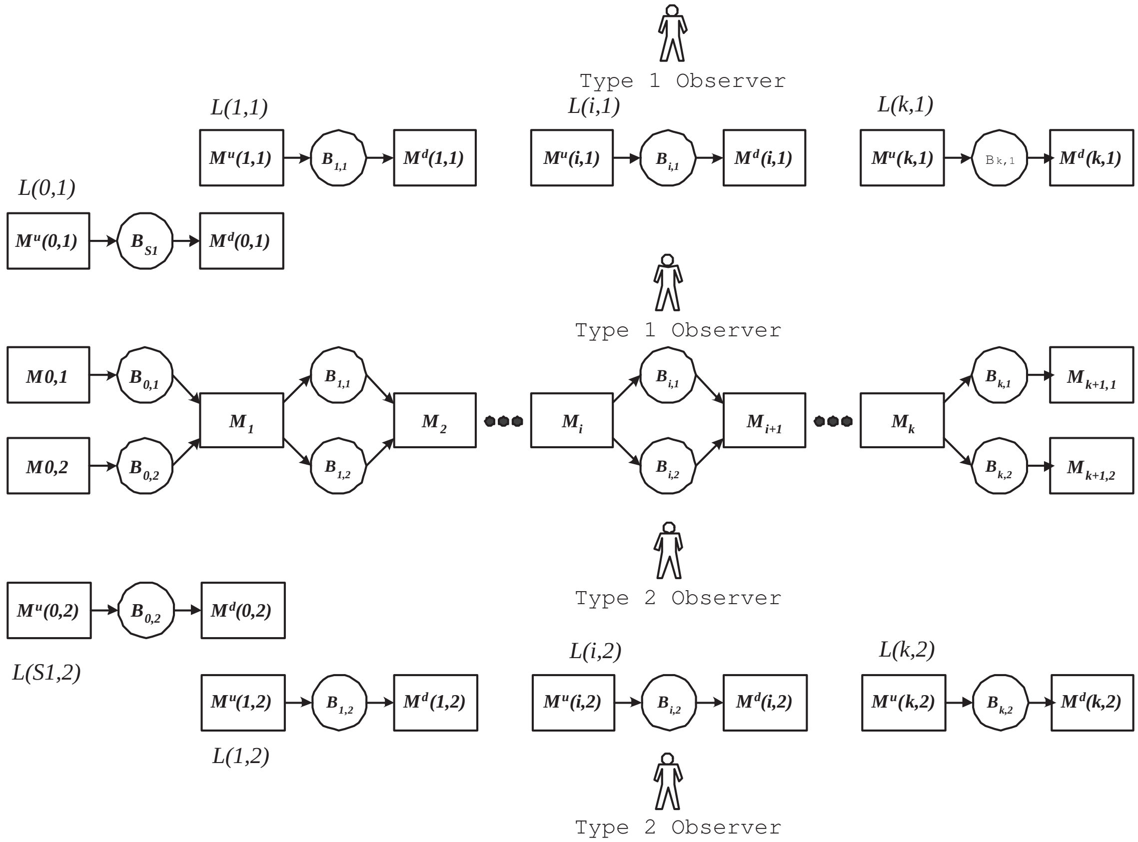

Fig. 2. The decomposition of a line into two-machine lines

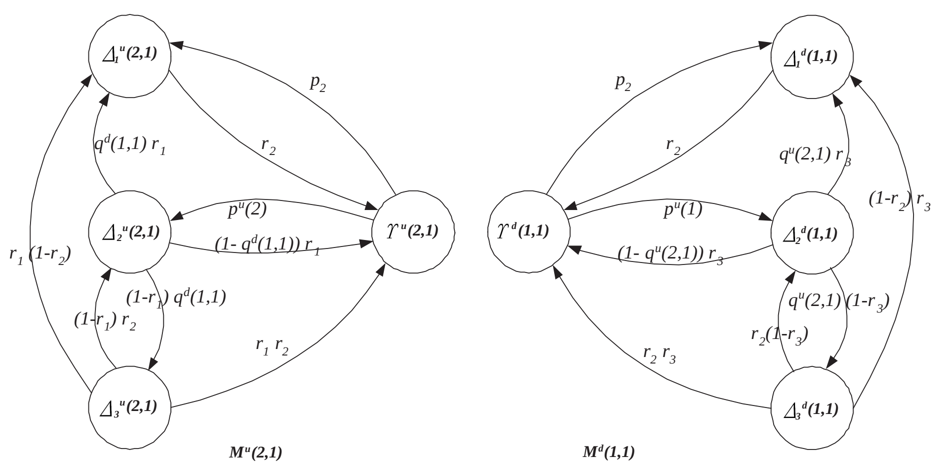

Fig. 3. Example of Markov states of M^d with three failure modes

$$
\begin{aligned}
r^\dagger_{j,i} &= Pr[\alpha^\dagger(t+1) = \Upsilon^\dagger_i \mid \alpha^\dagger(t) = \Delta^\dagger_j] \\
p^\dagger_{i,j} &= Pr[\alpha^\dagger(t+1) = \Delta^\dagger_j \mid \alpha^\dagger(t) = \Upsilon^\dagger_i \text{ and } n(t) < N] \\
q^\dagger_{i,j} &= Pr[\alpha^\dagger(t+1) = \Delta^\dagger_j \mid \alpha^\dagger(t) = \Upsilon^\dagger_i \text{ and } n(t) = N] \\
z^\dagger_{j,j'} &= Pr[\alpha^\dagger(t+1) = \Delta^\dagger_{j'} \mid \alpha^\dagger(t) = \Delta^\dagger_j] \\
& \quad \text{for } \dagger = u \text{ and } d
\end{aligned}
$$

We also define P^u and P^d such that

$$P^u = \sum_{j=1}^{J} p^u_j \qquad \text{and} \qquad P^d = \sum_{l=1}^{L} p^d_l$$

where J and L are the numbers of failure modes for the upstream machine and downstream machine respectively. The set of parameters p^u_j and p^d_l must be such that $P^u < 1$ and $P^d < 1$. We define

$$Q^u = \sum_{j=1}^{J} q^u_j \qquad \text{and} \qquad Q^d = \sum_{l=1}^{L} q^d_l$$

and again, the set of parameters q^u_j and q^d_l must be such that $Q^u < 1$ and $Q^d < 1$.

C. Efficiency with Idleness Failure

For every machine failure, there is also a repair. That is, for both the up- and down-stream machines in the two-machine line, the conditional probability that a machine is repaired, given it is down, times the probability that it is down is equal to the conditional probability that the machine fails times the probability that the machine is up. For the upstream machines, that is expressed as

$$\begin{aligned} & r_j^u(Pr[\alpha^u = \Delta_j^u \cap n < N] + Pr[\alpha^u = \Delta_j^u \cap n = N]) \\ & \quad = p_j^u Pr[\alpha^u = \Upsilon_j^u \cap n < N] + q_j^u Pr[\alpha^u = \Upsilon_j^u \cap n = N] \end{aligned} \tag{4}$$

Likewise, for the downstream machine,

$$\begin{aligned} & r_l^d(Pr[\alpha^d = \Delta_l^d \cap n > 0] + Pr[\alpha^d = \Delta_l^d \cap n = 0]) \\ & \quad = p_l^d Pr[\alpha^d = \Upsilon_l^d \cap n > 0] + q_l^d Pr[\alpha^d = \Upsilon_l^d \cap n = 0] \end{aligned} \tag{5}$$

We can use (4) and (5) to derive expressions for efficiencies for upstream and downstream machines. The upstream machine produces a part in time step $t+1$ if it is up at the end of time step $t+1$, and was not blocked at the end of time step t. We can then write E^u as follows:

$$E^u = Pr[\alpha^u(t+1) = \Upsilon^u \cap n(t) < N] \tag{6}$$

Observe that this expression has both time step $t+1$ and time step t in it. We proceed by conditioning on events occurring time step t to write (6) in terms of events occurring entirely in time step t. By doing so, we will be able to express the production rate of the upstream machine entirely in terms of the state probabilities, which are defined only on one time step.

$$\begin{aligned} E^u &= Pr[\alpha^u(t+1) = \Upsilon^u \cap n(t) < N] \\ &= Pr[\alpha^u(t+1) = \Upsilon^u | \alpha^u(t) = \Upsilon^u \cap n(t) < N] \\ &\quad \times Pr[\alpha^u(t) = \Upsilon^u \cap n(t) < N] \\ &\quad + \sum_{j=1}^{J} (Pr[\alpha^u(t+1) = \Upsilon^u | \alpha^u(t) = \Delta_j^u \cap n(t) < N] \\ &\quad \times Pr[\alpha^u(t) = \Delta_j^u \cap n(t) < N]) \\ &= (1 - P^u) Pr[\alpha^u(t) = \Upsilon^u \cap n(t) < N] \\ &\quad + \sum_{j=1}^{J} r_j^u Pr[\alpha^u(t) = \Delta_j^u \cap n(t) < N] \end{aligned}$$

If we apply the fact that the repair frequency equals failure frequency expressed in (4), then E^u is

$$\begin{aligned} E^u &= Pr[\{\alpha^u(t) = \Upsilon^u\} \cap \{n(t) < N\}] \\ &+ Q^u Pr[\{\alpha^u(t) = \Upsilon^u\} \cap \{n(t) = N\}] \\ &- \sum_{j=1}^{J} r_j^u Pr[\{\alpha^u(t) = \Delta_j^u\} \cap \{n(t) = N\}] \end{aligned}$$

since

$$\begin{aligned} Pr[\{\alpha^u(t) = \Upsilon^u\} \cap \{n(t) = N\}] &= \sum_{l=1}^{L} Pr(N, \Upsilon^u, \Delta_l^d) \\ Pr[\{\alpha^u(t) = \Delta_j^u\} \cap \{n(t) = N\}] &= \sum_{l=1}^{L} Pr(N, \Delta_j^u, \Delta_l^d) \end{aligned}$$

