SOME ISSUES IN TRANSIT RELIABILITY

by

Daniel Olivier Bursaux

Ingenieur Diplome de l'Ecole Polytechnique
(1977)

SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE
DEGREE OF

MASTER OF SCIENCE

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

May 1979

Signature of Author ....
Department of Civil Engineering, May 21, 1979

Certified by ....
Thesis Supervisor

Accepted by ....
Chairman, Departmental Committee on Graduate Students of the Department of Civil Engineering

SOME ISSUES IN TRANSIT RELIABILITY

by

DANIEL OLIVIER BURSAUX

Submitted to the Department of Civil Engineering
on May 21, 1979 in partial fulfillment of the requirements
for the Degree of Master of Science

ABSTRACT

The general purpose of this thesis is to investigate strategies that can be used in order to improve service reliability in urban transit systems.

Initially, a very simple model is used to compute the probability of bunching of buses on a route. This result is then used to prove that adding a very large number of buses on an existing line does not necessarily significantly improve the level of service (Chapter II).

The second major part of the thesis deals with the problem of control on a bus route. A specific strategy is first selected based on selective holding of buses at a point on a route. A method using elementary calculus to find the best point on the line at which to control is then described (Chapter III).

Finally, a practical study is performed on the Harvard-Dudley bus route running between Cambridge and Boston. Using the model developed in Chapter II we prove that the headway standard deviation cannot be more than fifty percent above the mean headway, as is suggested by the data collected. Then we try to show how an operator can use the theoretical results developed previously to improve the service (Chapter IV).

THESIS SUPERVISOR: Nigel Wilson
TITLE: Professor of Civil Engineering

ACKNOWLEDGEMENTS

I wish to thank Professor Nigel Wilson for the great help he provided during the preparation of this thesis by the ideas he suggested and the solutions he proposed to solve occuring problems.

I wish also to thank the students of Course 1.26 who collected data on the Harvard-Dudley bus line no matter what the weather conditions were.

Finally, I want to express all my gratitude to Colleen Keough who typed this thesis with great skill, in spite of short delays and various other difficulties.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

CHAPTER I

INTRODUCTION

I.1 General Summary

The general purpose of this thesis is to investigate strategies that can be used in order to improve service reliability in urban transit systems.

Initially, a very simple model is used to compute the probability of bunching of buses on a route. This result is then used to prove that adding a very large number of buses on an existing line does not necessarily significantly improve the level of service.

The second major part of the thesis deals with the problem of control on a bus route. A specific strategy is first selected based on selective holding of buses at a point on a route. A method using elementary calculus to find the best point on the line at which to control is then described.

Finally, a practical study is performed on the Harvard-Dudley bus route running from Cambridge to Boston. In this example we try to show how an operator can use the theoretical results developed previously in order to improve the service.

I.2 Background and Motivation

Attitudinal surveys[8] performed in the Baltimore-Philadelphia area show reliability to be among the most important service attributes for all travelers. Reliability is considered more important than average travel time and cost; safety being often the only factor to be viewed as more important. In light of this it seems very important to investigate the reliability of service on a fixed bus route and to develop strategies to improve it.

The importance of this problem can be outlined on the Dudley-Harvard route. On March 5th, 1979, a survey was performed during three hours by MIT students at different stops. For a mean headway of about 6 minutes, the following standard deviations were found for Northbound buses:

Dudley:	3.47 mn
Auditorium:	4.93 mn
Central:	5.55 mn
Harvard:	5.97 mn

This proves that the standard deviation is not small compared to the headways and that it tends to increase along the route.

The basic inherent factor in causing bus unreliability is the instability of the headway distribution. This instability tends to become more pronounced further downstream.

Exogenous factors such as delays due to traffic or loading conditions trigger an initial deviation from scheduled headways. The inherent instability lies in the fact that any delay in arrivals results in an increased dwell time at that stop due to the increased passenger load. Thus, late buses get later and early buses get earlier, eventually resulting in bunching, imbalanced loading and generally poor reliability.

As will be shown in this thesis, adding a very large number of buses on an existing line, without implementing any control, does not always increase significantly the reliability, because of a bunching phenomenon. In order to improve reliability one must thus find some other methods which are easy to implement and not too expensive.

To improve reliability, many strategies have been suggested, including:

- Turning back buses to split bunches.
- Disaggregating the service by using different levels of service (local and express buses) or by splitting the line into two or more parts.
- Acting directly on the route design to try to reduce the waiting time at the intersections and the dwelling time at the stops; these two kinds of delays being sources of high variance.
- Control strategies.

Among these last kinds of strategies, holding strategies at a few control points proved to be particularly promising in controlling headway variations. For example, Barnett[1] has developed an optimal control point holding strategy which was applied to a model of the Northbound Red Line in Boston, at the busiest stop on the line, resulting in a mean wait time reduction of about 10%. This strategy has been selected for further study in this thesis.

I.3 Previous Work

There are basically two kinds of papers about the subjects with which we deal in this thesis: Some give empirical evidence on unreliability and its causes, others introduce and discuss different control strategies.

- Chapman et al[3], in a study about the sources of irregularity in bus transportation, found that in Newcastle upon Tyne the variance introduced by dwelling time at the stops is about 30% of the total variance, the variance introduced by different travel times between bus stops being 70% of this total variance, if one neglects the different times spend in queuing delays. There is no indication on the percentages introduced separately by traffic lights and other random delays along the line.

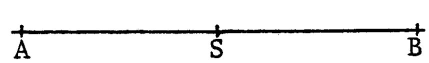
- Welding[9] indicates that waiting time at the intersections is about 20% of total travel time and dwelling time at the stops is 10%.

These results suggest that a good model to compute the variance of headways along the line must take into account both these components.

To choose a control strategy, we considered two works, which are among the most important in the literature.

- Newell[5] builds a model with two main simplifications: He assumes that deviations in travel time and dwelling time at the stops are small (which is certainly not generally true). He describes qualitatively, a method to decide when a bus must be held and by how much. One of his conclusions is that control should be applied as infrequently as possible. He also suggests that operators should introduce dynamically regulated headways throughout the system. He admits that his work does not give an easy to implement, therefore not effective, control strategy.

- Barnett's model is more realistic. The model deals with a linear route with several stops, one of which (S) is designated as a control stop.



Vehicles depart A at fixed intervals but by the time they arrive at point S, some clustering has occured. "Early" buses are held at S, whenever the preceding bus departed S behind schedule. Consequently, buses depart S for B with more regular headways. A small amount of bus holding at S will usually bring the expected waiting time quite close to its ideal value (one-half the scheduled headway) for most stops on the route.

The key simplification used by Barnett is to replace the continuous arrival time distribution with a two-point discrete distribution, which has one pike for "early" buses and one pike for "late" buses. The two-point distribution is used to construct a holding strategy: hold "early" buses at S for time x if the preceding one was late. The objective function used in this model determines x, the optimal length of time "early" buses should be held at S. This function recognizes the tradeoff between time spent waiting at a stop and time spent waiting on a bus held at S.

The problem with this strategy is that nothing is said about the choice of the control point. In the Red Line case, Barnett gives an empirical justification for the choice of Washington Street. One of our objectives is to give a general formulation and solution of this problem for any line configuration.

I.4 Thesis Contents

Now that we have discussed the main issues and motivations involved in this thesis, we can explain more precisely its contents.

Chapter II proves, with some assumptions, the most important of which being the absence of capacity constraints, that it is not possible to reduce to almost zero the waiting time of passengers on a line by only adding a very large number of buses, since heavy bunching can appear. To obtain this result, we first compute the probability of bunching on a line. In order to do so we use a very simple model, taking into account random traffic delays along the line and dwelling time at the stops. The computations prove that on a given route the probability of bunching is a decreasing function of the ratio (mean headway divided by headway's variance). Using this result, we show that even if we add a very large number of buses on a line, the waiting time of a passenger will have a lower bound, other than zero, if no action is taken to prevent bunching.

This is a good reason to implement some kind of control on a bus line to reduce the variance of headways and therefore the probability of bunching. Another reason is that, not taking into account the bunching phenomenon, one can easily prove that the waiting time of a passenger depends directly on the headway variance.

This is the matter which we address in Chapter III: We more specifically deal with Barnett's holding point method and try to give a general formulation allowing one to find the best point on a line to apply his strategy. To compute the headway variance at each stop, we use the same model as in Chapter II and assume that the variance at the terminus is zero. The result is the outcome of a double minimization problem; most of the basic concepts and parameters of it being introduced in Barnett's "On Controlling Randomness in Transit Operation". We then specialize these results to some very simple line configurations.

Chapter IV deals specifically with the Massachusetts Avenue bus line. We first describe the present operations and level of service and try to show what the causes of unreliability and the possible methods to reduce it are. We apply, together with an heuristic demonstration, the results of Chapter III to prove that the best control point is Harvard, if we only want to introduce one such point. In conclusion, we discuss the beneficial effects of this control and see whether it would be worthwhile to add some other control points on this route.

CHAPTER II

BUS BUNCHING

Bunching is a very well recognized phenomenon: two buses are bunched when they immediately follow each other. When there is no capacity problem, the second bus is useless. In general, when two buses are bunched, they will remain so along the rest of the line: this is due to the fact that traffic lights and other delays do not affect them independently anymore. Even if the second bus passes the first one, it will have to spend longer time at the stops to board passengers, and therefore will not be able to break the bunching. A bunched situation seems to be very stable. Theoretically, we shall say that two buses are bunched when their headway is zero, and we shall assume throughout this chapter the stability of this phenomenon.

This chapter will be divided into two parts:

In the first part we shall try to estimate the probability of bunching. In order to do so we shall first attempt to use a general model, however, it is shown to be analytically intractable. We shall then use simpler models to describe the causes of bunching: random delays along the route and queues at the stops will be introduced as independent causes of bunching. We shall then try to calibrate the model, considering the only source of random delays along the route to be in traffic lights.

In the second part, using the results of the first part, we will show that, because of bunching it is impossible to indefinitely reduce the waiting time of a passenger by only adding a large number of buses on a line. This will prove the necessity of some control on the line to reduce the variance of headways and therefore the probability of bunching.

II.1 How to Compute the Probability of Bunching

II.1.a A General Model

Turnquist[7] explains that a probability distribution of arrival times of a given bus on different days at a stop must have some characteristics due to the service. There is a definite earliest time of arrival dictated by the distance from the terminal to the stop, thus the distribution should be truncated to the left. It should also have a long tail to the right because there is a finite probability of the bus being very late. Increased dwell times at the stops, if the bus is late, due to larger boarding volumes than expected, introduce further delays. If the delay at the stop is proportional to the lateness arriving at that stop, we obtain a set of multiplicative effects.

A probability distribution consistent with all these characteristics is the lognormal. If the arrival time of a bus, t, is distributed lognormally, its density function may be expressed as follows:

$$f(t) = \frac{1}{t\sigma\sqrt{2\pi}} \exp\{-\frac{1}{2}[\frac{1}{\sigma}(\ell n t-\mu)^2]\}$$

where $\mu = E(\ell n t)$

$\sigma^2 = V(\ell n t)$

When we are at this point we would like to compute the probability of bunching.

This is: If t_1 is lognormal with (μ^1, σ^2) and t_2 is lognormal with (μ_2, σ^2) what is the probability that $t_1 - t_2 < 0$?

This probability is difficult to compute since it involves convolutions and we are going to develop another model, which will lead to normally distributed bus arrival times; much easier to deal with.

II.1.b A Computation Without Dwelling Time at the Stops

We consider that on the route there is a family of possible delays. $E(\xi)$ and $\sigma^2(\xi)$ are the mean and variance of this population.

The bus starts at time 0 and meets N delays $(\xi_1, \ldots \xi_N)$ on its route. We assume here that there is an average number n, of delays encountered per unit of time by the bus. Therefore, after time t, the bus has met $\underline{N = nt}$ delays. It is reasonable to think that this number will not depend on the day of operation: There is always the same number of traffic lights on the route, and the hazardous points remain pretty much the same.

We recall the central limit theorem: If $\overline{X}$ is the mean of a random sample of size N taken from a population, then as $N \to \infty$, the distribution of $\overline{Z} = \frac{\overline{X} - \mu}{\sigma \sqrt{N}}$ tends to $N(0,1)$, where μ and σ are the expectation and standard deviation of the population.

