AN INVESTIGATION OF METHODS AVAILABLE FOR INDICATING THE DIRECTION OF THE VERTICAL FROM MOVING BASES

By

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1941

Signature of Author ...........................................

Department of Physics, December 9, 1940

Signatures of Professors
in Charge of Research ........................................

Signature of Chairman of Department
Committee on Graduate Students ............................
December 9, 1940

Professor George W. Swett
Secretary of the Faculty
Massachusetts Institute of Technology
Cambridge, Massachusetts

Dear Professor Swett:

In accordance with the regulations of the faculty, I hereby submit a thesis entitled "An Investigation of Methods Available for Indicating the Direction of the Vertical from Moving Bases" in partial fulfillment of the degree of Doctor of Science in Applied Physics.

Sincerely,

Walter Wrigley
Walter Wrigley was born at Brockton, Massachusetts on March 26, 1913. He attended the public schools in Haverhill, Massachusetts, graduating from the Haverhill High School in 1930. In the summer of 1930 he was the representative of the State of Massachusetts in the second Thomas A. Edison national scholarship contest. He entered the Massachusetts Institute of Technology in the fall of 1930. As an undergraduate at M. I. T. he studied Physics with the major interest in optics, receiving the degree of Bachelor of Science in Physics in June 1934. From 1934 to 1937 he was employed as a research physicist by the Interchemical Corporation, first in their research laboratories in New York City and then in their subsidiary, the United Color and Pigment Company, of Newark, New Jersey. In the fall of 1937 he returned to M. I. T. for graduate study and was an assistant in Aeronautical Engineering from 1938 to 1940. He is now employed as an assistant project engineer by the Sperry Gyroscope Company of Brooklyn, New York.
ABSTRACT

A plumb bob consisting of a weight supported by a flexible string will serve as an instrument to show the direction of the vertical, if the point of support is stationary with respect to the Earth. However, if an indication of the vertical is required from an instrument carried on a base which is moving in an arbitrary fashion with respect to the Earth, a simple pendulum is no longer a satisfactory instrument. The essential difference between conditions on stationary and moving bases lies in the horizontal acceleration components produced on the moving base by the motion of the base. Such horizontal acceleration components do not exist on a stationary base.

The present thesis presents a discussion of the principles which can be used in equipment for indicating the vertical on a moving base. An analysis is carried out to determine the physical nature of such accelerations as may be encountered by an airplane flying over the Earth. The analysis then extends to an examination of the possibilities and limitations of a pendulum for overcoming the difficulties of indicating the vertical from a moving base. A similar analysis is carried out for the case of a gyroscope coupled to a pendulum. The motion of gyroscopes is treated by vac-
tor methods rather than the classical procedure. This vector method permits a quick and simplified treatment for predicting the behavior of a gyroscope under applied torques that can be used to advantage in practice.

Instruments for indicating the vertical from moving bases vary widely in their physical characteristics, but practical indicators of the vertical invariably consist of a self-contained system that is responsive to the resultant acceleration of the base carrying the instrument. The resultant acceleration of an instrument carried on a moving base has two physical properties on which instruments for indicating the vertical may function, namely magnitude and direction.

The magnitude of the resultant acceleration does not lend itself to practical determinations of the vertical. This is due to the fact that the magnitude of the resultant acceleration is but very slightly different from the magnitude of the acceleration of gravity when the two accelerations are almost coincident. For this reason, it is very difficult in practice to determine accurately small angular deviations from the vertical -- the vertical being defined as the direction of the acceleration of gravity. Furthermore the magnitude of the resultant acceleration is only capable of determining the angle between the acceleration of gravity and the resultant acceleration. This places the direction of the acceleration of gravity somewhere in a cone having the direction
of the resultant acceleration as its axis. Other equipment, e.g. a turn indicator, is required to determine just which generator of the cone is the vertical.

The direction of the resultant acceleration may be used to actuate practical indicators of the vertical, since the acceleration of gravity is the only acceleration which remains substantially constant in magnitude and direction in any region on the Earth. It follows that the average direction of the resultant acceleration will be the direction of the acceleration of gravity, and that self-contained instruments for indicating the vertical must contain an averaging device.

A self-contained instrument for indicating the vertical must contain a pendulous element to pick out the direction of the resultant acceleration. In addition any instruments to be used on a moving base must have an indicating system that will show the average orientation of the pendulous element. Indicators of the vertical then fall into two basic types. The first type is the long period pendulum in which the pendulous element and the averaging system are identical. In this class of instrument the pendulum tends to remain along the vertical. The other type is a short period pendulum coupled to a system which gives average indications of the position of the pendulum. The averaging is accomplished by mechanical, hydraulic, or electrical means. In this
class of instruments the pendulum tends to remain along the resultant acceleration.

A damped (long period) pendulum may be expected to give a satisfactory approximate indication of the vertical for cases in which the direction of the resultant acceleration differs from that of the vertical for periods of less than about five seconds. To be a satisfactory indicator of the vertical under such conditions a pendulum must have an undamped natural period of oscillation of from about twenty to fifty seconds. A simple pendulum having these natural periods must be several hundred feet long. Such an instrument is obviously unsuited for practical use in an airplane. On the other hand if the mass is distributed to form a physical pendulum, the center of gravity of the pendulum must be of the order of a thousandth of an inch from the pivot when the instrument has an overall dimension of only a few inches. Such an instrument is within the province of a skilled machinist, and may be used as an indicator of the vertical in airplanes.

To obtain an instrument that is suitable for general use in aircraft the indicating system used with a short period pendulum to form an indicator of the vertical must be able to average the position of the pendulum. Methods suitable for an averaging indicating system are accomplished in three different ways. One method is to make the mass determining the disturbing effect of accelera-
tions much smaller than the mass determining the inertia of the system (in the case of the long period pendulum the two masses are identical). This method is represented by a system controlled by a loose ballistic; that is, a ball free to move about on a spherical race which is eccentric with the pivot and contained in a nonpendulous larger body supported by the pivot. A second method is to make effective mass determining the inertia of the system much larger than the actual mass of the system (and incidentally much larger than the mass determining the disturbing effect of the accelerations). This method is represented by a gyroscope in which the effective mass is made large by causing a rotor to spin rapidly. The third method is to use a system with a long response lag (or characteristic time) in electric, hydraulic, pneumatic, etc., systems.

In general the construction difficulties probably would outweigh the improved performance as an indicator of the vertical in the case of a system controlled by a loose ballistic. This is due to the fact that the center of curvature of the race, in which the loose ballistic runs, must be machined and assembled to about one thousandth of an inch from the pivot in an instrument having overall dimensions of a few inches. In the case of the long period pendulum on the other hand, the center of gravity can be located close to the pivot after assembly by means of adjustable weights.
A pendulous gyro combines reasonably satisfactory performance as an indicator of the vertical with feasible mechanical properties. A pendulous gyro, however, possesses inherent characteristics undesirable in an instrument gyro, one of these being that a pendulous gyro precesses about the vertical and so erects; i.e. returns to the vertical, in a spiral path. In addition the rate of erection of a pendulous gyro depends on the damping which is generally small, whereas the rate at which a pendulous gyro leaves the vertical under the influence of horizontal accelerations is substantially independent of the damping. In rough air, then, a pendulous gyro would tend to leave the vertical more readily than it would tend to remain erect. An undamped pendulous gyro has no erecting means and hence is worthless as an indicator of the vertical. A gyroscope coupled by a servo-mechanism to a pendulum offers the most practical solution to the problem of indicating the vertical from moving bases. The looseness of coupling possible between the pendulum control and the gyroscope permits averaging of the pendulum's position so well that accuracy of indication of the vertical is possible without introducing prohibitive difficulties of construction. A well built gyroscopic indicator with servo-control can hold the vertical to within a few minutes of arc in the presence of rough air and typical airplane maneuvers.
The fundamental criterion for the return of a tilted gyroscope to the vertical is the automatic application of a force that is perpendicular to the displacement of the gyroscope from the vertical. In terms of torques this means that to return a tilted gyro to the vertical the erecting torque vector must lie in the vertical plane containing the spin axis of the gyro and be perpendicular to the spin axis. Many methods of erecting gyroscopes are available in practice, as is seen by a survey of the patent literature. However, all such methods, no matter how different they may appear to be superficially, must fulfill the criterion for erection given above.

It is immaterial from a practical viewpoint where an indicator of the vertical is located in an airplane. The instrument will indicate the vertical most accurately when located at the center of gravity of the airplane due to the absence there of linear acceleration effects from roll, pitch, and yaw. However, the slight improvement in performance will generally be overbalanced by considerations of convenience such as the desirability of placing indications of the vertical on the instrument board either by locating the complete instrument on the panel or by transmitting the readings to the panel.

The acceleration of Coriolis is of importance only when the vertical is to be determined to within a few minutes of arc. This is due to the fact that the shift in the apparent vertical from the true vertical due to Coriolis ef-
effects is only five minutes of arc for an airplane flying three hundred miles per hour in a latitude of forty-five degrees. Determinations of the vertical to a few minutes of arc are desired for celestial navigation, but are probably within the margin of error established by other factors for other purposes which require an accurate indication of the vertical.

It is desirable to say a word at this point about the method of treating forces used in this thesis. The point of view throughout is that of an observer located in a moving base, e.g. in an airplane flying completely blind, rather than the classical detached observer who is external to the whole system. The point of view is therefore subjective rather than objective. On this account the forces considered are the reaction forces of the body in question that are produced by accelerations of the pivot in the opposite direction. For example, in a turn, the pivot experiences a centripetal acceleration toward the center of the turn, while the pendulous element is then considered as experiencing a centrifugal force outward since the observer in the airplane is conscious only of the reaction effect. In other words, the concept of motion is that \( F = -ma \).

This thesis was carried out in conjunction with a research program for a commercial company. In view of this fact, the present international situation, and the subject matter of the thesis, the results that were considered to be of military or commercial value were not included. For this reason, the section dealing with practical aspects of indicators of the vertical that contain a gyroscope is rather sketchy.
The writer of this thesis wishes to acknowledge his gratitude and appreciation to the several persons who have rendered assistance, both material and otherwise, to the completion of this thesis.

To Professor C. S. Draper and Professor P. M. Morse for their continued inspiration and help during the complete project. To Professor J. A. Stratton for helpful suggestions and criticism, particularly in the theoretical aspects of the thesis.

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AN INVESTIGATION OF METHODS AVAILABLE FOR INDICATING
THE DIRECTION OF THE VERTICAL FROM MOVING BASES

OBJECT

The object of the work described in this thesis was to survey the theory and practice of instrumentation for indicating the direction of the vertical with particular attention to equipment designed for use on moving bases. This study has been divided into four parts:

a. A survey of the methods available for indicating the vertical from bases fixed relative to the Earth.

b. A survey of the difficulties introduced when an indication of the vertical is to be obtained on a base which is moving with respect to the Earth.

c. A survey of the methods available for overcoming the difficulties of indicating the vertical from moving bases.

d. A survey of the means which have been found useful in practice for indicating the vertical from moving bases.
INTRODUCTION

An airplane can maintain normal level flight only by creating, through its lifting surfaces, a force exactly equal in magnitude and opposite in direction to the force of gravity on the airplane, i.e. the weight of the airplane. Consequently in order to fly an airplane correctly it is essential to know at all times the direction of the force of gravity in order that the airplane's lifting surfaces may be kept in the proper relation to gravity. Even in maneuvers the direction of gravity is an essential item of knowledge to a pilot, since he must return to a reference plane determined by the direction of gravity, i.e. the horizontal plane, after the maneuver is completed.

Knowledge of the direction of gravity is essential to the flight of modern aircraft in several respects, for example:

1. As the normal determinant of a basic reference plane it permits manual flight in both contact and instrument flying.

2. The gravity-established reference plane in an automatic pilot system permits automatic or robot flight under all weather conditions.

3. In operations where an accurate reference to the Earth is required, e.g. aerial photography, mapping or bombing, it is essential to know the direction of the force of gravity.
4. In observations involving the measurement of the position of celestial bodies for navigation the basic plane to which the measurements are referred, i.e. the horizontal plane, must be known.

The simplest form of horizontal reference plane, the plane perpendicular to the direction of the force of gravity, is approximately determined as the plane including the airplane and the visual horizon. This method of determination, however, has three weaknesses:

1. The horizon must be visible, thereby limiting such determination to moderately clear weather only.

2. The observer must have continuous reference to it even when using instruments, such as cameras or sextants.

3. Angular measurements of reasonable accuracy are possible only in ideal flying conditions. Even then a dip error, due to the curvature of the Earth, should be taken into account when the airplane has more than a few feet altitude above the Earth, as is shown in Figure 1.

In order to provide a reliable and accurate vertical reference line, or more exactly a horizontal reference plane, several types of instruments have been devised. It is the purpose of this thesis to determine the essential requirements of a satisfactory indicator of the vertical, and to study mathematically the characteristics and performance of the various types of indicators available. The material will be presented in the form of curves suitable for engineering use in terms of dimensionless ratios that are charac-
DIP ANGLE — ANGLE BETWEEN THE HORIZONTAL PLANE AND THE VISUAL HORIZON — DUE TO CURVATURE OF THE EARTH AND ALTITUDE OF AN AIRPLANE ABOVE THE EARTH'S SURFACE.
teristic of the bodies being studied, in order to facilitate direct comparison between widely differing types of instruments.

Definition of the Vertical

Before studying the determination of the vertical reference line, or the horizontal reference plane perpendicular to the vertical line, whose importance to aircraft is shown above, it is necessary to define exactly what is meant by the term vertical. The vertical at a point on the Earth's surface is defined as the local direction of the force of gravity as indicated by a plumb bob hanging from a base that is stationary relative to the Earth. The horizontal plane is defined as the plane perpendicular to the vertical, or as the surface of a free liquid under the influence of gravity alone.

The force of gravity is not a simple force due to a single physical factor. In fact the force of gravity experienced by a body that is stationary on the surface of the Earth is a combination of at least two major forces and two minor forces. These forces are:

1. A force of mutual attraction, the force of universal gravitational attraction, between the Earth and the stationary body in question. The magnitude of this force is directly proportional to the product of the mass of the Earth and the mass of the body and is inversely proportional to the square of the distance between the Earth and the body. At
a first glance the latter condition appears to be indetermi-
nate. However, for an approximately spherical body such as
the Earth the force of attraction acts as if all the mass
were concentrated at the center of mass of the body, thereby
making the effective distance between the Earth and a body
on the surface of the Earth equal to the radius of the Earth.
It follows that a body on the surface of the Earth should be
attracted toward the center of the Earth.

2. A centrifugal force due to the daily rotation of the
Earth about its north-south axis. The magnitude of the
centrifugal force varies with the position of the body in
question on the Earth, being approximately proportional to
the cosine of the latitude, i.e. maximum at the equator and
zero at the poles. The direction of the centrifugal force
is always radially outward from the axis of the Earth's daily
rotation and in a plane parallel to the plane of the equator.
This force is the principal cause of the Earth's being
slightly oblate in shape rather than being a perfect sphere.

3. A force due to an angular acceleration of the Earth's
daily rotation is practically nonexistent, being due primari-
ly to the drag of the tides. A force due to a linear accel-
eration of the Earth as a whole in space is negligibly small
being due primarily to the centripetal acceleration of the
Earth's yearly orbit around the Sun.

The combination of attractive and centrifugal forces that
gives rise to the local force of gravity at any point on the
Earth's surface is shown in Figure 2 where are shown the re-
Fig. 2
CROSS-SECTION OF THE EARTH SHOWING FORCES ACTING TO PRODUCE THE DIRECTION OF THE VERTICAL
lations between the forces of attraction and centrifugal action, gravity, the vertical, the horizontal plane, and the spherical and oblate Earths.

The combination of attractive and centrifugal forces makes the resulting force of gravity fail to point toward the center of the Earth, except at the equator and poles. However, these same forces make the Earth into an oblate sphere, flattened at the poles, with the result that the force of gravity is approximately perpendicular to the general surface of the Earth (discounting such local irregularities as mountains, valleys, etc).

**Mechanism for Indicating the Direction of the Vertical**

Any practical self-contained instrument for indicating the direction of the vertical, no matter how complex it may appear to be, must contain a **pendulous element** as the fundamental indicator. A pendulous element is defined as a system free to rotate about a point and so constructed that it assumes a preferred orientation under the influence of the forces acting on it.

When the direction of the vertical is determined from a base that is stationary relative to the Earth, the measurements are extremely simple. A plumb bob will give as precise indications as can be desired; in fact the plumb bob is used to define the vertical. Measurements of this kind have been made for centuries in such ancient sciences as surveying, astronomy, and the construction of buildings.
When the measurements of the direction of the vertical are to be made by an instrument carried on a base that is moving with respect to the Earth, new complications arise because the motion of the base generally introduces horizontal acceleration components. A pendulous element will tend to indicate the direction of the resultant of all the forces acting on it rather than the direction of any force component. Now the effect of a force acting on a pendulous element is exactly the same as the effect that would be caused if the base carrying the pendulous element were accelerated in the opposite direction so far as the effects of rotation are concerned. Accordingly it is possible to replace the effects of a force by an equivalent acceleration of the proper magnitude and direction. This acceleration is parallel to the force but directed in the opposite sense.

The use of accelerations instead of forces in the study of the motion of a pendulous element carried on a moving base is advantageous since the accelerations are direct results of the maneuvers of the base whereas forces require also a knowledge of the masses involved. It is then possible to say that a pendulous element will tend to indicate the resultant direction of all the accelerations present. The fact that one of these acceleration components is due to gravity in no wise affects the situation. It is impossible to distinguish the acceleration of gravity from any linear acceleration by means of their instantaneous effects. This fact is one of the fundamental tenets of the general theory of relativity.  

*Superscript numbers in parentheses denote the number of a reference listed in the bibliography at the end of the text.
It is possible, however, to design an instrument able to distinguish between the acceleration of gravity and linear accelerations because gravity is the only acceleration that remains constant in both magnitude and direction at any point on the Earth. This characteristic indicates the desirability of so constructing a pendulous element that its response to accelerations is so slow that all accelerations except that of gravity have exerted their disturbing effect and disappeared before the pendulous element has had a chance to change its orientation very much. It is shown in this thesis that a slow-response pendulous element will average out the effect of all accelerations except that of gravity provided the element has certain physical characteristics. It is also shown that such necessary characteristics are not generally practical unless a gyroscope with its high effective inertia to mass ratio is included in the system.

Interest in instruments for indicating the direction of the vertical from moving bases has become important only in comparatively recent years. Until the advent of the high speed airplane the primary use for the indication of the vertical from a moving base was in the field of fire control on ships at sea. The earliest solution of this problem was a simple metal pendulum hung inside a metal ring which was parallel to the deck of the ship. When the tilt of the ship made the pendulum strike the ring, the resulting gong gave notice that the ship was in the proper attitude for firing a broadside. The use of guns whose elevation could be con-
trolled relative to the decks of a ship required the use of more complicated fire control mechanism.

The three dimensional freedom of motion, the high speed, and the necessity for a vertical reference when flying through thick weather all complicate the problem of identifying the direction of the force of gravity by means of instruments carried in aircraft, but make such an identification all the more necessary.

This thesis consists of a study of the difficulties encountered when indications of the vertical are made from a moving base, an analysis of the possibilities and limitations of methods available for overcoming these difficulties, and a survey of the actual devices available in practice for indicating the vertical from moving bases. The study of the difficulties encountered when indications of the vertical are made from a moving base involves a treatment of the origin, nature, and relative magnitudes of the accelerations present when a body is moving over the Earth. The analysis of the methods available for overcoming these difficulties shows the theoretical response of instruments based on these methods to the accelerations present in typical airplane maneuvers such as a perfectly banked turn, etc. The possibilities and limitations of such instruments are presented in the form of charts suitable for engineering use. The parameters of the charts are dimensionless ratios whenever possible in order to facilitate the comparison between instruments that are physically quite different. The survey of the actual devices available for indicating the vertical from moving bases involved a study of the patent literature.
SECTION I

FUNDAMENTALS OF THE PROBLEM OF INDICATING THE DIRECTION OF THE VERTICAL BY MEANS OF AN INSTRUMENT CARRIED ON A MOVING BASE

The problem of indicating the direction of the vertical from moving bases has been treated by individual parts in the literature with attention paid to specific problems rather than as a complete unit. The work of J. G. Gray and J. Gray\(^{(2)}\) in Scotland; and C. S. Draper\(^{(3)}\), L. S. Wasserman\(^{(4)}\), and S. H. Fang\(^{(5)}\) at the Massachusetts Institute of Technology are the principal sources of information.

The Grays\(^{(2)}\) were interested in the application of the gyroscope to the problem of indicating the direction of the vertical as a result of the World War of 1914-1918. They contrast the performance of a pendulous gyro with the performance of a gyro erected by a device of their own invention as an indicator of the vertical for the case of an airplane turning at a fixed speed. They also give a discussion of the requisite properties for the erection of a gyro, which the present writer does not feel is complete.

Wasserman\(^{(4)}\) has discussed the performance of the Sperry Artificial Horizon during turns. Fang\(^{(5)}\) treated the performance of both a pendulous gyro and the Sperry Artificial Horizon during airplane maneuvers. He also discussed the problem of stabilizing objects and lines of sight in airplanes.
by means of gyroscopes. The work of Draper\(^{(3)}\), in his class notes and lectures at M. I. T., is by far the most complete treatment of the problem of indicating the vertical from moving bases up to date, but it is unfortunately not yet available in printed form.

This thesis, using the work and nomenclature of Draper as a background, presents a unified treatment of the problem of indicating the vertical from moving bases. The material covered in the thesis discusses the difficulties encountered when determinations of the vertical are to be made from a moving base, possibilities and limitations of methods available for overcoming these difficulties, and a survey of the actual devices available in practice for indicating the vertical. The material presented in the thesis is summarized in the following outline:

1. A complete analysis of the accelerations acting on a pendulous element carried in an airplane flying over the Earth.

2. The possibility of using the magnitude of the resultant acceleration to indicate the vertical.

3. The possibilities and limitations of a pendulous element as an indicator of the vertical during a perfectly banked turn and other typical maneuvers including:

   a. The effect on performance by changing the location of the vertical indicator in an airplane.

   b. The possibility of using a system controlled by a loose ballistic as an indicator of the vertical.
4. A unified mathematical treatment suitable for engineering use, by means of vector methods, of the motion of a gyroscope due to applied torques; including a presentation of the requisite properties for erecting a gyroscope, and a study of nutation.

