SIMPLE MODELS FOR A SINGLE
ROUTE PUBLIC TRANSPORTATION SYSTEM

by

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ABSTRACT

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Submitted to the Sloan School of Management on February 24, 1978 in partial fulfillment of the requirements for the degree of Master of Science in Operations Research.

Several analytical models of the operation of a single route in a Public Transportation System are examined here. These models describe the system performance by the regularity of headways along the route. The first model assumes deterministic conditions and describes the effect of disturbances in the value of the parameters of the system on the regularity of the schedule. Dispatching headway and travel time disturbances are propagated and amplified at stops along the route by the passenger arrival and boarding process. Headway deviations of a vehicle will propagate to following vehicles. It is shown that the way the instability of schedule propagates to other vehicles depends on the value of the passenger arrival rate to boarding rate ratio.

The second and third models analyze the effect of random disturbances in dispatching headways and travel times on the regularity of headways along the route. Expressions predicting the variance of headways given the variance of the disturbances, are derived. Predictions of these models are compared with the results of a Simulation Program. The phenomenon of vehicle clumping is analyzed and explanations of its causes and effects are proposed.

The effect of traffic lights on trip time is examined in the last model. Expressions for the mean and variance of total delay based on the spatial distribution and operating characteristics of the traffic lights are proposed.

Thesis Supervisor: Richard C. Larson

Title Associate Professor of Electrical Engineering and Urban Studies
This research was inspired and motivated on suggestion of my advisor, Professor Richard Larson. His help and continuous feedback made possible the completion of this work. I appreciate the help of Janet Mayer in typing the thesis. My deepest sense of gratitude is centered on my wife Rina, who during all times, was a companion, friend, editor and occasional typist. To her, I dedicate this work.
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CHAPTER I

INTRODUCTION

A few decades ago, when a private vehicle was considered more of a luxury than a necessity, public transportation provided a convenient and inexpensive method of transportation. Traffic congestion was low and delays correspondingly so. A dramatic change in urban transportation followed with the culmination of World War II. During the war more attention was drawn towards automation as a means to improve the efficiency of the industrial production system. As a result, the automobile industry was benefited and private cars became available at lower cost, changing the ratio of private to public vehicles in an increasing fashion [1].

The increased mobility and convenience of using one's car was soon outweighted by traffic jams, unavailability of parking spaces and the longer trips for those who started moving out to the suburbia. Nowadays, with the need for a more rational consumption of energy and a less polluted environment, public transportation appears to be, once again, a better alternative to the private car user. However, buses and street cars are also affected by traffic congestion, services deteriorate, and the potential public transportation user keeps asking himself which is the worse alternative.

Public transportation companies, either privately or government owned, have to maintain operations at a level high enough as to produce the revenues needed to offset the large overhead costs characteristic of that type of enterprise. Good service, low waiting times and low fares would attract
more demand, producing revenues that would allow the management to improve the service further and so forth. This is not as easy to accomplish as it sounds, since there are many factors involved such as budgetary and technological constraints.

The optimization of the operation of a Public Transportation network may be viewed as a multi-objective hierarchial process with an optimizing stage at the aggregate level and another stage at the detailed level of operations.

In the first stage, long range planning decisions involving large investments layout are made. The main objective of this stage is the design of routes in the network and the allocation of resources (vehicles, equipment, manpower, etc.) that would best satisfy the estimated potential demand on the system. The addition of new routes to an existing network also falls within this stage. Operating costs are considered only at an aggregated level and occasionally function as a deterrent in the addition of a route with estimated insufficient generation of revenues.

The second stage involves the day to day, real time optimization of the operation of single routes in the network. Equipment and manpower capacity levels have been determined in the previous stage and the objective now is design and implementation of control measures that will allow the optimal utilization of the resources available in a probabilistic environment.

Transportation researchers and engineers have begun in recent years to pay more attention to this problem and much effort has been directed towards its solution. The inherent complexities of the problem, however, have hindered the development of a comprehensive analysis. Simulation and field experiments have been used to study particular situations. The main disadvantages of such techniques are very high set up costs and the failure
to provide answers general enough to maintain their validity in an ever changing environment.

As a viable alternative, the present work tries to initiate the development of a methodological framework that could provide an adequate solution to the real time optimization of the single route operation in a public transportation network.

The optimization of the operation of a single route can take different meanings according to the measures of effectiveness of one's choice. One obvious measure of performance is the operating cost of the service. Regular headways at all stops will cause an even distribution of passenger loads on all vehicles. This will keep at a minimum the number and capacity of vehicles needed to maintain the desired level of service.

From the prospective passenger point of view, good service is measured by the time waiting at stops and the comfort of the trip. Kulash [2] proved that minimum average waiting time for passengers at stops is obtained when the headways are most regular. Also, even loads in all vehicles increase the passenger's chance of finding a seat and improve the comfort of all passengers.

From the reasons stated in the above discussion, headways will be considered as the variable of most interest in this investigation. The overall organization of this analysis will stress the construction of modular analytical models centering on several aspects of headway behavior. The advantages of this approach are several. They allow us to analyze particular aspects of the problem isolating them from the rest. This, in turn, allows a deeper and more complete understanding of the issues involved. These models could afterwards be readily extended or adapted to cover other aspects of the situation and eventually describe the overall headway behavior.
As a first step in the analysis of headway behavior, it is necessary to identify the most relevant causes of irregular headways in the operation of the single public transportation route. This is done in Chapter II by assuming the operation to be free of random influences and observing the dynamic response of headway values to selected perturbations in an otherwise perfectly regular system. The chapter presents a new perspective from which to examine the behavior of the system in terms of graphical space-time representation. Expression for headways, waiting times and trip times are developed as functions or perturbations to the system.

The so devised deterministic model is not intended to represent a real life example of a public transportation route but it is designed to bring into clear perspective the functional relationships among the parameters of interest. The focus of the analysis is shifted in Chapter III to the problem of delays caused by traffic lights operations. Several surveys of urban public transportation systems have shown that delays at traffic signals amount to values between 15 and 40% of the total travel time between stops [3, 4]. Of special interest is the fact that in the particular situation of the Dudley-Harvard bus route in Boston, the variance of the delay at traffic lights amounts to almost 70% of the variance of the total trip time [5]. In this chapter several models of delays at isolated intersections controlled by traffic lights and an approximate model of total delay in a trip through an urban environment as a function of the spatial distribution of traffic lights is proposed.

Chapter IV extends the deterministic model in terms of random perturbations introduced into the operation of the system. A model of propagation of variance of perturbations in headways at consecutive stops is presented here and it is later extended to include the effect of delays due to traffic
lights. The extent of the prediction's validity of the models proposed are tested against the results of simulation. A correction to the models is then proposed when the service has deteriorated in such manner as to allow a very high irregularity of headways.

Finally, Chapter V presents a summary of the work done, conclusions and suggests lines of further work.
CHAPTER II

SINGLE ROUTE OPERATION: A DETERMINISTIC MODEL

It can be generally assumed that any Public Transportation System providing service to a typical urban area is divided in several distinct routes. The degree of interdependence in the operation of every route varies from system to system, i.e., a few routes may share several stops and therefore demand, or they may share vehicles, therefore sharing capacity.

It would seem rather simplistic to try to describe the operation of the whole system as an aggregate model of independent deterministic operations, one for each route. Nevertheless, there are several advantages in concentrating the analysis on the deterministic operation of a single route, as well as justifications. The main objective to be pursued is to find qualitative and quantitative causal relationships between the parameters of the system; that is, to develop a basic model that will serve as a starting point for the development of more complex systems. In order to obtain a clear understanding of the relationships involved, the more simpler example of a single route seems to best fit our needs.

Furthermore, although the effect of randomness will not be considered in this initial model, the structural integrity of the operation mechanism will be maintained as faithfully as possible; and it is expected that this will serve as a suitable basic framework for more realistic models.
Section 2.1: Description of Operation Parameters

The term "Transportation System Route" will be used throughout this thesis with the understanding that it refers to public transportation modes characteristic of urban environments and restricted to those technologies designed to handle fairly large passenger/vehicle ratios. The term vehicle, therefore, applies indiscriminately to buses, electric buses (trolleys) trams and rapid transit units, (subways, elevated rapid transits and monorails systems).

We will proceed to discuss the main aspects of the operation of a single route system and to provide typical ranges for these factors. The Basic Deterministic Model will then be proposed in terms of a set of assumptions on the characteristics of the operation.

2.1.1: General Characteristics of Operation

The set of operational characteristics of most interest to be considered in our analysis can be classified into three groups: System dependent, externally determined and controllable factors. Let us briefly describe each characteristic and indicate typical ranges for actual systems.

A) System Dependent Factors:

This category includes all those characteristics of a public transportation system that are dependent on long term system management decisions and therefore not susceptible to changes on a day to day basis. The factors involved comprise vehicular technology types and route geometry.
i) **Vehicular passenger capacity:** Table 2.1 shows typical ranges of passenger capacity according to the vehicular technology being used. Capacity values include seated and standing passengers. The table also shows typical values for the maximum number of cars per train, and resulting total passengers per train, for those vehicle-types that lend themselves to train operation.

ii) **Boarding and alighting times:** The main factors affecting boarding and alighting times are the fare collection system being utilized and the number of doors per vehicle (or train, where applicable). Table 2.1 shows some typical values. Fare collection is assumed to be carried out on boarding for buses and trams; and prior to boarding the vehicle in rapid transit system.

iii) **Vehicle performance:** Table 2.2 shows some typical values of maximum and platform speed for several types of vehicular technology. Platform speed is defined as the normal service speed, including passenger stop and normal delay-recovery time but excluding terminal layovers. Effective transit speeds are subject to delays caused by congestion in mixed traffic, passenger volume at stops and aboard vehicles and other factors.

iv) **Stop spacing:** The spacing of stops in the route has major effects in the door to door passenger travel time and system speeds and performance. Typical ranges of stop spacing are shown in Table 2.2 according to the type of vehicle. Research in optimum spacing of stops tends to favor the criterion of minimizing the total passenger time, defined as the sum of aggregated values of passenger access time and passenger time in the system.

Because of the general tendency towards overall reduction of door to door travel times and increase in vehicle speed and capacity, there exists a trend to increase the stop spacing distances on street routes and in the
### TABLE 2.1
CAPACITY OF PUBLIC TRANSPORTATION VEHICLES

**TYPICAL RANGES**

<table>
<thead>
<tr>
<th>Vehicle Type</th>
<th>Carrying Capacity</th>
<th>Transfer Capacity</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Per Single Unit</td>
<td>Per Maximum Train</td>
</tr>
<tr>
<td></td>
<td>Total passengers</td>
<td>Number of Units</td>
</tr>
<tr>
<td><strong>City/Suburban Bus</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Freewheeling</td>
<td>45-170</td>
<td>--</td>
</tr>
<tr>
<td><strong>Tram</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Rail Guided</td>
<td>80-450</td>
<td>2-6</td>
</tr>
<tr>
<td><strong>Urban/Regional Rapid Transit</strong></td>
<td>100-330</td>
<td>9-10</td>
</tr>
</tbody>
</table>

* Vehicles with fare boxes for incoming passenger
+ Per door lane

Source: Transportation and Traffic Engineering Handbook
## TABLE 2.2

VEHICLE PERFORMANCE AND STOP SPACINGS

TYPICAL VALUES

<table>
<thead>
<tr>
<th>Vehicle Type</th>
<th>Vehicle Velocities</th>
<th>Stop Linear Spacing (MTS)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Maximum Performance Speeds</td>
<td>Traditional Practice</td>
</tr>
<tr>
<td>Local Bus, Urban</td>
<td>80-105</td>
<td>150-250</td>
</tr>
<tr>
<td>Limited stop bus Urban</td>
<td>80-105</td>
<td>365-920</td>
</tr>
<tr>
<td>Local Tram Urban</td>
<td>65-100</td>
<td>150-250</td>
</tr>
<tr>
<td>Rapid Transit Urban</td>
<td>80-115</td>
<td>500-1100</td>
</tr>
<tr>
<td></td>
<td>13-23</td>
<td>19-29</td>
</tr>
</tbody>
</table>

Source: Transportation and Traffic Engineering Handbook
planning of new rapid transit systems.

Other factors that can be considered to be pre-determined in the day
to day operation of a single route are equipment and manpower resources,
such as number of units, number of vehicle operators, maintenance personnel,
etc.

B) Externally Determined Factors

Factors affecting the performance of the transportation system and
outside direct or indirect control of system management can generally be
described as external to the system and are usually characterized by prob-
abilistic influence.

Demand can be considered as the most important external factor affect-
ing the operation of the system. Demand variability is patently observable
in its main aspects: the distribution of demand during a cycle of operation
(usually one or one half day), the spatial distribution of demand along the
stops in the route; and the origin destination pattern of the average pas-
senger usually presents very high fluctuations in very short periods of
time, leaning heavily on the overall capability of the system.

Traffic congestion is another important factor affecting the perform-
ance of those systems that are operating in streets. Travel times during
rush hour can easily increase by a factor of two or three, and even more
compared to normal transit operation.

C) Controllable Factors

In order to meet predicted time patterns of demand, the operation mana-
ger can recur to several alternatives in order to modify the capacity of the
system. One option is to change the frequency of vehicle departures from
the main dispatching station at the beginning of the route or from selected

17.
intermediate depots. In only this case, passenger carrying capacity per unit is changed. The number of cars per train departing unit can be changed modifying the total passenger carrying capacity as well as the boarding and alighting rates.

As for real time control devices, technology sophistication and therefore layout and maintenance cost is the only deterrent. Presently available control systems include mid-route schedule control point, stop skipping, minimum headways warning devices (located either along the route, or aboard the vehicle and triggered by clocks at selected spots), traffic lights priority schemes (either mechanically or electronically activated) and electronic vehicle location monitoring system (AVM, etc.).

2.1.2: Deterministic Model Assumptions

The deterministic model of the operation of a Single Route Public Transportation System is succinctly illustrated in Figure 2.1. It consists of an infinite supply of vehicles leaving a dispatching station at regular intervals, travelling at constant speed along a fixed route, and allowing passengers to board and alight at every one of the regularly spaced stops in the route.

The above description can be formalized in the following set of assumptions:

ENVIRONMENTAL FACTORS:

1) The route to be travelled is the same for every vehicle in the system.

2) Times to travel between consecutive stops are constant and equal for all vehicles and pair of stops.
FIGURE 2.1

SINGLE ROUTE DETERMINISTIC MODEL

REGULAR DISPATCHING EVERY $H_0$ MINUTES

CONSTANT TRAVEL TIME BETWEEN STOPS

INFINITE SUPPLY OF VEHICLES

INFINITE SUPPLY OF PASSENGERS AT STOPS ($\lambda$)
OPERATING PROCEDURES:

3) Vehicles leave the dispatching station at regular intervals and the available supply of vehicle can be considered infinite.

4) Immediately upon arrival at a particular stop, alighting passengers will first vacate the vehicle, and only then, those waiting at the stop will be allowed to board it. When no more passengers remain at the stop, the vehicle will immediately proceed to the next stop along route.

5) The passenger alighting period is assumed of fixed length for all vehicles and stops, disregarding the amount of passengers alighting. The passenger boarding period, however, is assumed of length proportional to the number of passengers boarding the vehicle; the boarding rate being constant for all vehicles and stops.

6) Each vehicle has an infinite passenger carrying capacity.

DEMAND:

7) Prospective users of the system arrive at stops at a constant rate, identical for all stops.

8) If there is one or more vehicles at the stop at the moment of a passenger's arrival, that passenger will board only the first vehicle in line and no other.

9) If there are no vehicles at the stop at the moment of a passenger's arrival that passenger will wait for the next vehicle to arrive at the stop and board it, no matter how long the waiting time is.

Assumptions 1 and 4 are generally consistent with the operation of any public transportation route and need no further comment. Assumptions 2, 3, and 7 eliminate the effect of randomness in the system and are basic to our
model. Assumption 5 requires a more extensive justification. First of all, for any type of vehicle, boarding or alighting, do not start immediately after the bus comes to a halt at the stop; nor departure from the stop occurs immediately after the last passenger got on the bus. There is an overhead time that comprehends opening and closing of doors and some other actions by drivers and controllers.

Second of all, in most routes where the fare is paid at boarding, alighting can take place through more than one door. As for those systems such as rapid transit where the fare is paid before boarding the vehicle, alighting and boarding usually take place at the same time unless heavy crowding is in effect.

Adding the fact that alighting is generally faster than boarding (See Table 2.1) for the above mentioned system, it is reasonable to assume that the time a vehicle stays at a stop is proportional only to the passengers boarding the vehicle at that stop and to assume a constant time for alighting and other overhead operations.