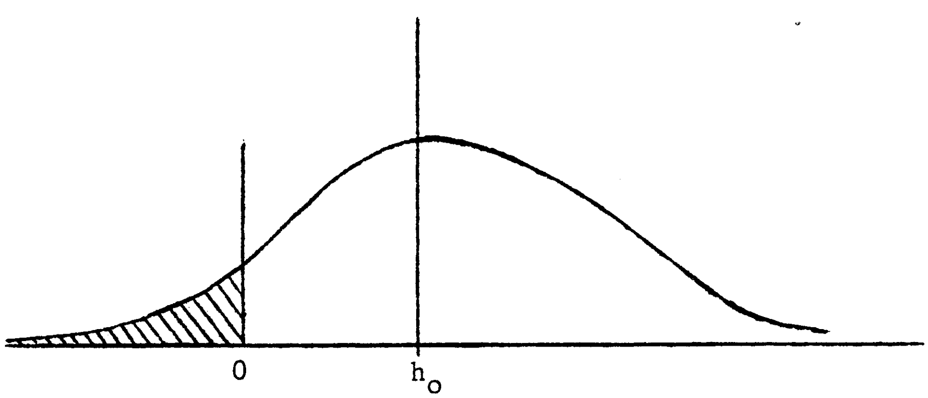
Therefore in our case, if we consider that N is large, the probability function of the deviation L of a bus from its schedule is:

$$f(L) = \frac{1}{\sqrt{2\pi}\,\sqrt{N\sigma^2(\xi)}}\, e^{-\frac{(L-NE(\xi))^2}{2N\sigma^2(\xi)}}$$

We introduce a new parameter $\sigma^2 = nt\ \sigma^2(\xi)$ and call h_o the normal headway.

We now consider two buses 1, and 2, and assume that their travel times are uncorrelated (this is certainly not perfectly true since the traffic situation facing two successive buses may be very similar).

The headway, which is the difference between two independent normally distributed variables should also be normally distributed with mean h_o and variance $s^2 = 2\sigma^2$.



But, with our assumption on bunching (2 buses which are bunched at one time remain bunched), we see that the probability density function for one headway is:

$$\begin{cases} f(x) = 0 & \text{for } x<0 \\ f(0) = \int_{-\infty}^{0} N(x, h_o, 2\sigma^2)\, dx & \text{which is our probability of bunching} \\ f(x) = N(x, h_o, 2\sigma^2) & \text{for } x>0. \end{cases}$$

$f(0)$ is the probability that the headway between two buses equals zero. If we consider that bunching only occurs by pairing, the probability of a given bus being bunched is $2f(0)$, but this is a restriction to one of our previous assumptions that all buses were independent. On the other hand, if we suppose that bunching can occur in larger groups, the probability of one bus bunching is $\phi f(0)$, where ϕ is a parameter such that $1 \leqslant \phi \leqslant 2$. Therefore, the probability B that one bus bunches with another is $B = \phi \int_{\frac{h_o}{\sigma\sqrt{2}}}^{\infty} N(t,0,1)\, dt$.

To see if this result is reasonable we use some data collected in March '75 on the Harvard-Dudley bus route. This data gives the arrival time of 73 buses at M.I.T. These buses were coming from Dudley, where they had an almost perfect headway h=5 minutes. This data gives us 70 headways with $h_o = 290$ seconds

and $s = 181$ seconds.

To compute σ we use the relationship $s^2 = 2\sigma^2$, which gives $\sigma = 128$.

We have then $\frac{\sigma}{h_o} = 0.45$ and we find a probability $B \simeq 10\%$ of bunching with our model, and $\phi=2$.

If we assume that a headway of less than 30 to 35 seconds means that two buses are effectively bunched, we find that there are about

DATA COLLECTED IN MARCH '75: Headways in Seconds

Headway Number	3/6	3/11	3/12
1	310	210	270
2	245	505	300
3	150	15	70
4	480	280	140
5	390	240	460
6	210	00	130
7	525	500	420
8	340	445	60
9	170	255	700
10	415	105	420
11	35	695	40
12	570	250	160
13	105	70	280
14	495	330	520
15	490	725	380
16	190	195	30
17	495	20	530
18	195	260	140
19	185	315	360
20	295	420	320
21	95	165	70
22	170	510	535
23	180	155	215
24		435	

TABLE 1

8-10 buses bunched, which is around 12%. We see that our model gives us a good approximation of the probability of bunching: This tends to prove that the truncated normal distribution is a good approximation of bus headways.

II.1.c Introducing Boarding Time at the Stops

It is very often assumed that the loading time, T, of a bus at a stop increases proportionally to the number of people waiting.

$$T = a + bW.$$

If the number of passengers arriving by unit of time is constant, k, we have $T = a + bkh$, where h is the headway between the considered bus and the preceding one.

The usual loading time is $T_o = a + bkh_o$. If the bus is late by Δh we have $T = T_o + bk\Delta h$. Therefore its lateness becomes $\Delta h\ (1 + bk)$. (After i stops, the lateness would be $\Delta h\ (1 + bk)^i$).

Now let us consider two buses at stop number one. If the first one is late by Δh, its lateness becomes $\Delta h\ (1 + bk)$. If the second was running on time it becomes early by $-\Delta hbk$.

We see that the fact of passing through a stop changes the headway between the two buses from an initial value of $h_o - \Delta h$ to $h_o - (1+2bk)\Delta h$. The fact of going through i stops would change it to $h_o - (1+2bk)^i\Delta h$, if nothing else happened. This is the multiplicative effect of the stops, problem of which we spoke in our introduction.

However, we have previously shown that the headway between two buses was normally distributed with parameters h_o and $s^2 = 2\sigma^2(t)$. We see that the passage through each stop has the effect of multiplying the variance by $(1 + 2bk)^2$, the headway still remaining normally distributed.

To compute the probability of bunching we used the parameter

$$s^2(t) = 2\sigma^2(t) = 2nt\ \sigma^2(\xi).$$

In order to simplify the problem, we are going to assume that the distance between two stops is constant L, and call V the average speed of the bus. We call $\sigma^2(i)$ the variance at stop i. The relationship above becomes:

$$\sigma^2(i) = \frac{mi}{2}$$

$$\text{where } \frac{m}{2} = \frac{nL}{V}\ \sigma^2(\xi)$$

But, if we now introduce the dwelling time, the sequence of $\sigma^2(i)$ must follow the relationship:

$$\sigma^2(i+1) = (\sigma^2(i) + \frac{m}{2})(1 + 2bk)^2$$

which takes into account both the previous arithmetic growth and the multiplicative effect of the stops.

In order to compute the general term of this sequence we introduce u such that $1+u = (1 + 2bk)^2$. The usual way is to look for α such that the sequence $\sigma^2(i) + \alpha$ is geometric with progression 1+u. If so:

$$\sigma^2(i+1) + \alpha = (1+u)(\sigma^2(i) + \alpha).$$

By identification we find that $\alpha u = \frac{m}{2}(1+u)$. As:

$$\sigma^2(i) + \alpha = (1+u)^i \, (\sigma^2(0) + \alpha)$$

we have:

$$\boxed{\sigma^2(i) = \frac{m}{2}\frac{(1+u)}{u}[(1+u)^i - 1]}$$

if u is small this relationship gives the usual relationship without dwelling time at the stops $\sigma^2(i) = \frac{mi}{2}$.

The probability of bunching after i stops will now be:

$$B_i = \phi \int_{\frac{h_o}{\sigma(i)\sqrt{2}}}^{\infty} N(t,0,1)\,dt.$$

Indeed, as $\sigma(i)$ is larger than σ, we see that the probability of bunching increases compared to the no dwelling time case. The following table shows the influence of u on the variance:

u	0	.05	.1	.2
$\sigma^2(5)$	5m	5.8m	6.7m	8.9m
$\sigma^2(10)$	10m	13.2m	17.5m	31m
$\sigma^2(20)$	20m	34.7m	63.5m	224m

On the average, on a line of 20 stops a reasonable value of u is u=.1 (Turnquist). We see that it is not possible to neglect the influence of u on the variance of the headway.

II.1.d Another Way to Compute the Probability of Bunching

Welding[3] has found that, on an average route, about 20% of the total travel time is spent by a bus waiting at the intersections, and that 10% is spent dwelling at bus stops. These percentage figures added to one suggested by Chapman et al that about 25% of the variance in buses' headways comes from the dwelling time at the stops encourage us to try to develop a new model which would only be based on the existence of delays due to traffic lights.

For example, we can consider each traffic light as a two points distribution, each of them having the same characteristics: a bus which arrives at a traffic light has a probability q of waiting 0 and a probability (1-q) of waiting time τ. We will assume here that the only delays in the traffic are due to the lights, which is surely an enormous simplification, and we are going to consider the headway between two buses after n lights.

The probability that bus 1 has waited time $x\tau$ is:

$$P_1(X=x) = \binom{x}{n}(1-q)^x q^{n-x}$$

The probability that bus 2 has waited time $y\tau$ is also:

$$P_2(Y=y) = \binom{y}{n}(1-q)^y q^{n-y}$$

Therefore, if we still assume independence between the progressions of the two buses we get:

$$P(X-Y=k) = \sum_{x-y=k} \binom{x}{n}\binom{y}{n}(1-q)^{x+y} q^{2n-(x+y)}$$

We therefore can compute the probability of bunching:

$$P(\text{Bunching for bus 1}) = \phi P(X-Y>\lambda_o)$$

where λ_o is the smallest λ such that $h_o-\lambda\tau<0$. The factor ϕ appears for the same reason as in the first model. Clearly, this method gives an algebraic formula, but it cannot be of much use in the practice. The first reason is that, contrary to the previous method, it cannot be assumed that n is a large number and so the central limit theorem does not apply.

The second reason, which is more important, is that, depending on the way we would adjust a two points model to the traffic lights, the results would be very different. For example, on the Harvard-MIT route there are 14 lights. During rush hours, the average length of a cycle is 85 seconds. The green phase is about 65% of that time. If we adjust a two points model by the moments method, we find $q=.85$ and $\tau=37$ seconds. For a headway $h_o=300$ seconds. This gives $\lambda=8$. A probability of bunching below 0.1%. If the adjusted parameters were $q=.80$ and $\tau=45$ seconds (a difference of about 20%) the probability of bunching would be around 5% (a difference of 5000%).

Therefore, it seems very difficult to adapt this method in order to find realistic results: the approach seems to be too simplified.

II.1.e Calibrating the Model of Section II.1.b and II.1.c

We have now arrived at a point where we have clearly shown, after having tried different methods to compute the probability of bunching on a bus route, that the most realistic one is developed in part II.1.b and II.1.c. We would now like to calibrate this model which seems difficult since the parameters $E(\xi)$ and $\sigma^2(\xi)$ are impossible to evaluate: they are the expectation and variance of the family of delays encountered by the bus on its route. These values can only be found by experiment.

However, we can try to estimate them by introducing a big simplification: Considering that the delays are all caused by the traffic lights. This will surely lead to a lower variance than the measured one.

We shall consider here a practical example: the MIT stop for the buses coming from Harvard, where we assume perfect dispatching. We shall deal with the afternoon hours and define:

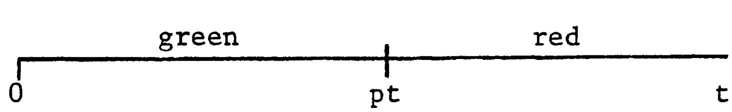
- N = the total number of traffic lights between Harvard and MIT.
- i = the number of stops.

We know that if we call $s^2(i)$ the variance of the headway after i stops, we have the relationship:

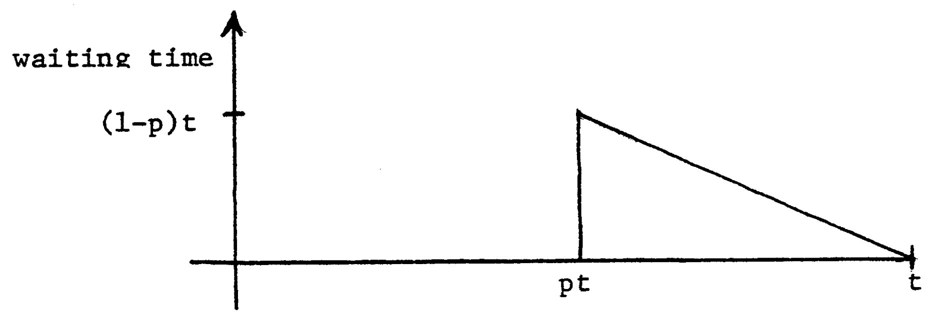
$$\frac{s^2(i)}{s^2} = \frac{(1+u)((1+u)^i-1)}{ui}$$

where s^2 does not take into account the dwelling times.