5. The possibilities and limitations of instruments containing a gyroscope as indicators of the vertical.

6. A survey of the present means available for indicating the vertical as revealed in the patent literature.

7. A study of the nature and practical effects of the acceleration of Coriolis.

The problem of indicating the vertical from any base, moving or fixed, involves primarily the study of directions in space, for which analysis the vector treatment is particularly useful. A vector is a physical quantity that has both magnitude and direction. It is represented by an arrow which points in the direction of the quantity and has a length proportional to the magnitude of the quantity. Examples of vector quantities are force, displacement, velocity, acceleration, angular velocity, angular momentum, torque, etc.

The use of the vector treatment, through its representation of a quantity by a single arrow, simplifies the analysis of the behavior of the quantity under forcing disturbances in three dimensional space since only a single equation of motion is required, instead of the three equations needed when an analysis is carried out in terms of components. A further value of the vector treatment lies in the fact that it
is relatively easy to visualize the representation of motion in space as a single arrow, but next to impossible to visualize simultaneous changes of three components. A treatment of the various laws of vector algebra and calculus necessary for the study of the indication of the vertical from moving bases is given in Appendix A, or may be found in many standard texts on the subject of vector analysis or mathematical physics such as the texts written by Coffin (6), Phillips (7), or Page (8).

The necessary mathematical and physical background required for the complete analysis of the problem of indicating the direction of the vertical from moving bases involves a treatment of the mechanics of rotation of a rigid body in moving reference systems and is given in Appendices A-G. The reader is also referred to standard texts on vector analysis or mathematical physics such as the texts written by Schaefer (9), Page (10), Coffin (11), Slater and Frank (12), Page (13), Osgood (14), Gray (15), Routh (16), Webster (17), Whittaker (18), and Appell (19), for further information.

**Accelerations Acting on a Pendulous Element Carried in an Airplane**

Newton's second law of motion relates an applied force to the resulting acceleration, but is strictly true only when referred to a system of coordinates that is unaccelerated relative to the fixed stars (10), i.e. to an inertial system.
It is generally convenient to express observed accelerations with reference to the Earth, but the Earth, rotating on its axis every day, does not constitute an inertial system. Accordingly the acceleration of a pendulous element carried in an airplane, while actually referred back to inertial space, will be expressed in terms of the more easily determined intermediate accelerations relative to the Earth and to the airplane when the motion of the pendulous element is to be studied by means of Newton's law.

The different coordinate systems used to give the intermediate accelerations required in the study of the motion of a pendulous element are shown in Figure 3. As seen in the figure these systems are:

1. An inertial system II.
2. A system EE centered at E, the center of the Earth, and fixed in the Earth.
3. A system 00 centered at 0 on the surface of the Earth, and nonrotating relative to the Earth. The system 00 is allowed to travel with the airplane, the point 0 being always directly below the center of gravity of the airplane. During any maneuver of the airplane that is being studied the system 00 is temporarily frozen on the Earth.
4. A system AA centered at the center of gravity A of the airplane and fixed in the airplane.
5. A system PP centered at the pivot P supporting the pendulous element, and fixed in the pendulous element. The point C represents the center of gravity of the pendulous element.
I is center of an inertial system - no acceleration relative to fixed space

E is center of the earth; origin of axes fixed in the earth

O is point on surface of earth; origin of axes fixed on the earth

A is center of gravity of airplane; origin of axes fixed in airplane

P is pivot supporting pendulous element; origin of axes parallel to set A

C is center of gravity of pendulous element; origin of axes fixed in element

\[ \vec{R}_{IC} = \vec{R}_{IE} + \vec{R}_{EO} + \vec{R}_{OA} + \vec{R}_{AP} + \vec{R}_{PC} \]

Fig. 3

Vector representation by components of the position of the center of gravity of a pendulous element supported by a pivot in an airplane flying over the earth
Fundamentally the accelerations in all five systems are referred back to system II. The method of referring motion in one coordinate system, e.g. system EE, in terms of a second coordinate system, e.g. system II, may be found in such texts as those written by Page(10) or Coffin(11). The present thesis extends this procedure to include the five above-mentioned coordinate systems, one on top of another. For example, the second derivative of the vector $\ddot{R}_{IC}$ in Figure 3 referred to system II is the fundamental acceleration, i.e. the acceleration of point C referred to fixed space. The second derivative of $\ddot{R}_{IC}$ referred to system II, i.e. $(\ddot{R}_{IC})_{II}$, may be thought of as being composed of five component accelerations when referred to system EE fixed at the center of the Earth:

1. $(\ddot{R}_{IE})_{II}$
   - acceleration of center of system EE with respect to fixed space,
2. $\dot{\omega}_{IE} \times (\ddot{R}_{EC})_{EE}$
   - tangential acceleration of point C due to a rotation $\dot{\omega}_{IE}$ of system EE in fixed space,
3. $\dot{\omega}_{IE} \times (\dot{\omega}_{IE} \times \ddot{R}_{EC})$
   - centripetal acceleration of point C due to a rotation $\dot{\omega}_{IE}$ of system EE in fixed space,
4. $2\dot{\omega}_{IE} \times (\dot{R}_{EC})_{EE}$
   - acceleration of Coriolis due to a velocity of point C in rotating system EE,
5. $(\ddot{R}_{EC})_{EE}$
   - acceleration of point C with respect to system EE.

Term 5) may then be used as a fundamental acceleration in system EE as was $(\ddot{R}_{IC})_{II}$ in system II and equated to five component accelerations referred to system OO. The process is continued, using the one "fundamental" acceleration in each
system for the coupling agent to the next system.

The complete derivation for the acceleration of the center of gravity C of a pendulous element carried in an airplane is given in Appendix D where it is shown that the acceleration referred to fixed space may be replaced by seventeen component accelerations referred to the intermediate coordinate systems. From the practical point of view, however, these seventeen accelerations may be reduced to but two major groups consisting of five component accelerations:

1. The effective acceleration of gravity, which for a moving base is composed of two parts:
   a. The local acceleration of gravity.
   b. The acceleration of Coriolis, which is due to the interaction of the velocity of the airplane and the daily rotation of the Earth.

2. The accelerations due to motions of the airplane relative to the Earth:
   a. The linear acceleration of the center of gravity of the airplane.
   b. The tangential acceleration due to a rotation of the airplane about its center of gravity.
   c. The centripetal acceleration due to a rotation of the airplane about its center of gravity.

The local force of gravity is illustrated in Figure 2 and as was discussed in the Introduction is composed of the vector resultant of the attractive gravitational force and the centrifugal force due to the daily rotation of the Earth.
It should be recalled here from the discussion in the Introduction that the rotational effects of a force may be replaced by the effects of an equivalent acceleration of the proper magnitude directed in the opposite sense. The local acceleration of gravity is accordingly directed upward antiparallel to the direction of the local force of gravity.

The origin of the acceleration of Coriolis is discussed in Appendix D; it will be sufficient to state here that this acceleration is required to maintain the conservation of angular momentum when referred to rotating coordinate systems.

The acceleration of Coriolis is probably not a familiar quantity, due mainly to the fact that the effects of this acceleration often become of importance only at speeds of several hundred miles per hour. True, the eastward drift of a falling body or the clockwise motion (in the Northern hemisphere) of ocean and atmospheric currents are readily observable at moderate speeds of only a few miles per hour. However high speed is essential for Coriolis effects to become important in a determination of the vertical. This fact is illustrated in Figure 4 where the angular deviation of the apparent vertical on a moving base due to Coriolis effects is plotted against the speed of the base as a function of the latitude. It is seen in the figure that the deviation of the apparent vertical is only five minutes of arc for a speed of three hundred miles per hour in a latitude of forty-five degrees. It follows that the effects of the acceleration of Coriolis can be neglected in all cases except those where a
\[
\theta_c = 2.29 \times 10^{-2} v \sin \lambda \text{ min.}
\]
\[
\lambda = \text{LATITUDE}
\]
\[
v = \text{VELOCITY IN MPH}
\]

\( v = \text{HORIZONTAL VELOCITY OF THE MOVING BASE IN MILES PER HOUR} \)

**Fig. 4**

TILT OF APPARENT VERTICAL ON A MOVING BASE DUE TO THE ACCELERATION OF CORIOLIS AS A FUNCTION OF THE VELOCITY OF THE BASE AND THE LATITUDE
determination of the vertical is to be made to within a few minutes of arc. The effect of the acceleration of Coriolis on precise determinations of the vertical is discussed in a later part of this report, and also in Appendix N.

The problem of determining the possibilities and limitations of various methods for indicating the vertical from moving bases is then to study the effects of the accelerations discussed in this section, and pertinent to a given airplane maneuver, on a pendulous element carried in the airplane flying over the Earth.
SECTION II

THE PENDULOUS ELEMENT AS AN INDICATOR OF THE VERTICAL

This section discusses the effect of subjecting a pendulous element to accelerations that would be encountered during typical airplane maneuvers. Draper\(^{(3)}\) has discussed the performance of a pendulous element located at the center of gravity of an airplane during a perfectly banked turn and during the initial stages of a perfectly banked turn. This thesis treats in addition the effect of location in an airplane on the performance of a pendulum as an indicator of the vertical, the performance of a pendulum as an indicator of the vertical during perfectly banked S-turns about a straight line of flight, the bubble cell as an indicator of the vertical, and a system controlled by a loose ballistic as an indicator of the vertical. The response of a pendulous element to typical disturbances of flight is shown by a series of charts in terms of dimensionless ratios characteristic of the element. The charts are then analyzed to ascertain the conditions affecting the performance of a pendulous element as an indicator of the vertical from a moving base.

A rigorous mathematical treatment of the performance of a pendulous element as an indicator of the vertical is given in Appendices H and I. The following sections state the conditions postulated to be present in each maneuver, and
analyze the performance of the pendulous element as an indicator of the vertical on the basis of dimensionless charts plotted from the mathematical results of Appendix I. In all cases it is assumed that the pivot supporting the pendulous element is located substantially in the median vertical longitudinal plane of the airplane.

A Perfectly Banked Turn

In a perfectly banked turn of an airplane about a vertical axis it is assumed that:

1. The resultant acceleration is always perpendicular to the general plane of the airplane's wings, i.e. the ball-bank indicator is in the center. This relates the centripetal acceleration of turning to the acceleration of gravity, as is shown in Figure 5.

2. The longitudinal axis of the airplane remains substantially horizontal.

3. The angle of bank remains constant and can be treated as a small angle.

4. The speed of the airplane remains substantially constant.

The steady state response of a pendulous element to a perfectly banked turn is shown graphically in Figure 6 where the logarithm of the ratio of the deviation from the vertical of the pendulum to the angle of bank of the airplane is plotted against the logarithm of the ratio of the undamped natural period of the pendulum to the time required for the airplane to
NORMAL TO PLANE OF AIRPLANE'S WINGS

RESULTANT ACCELERATION

(a) HORIZONTAL (CENTRIPETAL) ACCELERATION

\[ a_H = g \tan \theta_A \]

**Fig. 5**

ACCELERATIONS PRESENT IN AN AIRPLANE EXECUTING A PERFECTLY BANKED TURN
Fig. 6

Steady state amplitude of angular deviation from the vertical of a space damped pendulous element carried by an airplane during a perfectly banked turn — instrument mounted at center of gravity of the airplane.
complete a $360^\circ$ turn, for different values of the damping ratio (ratio of actual damping coefficient to the coefficient for critical damping).

The curves of Figure 6 show that the deviation of a pendulous element from the vertical during a perfectly banked turn is approximately equal to the angle of bank of the airplane when the undamped natural period of the pendulum is less than about one-half the time required for the airplane to complete a $360^\circ$ turn. In other words, a pendulous element under such conditions is an indicator of the resultant acceleration present in the airplane. When the undamped natural period of the element is longer than the time required for the $360^\circ$ turn, the deviation of the element from the vertical approaches zero as the natural period becomes longer. It follows that a pendulous element will act as an indicator of the vertical during a perfectly banked turn only when its undamped natural period is long compared with the time required to complete the $360^\circ$ turn.

It is seen from Figure 6 that for the deviation of the pendulous element to be only one percent of the angle of bank the undamped natural period of the element must be ten times the time required for the airplane to make a $360^\circ$ turn. For the standard blind flying turn of $360^\circ$ in two minutes at a speed of one hundred twenty miles per hour the angle of bank is sixteen degrees as is shown in Figure 7. In this case the deviation of the pendulous element from the vertical is about nine and one-half minutes of arc when the undamped natural period of the pendulum is twenty minutes.
Fig. 7

ANGLE OF BANK OF AN AIRPLANE IN A PERFECTLY BANKED TURN AS A FUNCTION OF THE SPEED OF THE AIRPLANE AND THE TIME REQUIRED FOR A 360° TURN
If the pendulous element has its mass distributed (the so-called **physical pendulum**) a reasonably sized body will have to be so mounted that its center of gravity is only a few millionths of an inch from the pivot, e.g. a disc of a three-inch radius can have a separation of only three ten-millionths of an inch between its center of gravity and the pivot, in order to obtain an undamped natural period of twenty minutes. Such precision of mounting is not mechanically feasible.

If the mass of the pendulous element is concentrated at a point (the so-called **simple pendulum**) the pendulous element must be about two hundred miles long to give an undamped natural period of twenty minutes. Such an instrument would obviously not be feasible for mounting in airplanes.

It is also seen from Figure 6 that damping has an effect only in the region of resonance, i.e. when the time for a \(360^\circ\) turn approximately equals the undamped natural period of the pendulous element.

It is shown in Appendix I that the effect of mounting the pendulous element in the airplane at points other than the center of gravity is negligible.

Note: It will be of interest to the readers familiar with mechanical vibrations to note here that the curves of Figure 6 are exactly the same as the response curves for the motion of a seismic element under the action of a sinusoidal acceleration as discussed by Draper and Wrigley (20).
Perfectly Banked S-Turns

In perfectly banked S-turns of an airplane about a straight line of flight as shown in Figure 8 it is assumed that:

1. The resultant acceleration is always perpendicular to the general plane of the airplane's wings, i.e. the ball-bank in the center. This relates the angle of roll to the rate of yaw of the airplane.

2. The deviation of the airplane in yaw from a straight flight path is small.

3. The longitudinal axis of the airplane remains substantially horizontal.

4. The angle of roll may be treated as a small angle.

5. The speed of the airplane remains substantially constant.

The steady state response of the pendulous element to perfectly banked S-turns is shown graphically in Figure 9, where the logarithm of the ratio of the maximum deviation of the pendulum from the vertical to the maximum angle of roll of the airplane is plotted against the logarithm of the ratio of the undamped natural period of the element to the period of yaw of the airplane (the periods of yaw and roll of the airplane are assumed to be identical). In Figure 9 the instrument is located at the center of gravity of the airplane, and shows the effect of changing the damping ratio. It is seen in the figure that damping has little effect except in the region of resonance. For the pendulous element to have a
FLIGHT PATH OF AIRPLANE EXECUTING PERFECTLY BANKED S-TURNS ABOUT A STRAIGHT LINE
Fig. 9

STEADY STATE AMPLITUDE OF ANGULAR DEVIATION FROM THE VERTICAL OF A SPACE DAMPED PENDULOUS ELEMENT CARRIED BY AN AIRPLANE WHEN THE AIRPLANE EXECUTES PERFECTLY BANKED S-TURNS ABOUT A STRAIGHT LINE - INSTRUMENT MOUNTED AT CENTER OF GRAVITY OF THE AIRPLANE
maximum deviation from the vertical of one percent of the maximum angle of roll it is necessary that the pendulum have an undamped natural period equal to ten times the period of yaw. For a period of yaw of two seconds and a maximum angle of roll of ten degrees the deviation of the pendulous element is about six minutes of arc and the undamped natural period of the element is twenty seconds. The shape of the curves are the same as for the case of a perfectly banked turn about the vertical.

For a physical pendulum consisting of a disc of a three-inch radius it is necessary that the separation between the pivot and the center of gravity be about one-thousandth of an inch. Such dimensions are practically feasible, but require very exacting work.

The pendulous element gives optimum performance as an indicator of the vertical when it is located at the center of gravity of the airplane. When the pendulum is located at points other than the center of gravity the performance falls off somewhat, but, as is shown in Appendix I, the effect is unimportant from a practical point of view.

Initial Stages of a Perfectly Banked Turn

During the initial stages of a perfectly banked turn it has been found experimentally by Mykytow, Pope, and Rieser\textsuperscript{(21)} that the airplane goes from a condition of straight and level flight to a condition of perfectly banked turning by a maneuver that can be represented analytically by a half-cosine
function as shown in Figure 10. In Figure 10 the angle of bank of the airplane is plotted against the time. During this maneuver it is assumed that:

1. The resultant acceleration is always perpendicular to the general plane of the airplane's wings, i.e. the ball-bank in the center. This relates the angle of roll to the rate of yaw of the airplane.

2. The deviation of the airplane in yaw from a straight flight path is small.

3. The longitudinal axis of the airplane remains substantially horizontal.

4. The angle of roll may be treated as a small angle.

5. The speed of the airplane remains substantially constant.

The response of the pendulous element to the initial stages of a perfectly banked turn is shown in Figure 11 where the ratio of the deviation of the pendulous element from the vertical to the full angle of bank of the airplane is plotted against the ratio of the time to the characteristic time of the pendulum, when the pivot is located at the center of gravity of the airplane. The characteristic time of the pendulous element is a particularly useful function for the study of transient effects (needed in this case since the complete solution of an equation involves both the transient and steady state regimes) because it includes both the undamped natural period of the pendulum and the damping ratio. The curves in Figure 11 are for different values of both the damping ratio.
\[ \theta_A = \frac{1}{2} \theta_{Ab} \left(1 - \cos \pi \frac{t}{T_\theta} \right) \text{ for } 0 < t < T_\theta \]

\[ \theta_{Ab} = \text{FULL ANGLE OF BANK} \]

\[ T_\theta = \text{TIME TO ATTAIN FULL ANGLE OF BANK} \]

**Fig. 10**

ANGLE OF BANK OF AN AIRPLANE IN THE INITIAL STAGES OF GOING FROM A STRAIGHT FLIGHT PATH INTO A PERFECTLY BANKED TURN
\[
\frac{\theta}{\theta_{ab}} = \frac{t}{\tau} = \frac{\text{TIME}}{\text{CHARACTERISTIC TIME OF PENDULOUS ELEMENT}}
\]

\[
\dot{\theta}_A = \left(\frac{\theta_{ab}}{2}\right) (1 - \cos \beta_{\tau} t / \tau)
\]

\[
\zeta = 0.1 \quad \zeta = 0.7 \quad \zeta = 0.4 \quad \zeta = 1.0
\]

\[\mu x z_{\tau} = 1\]

**Fig. 11**

*Initial stage of perfectly banked turn from straight flight - bank angle of pendulous element as fraction of full bank angle of airplane - pendulous element mounted at center of gravity of airplane*
and the ratio of the characteristic time of the pendulum to the time required to attain the full angle of bank for the airplane. It is seen in the figure that an increase in the damping ratio delays the tendency of the pendulous element to deviate from the vertical. It is also seen that the deviation of the pendulum from the vertical becomes smaller as the ratio of the characteristic time of the pendulum to the time required to attain the full angle of bank increases. In other words, the pendulous element indicates the direction of the resultant acceleration when the characteristic time of the pendulum is short compared to the time the airplane takes to attain the full angle of bank; but the pendulum remains substantially vertical when the time ratio is large. For all time ratios for which the deviation of the pendulum from the vertical is still substantially zero when the full angle of bank is attained the half-cosine function may be replaced for convenience by the step-function, i.e. by an instantaneous change from level flight to the full turn.

The time required to attain the full angle of bank is generally of about the same order of magnitude as are the times encountered in the case of perfectly banked S-turns, i.e. a few seconds. The pendulous element must accordingly have approximately the same physical characteristics to be a satisfactory indicator of the vertical in both S-turns and the initial stages of turns. It follows that the pendulous element used under these conditions must have an undamped natural period of about ten to twenty seconds, which requires a
separation between the pivot and the center of gravity of the element of only a few thousandths of an inch.

A BUBBLE CELL AS AN INDICATOR OF THE VERTICAL FROM MOVING BASES

A bubble cell is merely a cell containing a bubble of one fluid (usually air) floating on a denser fluid. The upper surface of a bubble cell consists of a glass having a spherical shape of a given radius of curvature. The motion, and hence the performance as an indicator of the vertical, of a bubble cell is identical with the motion of a simple pendulum whose length equals the radius of curvature of the cell.

One of the fundamentals of a stabilized reference system involving the use of optical equipment is that the radius of action of the stabilized reference must equal the focal length of the lens used to superimpose the stabilized reference and the image of object being observed. The radius of action of a bubble cell is the radius of curvature of the cell. Since instruments suitable for use in an airplane are limited in size it follows that the radius of curvature of a bubble cell used in airplanes cannot be more than a few inches in length.

A simple pendulum a foot long has an undamped natural period of about one second. Such a pendulum would only be a satisfactory indicator of the vertical in the presence of accelerations having a period of less than about one-tenth of a second; in other words during a straight and level flight in still air.
It is a common practice to take several sights of a celestial object in aerial navigation and to average the results. The more accurately the stabilized reference can indicate the vertical the fewer need be the number of observations to obtain a reliable answer to the position of the celestial body. True, skill of the navigator is a large factor in getting satisfactory celestial sights, but it stands to reason that even skill is hard put to offset the limitations of a mediocre stabilized reference such as a bubble cell.

COMPARISON OF THE PERFORMANCE OF A SYSTEM CONTROLLED BY A LOOSE BALLISTIC WITH THE PERFORMANCE OF A PENDULOUS ELEMENT

The treatment in this section showed that the undamped natural period of a pendulum must be long compared with the period of a disturbance in order that the pendulum be a satisfactory indicator of the vertical. In a pendulous element the disturbing torque (tending to deviate the element from the vertical) and the inertia of the element (tending to keep the element from moving) are both dependent on the mass of the element. It is shown in Appendix I that a long undamped natural period of the element is obtained by making the ratio of the distance between the pivot and the center of gravity of the element to the square of the radius of gyration of the pendulous element small, which is difficult to obtain in practice (the radius of gyration of a body is the radius of the ring having the same inertia as the body). The use of a
loose ballistic, i.e. a ball free to move around in a race, introduces the possibility of obtaining a long period element by making the mass appearing in the disturbing torque considerably less than the mass appearing in the inertia of the element, thereby increasing the practicability of obtaining a long period element.