If two vehicles are at the same stop at the same time, there is a general tendency among passengers to board the first vehicle in line to depart unless that vehicle is overcrowded. Assumptions 6 and 8 guarantee that even if an overcrowding situation exists the tendency to board the first vehicle is maintained. Both assumptions could be relaxed in the future for a more careful and realistic analysis of crowding conditions.

Finally, assumption 9 is included to eliminate the possibility of balking; which in any case is very rare since it is assumed that there are no immediately available alternative modes of transportation for the prospective user of the system.
Section 2.2: Operation of the Deterministic Single Route

Several studies and experiments [6, 7, 8] have shown that under technological and budgetary constraints, a regular schedule will minimize operation cost and maximize passenger comfort in any transportation system. Regularity of schedules of any given route is measured by the behavior of vehicle headways along the route. It seems justified, to describe the system operation in terms of headway behavior of the vehicles in the route.

2.2.1: Basic Relations Among Headways

Let us define the headway between two consecutive vehicles at a given stop as the difference in their departure times from that stop. If we use the variable $k$ to identify each stop on the route, and the variable $p$ to identify the vehicles in the order they left the dispatching station, headway is defined as

$$\phi_{k}^{p} = d_{k}^{p} - d_{k}^{p-1}$$

where $d_{k}^{p-1}$ and $d_{k}^{p}$ are the departure times of the $(p-1)$th and $p$th vehicle from stop $k$.

By the assumptions of the deterministic model, upon departure from a station, a vehicle will travel to the next station, halt to board passengers and depart immediately after. We can then relate the departure times of a vehicle from two consecutive stops in the following form

$$d_{k}^{p} = d_{k-1}^{p} + t_{k-1}^{p} + s_{k}^{p}$$

22.
where $t_{k-1}^p$ is the travel time between stops (k-1) and k for vehicle p, and $S_k^p$ is the time that vehicle p remains at stop k.

Inserting the relation described in 2.2 into the definition of headways, we obtain that

$$\rho_{k}^p = (d_{k-1}^p + t_{k-1}^p + S_k^p) - (d_{k-1}^{p-1} + t_{k-1}^{p-1} + S_k^{p-1})$$

$$= \rho_{k-1}^p + (t_{k-1}^p - t_{k-1}^{p-1}) + (S_k^p - S_k^{p-1})$$

This relationship is illustrated in Figure 2.2a. Assumption 2 of the deterministic model indicates that travel times between consecutive stops are equal and constant for all vehicles. The above relationship then becomes

$$\rho_{k}^p = \rho_{k-1}^p + (S_k^p - S_k^{p-1})$$

as shown in Figure 2.2b.

As previously stated in assumption 5, the amount of time a vehicle remains at a stop is equal to a minimum stopping time plus the time necessary to allow boarding of all passengers present or arriving at the stop while the vehicle is there. In mathematical terms, the time vehicle p remains at stop k is equal to

$$S_k^p = S_o + \frac{\rho_{k}^p}{\mu}$$

where $S_o$ is the minimum stopping time, $\mu$ is the boarding rate and $\rho_{k}^p$ is the number of passengers boarding vehicle p at stop k. Figure 2.3 illustrates
FIGURE 2.2
BASIC RELATIONS BETWEEN CONSECUTIVE HEADWAYS

FROM FIGURE:
\[ p^*_{k} = p^*_{k-1} + (s^p_k - s^p_{k-1}) \]

a. SPACE-TIME TRAJECTORIES OF VEHICLES: GENERAL CASE

FROM FIGURE:
\[ p^*_{k} + s^p_k = p^*_{k-1} + s^p_{k-1} \]

b. SPACE-TIME TRAJECTORIES OF VEHICLES:
EQUAL TRAVEL TIMES BETWEEN STOPS.
**FIGURE 2.3**

**PASSENGERS QUEUE SIZE AT STOP k**

\[ \begin{align*}
& a_k^p = \text{ARRIVAL TIME OF VEHICLE } p \text{ AT STOP } k \\
& d_k^p = \text{DEPARTURE TIME OF VEHICLE } p \text{ AT STOP } k 
\end{align*} \]
the behavior of the queue of passengers at stop \( k \) as a function of time. Passengers arrive at a constant rate \( \lambda \) at the stop, and while there is no vehicle allowing passengers aboard, the queue size grows at that rate. Upon arrival of a vehicle and after the minimum stopping time has elapsed, passengers in queue will get on the vehicle at a constant rate \( v \). Assuming that \( v \) is greater than \( \lambda \) and that prospective passengers arrive continuously at the stop, queue size is diminishing at a rate equal to \( (v - \lambda) \). As soon as there are no more passengers at the stop, the vehicle departs and the process repeats itself.

The number of passengers boarding vehicle \( p \) is therefore proportional to the interval of time between the departure of vehicle \( p \) and the previous vehicle departure. That is

\[
\eta_{k}^{p} = \lambda \cdot (d_{k}^{p} - d_{k}^{p-1}) = \lambda \cdot h_{k}^{p} \tag{2.6}
\]

Equation 2.4 then becomes

\[
h_{k}^{p} = h_{k-1}^{p} + (s_{o} + \frac{\lambda}{\lambda} h_{k}^{p}) - (s_{o} + \frac{\lambda}{\lambda} h_{k}^{p-1}) \tag{2.7}
\]

Let us define \( \lambda/\mu \) as the "capacity utilization ratio" of service and represent it by \( \rho \). Equation 2.7 can be rearranged to express the relation of the headway of the present vehicle at the current stop with the headway at the previous stop and with the previous vehicle headway at the current stop.
That is,

\[
\hat{h}_k^p = \frac{\hat{h}_{k-1}^p - \rho \hat{h}_k^{p-1}}{1 - \rho}
\]

Since we are dealing with a deterministic model, if we dispatch vehicles with constant headways \(h_o\) into the system, one could reasonably expect the vehicles to depart from each stop along the route with the same headway. In fact, it can be proved by induction on equation 2.4 that if the headways at the dispatching station take the value \(h_o\) for all vehicles, then the headways for all stops will take the same value. That is,

\[
\text{if } h_o^p = h_o \quad \text{for all } p,
\]

\[
\text{then } h_k^p = h_o \quad \text{for all } k \text{ and } p.
\]

2.2.2 Service Effectiveness Measures

The average waiting time of a passenger at a stop can be considered one of the most important measures of effectiveness of the system. Kulash [2] show that, for Poisson passenger arrivals at stops, the average waiting time of passengers is given by

\[
AWT = \frac{E\left[ h^2 \right]}{2 E\left[ h \right]} \tag{2.9}
\]

where \(E[h]\) and \(E[h^2]\) are the first two moments of the distribution of headways. For a regular schedule with headways at a constant value of \(h_o\) this expression reduces to
It is obvious that in case of regular arrivals, Kulash's expression is still valid.

The number of passengers boarding a vehicle with a corresponding headway of \( h \) is \( \lambda h \). Independently of his position in the queue, any passenger can be considered to wait at the stop until the vehicle's departure, even if the passenger is on board long before that. Since arrivals occur every \( 1/\lambda \) units of time, the average waiting time of passengers arriving during a headway of length \( h_0 \) reduces to the same expression derived by Kulash.

The experiments of passenger waiting times in London and Manchester, previously mentioned at the beginning of this section, showed that for short schedule headways, say, under 10 minutes, relation 2.9 was a good approximation of the measured values of waiting times. It is current practice, for longer headways, to publish service schedules and passengers with a knowledge of the schedule, tend to arrive at a stop shortly before a scheduled arrival. Equation 2.9 would then over-estimate the average passenger waiting time. In fact, it was also shown that the longer the headway, the greater the over-estimation.
Section 2.3: Perturbations to the Deterministic Model

It was shown in the previous section that if vehicles are dispatched at constant intervals into the route, then, headways at every stop will also remain constant and equal to the initial value.

However, what will happen if the schedule cannot be kept for a given vehicle and it is dispatched late? Since the vehicle is dispatched late, it will arrive at the first stop later than the scheduled arrival time. As a consequence, more passengers than usual will board the vehicle and therefore it will be delayed further. The next vehicle dispatched, assuming that the schedule is maintained, will encounter less passenger waiting at the first stop than the previous vehicle, so it will depart sooner than scheduled.

Several questions are raised by this behavior: will the deviation from schedule in stops along the route increase, remain constant or die away for subsequent vehicles? In other words, is the system stable? Will the deviation propagate to later stops in such a way as to make vehicles pair up before the end of the route? In order to answer these and other questions, let us examine closely the behavior of vehicles when a small disturbance is introduced into the dispatching schedule.

2.3.1: Propagation of Headway Perturbations for the First Vehicle

Let us assume that the system has been operating under a regular schedule for an infinite time, and that the dispatching headway for vehicle number 1 has been increased by a small amount $\Delta h_0$, and that thereafter all other vehicles are dispatched again with the same regular schedule headway.
That is,

\[ h_k^p = \begin{cases} 
   h_0 + \Delta h_0 & \text{for } p=1 \\
   h_0 & \text{otherwise}
\end{cases} \quad 2.11 \]

It seems clear that vehicles that have already departed before vehicle 1 should not be affected by the perturbation. That is,

\[ h_k^p = h_0 \quad \text{for } p < 1, \text{ all } k. \quad 2.12 \]

Figure 2.4 illustrates the change in headways for vehicle 1 at successive stops, due to the initial deviation from schedule. The vehicle arrives at the first stop with a delay equal to the dispatching headway deviation. This in turn, causes more passengers to board the vehicle, resulting in an increased delay in departure time. The situation repeats itself, to a larger extent, at stop 2 and the deviation increases very fast for further stops.

In order to get an expression for the delay at stop \( k \), let us refer to equation 2.8 evaluated at stop \( k \) for \( p = 1 \).

\[ h_k^1 = \frac{h_k^1 - \rho h_k^0}{1 - \rho} \quad 2.13 \]

From previous arguments (Equation 2.12),

\[ h_k^0 = h_0 \quad \text{for all } k. \]
FIGURE 2.4
HEADWAY DEVIATIONS OF FIRST VEHICLE

DISTANCE

VEHICLE ZERO

VEHICLE ONE

REGULAR SCHEDULE

ACTUAL SCHEDULE

STOP 5

STOP 4

STOP 3

STOP 2

STOP 1

0

TIME

$\Delta h_0$

$\Delta h_1$

$\Delta h_0 + \Delta h_1$
therefore,

\[ h_k^i = \frac{h_{k-1}^i}{1 - \rho} - \frac{1 - \rho}{1 - \rho} h_0 \]  

2.14

Applying 2.14 recursively for \( k \), we obtain that

\[ h_k^i = \frac{1}{(1 - \rho)^k} [h_0 + \Delta h_0] - \frac{\rho}{1 - \rho} \cdot \sum_{i=1}^{k} \frac{1}{(1 - \rho)^{k-i}} \]  

2.15

which reduces, after algebraical manipulation to

\[ h_k^i = h_0 + \Delta h_0 \cdot \frac{1}{(1 - \rho)^k} \]  

2.16

If the capacity utilization factor \( \rho \) is greater than one; that is, if the arrival rate is greater than the boarding rate, the first vehicle will not be able to leave stop 1. If, on the other hand, \( \mu \) is greater than \( \lambda \), the headway for vehicle 1 will increase exponentially along the route. This implies that the initial delay will result in increasingly larger delays for the vehicle along the route. Figure 2.5 illustrates this behavior for several values of \( \rho \).

As it can be seen from equation 2.16, the sign of the deviation is maintained for all stops. Therefore, if the vehicle is dispatched early instead of late (\( \Delta h_0 \) is negative), then it will be increasingly early along the route.

2.3.2: A Single Dispatching Perturbation: Headway History of the System

Having seen the effects of the initial disturbance on the schedule of the first vehicle, let us examine the effects on headways of subsequent vehicles.
FIGURE 2.5

PROPAGATION OF DEVIATIONS FOR VEHICLE ONE

\[ \Delta h_k \]

\[ \text{STOP k} \]
Let us examine the behavior of headway at stop 1. Using 2.13

\[ h^p_1 = \frac{1}{(1-p)} h^p_o - \frac{p}{(1-p)} h^p_{p-1} \tag{2.17} \]

Since \( h^p_o \) is constant for \( p \) greater than 1, then

\[ h^p_1 = \frac{1}{(1-p)} h^p_o - \frac{p}{(1-p)} \cdot h^p_{p-1}; \quad p \geq 2 \tag{2.18} \]

Expanding 2.18 in terms of \( p \) and recalling the result given by equation 2.16 for stop 1, we obtain

\[ h^p_1 = h^p_o + \Delta h^p_o \left( \frac{1}{(1-p)} \cdot \left( \frac{-p}{1-p} \right)^{p-1} \right); \quad p \geq 1 \tag{2.19} \]

Equation 2.19 uncovers some very important aspects of the deterministic single route system operation. First of all, the overall schedule of departures from the dispatching station was shifted by a small amount \( \Delta h_o \), affecting only the dispatching interval between vehicles 0 and 1. However, the schedule of departures from stop 1 does not shift in the same fashion. In fact, even if vehicles following vehicle 1 were made to depart with constant headways from the dispatching station, headways at stop 1 are still affected by the initial perturbations. As illustrated in Figure 2.6, headways at the first stop oscillate around the value corresponding to regular operation. Moreover, the way the instability of schedule propagates to other vehicles depends on the value of the capacity utilization factor \( \rho \).
FIGURE 2.6

PROPAGATION OF DEVIATIONS FOR STOP NUMBER ONE

$\Delta h_p$

VEHICLE (p)

$\rho = .45$

$\rho = .50$

$\rho = .55$
As indicated by equation 2.19 the ratio of deviations for two consecutive vehicles at the same stop is given by

\[ R(\Delta h_p) = \frac{\Delta h_{p-1}^P}{\Delta h_P^{p-1}} = -\frac{\rho}{1-\rho} \]  

2.20

where \( \Delta h_{p-1}^P \) and \( \Delta h_P^p \) are the headway deviations of vehicles \( p-1 \) and \( p \) respectively.

When \( \rho \) is less than 1/2 (the boarding rate is greater than twice the arrival rate) the ratio is less than one in absolute value, implying that the absolute value of deviations will constantly decrease and headways will asymptotically return to their equilibrium value. If, on the other hand, \( \rho \) is greater than 1/2 (the boarding rate is less than twice the arrival rate), the propagation factor is greater than one in absolute value and headways deviations grow exponentially. In other words, the system is now unstable.

This behavior would seem to be in contradiction with the fact that a service facility can maintain a stable operation as long as the service rate is greater than the arrival rate of prospective users.

However, the passenger's arrival and boarding at a stop, does not completely fit the standard queueing model of service facility. The difference lies in the fact that the server (the vehicle) remains unavailable part of the time, (i.e., when no boarding takes effect). Let us propose a possible interpretation of the situation.

Consider a queueing system where the server is available to arriving customers only during certain periods. Figure 2.7 shows the behavior of such system. During all times customers arrive at the facility at a constant rate \( \lambda \). The function of the number of customers that have arrived
FIGURE 2.7

QUEUEING PROCESS FOR PARTIALLY AVAILABLE SERVER

CUSTOMERS
N(t)

CUMULATIVE NUMBER OF ARRIVALS UNTIL TIME t

CUMULATIVE NUMBER OF CUSTOMERS SERVED UNTIL TIME t

SERVER NOT AVAILABLE TO CUSTOMERS

O

r

w

c

y

A

Mt

At

SECOND

QUEUE SIZE

w

f

I

f

(9)
at the facility at time $t$ is the straight line $OA$. The server alternates periods of "rest" and "work" of constant lengths. During a rest period the server is not available and arriving customers are forced to wait until the beginning of the next work period. When the server becomes available he will start serving customers in queue at a constant service rate until the queue is empty. After that, the server will serve any new customer entering the facility until this work period ends. At that point, the cycle repeats itself.

The function describing the total number of customers that have been served until time $t$ is indicated by the segmented line $OBCDE \ldots$ etc.

Let us define $w$ as the length of the working period, and $r$ as the length of the rest period. The total cycle length $c$ is then $r \oplus w$. The rest period is effectively diminishing the service capacity of the system. In fact, even for values of capacity utilization ratio ($\rho = \lambda/\mu$) of less than one, the rest period may be long enough as to completely fill up the capacity of the system (Figure 2.8a). It will be convenient then to redefine the capacity utilization ratio of the system in order to include the effect of server rest period in the system.

It is clear from Figure 2.7 that the server will be busy at the facility during all the intervals $(w - y)$. During $y$ the server will be busy serving only the customers arriving in that interval, since no more customers are left in queue.

During $y$, $\lambda y$ customers will be served, for a cumulative service time of $py$. 