If the average traffic light period is:



we have the following diagram for the waiting time of a bus at each traffic light depending on its arrival time: (and assuming that the bus is not subject to queuing delays).



We can then easily compute the parameters $E(\xi)$ and $\sigma(\xi)$ of such delays:

$$E(\xi) = 0 \times p(0) + \int_{pt}^{t} (t-x)\,\frac{dx}{t} = \frac{(1-p)^2}{2}t$$

$$\sigma^2(\xi) = p\left[\frac{(1-p)^2}{2}t\right] + \int_{pt}^{t} \left(x - \frac{(1-p)^2}{2}t\right)^2 \frac{dx}{t}$$

$$= \frac{(1-p)^2}{4}\left(\frac{1+2p}{3}\right)t^2$$

Therefore $\sigma^2 = N\left(\frac{1-p}{2}\right)^2 \left(\frac{1+2p}{3}\right)t^2$.

For the afternoon hours, the data given by the city of Cambridge and averaged over all the lights is: $\begin{cases} N=14 \\ t=85 \text{ seconds} \\ p=.64 \end{cases}$

Therefore:

$$\sigma^2 = 14(.18)^2(.760)\ 7225 \text{ sec}^2$$

$$\sigma^2 = 2490 \text{ sec}^2$$

$$s = \sqrt{2\sigma^2} = 70 \text{ sec.}$$

A group of MIT students have found that the dwelling time at the stops amounts to about 10% of the travel time. At the stops where very few people are waiting, the dwelling time is almost equal to the fixed term a. More than half the stops on the line are of this kind.

We can therefore take an average bk between 0.025 and 0.05 and u between 0.1 and 0.2.

For $u = .1$ we find $s(12) = 98$ sec.

$u = .2$ we find $s(12) = 139$ sec.

In both cases, this is less than the standard deviation measured at the stop which is 200 sec., but the magnitude order is reasonable, considering we only took into account traffic lights and dwelling times at the stops.

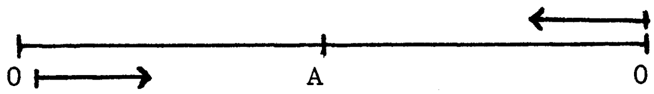
In conclusion, it seems that the model we chose is reasonable.

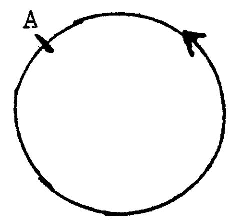
II.2 Bunching and Waiting Time

II.2.a Waiting Time Without Bunching

We are going to consider from now on that we never have any capacity problem for our buses. This means that we suppose that a bus which arrives at a stop has enough space to load all the people waiting at this stop. Indeed, this assumption is very important, and in many cases it will not be true. The reason for making it is the extreme difficulty of dealing analytically with the concept of capacity: There is almost no trace in the literature of analytical study of waiting time taking into account this capacity problem. Some further research is needed in this field.

We consider that we have a fixed number n of buses on our line, running in both directions, but we dispose of some control at one extremity so that we can make sure that the variance of headways at A is constant and does not increase with time: Then this will also be true for the probability of bunching, as far as the number of buses remains constant.





In fact, this makes our line look like a loop. We call ℓ its length, v the average running speed of the buses and we note that $t_o = \frac{\ell}{v}$.

We suppose that the buses are independent one from the other, and that they are distributed over the line with a flat probability density function. We call t_i the time at which the i^{th} bus will arrive at A, (i is the label of a bus, not its rank in arriving time) and we are going to compute under these conditions, the probability density function f(t) of the waiting time, and its expectation E(w).

We have:

$$P[\min(t_1,\ldots t_n) < t] = 1 - P[t_1 > t,\ldots t_n > t]$$

$$= 1 - P(t_1 > t)\ldots P(t_n > t)$$

$$= 1 - (1 - \frac{t}{t_o})^n$$

Therefore:

$$f(t) = \frac{dP[\min(t_1,\ldots t_n) < t]}{dt} = \frac{n}{t_o}(1-\frac{t}{t_o})^{n-1}$$

and

$$E(w) = \int_o^{t_o} t\, f(t)dt$$

$$= \int_o^{t_o} [-n(1-\frac{t}{t_o})(1-\frac{t}{t_o})^{n-1} + n(1-\frac{t}{t_o})^{n-1}]dt$$

$$= [\frac{nt_o}{n+1}(1-\frac{t}{t_o})^{n+1} - t_o(1-\frac{t}{t_o})^{n}]_o^{t_o}$$

$$\boxed{E(w) = \frac{t_o}{n+1}}$$

II.2.b Waiting Time With Bunching

We consider the same line: The buses are still independent, and are distributed with a flat probability density function, but some of them can now be bunched. If i buses are bunched, then there are only (n-i) effective buses on the line, since we do not consider capacity problems.

If B(i) is the probability that i bunches occur, the expectation of our waiting time is now:

$$E(w) = \sum_{i=0}^{n} B(i) \frac{t_o}{n-i+1}$$

but:

$$B(i) = (\tfrac{i}{n})\, B^i(1-B)^{n-i}$$

(Probability that out of n buses i are bunched, n-i are not, where B is the probability that one is bunched.)

Therefore:

$$E(w) = t_o \sum_{i=0}^{n} \frac{1}{n-i+1} \frac{n!}{i!(n-i)!} B^i (1-B)^{n-i}$$

$$= \frac{t_o}{n+1} \sum_{i=0}^{n} \frac{(n+1)!}{i!(n-i+1)!} B^i (1-B)^{n-i} \frac{(1-B)}{1-B}$$

$$\boxed{E(w) = \frac{t_o}{(n+1)(1-B)}(1-B^{n+1})}$$

Clearly, the waiting time has increased compared to the no bunching case.

We are now going to prove that E(w) has a limit as n grows larger and larger, in the worst situation for bunching when $\phi=2$.

We recall the value of B: $B = 2 \int_{\frac{h}{\sigma\sqrt{2}}}^{\infty} N(t,0,1)dt.$

As we explained, we consider that σ only depends on A, but we must not forget that $nh = t_o$ is constant and does not depend on the level of service.

We can also write:

$$B = \frac{2}{\sqrt{2\pi}} \int_{\frac{N_o h_o}{N\sigma\sqrt{2}}}^{\infty} e^{-\frac{x^2}{2}} dx = 1 - \frac{t_o}{n\sigma\sqrt{\pi}} + 0(\frac{1}{n})$$

where N_o, h_o represents the initial level of service.

Therefore, $\text{Log } B \simeq -\frac{t_o}{n\sigma\sqrt{\pi}}$

and $\text{Log } B^{n+1} = (n+1) \text{ Log } B$ has the limit $-\frac{t_o}{\sigma\sqrt{\pi}}$.

So $\lim_{n\to\infty}(1-p^{n+1}) = 1-e^{-\frac{t_o}{\sigma\sqrt{\pi}}}$.

In exactly the same way, we can prove that:

$$\lim_{n\to\infty}(n+1)(1-p) = \frac{t_o}{\sigma\sqrt{\pi}}$$

therefore:

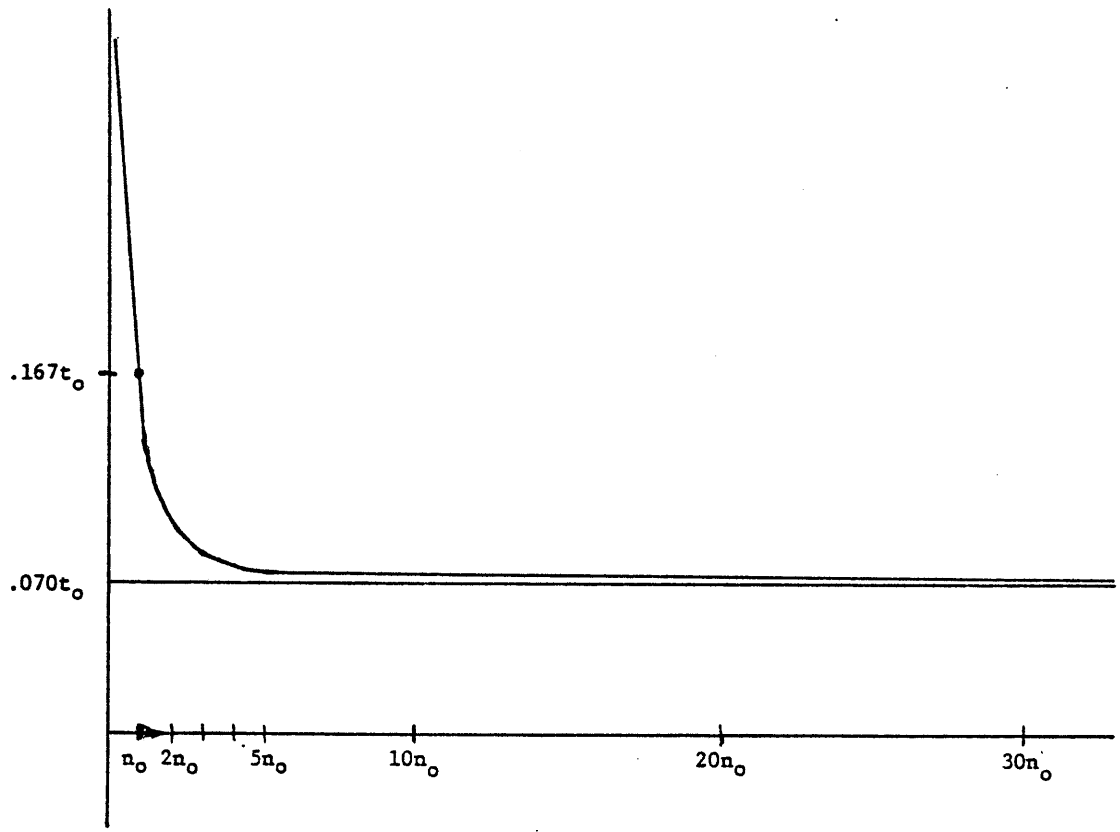
$$\boxed{\lim_{n\to\infty} E(w) = \sigma\sqrt{\pi}\left(1-e^{-\frac{t_o}{\sigma\sqrt{\pi}}}\right)}$$

The following example shows what the influence of n is on the waiting times. We start with a level of service where $n_o = 5$ and $\sigma = .04\ t_o$ which is in a reasonable range:

$\sigma/h = \frac{n\sigma}{t_o}$	$h/\sigma\sqrt{2}$	B	n	E(w)
.2	3.5	.00	n_o	$.167t_o$
.4	1.7	.07	$2n_o$	$.098t_o$
.6	1.2	.24	$3n_o$	$.082t_o$
.8	.88	.38	$4n_o$	$.077t_o$
1	.71	.48	$5n_o$	$.074t_o$
2	.35	.73	$10n_o$	$.073t_o$
4	.18	.86	$20n_o$	$.071t_o$
6	.11	.91	$30n_o$	$.070t_o$

Theoretical limit: Inf $E(w) = 0.71t_o$.

Indeed, at the points on the line where σ is smaller (close to the starter) the lower limit of waiting time which can be obtained will be



Waiting Time as a Function of the Number of Buses

FIGURE 1

smaller, since the probability of bunching will be smaller, and where σ is higher the probability of bunching will be higher.

II.3 Conclusion

In conclusion, we see that in the case where no capacity problem exists, adding a large number of buses to an existing fleet does not necessarily significantly improve the service, because of the bunching which then occurs.

However, some methods to prevent the bunching can be suggested:

- Establishing some control points, where one would try by selectively holding buses to decrease the variance.
- Adding an express line to the existing one: this would automatically break some bunching, since two bunched buses would not necessarily remain bunched.

These methods are generally cheaper to implement than the purchase and operation of new buses.

Some further research is needed to evaluate the effects of bunching when important capacity problems exist. It is clear that in this case the effect would not be as disastrous, since a bunched bus would no longer be a useless one.

CHAPTER III

OPTIMAL CONTROL POINT ON A LINE

III.1 General Formulation of the Problem

Many control strategies have been proposed in order to reduce the negative effects of the bus clustering, or bunching. From now on we are going to deal more specially with the method proposed by Barnett. Barnett used a stop on the line, where he intuitively thought the control should be implemented, because there the maximum would have utility for the user. He developed an objective function consisting of two parts: The first part represents the expected wait time for passengers boarding at the control point, $E(w)$. The second represents the expected delay to passengers aboard buses which are held at the control point, $E(d)$.