An element controlled by a loose ballistic is shown in Figure 12 where the large body is mounted nonpendulously. In the body is a spherical race, the center of curvature of the race being located a short distance from the pivot (the radius of the race is much larger than the distance between the pivot and the center of curvature of the race). A ball, i.e. the loose ballistic whose mass is small compared to the mass of the large element is free to run around on the race. When the airplane carrying the system is subject to maneuvers, the motion of the ball on the race is identical to the motion of a simple pendulum whose length equals the radius of curvature of the race.

When a force is applied to the system the ball is disturbed and climbs up on the race. The reaction force of the ball on the race, being normal to the race at the point of contact, i.e. along one of the radii of curvature of the race, acts through the pivot only when the point of contact, the center of curvature of the race, and the pivot are in a straight line; which condition is present only when the whole system is in equilibrium. Accordingly when the ball climbs up the race due to a disturbance, the force of the ball on the
Figure 12

DIAGRAM OF A SYSTEM CONTROLLED BY A LOOSE BALLISTIC
race produces a torque tending to rotate the element about
the pivot, as is shown in Figure 13. It is assumed that
during a maneuver the ball is in the dimple shown in Figures
12 and 13 so little of the time that the dimple can be ig-
nored. The use of the dimple will be discussed later.

The performance of the above system controlled by a
loose ballistic as an indicator of the vertical when mounted
at the center of gravity of an airplane during a perfectly
banked turn about the vertical is compared with the perform-
ance of a pendulous element for the same conditions in Appen-
dix J. The system controlled by the loose ballistic and the
pendulous element have the following characteristics:

1. The pendulous element and the system controlled by
the loose ballistic both have the same radius of gyration.

2. The distance between the pivot and the center of grav-
ity of the pendulous element equals the distance between the
pivot and the center of curvature of the race in the system
controlled by the loose ballistic.

3. The damping ratio is the same in both cases.

The comparison of the performance as indicators of the
vertical of the two systems is shown in Figure 14 where the
logarithm of the ratio of the deviation from the vertical of
the system controlled by the loose ballistic to the deviation
of the pendulous element is plotted against the ratio of the
undamped natural period of the pendulous element to the time
required for the airplane to complete a 360° turn. In the
figure the damping ratio is seven-tenths of critical, the
Fig. 13
FORCES ACTING ON A SYSTEM CONTROLLED BY A LOOSE BALLISTIC

\[ \bar{F}_B = m_B g \text{ sec} \theta_B \]

PIVOT AND CENTER OF GRAVITY
CENTER OF CURVATURE OF RACE
BALL OF MASS \( m_B \)
Dimple
RACE

\( \bar{F}_B = mg \text{ sec} \theta_B \)
REACTION FORCE OF BALL ON RACE
\[ \frac{\theta_L}{\theta_R} = \text{DEVIA\'TION OF LOOSE BALLISTIC SYSTEM FROM THE VERTICAL} \]

\[ \beta = \frac{T_n}{T_A} = \text{UNDAMPED NATURAL PERIOD OF PENDULOUS ELEMENT} \]

\[ \text{TIME FOR AIRPLANE TO MAKE 360° TURN} \]

\[ m_B = \text{MASS OF ROLLING BALL (LOOSE BALLISTIC)} \]

\[ m = \text{MASS OF LOOSE BALLISTIC SYSTEM} \]

\[ \ell = \text{PIVOT - CG SEPARATION OF PENDULOUS ELEMENT} \]

\[ \ell = \text{PIVOT - CENTER OF CURVATURE SEPARATION OF LOOSE BALLISTIC SYSTEM} \]

\[ R = \text{RADIUS OF CURVATURE OF LOOSE BALLISTIC SYSTEM} \]

**Fig. 14**

RATIO OF STEADY STATE ANGULAR DEVIATION FROM THE VERTICAL OF A SYSTEM CONTROLLED BY A LOOSE BALLISTIC TO THE STEADY STATE ANGULAR DEVIATION FROM THE VERTICAL OF A PENDULOUS ELEMENT — BOTH CARRIED IN AN AIRPLANE DURING A PERFECTLY BANKED TURN
ratio of the mass of the ball to the mass of the system is varied, and the ratio involving the radius of curvature is varied.

It is seen from Figure 14 that the deviation of the loose ballistically controlled system is less than the deviation of the pendulous element, and that the ratio of the deviations becomes very small for large values of the ratio of the undamped natural period of the pendulous element to the time for the $360^\circ$ turn. The decrease of the deviation ratio is directly proportional to the decrease in the ratio of the mass of the ball to the mass of the system. The effect of increasing the radius of curvature of the race is to increase the value of the period ratio at which the deviation ratio suddenly becomes rapidly smaller.

An increase in the distance between the pivot and the center of curvature of the race is shown in Appendix J to increase the disturbing torque acting on the system as is shown in Figure 13. Such an effect leads to the use of the dimple shown in Figures 12 and 13, since the dimple, having a short radius of curvature, must necessarily have the distance between its center of curvature and the pivot large, thereby producing a powerful torque. The net result is that during maneuvers the ball will be in the race of large radius most of the time, thereby producing only a small disturbing torque. On the other hand, when the system is substantially vertical, the ball will be in the dimple, thereby producing a powerful torque for keeping the system lined up along the vertical.
SUMMARY

1. A pendulum having an undamped natural period of ten times the period of a horizontal disturbance will indicate the vertical to within one percent of the angle of roll or pitch of the airplane causing this disturbance; for a smaller ratio between the natural period and the forcing period the accuracy with which the pendulum indicates the vertical is lessened.

2. A pendulum able to indicate the vertical to one percent of the angle of roll or pitch of the airplane must possess the following physical characteristics:

A. For a forcing period of five seconds (typical of rolling and pitching) the pendulum must have an undamped natural period of about twenty seconds, i.e.
   1. A well designed physical pendulum of a few inches overall dimensions must have the center of gravity a few thousandths of an inch from the pivot.
   2. A simple pendulum must be several hundred feet long.

B. For a forcing period of two minutes (typical of turning about the vertical) the pendulum must have an undamped natural period of about twenty minutes, i.e.
   1. A well designed physical pendulum of a few inches overall dimensions must have the center of gravity a few millionths of an inch from the pivot.
   2. A simple pendulum must be a few hundred miles long.
3. From a practical point of view it makes little difference whether the pendulum is located at the center of gravity of the airplane or not.

4. A simple pendulum that is only a few inches long will give excellent indications of the direction of the resultant acceleration.

5. A system that is controlled by a loose ballistic offers the possibility of obtaining more accurate indications of the vertical than are obtainable with a pendulum of similar physical dimensions, but involves many more difficulties of construction that probably outweigh the advantages.

CONCLUSION

It should be possible to construct a pendulous element which will give satisfactory approximate indications of the vertical when the period of the forcing disturbance is less than about five seconds. For longer forcing disturbances the center of gravity of the pendulum must be too close to the pivot for practical construction. This same difficulty is encountered when the pendulum is to be used for accurate indications of the vertical.
SECTION III

INDICATION OF THE VERTICAL BY THE MAGNITUDE
OF THE RESULTANT FORCE

If the resultant acceleration present in an airplane is perpendicular to the general plane of the airplane's wings, i.e., the condition for perfect banking as is shown in Figure 5, the magnitude of the resultant acceleration is a function of the angle which the resultant acceleration makes with the vertical. It follows that the magnitude of the resultant acceleration might be used to indicate the direction of the vertical. So far as the writer of this thesis can determine, this method for indicating the vertical has never been discussed in the literature.

For the condition of perfect banking shown in Figure 5, the magnitude of the resultant acceleration equals the magnitude of the acceleration of gravity multiplied by the secant of the angle that the resultant acceleration makes with the vertical. The ratio of the magnitude of the resultant acceleration to the magnitude of the acceleration of gravity is plotted in Figure 15 against the angle between the direction of the resultant acceleration and the vertical. It is seen in the figure that the resultant acceleration equals the acceleration of gravity when the angle between the two accelerations is zero, and increases with increasing angle, reaching a value of twice gravity at an angle of sixty degrees.
\[ a = g \sec \theta_A \]

\( a = \text{RESULTANT ACCELERATION} \)

\[ \theta_A = \text{ANGLE BETWEEN RESULTANT ACCELERATION AND THE VERTICAL} \]

**Fig. 15**

MAGNITUDE OF THE RESULTANT ACCELERATION PRESENT IN AN AIRPLANE AS A FUNCTION OF THE ANGLE BETWEEN THE RESULTANT ACCELERATION AND THE VERTICAL
An accelerometer mounted in the airplane so that its line of action is perpendicular to the general plane of the airplane's wings would then give a reading indicative of the bank angle of the airplane for the condition of perfect banking. If the indicating system of the accelerometer were set to balance out the effect of gravity, i.e. to read zero when the airplane is flying straight and level, the instrument would then read the difference between the magnitudes of the resultant acceleration and the acceleration of gravity. Such an arrangement could be accomplished by the use of an inductance bridge pick-up-like that described by Draper and Wrigley\(^{(20)}\), and is necessary in order to read small deviations from the vertical in the presence of gravity.

The reading of an accelerometer set up as described above is proportional to the ordinates of Figure 16 where the logarithm of the ratio of the difference between the magnitudes of the resultant acceleration and the acceleration of gravity to the magnitude of the acceleration of gravity is plotted against the angle between the two accelerations. It is seen in the figure that the difference ratio equals unity when the angle between the two accelerations equals sixty degrees and decreases to very small values when the angle approaches zero.

The angle between the directions of the resultant acceleration and the acceleration of gravity equals the bank angle of the airplane (and also the deviation of the line of action of the accelerometer from the vertical) for the condition of perfect banking. To the observer in the airplane the vertical then lies somewhere in the cone whose axis is the line of action of the accelerometer and whose half-angle equals the
\[ \frac{a - g}{g} = \sec \theta_a - 1 \]

\( \bar{a} \) = RESULTANT ACCELERATION
\( \bar{g} \) = ACCELERATION OF GRAVITY (THE VERTICAL)

\( \theta_a \) = ANGLE BETWEEN RESULTANT ACCELERATION AND THE VERTICAL

Fig. 16

angle of bank of the airplane. The problem of indicating
the direction of the vertical is then to identify just which
of the generators of the cone is the vertical. When the
longitudinal axis of the airplane is horizontal, the vertical
lies in the cross-ship plane in the direction given by the
turn indicator; otherwise it is very difficult to identify
the generator that is the vertical.

SUMMARY

1. The magnitude of the angle of bank of an airplane can
be found from the magnitude of the resultant acceleration
present in the airplane, for the condition of perfect banking,
by means of an accelerometer which is set to read the differ-
ence between the magnitude of the resultant acceleration and
the acceleration of gravity.

2. The magnitude of the quantity actuating the accelerom-
eter is very small for small angles of bank. This means
that an instrument capable of very precise measurements would
be required to indicate the vertical.

3. The direction of the vertical lies somewhere in a cone
of half-angle equal to the angle of bank. The direction of the
vertical in this cone can be indicated by a turn indicator, but
only when the longitudinal axis of the airplane is horizontal.

CONCLUSION

The magnitude of the resultant acceleration does not lend
itself to a practical indication of the vertical.
The treatment of the pendulous element showed the necessity of a slow rate of response to disturbances in instruments for indicating the vertical. One method of obtaining a slow rate of response is to use a body having a large inertia. In the case of a pendulous element, however, a large inertia is obtained only in conjunction with a large mass; and a large mass is undesirable both because weight is at a premium in airplanes and because weight increases the unavoidable friction present in the pivot required to support the mass. A gyroscopic element discussed in the subsequent sections offers by means of its rapid spin the possibility of obtaining a large effective inertia associated with a relatively small mass.

A gyroscopic element is defined as a system which has the following properties:

1. The system is free to rotate about a point; this is mechanically achieved by the use of a gimbal system.

2. The system is mounted so as to be non-pendulous, i.e. the center of gravity of the system is located at the pivot, which is the intersection of the lines of action of the gimbals.
3. The system has a rotor spinning at a high rate of speed about the axis of symmetry of the rotor.

4. The rotor is spinning at a constant speed.

5. All the angular momentum of the system is concentrated about the spin axis of the rotor.

A gyroscopic element has no preferred orientation under the influence of gravity. Accordingly a pendulous element must be used in conjunction with the gyroscopic element to form an indicator of the vertical. In some cases the gyroscope is mounted so as to be pendulous itself; this system is known as a pendulous gyro. In other cases the pendulous element is entirely separated from the gyroscopic element, the orientation of the pendulum acting through a servo-mechanism to control the orientation of the gyro, and the gyro furnishing the necessary inertia to make the system have a slow rate of response.

The process by which the gyroscopic element is caused to assume a preferred orientation through its coupling with the pendulous element is known as the erection of a gyroscope. The means used for erection is one of the most important features of any gyroscope-pendulum system for indicating the vertical.

An analysis of the motion of a gyroscopic element and the performance of indicators of the vertical containing a gyroscope is carried out in Appendices L and M. Draper\(^3\), Wasserman\(^4\), Coffin\(^11\), Slater and Frank\(^12\), Page\(^13\), Osgood\(^14\), Gray\(^15\), Routh\(^16\), Webster\(^17\), Whittaker\(^18\),
Ferry(22), Klein and Sommerfeld(23), Winkelmann ("Handbuch der Physik")(24) and Crabtree(25) have dealt with the theory of gyroscopic motion. Additional sources of information treating the performance of indicators of the vertical containing a gyroscope are the works of Gray and Gray(2), Draper(3), Wasserman(4) and Fang(5). Considerable work has been done in Germany on treating the motion of a gyroscope by means of novel form of calculation known as "Motorrechnung". While this method seems to be a powerful tool, it is difficult to learn and has the added disadvantage that all texts using it are written in German.

This thesis expands and unifies the vector treatment of gyroscopes as originated by Draper(3). As is shown subsequently it is possible to apply the vector cross-product very efficiently in practical cases to problems involving the motion of a gyroscope under the influence of forcing disturbances. The value of a simple means of calculating gyro response to disturbances in practice is readily appreciated by anyone who has attempted to solve such problems by the classical approach.

Motion of a Gyroscopic Element

A study of the motion of a gyroscopic element involves the use of three basic physical quantities, namely angular momentum, torque, and angular velocity. These three quantities all possess both magnitude and direction; hence they
are readily adaptable to the vector treatment.

Angular velocity is rate of rotation, and is expressed in radians per unit time. The angular velocity vector is considered as being parallel to the axis of rotation and directed by the right hand rule for rotation. Figure 17 illustrates the right hand rule for rotation. It is seen in Figure 17 that this rule may be illustrated by the motion of a right hand screw where the positive direction of the angular velocity vector represents the direction of advance of the screw. Velocity, or rate, of precession is an angular velocity.

Angular momentum, sometimes referred to in practice as gyroscopic moment, may be considered for a gyroscope as being equal to the product of the spin and the moment of inertia of the gyro rotor. The angular momentum vector is then directed parallel to the spin (angular velocity of rotation) vector. Rigorously angular momentum is a complicated quantity as may be seen in Appendix F.

Torque is the product of a force and the lever arm through which the force acts. The torque vector is directed perpendicular to the plane containing both the force and the lever arm according to the right hand rule for rotation. Figure 18 shows the relation between force and torque. The direction of the torque vector is the direction of the advance of a right hand screw when the screw is turned by the force.
ROTATIONAL VECTORS — ANGULAR VELOCITY, ANGULAR MOMENTUM, AND (SUBJECT TO RESTRICTIONS) ANGULAR DISPLACEMENT — ARE DIREC TED IN THE SENSE OF THE ADVANCE OF A RIGHT HAND SCREW TURNED WITH THE ROTATION

Fig. 17

REPRESENTATION OF ROTATIONAL VECTORS
TORQUE IS DIRECTED PERPENDICULAR TO THE PLANE CONTAINING THE FORCE AND LEVER ARM IN THE SENSE OF THE ADVANCE OF A RIGHT HAND SCREW

Fig. 18
VECTOR REPRESENTATION OF TORQUE
The complete expression for the motion of a gyroscopic element as given in texts on mathematical physics involves the use of energy methods, elliptic integrals, and cubic equations. Accordingly the final answer for the motion of the gyro is much too complex to be useful in practice. For engineering purposes, however, the five assumptions given at the start of the section on indicators of the vertical which contain a gyroscopic element give answers that are within the limits of practical tolerances, and give a relatively simple vector equation for the motion of the gyroscopic element. The equation is derived in Appendix L, and is given in Equation (220)

$$\bar{\omega}_p \times \bar{H} = \bar{M} - \bar{\omega}_{IE} \times \bar{H}$$

where

- $\bar{\omega}_p$ is the angular velocity of precession of the gyrooscope's axis of symmetry as seen by an observer on the Earth.
- $\bar{H}$ is the angular momentum of the gyroscope, and is parallel to the axis of spin of the gyro rotor (the axis of symmetry).
- $\bar{M}$ is the resultant torque applied to the gyrooscope.
- $\bar{\omega}_{IE}$ is the angular velocity of the Earth's daily rotation. Equation (220) has the following interpretation:

1. $\bar{\omega}_{IE} \times \bar{H}$ is the correction term that takes care of the fact that the precession $\bar{\omega}_p$ is observed from the Earth and not referred to an inertial system. When no torque is applied to the gyro, i.e. $\bar{M} = 0$, the velocity of precession of the gyro will be $\bar{\omega}_p = -\bar{\omega}_{IE}$; in other words the velocity of precession of the gyro equals the negative of the angu-
lar velocity of the Earth in its daily rotation. The gyro then appears to an Earth observer to rotate once a day; clockwise about a vertical axis when seen from above at the north pole, and clockwise about a north-south axis when seen from the north at the equator. The apparent daily rotation (precession) of a torque-free gyro as seen from the Earth is due to the fact that a torque-free gyro remains fixed in inertial space.

2. For a gyroscopic element to have no rotation of its spin axis relative to the Earth, i.e. for $\bar{\omega}_p = 0$, it is necessary to apply a torque equal to $\bar{\omega}_{IE} \times \bar{H}$ to the gyro. The required torque is directed to the east (to balance out the effect of the eastward motion of a point on the Earth) and has a magnitude that is proportional to the angular momentum of the gyro and to the cosine of the latitude (maximum at the equator and zero at the poles in the case of a vertical gyro). When the gyro is coupled to a pendulous element to form an indicator of the vertical, the torque required to keep the spin axis of the gyro from rotating relative to the Earth is produced in one of two ways:

a. A relative angular displacement between the spin axis of the gyro and the pendulum. In this case part of the torque intended to erect the gyro (erection of a gyro, i.e. lining the spin axis of the gyro up with the vertical is subsequently discussed) is diverted so as to hold the spin axis motionless relative to the Earth. In this situation the spin axis of the gyro is tilted slightly from the vertical. This tilt is known as the latitude error.
b. A compensating torque (either manually controlled or automatic) is applied to the gyro. In this situation the spin axis of the gyro is vertical.

3. In general the applied torque $\mathbf{M}$ is much larger than the correction term $\mathbf{\bar{w}}_{IP} \times \mathbf{H}$. Equation (220) then states that the applied torque is perpendicular to both the angular momentum (the axis of spin of the gyro rotor) of the gyro and to the velocity of precession (the axis about which the gyro precesses) of the gyro (the definition of the vector cross product is to be found in Appendix A). Figure 19 shows the relationship of the angular momentum, applied torque, and velocity of precession of a gyroscope. It is seen in the figure that the velocity of precession is so directed that when a torque is applied to a gyroscope element the gyro turns so as to line its angular momentum up parallel to the applied torque.

The last statement sounds simple, but by means of it it is possible to predict rapidly and accurately the motion of a gyroscope element under the action of any applied torque.

The velocity of precession $\bar{\omega}_p$ of the gyroscope element can be split up into two main parts:

1. An angular velocity about the axis about which the gyro deviates from the vertical. This angular velocity represents the rate of erection of the gyro.

2. An angular velocity about the vertical. This represents the precession of the gyro about the vertical.
VECTOR DIAGRAM SHOWING TENDENCY OF A GYROSCOPE TO LINE ITS ANGULAR MOMENTUM (SPIN AXIS) UP WITH AN APPLIED TORQUE

\[ \mathbf{m} = \mathbf{\omega}_p \times \mathbf{H} \]

Fig. 19
RELATIONSHIP OF ANGULAR MOMENTUM, APPLIED TORQUE, AND VELOCITY OF PRECESSION IN A GYROSCOPE
Erection of a Gyroscope

In studying the problem of the erection of a gyroscope, two important factors must be recognized:

1. When a torque is applied to a gyroscope the gyro precesses so as to try to line up its angular momentum, i.e. its axis of spin, with the applied torque, as previously discussed in relation to Equation (220).

2. In a gyroscope running at normal operating speed the torque required to drive the rotor is exactly balanced by the resistance torques due to windage and bearing friction. Furthermore a commercial gyroscope is so constructed that an externally applied torque which is parallel to the spin axis of the gyro must act on the case supporting the gyro and not on the rotor. The result is that there will be no unbalanced torque acting on a gyroscope parallel to the spin axis. It follows that a commercial gyroscope will spin at constant speed, as was postulated above.

The component of the applied torque perpendicular to the gyro spin axis must have the following properties to cause the gyro to erect:

1. The torque component must lie in the plane through the angular momentum and the vertical. The torque component is then perpendicular to the line of nodes, i.e. the axis about which the gyro is tilted away from the vertical as shown in Figure 20.
ERECTING TORQUE COMPONENT IS PERPENDICULAR TO ANGULAR MOMENTUM AND LIES IN THE PLANE CONTAINING THE ANGULAR MOMENTUM AND THE VERTICAL

**Fig. 20**

RESOLUTION OF AN APPLIED TORQUE ACTING ON A GYROSCOPE INTO ERECTING, STERILE, AND PRECESSING COMPONENTS
2. The torque must be so directed as to cause erection, not a further deviation from the vertical.

3. The magnitude of the torque component must equal zero when the gyro is vertical.

The problem of erecting a gyro is then to apply to the gyro a properly directed torque having a component perpendicular to the angular momentum of the gyro and lying in the vertical plane through the spin axis of the gyro.

Note: A gyro stops dead and will not overshoot when an applied torque is removed because in a gyro it is the angular velocity that is proportional to the torque; on the other hand a pendulum will overshoot because in this case it is the angular acceleration that is proportional to the torque.

Any component of the applied torque that does not form a part of either the erecting torque component or the "sterile" torque component parallel to the spin axis can be resolved about the line of nodes, and leads to a precession of the gyro about the vertical axis, e.g. the gravity torque acting on a pendulous gyro.