38.
FIGURE 2.8
SATURATED AND OVERSATURATED QUEUEING CASES

2.8-a \( \frac{\lambda}{C} = 1 - \rho \)

2.8-b \( \frac{\lambda}{C} > 1 - \rho \)
From queueing theory,

\[ \text{Prob (server is idle at the facility)} = 1 - \rho' \]

where \( \rho' \) is the effective capacity utilization ratio. Therefore,

\[ (1 - \rho') = \frac{y - \rho y}{c} = \frac{\text{[idle time, per cycle, at the facility]}}{\text{[cycle time]}} \]

From geometric relations

\[ y = \frac{w - \rho c}{1 - \rho} \]

then

\[ \rho' = \rho + (1 - \frac{w}{c}) = \rho + \frac{r}{c} \quad 2.21 \]

If the rest period is greater than \((1 - \rho)c\), the effective capacity ratio is greater than one and the queue grows indefinitely (Figure 2.8b).

When \( \rho' \) is equal to one, the server is working the least possible amount of time. That is, from equation \(2.21\) the minimum work to rest ratio is given by

\[ \left[ \frac{w}{r} \right]_{\text{min}} = \frac{\rho}{1 - \rho} \quad 2.22 \]

This is precisely the case of the deterministic model of vehicles arrival and departure process at stops. Vehicles are the server and the headway between departures is the length of the cycle. The working period corresponds to the boarding of passengers at the stops, and when the system is in equilibrium, the effective capacity ratio is one. The effect of a de-
lay in vehicle arrival is equivalent to a delay in the appearance of the server at the facility. Consider Figure 2.9. When the server is late by an amount \( \Delta r \), its rest period for that cycle has increased by the same amount. If the server were to leave the facility at the same time as usual, the effective capacity ratio would have changed to

\[
\rho' = \rho + \frac{r + \Delta r}{c} = 1 + \frac{\Delta r}{c} > 1
\]

and the system would be unstable.

Since the vehicle is forced to stay at the stop until all passengers have boarded (that is, to maintain an effective capacity utilization ratio of one), the initial delay \( \Delta r \) will actually cause the work period to increase by an amount \( \Delta w \), and

\[
\frac{\omega + \Delta \omega}{r + \Delta r} = \frac{\rho}{1 - \rho} = \frac{\omega}{r}
\]

giving

\[
\Delta w = \frac{\rho}{1 - \rho} \Delta r
\]

and increasing the total cycle length by

\[
\Delta c = \Delta r + \Delta w = \frac{1}{1 - \rho} \Delta r
\]

Since the assumption is that vehicles will continue to arrive at constant intervals (c) after the initial delayed arrival, the cycle length increase will actually affect the length of the next cycle. In fact, the second rest period is changed to
FIGURE 2.9

QUEUEING PROCESS FOR DELAYED SERVER ARRIVAL
\[ r' = r + \Delta r - \Delta c = r - \frac{p}{1-p} \Delta r \]

\[ = r + \Delta r' \]

and so, the next work period is changed, to maintain a unit capacity ratio, by

\[ \Delta w' = \frac{p}{1-p} \Delta r' = -\left(\frac{p}{1-p}\right)^2 \Delta r \]

The total length of the second cycle is then changed by an amount

\[ \Delta c' = \Delta w' + \Delta r' = \frac{1}{(1-p)} \left(\frac{p}{1-p}\right) \Delta r \]

The change in the length of the second cycle will cause an increase in the rest period of the following cycle. The process will then repeat itself alternatively. It is therefore clear that the minimum work to rest ratio is the fundamental value determining the limiting return to stable conditions, as it is the case in the vehicles arrival and departure process.

The results derived for the first stop are also extendible to other stops. The instability at the first stop will cause greater instability at the second, and so forth.

Generalizing for all stops and vehicles, it can be shown, using equation 2.8 recursively, that the headway deviation for vehicle \( p \) at stop \( k \), caused by the initial perturbation of the headway of vehicle \( 1 \), is given by,

\[ \Delta h^p_k = \frac{(p+k-2)!}{(p-1)!(k-1)!} \cdot \frac{1}{(1-p)^k} \cdot \left(\frac{-p}{1-p}\right)^{p-1} \cdot \Delta h^1_0 = \]

\[ \approx \sum_{k=1}^{p-1} \Delta h^p_k \]

43.
where $\varepsilon_k^{p-1}$ is the expansion factor of a single deviation for vehicle 1. The headway of vehicle $p$ at stop $k$, would then be given by

$$h_{k}^{p} = h_{o}^{p} + \Delta h_{k}^{p}$$  \hspace{1cm} (2.24)

Figure 2.10 illustrates an interesting analogy that can be used to interpret the propagation of headway deviations. The process can be equated to a two-dimensional grid. The output of a node with coordinates $p$ and $k$ would be equivalent to the headway deviation corresponding to vehicle $p$ at stop $k$. Links between nodes are unidirectional amplifiers of signals from the origin node. Vertical links correspond to stop transitions and horizontal links to vehicle transitions. The signal at any node would be the sum of the signals transmitted by each link entering the node.

If we input a signal at node $(0, 1)$ (corresponding to an initial headway deviation for the dispatching of vehicle 1) any path leading from this node to the $(k, p)$ node will amplify the signal by a factor

$$\frac{1}{(1-\rho)^{k-1}} \left(\frac{-\rho}{1-\rho}\right)^{p-1}$$

Since there are $\binom{k+p-2}{k-1}$ different possible paths, the actual signal observed at $(k, p)$, must then be multiplied by

$$\binom{k+p-2}{k-1} \frac{1}{(1-\rho)^{k-1}} \left(\frac{-\rho}{1-\rho}\right)^{p-1} = \varepsilon_k^{p-1}$$

which gives the same result expressed in 2.23.
Figure 2.10

Signal Grid Analogy

Output = 3 \cdot \frac{1}{(1-\rho)^2} \cdot \frac{1}{(1+\rho)^2}
Section 2.4: Effects of Vehicle Clumping

The results obtained during the analysis of the previous section assumed implicitly that the perturbations introduced into the system were small enough so that headways between successive vehicles never reached negative values before the end of the route. In general terms, the equations developed are valid up to the point where \( h_{k}^{p} \) for some \( p \) and \( k \) is equal to zero.

When the initial perturbation is large enough or when the service capacity ratio is close to .5, it is very likely that vehicles will deviate from their schedule in such a fashion as to be at the same stop at the same time. In current literature, this phenomenon is called clumping, bunching or pairing.

2.4.1: First Clumping Time

Let us examine some aspects of clumping. First of all, there is an initial tendency for vehicles to clump in pairs. Equation 2.21 tells us that headway deviations will alternate in sign for a given stop. For an initial positive deviation to vehicle 1, the first vehicle will initially tend to clump with the second, the third with the fourth, and so forth.

Secondly, since travel time along links connecting successive stops are identical for all vehicles in our simplified model, the actual clumping can only occur at stops. We can find the stop at which clumping first occurs for the first two vehicles by determining the smallest value of \( k^{*} \) for which \( h_{k^{*}}^{2} \) will be less or equal to zero. This implies solving the non-linear equation in terms of \( k^{*} \)
\[ h_o - k^* \cdot \Delta h_o \left( \frac{\rho}{1-\rho} \right) \cdot \frac{1}{(1-\rho) k^*} = 0. \]

Figure 2.11 illustrates the solution to this equation for several values of \( \Delta h_o/h_o \) and \( \rho \).

To analyze the behavior of the system for successive vehicles and stops other than the first \( k^* \) stops we must take into account two factors: the first one is that for service capacity ratio values of less than 0.5, the absolute value of the deviation is maximum for the first vehicle and steadily decreases for successive ones. This implies that vehicles following the first two will never clump at stops and times earlier than the occurrence for the first and second vehicle. The second factor is that system management policy and in some cases physical constraints (i.e., subways) rule out the overtaking alternative.

Let us concentrate only in the case where overtaking is not allowed. The second vehicle encounters the first one at stop \( k^* \), where there are still passengers waiting to board the first vehicle. From this point thereon, the second vehicle is forced to follow the exact behavior of the first one because no overtaking is allowed. Furthermore, because of the assumptions set in section 2.2, no more passengers will board the second vehicle as long as the first one is present at the same stop. The schedule of the first vehicle can still be predicted by equation 2.19 for further stops but headways for vehicle 2 will be set at zero for all stops after and including stop \( k^* \).

The behavior of following vehicles will not be affected by this clumping until they reach stop \( k^* \), and therefore their headways can still be predicted until that stop by equation 2.19. The second vehicle practically...
FIGURE 2.11
STOP OF FIRST CLUMPING

\[ \frac{\Delta h_2}{h_0} \]

\[ k^* \]

\( \rho = .05 \)

\( \rho = .10 \)

\( \rho = .20 \)

\( \rho = .50 \)
"disappears" from the system after stop k*. In terms of the "grid analogy" described in the previous section, all nodes corresponding to vehicle 2 for stops after k* are left unconnected to the grid. The links joining nodes corresponding to vehicle 1 and 2, connect directly to nodes corresponding to vehicle 3. (Figure 2.12).

It is clear that at this point an analytical description of the behavior of the system becomes, to say the least, cumbersome. Using a computer becomes a necessity and a graphical description will undoubtedly help to understand the behavior of the system.

2.4.2: A Graphical Approach to Describing Route Operation

A simple way to graphically describe the behavior of the system would be to use a distance vs time graphical relationship similar to that of Figures 2.2a and 2.2b, where the trajectory of every vehicle would be represented by a set of alternating horizontal (stop) and inclined (vehicle moving) straight line connected segments. Unfortunately, this graphical method does not provide a clear representation of the system's behavior. The method proposed, instead, is based in two concepts: establishment of departure times from stops as the time-space points of interest and the visual comparison of schedule deviation for different vehicles and parameters.

Consider an imaginary reference vehicle moving at a constant speed along a path that includes the route followed by the vehicles of the public transportation system. Furthermore, assume that the reference vehicle is at the dispatching station at time zero, and that the system has been in equilibrium for an infinite time. Let the k stop departure reference distance of vehicle p in the system, be the distance between vehicle p and
FIGURE 2.12
GRID ANALOGY FOR FIRST & SECOND VEHICLE CLUMPING
the reference vehicle measured at the time of vehicle p's departure from stop k.

If we plot all the stop-departure reference distances of a vehicle on schedule against the time of departure, the resulting graph should follow a straight line starting at the dispatching time of the vehicle and ending at the arrival time at the end of the route. If, furthermore, the speed of the reference vehicle is assumed equal to the equilibrium platform speed of vehicles (which includes times at stops), the trajectory of a vehicle on schedule should be parallel to the time axis. Figures 2.13 illustrates this relationship.

If a vehicle is behind schedule when departing from a stop, both the reference distance and the departure time will be greater than the value for an undelayed schedule. In case of an early vehicle, the converse is true and the values obtained will be less than what they should be for an undelayed schedule.

The slope of the curve is also important. A line curving towards the direction of increasing distance means increasing headway deviations. The converse means diminishing headway deviations.

Figures 2.14 and 2.15 illustrate the graphical description method for two systems with different service capacity ratio. In both cases the equilibrium headway is 10 minutes and the headway deviation of the first vehicle was 1 minute. Distances between stops were set at 500 meters and the travel time between stops at 2 minutes. Service rates in both cases were set at 10 passengers/minute and the arrival rates at 1 and 2 passengers/minute respectively.
**Figure 2.13**

**Departure Reference Distance Method**

\[
\text{Departure Reference Distance} = U \cdot q_k^b - \text{Dist}(\text{stop } k)
\]

- **V > V_p**
  - 0 stops
  - **V = V_p**
  - 1 stop
  - **V < V_p**
  - 2 stops

**Regular Headways Trajectories**

**Increasing Deviations**

**Decreasing Deviations**

**Perfect Schedule**

**Accumulated Delay in Trip**

52.
FIGURE 2.14
DRD - TRAJECTORIES

\( p = 0.1 \)
\( \Delta \theta_0 = +1 \ min \)
\( \theta_0 = 10 \ min \)

Vehicle trajectories depicted on a distance-time graph with vehicles numbered from 1 to 10. Key features include

- Vehicle stops indicated by "STOPS".
- A clump of vehicles (3-4) near the middle of the route.
- The end of the route is marked.
Figure 2.15
DRD - Trajectories

\[ \rho = 0.2 \]
\[ \Delta T = +1 \text{ min} \]
\[ T_0 = 10 \text{ min} \]
Minimum Stopping time was 1 minute in both cases. The platform speed was calculated according to the formula

\[ V_p = \frac{d_o}{t_o + t_s + \rho h_o} \]  \hspace{1cm} 2.25

where \( d_o \) and \( t_o \) are the interstop distance and travel time respectively, and \( t_s \) is the minimum stopping time. Platform speeds were found to be 250 meters/minute for \( \rho = 1 \) and 100 meters/minute for \( \rho = 2 \).

Each trajectory plotted corresponds to a single vehicle up to the point of clumping where it merges with the trajectory of the vehicle with which the clump occurs. Each vehicle trajectory was calculated sequentially starting with vehicle zero which was assumed to follow a perfect schedule and with dispatching time equal to zero. After that, each vehicle dispatching time was calculated sequentially according to

\[ d_o^p = d_o^{p-1} + h_o^p \]  \hspace{1cm} 2.26

The departure times from subsequent stops for each vehicle were found sequentially by relation 2.8 expressed in terms of headways

\[ d_k^p - d_k^{p-1} = \frac{d_k^p - d_{k-1}^{p-1}}{1 - \rho} + \frac{\rho}{1 - \rho} (d_k^{p-1} - d_k^{p-2}) \]

or

\[ d_k^p = \frac{1}{1 - \rho} (d_k^p - d_{k-1}^{p-1}) + \frac{1 - 2\rho}{1 - \rho} d_k^{p-1} + \frac{\rho}{1 - \rho} d_k^{p-2} \]
If vehicles previous to $p$ are clumped, there is a renumbering of vehicles until $p - 1$ and $p - 2$ represent the two nearest vehicles to $p$ that are not clumped. Points of departures in each trajectory are indicated by the stop number. Only even stops are shown for more clarity.

Comparing both figures we can immediately see the sharp difference on the effects of the service capacity ratio. When $p$ is equal to .1 almost regular intervals between departures from stop 10 are reached by the passage time of vehicle 5. When $p$ equals .2 this does not happen until the passage time of vehicle 13.
Section 2.5: Multiple Perturbations: Equivalence and Additive Property

The results obtained so far were limited to cases of single perturbations to the dispatching schedule. It is of obvious interest, however, the analysis of effects of perturbations to some other processes (i.e., travel time between stops, passengers arrival rate, etc.) and examples of multiple perturbations. It will be shown that, in fact, equations developed in previous sections are still applicable with minor adaptations to the more general problem indicated before.

2.5.1: Additivity Property of Perturbations

Let us examine the combined effects of perturbations to the dispatching headway of two consecutive vehicles. That is, let us analyze the system behavior when two deviations $\Delta h^1_o$ and $\Delta h^2_o$ are introduced into the dispatching headway of vehicle 1 and 2 respectively.

It should be clear that, assuming no clumping situation will occur because of the perturbations, vehicle one's behavior will not be affected by the behavior of vehicle two. Equation 2.16 describes exactly the behavior of the first vehicle.