E^u is therefore,

$$\begin{aligned} E^u &= \sum_{n=0}^{N-1} Pr(n, \Upsilon^u, \Upsilon^d) + \sum_{n=0}^{N-1} \sum_{l=1}^{L} Pr(n, \Upsilon^u, \Delta_j^d) \\ &+ Q^u \sum_{l=1}^{L} Pr(N, \Upsilon^u, \Delta_l^d) - \sum_{j=1}^{J} r_j^u \sum_{l=1}^{L} Pr(N, \Delta_j^u, \Delta_l^d) \end{aligned} \tag{7}$$

Similarly, for the downstream machine:

$$\begin{aligned} E^d &= \sum_{n=1}^{N} Pr(n, \Upsilon^u, \Upsilon^d) + \sum_{n=1}^{N} \sum_{j=1}^{J} Pr(n, \Delta_j^u, \Upsilon^d) \\ &+ Q^d \sum_{j=1}^{J} Pr(0, \Delta_j^u, \Upsilon^d) - \sum_{l=1}^{L} r_l^d \sum_{j=1}^{J} Pr(0, \Delta_j^u, \Delta_l^d) \end{aligned} \tag{8}$$

IV. Small Two-Part-Type Production Line

In this section, we introduce the concepts of the decomposition equations of a two-part-type long production line using a small production system. All of the fundamental concepts of the decomposition of the two-part-type production line can be described in terms of the small production line shown in Figure 4 without the algebraic difficulties of a longer production line. The small production line consists of two processing machines, two demand machines and four homogeneous buffers.

In Figure 4, M_1 and M_2 are processing machines — capable of processing two different part types with the priority rule described in Section II, while $M_{3,1}$ and $M_{3,2}$ are demand machines processing only Type 1 and Type 2 parts, respectively. Again, the buffers are homogeneous.

A. Model Assumptions and Notation for Two-Machine Lines

The decomposition of the system is also shown in Figure 4. There are four two-machine lines. Each line is denoted by $L(i,j)$. The line indices i and j indicate the ith two-machine line imitating the flow behavior of the jth part type in $B_{i,j}$. For example, $L(1,2)$ represents the first two-machine-line imitating the behavior of the second part type. The upstream and downstream machines in $L(i,j)$ are denoted by $M^u(i,j)$ and $M^d(i,j)$.

Although the actual system processes two different part types, the decomposed two-machine lines behave as though they are only processing a single part type. That is, lines $L(1,1)$ and $L(2,1)$ imitate the flow behavior of only Type 1 parts, while $L(1,2)$, and $L(2,2)$ imitate those of only Type 2 parts.

The machines are unreliable and they may have more than one failure mode. We assume that the machines can fail while they are either operating on a part or while they are idle, but we do not assume that the probabilities of failure are identical. In particular, we assume that the probability that $M^u(i,j)$ goes down in failure mode m while it is working on a part, given it is not blocked, is $p_m^u(i,j)$, and the probability that it fails into failure mode m while it is blocked is $q_m^u(i,j)$. Note that we do *not* assume that there is a new failure mode, but only that there is a new way of reaching failure modes. We define the quantities $p_m^d(i,j)$ and $q_m^d(i,j)$ for $M^d(i,j)$ similarly. Finally, we denote the probabilities that $M^u(i,j)$ and $M^d(i,j)$ are repaired

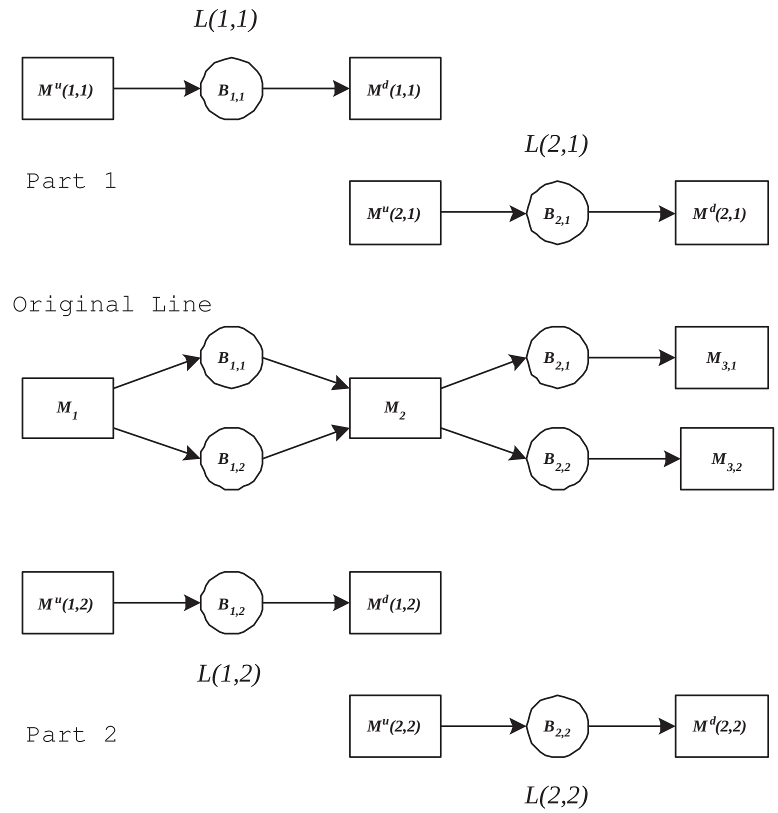

Fig. 4. Decomposition of the small production line processing two different part types

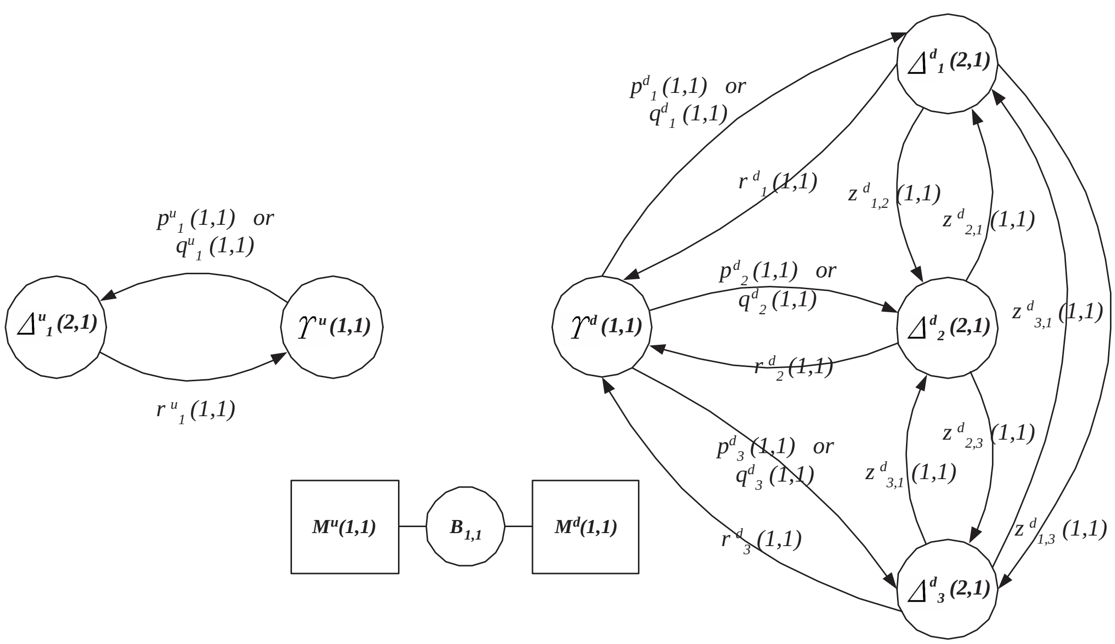

Fig. 5. Multiple failure modes and transition parameters of $L(1,1)$

while they are down in mode m by r_m^u and r_m^d, respectively. We call a failure that takes place while the machine is operating on a part an *operational failure*, and a failure that occurs when the machine is idle an *idleness failure*.

B. Approximation

In order to derive the decomposition, we need to make a crucial approximation. We will assume that the probability that a machine M_2 is simultaneously starved and blocked for a given part is negligible. That is, we assume that

$$\begin{aligned} Pr[n_{1,1}(t) = 0 \cap n_{2,1}(t) = N_{2,1}] \approx 0 \\ Pr[n_{1,2}(t) = 0 \cap n_{2,2}(t) = N_{2,2}] \approx 0 \end{aligned} \tag{9}$$

We justify this approximation with the following argument. In order for the machine to be both starved and blocked for a part simultaneously, it is necessary that at some point, the machine had exactly one part in the upstream buffer, and exactly one space in the downstream buffer. At the same time, the upstream machine must be unable to process parts to place in its downstream buffer, and the downstream machine must be unable to process parts, depleting the stores of its upstream buffer. Since the probability of a machine failing is assumed to be small — on the order of one percent — the probability that all three of these occurrences happen at the same time is likely to be quite low. In fact, testing this hypothesis using discrete event simulation has shown that the approximation holds in many systems with moderate sized buffers.

V. Decomposition Analysis for Type 1

We construct equations for Type 1 parts of the production line shown in Figure 4. As mentioned in Section II, the concept of the decomposition is to relate states of the real line with states in the corresponding two-machine lines of the decomposition. The two-part-type line flow behavior is much more complicated then the single part-type line. To simplify the presentation, we explain the decomposition equations for the two-machine-line in detail.

Throughout the rest of the section, we focus on $M^d(1,1)$ in $L(1,1)$. This downstream machine in the first two-machine line experiences all the critical flow behavior of Type 1 parts. Therefore, the decomposition equations related to $M^d(1,1)$ cover all the crucial concepts of the two-part-type behavior presented in this paper.