Figure 20 shows the three components of an applied torque acting on a gyro. The erecting component lies in the plane containing the vertical and the angular momentum of the gyro (remember that this component must be so directed as to erect, not deviate further from the vertical), and is perpendicular to the angular momentum. The
sterile component also lies in the plane containing the vertical and the angular momentum of the gyro, but is parallel (or anti-parallel) to the angular momentum. The precession component lies along the line of nodes, and is accordingly perpendicular to both the angular momentum of the gyro and to the vertical.

Figure 21 shows the couple acting so as to produce the erecting component of the applied torque. The couple, consisting of a lever arm and an applied force, lies in the plane containing the angular momentum of the gyro and the line of nodes.

Nutation

If the fifth requirement on the gyroscopic element given previously is not fulfilled, i.e., if all the angular momentum is not concentrated about the axis of spin of the rotor, the gyro is subject to an oscillation known as nutation. Nutation lends itself beautifully to mathematical treatment, but is undesirable in an instrument gyro.

The frequency of nutation is approximately the speed of the gyro rotor spin in revolutions for a disc shaped rotor. As the disc becomes thicker, the speed of nutation decreases, becoming zero for a sphere. A high speed gyro will then have any nutation damped out quickly, since a given ratio of actual damping to that required for critical damping will cause an oscillation to lose a given per-
THE VERTICAL

ANGULAR MOMENTUM

PIVOT

LINE OF NODES

PLANE CONTAINING ANGULAR MOMENTUM AND LINE OF NODES

LEVER ARM

ERECTING FORCE

ERECTING FORCE AND LEVER ARM THROUGH WHICH THIS FORCE ACTS BOTH LIE IN THE PLANE FORMED BY THE ANGULAR MOMENTUM AND THE LINE OF NODES

Fig. 21

COUPLE PRODUCING THE ERECTING TORQUE ON A GYROSCOPE
centage of its initial amplitude in the same number of cycles regardless of the frequency. In addition the initial amplitude of nutation for a given amount of energy decreases as the speed of rotation of the gyro increases. It follows that a high speed gyro will be less subject to the effects of nutation than a slower gyro. An analysis of nutation is carried out in Appendix L, where it is shown that it is necessary to give the gyro an angular velocity or a thrust of energy to cause nutation; an angular displacement alone cannot cause nutation.

That the assumption of all the angular momentum's being concentrated about the spin axis is justified for a commercial instrument gyro, e.g. the Sperry Artificial Horizon spinning at twelve thousand revolutions per minute, is shown in the fact that the angular momentum about the spin axis is approximately one million times the angular momentum about the other axes, see Appendix L.

Often times in practice a gyroscope of new design is observed to exhibit an undesirable oscillation or vibration when spinning. The vibration may be due to mass unbalance in the rotor due to the fact that the spin axis is not the axis of symmetry of the rotor; mass unbalance in the system in the gimbal mount; or nutation. Accordingly, a rapid and convenient method for determining the frequency of nutation of a given rotor would permit the observer to ascertain whether or not the vibration might be nutation. The frequency of nutation depends only on
the mass distribution and spin of the gyro rotor. The equation for the frequency of nutation is derived in Appendix L and is given in Equation (212):

$$\omega_{nu} = 2\pi n_{nu} = (1 - I/I_z)(I_z\omega_z/I)$$  \hspace{1cm} (212)

where \( \omega_{nu} \) = angular frequency of nutation in radians per second  
\( n_{nu} \) = frequency of nutation in cycles per second  
\( I_z \) = moment of inertia of rotor about spin axis  
\( I \) = moment of inertia of rotor about transverse axis  
\( \omega_z \) = angular velocity of rotor spin in radians per second  

It is seen from Equation (212) that the frequency of nutation is readily calculable from a knowledge of the moments of inertia and the spin of a rotor.

The Pendulous Gyro as an Indicator of the Vertical

It has been well known for many years that a gyroscope has a large inertia for a given mass. It is this property that has made the gyroscope so valuable in practical indicators of the vertical. In the earliest attempts to indicate the vertical from aircraft, about thirty years ago, the instruments used simply mounted the gyroscope such that its center of gravity was below the pivot. Such a system is called a pendulous gyro, or a gyropendulum. A pendulous gyro does not act merely as a pendulum with a large inertia, however. This fact is discussed at length in Appendix M
and illustrated in Figure 22. In Figure 22, where the angular momentum of the gyro, the pendulous torque, and the vertical are shown, it is seen that the pendulous torque is perpendicular to the vertical plane containing the angular momentum of the gyro. Hence there can be no component of the pendulous torque that can cause erection of the gyro. It follows that a pendulous gyro alone cannot serve as an indicator of the vertical since it has no means of seeking the vertical.

A pendulous gyro will attempt to line its angular momentum up with the pendulous torque, however. But as soon as the angular momentum starts to precess toward the pendulous torque the change in orientation of the gyro causes the pendulous torque to precess about the vertical axis just as fast as the gyro tilts toward the torque. This effect is discussed in Appendix M where it is shown that the net result is for the spin axis of the gyro to chase the pendulous torque forever and hence to trace out a cone of constant angle about the vertical. The rate of precession about the vertical is directly proportional to the pendulous torque and inversely proportional to the angular momentum of the gyro.

It is possible, however, for a damped pendulous gyro to form an indicator of the vertical since the damping friction can cause an erecting torque, as is discussed in Appendix M. Figure 23 illustrates the case of a damped pendulous gyro mounted on a pedestal. Precession of the gyro
PENDULOUS TORQUE IS DIRECTED ABOUT LINE OF NODES AND SO IS PERPENDICULAR TO PLANE CONTAINING THE ANGULAR MOMENTUM AND THE VERTICAL

**Fig. 22**
TORQUES ACTING ON A PENDULOUS GYRO
The vertical

Torque due to bearing friction in pedestal

Pendulous torque

Angular momentum

Sterile component of friction torque

$MC \sin \theta$

Erecting component of friction torque

Line of nodes

Precession about the vertical

Fig. 23

Torques acting on a damped pendulous gyro mounted on a pedestal

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about the vertical due to pendulosity causes friction in the pedestal bearing that is directed about the vertical. As is seen in Figure 23 the vertical friction torque has a component so directed as to cause erection of the gyro. A few of the early instruments for indicating the vertical embodied the idea of a pendulous gyro mounted on a pedestal, and erected by bearing friction.

It is generally customary to mount a gyro in gimbals rather than on a pedestal. Bearing friction in the gimbals due to the precession of the gyro about the vertical under the effect of pendulosity can also produce an erecting torque. The friction torque is directed about the line of action of a gimbal. By construction the line of action of the inner gimbal is perpendicular to the spin axis of the gyro while the line of action of the outer gimbal is horizontal, as is shown in Figure 24. Accordingly all the friction torque about the inner gimbal line of action is properly directed for erection. The erecting component of the friction torque about the outer gimbal line of action is the cosine (of the angle of tilt of the gyro from the vertical) component; the sine component is sterile. The torques, both pendulous and friction, acting on a damped pendulous gyro in a gimbal mount, are shown in Figure 25.

The performance of a damped pendulous gyro as an indicator of the vertical is discussed in Appendix M. Figure 26 shows the logarithm of the ratio of the maximum tilt of a pendulous gyro from the vertical to the angle of
Fig. 24

GIMBAL MOUNTING OF A GYROSCOPE
Fig. 25

TORIES ACTING ON A DAMPED PENDULOUS GYRO
MOUNTED IN GIMBALS
Fig. 26

MAXIMUM STEADY STATE ANGULAR DEVIATION FROM THE VERTICAL OF A PENDULOUS GYROSCOPE CARRIED BY AN AIRPLANE DURING A PERFECTLY BANKED TURN - INSTRUMENT MOUNTED AT CENTER OF GRAVITY OF THE AIRPLANE
bank of the airplane plotted against the logarithm of the ratio of the period of precession of the gyro about the vertical to the time for the airplane to complete a $360^\circ$ turn, when the gyro is mounted at the center of gravity of the airplane. It is seen in Figure 26 that the tilt of the gyro is two percent of the angle of bank of the airplane when the period of precession of the gyro is one hundred times the time required for the airplane to complete the $360^\circ$ turn. For a $360^\circ$ turn in two minutes the period of precession of the gyro would then have to be two hundred minutes. It is shown in Appendix M that a pendulous gyro having a rotor of a two-inch diameter and spinning at twelve thousand revolutions per minute will have a period of precession of two hundred minutes when the center of gravity of the gyro is about four thousandths of an inch below the pivot. For comparison, as is shown in Appendix I, a long-period physical pendulum of similar overall dimensions must have its center of gravity located three ten-millionths of an inch below its pivot to give similar performance during the $360^\circ$ turn in two minutes.

It is apparent that the construction features of a pendulous gyro (even when the problem of handling the spin of the rotor is considered) are much more feasible mechanically than are the features of a pendulum giving the same performance as an indicator of the vertical. Many proposed indicators of the vertical in the early days of airplanes
involved the use of a damped pendulous gyro. Although still proposed even today, the pendulous gyro is generally replaced now by a nonpendulous gyro that is servo-controlled by a pendulum in practical indicators of the vertical. The main objections to the pendulous gyro are:

1. The rate of erection of a pendulous gyro depends upon its damping, which is generally quite small. On the other hand the rate at which a pendulous gyro tends to leave the vertical under the action of horizontal disturbances is substantially independent of the damping. It follows that in rough air a pendulous gyro would show more tendency to leave the vertical than to remain along it.

2. The precession about the vertical (required to furnish the damping) causes a pendulous gyro to erect in a spiral. Such a feature is undesirable in an instrument gyro.

The two objections listed above for a pendulous gyro are discussed by Gray and Gray\(^2\) and in Appendix M. Considerable investigation has been carried out by such men as E. A. Sperry, Jr., M. M. Titterington, and M. F. Bates of the Sperry Gyroscope Company, and M. Schuler of Anschütz to damp the oscillations of a gyropendulum by means of additional features. An excellent example of this work is disclosed in U. S. patent 1,518,892 to M. F. Bates. In this invention a spherical pendulum attached to a pendulous gyro controlled the flow of air from jets in the gyro case. The reaction of the air flowing out of the jets damped the
oscillations of the gyro. It was soon found, however, that the gyro with air jet damping was a superior indicator of the vertical when it was made nonpendulous. Accordingly the pendulous gyro then gave way (approximately in 1925) to the nonpendulous gyro which was controlled through a servo-mechanism by a pendulum as a practical indicator of the vertical.

The Nonpendulous Gyro Servo-Controlled by a Pendulum as an Indicator of the Vertical

The damping of a pendulous gyro by air jets actually consisted merely of producing an erecting torque by means of the jet reaction. In general there is no torque component associated with the air jet reaction, so directed as to cause the gyro to precess about the vertical. Accordingly a gyro will erect radially, i.e. return to the vertical in a straight line, when the gyro is made nonpendulous and is controlled by the air jet reaction.

A good example of a gyro that is controlled by air jet reaction is the Sperry Artificial Horizon disclosed in U. S. patent 1,982,636 to B. G. Carlson and shown in Figure 27. In this instrument the action of the air jets is controlled by small flipper-like pendulums (the flippers serve the same purpose as the previously mentioned spherical pendulum of Bates, but are mechanically simpler and wear better). These flippers act differentially in pairs such that each jet is normally exactly half open. When
the gyro is tilted away from the vertical, however, one jet is covered more than the opposite one. The force of erection due to differential jet action is then directed opposite to the direction of the stream of air flowing out of the more open of the two jets; the lever arm through which the jet force acts is parallel to the spin axis of the gyro; this results, as shown in Figure 28, in the erecting torque's being so directed that all of the torque goes into erecting the gyro -- there are no sterile nor precessing components.

It is thus seen that the nonpendulous gyro servo-controlled by a pendulum can be so designed that all the torque produced, when the gyro tilts from the vertical, goes into erecting the gyro. This desirable feature was shown to be impossible for a pendulous gyro. It is not necessary that the erecting torque be produced by air jets, however. Many methods of accomplishing the coupling of a pendulous element to a gyroscopic element have been advanced, as is shown by a survey of the patent literature, such as the direct action of torque motors about the gimbal axes, rotating the gimbal frame, etc. But all methods for erecting gyroscopes have the same common basis, i.e. a pendulous element to follow the instantaneous direction of the resultant acceleration and a gyroscope (whose orientation is controlled by the pendulum) furnish a slow rate of response and so average the position of the pendulum.
NOTE:— THERE ARE NO STERILE NOR PRECESSING COMPONENTS TO THE JET TORQUE; THE COMPLETE TORQUE IS AVAILABLE FOR ERECTING THE GYRO

**Fig. 29**

DIAGRAMMATIC SKETCH OF SPERRY ARTIFICIAL HORIZON SHOWING JET FORCE AND JET TORQUE
A gyro servo-controlled by a pendulum will generally have two regimes of action:

1. The erecting torque is proportional to the angle of tilt of the gyro from the vertical. This regime exists when the gyro is approximately vertical.

2. The erecting torque is constant. This regime exists when the gyro has a tilt from the vertical greater than a predetermined angle. In some forms of erection this regime does not exist.

Since the pendulum, controlling the gyro, generally has a short natural period, it will tend to follow the direction of the resultant acceleration without appreciable time lag. It will accordingly be assumed for ease of calculation that the second regime, i.e. constant erecting torque, is the regime present during maneuvers of the airplane.

The rate of erection of a servo-controlled gyro when the erecting torque lies in the plane of the gyro rotor, i.e. the situation fulfilled by the "perpendicular" component of torque as was discussed earlier and illustrated in Figure 20, varies directly as the magnitude of the erecting torque and inversely as the angular momentum of the gyro (the angular momentum depending on the mass and the rate of spin of the rotor). For a typical erection rate of eight degrees per minute, e.g. the rate of the Sperry Artificial Horizon, a gyro will have a maximum deviation
from the vertical of approximately only sixteen minutes of arc in a maneuver having an eight-second period, as is discussed in Appendix M. For shorter periods of maneuver or for slower rates of erection the deviation will be less. A very slow rate of erection is undesirable, however, because then friction effects become important. Accordingly, a very accurate indicator of the vertical should have a faster rate of erection to get the gyro vertical initially and a slower rate of erection during operation.

A detailed treatment, both theoretical and experimental, of the performance of the Sperry Artificial Horizon during turns is discussed by Wasserman$^4$ and Fang$^5$. 

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SUMMARY

1. A damped pendulous gyro combines satisfactory performance as an indicator of the vertical with feasible physical properties. The pendulous gyro must be damped, generally by bearing friction, in order to be able to seek the vertical. Note: An undamped pendulous gyro has no erecting torque and so is useless as an indicator of the vertical.

2. The pendulous gyro has the undesirable property of precessing about the vertical while erecting; thereby approaching the vertical in a spiral. The pendulous gyro tends to leave the vertical more readily than it erects under the influence of horizontal disturbances. This is due to the fact that the erecting torque depends on the damping, which is generally small, while the disturbing torque is independent of the damping.

3. A gyroscopic element servo-controlled by a pendulum combines satisfactory performance as an indicator of the vertical with feasible physical properties. Such a system can be designed so that all the torque produced when the gyro tilts goes into erecting the gyro, i.e. there are no sterile nor precessing components. The system then erects in a straight line, i.e. radially.
4. The method and degree of coupling between the gyro and the pendulum may be varied to suit the particular needs of the problem at hand.

CONCLUSIONS

1. A damped pendulous gyro may be used as a practical indicator of the vertical, but contains certain inherent properties that are undesirable.

2. A gyroscope servo-controlled by a pendulum in general offers the most practical solution to the problem of indicating the direction of the vertical from moving bases.
SECTION V

EFFECTS OF THE ACCELERATION OF CORIOLIS

The acceleration of Coriolis is produced by the interaction of the velocity of a body moving over the Earth with the daily rotation of the Earth on its axis. This acceleration is generally an unfamiliar one because in airplanes it is practically unnoticeable until a body attains speeds of several hundred miles per hour\(^{(10)}\). A mathematical treatment of the acceleration of Coriolis may be found in several texts\(^{(10)(11)}\) or in Appendix D.

In brief, the acceleration of Coriolis maintains the conservation of angular momentum of one body moving on a rotating body.

The net effect of the acceleration of Coriolis due to a horizontal velocity of an airplane is to deviate the direction of the apparent vertical as seen from a moving base away from the true vertical, i.e. the direction of the force of gravity. In the northern hemisphere the deviation of the apparent vertical from the true vertical is always directed to the right (in the southern hemisphere the direction is to the left), and the magnitude of the deviation is independent of the heading of the airplane. The magnitude depends only on the velocity of the airplane and the latitude on the Earth. The magnitude of the hori-
zontal component of the acceleration of Coriolis is shown in Figure 4 as a function of the speed of the airplane and of the latitude, where it is seen that the deviation of the apparent vertical for a body moving at three hundred miles an hour in a latitude of forty-five degrees is about five minutes of arc. The above treatment of the acceleration of Coriolis is derived in Appendix D.

It is also shown in Appendix D that the vertical component of the acceleration of Coriolis, sometimes referred to as the centrifugal effect, is parallel to the acceleration of gravity and is negligibly small in comparison to gravity.

The acceleration of Coriolis due to a vertical motion of a body, i.e. the motion of falling, is shown in Appendix D, and in the literature\(^{10}\), to cause a falling body to deviate to the east at all times. The eastward deviation of a falling body is illustrated in Figure 29 as a function of the latitude and the height from which the body was dropped. It is seen in the figure that a body dropped from twenty thousand feet in latitude forty-five degrees has an eastward deviation of about sixty feet when it hits the ground.

Effect on Navigation

It follows from the fact that the effect of the Coriolis acceleration is to deviate the direction of the apparent vertical to the right (in the northern hemisphere, and
Fig. 29

Eastward deviation of a falling body as a function of the altitude from which the body is dropped and the latitude

\[ d = 0.383 \sqrt{h^3} \cos \lambda \] feet
\[ \lambda = \text{latitude} \]
\[ h = \text{altitude in thousands of feet} \]
to the left in the southern hemisphere) that celestial navigation sights taken with a bubble sextant from an airplane will be in error, since the bubble acts as a pendulum and so seeks the apparent vertical. It is seen in Figure 30 that the observed zenith distance is less than the true zenith distance when the celestial body is to the left of the heading of the airplane. This effect is zero dead ahead or astern, and the apparent zenith distance is greater than the true zenith distance when the celestial body is to the right of the heading of the airplane.

The magnitude of the deviation of the apparent vertical from the true vertical is given by the ratio of the horizontal Coriolis acceleration to the acceleration of gravity. In Figure 4 it is seen that the deviation of the apparent vertical from the true vertical is about five minutes of arc for an airplane moving three hundred miles per hour in the latitude of forty-five degrees north. Such a deviation can give rise to an error in the airplane's position, determined from celestial observations, of several miles.

The effect of the angular difference between the azimuth of the celestial body observed and the heading of the airplane is such as to introduce the sine of this angular difference to the equation for the difference between the observed zenith distance and the true zenith distance. The angular difference between the celestial body azimuth and airplane course is shown in Figure 31, where the angles are measured from north through east.
\( \theta_c = \) deviation of apparent vertical from true vertical due to Coriolis effects

\( \theta_T = \) true zenith distance of celestial body

\( \theta = \) apparent zenith distance of celestial body looking forward along the line of flight

Deviation of the apparent zenith distance of a celestial body from the true zenith distance due to Coriolis effects

**Fig. 30**

\[
\begin{align*}
\theta_c &= \text{deviation of apparent vertical from true vertical due to Coriolis effects} \\
\theta_T &= \text{true zenith distance of celestial body} \\
\theta &= \text{apparent zenith distance of celestial body looking forward along the line of flight}
\end{align*}
\]

Angular difference between heading of an airplane and azimuth of an observed celestial body

**Fig. 31**

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The correction factor for the difference between the observed and the true zenith distances as measured by a bubble sextant from an airplane moving over the Earth has been worked out by Stewart (26), and is also carried out in Appendix N.
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LIST OF SYMBOLS

\( \varphi, \psi, \theta \)  Euler's angles

\( X, Y, Z \)  Rectangular axes fixed on the Earth

\( x, y, z \)  Rectangular axes fixed in a body

\( X_A, Y_A, Z_A \)  Rectangular axes fixed in the airplane, or any moving base

\( \bar{g} \)  Acceleration of gravity

\( \bar{\omega}_{IE} \)  Angular velocity of the Earth due to its daily spin

\( \lambda \)  Latitude

\( \bar{F} \)  Force

\( \bar{M} \)  Torque

\( m \)  Mass

\( I \)  Moment of inertia

\( \bar{H} \)  Angular momentum

\( \bar{\omega} \)  Angular velocity

\( \bar{l} \)  Separation between the pivot and the center of gravity

\( \theta_A \)  Angle of bank of an airplane

\( T_A \)  Time for an airplane to complete a 360° turn about the vertical

\( \omega_n \)  Undamped natural angular frequency

\( T_n = 2\pi/\omega_n \)  Undamped natural period

\( c \)  Damping coefficient

\( c_C \)  Coefficient for critical (aperiodic) damping

\( \zeta = c/c_C \)  Damping ratio

\( T_f \)  Period of forcing disturbance

\( \beta = T_n/T_f \)  Frequency, or period, ratio

\( \mu = 1/(1-\beta^2)^{1/2}+2\zeta \beta \)  Amplitude factor

\( \phi_1 = \tan^{-1}2\beta\zeta/(1-\beta^2) \)  Phase angle
\( \varepsilon \) Base of natural logarithms
\( k \) Radius of gyration
\( T_\varphi \) Period of yaw of an airplane
\( v_A \) Velocity of an airplane
\( \tau = \frac{1}{\hat{\omega}_n} \) Characteristic time
\( X_{AAP} \) Separation between pivot and axis of yaw
\( Z_{AAP} \) Separation between pivot and axis of roll
\( \sigma_{\varphi_A}' = g_T^n/2\pi v_A \)
\( \sigma_X = 4\pi^2 X_{AAP}/g_T^2 \)
\( \sigma_Z = 4\pi^2 Z_{AAP}/g_T^2 \)
\( \mu_{XZ} = \sqrt{(\sigma_{\varphi_A}' \sigma_X)^2 + (1 - \sigma_Z^2)^2} \)
\( \psi_1 = \tan^{-1}(\sigma_{\varphi_A}' \sigma_X/(1 - \sigma_Z^2)) \)
\( \sigma_{\varphi_A}' \sigma_X = g_T^n/2\pi v_A \)
\( \sigma_{Xf} = 4\pi^2 X_{AAP}/g_T^2 \)
\( \sigma_{Zf} = 4\pi^2 Z_{AAP}/g_T^2 \)
\( \mu_{XZf} = \sqrt{(\sigma_{\varphi_A}' \sigma_X)^2 + (1 - \sigma_Z^2)^2} \)
\( \psi_{1f} = \tan^{-1}(\sigma_{\varphi_A}' \sigma_X)/(1 - \sigma_Z^2) \)
\( T_\Theta \) Time required to attain full angle of bank
\( \theta_{Ab} \) Full angle of bank (used in study of transients)
\( \dot{\beta}_\tau = \tau/T_\Theta \)
\( \sigma_{\varphi_A}' \sigma_X = g_T^n/\pi v_A \)
\( \sigma_{X\tau} = \pi^2 X_{AAP}/g_T^2 \)
\( \sigma_{Z\tau} = \pi^2 Z_{AAP}/g_T^2 \)
\[ \mu = \sqrt{(\sigma_{XZ \tau} \sigma_{A \tau} \sigma_{X \tau} \beta_{\tau})^2 + (1 - \sigma_{Z \tau} \beta_{\tau})^2} \]

\[ \psi_{\tau} = \tan^{-1}(\sigma_{X \tau} \beta_{\tau}) / (1 - \sigma_{Z \tau} \beta_{\tau}) \]

\[ \mu_{\tau} = 1 / \sqrt{(1 - (\pi \beta_{\tau})^2)^2 + (2 \pi \beta_{\tau})^2} \]

\[ \phi_{\tau} = \tan^{-1}(2 \pi \beta_{\tau}) / [1 - (\pi \beta_{\tau})^2] \]

\[ \Gamma = m_B / m \quad \text{Mass ratio of loose ballistic to supporting element} \]

\[ \omega_L = 1 / \sqrt{(\Gamma - \beta_{\tau})^2 + (2 \pi \beta_{\tau})^2} \quad \text{Loose ballistic amplification factor} \]

\[ \phi_L = \tan^{-1}(2 \pi \beta_{\tau} / (\Gamma - \beta_{\tau})^2) \quad \text{Loose ballistic phase angle} \]

\[ \sigma = \sqrt{\mu \beta / k^2} \quad \text{Loose ballistic period ratio} \]

\[ \omega_{\nu} \quad \text{Natural angular frequency of nutation} \]

\[ \psi_{\nu} \quad \text{Phase angle of nutation} \]

\[ \omega_p \quad \text{Angular velocity of precession of a gyroscope} \]

\[ T_p = 2 \pi / \omega_p \quad \text{Period of precession of a gyroscope} \]
Vector Analysis

A \textit{vector} is a physical quantity which has both \textit{magnitude} and \textit{direction}, e.g. displacement, momentum, force, etc., in contrast to a \textit{scalar} which has only \textit{magnitude}, e.g. mass, work, energy, etc. A vector is represented by an arrow which points in the direction of the quantity and has a length proportional to the magnitude of the quantity. Fig. 32 shows a typical vector: \( a \) is the tail and \( b \) the head of the vector \( \vec{F} \).