Let us examine the behavior of headways at the first stop. Dispatching (stop zero) headways are given by assumption, as

$$h^p_o = \begin{cases} h_o + \Delta h^1_o, & \text{for } p = 1 \\ h_o + \Delta h^2_o, & \text{for } p = 2 \\ h_o, & \text{otherwise.} \end{cases} \quad 2.28$$
Headways at stop 1 can be then calculated using the basic relation among headways (equation 2.8). That is, given that

\[
\begin{align*}
    h_k^p &= h_0 \quad \text{for } p < 1, \text{ all } k
\end{align*}
\]

then,

\[
\begin{align*}
    h_1^1 &= \frac{1}{1-p} h_0^1 - \frac{\rho}{1-p} h_1^0 \\
    &= h_0 + \Delta h_0 \frac{1}{1-p}
\end{align*}
\]

Similarly for \( p = 2, \)

\[
\begin{align*}
    h_1^2 &= \frac{1}{1-p} h_0^2 - \frac{\rho}{1-p} h_1^1 \\
    &= h_0 - \frac{\rho}{(1-\rho)^2} \Delta h_0^1 + \frac{1}{1-\rho} \Delta h_0^2
\end{align*}
\]

and \( p = 3, \)

\[
\begin{align*}
    h_1^3 &= \frac{1}{1-p} h_0^3 - \frac{\rho}{1-p} h_1^2 \\
    &= h_0 + \frac{\rho^2}{(1-\rho)^3} \cdot \Delta h_0^1 - \frac{\rho}{(1-\rho)^2} \Delta h_0^2
\end{align*}
\]

58.
Generalizing

\[ h_{1}^{p} = h_{0} + \Delta h_{0}^{1} \frac{1}{1 - \rho} \left( \frac{\mathbb{I}}{1 - \rho} \right)^{p-1} + \Delta h_{0}^{2} \frac{1}{1 - \rho} \left( \frac{\mathbb{I}}{1 - \rho} \right)^{p-2} \] 2.29

Two observations pertinent to the previous analysis can be made. First of all, if we define \( f (p) \) as the headway deviation of vehicle \( p \) at the first stop due to the application of an initial dispatching perturbation to vehicle \( q \), it follows from equation 2.29 that,

\[ f_{2} (p) = f_{1} (p - 1) \] 2.30

or generalizing for any vehicle \( q \),

\[ f_{q} (p) = f_{1} (p - q + 1) \] 2.31

Also, the deviation caused by the combined action of perturbations \( \Delta h_{0}^{1} \) and \( \Delta h_{0}^{2} \) is simply the sum of deviations resulting from the application of both perturbations singly. Therefore, if we define \( f_{1,2} (p) \) as the deviation of equilibrium headways at the first stop caused by \( \Delta h_{0}^{1} \) and \( \Delta h_{0}^{2} \), it follows from 2.29 and 2.30

\[ f_{1,2} (p) = f_{1} (p) + f_{2} (p - 1) \]

It is a simple task to generalize the above results for any set of dispatching perturbations. It then follows that the combined effects of an arbitrary set of dispatching deviations is given by

59.
where \( Q = \{ q \} \) is the set of vehicles whose dispatching headways are perturbed by an amount \( \Delta h^q_0 \) and \( f_q(p) \) is the headway deviation of vehicle \( p \).

The above results can be extended to any stop \( k \). Let us define \( g_q(p, k) \) as the headway deviation for vehicle \( p \) at stop \( k \) caused by the dispatching perturbation of vehicle \( q \), \( \Delta h^q_0 \). From previous sections when \( q = 1 \)

\[
g^q_1(p, k) = \begin{cases} 
\varepsilon_{k,p} \cdot \Delta h^1_0 & ; \ k, p \geq 1 \\
\Delta h^1_0 & ; \ k = 0 \text{ and } p = 1 \\
0 & ; \ \text{otherwise}
\end{cases}
\]

Using previous arguments, we have that

\[
g^q_k(p, k) = g^q_1(p-q+1, k)
\]

It can be shown, by induction on equation 2.8, that the headway deviation of vehicle \( p \) at stop \( k \) caused by a set of dispatching perturbations is given by

\[
g^q_Q(p, k) = \sum_{q \in Q} g^q_q(p, k) = \sum_{q \in Q} g^q_1(p-q+1, k)
\]

\( g^q_q(p, K) \) is the headway deviation of vehicle \( p \) at stop \( k \), and \( Q = \{ q \} \) the set of vehicles whose dispatching headways are perturbed by an amount \( \Delta h^q_0 \).
As for the case of single perturbations, vehicle clumping limits the validity of the relation expressed in equation 2.36. Here, however, clumping patterns are not as easily to predict.

When a single perturbation is introduced into the dispatching schedule of a vehicle, the earliest clumping along the route always involves that particular vehicle. If the initial dispatching headway deviation is negative, the vehicle dispatched would arrive increasingly early at every stop and eventually clump with the last previous vehicle dispatched. If the initial perturbation was positive, the vehicle would have been increasingly late and eventually clump with the next vehicle being dispatched. Given that the clumping of the first vehicle occurs at stop k, successive vehicle clumping will occur exclusively at later stops.

The application of the graphical method described previously to the case of multiple perturbation is not limited by the factors mentioned before. The methodology followed in case of single perturbations is maintained now with the only exception of the calculation of the initial dispatching time which will now consider more than one perturbation. That is, departure times from the dispatching stations are now calculated by

\[ d_o^p = d_o^{p-1} + h_o + \Delta h_o \]  \hspace{1cm} 2.37

Figures 2.16 and 2.17 illustrate the uses at which the graphical method can be applied. Assuming vehicle one was dispatched 5 minutes late, Figure 2.14 shows what would happen if the second vehicle is dispatched early as to maintain the original schedule of departures. It might have been better to let the second vehicle leave after a regular interval in order to pre-
FIGURE 2.16
DRD-TRAJECTORIES

\[ p = 1 \]

\[ \frac{\Delta p}{p_0} = 0.5 \] + for p=1

- for p=2

VEHICLE

VEHICLE

VEHICLE

VEHICLE

VEHICLE

VEHICLE

VEHICLE

VEHICLE

END OF ROUTE
FIGURE 2.17
DRD: COMPARISON OF TWO ALTERNATIVES

DISTANCE (KMS)

DISPATCHING DECISIONS FOR SECOND VEHICLES

1. RETURN TO NORMAL SCHEDULE
   $\Delta t^2 = -5 \text{ min}$

2. KEEP "LATE" SCHEDULE
   $\Delta t^2 = 0$

SECOND VEHICLE POSSIBLE TRAJECTORIES

FIRST VEHICLE TRAJECTORY

STOP OF CLUMPING FOR POLICY 1

STOP OF CLUMPING FOR POLICY 2

25. TIME (min)

10. 

20. 

25. 

30. 

35. 

40. 

45. 

50. 

55.
vent an early clumping. Figure 2.15 compares the effects of both measures.

2.5.2: Equivalence Among Perturbations

The previous analysis suggests a similar approach to the treatment of perturbations in other parts of the system other than dispatching. It would be very convenient to be able to express the effect of a perturbation in travel time, for example, by some simple modification of the propagation function for dispatching perturbations developed earlier in the section. In fact, it is claimed that practically any kind of perturbation can be examined in this fashion. Some examples now follow: Let us assume that vehicle \( p \) has been delayed by an amount \( \Delta t \) when travelling from stop \( k \) to \( k + 1 \). Since the system has been in equilibrium before this event takes place, previous vehicles will not be affected by the perturbation. Also, headways for subsequent vehicles will remain regular for stops previous to \( k + 1 \), since departure times at earlier stops are not affected by events that take place further along the route.

As illustrated in Figure 2.18, the delay \( \Delta t \) in the travel time from \( k \) to \( k + 1 \) of vehicle \( p \) has the same effect on the headways at stop \( k + 1 \) as the combination of two forced departure time deviations from stop \( k \) of value \( +\Delta t \) and \( -\Delta t \) for vehicles \( p \) and \( p + 1 \).

Headway for stops after stop \( k \) can be calculated by assuming stop \( k \) to be the dispatching station in a new shorter route beginning at \( k \), and using the relations developed earlier in the section. That is, if we define \( T^q_{i k} (p, k|\Delta t) \) as the headway deviation of vehicle \( p \) at stop \( k \) resulting from a travel time delay \( \Delta t \) of vehicle \( q \) between stops \( i \) and \( i + 1 \), it follows that,
**Figure 2.18**

Equivalence of Travel Time Delays and Dispatching Delays

Travel Time Delay for Vehicle $p$

Dispatching Delay Equivalence
\[ T(q(k, k | \Delta t)) = \begin{cases} g(q(k, k | \Delta t)) + g(q(k, k | \Delta t)) & k > l \ 
\text{otherwise} \end{cases} \]

where \( g_1 \) has been defined previously and with \( l \) dispatching headway deviation given by

\[ \Delta q^q_1 = + \Delta t \]
\[ \Delta q^q_1 = - \Delta t \]

Similar reasoning can be applied to the situation when vehicle \( p \), ready to depart from stop \( k \) is delayed by an amount \( \Delta t \). As shown in Figure 2.19, in every aspect of the operation of the route, other than queue behavior at the stop, the departure delay \( \Delta t \) is equivalent to a corresponding delay in vehicle arrival time of amount \((1 - \rho)\Delta t\). The analysis is repeated as previously. If we define \( D(q, p, k | \Delta t) \) as the headway deviation at stop \( k \) for vehicle \( p \), caused by a delay \( \Delta t \) in departure of vehicle \( q \) from stop \( i \), it then follows that

\[ D(q, p, k | \Delta t) = T(q(k, k | \Delta t)) \]

where \( T_1^q \) has previously been defined. Other types of perturbations, such as perturbations in the arrival process of passengers at stops, or boarding of passengers can be treated in similar fashion.
FIGURE 2.19

EQUIVALENCE OF
DEPARTURE AND TRAVEL TIME DELAYS

\[ \Delta a = \text{travel time delay from } k-1 \text{ to } k. \]
from figure \[ \Delta n = d \Delta a = \lambda (\Delta t - \Delta a) \]
\[ \Delta a = (1-p)\Delta t \]
A final comment is in order. It seems clear, that the material presented in this chapter has stirred more questions than it has answered. Time and funds have placed a limit on the extent of this investigation. Promising lines of research will be suggested in the final chapter.
CHAPTER III

TRAFFIC LIGHTS DELAYS IN AN URBAN ENVIRONMENT
Section 3.1: The Traffic Lights Problem

This chapter starts to devise a set of models that would relate the delay, due to traffic lights operation, experienced by a vehicle moving in an urban environment to the geographical distribution and cycle parameters setting of said traffic lights.

The set of models to be proposed here is certainly not exhaustive. As laid out in the introduction, the purpose of this work is to obtain simple and basic models that can be readily extended in future work. The purpose of these models is multiple: as a first application they could be used to isolate travel time variance due to traffic light effects, giving a better idea on the extent of influence of traffic congestion and interaction of public transportation schedules. Also, traffic lights synchronization measures could be analyzed with the models, including priority schemes for public transportation vehicles.

Although every model proposed will be thoroughly defined, I would like to comment on some parameters and assumptions common to all. The aspects to be covered can be grouped into three sets of assumptions: lights cycle settings, geographical distribution and vehicle movement.

A) Lights Cycle:

A typical traffic light functions with a cycle of three phases; green, yellow and red. The cycle length is the sum of the time length of each of the three phases. For the purpose of analysis we will consider a typical cycle to be made up of two phases; the effective green phase and the effected red phase.
The effective green time is defined as the time length out of every cycle during which cars are actually allowed to cross the intersection controlled by the traffic light. Effective green times are always shorter than the actual value of the green phase due to clearing and queueing delays caused by traffic congestion.

One other measure related to the cycle time is the offset of the cycle which consists in the waiting time for the start of a new cycle (generally indicated by the light turning green) relative to a specified time of the day. Offset setting for a traffic light sequence is sometimes critical in terms of flow saturation and road capacity in a heavy traffic situation.

Geographical Distributions

In this work, the geographical distribution of traffic lights will refer to overall distribution of traffic light controlled intersections encountered in the most heavily used routes in a specific public transportation network. A car trajectory can be modelled for the purpose of this job as an infinite path along which traffic lights are distributed with a given average density, \((\gamma)\).

The parameters of this distribution, of most interest in the analysis, will be the mean interspacing, \((1/\gamma)\) and the Coefficient of Variation \((Cv)\) defined as the ratio of the standard deviation to the mean of lights interspacing. For completely regular interspacing, the coefficient is zero and for completely random interspacing, \((\text{exponential distribution})\) the coefficient has the value of unity.

71.
Vehicle Movement

Throughout this work, vehicles will be assumed to move at a constant speed all along the route stopping only at traffic lights when required and doing it instantaneously. Constant speed is then resumed, also instantaneously, as soon as the light turns green and traffic allows it.
Section 3.2: Delays at an Isolated Traffic Light

There is an extensive literature dealing with the analysis of delays to motorists crossing an intersection, that is either uncontrolled or under the control of a traffic light. The objective that is being pursued is to determine control measures that will optimize a certain cost function, generally associated with the total delay to motorists and safety of both motorists and pedestrians using the intersection.

This section will focus in the particular case of intersections controlled by fixed cycle traffic lights. The problem can be briefly described as a system in which two (or more) sets of entities with different arrival patterns are requesting the use of a common facility (street space). Even if the service time at the facility can be considered to be same for both types of entities (time per vehicle to clear the intersection), there is a changeover time penalty for transferring use of the facility from one entity type to the other (driver's reaction time large, acceleration).

From this perspective, it becomes clear that the methods of analysis from queueing theory are ideally suited for the problem in hand. Several characteristics of the problem, however, makes its solution rather difficult to obtain.

First of all, several studies [9, 10, 11] have shown that the headway between arrivals of vehicles at the intersection are in general not exponentially distributed. Furthermore, the general form of the distribution changes with the traffic density and capacity characteristics of the street. Service time, on the other hand, can in general be considered constant.

Secondly, the first-in first-out discipline applies only in the case of one land roads. Other factors that contribute to the complexity of the
issue, include the treatment of right and left turns and the fact that service for a given approach to the traffic light is restricted to the intervals corresponding the green phases for that approach.

Adams [12], has treated the problem as an M/G/1 queue and shows that vehicle arrivals can be considered to be a Poisson Process when the traffic is light. More complex models [13, 14] have been used for different conditions of the road.

This section will examine two very simple models of the delay to a random vehicle crossing the intersection. In this approach, methods of queueing theory will not be used since both arrivals and departures will be considered to occur at regular intervals.

3.2.1: Delay at Very Low Traffic Density

The first model deals with the arrival of a vehicle to a traffic free intersection controlled by a fixed cycle traffic light. The cycle will be considered to be of length $c$ and to consist of two phases; effective red and effective green, of respective length $r$ and $g$. The yellow period can be considered part of either phase according to the characteristics of the intersections.

Assuming that the cycle starts with the red phase and letting $t$ be the time of arrival of the vehicle after the beginning of the cycle, the delay to the vehicle ($\delta$) can be expressed as a function of $t$ by:

$$\delta = \begin{cases} r - t & \text{if } 0 < t < r \\ 0 & \text{otherwise} \end{cases}$$
Since the arrival of the vehicle is a random incident, \( t \) has an uniform probability distribution in the interval \((0, c)\).

The expected amount of delay can be obtained by integration over the cycle time interval:

\[
E[\delta(t)] = \int_0^c \delta(t) \cdot p(t) \, dt = \int_0^r (r-t) \frac{1}{c} \, dt.
\]

yielding

\[
E[\delta] = \frac{r^2}{2c}
\]

The second moment of the delay can be obtained in a similar way, yielding:

\[
E[\delta^2] = \frac{r^3}{3c}
\]

The variance of the delay is found to be, from the previous results:

\[
\sqrt{\delta} = \frac{r^3}{c} \left( \frac{1}{3} - \frac{1}{4} \frac{r}{c} \right)
\]

3.2.2: Delays at Moderate Traffic Densities

The second model of delay at an isolated traffic light will include the effect of other vehicles travelling along the same street. When the vehicle approaches the traffic light, it may encounter a line of other vehicles waiting to cross the intersection at the end of a red phase. The arriving vehicle is forced to wait at the beginning of the green phase for all the other vehicles in front of it to cross the intersection before it can do so. Even if the arrival occurs during a green phase, the vehicle
May find a few other vehicles still awaiting to cross the intersection.

May [15] developed a continuous stream model which is illustrated in Figure 3.1. Vehicles arrive in a single lane street at a fixed cycle traffic light at a rate equal to the traffic flow \( q \) of the road. If the light is green and there are no other vehicles at the intersection the arriving vehicles cross the intersection without any delay. As soon as the light enters the red phase vehicles queue up at the intersection. When the light turns green the line of vehicles begin to clear the intersection at a rate equal to the saturation flow of the road (time lag in drivers' reaction of the first few vehicles can be taken care of by including the yellow period in the green phase or by adding an extra amount to the red phase length). While the vehicles that had arrived during the red phase are clearing the intersection, new arrivals are still forced to wait until eventually there are no more vehicles waiting (interval \( r' \)), and arriving vehicles will suffer no delay (interval \( g' \)).

The values of the incoming traffic flow and the saturation flow of the street determine the length of the clearing interval \( r' \) and therefore limit the range of settings of the traffic light. Using geometric relationships we obtain from Figure 3.1

\[
S \cdot (r + r') = q \cdot r
\]

or

\[
r' = \frac{q}{1 - p} \cdot r
\]

or

\[
r' = \frac{\rho}{1 - \rho} \cdot r
\]

76.
FIGURE 3.1

QUEUEING AT A TRAFFIC LIGHT

departures = $5t$

arrivals = $9t$

time($t$)

queue size

$r'$

$r$

$g'$

$g$
where $\rho = \frac{q}{s}$ is the capacity utilization ratio of the road. In order to prevent an infinite queue from forming at the traffic light, the green phase length must be long enough as to allow complete clearance of the intersection. That is,

$$g \geq r' = \frac{\rho}{1-\rho} r$$

or

$$\frac{g}{c} \geq \rho$$  \hspace{1cm} (3.5)

Given this condition, the mean and variance of the delay to a random vehicle crossing the intersection can be obtained by the same methods used in the previous section.