1) State and Parameter Definitions: The downstream machine $M^d(1,1)$ represents all the downstream Type 1 flow behavior from $B_{1,1}$. The $M^d(1,1)$ up state occurs when M_2 is up and is not blocked for a Type 1 part.

$$\Upsilon^d(1,1) = \{\alpha_2 = 1 \cap n_{2,1} < N_{2,1}\}$$

That is, the observer in $B_{1,1}$ will see a part moving out of the buffer, when M_2 is up is not blocked for Type 1. There are several states in which the observer does not see a part moving out of the buffer and therefore believes that $M^d(1,1)$ is down. These states are:

- M_2 is down, or
- M_2 is up but blocked due to the failure of $M_{3,1}$, or
- M_2 is down and also blocked due to the failure of $M_{3,1}$

When the system is not in one of the these states, the observer believes $M^d(1,1)$ is working. The first down state represents the local failure, while the second indicates a remote failure. These down states are typical and they can be seen in two-machine lines of the single-part-type line decomposition.

However, the last down state can happen when machines are processing multiple part types. This down state is a mixture of local and remote failures. This can occur when the following sequence of failures occurs: suppose $M_{3,1}$ is down. If the failure persists long enough, it will make $B_{2,1}$ full, causing the blockage of M_2. Now, M_2 is blocked for Type 1 parts, and $M^d(1,1)$ will be down in the second down state described above. While M_2 is blocked for Type 1, it may work on a Type 2 part. Let us consider a situation in which M_2 fails while it is working on a Type 2 part. At this moment, $M_{3,1}$ is down and $B_{2,1}$ is full and M_2 is also down. The observer in $B_{1,1}$ sees that its downstream machine is not only down but also blocked for a Type 1 part. Again, this down state is the combination of the local and remote failure.

In order to get into this down state, the remote failure must occur first before the local failure takes place. This is because the blockage cannot occur when the machine is down already. Therefore, from the observer's view point, the third down state can be reached only from the second down state.

The state definitions for $M_d(1,1)$ are:

$$\begin{aligned} \Upsilon^d(1,1) &= \{\alpha_2 = 1 \cap n_{2,1} < N_{2,1}\} \\ \Delta_1^d(1,1) &= \{\alpha_2 = 0 \cap n_{2,1} < N_{2,1}\} \\ \Delta_2^d(1,1) &= \{\alpha_2 = 1 \cap n_{2,1} = N_{2,1}\} \\ \Delta_3^d(1,1) &= \{\alpha_2 = 0 \cap n_{2,1} = N_{2,1}\} \end{aligned} \tag{10}$$

Similarly, the state definitions for $M^u(2,1)$ are

$$\begin{aligned} \Upsilon^u(2,1) &= \{\alpha_2 = 1 \cap n_{2,1} > 0\} \\ \Delta_1^u(2,1) &= \{\alpha_2 = 0 \cap n_{2,1} > 0\} \\ \Delta_2^u(2,1) &= \{\alpha_2 = 1 \cap n_{2,1} = 0\} \\ \Delta_3^u(2,1) &= \{\alpha_2 = 0 \cap n_{2,1} = 0\} \end{aligned} \tag{11}$$

Note that the second and the third state definitions implicitly express that they are not starved for a Type 1 part because of our approximation (9).

As we explained in the earlier section, there are transition probabilities between these states. Transitions in $M^d(1,1)$ are denoted as follows:

$$\begin{aligned} r_j^d(1,1) &= Pr[\Upsilon^d(1,1) \text{ at } t+1 \,|\Delta_j^d(1,1) \text{ at } t\,] \\ p_j^d(1,1) &= Pr[\Delta_j^d(1,1) \text{ at } t+1 \,| \\ &\qquad \Upsilon^d(1,1) \cap n_{2,1}(t) < N_{2,1} \text{ at } t\,] \\ q_j^d(1,1) &= Pr[\Delta_j^d(1,1) \text{ at } t+1 \,| \\ &\qquad \Upsilon^d(1,1) \cap n_{2,1}(t) = N_{2,1} \text{ at } t\,] \\ z_{j,j'}^d(1,1) &= Pr[\Delta_{j'}^d(1,1) \text{ at } t+1 \,|\Delta_j^d(1,1) \text{ at } t\,] \end{aligned} \tag{12}$$

$$\text{for } j = \{1,2,3\}\ j' = \{1,2,3\}, \text{ and } j \neq j'$$

Likewise, for $M^u(2,1)$,

$$
\begin{aligned}
r_j^u(2,1) &= Pr[\Upsilon^u(2,1) \text{ at } t+1 \mid \Delta_j^u(2,1) \text{ at } t\ ] \\
p_j^u(2,1) &= Pr[\Delta_j^u(2,1) \text{ at } t+1 \mid \\
&\qquad \Upsilon^u(2,1) \cap n_{2,1}(t) > 0 \text{ at } t\ ] \\
q_j^u(2,1) &= Pr[\Delta_j^u(2,1) \text{ at } t+1 \mid \\
&\qquad \Upsilon^u(2,1) \cap n_{2,1}(t) = 0 \text{ at } t\ ] \\
z_{j,j'}^u(2,1) &= Pr[\Delta_{j'}^u(2,1) \text{ at } t+1 \mid \Delta_j^d(2,1) \text{ at } t\ ] \\
&\quad \text{for } j = \{1,2,3\}\ j' = \{1,2,3\}, \text{ and } j \neq j'
\end{aligned} \tag{13}
$$

2) Equalities: For convenience, we define the following two-machine-line probabilities:

$$
\begin{aligned}
W^u(i,j) &= Pr[\Upsilon^u(i,j) \cap n_{i,j} < N_{i,j}] \\
W^d(i,j) &= Pr[\Upsilon^d(i,j) \cap n_{i,j} > 0] \\
X_m^u(i,j) &= Pr[\Delta_m^u(i,j)] \\
X_n^d(i,j) &= Pr[\Delta_n^d(i,j)] \\
Pb(i,j) &= Pr[\Upsilon^u(i,j) \cap n_{i,j} = N_{i,j}] \\
Ps(i,j) &= Pr[\Upsilon^d(i,j) \cap n_{i,j} = 0] \\
Db(i,j) &= Pr[\Delta_1(i,j) \cap n_{i,j} = N_{i,j}] \\
Ds(i,j) &= Pr[\Delta_1(i,j) \cap n_{i,j} = 0]
\end{aligned} \tag{14}
$$

Observe that all the events in the above expressions are evaluated at the same time step. These quantities have the following interpretations. $W^u(i,j)$ and $W^d(i,j)$ are probabilities that $M^u(i,j)$ and $M^d(i,j)$ are up, and not blocked and are not starved respectively. The quantities $X_m^u(i,j)$ and $X_n^d(i,j)$ are probabilities of upstream and downstream events $\Delta_m^u(i,j)$ and $\Delta_n^d(i,j)$. $P_s(i,j)$ and $P_b(i,j)$ are probabilities that upstream and downstream machines are up, but idle because of blockage or starvation. On the other hand, $D_b(i,j)$ and $D_s(i,j)$ are probabilities that machines are down and also starved or blocked.

$W^d(1,1)$ indicates that M_2 is up and neither starved nor blocked for Type 2, because of the definition of Υ^d in (10). If we related this two-machine line probability with the real line then

$$
\begin{aligned}
W^d(1,1) &= Pr[\Upsilon^d(1,1) \cap n_{i,j} > 0] \\
&= Pr[\alpha_2 = 1 \cap n_{1,1} > 0 \cap n_{2,1} < N_{2,1}]
\end{aligned}
$$

Notice that that quantity is also equivalent to

$$
\begin{aligned}
&Pr[\alpha_2 = 1 \cap n_{1,1} > 0 \cap n_{2,1} < N_{2,1}] \\
&\qquad = Pr[\Upsilon^u(2,1) \cap n_{2,1} < N_{2,1}] \\
&\qquad = W^u(2,1)
\end{aligned}
$$

Therefore,

$$
W^d(1,1) = W^u(2,1) \tag{15}
$$

Next, $X_2^d(1,1)$ is the probability that downstream machine is down at mode 2 in $L(1,1)$. From the definition (10), it is

$$
X_2^d(1,1) = Pr[\alpha_2 = 1 \cap n_{2,1} = N_{2,1}]
$$

Since we approximate that the probability of a machine being blocked and starved at the same time step is zero, this equation implies that

$$
\begin{aligned}
X_2^d(1,1) &= Pr[\alpha_2 = 1 \cap n_{1,1} > 0 \cap n_{2,1} = N_{2,1}] \\
&= Pr[\Upsilon^u(2,1) \cap n_{2,1} = N_{2,1}] \\
&= P_b(2,1)
\end{aligned}
$$

A similar equality can be derived for $X_2^u(2,1)$. Therefore,

$$
X_2^d(1,1) = P_b(2,1) \tag{16}
$$

$$
X_2^u(2,1) = P_s(1,1) \tag{17}
$$

Last,

$$
\begin{aligned}
X_3^d(1,1) &= Pr[\alpha_2 = 0 \cap n_{1,1} > 0 \cap n_{2,1} = N_{2,1}] \\
&= Pr[\Delta_1^u(2,1) \cap n_{2,1} = N_{2,1}] \\
&= D_b(2,1)
\end{aligned}
$$