In this paper the vector notation is that of Gibbs. Since the fundamental laws of vector algebra and calculus can be found in any standard text on vector analysis such as texts written by Coffin\(^{(6)}\), Phillips\(^{(7)}\), or Page\(^{(8)}\), no attempt will be made here to derive the mathematical rules; they will simply be stated, giving the nomenclature used and the fundamental properties.

The following are a few important definitions concerning vectors:

1. \textit{Unit vector}: a unit vector is a vector whose magnitude is unity. The most important unit vectors are \( i, j, k \), parallel respectively to the rectangular \( x, y, z \) axes. \textit{Note}: rectangular axes are taken so as to constitute a right hand set, i.e. the positive direction of the \( z \) axis is determined by the sense of the advance of a right hand screw rotated in the \( xy \) plane from the positive direction of \( x \) to that of \( y \) through the right angle between them.

2. \textit{Equal vectors}: two vectors are said to be equal when they have the same magnitude and the same direction.

3. \textit{Negative vector}: the negative of a vector is the vector taken with the arrow pointing in the opposite direction.
\[ \vec{b} \text{ is the tail of the vector } \vec{p} \]
\[ \vec{a} \text{ is the head of the vector } \vec{p} \]

**REPRESENTATION OF A VECTOR BY AN ARROW**

**Fig. 32**

\[ \vec{R} = \vec{p} + \vec{q} \]

**Fig. 33**

**ADDITION OF TWO VECTORS**
A vector will be represented mathematically by a symbol having a bar over it, e.g. \( \vec{P} \). The magnitude alone of such a vector will be represented by the same symbol without the bar, e.g. \( P \). The vector may also be represented by the combination of a magnitude and a unit vector parallel to the original vector, e.g. \( \vec{P} = lP \).

**Multiplication of a vector by a scalar**

To multiply a vector by a scalar \( A \) multiply the magnitude of the vector by \( A \) leaving the direction of the vector unchanged.

**Addition of vectors**

To add two vectors \( \vec{P} \) and \( \vec{Q} \) place the tail of \( \vec{Q} \) at the head of \( \vec{P} \). The vector \( \vec{R} \) drawn from the tail of \( \vec{P} \) to the head of \( \vec{Q} \) in Fig.33 is the vector sum \( \vec{P} + \vec{Q} \). It is seen in Fig.33 that the sum \( \vec{P} + \vec{Q} \) is identical with the sum \( \vec{Q} + \vec{P} \), therefore the commutative law holds for vector addition. It is also seen in Fig.33 that the vector \( \vec{R} \) is merely the diagonal of the parallelogram of which \( \vec{P} \) and \( \vec{Q} \) form two adjacent sides. This fact gives rise to vector addition's being known as the parallelogram law.

**Subtraction of vectors**

To subtract the vector \( \vec{Q} \) from the vector \( \vec{P} \) add \(-\vec{Q}\) to \( \vec{P} \).

**Components of a vector**

The components of a vector \( \vec{P} \) are the vectors whose sum is \( \vec{P} \). The most generally used components are the rectangular ones parallel to the x,y,z, axes. Thus

\[
\vec{P} = lP_x + jP_y + kP_z
\]  

(1)

where \( P_x, P_y, P_z \) are the projections of \( \vec{P} \) on the x,y,z axes respectively.

To add two vectors simply add their components.

\[
\vec{P} + \vec{Q} = (l(P_x + Q_x) + j(P_y + Q_y) + k(P_z + Q_z)
\]

(2)
**Multiplication of vectors - the scalar or dot product**

The scalar or dot product of two vectors is defined as the scalar equal in magnitude to the product of the magnitudes of the two vectors and the cosine of the smaller angle between them; thus

\[ \vec{P} \cdot \vec{Q} = PQ \cos(PQ) \]  

(3)

Eq. (3) also shows that the dot product of two vectors equals the product of the magnitude of one vector and the component of the second vector on the first.

The commutative law holds for the dot product, i.e. \( \vec{P} \cdot \vec{Q} = \vec{Q} \cdot \vec{P} \). The distributive law also holds, i.e. \( (\vec{P}+\vec{Q}) \cdot \vec{R} = \vec{P} \cdot \vec{R} + \vec{Q} \cdot \vec{R} \). The unit vectors \( \hat{i}, \hat{j}, \hat{k} \) give the relations \( \hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1 \) and \( \hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0 \). In terms of rectangular components the dot product is

\[ \vec{P} \cdot \vec{Q} = (P_x \hat{i} + P_y \hat{j} + P_z \hat{k}) \cdot (Q_x \hat{i} + Q_y \hat{j} + Q_z \hat{k}) = P_x Q_x + P_y Q_y + P_z Q_z \]  

(4)

The square of a vector is \( \vec{P}^2 = \vec{P} \cdot \vec{P} = P_x^2 + P_y^2 + P_z^2 \).

**Multiplication of vectors - the vector or cross product**

The vector or cross product of two vectors is defined as the vector perpendicular to the plane of the two initial vectors and directed in the sense of the advance of a right hand screw rotated from the first vector to the second through the smaller angle between them. The cross product is equal in magnitude to the product of the magnitudes of the two vectors and the sine of the smaller angle between them; thus

\[ \vec{P} \times \vec{Q} = PQ \sin(PQ) \]  

(5)

Eq. (5) also shows that the magnitude of a cross product equals the area of the parallelogram of which \( \vec{P} \) and \( \vec{Q} \) are two adjacent sides. The cross product is shown diagramatically in Fig. 34.

The commutative law does not hold for the cross product since

\[ \vec{Q} \times \vec{P} = -(\vec{P} \times \vec{Q}) \]. The distributive law does hold however, i.e. \( (\vec{P}+\vec{Q}) \times \vec{R} = \)
VECTOR OR CROSS PRODUCT OF TWO VECTORS

Fig. 34

VECTOR FUNCTION OF A SCALAR

Fig. 35
(\overline{P}X\overline{R})+(\overline{Q}X\overline{R})$. The unit vectors $i, j, k$ give the relations $iXl = jXj = kXk = 0$ and $iXj = k, jXk = l, kXi = j$. In terms of rectangular components the cross product is

$$\overline{P}X\overline{Q} = (iPx+jPy+kP_z)X(iQ_x+jQ_y+kQ_z)$$

$$= i(P_yQ_z-P_zQ_y)+j(P_zQ_x-P_xQ_z)+k(P_xQ_y-P_yQ_x)$$

(6)

which may be more compactly expressed by the determinant

$$\overline{P}X\overline{Q} = \begin{vmatrix}
i & j & k \\
P_x & P_y & P_z \\
Q_x & Q_y & Q_z \\
\end{vmatrix}$$

A vector crossed with itself is zero, i.e. $\overline{P}X\overline{P} = 0$

The triple cross product

$$\overline{R}X(\overline{P}X\overline{Q}) = \overline{P}(\overline{R}X\overline{Q}) - \overline{Q}(\overline{R}X\overline{P})$$

(7)

**Division of vectors**

Vectors cannot be divided, hence there are no laws of vector division.

**Differentiation of vectors**

Let the vector $\overline{F}$ in Fig. 35 be given by $\overline{F} = ix+jy+kz$ where $x, y, z$, are functions of the scalar $t$ such as $x = x(t), y = y(t), z = z(t)$. Then the vector $\overline{F}$ is said to be a vector function of the scalar $t$. If the tail of $\overline{F}$ is kept fixed at the origin of coordinates and $t$ is varied then the head of $\overline{F}$ describes a curve. Consider on the curve in Fig. 35 the two points $a$ and $b$ whose position vectors are $\overline{F}$ and $\overline{F}_1$ respectively. Then the change $\overline{\Delta r}$ in $\overline{F}$ is $\overline{\Delta r} = \overline{F}_1 - \overline{F}$ and has the direction of the secant $ab$.

For infinitesimal values of $\overline{\Delta r}$ the secant approaches the tangent; thus the immediate change in $\overline{F}$ is along the tangent to the curve. In the limit of vanishingly small change in $t$ the change in $\overline{F}$ divided by the change in $t$ is the derivative of $\overline{F}$ with respect to $t$ and is written as $d\overline{F}/dt$.

If $\overline{F}_1 = ix_1+jy_1+kz_1$ then $\overline{\Delta r} = t(x_1-x)+j(y_1-y)+k(z_1-z) = t\Delta x+j\Delta y+k\Delta z$. 
So \[ \frac{d\mathbf{r}}{dt} = t\frac{dx}{dt} + \frac{dy}{dt} + \frac{dz}{dt} \] (8)

If \( t \) is the time then \( d\mathbf{r}/dt = \mathbf{v} \) the vector velocity.

The derivative of a dot product is

\[ \frac{d}{dt}(\mathbf{P} \cdot \mathbf{Q}) = \frac{d\mathbf{P}}{dt} \cdot \mathbf{Q} + \mathbf{P} \cdot \frac{d\mathbf{Q}}{dt} \] (9)

Eq. (9) shows that the derivative of a dot product is exactly the same as the derivative of an algebraic product, the order of terms being immaterial.

The derivative of a cross product is

\[ \frac{d}{dt}(\mathbf{P} \times \mathbf{Q}) = \frac{d\mathbf{P}}{dt} \times \mathbf{Q} + \mathbf{P} \times \frac{d\mathbf{Q}}{dt} \] (10)

Eq. (10) shows that the derivative of a cross product is the same as the derivative of an algebraic product, but the order of the vectors must be retained.
APPENDIX B

Vector Treatment of "Small" Angles

The use of vectors to represent physical quantities in equations of motion, etc. affords a very powerful and convenient method of handling such problems. However one very important quantity used in the study of the indication of the vertical cannot be rigorously treated as a vector, viz. angular displacements. The failure of angular displacements to act as vectors is due to the fact that the final position of a vector after a series of rotations about different axes depends on the order in which the rotations were carried out, e.g. a vector along the +x axis rotated 90° about the +y axis, 90° about the +x axis, and 90° about the +z axis in that order ends up along the -x axis; whereas the same vector rotated 90° about the +z axis, 90° about the +x axis, and 90° about the +y axis in that order ends up along the +x axis. This appendix is intended to show that angular displacements can be treated as vectors for practical purposes however, provided certain conditions are fulfilled.

Let an angular displacement be represented by a vector whose length gives the size of the angle and whose direction is along the axis about which the rotation is made. The direction is treated as positive by the right hand rule for rotation: a rotation, i.e. an angular velocity or an angular displacement, is positive in the sense of the advance of a right hand screw and is directed along the axis of rotation as shown in Fig. 17. Take a linear displacement A directed along the +y axis; rotate this about the x axis as shown in Fig. 36a. By trigonometry the new displacement is

\[ \overrightarrow{D_1} = jA \cos \theta_x + kA \sin \theta_x \]  

(11)
ROTATION ABOUT X AXIS

ROTATION ABOUT X AND Y AXES

ROTATION ABOUT X, Y, AND Z AXES

COMPOSITION OF ROTATIONS ABOUT SEVERAL AXES LEADING UP TO TREATMENT OF SMALL ANGLES AS VECTORS
By vector treatment the new displacement is

$$\vec{D} = \vec{O} + jA = t\theta_x X jA + jA = jA + kA\theta_x$$  \hspace{1cm} (12)

For Eqs. (11) and (12) to be identical the following conditions must evidently hold:  \(\cos \theta_x \neq 1\) and \(\sin \theta_x \neq \theta_x\) \hspace{1cm} (A)

Now rotate the \(z\) component of Eq. (11) about the \(y\) axis as shown in Fig. 36b obtaining for the displacement

$$\vec{D}_z = tA \sin \theta_x \sin \theta_y + jA \cos \theta_x + kA \sin \theta_x \cos \theta_y$$  \hspace{1cm} (13)

Obviously the \(y\) component of Eq. (11) cannot be rotated about the \(y\) axis. Now rotate the \(x\) and \(y\) components of Eq. (13) about the \(z\) axis as shown in Fig. 36c obtaining for the displacement

$$\vec{D}_n = tA(\sin \theta_x \sin \theta_y \cos \theta_z - \cos \theta_x \sin \theta_z) + jA(\sin \theta_x \sin \theta_y \sin \theta_z + \cos \theta_x \cos \theta_z) + kA \sin \theta_x \cos \theta_y$$  \hspace{1cm} (14)

Letting the \(\cos \theta \neq 1\) and \(\sin \theta \neq \theta\) in Eq. (14) gives

$$\vec{D}_n \approx tA(\theta_x \theta_y - \theta_z) + jA(\theta_x \theta_y + \theta_z) + kA \theta_x$$  \hspace{1cm} (15)

By vector treatment the total displacement is

$$\vec{D} = jA + \vec{O} = jA + (t\theta_x + j\theta_y + k\theta_z)X jA = tA \theta_x + jA + kA \theta_x$$  \hspace{1cm} (16)

For Eqs. (15) and (16) to be identical the following conditions must also hold: the double or triple product of angles must be negligibly small in comparison with the angles themselves. \hspace{1cm} (B)

When conditions (A) and (B) hold it is permissible to treat angles as vector quantities. The limiting conditions are shown in Fig. 37 where it is seen that for the

1. Sine condition

   Angles up to 45° may be used with a 10% tolerance
   " " " 32° " " " 10% "
   " " " 14° " " " 1% "

2. Cosine condition

   Angles up to 25° may be used with a 10% tolerance
   " " " 18° " " " 5% "
   " " " 8° " " " 1% "
Fig. 37
MAGNITUDE OF PERCENT ERROR IN SINE, COSINE, AND PRODUCT CONDITIONS INTRODUCED WHEN "SMALL" ANGLES ARE TREATED AS VECTORS
3. Product condition

Angles up to 6° may be used with a 10% tolerance

3° 3° 35° 5% 1%

The product condition is evidently the most restricting condition.
APPENDIX C

Euler’s Angles

The failure of angular displacements to act rigorously as vectors was noted in Appendix B where it was seen that the failure is due to the fact that the final displacement after a series of rotations about different axes depends on the order in which the rotations were carried out. It follows that this dependence on order of rotation of angular displacements makes the angles $\theta_x, \theta_y, \theta_z$ unsuitable for use in transforming vectors from one set of coordinate axes to another set having a common origin. As a result a peculiar set of three angles known as Euler’s angles are used in transformations between sets of axes.

The Euler angles $\varphi, \psi, \theta$ and the attendant nomenclature used here are that of Schaefer(9) and are shown in Fig.38: where

1. $X,Y,Z$ are one set of axes
2. $x,y,z$ are a second set of axes
3. $O$ is the common origin of coordinates
4. Originally $x,y,z$ and $X,Y,Z$ are coincident
5. LON is the line of nodes and is the intersection of the xy and XY planes
6. $Z$ is normal to the XY plane, and $z$ is normal to the xy plane; both $z$ and $Z$ are positive according to the right hand rule.
7. $\varphi$ is the angle of rotation (precession) of the line of nodes from the $X$ axis, and is positive about the $+Z$ axis.
8. $\psi$ is the angle of rotation of the $x$ axis from the line of nodes, and is positive about the $+z$ axis.
9. $\theta$ is the angle between $z$ and $Z$, and is positive about the line of nodes; the positive direction of the line of nodes originally
XYZ are one set of axes
xyz are a second set of axes
O is the common origin
LON is the line of nodes - intersection of XY and xy planes
φψθ are Euler's angles

Fig. 38
Euler's angles for transformations between coordinate systems
coincides with the positive x and X axes.

To transform from xyz to XYZ and vice versa the following tensor components are used

\[ x = a_{11}X + a_{12}Y + a_{13}Z \]
\[ y = a_{21}X + a_{22}Y + a_{23}Z \]
\[ z = a_{31}X + a_{32}Y + a_{33}Z \]  
(17)

\[ X = a_{11}x + a_{21}y + a_{31}z \]
\[ Y = a_{12}x + a_{22}y + a_{32}z \]
\[ Z = a_{13}x + a_{23}y + a_{33}z \]  
(18)

Putting the values obtained trigonometrically from Fig. 38 for the coefficients \( a_{11}, \) etc. of Eqs. (17) and (18) gives

\[ a_{11} = \cos\phi\cos\psi - \sin\phi\sin\psi\cos\theta \]
\[ a_{12} = \sin\phi\cos\psi + \cos\phi\sin\psi\cos\theta \]
\[ a_{13} = \sin\psi\sin\theta \]
\[ a_{21} = -\cos\phi\sin\psi - \sin\phi\cos\psi\cos\theta \]
\[ a_{22} = -\sin\phi\sin\psi + \cos\phi\cos\psi\cos\theta \]
\[ a_{23} = \cos\psi\sin\theta \]
\[ a_{31} = \sin\phi\sin\theta \]
\[ a_{32} = -\cos\phi\sin\theta \]
\[ a_{33} = \cos\theta \]  
(19)

To transform any vector let its xyz and XYZ components replace the simple xyz and XYZ in Eqs. (17) and (18), e.g.

\[ P_x = a_{11}P_X + a_{12}P_Y + a_{13}P_Z \] etc.
APPENDIX D

Velocity and Acceleration Referred to Moving Coordinate Systems

In many types of problems it is desirable to express a velocity or an acceleration in one system of axes, e.g. a set fixed in the Earth, when the values of these vectors are given in some other system of axes, e.g. a set fixed in an airplane. It is the purpose of this appendix to derive the equations required for the transformation of vector changes between different sets of axes.

Transformation of Velocities or First Derivatives

One type of transformation required is that between two frames of reference, i.e. sets of axes, that do not have a common origin and have no rotation relative to each other, as is shown in Fig. 39. In this figure

\[ \mathbf{r}_{0_2} \] is the position vector of point \( 0_2 \) referred to system 1
\[ \mathbf{r}_{0_1} \] is the position vector of point \( 0_1 \) referred to system 1
\[ \mathbf{r}_{0_2} \] is the position vector of point \( 0_2 \) referred to system 1
\[ \mathbf{r}_{0_1} \] is the position vector of point \( 0_1 \) referred to system 1

By the law of vector addition

\[ \mathbf{r}_{0_1} = \mathbf{r}_{0_1} + \mathbf{r}_{0_2} \] \hspace{1cm} (20)

Taking the time derivative of Eq. (20) gives

\[ \frac{\mathbf{r}_{0_1}}{\mathbf{t}} = \frac{\mathbf{r}_{0_1} + \mathbf{r}_{0_2}}{\mathbf{t}} \] \hspace{1cm} (21)

where \[ \frac{\mathbf{r}_{0_1}}{\mathbf{t}} = \frac{d\mathbf{r}_{0_1}}{dt} \], etc.

Eq. (21) shows that when the frames of reference are non-rotating with respect to each other the rate of change of a position vector referred
$x_1y_1z_1$ and $x_2y_2z_2$ are two mutually non-rotating reference systems.

$\vec{R}_{0102}$ is position vector of point $O_2$ in $x_1y_1z_1$ system.

$\vec{R}_{01P}$ is position vector of point $P$ in $x_1y_1z_1$ system.

$\vec{R}_{02P}$ is position vector of point $P$ in $x_2y_2z_2$ system.

\[
\vec{R}_{01P} = \vec{R}_{0102} + \vec{R}_{02P}
\]

or

\[
\left(\frac{d\vec{R}_{01P}}{dt}\right)_{01} = \left(\frac{d\vec{R}_{0102}}{dt}\right)_{01} + \left(\frac{d\vec{R}_{02P}}{dt}\right)_{02}
\]

*Fig. 39*

Rate of change of a vector referred to two sets of axes having separate origins and no mutual rotation.
to one frame equals the rate of change of the position vector of the origin of the second frame referred to the first frame plus the rate of change of the original position vector referred to the second frame.