He combines $E(d)$ and $E(w)$ in a linear objective function $F = \gamma E(d) + (1-\gamma)E(w)$, where γ is a parameter used to indicate the relative importance of "in vehicle delays" versus "out of vehicle delays". The problem with this method is that the objective function does not take into account the lower part of the line and takes into account the upper part of the line only to the extent that some passengers are still on board at the considered stop.

We are going to try to generalize this idea, and find, in the case of some simple configurations, the best point on the line at which to apply a control strategy directly derived from his.

We model our line in a very simple way

There are n stops on the line. For the i^{th} stop we consider the following average values:

W_i = Number of people waiting to board

D_i = Number of people who would be delayed if bus is held at stop i

Our objective will be:

$$\theta = \Gamma_i E_i(d) + \sum_{j=1}^{n} \gamma_j E_j^i(w)$$

θ represents the total waiting time for all the passengers of the line, supposing the bus is held at i. The parameters Γ_i and γ_j can be chosen once one has decided the relative importance of "in vehicle delays" and "out of vehicle delays". For simplicity we shall decide that the importance of in vehicle and out of vehicle delays are the same on a wait time basis:

Therefore, we shall take: $\Gamma_i = \frac{D_i}{N'}$

$$\gamma_j = \frac{W_j}{N'}$$

where N'is the total average number of people waiting for a bus on the line

$$N' = \sum_{j=1}^{n} W_j, \text{ thus } \sum_{j=1}^{n} \gamma_j = 1.$$

Indeed, these coefficients could be modified in a farther analysis. To find the optimal value of the delay to impose, we need a first minimization of θ at each stop, and then we must search for the stop where this minimum value is itself minimum: this will give us the result which we are looking for.

Newell[5] has shown that the expected wait time can be expressed in terms of the headway between buses in the following manner:

$$E_j(w) = \frac{E_j(\bar{H}^2)}{E_j(\bar{H})} \quad \text{where } \bar{H} \text{ is the headway at the stop } j.$$

This is also $E_j(w) = \frac{H}{2} + \frac{1}{2}\frac{\text{var}_j(\bar{H})}{\bar{H}}$ where H is the mean headway.

We now recall the model which we developed in part II.1.c of this study. We proved with the assumptions that there is a smooth average number of delays encountered per unit time, a constant speed and equidistant stops that after i stops we have:

$$\text{Var}_i(\bar{H}) = m\frac{(1+u)}{u}[(1+u)^i - 1]$$

or $\text{Var}_i(\bar{H}) = mi$ if we neglect the dwell time at the stops. Now let us suppose we exert control at stop i. We call, in this case, $\text{Var}\,C_{i,j}(\bar{H})$ the variance of the headway at stop j.

For $j<i$ we have $\text{Var}\,C_{i,j}(\bar{H}) = \text{Var}_j(\bar{H})$

If we consider the same model as in part II.1.c, taking into account both the linear and geometric effects, we must have the following relationship for $j \geq i$.

$$\sigma^2_{j+1} = (\sigma^2_j + \frac{m}{2})(1+u)$$

Using exactly the same method as in Section II.1.c, we find for $j \geq i$:

$$\text{Var}C_{i,j}(\bar{H}) = \text{Var}C_{i,i}(1+u)^{j-i} + \frac{m(1+u)}{u}[(1+u)^{j-i} - 1].$$

For u "small" this gives $\text{VarC}_{j,i}(\bar{H}) = \text{VarC}_{i,i} + m(j-i)$.

If we introduce these expressions in our objective function we find:

$$\theta = \Gamma_i E_i(d) + \sum_{j\geq i} \frac{\gamma_j}{2H} [H^2 + (1+u)^{j-i}\text{VarC}_{i,i} + \frac{m(1+u)}{u}[(1+u)^{j-i}-1]]$$

$$+ \sum_{j<i} \frac{\gamma_j}{2H}[H^2 + \frac{m(1+u)}{u}[(1+u)^j-1]]$$

This equality can also be written:

$$\theta = \Gamma_i E_i(d) + \frac{H}{2} + \sum_{j\geq i} \frac{\gamma_j}{2H}[(1+u)^{j-i}\text{VarC}_{i,i} + \sum_{j=0}^{n} \frac{\gamma_j}{2H} m\frac{(1+u)}{u}[(1+u)^j-1]]$$

$$+ \sum_{j\geq i} \frac{\gamma_j}{2H} m\frac{(1+u)}{u}[(1+u)^{j-i}-1 - [(1+u)^j-1]]$$

which is also:

$$\theta = \Gamma_i E_i(d) + \frac{H}{2} + \sum_{j=0}^{n} \frac{\gamma_j}{2H} m\frac{(1+u)}{u}[(1+u)^j-1]$$

$$+ \sum_{j\geq i} \frac{\gamma_j}{2H}(1+u)^{j-i}[\text{VarC}_{i,i} - \frac{m(1+u)}{u}[(1+u)^i-1]]$$

We can break this equality into two terms, which are easy to explain:

$$* \quad \frac{H}{2} + \sum_{j=0}^{n} \frac{\gamma_j}{2H} m\frac{(1+u)}{u}[(1+u)^j-1] = \frac{H}{2} + \sum_{j=0}^{n} \frac{\gamma_j}{2H} \text{Var}_j(\bar{H})$$

is the normal waiting time if no control is operated on the line.

If we call x_i the optimal hold time at stop i, we shall use Barnett's results:

$$E_i(d) = (1-p)p_{dc}x_i$$

$$\mathrm{Var}C_{i,i}(\bar{H}) = 2pp_{cd}(L_i^2 + x_i^2 - L_i x_i(1+p_{cd})$$

$$\mathrm{Var}_i(\bar{H}) = 2pp_{cd}L_i^2$$

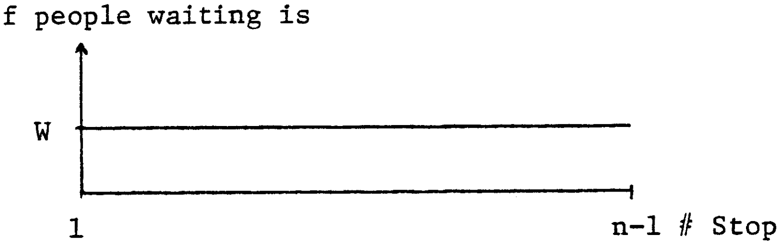
Now the problem is completely formulated in terms of a double minimization problem. We are going to apply it in some very simple line configurations.

III.2 Study of Some Simple Configurations

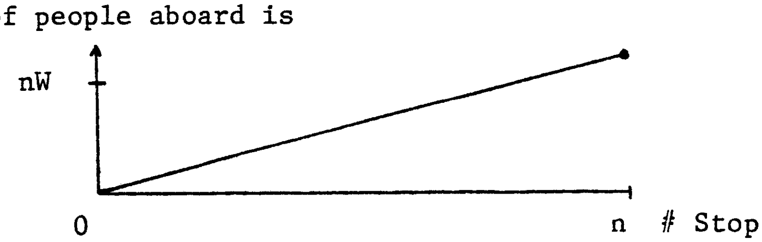
III.2.a Common Destination Line: No Dwell Time at the Stops

In this first example we are going to consider a line on which the same average number of passengers wait at each stop and where no one gets down before the last stop. This kind of model can accurately represent a line coming from the suburbs to the center of a city.

The number of people waiting is



The number of people aboard is



$$* \Gamma_i E_i(d) + \sum_{j\geq i} \frac{\gamma_j}{2H}(1+u)^{j-i}[\text{Var}C_{i,i} - \frac{m(1+u)}{u}[(1+u)^i-1]]$$

$$= \Gamma_i E_i(d) + \sum_{j\geq i} \frac{\gamma_j}{2H}(1+u)^{j-i}[\text{Var}C_{i,i} - \text{Var}_i(\overline{H})] = \hat{\theta}.$$

is the time (negative) which shall be gained by the control operated at stop i. We want to maximize this time, that is to say, minimize this term $\hat{\theta}$.

We are going to apply exactly the same scheme as Barnett:

At a stop we shall consider only the first and second moments; then we can introduce a discrete lateness distribution with only two points. We use Barnett's technology:

c_i = Lateness of a bus which is relatively early when it reaches i.

d_i = Lateness of a bus which is relatively late when it reaches i.

p_i = Probability of an early bus at s.

$L_i = d_i - c_i$.

p_{cdi} = Probability of a late bus given that the preceding was early.

p_{dci} = defined in a similar manner.

We shall assume throughout this thesis that p_i, p_{cdi}, p_{dci} do not depend on i. This seems reasonable since the lateness interval L_i, for example, will surely increase with i, but the probability of a late bus, given that the previous one was early, will not change very much, depending on the stop. We shall therefore drop the subscript i for these three parameters and introduce p, p_{cd}, p_{dc}.

Clearly, in this case we have:

$$\Gamma_i = \frac{i}{n} \qquad \gamma_i = \frac{1}{n}$$

In this section we take u=0. Therefore, $Var_i(\bar{H})=mi$.

In this case:

$$\theta = \Gamma_i E_i(d) + \sum_{j=1}^{n-1} \frac{\gamma_j}{2H}[VarC_{i,i}(\bar{H}) - Var_i(\bar{H})]$$

$$= \frac{1}{n}[(1-p)p_{dc}i\ x_i + \frac{n-i}{2H}(2pp_{cd})(x_i^2 - L_i x_i(1+p_{cd}))]$$

We are looking for $\min\limits_{i} \min\limits_{x_i} \hat{\theta}$.

x_i^* is obtained by setting $\frac{\partial\hat{\theta}}{\partial x_i}(x_i^*)=0$.

If the value found this way is negative, then we take $x_i^*=0$.

$$x_i^* = \max\left\{\frac{\frac{1}{2H}(n-i)2pp_{cd}L_i(1+p_{cd})-(1-p)p_{dc}i}{\frac{1}{H}(n-i)2pp_{cd}},\ 0\right\}$$

Since our function $\hat{\theta}_i$ was of the form $Ax_i^2 - Bx_i$, the minimum is at $x_i^* = \frac{B}{2A}$ and we have:

$$\hat{\theta}_i(x_i^*) = -\frac{B^2}{4A}.$$

Therefore:

$$n\theta_i^* = -\frac{[(1-p)p_{dc}i - 2pp_{cd}\frac{1}{2H}(n-i)L_i(1+p_{cd})]^2}{\frac{2}{H}(n-i)2pp_{cd}}$$

We now recall $L_i = \sqrt{\dfrac{mi}{2pp_{cd}}}$ and we look for i such that:

$$f(i) = i(n-i)\left[\frac{(1-p)p_{dc}\sqrt{i}}{n-i} - \sqrt{2mpp_{cd}}\,\frac{(1+p_{cd})}{2H}\right]^2 \text{ is maximum,}$$

not forgetting that, on our field, we have:

$$\frac{(1-p)p_{dc}\sqrt{i}}{n-i} - \sqrt{2mpp_{cd}}\,\frac{(1+p_{cd})}{2H} \leqslant 0$$

In order to facilitate the computations, we introduce new coefficients:

$$A = (1-p)p_{dc} \; ; \qquad B = \frac{1+p_{cd}}{2H}\sqrt{2mpp_{cd}}$$

$$g(i) = \frac{\sqrt{i}}{n-i} \quad ; \text{ then } g'(i) = \frac{n+i}{2\sqrt{i}(n-i)^2}$$

Our problem now is:

$$\begin{cases} \max f(i) = i(n-i)(Ag(i)-B)^2 \\ Ag(i)-B \leqslant 0 \\ 1 \leqslant i \leqslant n \end{cases}$$

We have:

$$\begin{aligned} f'(i) &= (Ag(i)-B)[(n-2i)(Ag(i)-B)+2i(n-i)Ag'(i)] \\ &= (Ag(i)-B)(n-2i)[-B+A(g(i) + \frac{2i(n-i)g'(i)}{n-2i})] \\ &= (n-2i)(Ag(i)-B)[-B+Ag(i)(1 + \frac{n+i}{n-2i})] \\ &= (n-2i)(Ag(i)-B)(-B+Ag(i)\frac{2n-i}{n-2i}) \end{aligned}$$

The function $i \to \dfrac{2n-i}{n-2i}$ is increasing with i.

Since g(i) is also increasing with i, the function $g(i)\frac{2n-i}{n-2i}$ increases with i.