Again, $X_3^u(2,1)$ can be derived in the similar way. Therefore,

$$
X_3^d(1,1) = D_b(2,1) \tag{18}
$$

$$
X_3^u(2,1) = D_s(1,1) \tag{19}
$$

3) Resumption of Flow: The resumption of flow is the transition probability from a down state to the up state. Since there are three down states for $M^d(1,1)$, three transitions exist. First, $r_1^d(1,1)$ the transition from the first down state to the up state and

$$
\begin{aligned}
r_1^d(1,1) &= Pr[\Upsilon^d(1,1) \text{ at } t+1 \mid \Delta_1^d(1,1) \text{ at } t] \\
&= Pr[\alpha_2(t+1) = 1 \cap \\
&\qquad n_{2,1}(t+1) < N_{2,1} | \alpha_2(t) = 0 \cap n_{2,1}(t) < N_{2,1}] \\
&= r_2
\end{aligned} \tag{20}
$$

Since $\Delta_1^d(1,1)$ represents the local failure, its repair probability is equal to the repair probability of the real machine M_2. Next, the repair probability of the remote down state is

$$
\begin{aligned}
r_2^d(1,1) &= Pr[\Upsilon^d(1,1) \text{ at } t+1 \mid \Delta_2^d(1,1) \text{ at } t] \\
&= Pr[\alpha_2(t+1) = 1 \cap n_{2,1}(t+1) < N_{2,1} | \\
&\qquad \alpha_2(t) = 1 \cap n_{2,1}(t) = N_{2,1}] \\
&= \Big(1 - q_1^u(2,1)\Big) r_3
\end{aligned} \tag{21}
$$

This is because remote failure occurred due to the blockage of $B_{2,1}$ which was caused by the failure of M_3. Therefore, M_3 must be repaired in order to resume the flow of $M^d(1,1)$. There is also one more condition required for the resumption of the flow. Notice that at time t, M_2 is blocked and therefore, it could work on a Type 2 part. The resumption of the flow happens at the next time step when it does not go down while it is working on Type 2. The probability that M_2 goes down while it is blocked and processing a Type 1 part is $q_1^u(1,1)$ by the definition. Therefore, the transition probability from $\Delta_2^d(1,1)$ to $\Upsilon^d(1,1)$ is $\Big(1 - q_1^u(2,1)\Big) r_3$.

For the third repair probability, $r_3^d(1,1)$ is the transition probability from $\Delta_3^d(1,1)$ to $\Upsilon^d(1,1)$. We argue that this is

$$
\begin{aligned}
r_3^d(1,1) &= Pr[\Upsilon^d(1,1) \text{ at } t+1 \mid \Delta_3^d(1,1) \text{ at } t] \\
&= Pr[\alpha_2(t+1) = 1 \cap n_{2,1}(t+1) < N_{2,1} | \\
&\qquad \alpha_2(t) = 0 \cap n_{2,1}(t) = N_{2,1}] \\
&= r_2 r_3
\end{aligned} \tag{22}
$$

As shown in the equation, M_3 is down at t, causing $B_{2,1}$ to be full, and also M_2 is down at time t. Therefore, the both machines must be repaired for the flow of a Type 1 part to resume.

Similarly, the resumption of flow equations for $M^u(2,1)$ are:

$$r_1^u(2,1) = r_2 \quad (23)$$

$$r_2^u(2,1) = \Big(1 - q^d(1,1)\Big)r_1 \quad (24)$$

$$r_3^u(2,1) = r_1 r_2 \quad (25)$$

4) Failure Mode Change: Failure mode changes are transitions taking place between down states. First, we consider transitions from $\Delta_1^d(1,1)$ to other down states. This down state is the failure of M_2. Since the local machine is down and is not blocked, any state change further downstream of this local machine will not change the state of $M^d(1,1)$. The only event that will change the state of Δ_1^d is the resumption of flow. Therefore, there is no transition from $\Delta_1^d(1,1)$ to the rest of the down states. That is,

$$z_{1,2}^d(1,1) = 0 \quad (26)$$

$$z_{1,3}^d(1,1) = 0 \quad (27)$$

Next, consider $\Delta_2^d(1,1)$, the down state caused by the failure of M_3. In this down state, M_2 is up but is blocked for a Type 1 part. We can think of two possible other down states that can be reached from this one. First, consider the case in which M_2 goes down, while M_3 is still down. In this case, M_2 will be down and blocked at the same time. Therefore $M^d(1,1)$ will move $\Delta_3^d(1,1)$. Since M_2 goes down while it is blocked and M_3 is not repaired, the transition probability from $\Delta_2^d(1,1)$ to $\Delta_3^d(1,1)$ is $q_1^u(2,1)\Big(1-r_3\Big)$. If we express this failure mode change, then

$$\begin{aligned} z_{2,3}^d(1,1) &= Pr[\Delta_3^d(1,1) \text{ at } t+1 \mid \Delta_2^d(1,1) \text{ at } t\,] \quad (28)\\ &= Pr[\alpha_2(t+1)=0 \cap n_{2,1}(t+1)=N_{2,1} \cap \\ &\qquad \alpha_3(t+1)=0 | \\ &\qquad \alpha_2(t)=1 \cap n_{2,1}(t)=N_{2,1} \cap \alpha_3(t)=0] \\ &= q_1^u(2,1)\Big(1-r_3\Big) \end{aligned}$$

The second failure mode change from $\Delta_2^d(1,1)$ is the case that M_2 gets down while it is blocked, but M_3 gets repaired. In this case, M_2 will be no longer blocked, but will move to the local failure mode. Therefore, with the transition probability of $q^u(2,1)r_3$ $M^d(1,1)$ will move from $\Delta_2^d(1,1)$ to $\Delta_1^d(1,1)$. That is,

$$\begin{aligned} z_{2,1}^d(1,1) &= Pr[\Delta_1^d(1,1) \text{ at } t+1 \mid \Delta_2^d(1,1) \text{ at } t\,] \quad (29)\\ &= Pr[\alpha_2(t+1)=0 \cap n_{2,1}(t+1)<N_{2,1} \cap \\ &\qquad \alpha_3(t+1)=1 | \\ &\qquad \alpha_2(t)=1 \cap n_{2,1}(t)=N_{2,1} \cap \alpha_3(t)=0] \\ &= q_1^u(2,1)r_3 \end{aligned}$$

The last down state to be considered is Δ_3^d. In this state, $M^d(1,1)$ is down because of the local failure of M_2 and the blockage caused by the failure of M_3. The failure mode change to $\Delta_2^d(1,1)$ happens when M_2 is up, but M_3 remains down. That is,

$$\begin{aligned} z_{3,2}^d(1,1) &= Pr[\Delta_2^d(1,1) \text{ at } t+1 \mid \Delta_3^d(1,1) \text{ at } t\,] \quad (30)\\ &= Pr[\alpha_2(t+1)=1 \cap n_{2,1}(t+1)=N_{2,1} \\ &\qquad \cap \alpha_3(t+1)=0 | \\ &\qquad \alpha_2(t)=0 \cap n_{2,1}(t)=N_{2,1} \cap \alpha_3(t)=0] \\ &= r_2\Big(1-r_3\Big) \end{aligned}$$

On the other hand, if M_3 gets fixed, while M_2 remains down when $M^d(1,1)$ is in $\Delta_3^d(1,1)$, M_2 will be no longer gets blocked for a Type 1 part, but will be in a local failure mode. Therefore,

$$\begin{aligned} z_{3,1}^d(1,1) &= Pr[\Delta_1^d(1,1) \text{ at } t+1 \mid \Delta_3^d(1,1) \text{ at } t\,] \quad (31)\\ &= Pr[\alpha_2(t+1)=0 \cap n_{2,1}(t+1)<N_{2,1} \\ &\qquad \cap \alpha_3(t+1)=1 | \\ &\qquad \alpha_2(t)=0 \cap n_{2,1}(t)=N_{2,1} \cap \alpha_3(t)=0] \\ &= \Big(1-r_2\Big)r_3 \end{aligned}$$

With similar approaches, we also can derive the failure mode change quantities for $M^u(2,1)$:

$$z_{1,2}^u(2,1) = 0 \quad (32)$$

$$z_{1,3}^u(2,1) = 0 \quad (33)$$

$$z_{2,1}^u(2,1) = q_1^d(1,1)r_1 \quad (34)$$

$$z_{2,3}^u(2,1) = \Big(1-r_1\Big)q_1^d(1,1) \quad (35)$$

$$z_{2,1}^u(2,1) = q_1^d(1,1)r_1 \quad (36)$$

$$z_{3,1}^u(2,1) = r_1\Big(1-r_2\Big) \quad (37)$$

$$z_{3,2}^u(2,1) = \Big(1-r_1\Big)r_2 \quad (38)$$

5) Interruption of Flow: The interruption of flows are failures of $M^d(1,1)$. The probability $p_1^d(1,1)$ is the transition from $\Upsilon^d(1,1)$ to $\Delta_1^d(1,1)$, which represents the failure of M_2 when $B_{1,1}$ is not empty. This transition probability is

$$\begin{aligned} p_1^d(1,1) &= Pr[\Delta_1^d(1,1) \text{ at } t+1 \mid \quad (39)\\ &\qquad \Upsilon^d(1,1) \cap n_{1,1}>0 \text{ at } t] \\ &= Pr[\alpha_2(t+1)=0 \cap n_{2,1}(t+1)<N_{2,1} | \\ &\qquad \alpha_2(t)=1 \cap n_{1,1}(t)>0 \cap n_{2,1}(t)<N_{2,1}] \\ &= p_2 \end{aligned}$$

Next, we derive the interruption of flow equation for $p_2^d(1,1)$, the transition probability from the up state to the state in which M_3 is down and $B_{2,1}$ is full. This transition probability is,

$$p_2^d(1,1) = Pr[\Delta_2^d(1,1) \text{ at } t+1 \mid \Upsilon^d(1,1) \cap n_{1,1}>0 \text{ at } t]$$

We first start with the derivation of this equation by applying the fact that the probability of going out of a state is equal to the probability of going into that state.