Assuming a random arrival and letting $t$ be the time elapsed since the latest start of a red phase when the arrival occurs, then $t$ can be considered uniformly distributed in the interval $(0, c)$.

If the arrival occurs during the interval $g'$, the vehicle will not be delayed at all. If the arrival occurs during the red phase (interval $r$), the vehicle will have to wait until the end of the phase and then until all the vehicles in front of it ($qt$) clear the intersection. If the vehicle arrives during the clearing period ($r'$), however, it has to wait only until the vehicles still remaining in front of it ($qt (t - r)s$) clear the intersection in order to proceed.

From the above discussion and using algebraic manipulation, the delay at the intersection can be expressed as a function of $t$ by

$$j(t) = \begin{cases} 
  r - t (1 - \rho) & \text{if } 0 \leq t < r + r' = \frac{r}{1-\rho} \\
  0 & \text{if } \frac{r}{1-\rho} \leq t < c 
\end{cases}$$
The expected delay is again derived by integration over the cycle period and yields

\[ E[\delta] = \frac{r^2}{2c (1-c)} \]  
3.6

Similarly, for the second moment,

\[ E[\delta^2] = \frac{r^3}{3c (1-c)^2} \]  
3.7

The variance can then be computed as

\[ \text{V}[\delta] = \frac{r^3}{c(1-c)} \left[ \frac{1}{3} - \frac{1}{4} \frac{r}{c} \right] \]  
3.8

Equations 3.6 to 3.8 reduce to the results for the previous model when the capacity utilization ratio goes to zero, as one would reasonably expect.
Section 3.3: Total Delay at Traffic Lights for a Journey through an Urban Environment

The analysis carried out in the previous section can be extended to the problem of finding an expression for the amount of traffic lights delay experienced by a motorist that is making a trip through an urban environment. Depending on the focus of interest, solutions may be found for particular situations as a bus of a public transportation system, a dial-a-ride minibus, or simply a private vehicle.

Let us assume that the trip covers a distance \(d\), and that the vehicle travels at a constant speed when not stopped at traffic lights. The total delay for the trip may be calculated as the sum of \(n\) delays, each one at a particular light, where \(n\) is the number of lights encountered along the route of length \(d\) chosen by the driver. Considering the results of the previous section the task is then reduced to finding expressions for \(n\), based on the distance travelled and the characteristics of the environment.

Traffic lights are not regularly distributed throughout the area of a particular city. They are almost always located where they are needed the most: arteries with a high traffic density. It is not realistic, therefore, to speak of a "general" distribution of lights for the whole city. Nevertheless, it is always possible to subdivide a city in several areas, each of them characterized by a particular pattern of lights distribution. A particular trip may then be divided for purpose of analysis into several "legs", each one traversing a different area.

In most applications it may not even be needed to break down the journey; the trip itself may be restricted to a given area. Consider the case of a particular route in a public bus system. Bus routes are generally de-
signed to traverse areas of high demand potential and to allow minimum assessibility for the potential user. Because of these reasons, bus routes are generally constrained to the main city arteries, which usually satisfy both conditions.

This section will analyze several examples of possible patterns of traffic light interspacing. The examples will be confined to cases for which the coefficient of variation is less than or equal to one. A value of the coefficient of variation greater than one implies that traffic lights are grouped in "bunches" in the area. The area could then be subdivided into smaller areas, each with a reduced coefficient of variation of lights interspacing.

3.3.1: Traffic Lights at Constant Intervals

Let us assume the simplest of situation: traffic lights spaced at constant intervals in a distance \( d \). The number of traffic lights in \( d \) is simply

\[
\eta_d = \gamma d
\]

where \( 1/\gamma \) is the spacing between two consecutive traffic lights. The total delay at traffic lights can be expressed as:

\[
\Delta_d = \sum_{i=1}^{\eta_d} \delta_i
\]

where \( \Delta_d \) is the total delay and \( \delta_i \) is the delay at the \( i \)-th light encountered by the vehicle.

The expected value of \( \Delta_d \) is then the sum of the expected delay at each traffic light. If, for simplicity we assume that the lights have the same operating settings, we obtain
The variance of total delay, however, can be expressed in similar simpler terms only if we add the assumption that the individual delays are mutually independent random variables. A simple condition (although unrealistic) that guarantees the previous assumption is that traffic lights have relative offsets that are uniformly distributed in an interval of length equal to the length of the traffic light cycle.

Assuming then mutual independent delays, the variance of total delay is the sum of the variances for the delays at each traffic light in d. Again, assuming the same operating characteristics for all lights, we obtain

\[ \text{3.11} \]

\[ \text{3.12} \]

3.3.2: Traffic Lights in a Random Pattern: The Exponential Distribution

When we have regularly spaced traffic lights, we can exactly predict the distance remaining to reach the next light if we know how far behind is the light previously passed. Let us consider the opposite extreme; that is, the case where any information about the distance travelled so far since the last traffic light is as good as no information at all in predicting the distance to the next light. This would be exactly the situation if we assume the distances between consecutive traffic lights to be indentically exponentially distributed random variables. The probabil-
ity density function for each interspacing would be given by

\[ f(d) = \gamma e^{-\gamma d} \]  

3.13

where \(1/\gamma\) is the mean interspacing of consecutive lights.

The number of traffic lights encountered in \(d\) is then a Poisson process with parameters \((\gamma d)\). That is, the probability of finding \(n\) lights in \(d\) is given by

\[ P_{\eta_d}(n) = \frac{(\gamma d)^n e^{-\gamma d}}{n!}; \quad n = 0, 1, \ldots \]  

3.14

The mean and variance of a Poisson process are equal, and in our case

\[ E[\eta_d] = V[\eta_d] = \frac{\gamma d}{\gamma} \]  

3.15

If we maintain the same assumptions of independence of delays and equal operational parameters of traffic lights as in the previous case, the total delay in travelling a distance \(d\) is the sum of \(n\) independent identically distributed random variables where \(n\) is also a random variable. Drake [16] shows that when this is the case the mean and variance of the total delay can be expressed by

\[ E[\Delta_d] = E[\eta_d] \cdot E[\delta] \]  

3.16

\[ V[\Delta_d] = E[\eta_d] \cdot V[\delta] + V[\eta_d] \cdot E[\delta]^2 \]  

3.17
For our Poisson process, the expressions above become

\[
E[\Delta_d] = \gamma d \cdot E[\delta]
\]

\[
\sqrt{\Delta_d} = \gamma d \left\{ \sqrt{\delta} + \frac{E[\delta]}{\sqrt{E[\delta]^2}} \right\} = \gamma d \cdot E[d^2]
\]

3.18

3.19

3.3.3: A More General Pattern of Interspacing: The Gamma Distribution

The distribution of traffic lights examined so far, represent the two extremes of the range of interest of the coefficient of variation of light interspacing, mentioned before. Examining equation 3.17 in a different format,

\[
\sqrt{\Delta_d} = E[\eta_d] \left\{ \sqrt{\delta} + \frac{\sqrt{\eta_d}}{E[\eta_d]} \cdot E[\delta]^2 \right\}
\]

3.20

it can be noted that a value of zero for the coefficient of variation of interspacings (regularly spaced lights) corresponds to a null value of the variance to mean ratio of the corresponding distribution of the number of lights in a distance d. Also, a value of 1 for Cv (exponentially distributed interspacings) corresponds to a value of one for the variance to mean ratio of the distribution of the number of lights. It would be desirable to obtain a general expression for the case when this ratio is in between both extremes. This suggests using an interspacing distribution with a coefficient of variation adjustable to values inside the (0, 1) range to describe an intermediate situation to the ones previously described.
We can view the distribution of traffic lights along a given route as a renewal process for which the time flow is measured in units of distance and where renewals consist in consecutive locations of traffic lights along the distance axis. The distance between consecutive traffic lights are assumed to be independent identically distributed random variables and are equivalent to the interarrival times of the formal renewal process.

Let us assume that the distance between traffic lights are random variables with a Gamma distribution. The probability density function is given by

\[ g(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \quad x \geq 0, \quad \alpha \geq 1, \quad \beta > 0 \]  

3.21

The expectation and standard deviation of this distribution are given by

\[ E[x] = \frac{\alpha}{\beta} = \frac{1}{\delta} \]  

3.22

\[ \sigma_x = \frac{\sqrt{\alpha}}{\beta} = \frac{1}{\sqrt{\alpha}} \cdot \frac{1}{\delta} \]  

3.23

which gives a coefficient of variation of

\[ C_v(x) = \frac{1}{\sqrt{\alpha}} \]  

3.24

Note that if we keep the mean constant, the extremes of the range of values \( \alpha \) correspond to the limits of the range of interest of \( C_v \). In fact, when \( \alpha = 1 \) the gamma density reduces to the exponential, and when \( \alpha \) tends to infinity the gamma density tends to reduce to an impulse of area one at \( X = 1/\delta \), which represents the case of regular interspacing. Figure 3.2 illustrates these relationships.
FIGURE 3.2
GAMMA DISTRIBUTION
WITH MEAN $\frac{\alpha}{\beta} = 1.$
Identifying $n_d$ as the counting variable in the renewal process, that is, the number of renewals in an interval of length $d$, Karlin [17] shows that the limiting values of the mean and variance of $n_d$ can be related to the moments of the interarrival distribution in the following manner:

$$\lim_{d \to \infty} \frac{E[n_d]}{d} = \frac{1}{E[x]} = \gamma$$

$$\lim_{d \to \infty} \frac{V[n_d]}{d} = \frac{V[x]}{E[x]^3} = \frac{\delta}{\alpha}$$

The convenience of adopting the Gamma distribution as the general probability distribution for the interspacing of lights becomes evident when we substitute the values obtained by equations 3.24 and 3.25 in the formulas for the mean and variance of total delay,

$$E[\Delta_d] = \gamma d \cdot E[x]$$

$$V[\Delta_d] = \gamma d \left( V[x] + \frac{1}{\alpha} E[x^2] \right) = \gamma d \left( V[x] + \left( \frac{E[x]}{c_V(x)} \right)^2 \right)$$

We can then approximate any real distribution of traffic lights in a given area by a Gamma PDF and identify the area simply by the mean interspacing and use the coefficient of variation as an indication of the
"irregularity" of traffic light interspacing. Equations 3.26 and 3.27 can then be used to obtain estimates of the delay for a random trip through the area.
Section 3.4: The Probability Distribution of Total Delay: Normal Approximation

There will be occasions, in the analysis of problems related to the delay of traffic lights, when the value of the first moments will fail to provide the answer. As an example, we may be interested in finding the probability that a given vehicle be able to cover a certain distance in less than a certain time. In order to answer that type of question we are in need of an expression for the probability distribution of the delay.

We defined the total delay $\Delta$ as the sum of $n$ independent identically distributed random variables $S, i$ where $n$ is also a random variable. Drake [16], shows that the exponential transform of the total delay is equal to the geometrical transform for the number of lights encountered, evaluated at $z$ (the geometrical transform variable) equal to the exponential transform for the delay at a single light.

In mathematical notation:

$$
\mathcal{P}^{\Delta e}(s) = \mathcal{P}^{g}(z) \bigg| z = \mathcal{P}^{e}(s)
$$

Consider the model of delay for a single traffic light developed in 3.3.2. The exponential transform for the delay is given by

$$
\mathcal{P}^{e}_{\delta}(s) = \left(1 - \frac{r}{1 - \rho}\right) + \frac{1}{c} \left(1 - e^{-\frac{r}{1 - \rho}s}\right)
$$

When the traffic lights are spaced according to an exponential distribution, the number of traffic lights in $d$ is a Poisson process with a geometrical transform.
Inserting the last two results into equation 3.28, we get

\[ \tilde{g}_d(z) = e^{\gamma_d(z-1)} \]  

3.30

Expanding the inside exponential in terms of \( s \) and rearranging terms we obtain for the transform of total delay:

\[ \tilde{f}_d^e(s) = \exp \left\{ \gamma_d \left[ 1 - \frac{\gamma c}{1-p} + \frac{1}{c s} (1 - e^{-\frac{\gamma c}{1-p} s}) - 1 \right] \right\} \]  

3.31

where \( O(s^3) \) represents the sum of all the terms of powers of \( s \) greater than 2. Since for any function \( f \) and its transform \( f^e \) the following relation holds:

\[ \lim_{t \to \infty} f(t) = \lim_{s \to 0} f^e(s) \]  

3.32

we can assume that for large values of total delay, its transform can be approximated by

\[ \tilde{f}_d^e(s) = \exp \left\{ -\frac{\gamma d \sigma^2}{2c(1-p)^2} s + \frac{\gamma d \sigma^3}{3c(1-p)^3} \frac{s^2}{2} \right\} \]  

3.33

where we have neglected the terms in \( O(s^3) \).

The expression in 3.34 is the exponential transform of a normal pdf with mean and variance given by
which are exactly the moments of total delay obtained in the previous section.

The above argument leads to the suggestion that for large values of \( d \) we can approximate the total delay distribution by a normal pdf. The analysis for interspacing distributions other than the exponential is more involved but it seems reasonable to generalize the approximation to the case of a Gamma interspacing density.

In order to test the validity of the normal approximation of total delay, simulation runs were carried out for the cases of regular distributions (Cv = 0), exponential distribution (Cv = 1), and gamma distributions with \( a = 2 \) (Cv = .5).

Sample cumulative distributions were obtained in each case for total travel time. The results are illustrated in Figures 3.3, 3.4, and 3.5 against the assumed normal cumulative distribution. For every case, the average interspacing was assumed to be 250 meters and the test distance 10,000 meters. Traffic light model used in the simulation was the one developed in this chapter. Effective green phase length was 60 seconds and cycle time was 120 seconds in all cases. The capacity utilization ratio was assumed to be zero for the case of the regularly spaced traffic lights and .25 for the other two cases. The vehicle was assumed to move at a constant speed of 10 meters/second when not stopped at lights.
FIGURE 3.3
CUMULATIVE SAMPLE DISTRIBUTION OF TRAVEL TIME
- REGULARLY SPACED LIGHTS -

\( \alpha = \infty \)

\( M = 1583 \)
\( \sigma = 144.3 \)

\( M = 1600 \)
\( \sigma = 122.6 \)
FIGURE 3.4
CUMULATIVE SAMPLE DISTRIBUTION OF TRAVEL TIME

EXPONENTIALLY DISTRIBUTED LIGHTS

\[ \alpha = 1. \]

\( \mu = 1600, \sigma = 154.9 \)NORMAL DISTRIBUTION

\( \mu = 1629.4, \sigma = 168.1 \)SAMPLE DISTRIBUTION

\( t_d \) (sec)
**FIGURE 3.5**

Cumulative Sample Distribution of Travel Time

$F(t)$

Gamma Distributed Lights

$\alpha = 2$

$\mu = 1595.4$

$\sigma = 148.3$

Sample Distribution

Normal Distribution

$\mu = 1600$

$\sigma = 139.6$

$t_d$ (sec)
A chi square test was carried out for each case to test the hypothesis of normal distribution. The sample mean and variance were in close agreement with the predicted values. The $X^2$ test, however, suggests a rejection of the hypothesis of normal distribution fit at the .05 level for the case of exponentially and gamma distributed interspacing. In the case of regular interspacing, the test is inconclusive. The results indicate that further tests are necessary to limit the validity of normal approximation.
| Intercalation Method         | \( \chi^2 \)   | \( d.f. = 7 \)  | \( F_{\chi^2} \) | \\
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1) Regular Intercalation, ( \alpha = \infty )</td>
<td>5.13</td>
<td>7</td>
<td>0.35 (unclear)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>2) Exponential Intercalation, ( \alpha = 1 )</td>
<td>19.09</td>
<td>11</td>
<td>0.98 (reject at 5% level)</td>
<td></td>
</tr>
<tr>
<td></td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3) Gamma Intercalation, ( \alpha = 2 )</td>
<td>22.15</td>
<td>11</td>
<td>0.95 (reject at 5% level)</td>
<td></td>
</tr>
</tbody>
</table>

**Note:** # in sample = 100

---

**96.**
CHAPTER IV

In the context of this chapter, I will examine the behavior of the single route public transportation system discussed in Chapter One when random deviations are introduced into the system.

As we saw in Chapter One a small deviation from schedule in the departure of a vehicle from the dispatching station expands exponentially for that particular vehicle along the route and its effects on subsequent buses at a particular stop will die away if the capacity utilization ratio is less than .5. If a random deviation of similar pattern is introduced in the dispatching headway of every bus, it is reasonable to expect that the effect of these deviations will increase along the route and produce deviations at a particular stop that would be random variables with the same probability distribution characteristic of the stop.