We can now draw the variations of the functions involved.

i	0		n/2		n
Ag(i)-B		—		—	
n-2i		+	0	—	
$-B+Ag(i)\frac{2n-i}{n-2i}$	-B	↗	+∞ ‖	—	
f'(i)	+	0	—	—	
f(i)	↗	↘		↘	

We see that there is only one possibility: The control is the point closest to the unique λ such that:

$$B = Ag(\lambda)\frac{2n-\lambda}{n-2\lambda}.$$

This is also:

$$\frac{1+p_{cd}}{2H}\sqrt{2mpp_{cd}} = (1-p)p_{dc}\frac{\sqrt{\lambda}}{n-\lambda}\cdot\frac{2n-\lambda}{n-2\lambda}.$$

This point will always be before the middle of the line.

It is close to the start if B is relatively small, (that is to say, if the rate of increase of the variance, m, is relatively small) and it goes to the middle if m is larger. We can show how this applies in a hypothetical example.

We consider a line with 20 stops.

We take $s^2 = \frac{H^2}{50} i$

and some values found by Barnett for the Red Line:

$$\frac{B}{A} = \frac{\frac{1+p_{cd}}{2H}\sqrt{2mpp_{cd}}}{(1-p)p_{dc}} \simeq \frac{2\sqrt{m}}{H} = \frac{2}{\sqrt{50}} = 0.28$$

We want λ such that $\frac{B}{A} = \frac{\sqrt{\lambda}}{n-\lambda}\frac{2n-\lambda}{n-2\lambda} = \ell(\lambda)$.

The best way is to try for different values of λ:

$$\left.\begin{array}{l} \ell(1) = .114 \\ \ell(2) = .166 \\ \ell(3) = .269 \\ \ell(4) = .375 \end{array}\right\}$$

In this case the best control point would be the third stop.

III.2.b Common Destination; Dwell Time at the Stops

This example starts exactly as example III.2.a. We still want to minimize:

$$n\hat{\theta}_i = (1-p)p_{dc} i \, x_i + \frac{(1+u)^{n-i}-1}{u}\frac{1}{2H}\left[x_i^2 - L_i x_i (1+p_{cd})\right]$$

and we come to the maximization of the function:

$$f(i) = \frac{\left[(1-p)p_{dc} i - 2pp_{cd}\frac{1}{2H}\left(\frac{(1+u)^{n-i}-1}{u}\right)L_i(1+p_{cd})\right]^2}{\frac{(1+u)^{n-i}-1}{u}}$$

where, this time $L_i = \sqrt{\frac{m(1+u)}{2pp_{cd}u}[(1+u)^i-1]}$

$$f(i) \equiv ((1+u)^{n-i}-1)(1+u)^i-1)\left[\frac{(1-p)p_{dc}i}{[(1+u)^{n-i}-1]\sqrt{(1+u)^i-1}} - \frac{1\sqrt{2pp_{cd}}m(1+u)(1+p_{cd})}{2Hu\sqrt{u}}\right]^2$$

This function can be written in the same form as in example III.2.a:

$$f(i) = [(1+u)^{n-i}-1][(1+u)^i-1][Ag(i)-B]^2$$

where $g'(i) > 0$. We have:

$$f'(i) = (Ag(i)-B)(\text{Log}(1+u))((1+u)^{n-i}-(1+u)^i)[-B+A[g(i)+2g'(i)$$

$$\left.\frac{[(1+u)^{n-i}-1][(1+u)^i-1]}{[(1+u)^{n-i}-(1+u)^i]\text{Log}(1+u)}\right]$$

This proves, exactly as in III.2.a, that the control still has to be implemented before the middle stop $\{(1+u)^{n-i}-(1+u)^i \leqslant 0 \text{ for } i \geqslant \frac{n}{2}\}$. It is also possible to see that as u gets larger, the control has to be closer to the middle. This result is consistent with the fact that the bigger m was in example III.2.a, the closer to the middle the stop had to be: In simple terms, the faster the variance increases, the closer to the middle the control has to be.

As no easy algebraic formula exists, the best way to find i_{opt} is to try successively f(1), f(2)... . We can deal with the same example as in III.2.a and introduce some values of u.

* For u = .2 we have:

$$f(i) = [(1.2)^{20-i}-1][(1.2)^{i}-1][\frac{.281i}{[(1.2)^{20-i}-1]\sqrt{(1.2)^{i}-1}} - 1.021]^2$$

We find that:

f(4) = 17.266; f(5) = 18.989; f(6) = 19.399; f(7) = 20.062:

f(8) = 19.530; f(9) = 18.345; f(10) = 16.565.

Therefore, in this case, the best holding point would be at i=7. However, we can see that the maximum of the function f is rather "flat". Taking i=10 instead of i=7 only makes a difference of 15% in the value of the objective function.

* For u = .3 we would find i=8.

* For u = .1 we would find i=5.

This suggests that u has quite a large potential impact on the shifting of the optimal control point towards the right.

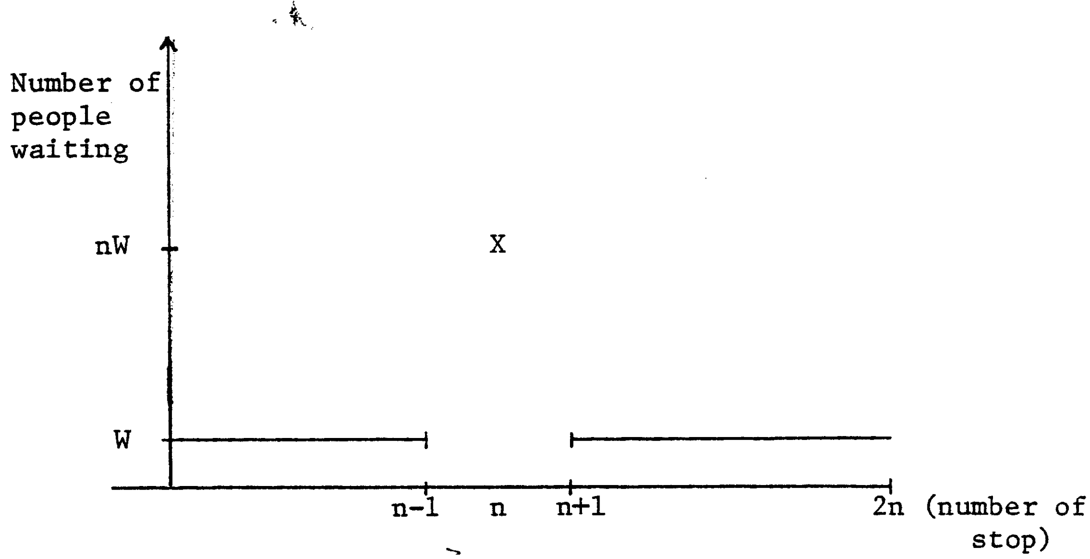
III.2.c Singular Point at the Middle of the Line

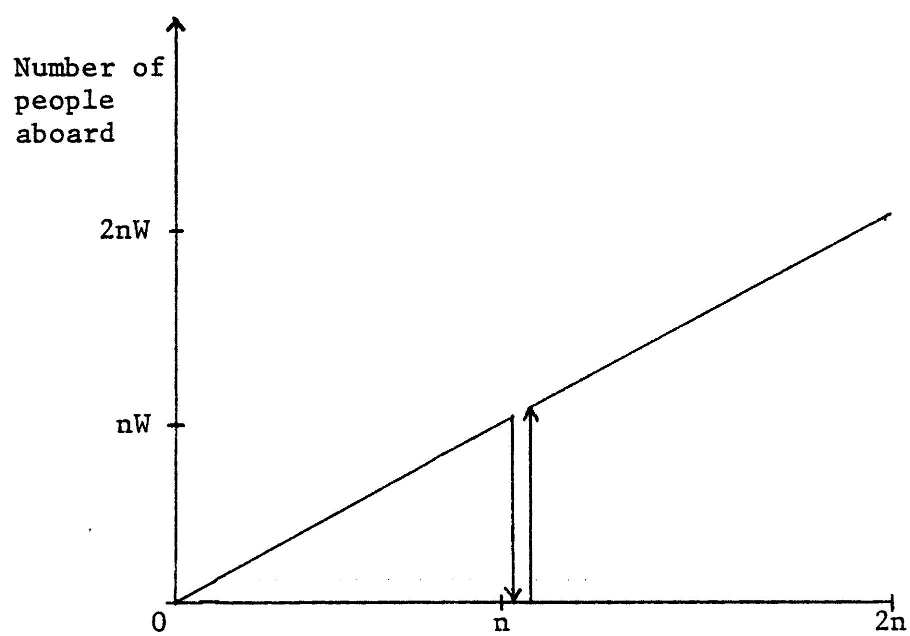
In this section we are going to deal with two models having a singular point at the middle of the line. The first one is not very realistic, but will help dealing with the second one.

A simple configuration is a line with 2n stops where:

- The same number of people W are waiting everywhere, except at the middle stop, where this number is nW.

- Nobody deboards except at the middle stop, where everyone deboards.





This time we have:

$$\gamma_j = \frac{1}{3n} \quad \text{for} \quad j\neq n \qquad \gamma_n = \frac{1}{3}$$

$$\Gamma_j = \frac{i}{3n} \quad \text{for} \quad j\neq n \qquad \Gamma_n = 0$$

We suppose at first that u=0.

The formulation of $\hat{\theta}$ now depends on the position of the control stop i:

- For i<n we have:

$$3n\hat{\theta}_i^a = (1-p)p_{dc}i\ x_i + \frac{1}{2H}\sum_{\substack{j=i \\ j\neq n}}^{2n}(2pp_{cd})(x_i^2-L_i x_i(1+p_{cd}))+\frac{n}{2H}(2pp_{cd})(x_i^2-L_i x_i(1+p_{cd}))$$

which gives:

$$3n\hat{\theta}_i^a = (1-p)p_{dc}i\ x_i + \frac{3n-i}{2H}(2pp_{cd})(x_i^2-L_i x_i(1+p_{cd}))$$

- For i=n we have:

$$3n\hat{\theta}_n = \frac{1}{2H}(\sum_{j=n}^{2n}(2pp_{cd})(x_i^2-L_i x_i(1+p_{cd})) + \frac{n}{2H}(2pp_{cd})(x_i^2-L_i x_i(1+p_{cd})$$

$$= \frac{2n}{2H}(2pp_{cd})(x_i^2-L_i x_i(1+p_{cd}))$$

- and for i>n we have:

$$3n\hat{\theta}_i^b = (1-p)p_{dc}i\ x_i + \frac{2n-i}{2H}(x_i^2-L_i x_i(1+p_{cd}))(2pp_{cd}).$$

We are now, as usual, looking for $\min_{i} \min_{x_i} \hat{\theta}_i(x_i)$

First we can easily see that the control has to be operated at $i \leqslant n$. Indeed, using the results of part III.2.a, we see that:

$$3n\hat{\theta}_i^b = \frac{[(1-p)p_{dc}i - 2pp_{cd}\frac{1}{2H}(2n-i)L_i(1+p_{cd})]^2}{\frac{2}{H}(2n-i)2pp_{cd}}$$

and we proved in this part that this function of i was decreasing with i as soon as i was superior or equal to $\frac{2n}{2} = n$.

* To find the optimal value of i, we now have to compare:

$$-3n\hat{\theta}_n^* = +(\frac{2n}{2H})(2pp_{cd})\frac{1}{4}(L_n(1+p_{cd}))^2$$

and the maximum for $i<n$ of the function:

$$-3n\hat{\theta}_i^{*a} = +\frac{[(1-p)p_{dc}i - 2pp_{cd}\frac{1}{2H}(3n-i)(L_i(1+p_{cd})]^2}{\frac{2}{H}(3n-i)2pp_{cd}}$$

We are now going to use $L_i = \sqrt{\frac{mi}{2pp_{cd}}}$ and operate the same kind of transformation as we did before.

We have:

$$3n\hat{\theta}_i^{*a} = \frac{\frac{i(3n-i)}{4H^2}(2pp_{cd})}{\frac{2}{H}(2pp_{cd})}[\sqrt{m}(1+p_{cd}) - \frac{(1-p)p_{dc}\sqrt{i}}{(3n-i)\sqrt{2pp_{cd}}}]$$

and we must not forget that:

$$\sqrt{m}(1+p_{cd}) - \frac{(1-p)p_{dc}\sqrt{i}}{(3n-i)\sqrt{2pp_{cd}}} \geqslant 0,$$

which is a necessary condition for operating a control at stop i.

$$- \quad 3n\hat{\theta}_n^* = \frac{mn^2(1+p_{cd})^2}{4H}.$$

There is no sign condition here since everyone deboards the bus at stop n.