$$\begin{aligned} X_2^d(1,1)\bigg(\Big(1-q^u(2,1)\Big)r_3 + q^u(2,1)r_3 + q^u(2,1)\Big(1-r_3\Big)\bigg) \\ = W^d(1,1)p_2^d(1,1) + X_3^d(1,1)r_2(1-r_3) \end{aligned}$$

If we simplify this equation, then

$$X_2^d(1,1)\left(r_3+q^u(2,1)\left(1-r_3\right)\right)$$
$$=W^d(1,1)p_2^d(1,1)+X_3^d(1,1)r_2\left(1-r_3\right)$$

That is,

$$p_2^d(1,1)=\frac{1}{W^d(1,1)}\times \tag{40}$$
$$\left[X_2^d(1,1)\left(r_3+q^u(2,1)\left(1-r_3\right)\right)-X_3^d(1,1)r_2(1-r_3)\right]$$

Next, the interruption of flow in which equation for the third down state, $\Delta_3^d(1,1)$. This is the transition that goes to the state M_2 is down and blocked for a Type 1 part. Note that when M_2 is up and not blocked for a Type 1 part, it is impossible for M_2 to be down and blocked at the next time step because once the machine is down, it will not move a part into the downstream buffer. Therefore,

$$\begin{aligned} p_3^d(1,1) &= Pr[\Delta_3^d(1,1) \text{ at } t+1 \mid \\ &\qquad \Upsilon^d(1,1)\cap n_{1,1}>0 \text{ at } t]=0 \end{aligned} \tag{41}$$

Similarly, the interruption of flow equations for $M^u(2,1)$ are

$$\begin{aligned} &p_1^u(2,1)=p_2 \\ &p_2^u(2,1)=\frac{1}{W^u(2,1)}\times \\ &\left[X_2^d(2,1)\left(r_1+\left(1-r_1\right)q^d(1,1)\right)-X_2^d(2,1)\left(1-r_1\right)r_2\right] \\ &p_3^u(2,1)=0 \end{aligned} \tag{42}$$

As a summary, Figure 6 shows all the transitions of $M^d(1,1)$ and $M^u(2,1)$.

6) Idleness Failure of Type One: Now we need to derive expressions for the idleness failure of the two-machine lines. The idleness failure is the transition from an up state to a down state in pseudo-machine while it is idle. The probability $q_1^d(1,1)$ represents the probability that pseudo-machine M^d(1,1) is down at $t+1$ given that it was up and starved at t. That is,

$$\begin{aligned} q_1^d(1,1) &= Pr[\Delta_1^d(1,1) \text{ at } t+1 \mid \Upsilon^d(1,1)\cap n_{1,1}=0 \text{ at } t] \\ &= Pr[\alpha_2(t+1)=0 \mid \\ &\qquad \alpha(t)=1\cap n_{1,1}(t)=0\cap n_{2,1}<N_{2,1}] \end{aligned}$$

The only way that this is possible is if the processing machine M_1 failed while making a Type 2 part in time step t. Therefore,

$$\begin{aligned} q_1^d(1,1) &= p_2 Pr[\alpha_2(t)=1\cap n_{1,1}(t)=0 \\ &\qquad \cap n_{1,2}(t)>0\cap n_{2,2}(t)<N_{2,2}] \\ &= p_2 W^d(1,2)\frac{P_s(1,1)}{P_b(2,1)+P_s(1,1)} \end{aligned} \tag{43}$$

According to the definition, $q_2^d(1,1)$ is

$$q_2^d(1,1)=Pr[\Delta_2^d(1,1) \text{ at } t+1 \mid \Upsilon^d(1,1)\cap n_{1,1}=0 \text{ at } t]$$

If we expand this equation using (10),then

$$\begin{aligned} q_2^d(1,1) &= Pr[\alpha_2(t+1)=0\cap n_{1,1}(t+1)=0\cap \\ &\qquad n_{2,1}(t+1)=N_{2,1} \mid \\ &\qquad \alpha(t)=1\cap n_{1,1}(t)=0\cap n_{2,1}<N_{2,1}] \\ &= 0 \end{aligned} \tag{44}$$

This is because in our assumption in (9) states that the probability that a machine is starved and blocked at the same time step is zero. For the same reason,

$$q_3^d(1,1)=0 \tag{45}$$

Similarly,

$$q_1^u(2,1) = p_2 W^u(2,2)\frac{P_b(2,1)}{P_b(2,1)+P_s(1,1)} \tag{46}$$
$$q_2^u(2,1) = 0 \tag{47}$$
$$q_3^u(2,1) = 0 \tag{48}$$

Note that since $W^u(2,1)=W^d(1,1)$, if we add $q_1^u(2,1)$ and $q_1^d(1,1)$ then we will have

$$q^u(2,1)+q^d(1,1)=p_2W^u(2,2) \tag{49}$$

This equality makes sense because by the definition, the idleness failure is the probability that a machine gets failed when a machine is working a Type 2 part.

VI. Type 2 Decomposition Analysis

The existence of Type 2 parts is made apparent to Type 1 parts only through the existence of idleness failures — which requires only a minor modification to the decomposition equations. However, the existence of Type 1 parts has a major influence on the production of Type 2 parts, and therefore the derivation for the Type 2 part decomposition is much more complicated. In particular, we now have to account for the possibility that an observer in a Type 2 buffer will see flow into his buffer cease because the upstream machine switched from making Type 2 to Type 1 parts. Furthermore, we still have to account for idleness failures which can occur if, for example, a Type 2 buffer fills up and the machine begins to produce Type 1 parts while the buffer is still full.

In analyzing the decomposition equations for Type 2, we also use the same approach we made for the Type 1 part: defining states of two-machine lines, constructing transition equations and relating two-machine line parameters to those of the real line and other two-machine lines. When we construct decomposition equations for Type 2 parts, we not only have to consider blockages and starvations of the Type 2 flow, but also have to consider the interruption and resumption of the flow due to the Type 1 flow.

In order to reduce the complexity, we separate our analysis into two parts. The first analysis is of the impact of Type 1 flow on Type 2. The second analysis is of the interruption and resumption of flow within Type 2 flow. The first analysis focuses on how the flow of Type 1 parts affect the flow of Type 2 parts regardless of the buffer state for Type 2. The second analysis concentrates on the buffer situations for Type 2 parts that affect the flow of Type 2. We call the first analysis *Type 2 availability analysis*. Then we construct the decomposition equations for Type 2, by combining the analysis of the buffer situations and the availability analysis.

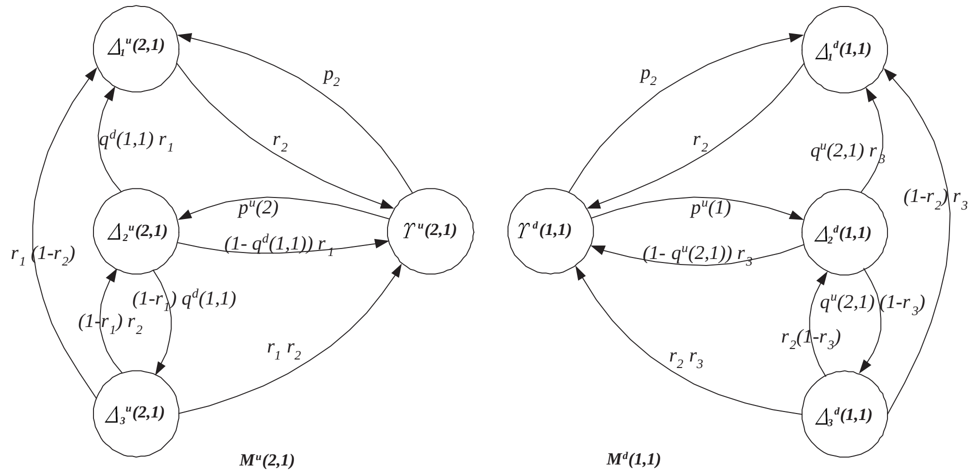

Fig. 6. Markov Chain of $M^u(2,1)$ and $M^d(1,1)$

7) Availability of Type 2: We begin our analysis by defining a new machine state variable for Type 2 parts. We say a machine is *available for Type 2*, when it is up, and is either starved or blocked for a Type 1 part. This availability for Type 2 of machine M_i is denoted by β_i. That is, if $\beta_i = 1$, real machine M_i is up and it is starved or blocked for Type 1 parts. If $\beta_i = 0$, M_i is down, or processing Type 1 parts. This can be expressed as follows:

$$\beta_i = \begin{cases} 1 & \text{if } \{\alpha_i = 1\} \cap \{n_{i,1} = 0 \cup n_{i+1,1} = N_{i+1,1}\} \\ 0 & \text{if } \{\alpha_i = 0\} \cup \{\alpha_i = 1 \cap n_{i,1} > 0 \cap n_{i+1,1} < N_{i+1,1}\} \end{cases} \tag{50}$$

Recall that there are two conditions under which a machine can produce Type 2 parts: first, the machine is blocked or starved for Type 1 parts, and second, the machine is neither blocked nor starved for Type 2 parts. Therefore, β_i is the indicator of the first condition that must hold in order to produce Type 2 parts.

8) Type 2 Availability Probability: Now, we introduce new parameters for the Type 2 decomposition: r_2^* and p_2^*. The parameter r_2^* is the transition probability that M_2 is available to process Type 2 at time $t+1$, given that it was not available for Type 2 at time t. We call this quantity *Type 2 availability transition probability*.

$$\begin{aligned} r_2^* \quad = \quad & Pr[\beta_2 = 1 \cap \\ & n_{1,2} > 0 \cap n_{2,2} < N_{2,2} \text{ at } t+1 | \beta_i = 0 \text{ at } t] \end{aligned}$$

There are several states in which M_2 is not available for Type 2 at time t. These states fall into the following three categories:

(a) M_2 is up, and producing a Type 1 part at time t.
(b) M_2 is down and starved for a Type 1 part at time t, and will be available for Type 2 when it comes up.
(c) M_2 is down and blocked for a Type 1 part at time t, and will be available for Type 2 when it comes up.