For any vector other than position and its derivatives (velocity and acceleration) the first term on the right side of Eq. (21) becomes zero, from which it follows that the rate of change of a vector (except position, etc.) is the same when referred to any set of mutually non-rotating frames.

A second type of transformation is that required between two frames having a common origin, but possessing rotation relative to each other, as shown in Fig. 40. This transformation is known as the theorem of Coriolis11.

\[ x_1y_1z_1 \text{ is one reference frame centered at } 0 \]
\[ x_2y_2z_2 \text{ is a second reference frame centered at } 0 \]
\[ x_3y_2z_3 \text{ is rotating with an angular velocity } \omega_{12} \text{ with respect to } x_1y_1z_1 \text{ about an axis coincident with the vector } \omega_{12} \]
\[ d\mathbf{R}_{01} \text{ is the change in the vector } \mathbf{R} \text{ referred to the "stationary" frame 1} \]
\[ d\mathbf{R}_{02} \text{ is the change in the vector } \mathbf{R} \text{ referred to the "rotating" frame 2} \]
\[ \omega_{12}dtXR \text{ is the distance the point of observation, which was originally at the head of vector } \mathbf{R}, \text{ has moved by virtue of its remaining in frame 2} \]

By the law of vector addition

\[ (d\mathbf{R})_{01} = (d\mathbf{R})_{02} + \omega_{12}dtXR \]

or

\[ \dot{(\mathbf{R})}_{01} = \dot{(\mathbf{R})}_{02} + \omega_{12}XR \]

Eq. (22) shows that the rate of change of a vector referred to a "fixed"

\( \omega_{12} \) is the angular velocity of the system \( x_2y_2z_2 \) referred to the system \( x_1y_1z_1 \).

\((d\mathbf{R})_{01}\) is the change in vector \( \mathbf{R} \) referred to the system \( x_1y_1z_1 \).

\((d\mathbf{R})_{02}\) is the change in vector \( \mathbf{R} \) referred to the system \( x_2y_2z_2 \).

\[
(d\mathbf{R})_{01} = (d\mathbf{R})_{02} + \omega_{12} dt \times \mathbf{R}
\]

or

\[
\frac{d\mathbf{R}}{dt}_{01} = \frac{d\mathbf{R}}{dt}_{02} + \omega_{12} \times \mathbf{R}
\]

Fig. 40

Rate of change of a vector referred to two sets of axes having a common origin and mutual rotation.
frame equals the rate of change of the vector referred to a "moving" frame plus the cross product of the angular velocity, between the two frames, and the vector.

Combining Eqs. (21) and (22) to get the rate of change of a position vector in a "fixed" frame in terms of rate of change of the vector in a "moving" frame gives

$$\frac{d}{dt}(\mathbf{R}_{01}P)_{01} = (\mathbf{R}_{01}O_2)_{01} + (\mathbf{R}_{O2}P)_{02} + \mathbf{\omega}_{12} \times \mathbf{R}_{O2}P$$

(23)

The following vector notation has been used in Eqs. (20)-(23) and will be used throughout this paper:

- $\mathbf{R}_{ab}$ is the position vector from point $a$ to point $b$
- $\mathbf{\omega}_{ab}$ is the angular velocity of a set of axes $b$ centered at point $b$ with respect to a set of axes $a$ centered at point $a$
- $\mathbf{a}$ is a vector referred to a set of axes $a$ centered at point $a$

**Transformation of accelerations or second derivatives**

To make the transformation of accelerations between two mutually translating and rotating systems first write $d\mathbf{R}/dt = \mathbf{v}$ for convenience. With this substitution Eq. (23) becomes

$$\frac{d}{dt}(\mathbf{v}_{01}P)_{01} = (\mathbf{v}_{01}O_2)_{01} + (\mathbf{v}_{O2}P)_{02} + \mathbf{\omega}_{12} \times \mathbf{R}_{O2}P$$

(24)

Now acceleration is merely the time derivative of velocity, so the time derivative of Eq. (24) with respect to system 1 gives

$$\left[\frac{d}{dt}(\mathbf{v}_{01}P)_{01}\right]_{01} = \left[\frac{d}{dt}(\mathbf{v}_{01}O_2)_{01}\right]_{01} + \left[\frac{d}{dt}(\mathbf{v}_{O2}P)_{02}\right]_{01} + \left[\frac{d}{dt}(\mathbf{\omega}_{12} \times \mathbf{R}_{O2}P)\right]_{01}$$

(25)

The first and third terms on the right side of Eq. (25) are referred only to system 1, but the second term is referred to both systems 1 and 2. By use of Eq. (23) the second term can be referred to system 2 alone.

Applying Eq. (23) to the second term and expanding the third term by Eq. (10) makes Eq. (25) become

$$\left[\frac{d}{dt}(\mathbf{v}_{01}P)_{01}\right]_{01} = \left(\mathbf{v}_{01}O_2\right)_{01} + \left(\mathbf{v}_{02}P\right)_{02} + \left[\mathbf{\omega}_{12} \times \left(\mathbf{v}_{O2}P\right)_{02} + \mathbf{\omega}_{12} \times \mathbf{R}_{O2}P\right]_{01}$$

(26)
Again making use of the fact that \( \ddot{v} = \frac{d\ddot{R}}{dt} \) gives for Eq. (26)

\[
\ddot{(R_{01}P)}_{01} = \ddot{(R_{01}O_2)}_{01} + \dddot{(R_{02}P)}_{02} + 2\dddot{\omega}_{12} \times (R_{02}P)_{02} + \dddot{\omega}_{12} \times R_{02}P + \dddot{\omega}_{12} \times (\dddot{\omega}_{12} \times R_{02}P) \tag{27}
\]

where \( \dddot{\omega}_{12} = \frac{d^2R_{01}P}{dt^2} \), etc.

Eq. (27) gives the acceleration of a point \( P \) referred to a fixed system in terms of the acceleration referred to a moving system plus the necessary correction terms to take care of the relative motion between the two systems. The following is an analysis of the separate terms in Eq. (27)

- \( \dddot{(R_{01}P)}_{01} \) is the acceleration of the point \( P \) referred to system 1
- \( \dddot{(R_{01}O_2)}_{01} \) is the acceleration of the origin of system 2 referred to system 1
- \( \dddot{(R_{02}P)}_{02} \) is the acceleration of point \( P \) referred to system 2
- \( 2\dddot{\omega}_{12} \times (R_{02}P)_{02} \) is the acceleration of Coriolis due to the fact that the point \( P \) has a component of linear velocity referred to the second system that is perpendicular to the angular velocity of the second system referred to the first system
- \( \dddot{\omega}_{12} \times R_{02}P \) is the tangential acceleration due to the rotation of system 2 referred to system 1
- \( \dddot{\omega}_{12} \times (\dddot{\omega}_{12} \times R_{02}P) \) is the centripetal acceleration due to the rotation of system 2 referred to system 1

**Acceleration of a pendulous element carried in an airplane**

The work of the previous sections on the transformations of velocities and accelerations between frames of reference has a particularly valuable application in the study of the motion of a body on the Earth. The measurable accelerations are all referred to the Earth, but
Newton's law of motion which relates forces and corresponding accelerations is strictly valid only in a frame of reference which is unaccelerated with respect to the fixed stars\(^{(10)}\), i.e. to an inertial system. The following treatment builds up the acceleration, referred to an inertial system, of a pendulous element carried in an airplane, in terms of the intermediate accelerations which can be more easily measured.

For this problem of studying the acceleration of a pendulous element carried by an airplane six different frames of reference are needed. These frames are shown in Fig. 3 and are

1. II centered at I and fixed in an inertial system
2. EE centered at E, the center of the Earth, and fixed in the Earth
3. OO centered at O, on the surface of the Earth, and non-rotating with respect to system EE
4. AA centered at A, the center of gravity of the airplane carrying the pendulous element, and fixed in the airplane
5. PP centered at P, the pivot supporting the pendulous element, and fixed in the pendulous element

Point C is the center of gravity of the pendulous element.

The inertial system II may be arbitrarily chosen and the point I arbitrarily located, because an acceleration will be measured identically in any frame of reference that is unaccelerated with respect to fixed space. This statement is one of the fundamentals of the restricted theory of relativity\(^{(1)}\).

The system OO travels with the airplane such that the point O is always directly below the point A. During any maneuver in which the motion of the pendulous element is studied the system OO is temporarily "frozen" on the surface of the Earth.
In system PP the acceleration of the center of gravity \( C \) of the pendulous element is, according to Eq. (27)
\[
\begin{align*}
\frac{\ddot{u}}{} (R_{PC})_{PP} &= (R_{PC})_{PP} \\
\end{align*}
\]  
(28a)

In system AA the acceleration of point \( C \) is
\[
\begin{align*}
\frac{\ddot{u}}{} (R_{AC})_{AA} &= (R_{AP})_{AA} + \dot{\omega}_{AP} \times (R_{PC})_{PP} + \dot{\omega}_{AP} \times (\dot{\omega}_{AF} \times \vec{R}_{PC}) + (R_{PC})_{PP} \\
\end{align*}
\]  
(28b)

In system 00 the acceleration of point \( C \) is
\[
\begin{align*}
\frac{\ddot{u}}{} (R_{OC})_{00} &= (R_{OA})_{00} + 2 \dot{\omega}_{OA} \times (R_{AC})_{AA} + \dot{\omega}_{OA} \times (\dot{\omega}_{OA} \times \vec{R}_{AC}) + (R_{AC})_{AA} \\
\end{align*}
\]  
(28c)

In system EE the acceleration of point \( C \) is
\[
\begin{align*}
\frac{\ddot{u}}{} (R_{EC})_{EE} &= (R_{EO})_{EE} + 2 \dot{\omega}_{EO} \times (R_{OC})_{00} + \dot{\omega}_{EO} \times (\dot{\omega}_{EO} \times \vec{R}_{OC}) + (R_{OC})_{00} \\
\end{align*}
\]  
(28d)

In system II the acceleration of point \( C \) is
\[
\begin{align*}
\frac{\ddot{u}}{} (R_{IC})_{II} &= (R_{IE})_{II} + 2 \dot{\omega}_{IE} \times (R_{EC})_{EE} + \dot{\omega}_{IE} \times (\dot{\omega}_{IE} \times \vec{R}_{EC}) + (R_{EC})_{EE} \\
\end{align*}
\]  
(28e)

Now by substituting Eq. (28d) for the last term of Eq. (28e), substituting Eq. (28c) for the last term of Eq. (28d), etc. back to Eq. (28a) it is possible to express the acceleration of the center of gravity \( C \) of the pendulous element referred to the intermediate system II in terms of the accelerations referred to the intermediate systems. Such substitution gives
\[
\begin{align*}
\frac{\ddot{u}}{} (R_{IC})_{II} &= (R_{IE})_{II} + 2 \dot{\omega}_{IE} \times (R_{EC})_{EE} + \dot{\omega}_{IE} \times (\dot{\omega}_{IE} \times \vec{R}_{EC}) + \\
&\quad (R_{EO})_{EE} + 2 \dot{\omega}_{EO} \times (R_{OC})_{00} \dot{\omega}_{EO} \times (\dot{\omega}_{EO} \times \vec{R}_{OC}) + \\
&\quad (R_{OA})_{00} + 2 \dot{\omega}_{OA} \times (R_{AC})_{AA} + \dot{\omega}_{OA} \times (\dot{\omega}_{OA} \times \vec{R}_{AC}) + \\
&\quad (R_{AP})_{AA} + 2 \dot{\omega}_{AP} \times (R_{PC})_{PP} + \dot{\omega}_{AP} \times (\dot{\omega}_{AP} \times \vec{R}_{PC}) + \\
&\quad (R_{PC})_{PP} \\
\end{align*}
\]  
(29)

Eq. (29) gives the complete and rigorous expression for the acceleration of the center of gravity of a pendulous element, but is of such length as to be unwieldy. Fortunately this equation is subject to considerable simplification in practical cases as is shown in the following analysis of the component terms

1. \((R_{IE})_{II}\) is the acceleration of the center of the Earth in fixed
space. This term is negligibly small inasmuch as the centripetal acceleration of the Earth in its annual orbit around the Sun (the only known astronomical acceleration affecting the Earth) is about 0.05% of the acceleration of gravity.

2. \( 2\mathbf{w}_{IE}\times (\mathbf{R}_{EC})_{EE} \) is the Coriolis acceleration due to the daily rotation of the Earth crossed with the velocity of the airplane over the surface of the Earth. Being a maximum of 0.2% of the acceleration of gravity when the airplane is doing 400 mph (and proportionally less for lower speeds) this term is generally negligible except in cases where extreme accuracy of measurement is desirable. This term will be retained, however, and will be grouped with the acceleration of gravity, which is discussed later, to form an apparent acceleration of gravity for moving bases.

3. \( \frac{1}{\mathbf{w}_{IE}}\mathbf{r}_{EC} \) is the tangential acceleration due to the daily rotation of the Earth and is approximately zero since the daily rotation of the Earth is substantially constant.

4. \( \mathbf{w}_{IE}\times (\mathbf{w}_{IE}\times \mathbf{r}_{EC}) \) is the centripetal acceleration due to the daily rotation of the Earth. This term is coupled with the acceleration of universal gravitation, which is discussed later, to form the local acceleration of gravity.

5. \( (\mathbf{R}_{DO})_{EE} \) is zero since system 00 is frozen on the Earth during any maneuver being studied.

6. \( 2\mathbf{w}_{DO}\times (\mathbf{R}_{OC})_{00} = \frac{1}{\mathbf{w}_{DO}}\mathbf{r}_{OC} = \mathbf{w}_{DO}\times (\mathbf{w}_{DO}\times \mathbf{r}_{OC}) = 0 \) since system 00 is non-rotating with respect to system EE, making \( \mathbf{w}_{DO} = 0 \) and \( \mathbf{r}_{DO} = 0 \)

7. \( (\mathbf{R}_{OA})_{00} \) is the acceleration of the center of gravity of the airplane referred to system 00 fixed on the Earth. Since system 00 is non-rotating with respect to system EE then from Eq. (21) \( (\mathbf{R}_{OA})_{00} = (\mathbf{R}_{OA})_{EE} \).
The form \( (\mathbf{R}_{OA})_{EE} \) will be used hereafter. Also since system 00 is non-rotating relative to system EE then \( \omega_{OA} = \omega_{EA} \), the latter form \( \omega_{EA} \) here will also be the one used. The term 7. is a very important one.

8. \( 2\omega_{OA} \times (\mathbf{R}_{AC})_{AA} \) is the Coriolis acceleration due to rotation of the airplane axes AA crossed with the velocity of the center of gravity of the pendulous element with respect to system AA. The last named velocity is sufficiently small in any satisfactory indicator of the vertical to render this whole term negligible.

9. \( \dot{\omega}_{OA} \times \mathbf{R}_{AC} \) is the tangential acceleration due to a rotation of the airplane relative to the Earth. The position vector \( \mathbf{R}_{AC} = \mathbf{R}_{AP} + \mathbf{R}_{PC} \); \( \mathbf{R}_{AP} \) is the distance between the center of gravity of the airplane A and the pivot P supporting the pendulous element while \( \mathbf{R}_{PC} \) is the distance between the pivot and the center of gravity C of the pendulous element.

In most combinations of airplanes and indicators of the vertical \( \mathbf{R}_{AP} \gg \mathbf{R}_{PC} \); thus \( \mathbf{R}_{AC} \approx \mathbf{R}_{AP} \). The term 9. is a very important one.

10. \( \mathbf{w}_{OA} \times (\mathbf{w}_{OA} \times \mathbf{R}_{AC}) \) is the radial acceleration due to a rotation of the airplane relative to the Earth; as above \( \mathbf{R}_{AC} \approx \mathbf{R}_{AP} \). The term 10. is an important one.

11. \( (\mathbf{R}_{AP})_{AA} \) is the acceleration of the pivot relative to the airplane. This term will generally be zero, except for such cases as the pivot's being spring mounted, and so is considered as being negligible.

12. \( 2\mathbf{w}_{AP} \times (\mathbf{R}_{PC})_{PP} \) is generally zero, except for such cases as the pendulous element's being spring mounted from the pivot, and so is negligible. The same is true of term \( (\mathbf{R}_{PC})_{PP} \).

13. \( \dot{\omega}_{AP} \times \mathbf{R}_{PC} \) and \( \mathbf{w}_{AP} \times (\dot{\omega}_{AP} \times \mathbf{R}_{PC}) \) are both negligible since \( \mathbf{R}_{PC} \) is generally a very small distance.

On the basis of the above analysis it is seen that only terms 2., 4.,
7., 9., and 10. need be retained in the practical case of an indicator of the vertical mounted in an airplane. It was also shown in the analysis that \( \omega_{OA} = \omega_{EA} \) and that \( \overrightarrow{RA_C} \neq \overrightarrow{RA_P} \). In addition a further reduction is possible in the case of 2. above; \( \frac{\dot{\overrightarrow{RE}}_{EE}}{\overrightarrow{RE}}_{EE} = \frac{\dot{\overrightarrow{RE}}_{EO}}{\overrightarrow{RE}}_{EO} + \frac{\dot{\overrightarrow{RE}}_{OA}}{\overrightarrow{RE}}_{OA} + \frac{\dot{\overrightarrow{RE}}_{AC}}{\overrightarrow{RE}}_{AC} \) where \( \frac{\dot{\overrightarrow{RE}}_{EO}}{\overrightarrow{RE}}_{EO} = 0 \) since system 00 is frozen relative to system EE during any maneuver, and \( \frac{\dot{\overrightarrow{RE}}_{OA}}{\overrightarrow{RE}}_{OA} \) the velocity of the airplane relative to the Earth is much greater than \( \frac{\dot{\overrightarrow{RE}}_{AC}}{\overrightarrow{RE}}_{AC} \) the velocity of the pendulous element relative to the airplane. It follows that \( \frac{\dot{\overrightarrow{RE}}_{EO}}{\overrightarrow{RE}}_{EO} \neq \frac{\dot{\overrightarrow{RE}}_{OA}}{\overrightarrow{RE}}_{OA} \).

On the basis of the above analysis Eq. (29) reduces to

\[
\frac{\ddot{\overrightarrow{RI}}_{II}}{\overrightarrow{RE}}_{II} = \frac{\dot{\overrightarrow{RE}}_{EO}}{\overrightarrow{RE}}_{EO} + \frac{\omega_{EA} \times \overrightarrow{RAF}}{\overrightarrow{RE}}_{EA} + \frac{\omega_{EA} \times \overrightarrow{RA}}{\overrightarrow{RE}}_{EA} + 2 \frac{\omega_{IE} \times \overrightarrow{RE}}{\overrightarrow{RE}}_{IE} + \frac{\dot{\overrightarrow{RE}}_{OA}}{\overrightarrow{RE}}_{OA} 
\]

(30)

Eq. (30) gives the acceleration of the center of gravity of the pendulous element. Eq. (30) also has the following interpretation; \( \frac{\dot{\overrightarrow{RI}}_{II}}{\overrightarrow{RE}}_{II} \) is the acceleration that must be possessed by the center of gravity of the pendulous element in order that the pendulous element remain attached to the pivot P in the airplane while the airplane executes maneuvers over the Earth. However, to an observer in an inertial system, the motion of the pendulous element, by virtue of its remaining with the pivot P, is exactly the same as the motion produced by the application of a force \( \overrightarrow{F_k} \) applied to the element, where

\[
\overrightarrow{F_k} = m \frac{\ddot{\overrightarrow{RI}}_{II}}{\overrightarrow{RE}}_{II} 
\]

(31)

and \( m = \) mass of the pendulous element. It follows that as far as the inertial system II is concerned a force \( \overrightarrow{F_k} \) must be applied to the pendulous element in order that the element remain with the pivot P.

It should be recalled here that all this interest in inertial systems is due to the fact that Newton's law of motion, which states that the applied force equals the product of the mass of a body and the resulting
acceleration, is valid only in an inertial system. In other words the equations governing the motion of bodies under the action of applied forces can be formulated only in an inertial system.

Eq. (30) expresses the accelerations of a pendulous element in an airplane due to the motions of the different coordinate systems only. In addition to these accelerations any body on the Earth will also be subject to the acceleration of universal gravitational attraction of the mass of the Earth; and the gravitational force \( F_{\text{grav}} \) will be partly responsible for the motion of such a body. It follows then that the force \( F \) remaining to hold the pendulous element to the pivot in the presence of the Earth's gravitational field will be

\[
F = F_k - F_{\text{grav}}
\]  

(32)

Now the force of gravitational attraction on the Earth is

\[
F_{\text{grav}} = -\frac{GMm}{RE^2}
\]  

(33)

where \( M \) = mass of the Earth \( G \) = universal gravitational constant

Combining Eqs. (31) and (33) in Eq. (32) gives

\[
F = m\left[(\hat{R}_{IC})_{II} + \frac{GM}{RE^2}\right]
\]  

(34)

Using the value of \( (\hat{R}_{IC})_{II} \) given in Eq. (30) gives

\[
F = m \left[ (\hat{R}_{OA})_{EE} + \hat{\omega}_{EA}\hat{X}_{AP} + \hat{\omega}_{EA}(\hat{\omega}_{EA}\hat{X}_{AP}) \right] + \frac{2\hat{\omega}_{IE}(\hat{R}_{OA})_{EE}}{1} + \frac{1}{\hat{\omega}_{IE}(\hat{\omega}_{IE}\hat{X}_{EE}) + \frac{GM}{RE^2}}
\]  

(35)

In the second bracket of Eq. (35) the centripetal acceleration due to the daily rotation of the Earth can be combined with the acceleration of universal gravitation to give the observed local acceleration of gravity as shown in Fig. 2.

\[
\vec{g} = \hat{\omega}_{IE}(\hat{\omega}_{IE}\hat{X}_{EE}) + \frac{GM}{RE^2}
\]  

(36)

The magnitude of \( \vec{g} \) in Eq. (36) is the magnitude of the acceleration of
gravity that would be measured at any desired point on the Earth's surface with a pendulum, etc. while the direction of \( \overline{g} \) is the direction indicated by a plumb bob.