As for $i \leqslant n$, the following inequality holds: $i(3n-1) \leqslant 2n^2$, we now see that $(\forall i \leqslant n)$ $(-\hat{\theta}_i^{*a} \leqslant \hat{\theta}_n^*)$. Therefore, for such a line, the optimal holding point is the center of the line, whatever the value of m is.

In this case we can give an evaluation of the relative importance of the reduction in waiting time due to the control. Without control our generalized waiting time is:

$$\theta = \frac{H}{2} + \sum_{\substack{j=0 \\ j \neq n}}^{j=2n} \frac{1}{3n} \cdot \frac{1}{2H} \cdot mj + \frac{1}{3.2H} \cdot mn \simeq \frac{H}{2} + \frac{5nm}{6H}$$

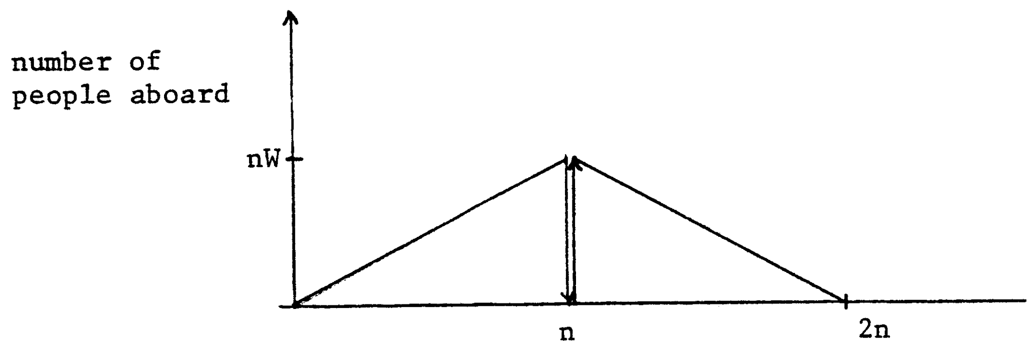
The reduction due to control is $\hat{\theta}_n^* = \dfrac{mn(1+p_{cd})^2}{12H}$.

If we consider, for example, the same values of the parameters as in example III.2.a, we find:

$$\theta = 0.5H + 0.33H; \quad \hat{\theta}_n^* = 0.08H.$$

Therefore, the relative reduction of θ due to control is about 10%, which is far from being negligible.

A very similar loading profile is the following:



The computation for this example is roughly the same. First we see that the control has to be operated for i<n and then that i=n is the best point. Here the holding point is still the middle of the line.

This is a very realistic model since a line through the center of a city can often be modeled with some accuracy in this way: on a shopping day passengers go to shop downtown, coming from the suburbs, and about the same number go back home, after having done their shopping. This is especially the kind of loading profile one can have in Boston on the Red Line; the singular point being Washington. Barnett, who applied his method to this particular line, decided directly that the best holding point was this station, which indeed our results tend to support.

A few comments can be added to this example:

* The fact that nW people board at stop n is very important. If this number were higher, the result would not be modified. If this number were smaller, the result would be modified eventually under a lower bound and the optimal holding point would shift towards the left.

* If we want to introduce the loading time at the stops, the same kind of modification appears as in example III.2.b. The analogy between the computation shows that the control point cannot be after the middle of the line. The fact of introducing dwelling times having the effect of shifting the optimal point to the right, the result will still hold: The optimal holding point is still the middle of the line.

III.2.d Common Origin

Indeed, this is the simpler model and does not need any computation. If no one boards the bus after the first stop, but people only deboard, then clearly the control has to be operated at the starter; since afterwards it will only delay some passengers and not reduce the waiting time of anyone.

CHAPTER IV

A CASE STUDY: THE HARVARD-DUDLEY BUS LINE

In this chapter we are going to consider a particular bus line: the Harvard-Dudley bus line, which runs between Cambridge and Boston. We shall discuss the hypothesis, strategies and results of Chapters II and III of this thesis. We shall also make some recommendations for the improvement of the service on this line.

IV.1 Present Operations

The present line is characterized by 31 stops northbound and 28 stops southbound. During the afternoon hours, six of these stops, each way, have a demand level higher than 10 passengers per bus (average number of passengers boarding and deboarding at a stop). On March 5th, the measured demands at these stops were: Harvard (28), Dudley (21), Auditorium (20), Huntington (13), Central (13) and MIT (12) for northbound trips. Fifteen stops have a demand level lower than three passengers per bus: demand averaged over all stops being approximately six passengers per bus.

A regression analysis performed on a sample of 150 dwell times measured at different stops during different days gave:

$T(\text{seconds}) = 16.5 + 2.5W$ where T is the dwelling time at a stop and W the number of passengers boarding at that stop.

At the low demand stops, the multiplicative effect on the dwelling time defined by $u = (1+2bk)^2-1$ is approximately: $\frac{4 \times .8 \times .25}{360} = 0.02$.

For the high demand stops, where the average number of passengers boarding is 9, $u = 0.26$.

If we reduce our line only to the important stops, as there are, on the average, six low demand stops between two of them, we shall consider that a good approximation of u is $u = (1.26)(1.02)^6-1 = .42$.

There are 32 traffic lights northbound and 26 southbound. During the afternoon hours (3PM-6PM) these lights have a mean total cycle length of 85 seconds, the green phase being approximately 64% of this cycle time.

The route is characterized by a highly unreliable level of service as is shown by the following figures:

- An average 15% of the buses are bunched at the MIT stop.
- On a weekday afternoon, the standard deviation of the headway of buses arriving at Harvard is nearly equal to the mean headway.
- Even at Dudley, after dispatching, this deviation is far from being zero.
- There is, especially at the MIT stop, a capacity problem: on March 5, between 3 and 6PM, 146 passengers boarded the first bus which came by, while 76 had to wait for another one at MIT.

Some reasons for the existence of such unreliability can be easily seen:

- Ineffective passing policy.
- Insufficient number of buses, combined with the fact that often some buses are removed from the line to serve elsewhere.
- Over ambitious scheduling (some days less than 50% of the trips are run in the scheduled time.)

- Little use made of the rear door.
- Randomness in passenger dwell times.
- Link delays.
- Existence of only one external dispatch point: at Harvard the drivers dispatch themselves, resulting in the fact that the headway variance of buses departing Harvard is sometimes almost as high as for arriving buses.

We are now going to consider some possible ways of reducing this unreliability by suppressing the different causes mentioned above.

IV.2 Improving the Reliability of Service

IV.2.a Passing Policy

A previous discussion already led to the conclusion that a bunched situation is stable and that no passing policy can really affect it. This is confirmed by the data collected on the line, on different days and times. Most of the buses which were bunched at a point, remained bunched down the line, even when some passing occured. (97% with no passing, 70% with passing).

On the average, more than 20% of the buses are bunched during peak hours on the northbound trip. To reduce this bunching, we could let the second bus of a bunched couple pass the first one. The second bus would spend less time at the stations to pick up passengers. This would make the two buses stick together, but could reduce the capacity problems, since an empty bus would come ahead of a crowded one.

This would also increase the speed of the pair because of reduced dwell times on the less crowded bus leader.

A regression analysis performed on buses with standees or crowded conditions give T = 27 + 2.2W, while the dwelling time for less crowded buses was found to be T = 12 + 2.5W. Therefore, this strategy, which seems to be useful, should be implemented as often as possible.

IV.2.b Increasing the Number of Buses

Except at the MIT stop, there are no real capacity problems. Very few passengers are generally left behind. Therefore, we can apply, for each stop, other than MIT, the theory developed in Chapter II with the parameters n_o, h_o and σ measured on March 5th. We shall suppose that at MIT, even bunched buses are useful.

The following table gives the measured bunching probabilities (B_{meas}) at four stops and the same probabilities computed with the theory developed in Chapter II: ($B=\phi \int_{h_o/\sigma\sqrt{2}}^{\infty} N(t,0,1)dt$) in three different cases. In the real case (B_o) with n_o=10 buses operating, with 12 (B_1) and 15 (B_2) buses operating. The weighted importance of the stops, as described above, is also given in order to compute the improvement due to additional buses. The waiting time is computed with the relationship: $E(w) = t_o/(n+1)(1-B)$, and ϕ=2.

	Weight	B_{meas}	B_0	B_1	B_2	$\frac{E_0(w)}{t_o}$	$\frac{E_1(w)}{t_o}$	$\frac{E_2(w)}{t_o}$
Dudley	.27	.16	.24	.30	.40	.120	.110	.104
Aud. & Hunt.	.41	.30	.38	.45	.54	.147	.140	.136
MIT	.16					.100	.083	.067
Central	.15	.40	.45	.51	.60	.165	.159	.156
Total	1.00					.133	.124	.118

We see that in all the cases, B_{meas} is smaller than B_0.

A part of the deviation from our model can probably be explained by the fact that the drivers generally try to reduce the bunching. They leave a 2 minutes headway instead of running exactly at the same time, when bunching begins to occur. It is difficult to introduce this psychological factor into our model.

However, our results are acceptable and give a good order of magnitude. Our model shows that adding two buses to the fleet would reduce the average waiting time by about 7% on the average, while adding 5 buses would lead to a reduction of about 11%.

Even if we consider that our estimation is too pessimistic by about 25%, we see that bunching is a real problem. Adding 20% (50%) of buses to the fleet only reduces the waiting time by 8% (13%) when the optimal value would be 17% (35%). On the southbound trips the effectiveness of added buses would be even lower.

This induces me to think that we are at the point where the benefits of adding new buses begin to be low compared to the costs. Therefore, as long as no capacity problems appear at stops other than MIT, I would not strongly recommend adding new buses.

On the other hand, reducing the number of buses from 10 to 9 or 8 has relatively more importance on the average waiting time. Thus, I think that switching some buses from this line to other lines should be viewed with caution.

IV.2.c Improving the Schedules

Since many buses are unable to perform the round trip in the scheduled time (70 minutes, including layover) it would be wise to change the schedules in order to reflect more accurately the trip length.

Figure 2, which is based on the survey of 30 trips (travel times only) suggests possible changes.

If a bus starts at t=0 from Dudley, then it must leave Harvard at t=43 minutes, whenever possible, (average waiting time 7 minutes) and leave Dudley again at t=80 minutes (same average waiting time). The round trip time would then be 80 minutes and more than 90% of the buses should be able to fall in this time. It is interesting to note that these times are clearly breakpoints on the diagram.

It is difficult to give an evaluation of the time which would be gained (if any) by this very simple action, but among other results, it should strongly reduce the headway variances at Dudley and Harvard and allow the passengers to better forecast their travel time.

IV.2.d Variance Along the Line

Before applying the method developed in Chapter III, we are going to see whether the relationship which we derived for the evolution of the headway variance along the line applies here or not.