Category (a) is the set of states in which M_2 is processing Type 1, and therefore it is not available for a Type 2 part. This state is

$$V = \{\alpha_2(t) = 1 \cap n_{1,1}(t) > 0 \cap n_{2,1}(t) < N\} \tag{51}$$

Next, we consider the two sets of states that represent M_2 being down in step t and available for a Type 2 part at $t+1$, as described in categories (b) and (c). This happens if the machine is either blocked or stared for Type 1. Since we assume that the machine cannot simultaneously blocked and starved for either part type, these events are represented as follows:

$$\begin{aligned} D_s &= \{\alpha_2(t) = 0 \cap n_{1,1}(t) = 0\} \\ D_b &= \{\alpha_2(t) = 0 \cap n_{2,1}(t) = N\} \end{aligned} \tag{52}$$

With these events defined, we are now ready to begin our analysis of r_2^*. We begin by observing that the three events defined above are disjoint. Define the event, called U, that M_2 is available to process Type 2 at $t+1$. Then

$$\begin{aligned} U \quad = \quad & \Big\{\alpha_2(t+1) = 1 \cap \Big(n_{1,1}(t+1) = 0 \\ & \cup n_{2,1}(t+1) = N_{2,1}\Big)\Big\} \end{aligned}$$

That is, U is the event that M_2 is up, and is blocked or starved for Type 1 at $t+1$. Under these conditions, M_2 is available for Type 2. Then, since the three conditioning events are disjoint, we can write the following:

$$\begin{aligned} r_2^* &= Pr[U|V \cup D_s \cup D_b] \qquad (53) \\ &= \frac{Pr[U|V]Pr[V] + Pr[U|D_s]Pr[D_s] + Pr[U|D_b]Pr[D_b]}{Pr[V \cup D_s \cup D_b]} \end{aligned}$$

It remains for us to calculate the individual probabilities. We first calculate the probability of event V, D_s, and D_b. First, V is the event that M_2 is working on a Type 1 part. The probability of V is $W^d(1,1)$ or $W^u(2,1)$ because of the equality we stated in (15).

$$Pr[V] = W^d(1,1) = W^u(2,1) \tag{54}$$

Next, the events D_s and D_b are expressed with the definitions we made in (10) and (14),

$$\begin{aligned} Pr[D_s] &= X_3^u(2,1) \\ Pr[D_b] &= X_3^d(1,1) \end{aligned} \tag{55}$$

Since events V, D_s, and D_b are disjoint,

$$Pr[V \cup D_s \cup D_b] = W^d(1,1) + X_3^u(2,1) + X_3^d(1,1) \quad (56)$$

Now we calculate each conditional probability in (53). We begin with the conditional probability, $Pr[U|V]$. This is the conditional probability that M_2 is available for a Type 2 part at time $t+1$ due to the starvation or blockage of a Type 1 part, given that it was processing Type 1. Since we approximated in Section(II) the probability that a machine is blocked and starved at the same time step as zero, we can state that, approximately,

$$\begin{aligned} &Pr[U|V] = \quad (57)\\ &\quad Pr[\alpha_2(t+1) = 1 \cap n_{1,1}(t+1) = 0|V] \\ &\quad +Pr[\alpha_2(t+1) = 1 \cap n_{2,1}(t+1) = N_{2,1}|V] \end{aligned}$$

The first term on the right hand side of (57) is the conditional probability that M_2 is starved for a Type 1 part at $t+1$, given that it was processing a Type 1 part at t. This is equivalent to saying that $M^u(2,1)$ goes to down mode $\Delta_2^u(2,1)$ at $t+1$, given that it was up at t. Similarly, the second term on the righthand side of (57) is equal to the probability that $M^d(1,1)$ goes down to the down mode $\Delta_b^D(1,1)$ at $t+1$, given that it was up at t. We can then write

$$\begin{aligned} &Pr[U|V] = \quad (58)\\ &\quad Pr[M^u(2,1) = \Delta_2^u(2,1) \text{ at } t+1 | M^u(2,1) = \Upsilon^u(2,1) \text{ at } t] \\ &\quad +Pr[M^d(1,1) = \Delta_b^D(1,1) \text{ at } t+1 | M^d(1,1) = \Upsilon^d(1,1) \text{ at } t] \\ &\quad = p_s^u(2,1) + p_b^D(1,1) \end{aligned}$$

Next, we need to derive an expression $Pr[U|D_s]$. In this case, M_2 is down and starved for a Type 1 part at time t, and is up but it remains starved at the next time step. Note that this transition is the same as the transition in which $M^u(2,1)$ is down at mode $\Delta_2^u(2,1)$ at $t+1$, given that it was down at mode $\Delta_3^u(2,1)$ at t. Therefore,

$$\begin{aligned} &Pr[U|D_s] = \\ &\quad Pr[M^u(2,1) = \Delta_2^u(2,1) \text{ at } t+1| \\ &\qquad M^u(2,1) = \Delta_3^u(2,1) \text{ at } t] = (1-r_1)r_2 \end{aligned}$$

Similarly, $Pr[U|D_b]$ is equal to the probability that $M^d(1,1)$ is down at mode $\Delta_b^D(1,1)$ at $t+1$, given that it was down at mode $\Delta_3^d(1,1)$ at t. Therefore,

$$\begin{aligned} &Pr[U|D_b] = \\ &\quad Pr[M^d(1,1) = \Delta_b^D(1,1) \text{ at } t+1| \quad (59)\\ &\qquad M^d(1,1) = \Delta_3^d(1,1) \text{ at } t] = r_2(1-r_3) \end{aligned}$$

Putting everything together, we have

$$\begin{aligned} r_2^* \;=\; & \frac{1}{W^d(1,1) + X_3^u(2,1) + X_3^d(1,1)} \times \quad (60)\\ & \Bigg(W^d(1,1)\Big(p^u(2,1) + p^d(1,1)\Big) + \\ & \quad X_3^u(2,1)\Big(1-r_1\Big)r_2 + X_3^d(1,1)r_2\Big(1-r_3\Big) \Bigg) \end{aligned}$$

9) Type 2 Unavailability Probability: The quantity p_2^* is the conditional probability that M_2 is not available for Type 2 at time $t+1$, but it was available and was not blocked and not starved at time t. We call this quantity *the Type 2 unavailability transition probability*. We can write this as

$$\begin{aligned} p_2^* \;=\; & Pr[\beta_2(t+1) = 0 | \beta_2(t) = 1 \cap n_{1,2}(t) > 0 \\ & \cap n_{2,2}(t) < N_{2,2}] \end{aligned}$$

There are two possible cases that prevents M_2 from processing Type 2 parts. The first case is that M_2 is down while it is processing Type 2 parts. The second case is that M_2 is no longer blocked or starved for Type 1 parts and therefore processing Type 1 parts. Then p_2^* is

$$\begin{aligned} p_2^* \;=\; Pr\Bigg[& \{\alpha_2 = 1 \cap n_{1,1} > 0 \cap n_{2,1} < N_{2,1}\} \cup \quad (61)\\ & \{\alpha_2 = 0 \cap n_{1,1} = 0\} \cup \{\alpha_2 = 0 \cap n_{2,1} = N_{2,1}\} \text{ at t+1} \Bigg| \\ & \Big\{ \{\alpha_2 = 1 \cap n_{1,1} = 0\} \cup \{\alpha_2 = 1 \cap n_{2,1} = N_{2,1}\} \Big\} \\ & \cap \Big\{ \alpha_2 = 1 \cap n_{1,2} > 0 \cup n_{2,2} < N_{2,2} \Big\} \text{ at } t \Bigg] \end{aligned}$$

Let us define the following events:

$$\begin{aligned} B &= \{\alpha_2 = 1 \cap n_{2,1} = N_{2,1}\} \quad (62)\\ S &= \{\alpha_2 = 1 \cap n_{1,1} = 0\} \\ W_2 &= \{\alpha_2 = 1 \cap n_{1,2} > 0 \cap n_{2,2} < N_{2,2}\} \end{aligned}$$

The event B is the set of states in which M_2 is blocked for a Type 1 part, while S is the set of states in which M_2 is starved for a Type 1 part. These two sets are disjoint events because of the approximation that probability of a machine being blocked and starved is zero as we stated in (9). If we apply definitions in (51) and (52), then

$$p_2^* = Pr\Big[V \cup D_s \cup D_b \Big| \{S \cap W_2\} \cup \{B \cap W_2\}\Big] \quad (63)$$

By expanding this, we have

$$\begin{aligned} p_2^* = & \quad (64)\\ & \frac{Pr[S \cap W_2]}{Pr[B \cap W_2] + Pr[S \cap W_2]} \times \\ & \quad \Big(Pr[V|S \cap W_2] + Pr[D_s|S \cap W_2] + Pr[D_b|S \cap W_2] \Big) \\ & + \frac{Pr[B \cap W_2]}{Pr[B \cap W_2] + Pr[S \cap W_2]} \times \\ & \quad \Big(Pr[V|B \cap W_2] + Pr[D_s|B \cap W_2] + Pr[D_b|B \cap W_2] \Big) \end{aligned}$$

The fractions in the equations can be approximated as follows

$$\begin{aligned} \frac{Pr[S \cap W_2]}{Pr[B \cap W_2] + Pr[S \cap W_2]} &\approx \frac{Pr[S]}{Pr[B] + Pr[S]} \quad (65)\\ \frac{Pr[B \cap W_2]}{Pr[B \cap W_2] + Pr[S \cap W_2]} &\approx \frac{Pr[B]}{Pr[B] + Pr[S]} \end{aligned}$$

It remains for us to calculate the individual probabilities. We already have defined in (16) that M being up and starved is the same event as $M^u(2,1)$ being down at mode Δ_2^u. Also it is stated that M being up and blocked is equivalent to $M^d(2,1)$ being down at mode Δ_s^D. Therefore,

$$\begin{aligned} Pr[S] &= X_2^u(2,1) \\ Pr[B] &= X_b^D(1,1) \end{aligned} \tag{66}$$

Now, we need to calculate conditional probabilities in (64). First, note that the following conditional probabilities are zero:

$$\begin{aligned} Pr[D_s|B\cap W_2] &= 0 \\ Pr[D_b|S\cap W_2] &= 0 \end{aligned} \tag{67}$$

This is due to our approximation in (9).