The first section of this chapter will examine the relation between probabilistic dispatching headways and the headways at stops along the route. It will also introduce an expression relating the variance of deviations at a particular stop to the variance of the dispatching disturbance.

Operation of traffic lights distributed along the route will produce additional variation for headways at stops located after the traffic lights. Delays to two consecutive buses coming across the same traffic light combine to produce a difference in travel times between stops for the two buses.

97.
Section two will prove the equivalence of this difference to a deviation of headway of the same magnitude of the departing times of the pair of vehicles from the first stop. Expressions will then be obtained for the propagation of the variability of this disturbance in terms of the variability of delays at the traffic lights operating along the route.

The expressions obtained from the analysis of these two sections were tested in a series of simulation runs. The results are reported in section three. The effect of clumping is noted and a method to account for this is proposed based on the discussion of the previous section.

The last section will consider the effect of clumped buses in the propagation of the variability of previous headways. Clumped buses are forced to move as single units and therefore the headway between two consecutive buses that are clumped is deterministically zero. This fact will dampen the rate of growth of the variability of headways as measured by taking into account all the headways, including those between clumped buses. This section will introduce a method for estimating the amount of clumping at a particular stop and the correction to be added to the previous models.
Section 4.1: Probabilistic Dispatching Headways

It is practically impossible for any public transportation system to maintain a perfectly regular dispatching schedule. Severe weather conditions and other unforeseen circumstances can sometimes affect the dispatching schedule to a great extent. Even if one could disregard such uncommon events, operations under normal circumstances still sustain a certain amount of variability. Trivial events (such as temporary unavailability of drivers or vehicles) could cause small variations in schedule. One could consider these small deviations (a minute or so) as being perfectly normal. A one minute deviation in a dispatching schedule at five minute intervals represent, however, a twenty percent deviation from regular headways. The importance of a one minute delay is then obvious, especially if one considers that short headways imply high demand and therefore, a high capacity utilization ratio.

Let us consider the dispatching problem stated previously in terms of the deterministic single route model described in chapter two. Relaxing the assumption of perfectly regular dispatching headways, let us assume that a small random disturbance is introduced in the intervals between consecutive vehicle departures. If we maintain all the other assumptions of the deterministic model one should expect that these disturbances would create headway deviations along the route increasing with the stop number, similarly to the results obtained in the deterministic case.

Furthermore, if the initial headway deviations are small and identically distributed random variables, one could reasonably expect that after an initial transient time, headway deviations at all stops could also be considered non-independent, identically distributed, random variables, with
distribution parameters related to the parameters of the dispatching distribution.

4.1.1: Propagation of a Single Random Disturbance

Consider the case of a single headway disturbance introduced in the dispatching operation. If the amount of the initial deviation is known, headway deviations at every stop can be exactly predicted by the relations derived in chapter two. Consider the particular case where vehicle one has been dispatched with a headway equal to $h_0 + \Delta h_0^1$. If the initial headway deviation, $\Delta h_0^1$, is small and the demand is reasonably low, clumping will not occur at an early stop; and for such a stop, vehicle headway deviations can be predicted by

$$\Delta h_k^p = \varepsilon_{k}^{p-1} \cdot \Delta h_0^1 ; \quad p \geq 1 , \quad k \leq k^*$$

where $\varepsilon_{k}^{p-1}$ is the disturbance expansion factor defined in chapter two.

If, however, the initial deviation is a random variable with a known distribution, headway deviations at stops are no longer deterministic even if the initial disturbance is the only probabilistic factor externally affecting the system.

To find the distribution of these deviations is nevertheless a simple task, as shown in Figure 4.1, since they are directly proportional to the initial deviation.

100.
FIGURE 4.1

TRANSMISSION OF INITIAL DISTRIBUTION
OF DEVIATIONS

AFTER S STOPS,
FOR NEXT VEHICLE

AFTER K STOPS,
FOR THE SAME VEHICLE
In fact, if $\Delta h_o^1$ is distributed with moments given by

\[ E[\Delta h_o^1] = \mu_o^1 \]
\[ \sqrt{E[\Delta h_o^1]} = (\sigma_o^1)^2 \]  

it follows from 4.1 that the moments of the headway deviation for vehicle p at stop k are given by

\[ E[\Delta h_k^b] = \mu_k^b = \varepsilon_k^{b-1} \cdot \mu_o^1 \]
\[ \sqrt{E[\Delta h_k^b]} = (\sigma_k^b)^2 = (\varepsilon_k^{b-1} \cdot \sigma_o^1)^2 \] 

where,

\[ \varepsilon_s^j = \frac{(s+j-1)!}{(s-1)! \cdot j!} \cdot \frac{1}{(1-c)^s \cdot (1-c)^j} \]

The above expression is valid only until the first vehicle clumps. As opposed to the deterministic situation, one cannot now exactly determine the stop at which clumping occurs. Instead, one could obtain the probability that the clumping will happen before, or at a certain stop k.

As an example, let us refer to Figure 4.2. The dispatching deviation of vehicle one is assumed to be uniformly distributed in the interval $(-a,a)$. The headway deviation of vehicle one at stop k is therefore uniformly distributed in the interval $(-\varepsilon_k^o a + \varepsilon_k^o a)$. As long as the range of the interval precludes values of deviation less than $-h_o$ (minus the regular headway), the probability that vehicle will clump with the previous vehicle, remains zero. Otherwise, the probability of clumping at or before stop k is given by
Figure 4.2

Probability of Clumping for First Vehicle

\[ P_0'(\Delta h_0') \]

\[ \frac{1}{2a} \]

\[ -a \]

\[ +a \]

\[ \Delta h_0' \]

Prob\{1 4 2 clump before \( k+1 \)\}

\[ f_k'(\Delta h_{nr}) \]

\[ \frac{1}{2a|e_k|} \]

\[ -|e_k| -h_0 \]

\[ +|e_k| +h_0 \]

\[ \Delta h_{nr} \]

Prob\{4 do clump before \( k+1 \)\}

\[ f_k^2(\Delta h_{nr}^2) \]

\[ \frac{1}{2a|e_k|} \]

\[ -|e_k| -h_0 \]

\[ +|e_k| +h_0 \]

\[ \Delta h_{nr}^2 \]
The calculation of the probability of vehicle one clumping with vehicle two follows the same lines. The event will occur if and when the headway deviation of vehicle two is less than minus the dispatching headway $-h_0$. Again, the probability of vehicles one and two clumping is zero as long as the range of the distribution for vehicle two's headway at stop $k$ does not include values smaller than $-h_0$. Otherwise,

$$
\text{Prob \{ vehicles 1 \& 2 clump before } k+1 \} = \frac{1}{2} \left[ 1 - \frac{h_0}{\epsilon_k^0 a} \right]^{4.5}
$$

The previously described events are mutually exclusive; that is, vehicle one will either clump with vehicle zero or with vehicle two. The contrary is impossible since headway deviations alternate in sign for consecutive vehicles. Therefore, the probability of vehicle one clumping with either vehicles zero or two at or before stop $k$ is given by

$$
P_k^0 = \begin{cases} 
0 & \text{if } (\epsilon_k^0 \leq \frac{h_0}{a}) \\
\frac{1}{2} \left[ 1 - \frac{h_0}{\epsilon_k^0 a} \right] & \text{if } (\epsilon_k^0 > \frac{h_0}{a}) \& (-\epsilon_k^0 \leq \frac{h_0}{a}) \\
1 - \frac{h_0}{2a} \left[ \frac{1}{\epsilon_k^0} - \frac{1}{\epsilon_k^1} \right] & \text{otherwise}
\end{cases}
$$

where

$$
P_k = \text{Prob \{ vehicles one clumps before } k + 1 \}$$
4.1.2: Combined Effects of Continuous Random Disturbances

Let us assume now that the headway of every vehicle dispatched after vehicle one is similarly disturbed. That is, the dispatching headway of vehicle $p$ is no longer $h_o$ but

$$h^p_o = h_o + \Delta h^p_o$$

As for the case of deterministic disturbances the time of dispatching of vehicle $p$ is simply the time of dispatching of vehicle $p-1$ plus the headway of vehicle $p$. That is

$$d^p_o = d^{p-1}_o + h^p_o$$

In section 2.5 it was shown that the effects of multiple dispatching perturbations is additive. Assuming continuous disturbances to headways of vehicles starting with vehicle one, the headway deviation of vehicle $p$ at stop $k$ is equal to

$$\Delta h^p_k = \sum_{j=1}^{b} \epsilon^{p-j} \cdot \Delta h^j_o, \quad p \geq 1$$

where $\epsilon^{p-j} \cdot \Delta h^j_o$ is the contribution of the dispatching disturbance of vehicle $j$ to the total headway deviation of vehicle $p$ at stop $k$.

The expectation of the headway deviation of vehicle $p$ at stop $k$ is therefore the sum of the expected values of the initial disturbances, weighted by the respective expansion factors,
4.9

\[ E[\Delta b_{k,p}^p] = \sum_{j=1}^{b_p} E[E^{p,j} \cdot \Delta b_{o}^j] = \]

\[ = \sum_{j=1}^{b_p} E^{p,j} \cdot E[\Delta b_{o}^j] \]

Since the dispatching deviations are identically distributed random variables with mean zero, the expectation of the headway deviation of every vehicle will also be zero,

\[ E[\Delta b_{k,p}^p] = E[\Delta b_{o}^p] \cdot \sum_{j=1}^{b_p} E^{p,j} = 0. \]  

4.10

Given the assumption of mutual independence among dispatching headways, the variance of the headway deviation of vehicle \( p \) at stop \( k \) is then simply the sum of the variance of the initial deviations weighted by the square of the corresponding expansion factor. Since the dispatching variables are identically distributed, then

\[ \sqrt{\Delta b_{k,p}^p} = \sum_{j=1}^{b_p} (E^{p,j})^2 \sqrt{\Delta b_{o}^j} = \]

\[ = C_o^2 \cdot \sum_{j=1}^{b_p} (E^{p,j})^2 \]

4.11

The above relation is only valid when the probability of clumping is very low.

Let us observe equation 4.11 in more detail. If the capacity utilization ratio of the system \( \rho \), is small, the expansion factor \( E^{i,k} \) diminishes rapidly for large \( i \). When \( p \) is large, the major contributors to the sum expressed in 4.11 are the expansion factors corresponding to vehicles de-
parting immediately before vehicle \( p \). In fact, the expansion factors corresponding to early dispatched vehicles contribute almost nothing to the sum. This argument suggest that after a transient period the process should reach a limiting state and the variance for consecutive vehicles should be approximately the same. We, then define \( \sigma_k^2 \) as the limiting value of the variance of headway deviations at stop \( k \) and it is given by

\[
\left( \sigma_R \right)^2 = \left( \sigma_o \right)^2 \sum_{j=0}^{\infty} \left( \xi_j^d \right)^2 = \left( \sigma_o \right)^2 y_h(p, k)
\]

where \( y_h(p, k) \) is defined as the limiting expansion factor of the variance of dispatching deviations at stop \( k \) for system capacity utilization ratio. Figure 4.3 illustrates the behavior of the expansion factor for several stops, over the range of the capacity utilization factor of the system. As expected, for values of \( p \) near .5, the series defining \( y_h \) shows very slow convergence and increases very rapidly. This fact is in close agreement with the behavior of the system under deterministic conditions. Also, for values of \( p \) near zero and the first stop, \( y_h \) can be approximated by

\[
\frac{1}{(1-2p)^k}
\]

In fact, for \( k = 8 \) and \( p = .1 \), the expression indicated above differs from the true value of \( y_h \) by less of 10%.
FIGURE 4.3
LIMITING VARIANCE EXPANSION FACTOR

\[ \chi_{p,k} \]

\( k = 8 \)
\( k = 4 \)
\( k = 2 \)
\( k = 1 \)

\( P \)
\( 0 \) to 0.5

\( 0 \) to 100.
Section 4.2: Probabilistic Travel Times Between Stops

In the deterministic model of chapter two, travel times between stops were assumed to be equal and constant for all vehicles in the system. However, this is not the case for any public transportation system. The assumption of constant travel times may be a reasonable approximation for rapid transit routes where the interaction with other vehicles is kept at a minimum. In most surfact routes (buses, trolleys, etc.,) however, traffic congestion and the operation of traffic lights cause a high degree of variability in the travel times of vehicles.

Traffic tends to follow a constant time dependent pattern in most areas (peak hours) and therefore, the overall change in effective vehicle speed is somewhat predictable. During light traffic hours, public transportation vehicles will most of the time travel undisturbed by other vehicles. Traffic lights, however, are at all times, a constant source of potential delays. Lights synchronization does not usually matter for public transportation vehicles because they have to halt at stops, in between traffic lights.

Let us then expand the introduction of randomness into the single route model as to include the case where travel times are random variables; and in particular, when the randomness is mainly due to the operation of traffic lights located along the route.

4.2.1: Equivalence of Travel Time Delays and Headway Deviations

The interstop travel time of a public transportation vehicle can be always expressed as the sum of two components. If the vehicle is allowed to cover the distance between stops at the maximum cruising speed allowed by
the characteristics of the environment, the travel time will be at its minimum (undelayed travel time). Interaction with other vehicles and traffic lights operations will effectively reduce the average speed and increase travel time. In mathematical terms,

\[ t_k^p = u_k + \delta_k^p \]

where \( t_k^p \) is the travel time of vehicle \( p \) between stops \( k \) and \( k+1 \), \( u_k \) is the undelayed or minimum travel time between the stops and \( \delta_k^p \) is the time delay forced on vehicle \( p \) in between the stops.

The relation between consecutive headways at consecutive stops (equation 2.4) can be modified to take into account the possible delays. The relation is now expressed as

\[ h_k^p = h_{k-1}^p + (S_k^b - S_{k-1}^b) + (\delta_{k-1}^p - \delta_{k-1}^{p'}) \]

where \( h_k^p \) and \( S_k^p \) are headway and stopping time at stop \( k \) and \( (\delta_{k-1}^p - \delta_{k-1}^{p-1}) \) represents the difference of delays in the travel time of conservative vehicles between stops \( k-1 \) and \( k \).

The basic relation between headways of consecutive vehicles transforms to

\[ h_k^p = \frac{h_{k-1}^p - h_k^p + (\delta_{k-1}^p - \delta_{k-1}^{p-1})}{1-p} \]
or in terms of headway deviations:

\[
\Delta h_k^p = \frac{1}{1 - \rho} \left[ \Delta h_{k-1}^p + \Delta l_{k-1}^p \right] - \frac{\rho}{1 - \rho} \Delta h_{k-1}^{p-1}
\]

4.16

where

\[
\Delta l_k^p = \delta_k^p - \delta_{k-1}^p
\]

4.17

The definition of equivalence between travel time delays and headway deviations is based upon the above relations. To an observer at stop \( k \), the term \( [\Delta h_{k-1}^p + \Delta l_{k-1}^p] \) represents the headway deviations coming from stop \( k - 1 \). To the observer at \( k \), a deviation caused by travel time delays is undistinguishable from a departure headway deviation of the same magnitude at stop \( k - 1 \). The effects of both deviations on the rest of the route are, therefore, identical.

It was shown in chapter two that the effects of multiple dispatching disturbances is additive. The equivalence argument mentioned before leads us to generalize the additivity property for the case of travel time disturbances. In fact, the analysis of the effects of a deterministic set of travel time delays between stops \( k - 1 \) and \( k \) is identical to the analysis of the effects of an equivalent set of deterministic dispatching disturbances for a shorter route with the dispatching station located at \( k - 1 \).

Each dispatching disturbance would be the corresponding difference in travel time delays expressed by 4.17.

The probabilistic case is somewhat different. The analysis of dispatching disturbances assumed that these were independent, identically distributed random variables. Consecutive differences of travel time delays are not
independent, and in some extreme cases may not even be identically distributed. As an example, consider the case when travel time delays are solely determined by the operation of traffic lights between the dispatching station and the first stop. Assume that the traffic light is operating with a constant cycle length and equal split in green and red phases. If the dispatching headway is an even multiple of the cycle time, consecutive vehicles will experiment exactly the same delay. Delay differences will therefore be identically distributed (all equal to zero), but certainly not independent. Furthermore, if the dispatching headway is an odd multiple of half the length of the cycle, consecutive vehicle delays will alternate between zero and a constant non-zero value. Delay differences in this case would not even be identically distributed.

It may be argued, however, that assuming a constant average traffic density, traffic delays could be considered identically distributed random variables and independence can be reasonably claimed. If the headways between consecutive vehicles are large enough, factors causing delays to a vehicle will not necessarily affect the next consecutive vehicle.