On March 5th, the variance of the headway distributions, during the afternoon hours, were measured at five stops for buses operating northbound. The results were:

ONE WAY TRAVEL TIME

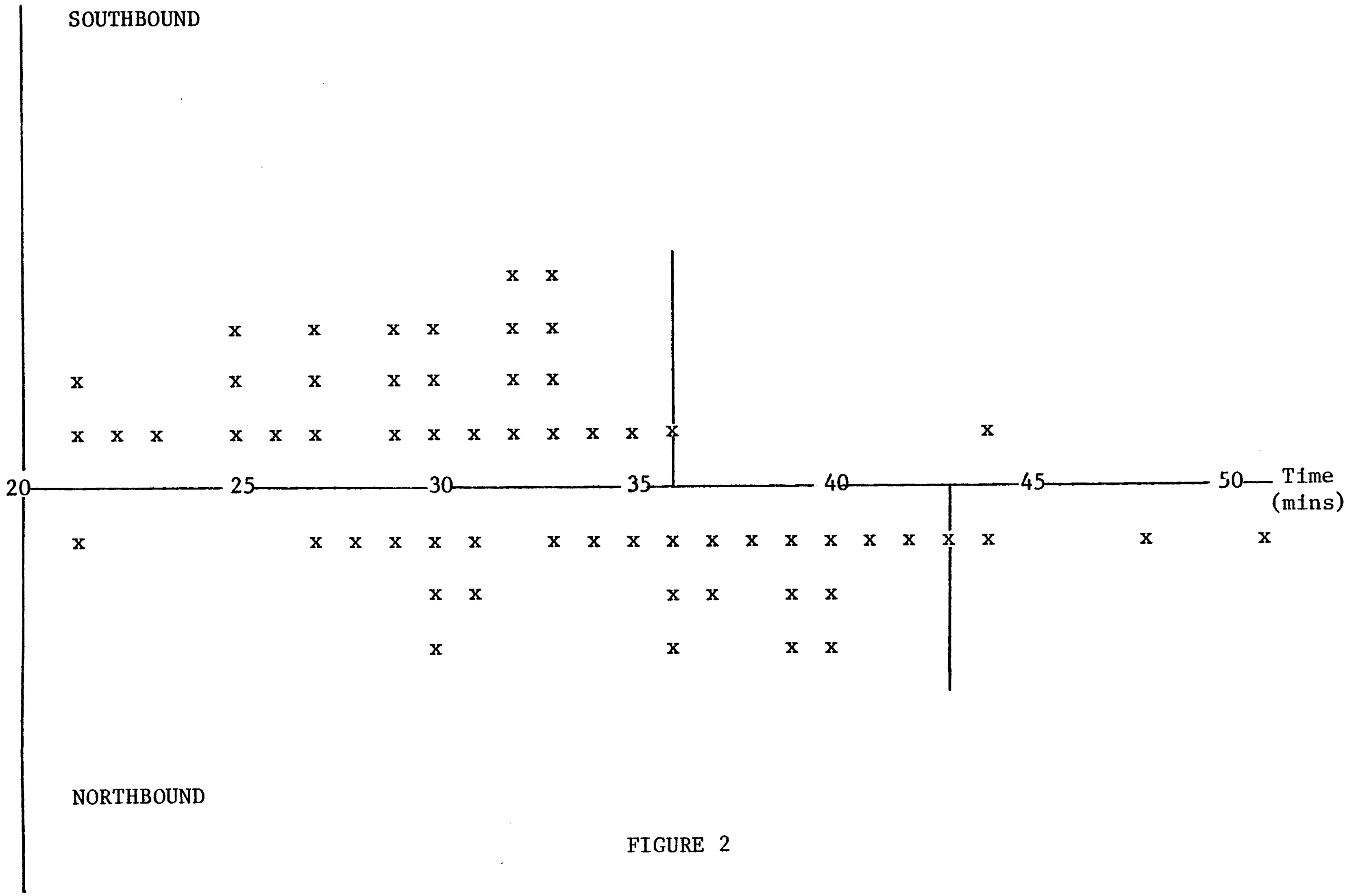


FIGURE 2

Dudley	$s^2 = 2\sigma^2 = 12\ min^2$	(stop 0, departing)
Auditorium	$s^2 = 24\ min^2$	(stop 1, arriving)
MIT	$s^2 = 25\ min^2$	(stop 2, arriving)
Central	$s^2 = 31\ min^2$	(stop 3, arriving)
Harvard	$s^2 = 36\ min^2$	(stop 4, arriving)

As there are, on the average, 6 traffic lights between each of these stops, the increase in variance between two stops, due only to traffic lights, computed as in Section II.1.e, is $m = 0.60\ min^2$.

The relationship:

$$s^2(i) = \frac{1}{1+u}[s^2(0)(1+u)^i + \frac{m(1+u)}{u}((1+u)^i)]$$

$$= s^2(0)(1+u)^{i-1} + \frac{m}{u}((1+u)^i-1)$$

would give the follwing results for the headway variances of arriving buses (except at i=0 where $s^2(0)$ is the variance of departing buses):

Dudley	$s^2(0) = 12\ min^2$
Auditorium	$s^2(1) = 12.60\ min^2$
MIT	$s^2(2) = 18.49\ min^2$
Central	$s^2(3) = 26.85\ min^2$
Harvard	$s^2(4) = 38.73\ min^2$

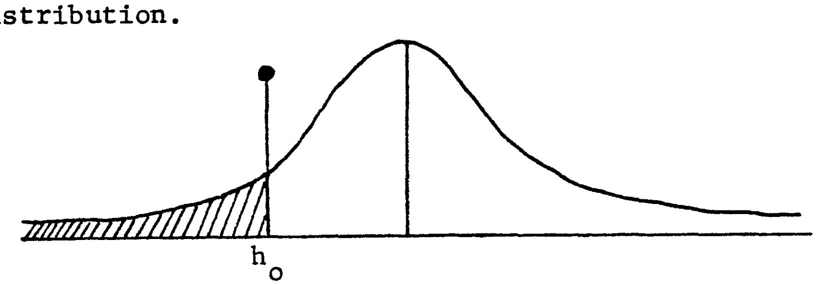
The relationship proves that, in fact, most of the variance comes from dwelling times. On this northbound trip, our model gives a rather good approximation of the evolution of the variance. It is normal to find a smaller variance than measured, since traffic lights are not the only sources of delays.

On the southbound trip, on March 6th, this variance was between 25 and 30 min^2 for each stop. The mean headway was 6 minutes. These results clearly contradict our model.

The explanation of this contradiction is easy to see: To derive the lateness distribution in arriving at a stop we used the central limit theorem. We found a normal distribution with a variance $\sigma^2(i)$ increasing linearly and geometrically along the line. Then we said that the headway distribution should be normally distributed with variance $s^2(i) = 2\sigma^2(i)$ and a non-discrete point for h=0.

Indeed, when P(h=0) gets relatively important, this result is distorted and does not hold anymore. This is the case when s is close to, or above value $\frac{h_o}{2}$.

We are going to compute the real mean h' and variance s'^2 of our headway distribution.



We shall call $u(x) = \frac{1}{s\sqrt{2\pi}} e^{-\frac{(x-h_o)^2}{2s^2}}$

The following relationships hold: $\frac{B}{2} = \int_{-\infty}^{0} u(x)\,dx$

$$1 = \int_{-\infty}^{\infty} u(x)\,dx$$

$$(x-h_o)u(x) = -s^2\, u'(x)$$

We have:

$$h' = 0 + \int_o^\infty x\ u(x)\,dx$$

$$= \int_o^\infty (x-h_o)u(x)\,dx + \int_o^\infty h_o u(x)\,dx$$

$$= \int_o^\infty -s^2 u'(x)\,dx + h_o(1-\frac{B}{2})$$

Therefore:

$$h' = [-s^2\ u(x)]_o^\infty + h_o(1 - \frac{B}{2})$$

$$h' = \frac{s}{\sqrt{2\pi}} e^{-\frac{h_o^2}{2s^2}} + h_o(1 - \frac{B}{2})$$

We also have:

$$s'^2 = \frac{B}{2}\, h'^2 + \int_o^\infty (x-h')u(x)\,dx$$

$$= \frac{B}{2}\, h'^2 + \int_o^\infty (x-h_o+h_o-h')^2\ u(x)\,dx$$

By developing the square inside the integral, we have three terms to compute:

* $\int_o^\infty (h_o-h')^2 u(x)\,dx = (h_o-h')^2(1 - \frac{B}{2})$

* $2\int_o^\infty (x-h_o)(h_o-h')u(x)\,dx = 2(h_o-h')\ \frac{s}{\sqrt{2\pi}}\ e^{-\frac{h_o^2}{2s^2}}$

* $\int_o^\infty (x-h_o)^2 u(x)\,dx = \int_o^\infty -(x-h_o)s^2\ u'(x)\,dx.$

We integrate by parts. This integral is also:

$$[-(x-h_o)s^2\ u(x)]_o^\infty + \int_o^\infty s^2\ u(x)\,dx$$

$$= -\frac{s\,h_o}{\sqrt{2\pi}}\, e^{-\frac{h_o^2}{2s^2}} + s^2(1-\frac{B}{2})$$

Therefore:

$$s'^2 = (1-\frac{B}{2})[(h_o-h')^2 + s^2] + \frac{s}{\sqrt{2\pi}}\, e^{-\frac{h_o^2}{2s^2}}(h_o-2h') + \frac{B}{2}\, h'^2$$

For large s we have $B \to 1$ and $h' \sim \frac{s}{\sqrt{2\pi}}$. We also have:

$$s'^2 \sim \frac{s^2}{2}[+\frac{2}{2\pi} + 1] - \frac{2s^2}{2\pi} = \frac{s^2}{2}(1-\frac{1}{\pi})$$

and:

$$\frac{s'}{h'} \sim \sqrt{\pi - 1} \sim 1.46.$$

Therefore, the ratio $\frac{s'}{h'}$ is limited.

As s' and h' are, in fact, the two parameters which we measure, we should not expect them to be such that $s'/h' > 1.46$.

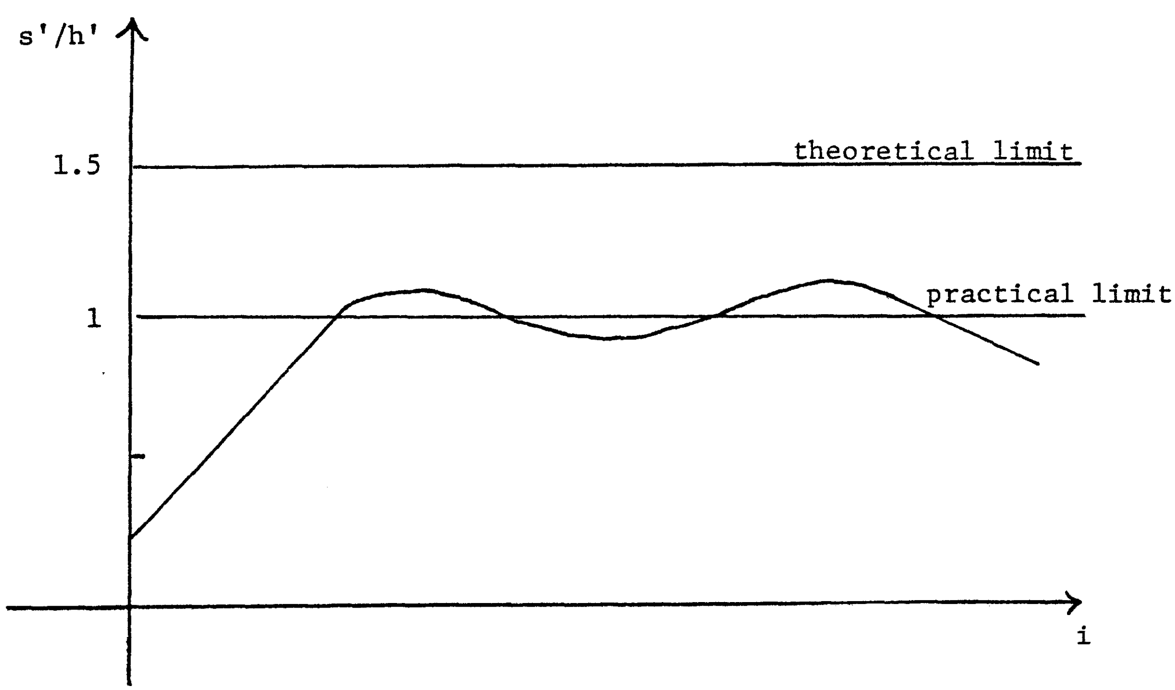
On the Harvard-Dudley bus line, we have for arriving buses, during the afternoon, the following values of s'/h':

	HARVARD	DUDLEY
March 5	0.80	0.95
March 21	0.93	1.03
March 22	1.04	1.19

As these ratios are relatively high, it seems normal not to find a linear and geometric progression for the evolution of the variance along the line.

However, we can perform an empirical study: On three weekday afternoons, the standard deviation along the line increased very quickly from Dudley to Auditorium and from Harvard to Central, and remained approximately constant, with some variations up and down, as soon as it reached the level 0.8h'. This suggests the following hypothesis:

If at a busy stop the standard deviation is reduced to less than 0.5h', then at the next busy stop it will be back in the range 0.8h'-1.2h'. The eventual reduction of variance does not propagate along the line, but there is also an upper limit to the instability of the system. We found a theoretical 1.5h' and practically it seems to be around 1.2h'. The use of a lognormal distribution would probably have led to another limit.