Putting Eq. (36) in Eq. (35) gives

\[
\overline{F} = m \left[ (R_{OA})_{EE} + \overline{w}_{EA} \times R_{AP} + \overline{w} \times \overline{w} (R_{EA} \times R_{AP}) \right] + \left[ 2 \dot{\omega} \times (R_{OA})_{EE} + \overline{g} \right]
\]

(37)

The force \( \overline{F} \) in Eq. (37) is the total force applied to the pendulous element by the pivot in order to keep the element attached to the pivot and in the presence of the Earth's gravitational field. The three terms in the first bracket of Eq. (37) represent the accelerations applied to the pendulous element by the motion of the airplane carrying the element, i.e. the acceleration of the center of gravity of the airplane and rotational accelerations about the center of gravity of the airplane. The two terms in the second bracket of Eq. (37) represent the Coriolis acceleration due to the Earth's daily rotation, and the acceleration of gravity; the former serving to cause a deviation in the apparent acceleration of gravity from the true (which is \( \overline{g} \) alone) when on a moving base.

If the pendulous element were fixed relative to the Earth all the terms in Eq. (37) except the acceleration of gravity \( \overline{g} \) become zero. The force \( \overline{F} \) required to maintain the element fixed on the Earth in the presence of gravitational and centripetal forces is then \( m \overline{g} \), and is directed upwards.

**The Acceleration of Coriolis**

The acceleration of Coriolis was discussed in part 2. of the analysis of Eq. (29) and was shown to be

\[
\overline{a}_C = 2 \dot{\omega} \times (R_{OA})_{EE}
\]

(38)
where \( \omega_{IE} \) = the daily angular velocity of spin of the Earth 
\( (\overrightarrow{ROA})_{EE} \) = the velocity of the center of gravity of the airplane referred to axes fixed in the Earth.

Eq. (38) represents the acceleration \( \overline{a}_C \) acting on the pivot to keep the pivot fixed in the airplane when the airplane is constrained to fly a straight course.

With Earth axes \( 00 \), which are parallel to Earth-center axes \( EE \), taken so that
- \( X \) points toward the geographical north
- \( Y \) points toward the geographical west
- \( Z \) points vertically upward parallel to the acceleration of gravity \( g \)

the local components of the Earth's angular velocity \( \omega_{IE} \) lie in the meridian plane, i.e. \( \omega_{IEY} = 0 \), as is shown in Fig. 41. Then

\[
\begin{align*}
\omega_{IEX} &= \omega_{IE\cos\lambda} \\
\omega_{IEZ} &= \omega_{IE\sin\lambda}
\end{align*}
\]

(39)

where \( \lambda \) = the latitude on the Earth

Let the XYZ components of \( (\overrightarrow{ROA})_{EE} \) be

\( (\overrightarrow{ROA})_{EE} = t\overrightarrow{X} + j\overrightarrow{Y} + k\overrightarrow{Z} \)

(40)

Putting Eqs. (39) and (40) in Eq. (38) gives

\[
\begin{align*}
2\omega_{IEX}(\overrightarrow{ROA})_{EE} &= \begin{vmatrix} t & j & k \\ 2\omega_{IE\cos\lambda} & 0 & 2\omega_{IE\sin\lambda} \\ \dot{X} & \dot{Y} & \dot{Z} \end{vmatrix} \\
&= 2\omega_{IE}[t\dot{Y}\sin\lambda + j(X\sin\lambda - Z\cos\lambda) + k\dot{Z}\cos\lambda]
\end{align*}
\]

(41)

For an airplane flying in a horizontal plane \( Z = 0 \), and the Coriolis acceleration of a pendulous element carried by the airplane is, from Eq. (41)

\[
\overline{a}_C = 2\omega_{IE}[-t\dot{Y}\sin\lambda + jX\sin\lambda + k\dot{Z}\cos\lambda]
\]

(42)

and the horizontal component is
Fig. 41

COMPONENTS AT ANY POINT ON THE EARTH OF THE DAILY SPIN VELOCITY OF THE EARTH IN SPACE AS A FUNCTION OF THE LATITUDE OF THE POINT
\[ \ddot{a}_{CH} = -2\omega I_{HE} \sin \lambda (iY - jX) \]  

(43)

The horizontal velocity \( \dot{V}_H \) of the airplane is given in Eq. (40) as

\[ \dot{V}_H = iX + jY \]  

(44)

Now if Eq. (43) is treated by the method of the complex plane\(^{(12)}\), where the \( X \) axis is the axis of real quantities and the \( Y \) axis is the axis of imaginary quantities

\[ iY - jX \rightarrow Y - jX \]  

(45)

where \( i = 1 \) and \( j = \sqrt{-1} \). Now multiply and divide by \( j \)

\[ iY - jX = j(Y - jX) e^{-j\pi/2} \]  

(46)

since \( 1/j = e^{-j\pi/2} \)

\[ iY - jX = (X + jY) e^{-j\pi/2} = \dot{V}_H e^{-j\pi/2} \]  

(47)

since \( j^2 = -1 \) and \( \dot{X} + j\dot{Y} = \ddot{V}_H \)

Combining Eqs. (44) - (47) makes Eq. (43) become

\[ \ddot{a}_{CH} = -2\dot{V}_H \omega I_{HE} \sin \lambda e^{-j\pi/2} \]  

(48)

It is shown later in Appendix E that the acceleration tending to make a pendulous element rotate around its pivot is the negative of the acceleration \( \ddot{a}_{CH} \) of the pivot. It follows that the horizontal component of the Coriolis acceleration that is to be coupled with the acceleration of gravity to form an apparent vertical for moving bodies is the negative of the acceleration \( \ddot{a}_{CH} \) given in Eq. (48) so that

\[ \ddot{a}_{CHF} = 2\dot{V}_H \omega I_{HE} \sin \lambda e^{-j\pi/2} \]  

(49)

where \( \ddot{a}_{CHF} \) is the horizontal component of the acceleration of Coriolis that tends to cause a pendulous element, or any free body such as atmospheric and ocean currents, to deviate in a horizontal direction.

Eq. (49) shows that the horizontal component of the acceleration of Coriolis has a magnitude that depends only on the speed \( \dot{V}_H \) of the airplane and the sine of the latitude \( \lambda \) (maximum at the poles and zero at
the equator), but is independent of the heading of the airplane since neither \( \dot{X} \) nor \( \dot{Y} \) appear in the final answer. The magnitude of the horizontal component of the Coriolis acceleration is shown in Fig. 4 as a function of the speed of the body and of the latitude, where it is seen that the deviation of the apparent vertical for a body moving 300 mph in latitude 45° is about 5 minutes of arc. The direction of the acceleration acting on pendulous or free bodies is always at right angles to the heading of the airplane, and directed to the right side in the northern hemisphere, as is shown in Fig. 42. In the southern hemisphere the acceleration is directed to the left since the sine of the latitude changes algebraic sign.

The rightward direction of the acceleration of Coriolis gives rise to:

1. The drifting to the right of ocean currents such as the Gulf Stream, and the counter-clockwise rotation of winds blowing in toward a low pressure area in the cyclonic motion of the atmospheric disturbances; both examples cited for the northern hemisphere.

2. The shift to the right of the apparent plumb direction, or the shift to the left of the apparent zenith, when measured from a moving body; also cited for the northern hemisphere.

The vertical component \( \bar{a}_{CV} \) of the acceleration of Coriolis in Eq. (42) is given by

\[
\bar{a}_{CV} = 2\omega IE\dot{Y}\cos\lambda
\]  

(50)

As in the treatment of the horizontal component it is the negative of the acceleration \( \bar{a}_{CV} \) that acts on pendulous or free bodies, making Eq. (50) become

\[
\bar{a}_{CVP} = -2\omega IE\dot{Y}\cos\lambda
\]  

(51)

Eq. (51) shows that the magnitude of the vertical component of the acceleration of Coriolis varies with the heading of the airplane (maximum
Fig. 42

Direction of the horizontal component of Coriolis acceleration due to horizontal velocity of an airplane for any heading of the airplane.
when east or west and zero when north or south), the velocity $v_H$ of the airplane, and the cosine of the latitude $\lambda$ (maximum at the equator and zero at the poles). The direction is the same in both the northern and southern hemisphere since the cosine of the latitude does not change algebraic sign. The magnitude of this term is so small compared with gravity (only one-third of one percent of gravity at the equator for an airplane flying 500 mph east or west) that it can be neglected, since it is always either parallel or anti-parallel to gravity. Eq. (51) is sometimes called the centrifugal effect, but it is seen to be merely the vertical component of the Coriolis effect.

The effect of the acceleration of Coriolis on a falling body, i.e. $\dot{x} = \dot{y} = 0$, takes care of the remaining term of Eq. (41) and is

$$\ddot{a}_{CZ} = -j2\omega I_E Z \cos \lambda$$

(52)

As in the case of the horizontal components it is the negative of the acceleration of the pivot given in Eq. (52) that acts on a free body. Therefore

$$\ddot{a}_{CZP} = j2\omega I_E Z \cos \lambda$$

(53)

Eq. (53) shows that the magnitude of the deviation of a falling body due to Coriolis effects depends on the speed of falling $Z$ (hence on the distance the body has fallen) and on the cosine of the latitude (maximum at the equator and zero at the poles). The direction of the deviation is along an east-west line, and since for a falling body $Z$ has a negative algebraic sign the quantity $\ddot{a}_{CZP}$ has the direction $-j$. It follows that a falling body always deviates eastward, in both hemispheres. The magnitude of this eastward deviation is shown in Fig. 29 as a function of the altitude from which the body is dropped and the latitude; where it is seen that a body dropped from twenty thousand feet in latitude $45^\circ$ has an eastward deviation of about sixty feet when it hits the ground.
APPENDIX E

Mass and Buoyancy Effects

The force $\mathbf{F}$ in Eq. (37) required to keep a pendulous element attached to a pivot in the presence of gravitational and other accelerations is applied to the element by the pivot itself. It is desirable, however, to know what force is acting at the center of gravity $C$ of the pendulous element, because the motion of a rigid body can be most effectively expressed as a combination of a translation of the center of gravity and a rotation about the center of gravity. The addition of two equal and opposite forces $\mathbf{F}$ and $-\mathbf{F}$ (each equal to the $\mathbf{F}$ of Eq. (37)) at the center of gravity $C$ as shown in Fig. 43 in no way affects the total force acting on the element, i.e. the force $\mathbf{F}$ acting at the pivot. The two new forces acting at the point $C$ together with the force acting at the pivot may be resolved into a force $\mathbf{F}$ acting to translate the point C plus a torque (moment or couple) $\tau X \mathbf{F}$ acting to rotate the body about the point $C$ where $\ell$ is the distance between the pivot $P$ and the center of gravity $C$ of the pendulous element. The torque is also equal to the torque $\tau X (-\mathbf{F})$ acting to rotate the body about the pivot $P$. The translation of the element as a whole is of no interest to the problem of indicating the direction of the vertical, but the rotation of the body about the pivot is the whole story. The torque tending to cause the rotation of the element about the pivot is

$$\tau = \ell X (-\mathbf{F})$$

(54)

Eq. (54) shows that the torque tending to rotate a pendulous element about a pivot when the pivot is accelerated is the *inertia reaction* torque of the element.

Since the force $\mathbf{F}$ in Eq. (37) represents *all* the forces acting on
ROTATION OF A PENDULOUS ELEMENT ABOUT A PIVOT

Fig. 43

EFFECT OF MASS AND BUOYANCY FORCES ON THE ROTATION OF A PENDULOUS ELEMENT ABOUT A PIVOT

Fig. 44

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the pivot, referred to an inertial system, the point P may be treated as though it were a point fixed in an inertial system; and so used as the origin of coordinates in the study of the rotation of the pendulous element.

It was shown in Eq. (54) that the inertia force tending to rotate a pendulous element about its pivot when the pivot is accelerated with an acceleration $\ddot{a}$ is $\mathbf{F} = m(-\ddot{a})$. This force acts at the center of gravity through a lever arm $\ell_{cg}$, the distance between the pivot and the center of gravity. The torque due to inertia reaction tending to rotate the element about the pivot is given from Eq. (54) as

$$
\mathbf{M}_g = \ell_{cg} x_m(-\ddot{a}) = -m\ell_{cg}x\ddot{a}
$$

(55)

If the element is immersed in a fluid buoyancy forces must also be considered. By analogy with gravity, where the force of gravity is directed downward while the force of buoyancy is directed upward, it is seen that the buoyancy force relative to any acceleration acts in the direction opposite to the inertia force, as is shown in Fig. 44. It follows that the buoyancy force attending an acceleration $\ddot{a}$ is given by $m_L \ddot{a}$, where $m_L$ is the mass of the fluid displaced by the element (Archimedes' principle). This buoyancy force acts at the center of buoyancy through a lever arm $\ell_{cb}$, the distance between the pivot and the center of buoyancy. The torque due to buoyancy tending to rotate the element about the pivot is

$$
\mathbf{M}_b = \ell_{cb} x_{m_L} \ddot{a} = m_L \ell_{cb} x\ddot{a}
$$

(56)

The mass of the fluid displaced by the element is found by virtue of the fact that the volume of the fluid displaced equals the volume of the element. The volume of the element is

$$
V = m/\rho
$$

where $\rho$ is the density of the element.
The volume of the fluid displaced is

\[ V_L = \frac{m_L}{\rho_L} \]

where \( \rho_L \) is the density of the fluid.

Now \( V_L = V \); so \( m/\rho = m_L/\rho_L \), and

\[ m_L = \frac{m \rho_L}{\rho} \]  

(57)

Putting the value of Eq. (57) in Eq. (56) gives

\[ M_b = m (\rho_L/\rho) \vec{\ell}_{cb} \vec{a} \]

(58)

Combining Eqs. (55) and (58) gives the total torque, inertia and buoyancy, tending to rotate the element about the pivot.

\[ \vec{M} = \vec{M}_g + \vec{M}_b = -m \vec{\ell}_{cg} \vec{a} + m (\rho_L/\rho) \vec{\ell}_{cb} \vec{a} \]

\[ = m (\vec{\ell}_{cg} + \vec{\ell}_{cb} \rho_L/\rho) \vec{a} \]

(59)

Now let \( \vec{\ell} = \vec{\ell}_{cg} - \vec{\ell}_{cb} \rho_L/\rho \); then Eq. (59) becomes

\[ \vec{M} = -m \vec{\ell} \vec{a} \]

(60)

Eq. (60) shows that the important lever arm in the torque tending to rotate a pendulous element about its pivot is neither the distance between the pivot and the center of gravity alone nor the distance between the pivot and the center of buoyancy alone, but the combination of the two distances given by the effective lever arm \( \vec{\ell} \). To study the effect of the distance \( \vec{\ell} \) on the stability of the pendulous element assume the torque \( \vec{M} \) in Eq. (60) is balanced only by the inertia of the element, i.e. by \( I \theta \), where

\[ I \] is the moment of inertia of the element about the pivot

\[ \theta \] is the angular acceleration of the element about the pivot

Then

\[ I \vec{\theta} = -m \vec{\ell} \vec{a} \]

(61)

If the acceleration \( \vec{a} \) is merely the acceleration of gravity then Eq. (61) is

\[ I \vec{\theta} = -m \vec{\ell} \vec{a} (-g) = m \vec{\ell} \vec{a} g = mg \vec{a} \sin \theta \neq mg \vec{\theta} \]

(62)

where \( \theta \) is the angular deviation of the element from the vertical. Then
Eq. (63) gives as its solution a stable oscillation only when the second term is positive. It follows that for stability \( \ddot{\ell} \) must be negative, i.e. the "effective center of gravity" must be below the pivot. Extension of this statement to the components of \( \ddot{\ell} \) shows that stability exists when the center of gravity is below the pivot, for no fluid effects; and when the center of buoyancy is above the pivot, for the pivot and the center of gravity coincident. For the general case, however, stability exists when the vector sum \( (\ddot{z}_{cg} - \ddot{z}_{cb} pL/p) \) is below the pivot.

It is possible on the above basis to make a body immersed in a fluid nonpendulous, i.e. give it neutral equilibrium, by making both the center of gravity and the center of buoyancy lie on the same side of the pivot, e.g. either both above or both below, and making \( \ddot{z}_{cg} = \ddot{z}_{cb} pL/p \) in magnitude. It is not possible to make the body non-pendulous by making the center of gravity lie just as far below the pivot as the center of buoyancy lies above, in spite of opinions to the contrary.

Note: the quantity \( \ddot{\ell} \), which is negative for stability, is analogous to the metacentric height, which is positive for stability, used in the studies of the stability of marine craft.
APPENDIX F

Rotation of a Rigid Body

Equations of motion for rotation may be derived either from Newton's second law of motion, which states that an applied force equals the product of the mass of the body acted on and the resulting acceleration, or from d'Alembert's principle concerning restraints. The following treatment of the rotation of a rigid body will be based on the latter authority. Suitable references for a study of rotation are any text on vector analysis or advanced mechanics (11)(12)(13)(14)(15)(16)(17)(18)(19).

Equation of Motion

D'Alembert's principle states

$$\Sigma (m\ddot{r} - F) \cdot \delta \vec{r} = 0$$

(64)

where  
- $m$ is the mass of any individual component or particle of the whole system
- $\ddot{r}$ is the acceleration of mass component $m$
- $\vec{F}$ is the force acting on mass component $m$
- $\Sigma$ denotes a summation over all the mass components $m$
- $\delta \vec{r}$ is any arbitrary displacement consistent with the restraints imposed on the system

For the case of rotation let

$$\delta \vec{r} = \delta \vec{\theta} \vec{r}$$

(65)

where  
- $\delta \vec{\theta}$ is any elementary rotation. Putting Eq. (65) in Eq. (64)
gives

$$\Sigma m \ddot{r} \cdot \delta \vec{\theta} \vec{r} = \Sigma \vec{F} \cdot \delta \vec{\theta} \vec{r}$$

or

$$\Sigma m \ddot{\vec{r}} = \Sigma \vec{F} \cdot \delta \vec{\theta}$$

(66)

For a rigid body $\delta \vec{\theta}$ is the same for all particles, and so may be elimin-
ated from both sides of Eq. (66). Also
\[ \vec{F}x_1^t = d(\vec{F}x_1^t)/dt \quad \text{since} \quad \dot{\vec{r}}x_1^t = 0 \]

With the last two conditions Eq. (66) becomes
\[ d(\Sigma m\vec{F}x)^t)/dt = \Sigma \vec{F}x^t \quad (67) \]

Now let \[ \vec{H} = \Sigma m\vec{F}x = \text{angular momentum about axis of rotation} \]
\[ \vec{M} = \Sigma \vec{F}x^t = \text{torque applied about axis of rotation} \]

With these definitions Eq. (67) becomes
\[ \frac{d\vec{H}}{dt} = \vec{M} \quad (68) \]

Eq. (68) states that the time rate of change of angular momentum about an axis of rotation equals the torque applied about that same axis.

Eq. (68) is to the study of rotation what Newton's second law of motion is to the study of translation. As in Newton's law Eq. (68) is rigorously valid only for an inertial system.

**Angular momentum and the ellipsoid of inertia**

The angular momentum \( \vec{H} \) of a body about an axis of rotation has been defined as
\[ \vec{H} = \Sigma m\vec{F}x = \Sigma m\vec{F}x (\omega \vec{x}) \quad (69) \]

That \( \vec{r} = \omega \vec{x} \) is shown in Eq. (22) where system 2 is fixed in the rigid body, thereby making \( \vec{H}_2 \) equal zero. Now in terms of their rectangular components \( \vec{F} = ix+fy+kz \) and \( \vec{\omega} = i\omega_x+j\omega_y+k\omega_z \). Expanding Eq. (69) in terms of the components of \( \vec{F} \) and \( \vec{\omega} \) gives
\[ \vec{H} = i[\Sigma m\omega_x(x^2+y^2+z^2) - \Sigma m x(x^2+y^2+z^2)] + \\
   j[\Sigma m\omega_y(x^2+y^2+z^2) - \Sigma m y(x^2+y^2+z^2)] + \\
   k[\Sigma m\omega_z(x^2+y^2+z^2) - \Sigma m z(x^2+y^2+z^2)] \\
   = i[\omega_x\Sigma m(x^2+y^2+z^2) - \omega_y\Sigma mxz - \omega_z\Sigma mxy] + \\
   j[-\omega_x\Sigma mxy + \omega_z\Sigma m(y^2+z^2) - \omega_z\Sigma mzx] + \\
   k[-\omega_x\Sigma mzx + \omega_y\Sigma mzy + \omega_z\Sigma m(x^2+y^2)] \quad (70) \]
Now define

\[ I_{xw} = E m(y^2 + z^2) \]
\[ I_{yw} = E m(z^2 + x^2) \]
\[ I_{zw} = E m(x^2 + y^2) \]

where \( I_{xw}, I_{yw}, I_{zw} \) are called moments of inertia of the body about the axes \( x_w, y_w, z_w \) respectively; and \( K_{yzw}, K_{zxw}, K_{xyw} \) are called the products of inertia of the body with respect to the planes \( yzw, zxw, xyw \) respectively. Using the identities of Eq. (71) in Eq. (70) gives

\[
\vec{H} = i(I_{xw}\omega_{xw} - K_{xyw}\omega_{yw} - K_{zxw}\omega_{zw}) + j(-K_{xyw}\omega_{xw} + I_{yw}\omega_{yw} - K_{yzw}\omega_{zw}) + k(-K_{zxw}\omega_{xw} - K_{yzw}\omega_{yw} + I_{zw}\omega_{zw})
\]

Eq. (72) shows that the angular momentum vector \( \vec{H} \) is a linear vector function of the angular velocity vector \( \vec{\omega} \). Let \( \varphi \) be this linear vector function; then

\[
\vec{H} = \varphi \vec{\omega}
\]

where Eq. (73) is merely a more convenient way of writing Eq. (72). The form of Eq. (73) is very convenient in leading up to the so-called ellipsoid of inertia, which is developed in the following work. The expression

\[
\vec{\omega} \cdot \vec{H} = \varphi \omega \cdot \vec{\omega} = E m \vec{\omega} \cdot [\vec{\omega} \times (\vec{\omega} \times \vec{F})] = E m (\vec{\omega} \times \vec{F}) \cdot (\vec{\omega} \times \vec{F}) = E m (\vec{\omega} \times \vec{F})^2
\]

\[
= I_{xw}\omega_{xw}^2 + I_{yw}\omega_{yw}^2 + I_{zw}\omega_{zw}^2 - 2K_{yzw}\omega_{yw}\omega_{zw} - 2K_{zxw}\omega_{xw}\omega_{zw} - 2K_{xyw}\omega_{xw}\omega_{yw} = 2T
\]

where \( T \) = kinetic energy of the rotating body. The kinetic energy \( T = (\vec{\omega} \cdot \vec{\omega})/2 \) is seen to be a quadratic scalar function of the angular velocity vector \( \vec{\omega} \); and so represents a quadric surface when the energy is constant.