It may also be argued that during normal operations, headway deviations and traffic delays contribute to produce vehicle arrival times at traffic lights that could be considered almost random incidents. If we accept these arguments, and make the exception that the results henceforth derived will be restricted by such conditions, we can consider travel time delays to be independent, identically distributed random variables. Travel time difference will also be identically distributed random variables.

Non consecutive delay differences are previously uncorrelated; that is, the covariance between non consecutive delays is zero. In fact, for any
pair \( i, j \), \( j \) is greater than \( i \),

\[
\text{Cov} [\Delta e_k^i, \Delta e_k^j] = E[\Delta e_k^i \cdot \Delta e_k^j] - E[\Delta e_k^i] \cdot E[\Delta e_k^j]
\]

\[
= E[(\delta_k^{i'} - \delta_k^{i-1}) (\delta_k^{j'} - \delta_k^{j-1})] - E[\delta_k^{i'} - \delta_k^{i-1}] \cdot E[\delta_k^{j'} - \delta_k^{j-1}]
\]  \hspace{1cm} 4.18

For non-consecutive delay differences; that is, \( j \) is greater than \( i - 1 \), we have

\[
E[\Delta e_k^i \cdot \Delta e_k^j] = E[\delta_k^{i'}] E[\delta_k^{j'}] - E[\delta_k^{i'}] E[\delta_k^{j'-1}] +
\]

\[
+ E[\delta_k^{i'-1}] E[\delta_k^{j'-1}] - E[\delta_k^{i'-1}] E[\delta_k^{j'}]
\]  \hspace{1cm} 4.19

\[
= E[\delta_k^{i'} - \delta_k^{i-1}] \cdot E[\delta_k^{j'} - \delta_k^{j-1}]
\]

\[
= E[\Delta e_k^i] \cdot E[\Delta e_k^j]
\]

For consecutive differences, however, \((j = i + 1)\), and

\[
E[\Delta e_k^i \cdot \Delta e_{k+1}^{i+1}] = E[\delta_k^{i'}] E[\delta_k^{i'+1}] + E[\delta_k^{i'}] E[\delta_k^{i'-1}] -
\]

\[
- E[\delta_k^{i'-1}] E[\delta_k^{i'+1}] - E[\delta_k^{i'}]^2
\]  \hspace{1cm} 4.20

\[
= E[\Delta e_k^i] E[\Delta e_{k+1}^{i+1}] - V[\delta_k^{i'}]
\]

where \( V[\delta_k^{i'}] \) is the variance of the delay to vehicle \( i \).

4.2.2: Headway Deviations due to Single Source of Delays

The additivity property of deviation effects conveniently allows us to divide the analysis of delay disturbances along the route into a series
of simpler problems; each of the simpler problems concentrating in the delays occurring between a specific pair of stops.

Let us then assume that travel time delays along the route are restricted to the segment of the route located between the dispatching station and stop one. Furthermore, let us assume that there are no other sources of disturbances in the system. Let us also assume that the first vehicle delayed is vehicle one and that the total delay in the route segment for consecutive vehicles can be considered independent identically distributed random variables with expectation and variance given by

\[
E[\delta^i] = \delta_o \\
\sqrt{\text{Var}[\delta^i]} = (\sigma)^2
\]

Before the dispatching of vehicle one, the system was not subjected to any kind of disturbances and therefore, headways were regular. Starting with vehicle one, headways are no longer regular, and the headway deviation for particular vehicles and stops is given by

\[
\Delta h_k^p = \sum_{j=1}^{p} c_j \Delta e_k^j
\]

where \(\Delta h_k^p\) is the headway deviation of vehicle \(p\) at stop \(k\), and is the expansion factor corresponding to the \(j\)-th delay difference and previously defined in the chapter, and

\[
\Delta e_k^j = \begin{cases} 
\delta_o & \text{if } j=1 \\
\delta_o - \delta_o^{j-1} & \text{if } j > 1
\end{cases}
\]
The expectation of headway deviation is therefore
\[ E[\Delta h_k] = \sum_{j=1}^{b} \epsilon_k \cdot E[\Delta \epsilon^j] \]
where the expectation of the j-th delay difference is,
\[
E[\Delta \epsilon^j] = \begin{cases} 
E[\epsilon_j - \epsilon_{j-1}] = 0 & \text{for } j \neq 1 \\
E[\epsilon^1] = \delta_0 & \text{for } j = 1
\end{cases}
\]
The limiting value of the expected deviation at stop k is determined by the value of the capacity utilization ratio. Since,
\[
E[\Delta h_k] = \lim_{p \to \infty} \sum_{j=1}^{b} \epsilon_k \cdot E[\Delta \epsilon^j] = \\
= \delta_0 \lim_{p \to \infty} \epsilon_k \left( \begin{array}{c} \epsilon_{b+1} \\ \vdots \\ \epsilon_k \end{array} \right) = \\
= \begin{cases} 
0 & \text{for } p < 0.5 \\
\infty & \text{otherwise}
\end{cases}
\]
In order to calculate the variance of the headway deviation, it has to be noted that consecutive delay differences are not independent. Therefore,
\[
V[\Delta h_k] = \sum_{j=1}^{b} V[\Delta \epsilon^j \cdot \epsilon_k] + \\
+ 2 \sum_{j=1}^{b-1} \sum_{i=j+1}^{b} \text{COV}[\epsilon_k \cdot \Delta \epsilon^j, \epsilon_k \cdot \Delta \epsilon^i]
\]
The variance of a delay difference is simply the sum of the variance of the delays. Then,

\[
V[\varepsilon_k^{p,j} \cdot \Delta t_k^j] = (\varepsilon_k^{p,j})^2 \cdot V[\Delta t_k^j] = \begin{cases} 
(\varepsilon_k^{p,j})^2 \sigma^2, & j = 1 \\
2(\varepsilon_k^{p,j})^2 \sigma^2, & j > 1
\end{cases}
\]

4.28

The covariance terms can be expressed as,

\[
\text{COV}[\varepsilon_k^{p,j} \cdot \Delta t_k^j, \varepsilon_k^{p,i} \cdot \Delta t_k^i] = (\varepsilon_k^{p,j} \cdot \varepsilon_k^{p,i}) \left\{ E[\Delta t_k^j \cdot \Delta t_k^i] - E[\Delta t_k^j] E[\Delta t_k^i] \right\}
\]

4.29

From previous arguments in this section, the covariance term reduces to zero for non-consecutive delay differences, and for consecutive differences, we have

\[
\text{COV}[\varepsilon_k^{p,j} \cdot \Delta t_k^j, \varepsilon_k^{p,j-1} \cdot \Delta t_k^{j-1}] = - \left\{ \varepsilon_k^{p,j} \cdot \varepsilon_k^{p,j-1} \cdot V[\varepsilon_k^j] \right\}
\]

4.30

The variance of headway deviations is then given by,

\[
V[\Delta h_k^p] = (\bar{\varepsilon}_o)^2 \left\{ (\varepsilon_k^{p-1})^2 + 2 \sum_{j=2}^{b} (\varepsilon_k^{p,j})^2 - 2 \sum_{j=1}^{b} (\varepsilon_k^{p,j} \cdot \varepsilon_k^{p,j-1}) \right\}
\]

4.31

Redefining index variables and using algebraic transformations, the relation above reduces to

\[
V[\Delta h_k^p] = (\bar{\varepsilon}_o)^2 \left\{ (\varepsilon_k^{p-1})^2 + \sum_{j=1}^{b-1} (\varepsilon_k^{j} - \varepsilon_k^{j-1})^2 \right\}
\]

4.32
As in the analysis of dispatching disturbances, it is of obvious interest to obtain the limiting value of the variance of headway deviations at stop \( k \). Then,

\[
\begin{align*}
\sqrt{\sum_{j=1}^{\infty} (\epsilon_{kj}^j - \epsilon_{kj}^{j-1})^2} = \lim_{p \to \infty} \sqrt{\sum_{j=1}^{\infty} (\epsilon_{kj}^j - \epsilon_{kj}^{j-1})^2}
\end{align*}
\]

4.33

The sum will converge if

\[
\left| \frac{\epsilon_{kj}^j - \epsilon_{kj}^{j-1}}{\epsilon_{kj}^{j-1} - \epsilon_{kj}^{j-2}} \right| < 1
\]

It can be shown that, for \( j > 1 \)

\[
\left| \frac{\epsilon_{kj}^j - \epsilon_{kj}^{j-1}}{\epsilon_{kj}^{j-1} - \epsilon_{kj}^{j-2}} \right| < \frac{|\epsilon_{kj}^{j-1}|}{|\epsilon_{kj}^{j-2}|}
\]

The absolute value of the expansion factor diminishes for consecutive vehicles only if \( p \) is less than \( .5 \). Therefore, for the previous range of the capacity utilization factor, the limit of the variance exists and it is given by

\[
\begin{align*}
\sqrt{\sum_{j=1}^{\infty} (\epsilon_{kj}^j - \epsilon_{kj}^{j-1})^2} = (\text{some expression})
\end{align*}
\]

4.34

where

\[
\sum_{j=1}^{\infty} (\epsilon_{kj}^j - \epsilon_{kj}^{j-1})^2 = \text{expression}
\]

4.35

is defined as the expansion factor for headway deviations due to travel time delays between zero (dispatching station) and one.
4.2.3: Combined Effects of Several Sources of Delay

We can generalize the results of the analysis of the single source of delay for any pair of stops. Let us assume that the source of travel time delays is located between stops S and S + 1. Headways will remain regular for stops before k. Headway deviations for any stop thereafter can be obtained using expression 4.34 simply by shifting the stop number by S, i.e., stop k becomes k - S, etc.

The variance of headways at stop k is then given by

\[
\sqrt{\left[\Delta h_k\right]} = \begin{cases} 
\left(\sigma_S^2 \gamma_S(p, k-S)\right) & \text{for } k > S \\
0 & \text{otherwise}
\end{cases} 
\]

where \(\sigma_S^2\) is the variance of delays between stops S and S + 1.

Since the effects of several sources of delay are additive, the limiting value of the variance of headways at stop k is simply the sum of the individual contributions of sources of delays located before k. That is,

\[
\sqrt{\left[\Delta h_k\right]} = \sum_{S=0}^{k-1} \left(\sigma_S^2 \gamma_S(p, k-S)\right)
\]

The contribution of each source of variability is weighted by a factor proportional to the distance of the source. The further the source of delay the greater the contribution.

If delays for all sources have the same variance, expression 4.37 becomes

\[
\sqrt{\left[\Delta h_k\right]} = \sigma_S^2 \sum_{S=0}^{k-1} \gamma_S(p, k-S)
\]
We can now fully appreciate the strength of arguments on equivalence and additivity of disturbances. Equation 4.37 does not imply any specific causes of delays as long as they can be considered mutually independent. The term delay could therefore be indiscriminately applied to delays caused by pedestrians crossings, non signalized intersections and traffic lights operations.

Also, any other type of disturbance could be incorporated into the model simply by adding a term to equation 4.37, if independence between these disturbances and traffic delay could be reasonably assumed. In fact, the variance of headway deviations in a system where both travel time delays and dispatching disturbances are present is simply found by combining equations 4.12 and 4.37.

\[
\sqrt{\Delta h_k^p} = \sqrt{\Delta h_o} \cdot y_h(p,k) + \sum_{s=0}^{k-1} (\sigma_x^2)^2 y_{\delta}(p,k,s)
\]

4.39

where \(y_h\) and \(y_{\delta}\) are the respective expansion factors of dispatching and travel time variability.
4.3: Testing the Probabilistic Model

Any hypothetical working model of a system implies a certain degree of oversimplification of the system, and even if the model is an accurate representation, it will have a limited range of validity. It is generally deemed necessary to test the validity of model's prediction against field data, and the models developed earlier are not exceptions. Meaningful amounts of data gathering would have exceeded the time limitations of this work. It was instead decided to test the validity of the models against the statistics generated by a computer simulation of a public transportation route.

It should be emphasized that the series of test carried out do not represent an exhaustive set. The particular examples were chosen as to give a feeling of the "goodness" of fit of the models. Time and funds limited further examination.

4.3.1: Simulation of the Single Route System

The simulation of operations in the single route system follows the basic structural description of the models developed throughout this work and stated in Section 2.2. The fundamental aspects concerning these models could be considered as follows:

The first aspect concerns the dispatching policy. It is assumed that vehicles leave the dispatching station at intervals of constant average length. The actual value of each interval however, is considered to vary from the average value by a random fluctuation. Fluctuations of consecutive vehicles are assumed to have identical probability distribution and to be mutually independent.
The second aspect involves the movement of vehicles along the route. It is assumed that under ideal conditions travel times between stops would be constant and equal for all vehicles. Traffic congestion and traffic lights operation introduce a factor of randomness into travel times. It is therefore assumed that the total travel time between stops will be affected by random delays whose probability distribution depends on the pattern of traffic and traffic lights in the particular link connecting two consecutive stops. Delays between consecutive vehicles are deemed independent unless the vehicles are clumped, in which case they move as a single unit. Also, delays in two adjacent links are also considered to be independent.

Finally, the third aspect to be considered is the series of events occurring at stops along the route. It is assumed that passengers arrive at stops at the same constant rate typical for each particular stop. When a vehicle arrives at a stop, it will first go through an unloading period constant for all stops and regardless of the number of alighting passengers. After this, passengers waiting at the stop or arriving while the vehicle is there will board it at a constant rate, equal for all vehicles and stops. The boarding process will continue, regardless of the capacity of the vehicle, until no more passengers are left at the stop. The vehicle will then immediately initiate the move to the next stop. Vehicles arriving at a stop while another vehicle is present will go through their unloading phase undisturbed. If the vehicle that was originally at the stop is still boarding passengers, the second vehicle will then clump and follow the exact movements of the first one for the rest of the route.

It is also assumed that if a passenger arrives at a stop where more than one vehicle are present, he will only be allowed to board the first vehicle in the line, and no other.
The simulation program was designed to execute the operations described above in a sequence of discrete events. A diagram of this sequence is illustrated in Figure 4.4.

As a first step, values are assigned to the parameters used to describe the system. The set of defined parameters includes the parameters of the dispatching disturbance distribution, the passenger arrival and boarding rate and unloading time, the number of stops and interstops minimum travel times, the location and operation characteristic of traffic lights along the route, and finally the parameters of the probability distribution of traffic delays along the route.

The program assumes that the system is operating in perfect schedule until time zero. When the last vehicle to be on schedule is dispatched (Vehicle zero) arrival and departure times for all stops are calculated for this vehicle assuming traffic and traffic lights delays set at their mean values, and boarding times of an undelayed, regular dispatching system.

Each traffic light along the route is assumed to start operating with the passage of the vehicle zero. Traffic lights offsets relative to the starting time are calculated from random variates of a uniform distribution with an interval length equal to the cycle time of the light.

The history of each vehicle is then simulated as a sequence of events through which the vehicle passes once it starts to travel the route. The time between successive vehicles, or initial headways are determined by the dispatching policy. Departure time from the dispatching station are calculated as the sum of the departure time of the previous vehicle plus a random deviate from the probabilistic distribution defined by the dispatching parameters.
FIGURE 4.4.
SIMULATION OF THE SINGLE ROUTE SYSTEM

START

INPUT PARAMETERS

INITIALIZATION
- RUN CONTROL VEHICLE
- SET TRAFFIC LIGHTS

END OF SIMULATION

NEXT EVENT?

NEW VEHICLE DISPATCHED

VEHICLE ARRIVES AT STOP

LOAD/UNLOAD
SET DEPARTURE TIME

GENERATE ARRIVAL TIME TO NEXT STOP

GENERATE TRAVEL TIME TO STOP 1

END OF ROUTE?

YES

RETIRE VEHICLE

NO

END OF PRESIMULATION?

YES

START COLLECTING STATISTICS

NO

OUTPUT STATISTICS

FINISH
The time needed to travel from stop $k - 1$ to stop $k$ is also treated as a random variable. The value for each vehicle traversing the link between the stop is calculated as the sum of minimum constant time plus a total delay at traffic lights located between the stops. Delays explicitly caused by traffic congestion are not specifically considered in the program because it was argued that they did not represent a large source of variability compared with traffic lights (Section 4.1).

However, traffic effects are implicitly considered in the delays at traffic lights. It was assumed that the delay at a light followed the continuous flow model described in chapter three, where the average traffic density of the road placed an additional delay at the intersection. The delay for a particular vehicle was calculated in base of the arrival time at the light and the relative offset of the light in question. The arrival at the light was calculated from the departure time at the stop previous to the light, assuming constant speed and accounting for other delays at other lights located in between.

At the arrival of a vehicle to a stop, the departure time of the vehicle was calculated in relation to the departure time of the previous vehicle. Since the passenger arrival process is assumed deterministic, the departure time could be calculated by a simple equation. Successive travel and stopping times are calculated sequentially on this same fashion until the vehicle reaches the end of the route. At that point, the vehicle leaves the system.