IV.2.e Introducing a Control Point

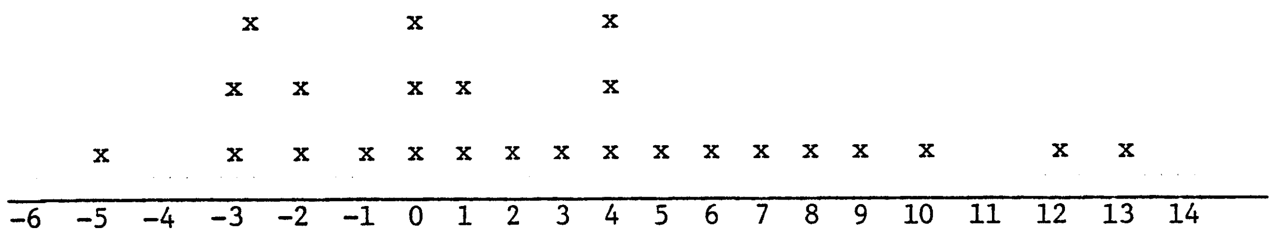
The previous section shows that we cannot directly apply the model developed in Chapter III to find the best holding point and the reduction in waiting time, which we can expect from a control strategy. We are going to perform a study based on the conclusion of IV.2.d and the data of March 5th, which is represented in the following diagrams.

If we apply Barnett's control strategy, the only point where some reduction in waiting time can be expected is the holding point. As we assume that the variance is about the same all along the line, the best control point is clearly the point at which the trade-off between delayed people and people waiting to board is optimal.

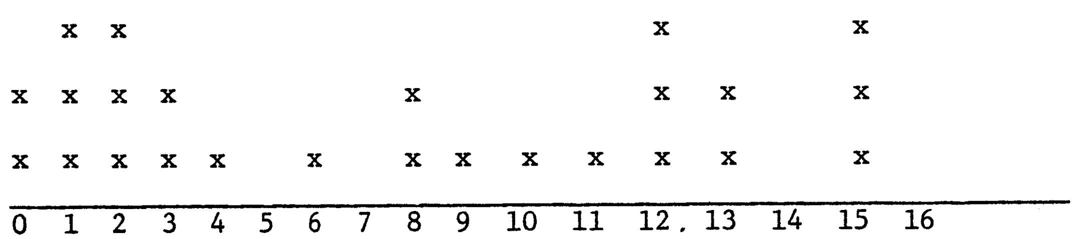
On March 5th, the following numbers of passengers got on and off the buses during the afternoon hours:

	OFF	ON
Dudley		500
Auditorium	80	340
MIT	70	200
Central	280	150
Harvard	750	570
Central	180	50
MIT	30	210
Auditorium	110	200
Dudley	700	

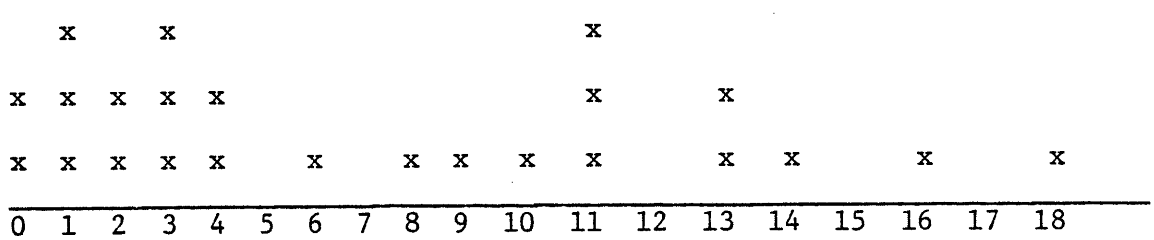
BUS ARRIVAL TIME DISTRIBUTIONS



Distribution of Lateness Arriving at Harvard



Distribution of Headway Arriving at Harvard



Distribution of Headway Arriving at MIT

FIGURE 3

With no doubt, the optimal control point is Harvard, where no one is delayed and where a large number of passengers are waiting to board.

It is interesting to note that, with such a profile, our model would have given the same optimal control point in the case of a linear increase of the variance.

In order to define a control strategy, we must fit a two-point distribution with the headway distribution at Harvard. On Figure 3 above, we can see that such a distribution is realistic for both Harvard and MIT.

For Harvard we decide to choose the breakpoint at 6 minutes. The sequence of headway during the afternoon was (in minutes):

8, 3, 15, 1, 2, 15, 1, 13, 2, 11, 0, 10, 4, 12, 2, 15, 9, 6, 12, 13, 2, 2, 8, 1, 3.

This gives:

$$\begin{cases} p = .50 \\ p_{cd} = .75 \\ p_{dc} = .83 \end{cases}$$

The same survey at MIT would give $p = .5$; $p_{cd} = .73$; $p_{dc} = .75$. These results would tend to justify, even on this chaotic line, our hypothesis on the constance of p, p_{cd} and p_{dc} along the line.

In order to find c, d and L we use a moment method. The equations are:

$$\begin{cases} 0.80 = .5(c+d) \\ 28 = .5(c^2+d^2) \end{cases}$$

We thus have: $c = -2.9$; $d = 4.5$; $L = 7.4 > a = 6.8$. Here again, because of the high variance we are not in the usual case of application of Barnett's method, which supposes $L < a$. However, Chalstrom and Terziev[2] constructed an extension of Barnett's method for this case.

At Harvard we can take $\gamma=0$, since no one is delayed. The objective function computed in this case is:

$$\theta = \frac{E(H^2)}{2E(H)} = \frac{a^2+2pp_{cd}(L^2+x^2+Lx-2ax-Lxp_{cd})}{a+2pp_{cd}(L-a)}$$

The control time x^* is given by $\frac{\partial\theta}{\partial x}(x^*)=0$.

$$x^* = a - \frac{L(1-p_{cd})}{2}$$

Using the parameters computed above, we have $x^* = 5.8$, and the relative improvement of the waiting time at Harvard is:

$$1 - \frac{\theta(n^*)}{\theta(0)} = 1 - \frac{61}{87} \simeq 30\%$$

due to a variance reduction of 63%.

We see that this improvement is considerable, due to the fact that nobody is on board passing through this stop. This variance reduction must be compared to the present one due to self-dispatching which is about 35%.

As we assumed that this improvement was not propagating along the line, the reduction in waiting time for all the users of this route is about a fifth of this, since users at Harvard represent about a fifth of all users. This would lead to a global improvement of 3%.

I therefore strongly recommend the application of Barnett's holding point method at Harvard. This method is simple and should be effective.

If we had to choose another point, the control would not be so effective: There are at every stop, except Harvard, as many passengers on board (or more) as passengers waiting to board. The number of people benefitted is also rather small since the decrease of the variance at one stop does not propagate far down the line.

The relative improvement would not be more than 10% at the stop and probably less than 1% altogether. Therefore, I do not recommend more than one control point.

CHAPTER V

CONCLUSION

V.1 Summary

In this thesis, we have investigated some strategies that can be used in order to improve service reliability in urban transit systems.

In the first part, using a very simple model based on the assumption that the arrival time of a bus at a stop is normally distributed, we computed the probability of bunching on a bus route. Using this result, we proved that adding a large number of buses on an existing line does not necessarily significantly improve the level of service. This conclusion is interesting since it goes against a commonly admitted idea: the first solution one would consider in order to reduce the passengers' waiting time is to add new buses. Unfortunately, this is always expensive and is not always very effective.

Next we investigated another strategy based on the selective holding of buses at a point on a route. Assuming that the headway variance has a mixed arithmetic and geometric propagation along the line, we found, for some simple line configurations, the best point at which to apply such a strategy. Theoretically, the implementation of such a strategy at the optimal point can, in some cases, improve by 10% or more the value of an objective function which takes into account both the waiting time of passengers and the delays inflicted on those who are on board.

Finally, we tried to apply these theories to a real case: the Harvard-Dudley bus line. We found that apparently the theory developed to predict bunching did accurately reflect the phenomenon observed.

Unfortunately, the observed variance pattern along the route did not agree with the theoretical predictions. However, the use of our model based on a normal headway distribution, together with a more sophisticated approach in the case of large variance, led us to an interesting result: in fact, there is an asymptotic limit to the headway standard deviation which is about fifty percent above the mean headway. So, the variance cannot have the previously estimated evolution, except for small values. Taking into account this result, we saw that the best holding point was Harvard, and that we could expect an overall reduction in waiting time of about 3% by implementing just this strategy.

V.2 Recommended Future Work

Indeed, some further research would be interesting.

- We never introduced the capacity of buses as a factor of strategy choice; it would be interesting to introduce it in an analytical model dealing with this problem. This should reduce the adverse effects of bunching on waiting time.

- We suggested the implementation of a control strategy and the improvement of scheduling at the same time, but we did not try to compute the benefits which would occur from better scheduling. It would be interesting to find out whether it would be worth implementing both strategies and, if not, which one should be selected.

- The limitation which was found for the standard deviation does not seem to be equal to the one observed. It would be interesting to know why, and to find some heuristics to explain the existence of such limits.

- Our method for finding the best control point was not used in the case study because of the extreme unreliability of the route. It would be interesting to try to apply this method to a more reliable route, for example a subway.

- Calculus and analytical approachs are efficient to deal with simple problems and can give good models, but obviously they are not powerful enough to allow in-depth investigations. More data and simulation models are needed to better test our theories.

APPENDIX: GLOSSARY

$E(\xi)$: mean value of possible delays

$\sigma^2(\xi)$: variance of possible delays

t: time since the bus departed

n: number of delays encountered per unit of time

$N=nt$: number of delays encountered after time t

$\sigma^2=nt\sigma^2(\xi)$: variance of the lateness distribution

$s^2=2\sigma^2$: variance of the headway distribution

h_o: mean headway

B: probability of bunching of one bus

ϕ: correcting parameter

T: dwelling time at a stop

W, W_i number of passengers boarding at a stop; at stop i

a,b: dwelling time parameters

k: number of passengers arriving per unit of time

L: distance between two stops

V: average speed of a bus

$m = \frac{2nL}{V}\sigma^2(\xi)$

$u = (1+2bk)^2-1$

$\sigma^2(i)$: variance of the lateness distribution after stop i

$s^2(i)$: variance of the headway distribution after stop i

B_i: probability of bunching after i stops

A: given point on a line

p, q, τ: characteristics of traffic lights

λ_o: smallest λ such that $h_o - \lambda\tau < 0$

t_i: time at which the ith bus arrives at A

$E(w)$: expected waiting time of a passenger at A

ℓ: length of a loop

$t_o = \frac{\ell}{V}$

$B(i)$: probability that i bunches occured at A

n_o, n initial and actual number of buses

$E(d)$: expected delay to passengers

$\gamma, \gamma_j, \Gamma_i$: weighting parameters

D_i: number of people delayed if bus is held at stop i

θ: objective function

N': total average number of people waiting on the line

$\overline{H}$: headway

H: mean headway

$\text{Var}\, C_{i,j}(\overline{H})$: headway variance at stop j if we exert control at stop i

$\text{Var}_j(\overline{H})$: headway variance at stop j without control

$\hat{\theta}$: reduced objective function

c_i: lateness of a bus which is relatively early at i

d_i: lateness of a bus which is relatively late at i

p_i: probability of an early bus at i

$L_i = d_i - c_i$

P_{cdi}: probability of a late bus given that the preceding was early

P_{dci}: defined in a similar manner

P, P_{cd}, P_{dc}: same parameters as above, but do not depend on i

x, x_i optimal holding times

A, B, f, g, ℓ: parameters and functions used to simplify the notations

h': real mean of the headway distribution

s'^2: real variance of the headway distribution

a: mean headway at Harvard

REFERENCES

1. BARNETT, A., "On Controlling Randomness in Transit Operations", (TRANSPORTATION SCIENCE, May 1974).

2. CHALSTROM, BROWNELL and TERZIEV, "Control Strategies for Reducing Randomness in Bus Operations" (MIT, May 1975).

3. CHAPMAN, GAULT and JENKINS, "The Operation of Urban Bus Routes", (TRAFFIC ENGINEERING AND CONTROL, June 1977).

4. KULASH, D., "Routing and Scheduling in Public Transit Systems", (MIT, Phd Thesis, 1971).

5. NEWELL, G. F., "Control of Pairing of Vehicles on a Public Transportation Route", (TRANSPORTATION SCIENCE, August 1974).

6. TURNQUIST, M. A., "Measures of Reliability for Urban Transit Systems: User Perspective", (TRANSPORTATION SCIENCE, November 1976).

7. TURNQUIST, M. A., "A Model for Investigating the Effects of Service Frequencies and Reliability on Bus Passengers' Waiting Time"

8. TSC Urban and Regional Series, "Transit Service Reliability" (Final Report, December 1978).

9. WELDING, P. I., "The Instability of a Close Interval Service", (OPERATION RESEARCH QUARTERLY (1957), 8(3)).