Next, $Pr[V|S\cap W_2]$ is the probability that M_2 is working on a Type 1 part at time $t+1$, given that it was up and starved for Type 1, but not starved nor blocked for Type 2 at time t. This probability is the same as the probability that $M^u(2,1)$ is up and not blocked in time $t+1$ give that it was down at mode Δ_2^u. Therefore, this conditional probability is the transition probability from Δ_b^D to Υ^d, which is given by.

$$\begin{aligned} & Pr[V|S\cap W_2] = \\ & \quad Pr[M^u(2,1) = \Upsilon^u \cap n_{2,1} < N_{2,1} \text{ at } t+1 | \\ & \quad\quad M^u(2,1) = \Delta_2^u \text{ at } t] \\ & \quad = r_1\Big(1-q^d(1,1)\Big) \end{aligned} \tag{68}$$

In a similar manner, $P[V|B\cap W_2]$, the conditional probability that M_2 is working on Type 1 part at $t+1$, given that M_2 was blocked, but was not blocked nor starved for Type 2 at t, can be written with two-machine-line parameters such that

$$\begin{aligned} & Pr[V|B\cap W_2] = \\ & \quad Pr[M^d(1,1) = \Upsilon^d \cap n_{1,1} > 0 \text{ at } t+1 | \\ & \quad\quad M^d(1,1) = \Delta_2^d \text{ at } t] \\ & \quad = \Big(1-q^u(2,1)\Big)r_3 \end{aligned} \tag{69}$$

This is because when M_3 is repaired and M_2 does not go into the idleness failure — fail while it is blocked — at the end of time step t, M_2 will be no longer blocked and process a Type 1 part at time $t+1$.

Next $P[D_s|S\cap W_2]$ is the probability that M_2 goes down while it is working on Type 2. This is equivalent that $M^u(2,1)$ is initially at the down mode Δ_2^u and $n_{2,1} < N_{2,1}$, but the down mode change takes place from Δ_2^u to Δ_3^u at the end of time step. Therefore,

$$\begin{aligned} & Pr[D_s|S\cap W_2] = \\ & \quad Pr[M^u(2,1) = \Delta_3^u \cap n_{2,1} < N_{2,1} \text{ at } t+1 | \\ & \quad\quad M^u(2,1) = \Delta_2^u \text{ at } t] \\ & \quad = \Big(1-r_1\Big)q^d(1,1) \end{aligned} \tag{70}$$

For the similar reason, $P[D_b|B\cap W_2]$ is

$$\begin{aligned} & Pr[D_b|B\cap W_2] = \\ & \quad Pr[M^d(1,1) = \Delta_3^d \cap n_{1,1} > 0 \text{ at } t+1 | \\ & \quad\quad M^d(1,1) = \Delta_2^d \text{ at } t] \\ & \quad = q^u(2,1)\Big(1-r_3\Big) \end{aligned} \tag{71}$$

Putting everything together, we have

$$\begin{aligned} p_2^* = & \frac{1}{Ps(1,1)+Pb(2,1)} \times \\ & \Bigg[Ps(1,1)\bigg((1-r_1)q^d(1,1) + r_1\Big(1-q^d(1,1)\Big)\bigg) \\ & +Pb(2,1)\bigg(q^u(2,1)\Big(1-r_3\Big) + \Big(1-q^u(2,1)\Big)r_3\bigg)\Bigg] \end{aligned} \tag{72}$$

10) Decomposition of Type 2: We have so far discussed the machine availability for Type 2. However, in order to process a Type 2 part, a part and a space also have to be in the upstream and downstream buffers. The decomposition equations presented here integrate the availability analysis and the buffer state analysis.

We introduce up and down states for Type 2 two-machine-line. As we do for the Type 1 case, we also explain all the states and transitions of the Type 2 two-machine-line with the example of $M^d(1,2)$ in detail.

All the states for $M^d(1,2)$ are

$$\begin{aligned} \Upsilon^d(1,2) &= \{\beta_2 = 1 \cap n_{2,2} < N_{2,2}\} \\ \Delta_1^d(1,2) &= \{\beta_2 = 0 \cap n_{2,2} < N_{2,2}\} \\ \Delta_2^d(1,2) &= \{\beta_2 = 1 \cap n_{2,2} = N_{2,2}\} \\ \Delta_3^d(1,2) &= \{\beta_2 = 0 \cap n_{2,2} = N_{2,2}\} \end{aligned} \tag{73}$$

Likewise, the states for $M^u(2,2)$ are

$$\begin{aligned} \Upsilon^u(2,2) &= \{\beta_2 = 1 \cap n_{2,2} > 0\} \\ \Delta_1^u(2,2) &= \{\beta_2 = 0 \cap n_{2,2} > 0\} \\ \Delta_2^u(2,2) &= \{\beta_2 = 1 \cap n_{2,2} = 0\} \\ \Delta_3^u(2,2) &= \{\beta_2 = 0 \cap n_{2,2} = 0\} \end{aligned} \tag{74}$$

The state $\Upsilon^d(1,1)$ is the in which M_2 is up and is either blocked or starved for Type 1. It is also not blocked for Type 2. Therefore, if $M^d(1,2)$ is in this state M_2 works on Type 2.

The state$\Delta_1^d(1,2)$ is the state in which M_2 is either down, or up and not blocked and starved for Type 1. The second down state, $\Delta_2^d(1,2)$, represents the case in which $M2$ is available for processing Type 2, however it is blocked for Type 2. Therefore, M_2 is either blocked or starved for Type 1 and at the same time it is blocked for Type 2 — M_2 is therefore idle for both part types. The last down state, $\Delta_3^d(1,2)$, is the state in which M_2 is not available for Type 2 and it also blocked for Type 2 parts.

Now, we need to derive the interruption of flow, resumption of flow, and failure mode change equations for Type 2 parts. Before we construct equations as we have done for the Type 1 case, notice that the Type 2 availability, β, seen by a Type 2 observer, gives us the identical behavior with α in Type 1 two-machine-line. For instance, an observer in $n_{1,1}$, who believes that she is in a two-machine-line, processing a single part type, will see a Type 2 part moving out of the buffer when there is a part that is, $n_{1,1} > 0$ and at the same time, $\beta_2 = 1$. If β_2 goes from 1 to 0 when there is no part at $n_{1,1}$ will believe that there is an idleness failure occurred. Since she believes that the machines are processing only one part type, the parameter β gives the exactly same meaning to the observer in a Type 2 flow as α does to its Type 1 observer. Therefore, for the Type 2 observer's point of view, the parameters p^* and r^* are seen the same way as p and r seen by a Type 1 observer. For this reason, if we apply the same logic in constructing decomposition equations we have done for a Type 1 part, we can derive all the decomposition equations for a Type 2 part. The resulting decomposition equations for $M^d(1,1)$ and $M^u(2,1)$ are

$$p_1^d(1,2) = p_2^* \tag{75}$$

$$p_2^d(1,2) = \frac{1}{W^d(1,2)} \times \left[X_2^d(1,2)\left(r_{3,2} + q_1^u(2,2)\Big(1 - r_{3,2}\Big)\right) - X_3^d(1,2) r_2 (1 - r_{3,2}) \right] \tag{76}$$

$$p_3^d(1,3) = 0 \tag{77}$$

$$r_1^d(1,2) = r_2^* \tag{78}$$

$$r_2^d(1,2) = \Big(1 - q_1^u(2,2)\Big) r_{3,2} \tag{79}$$

$$r_3^d(1,2) = 0 \tag{80}$$

$$z_{1,2}^d(1,2) = 0 \tag{81}$$

$$z_{1,3}^d(1,2) = 0 \tag{82}$$

$$z_{2,1}^d(1,2) = q_1^u(2,2) r_3^* \tag{83}$$

$$z_{2,3}^d(1,2) = q_1^u(2,2)\Big(1 - r_{3,2}\Big) \tag{84}$$

$$z_{3,1}^d(1,2) = \Big(1 - r_2^*\Big) r_{3,2} \tag{85}$$

$$z_{3,2}^d(1,2) = r_2^*\Big(1 - r_{3,2}\Big) \tag{86}$$

11) Idleness Failure of Type Two: The idleness failure probability for Type 2 is

$$\begin{aligned} q_1^d(1,2) &= Pr[\Delta_1^d(1,2) \text{ at } t+1 \,|\Upsilon^d(1,2) \cap n_{1,2} = 0 \text{ at } t] \\ &= Pr[\beta_2(t+1) = 0 \cap n_{2,2}(t+1) < N_{2,2}| \\ &\qquad \beta_2(t) = 1 \cap n_{1,2}(t) = 0] \end{aligned}$$

This equation is expressed based on the definition of $\Delta_1^d(1,2)$ and $\Upsilon^d(1,2)$. In the equation $n_{1,2}(t) = 0$ implies that $n_{2,2} < N_{2,2}$ due to approximation (9). Therefore, at the next time step, when the machine does not process a Type 2 part, $B_{2,2}$ must remain $n_{2,2} < N_{2,2}$. If we rewrite this, then

$$q_1^d(1,2) = Pr[\beta_2(t+1) = 0 | \beta_2(t) = 1 \cap n_{1,2}(t) = 0] \tag{87}$$