If the calculated departure time of a vehicle is less than the departure time of the previous vehicle from the same stop, a clumping occurs. From that stop on, arrival and departure times for that vehicle are assumed to be equal to those of the vehicle preceeding it.
Arrival and departure times are recorded for each vehicle. After a specified number of vehicles have travelled the route, the simulation ends. Finally, statistics of the performance of the system are calculated and printed.

4.3.2: Results of Simulation Tests

Simulation runs were separated in two groups. The object of the first group was to test the predictions of the random dispatching model analyzed in Section 4.1. The group consisted of three runs, each for a different level of capacity utilization ratio of the system, \( \rho = 0.1, 0.2, \) and \( 0.4 \). In this set, travel times between stops were assumed to be equal and constant for all vehicles. The random dispatching headways were sampled from an uniform distribution in the intervals of 10 ± 1 minutes.

Two hundred and five vehicles were dispatched into a route of ten stops. The first five buses were considered to cover the transient period of the system. Tables of departure headways were produced for each stop from the history of the remaining 200 vehicles. Averages and standard deviations were calculated for every stop.

Two runs formed the second group; each at a different level of the capacity utilization ratio of the system \( \rho = 0.1, 0.2 \). In this run, 105 vehicles were sent into a route of 20 stops. The first five vehicles constitute a pre-simulation or transient period.

In this set of runs, travel times between stops were affected by delays at traffic lights located along the route. It was assumed that each street link connecting two consecutive stops, contained a traffic light at the mid point of the link. Operational parameters of the traffic lights were assumed to be equal for all lights. Offsets were randomly determined at the
The lights were assumed to be working on an even split cycle and the delays were to be calculated according to the continuous flow model of chapter three for a capacity ratio of .25. As in the first model, tables and statistics were calculated for each stop along the route.

According to predictions, the sample mean headway did not change appreciably at any stop for any case. Figure 4.5 and 4.6 compare the sample standard deviation obtained from the simulations against the predicted values for several stops. The results proved to be quite interesting and there are several details worth mentioning.

As it can be noted in both figures, the range of validity of predictions appears to be independent of the stop number or the level of capacity utilization. In fact, for all five runs, sample standard deviations begin to depart from predicted values at about the same threshold value (~ 5. or half of headway mean). For predicted values greater than this threshold value corresponding sample standard deviations follow a quite distinct pattern from what it was expected.

Figure 4.7 confirms the previous reasoning. In here, it can be noted that the error in predicting standard deviations is proportional only at the value of the prediction itself. The nature of the delay or the level of capacity utilization ratio of the system do not have any role in calculating the correction.

From tables of recording headways at each stop, it was possible to obtain a measure of the disruption of service in terms of the proportion of vehicles clumped together at each stop.
FIGURE 4.5

PROPAGATION OF DISPATCHING DEVIATIONS
PREDICTIONS & SIMULATIONS RESULTS
FIGURE 4.6

PROPAGATION OF HEADWAY DEVIATIONS
DUE TO DISPATCHING & TRAFFIC LIGHTS
DISTURBANCES

CODE:

△ OBSERVED

— PREDICTED

HEADWAY
STANDARD DEVIATION
$\Gamma_k$
(minutes)

STOP #
FIGURE 4.7

ERROR IN ESTIMATION OF STANDARD DEVIATION

CODE:
+ $p = .2$, no lights
* $p = .4$, no lights
$\Delta$ $p = .2$, traffic lights
As it can be seen in Figure 4.8 the amount clumping is highly correlated to the standard deviation of the headways at a particular stop. Also, the relation between standard deviation of headways and clumping seems to be independent of the level of the capacity utilization ratio and nature of delays along the route.

The next section will try to examine the nature of the relation between the observed variance and the amount of clumping.
FIGURE 4.8

RATIO OF NON CLUMPED VEHICLES VS. STANDARD DEVIATION

\[ 1 - p_k \]

\[ \text{CODE} \]

\[ 0 \rightarrow p = 0.1 \text{ (with traffic lights)} \]

\[ + \rightarrow p = 0.2 \]

\[ G_k \text{ (PRED.)} \]
Section 4.4: Clumping in the Probabilistic Model

The analysis carried out in the first two sections of this chapter assumed that deviations were not so large as to allow a situation when two vehicles would meet along the route before they reached destination. In fact, simulation results show that the equations developed for variances of headways are valid and good approximations only for those early stops where the probability of clumping is small. It is intuitively clear and the results confirm it, that the variance of early deviations should magnify into larger variances for stops further along the route, which in turn implies a larger probability that a pair of vehicles would clump.

When two vehicles clump, their headway is zero, and since the second vehicle is forced to follow the movements of the first one, a third vehicle behind will be affected only by the behavior of the first one. In terms of transmission of perturbations, it is in fact like the second vehicle "disappeared from the system".

The overall average headway should remain the same for every stop since we are assuming that the system reaches a stable condition. If this were not the case, headways would grow with time, and vehicles departing from the dispatching station would take an increasingly long time to reach destination.

The average headway between non clumped vehicles, on the other hand, must increase with the clumping probability in order to maintain the overall average headway. A probability $p_k$ of clumping means that, on the average, a fraction $p_k$ of vehicles will have zero headways at stop $k$, and then the average headway for non clumped units must be proportionally greater.
To express this notion in mathematical terms, let us define $h_0$ as the overall mean headway for the system (determined by the dispatching policy), and let $p_k$ be the probability that an observer at stop $k$ will measure a zero headway (a "clumping"). If we call $h^c_k$ the average headway between non clumped units, we have that

$$h_0 = E[h_k] = p_k \cdot 0 + (1-p_k) h^c_k$$

implying that

$$h^c_k = \frac{h_0}{1-p_k}$$  \hspace{1cm}  (4.40)

The overall variance at each stop is also affected by the amount of clumped vehicles. The vehicles that have clumped, no longer present variability in their headways, but the value of their zero headway is recorded in any estimation of the overall variance. To examine this effect more closely, let us calculate the relation between the overall or "observed" variance and the variance of the unclumped units.

The variance of the headways can be calculated as the difference of the mean square headway and the square of the overall mean headway. That is,

$$\sqrt{\mathbb{E}[h_k]} = \mathbb{E}[(h_k)^2] - \mathbb{E}[h_k]^2$$  \hspace{1cm}  (4.41)

The value of the mean of square headways can be calculated, using arguments similar to the calculation of the mean headway, as
\[ E[h_{k}^2] = p_k^2 \cdot \mathbb{E}[h_{k}^2] + (1-p_k) \cdot \mathbb{E}[h_{k}^2 | h_k > 0] = \]
\[ = (1-p_k) \cdot \mathbb{E}[h_{k}^2 | h_k > 0] \]

Equation 4.41 then becomes
\[ \mathbb{V}[h_{k}^2] = (1-p_k) \cdot \mathbb{E}[h_{k}^2 | h_k > 0] - \sigma_o^2 \]

4.43

If we let \( \tau_k^2 \) denote the variance of the unclumped unit headways, we obtain by algebraic manipulation, that the variance of headways at stop \( k \), is given by
\[ \mathbb{V}[h_{k}^2] = (1-p_k) \cdot \mathbb{E}[h_{k}^2 | h_k > 0] - \sigma_c^2 - \sigma_o^2 + \frac{\sigma_o^2}{(1-p_k)} = \]
\[ = (1-p_k) \cdot \tau_k^2 + \frac{p_k}{1-p_k} \cdot \sigma_o^2 \]

4.44

In order to apply the above correction to the predictions of the model, it is necessary, however, to devise a way to estimate the clumping probability \( p_k \).

We defined \( p_k \) at the beginning of the chapter as
\[ p_k = \text{Prob \{Vehicle clumps at or before stop } k \} \]

therefore
\[ \Delta p_k = p_k - p_{k-1} = \text{Prob \{Vehicle clumps at } k \} \]

We could then treat the process of clumping as a sequential problem. That is, assume that we know \( p_k \) and find \( p_{k+1} \) as
\[ p_{k+1} = p_k + \Delta p_k \]

where \( \Delta p_k \) is the probability that any vehicle will clump between stops \( K - 1 \) and \( k \). Since a vehicle can either clump with the one ahead or behind,

\[ \Delta p_k = 2 \cdot q_k \]

where \( q_k \) is the probability that a given headway greater than zero before stop \( k \), becomes zero thereafter.

How can we calculate \( q_k \)? Let us assume that the headway distribution at stop \( k \) has a general form similar to the one described in Figure 4.9 with mean \( \mu_k \) and variance \( \sigma_k^2 \). The distribution must be asymmetrical because negative headways are not allowed.

Consider two consecutive vehicles not clumped and their headway at stops \( k \) and \( k + 1 \). Since they are not clumped at \( k \), their headway must be distributed according to \( f_k(h_k) \) with mean \( \mu_k \) and variance \( \sigma_k^2 \). As the two vehicles travel to the next stop, the variability of the headway increases according to the expansion rule developed in previous sections. The expected headway, however, cannot change even if the two vehicles clump because of stability reasons explained before. Since the mean stays the same but the variance is larger, it is reasonable to assume that the function spreads out evenly. Granting this, the value of \( q_k \) is the area under the expanded function that covers negative values. The cumulative probability of clumping, is then updated, and the new distribution function is obtained from the old one by erasing the negative part and normalizing the rest of the curve.

We are however, at a stand still, since the distribution \( f_k \) is not known. At this point it is clear that further analysis is needed.
Let us comment on an additional factor that contributes to the discrepancies between predicted and sample variances. Let us assume that vehicle $p$ and $p - 1$ clump at stop $k$.

Since a clumping occurs when a headway becomes negative then,

$$\frac{h^p_k}{1 - p} < 0$$

therefore,

$$h^{p-1}_k > \frac{1}{p} h^p_{k-1}$$

For small values of $p$ this mean that

$$h^{p-1}_k >> h^p_{k-1}$$

Before vehicle $p$ clumps, the headway of vehicle $p + 1$, say at stop $S < k$ was governed by

$$h^{p+1}_s = \frac{h^{p+1}_{s-1} - p h^p_s}{(1 - p)}$$

Since $h^p_s$ is growing smaller with increasing $S$ (it will become zero at stop $k$), the headway of vehicle $p + 1$ must be large. Actually, the smaller $h^p_s$ becomes, the larger $h^{p+1}_s$ grows.

After vehicle $p$ is clumped, its headway no longer affect the headway of vehicle $p + 1$, and the above relation is no longer valid. The headway of bus $p - 1$ replaces the headway of bus $p$ and the above relation becomes

$$h^{p+1}_c = \frac{h^{p+1}_{c-1} - p h^p_c}{1 - p}$$

Since $h^p_{c}$ is large, the clumping of vehicle $p$ will not only set its headway at zero but also reduces the variance of headways at stops after $k$ since
headways that were growing large ($h_{k}^{b+1}$) will no longer do it.

The overall reduction in variance is therefore proportional to the number of vehicles that have clumped. From the results of the simulation it seems however that this effect does not completely eliminate the growing of the variance but only limits it. It is clear at this point that a more careful analysis is needed before any other conclusions are drawn.
CHAPTER V

CONCLUSIONS AND RECOMMENDATIONS

This thesis has examined various aspects of the operation of a single route of a Public Transportation System. The main objective in the investigation has been to construct an analytical framework suitable for the evaluation and optimization of system service.

This chapter summarizes the major findings and conclusions of this investigation and underlines the different aspects that deserve further analysis. It also comments on several possible applications of the research in the field of public transportation.
Section 5.1: Findings and Conclusions

The thesis began by disregarding factors of random nature affecting the operation of the system, and examining the causality relationships in its structure. A deterministic model which defined the nature of the dependency among consecutive headways was developed in chapter two. Some important findings were obtained based on the analysis of this model.

First of all, it was found that the factor that solely determines the response of the system to external perturbations is the ratio of passenger demand rate and boarding rate capacity. This response can be described in terms of service stability of vehicles and stops. Any isolated disturbance affecting a particular vehicle at a point in its trip, will inevitably deteriorate the service that this vehicle can provide for the remainder of the route. The amount of disruption is proportional to the demand on the system. Furthermore, even if the original disturbance does not involve any other vehicle, it will nevertheless affect all the following vehicles. The amount of service disruption at a particular stop will propagate for consecutive vehicles at a rate proportional to the demand on the system. The effect caused by the original disturbance will disappear eventually, only if the ratio of demand at a stop to boarding rate is less than one half.

Second of all, it was determined that the response of the system to multiple disturbances is the sum of the responses to each individual disturbance. That is, the effect of a particular disturbance, is independent of the response of the system to previous disturbances. This fundamental for the development of probabilistic models of the system.

Thirdly, it was found that there exists an equivalence relation among disturbances to different aspects of route operation (dispatching travel times, etc.). Two sets of disturbances of different nature can be termed
equivalent in terms of service disruption, if they induce the same response. To this effect, for any disturbance to a particular aspect of the system there is a unique equivalent set of disturbances to some other particular aspect.

The analysis of chapter three was motivated by several studies indicating that traffic lights operation was one of the major causes of the variability of system performance. The problem was formulated in terms of a distance renewal process, in which the point of renewal were delays at a particular light and the intervals between renewals were the distances between consecutive traffic lights in a path through an urban environment. Expectation and variance of total delay in travel times for equal distances were found to be dependent only on the operational settings of the lights and the density and the interspacing regularity of traffic lights in the area to which the trip was limited.

An attempt was made in modeling the problem as a Wiener Process, assuming a normal distribution for the total delay. The assumptions were not tested against field data since these data were not readily available. A computer simulation of the problem seemed to reject the hypothesis of normality but the result cannot be considered conclusive.

Chapter four was dedicated to the analysis of some probabilistic aspects of the single route operation. Irregularity in dispatching and travel time were modelled as a continuous stream of random disturbances to the system. Expressions relating the variability of headways at stops to the variability of the disturbances were found using the relations of the model proposed in chapter two. Field testing of the models was not possible and a computer simulation of several hypothetical cases were instead designed.
In all cases simulated, the results showed agreement with the predicted values and provided a measure of the range of validity of the models. Simulation results indicated that above a certain threshold of variability, the clumping of vehicles introduces a new dimension to the problem and causes deviations in the models proposed. Results also suggested that the variability threshold was related to the mean headway of vehicles dispatched into the route. The relation of clumping and mean dispatching headway to the variability of service was not examined in close detail but several suggestions on its nature were formulated.
Section 5.2: Proposed Lines of Further Research

It is proposed that this investigation could be extended by introducing the effects of some other aspects of the operation of a single route.

Some suggestions of possible aspects are indicated as follows.

1. Passenger arrivals at stops could be treated as random events. The models proposed in this work could be extended to include a Poisson formulation of the arrival process. Expressions predicting the variability of headways could be developed using the relations developed in chapter two.

2. The clumping process should be analyzed in closer detail. Possible lines of approach to the analysis were indicated in the discussion of clumping in chapter four.

3. The models proposed in this work assumed a very simplified description of a single route in a Public Transportation System. Efforts should be directed in extending the results of this investigation to more general descriptions. Passenger demand is not generally the same at every stop along the route, nor remains constant at all times. The models could be extended to include a non uniform distribution of demands at stops and the time dependence of the demand rate. Also, the assumption of infinite vehicle capacity and unloading times could be changed to provide a more realistic view of the operation of the system.
4. The models proposed here are of a descriptive nature.
   Extensions of the model should include the evaluation of real
time control measures. These measures could be formulated as
additional external disturbances triggered by the presence of
headways below a desired level of service. The same methods
used here could be applied to obtain procedures for control
decisions.

5. Finally, perform field experiments in order to corroborate
   the results obtained.
Section 5.3: Contributions and Applications of the Research

The Reference Distance Departure Time Method (RDDT) was developed in chapter two in order to provide a clear overview of the history of service performance. This graphic method allows an immediate identification of early and late vehicles at a given point in time and provides an indication of the rate of change of the schedule deviation of particular vehicles. An AVL system could be used in conjunction with RDDT to provide a control system with real time decisions capability. AVL systems can provide a central system control supervisor with an up to date location of vehicles. By using a CRT screen or some other suitable visual device, the departure time history of the system can be observed in the screen using the RDDT method. "Trouble spots" could be immediately identified and control decisions could be evaluated by extrapolating the system response to the control measure under the actual state of the system.

The delay model proposed in chapter three could perhaps be used in the evaluation of response time of DIAL-A-RIDE or other similar systems of public transportation.

Finally, the models proposed in this investigation could be used to evaluate alternatives in the implementation of new operating procedures, the incorporation of new routes and the creation of new transportation systems.
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