In this equation, $\beta_2 = 1$ means M_2 has been either blocked or starved for a Type 1 part. Also, $n_{1,2} = 0$ indicates that M_2 has been starved for a Type 2 part. That is, M_2 is begin idle at t. Remember that the real machine cannot fail while it is idle. Therefore, the idleness failure probability is

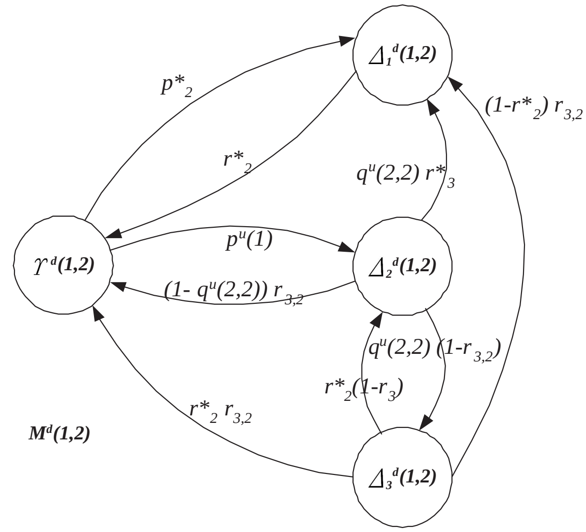

Fig. 7. Markov Chain of $M^d(1,2)$

$$\begin{aligned} q_1^d(1,2) &= Pr[\alpha_2(t+1) = 1 \cap \\ &\qquad n_{1,1}(t+1) > 0 \cap n_{2,1}(t+1) < N_{2,1}| \\ &\qquad \alpha_2(t) = 1 \cap (n_{1,1}(t) = 0 \cup n_{2,1}(t) = N_{i,1})] \end{aligned} \tag{88}$$

This expression is the probability that M_2 resumes processing a Type 1 part at $t+1$, after having been idle for the both part types. Then,

$$q_1^d(1,2) = \frac{P_b(2,1)\Big(1 - q^u(2,1)\Big) r_3 + P_s(1,1) r_1 \Big(1 - q^d(1,1)\Big)}{P_b(2,1) + P_s(1,1)} \tag{89}$$

Since we approximate the probability that a machine gets blocked and starved at the same time is zero, we can state that,

$$q_2^d(1,2) = 0 \tag{90}$$

$$q_3^d(1,2) = 0 \tag{91}$$

In a similar manner, we can derive

$$q_1^u(2,2) = \frac{P_b(2,1)\Big(1 - q^u(2,1)\Big) r_3 + P_s(1,1) r_1 \Big(1 - q^d(1,1)\Big)}{P_b(2,1) + P_s(1,1)} \tag{92}$$

$$q_2^u(2,2) = 0 \tag{93}$$

$$q_3^u(2,2) = 0 \tag{94}$$

The equations for $q_1^d(1,2)$ and $q_1^u(2,2)$ are identical. This is because whenever $M^d(1,2)$ is starved or $M^u(2,2)$ is blocked, M_2 goes back processing a Type 1 part, they goes to the idleness failure.

VII. Algorithm and Results

A. Algorithm

We present an algorithm for solving the decomposition equations derived in Section V and Section VI. The basic idea of the algorithm is a generalization of the DDX algorithm for the single-part case described in [5]. In this case, we first sweep down the line calculating the upstream two-machine parameters for Type 1 using the parameters of the previous two-machine line. Then we sweep up the line to calculate the downstream two-machine line parameters for Type 1. We then repeat the process for Type 2. The termination conditions for the algorithm are such that

$$\|E(i,j) - E(0,j)\|$$

for $i = 1, \ldots, k$, is less than some specified ϵ for each Type j part. This method exploits the recursive nature of the interruption and resumption of flow equations.

B. Randomly Generated Cases

Since we have not proved the convergence of the algorithm analytically, we follow the procedure described in [2] and test the algorithm on multiple randomly generated cases where the parameters of the random systems are within certain tolerances. The random cases we generated have machines that have similar though not identical characteristics. We allow for the machines to have different isolated efficiency rates, but we do not generate lines with an extreme bottleneck machine.

The isolated production rates of the processing machines vary from 0.85 to 0.95. For the demand machines we typically generate repair probabilities that are of the same order of magnitude as those generated for the processing machines. However, the failure probabilities of the demand machines are higher; rather than being an order of magnitude smaller than the repair probability, they are of the same order of magnitude. This ensures that the demand rates for both part types are individually below the capacity of the line. This is because a system in which the Type 1 demand machine has an isolated efficiency similar to that of the other machines in the line tends to be uninteresting as the line spends all of its time producing Type 1 parts. We also ensure that the demand rate for Type 1 and Type 2 are such that, combined, the processing line would not have the capacity to meet demand for both Type 1 and Type 2 parts. This is because if the line has the capacity to meet demand for both part types then the estimation process is trivial; production rate would be equal to demand rates and all intermediate buffers will be nearly full. These restriction on demand rates are expressed such that:

$$\begin{aligned} e_{d1} < \quad e_i \quad &< e_{d1} + e_{d2} \\ e_{d2} < \quad e_i \quad &< e_{d1} + e_{d2} \\ &\text{,where } i = 1, \ldots, k \end{aligned} \tag{95}$$

We generate 300 random lines. The first 100 random lines are lines where demand for part Type 1 and Type 2 are roughly the same. The second 100 cases are of the line where the demand for part Type 1 exceeds that of part Type 2 by up to 30%. The remaining 100 cases are of the line where the demand for part Type 2 exceeds that of part Type 1 by up to 30%. Buffers size vary from 5 to 20. For production rates, we calculate the percent error of the approximated production rate from the simulated production rate in the following manner.

$$\%Error = 100 \times \frac{E_{decomp} - E_{sim}}{E_{sim}} \tag{96}$$

For average buffer levels, we calculate the percent error of the approximated average buffer level as follows:

$$\%Error = 100 \times \frac{\bar{N}_{decomp} - \bar{N}_{sim}}{N} \tag{97}$$

These measurements are standard in the literature cited.

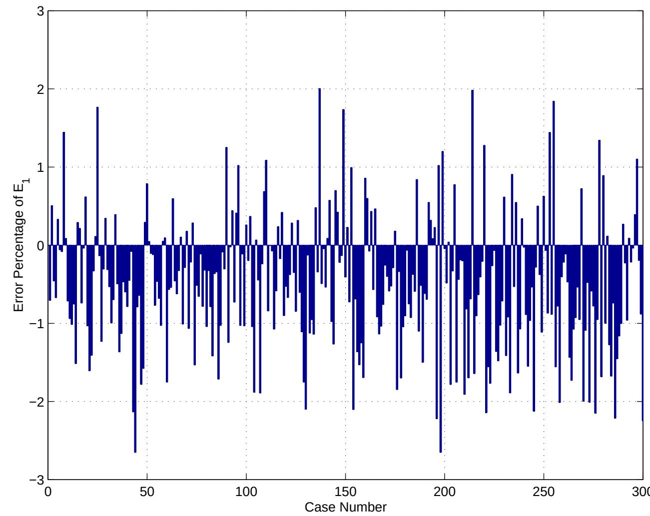

Fig. 8. The errors in the decomposition approximation for Type 1 production rates

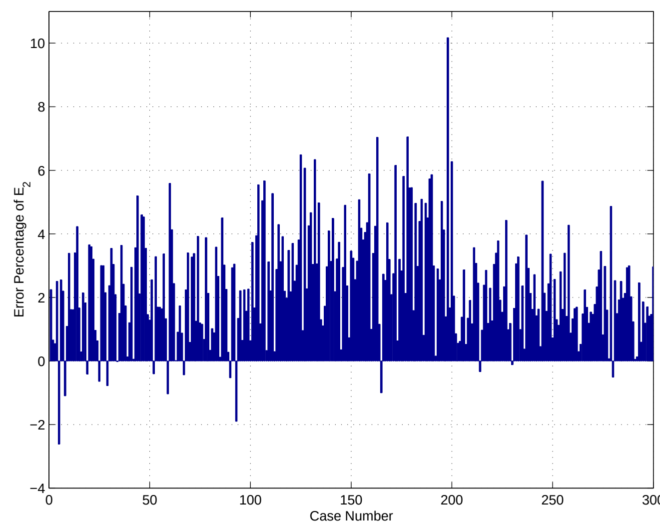

Fig. 9. The errors in the decomposition approximation for Type 2 production rates

C. Numerical Results

The percent errors calculated for all 300 cases are shown in Figure 8 and 9. The average absolute errors for Type 1 is 0.21% while Type 2 is 1.8%. The average error for buffer levels is 6.2%. As shown in the figures, the algorithm tends to slightly underestimate the Type 1 production rate, while overestimating for Type 2 parts. The behavior of the algorithm varies depending on the demand rates for Type 1 and Type 2. In the case where the demand rates for Type 1 is low and Type 2 is high — line 201 - 300 — the algorithm tends to give the most accurate results. We attribute this to the increase of randomness behavior for Type 2 parts. Since the demand rate for Type 1 parts is relatively low, there will be more blockage of Type 1 parts, leading to a longer period in which Type 2 can be processed. This will make the up and down time of pseudo machines in two-machine lines for Type 2 be distributed more nearly geometrically.

VIII. Conclusion and Future Research

We have found that the algorithm seems to converge most reliably, even when the mean time to failure and mean time to repair of all the machines in the line are of radically different order of magnitude.

Also the accuracy of the algorithm with respect to the simulation results for production rate and average buffer levels was satisfiable. We noted that the algorithm tended to overestimate the production rates for Type 2 parts. This suggested that improvements can be made to the decomposition to increase the accuracy. For our future research, we will model for a longer line with more part types. Also the idea of decomposition will be modified to analyze a system with re-entrance flow.

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