Now the kinetic energy of rotation of a rigid body about an axis
is known to be
\[ T = \frac{(I_\omega \bar{\omega}^2)}{2} \quad (75) \]
where \( I_\omega \) is the moment of inertia of the body about the axis of rotation
\( \bar{\omega} \) is the angular velocity about the same axis

Since no rotating finite body can have either zero or infinite moment of inertia about any axis it follows that the radius vector \( \bar{\omega} \) of the quadric \( 2T \) must have a finite maximum and minimum. Therefore the quadric is a closed surface, i.e. an ellipsoid. This ellipsoid is known as Poinsot's ellipsoid of inertia. The ellipsoid has the property that the moment of inertia \( I_\omega \) about any axis of rotation \( \bar{\omega} \) through the origin of coordinates is inversely proportional in magnitude to the square of the radius vector \( \bar{\omega} \) along that axis.

It is a known law of analytic geometry that the cross product terms (in this case of the ellipsoid of inertia the products of inertia \( K \)) can be made to vanish by the proper choice of axes. Axes which eliminate the cross products are called the principal axes, in this case the principal axes of inertia. In terms of these principal axes Eq. (72) becomes
\[ \bar{H} = I_x \omega_x + I_y \omega_y + I_z \omega_z \quad (76) \]
where \( I_x, I_y, I_z \) and \( \omega_x, \omega_y, \omega_z \) represent the moments of inertia and angular velocities of the body about the principal axes. Any axis of symmetry is a principal axis of inertia.

The above treatment of the ellipsoid of inertia was taken for one specific point as the center of rotation. If the center of rotation is changed the ellipsoid is also different.
Principal axes of inertia

The principal axes of a quadric are also defined as those axes for which the magnitude of the quadratic scalar function of the radius vector is a maximum or a minimum, i.e. \( d(\vec{\omega}^2) = 0 \). This also means that the radius function \( \vec{\omega}^2 \) equals a constant for these axes. Now multiply the radius function \( \vec{\omega}^2 \) by an arbitrary multiplier \(-\lambda\) which will later determine the properties of the radius vector to locate the principal axes. Thus

\[
-\lambda \vec{\omega}^2 = \text{const.} \tag{77}
\]

It is also necessary to have

\[
\vec{\omega} \cdot \vec{\omega} = \text{const.} \tag{78}
\]

i.e. the kinetic energy is constant. Adding Eqs. (77) and (78) gives

\[
\vec{\omega} \cdot (\vec{\omega} - \lambda \vec{\omega}) = \text{const.} \tag{79}
\]

The derivative of Eq. (79) must be zero for principal axes, i.e. the condition of a maximum or a minimum, so

\[
d\vec{\omega} \cdot (\vec{\omega} - \lambda \vec{\omega}) + \vec{\omega} \cdot (\vec{\omega} - \lambda d\vec{\omega}) = 2d\vec{\omega} \cdot (\vec{\omega} - \lambda \vec{\omega}) = 0 \tag{80}
\]

The necessary condition for the radius vector in principal axes is then

\[
\vec{\omega} = \lambda \vec{\omega} \tag{81}
\]

Eq. (81) states that the angular momentum vector \( \vec{\omega} = \vec{H} \) is parallel to the radius vector \( \vec{\omega} \) for principal axes. By combining Eqs. (72) and (81)

\[
(I_x - \lambda) \omega_x - K_{xy} \omega_y - K_{xz} \omega_z = 0
\]

\[
- K_{xy} \omega_x + (I_y - \lambda) \omega_y - K_{yz} \omega_z = 0
\]

\[
- K_{xz} \omega_x - K_{yz} \omega_y + (I_z - \lambda) \omega_z = 0 \tag{82}
\]

Eq. (82) is true only for values of \( \omega_x, \omega_y, \omega_z \neq 0 \) when the determinant

\[
\begin{vmatrix}
I_x - \lambda & -K_{xy} & -K_{xz} \\
-K_{xy} & I_y - \lambda & -K_{yz} \\
-K_{xz} & -K_{yz} & I_z - \lambda
\end{vmatrix} = 0 \tag{83}
\]
The solution of Eq. (83) gives rise to a cubic in \( \lambda \), showing that a quadric always has three principal axes. Let \( \lambda_x, \lambda_y, \lambda_z \) be the three solutions of Eq. (83), and \( \omega_x, \omega_y, \omega_z \) the corresponding angular velocities about the principal axes. Then from Eq. (83)

\[
\begin{align*}
\varphi \omega_x &= \lambda_x \omega_x \\
\varphi \omega_y &= \lambda_y \omega_y
\end{align*}
\]

(84)

Multiply the first equation by \( \omega_y \), and the second equation by \( \omega_x \); subtract

\[
\omega_y \cdot \varphi \omega_x - \omega_x \cdot \varphi \omega_y = \omega_y \cdot \lambda_x \omega_x - \omega_x \cdot \lambda_y \omega_y
\]

(85)

now

\[
\omega_y \cdot \varphi \omega_x = \omega_x \cdot \varphi \omega_y
\]

so

\[
(\lambda_x - \lambda_y)(\omega_x \cdot \omega_y) = 0
\]

(86)

Eq. (86) means that unless \( \lambda_x = \lambda_y \), which is in general not true, the dot product \( \omega_x \cdot \omega_y \) must be zero. It follows that axes \( x \) and \( y \) must then be mutually perpendicular. Similar reasoning holds between both axis \( x \) and axis \( y \) with axis \( z \). From this it is seen that principal axes are mutually orthogonal.

**The angular momentum**

The vector \( \vec{\omega} \) is the director of the radius of the quadric \( 2T = \vec{\omega} \cdot \varphi \vec{\omega} \) discussed earlier. Let \( \vec{\omega} \cdot \varphi \vec{\omega} \) equal unity for convenience, and differentiate

\[
\vec{\omega} \cdot \varphi \vec{\omega} = 1 = \text{constant}
\]

(87)

\[
d(\vec{\omega} \cdot \varphi \vec{\omega}) = d\vec{\omega} \cdot \varphi \vec{\omega} + \vec{\omega} \cdot \varphi d\vec{\omega} = 0
\]

(88)

Now

\[
\vec{\omega} \cdot \varphi d\vec{\omega} = d\vec{\omega} \cdot \varphi \vec{\omega}, \quad \text{so}
\]

\[
d(\vec{\omega} \cdot \varphi \vec{\omega}) = 2d\vec{\omega} \cdot \varphi \vec{\omega} = 0
\]

(89)

Since the dot product in Eq. (89) equals zero \( \varphi \vec{\omega} \) (or \( \vec{H} \)) must be perpendicular to \( d\vec{\omega} \). But \( d\vec{\omega} \) is a small vector at the extremity of \( \vec{\omega} \) and must lie in the plane tangent to the ellipsoid \( \vec{\omega} \cdot \varphi \vec{\omega} \) at the tip of \( \vec{\omega} \) (see
discussion in Appendix A on the differentiation of vectors). It follows that the angular momentum vector $\vec{H}$ is perpendicular from the origin of the ellipsoid, to the plane tangent to the ellipsoid of inertia at the tip of the angular velocity vector $\vec{\omega}$. This is illustrated in Fig. 45.

**Torque-free motion**

From Eq. (68) it is seen that if no torque is applied to a body $\frac{d}{dt} \vec{H} = 0$ (90)

which means that $\vec{H}$ is a constant vector quantity, changing in neither magnitude nor direction. It follows that the plane, tangent to the ellipsoid of inertia, to which the angular momentum is always perpendicular must remain fixed in space. This plane is called the *invariable plane*. The point where the invariable plane contacts the ellipsoid is at the extremity of the instantaneous axis of rotation $\vec{\omega}$, or pole; so the ellipsoid is always rolling without sliding on the invariable plane. This means that having constructed the ellipsoid of inertia and determined the position of the invariable plane in space, the motion of the body in question is exactly the same as if the body were rigidly attached to the ellipsoid which is rolling on the invariable plane. The velocity of rotation $\vec{\omega}$ is proportional to the length of the radius vector from the origin to the point of contact between the ellipsoid and the plane. Since the angular momentum is constant the perpendicular from the origin to the tangent plane must remain constant in magnitude and direction.

The radius vector $\vec{\omega}$ of the ellipsoid always passes through a fixed point, the origin, and therefore must describe a cone in space about the vector $\vec{H}$ as the axis of the cone. The vector $\vec{H}$, which remains fixed in space, is called the *invariable line*. The radius vector $\vec{\omega}$ also traces
RELATION BETWEEN ANGULAR MOMENTUM AND ANGULAR VELOCITY ON THE ELLIPSOID OF INERTIA

Fig. 45

ELLIPSOID ROLLS ON INVARIABLE PLANE

Fig. 46

POINSOT ELLIPSOID REPRESENTATION OF THE MOTION OF A RIGID BODY WHEN NO TORQUES ARE APPLIED
out a cone in the ellipsoid. The description of the motion of the body by means of the space and body cones traced out by the radius vector \( \mathbf{\overline{r}} \) was originally due to Poinsot, and is accordingly referred to as *Poinsot motion*. The path traced out by the point of contact between the ellipsoid and the invariable plane is shown in Fig. 46 and is as follows:

1. on the invariable plane - is called the *herpolhode* and is the intersection of the invariable plane and the herpolhode cone, i.e. the cone traced out in space.

2. on the ellipsoid - is called the *polhode* and is the intersection of the ellipsoid and the polhode cone, i.e. the cone traced out in the ellipsoid.

In three cases, viz. those of the three principal axes where the angular momentum \( \mathbf{\overline{H}} \) and the angular velocity \( \mathbf{\overline{\omega}} \) are parallel, the herpolhode and polhode reduce to a single point. The ellipsoid rotates without rolling about these axes permanently, once the motion is started. Thus there are three directions about any point about which, if the body is started rotating, the body will continue to rotate forever. Two of these permanent axes are stable, the ones of maximum and minimum moments of inertia, the intermediate moment of inertia axis being unstable for rotation. The axis about which the moment of inertia is minimum is the most stable. If there is an axis of symmetry the rotation is most stable about the axis of symmetry, regardless of whether this axis has a greater or smaller moment of inertia about it than have the other two axes.

*Applications*

The problem of constraining a body to rotate about one of its principal axes is exactly the problem of balancing a gyroscope. A
gyroscope is constrained by its bearings to rotate about a given axis. The problem of balancing the gyroscope consists then of so distributing the mass of the gyroscope that the constrained axis of rotation coincides with one of the principal axes of inertia of the gyroscope. Then, and only then, will the gyroscope spin smoothly. Otherwise the angular momentum, which remains fixed in space, will appear to move relative to the gyroscope and so give rise to a more or less violent wobbling, i.e. the polhode is not a point.
APPENDIX G

Euler's Equations of Rotation

The fundamental equation of motion for rotation relating the applied torque to the resulting rate of change in angular momentum was given in Eq. (68) as

\[ \dot{\bar{M}} = (\bar{H})_{fs} \]  

(68)

where \((\cdot)_{fs}\) denotes the change as referred to inertial or fixed space. It was shown in Eq. (90) that for no applied torque \(\bar{M} = 0\) and so \((\bar{H})_{fs} = 0\)

(90)

Eq. (90) shows that for no applied torque the angular momentum remains constant in both magnitude and direction in fixed space, no matter how peculiar the motion of the body may appear to an observer in a moving space.

Making use of Eq. (22) in connection with Eq. (68) gives for the equation of motion for rotation of a rigid body referred to moving space

\[ \dot{(\bar{H})_{fs}} = (\bar{H})_{ms} + \bar{w}_{fsms} \times \bar{H} = \bar{M} \]  

(91)

where \((\cdot)_{ms}\) denotes the change as referred to the moving space, and \(\bar{w}_{fsms}\) is the angular velocity of the moving space referred to inertial space.

If there is no applied torque Eq. (91) becomes

\[ \dot{(\bar{H})_{ms}} = -\bar{w}_{fsms} \times \bar{H} \]  

(92)

where \(-\bar{w}_{fsms} \times \bar{H}\) is called the centrifugal couple, and is perpendicular to both \(\bar{w}_{fsms}\) and \(\bar{H}\). Eq. (92) states that for the torque-free condition the rate of change of angular momentum referred to a moving space equals the centrifugal couple acting. Since the change in \(\bar{H}\) is parallel to the centrifugal couple it must be perpendicular to \(\bar{H}\) itself. It follows that \(\bar{H}\) can then change only in direction, not in magnitude. Thus \(\bar{H}\)
describes a cone in the moving space even though remaining fixed in fixed space. If \( \vec{\omega}_{fms} \) and \( \vec{H} \) are parallel then \( \vec{\omega}_{fms} \times \vec{H} \) and hence \( (\vec{H})_{ms} \) equals zero; then \( \vec{H} \) remains fixed in the moving space too. If the moving space were fixed in the rotating body the fixture of \( \vec{H} \) in the moving space becomes the case of the balancing of a gyroscope, which was discussed in Appendix F.

Eq. (91) is the vector form of Euler's equation for rotation, if the moving space is fixed in the rotating body. Expanding Eq. (91) in terms of the body's principal axes and using Eq. (76) gives

\[
\begin{align*}
I_x \omega_x - (I_y - I_x) \omega_y \omega_z &= M_x, \\
I_y \omega_y - (I_z - I_x) \omega_z \omega_x &= M_y, \\
I_z \omega_z - (I_x - I_y) \omega_x \omega_y &= M_z.
\end{align*}
\]

Euler's equations are usually expressed in the form of components as in Eq. (93), but it must be remembered that Eqs. (91) and (93) are merely two ways of saying the same thing; one in vector form, the other in components.

The torques in Eq. (93) are the components of the total torque about the principal axes fixed in the rotating body, while \( \omega_x, \omega_y, \omega_z \) are the instantaneous values of the angular velocity components about the principal axes. The chief value of Eq. (93) over a component form of Eq. (68) lies in the fact that the moments of inertia are constant and the products of inertia are zero about principal axes fixed in the body, whereas both the moments and products would have finite and varying values if referred to axes fixed in inertial space.

Strictly speaking Eqs. (91) or (93) are valid only when the motion is referred to a point that is either fixed in inertial space, or is referred to the center of gravity of the rotating body. However if the
total force acting on the body, i.e. the force \( \bar{F} \) in Eq. (37), is used
the point about which the body rotates, even though this point is
moving relative to inertial space, may be used as the origin of co-
ordinates for Eq. (93). In the case of a pendulous element carried in
an airplane the pivot supporting the element is such a point.

The concept of principal axes as discussed in Appendix F made no
limitation as to the location of the point that served as the origin
of coordinates. It follows that the pivot is just as good an origin
for the solution of Eq. (93) as is the center of gravity of the rotat-
ing body. It must be remembered that the principal axes so used are
now those centered at the pivot.

**Stability of rotation about different axes**

The problem of stability is one concerned only with the inherent
properties of the body involved and not on the external torques applied.
Accordingly let the torque components in Eq. (93) be zero, from which

\[
\begin{align*}
I_x \dot{\omega}_x &= (I_y - I_z) \omega_y \omega_z \\
I_y \dot{\omega}_y &= (I_z - I_x) \omega_z \omega_x \\
I_z \dot{\omega}_z &= (I_x - I_y) \omega_x \omega_y
\end{align*}
\]

Let the x axis be the axis about which the moment of inertia of the body
is least, the z axis the axis about which the moment of inertia is
greatest, and the y axis the axis about which the moment of inertia is
intermediate, i.e. \( I_x < I_y < I_z \). To study the effect of a small angular
velocity \( \omega_y \) about the y axis on the stability of rotation of the body
about the x axis assume \( \omega_y \) is much smaller than \( \omega_x \) and that \( \omega_z \) is zero.

Differentiate the second term of Eq. (94)

\[
I_y \ddot{\omega}_y = (I_z - I_x)(\omega_x \dot{\omega}_z + \omega_z \dot{\omega}_z)
\]

Now \( \dot{\omega}_z = 0 \) and from Eq. (94)
\[ \dot{\omega}_z = (I_x - I_y)\omega_x \omega_y / I_z \]

with these values Eq. (95) becomes

\[ \ddot{\omega}_y = (I_z - I_x)(I_x - I_y)\omega_x^2 \omega_y / I_y I_z = -A^2 \omega_y \quad (96) \]

Now \((I_z - I_x)\) is positive and \((I_x - I_y)\) is negative, therefore their product is negative, from which it is seen that the coefficient \(A^2\) on the right hand side of Eq. (96) must be positive. Rewriting Eq. (96) gives

\[ \omega_y + A^2 \omega_y = 0 \]

(97)

which gives a stable (trigonometric) solution as its answer. Therefore a small angular velocity about the \(y\) axis remains small and so does not cause any instability of the much larger rotation about the \(x\) axis. By similar reasoning it is found that a small angular velocity \(\omega_z\) about the \(z\) axis does not cause any instability in the rotation about the \(x\) axis. It follows that rotation about the \(x\) axis, the axis of least inertia, is stable. Extension of this analysis to the \(y\) and \(z\) axes shows that rotation about the \(z\) axis, the axis of most inertia, is also stable; but that rotation about the \(y\) axis, the axis of intermediate inertia, is unstable.

If one of the axes is an axis of symmetry the stability problem is a bit different. Let the \(z\) axis be the axis of symmetry, then \(I_x = I_y\). When the procedure of Eqs. (95) through (97) is applied in this case it is found that motion about the \(z\) axis is stable but that the coefficient \(A^2\) in Eq. (97) is zero in the study of stability about the \(x\) and \(y\) axes. It follows that stability about the \(x\) and \(y\) axes is neutral.

An interesting result of the effect of symmetry on the motion of the rotating body arises when the airplane carrying the body is subject to vibrations. These vibrations of the airplane are transmitted through the
pivot and tend to rotate the body about the pivot. If the body is mounted on the pivot such that the pivot is on the axis of symmetry, i.e. the z axis, any rotation about either the x or the y axis will continue to remain about that axis. On the other hand if the body does not have an axis of symmetry, or if the body does have an axis of symmetry but the pivot does not lie on that axis, any vibration-induced rotation about the intermediate axis, i.e. the y axis, will be unstable and the body will rotate so as to bring the vibration-induced rotation about the stable x axis, thus giving rise to a rotation about the z axis in the process. Such procedure is not of too serious consequence to the problem of indicating the vertical, but is extremely serious to the problem of indicating azimuth, e.g. to a magnetic compass.
APPENDIX H

Equations of Motion of a Pendulous Element

The work of the previous appendices has developed the fundamentals necessary to the treatment of a pendulous element as an indicator of the vertical. Such requisite information as the forces acting on the element when the element is carried on a pivot in an airplane, and the methods of Euler for writing the equations of rotation of a rigid body in moving coordinate systems have been developed from the basic fundamentals. It is the purpose of this appendix to apply these methods so developed to the problem of setting up the equations of motion of a pendulous element.

The torque acting to rotate a pendulous element about its pivot due to the combined effect of inertia reaction and buoyancy forces was shown in Eq. (60) to be

\[ \bar{M} = -m\ddot{x}\bar{a} = \ddot{x}(\bar{F}) \]

(60)

where \( \bar{a} \) is the distance between the pivot and an effective center of gravity, which takes into account both inertia and buoyancy effects.

In Appendix G it was shown by Euler's equations of rotation that the equations of motion of a rigid body are most conveniently handled in the principal axes of the rotating body, because then the moments of inertia of the body are constant and the products of inertia are zero. The lever arm \( \bar{a} \) of Eq. (60) is best given in terms of the body's principal axes, and is

\[ \bar{a} = i\ell_x + j\ell_y + k\ell_z \]

(98)

The force \( \bar{F} \) of Eq. (60) is generally given in terms of axes fixed on the surface of the Earth, i.e. the system \( \text{OO} \) of Appendix D, which will be designated by \( \text{XYZ} \) where
X is positive toward the geographical north
Y is positive toward the geographical west
Z is positive vertically up

The force $\mathbf{F}$ is then given by

$$\mathbf{F} = \mathbf{i}F_X + \mathbf{j}F_Y + \mathbf{k}F_Z$$

(99)

Now in order to substitute Eqs. (98) and (99) in Eq. (60) it is necessary that both equations be expressed in terms of the same set of axes. Since it was shown that the principal axes $xyz$ of the pendulous element are the most convenient set of axes to use it follows that Eq. (99) must be rewritten in terms of the $xyz$ axes. To transform the components of Eq. (99) use a set of Euler's angles $\varphi, \psi, \theta$ in the tensor components of Eqs. (17) and (19), from which

$$F_X = F_X(\cos \varphi \cos \psi - \sin \varphi \sin \psi \cos \theta) + F_Y(\sin \varphi \cos \psi + \cos \varphi \sin \psi \cos \theta) + F_Z \sin \varphi \sin \theta$$

$$F_Y = -F_X(\cos \varphi \sin \psi + \sin \varphi \cos \psi \cos \theta) + F_Y(-\sin \varphi \sin \psi + \cos \varphi \cos \psi \cos \theta) + F_Z \cos \varphi \sin \theta$$

$$F_Z = F_X \sin \varphi \sin \theta - F_Y \cos \varphi \sin \theta + F_Z \cos \theta$$

(100)

To find the values of $F_X, F_Y, F_Z$ to put in the right hand side of Eq. (100) expand the total force equation Eq. (37) giving

$$F_X = m\left\{ \ddot{x} + (\omega EAY Z_{AP} - \omega EAZ X_{AP}) + [\omega EAX(\omega EAY X_{AP} + \omega EAZ Z_{AP}) - X_{AP}(\omega EAX + \omega EAZ)] + 2 \omega E \sin \lambda \right\}$$

$$F_Y = m\left\{ \ddot{y} + (\omega EAZ X_{AP} - \omega EAZ Z_{AP}) + [\omega EAY(\omega EAZ X_{AP} + \omega EAX X_{AP}) - Y_{AP}(\omega EAZ + \omega EAX)] + 2 \omega E \sin \lambda - 2 \omega E \cos \lambda \right\}$$

$$F_Z = m\left\{ \ddot{z} + (\omega EAX X_{AP} - \omega EAX Y_{AP}) + [\omega EAZ(\omega EAX X_{AP} + \omega EAY Y_{AP}) - Z_{AP}(\omega EAX + \omega EAY)] + 2 \omega E \cos \lambda + g \right\}$$

(101)

where $\ddot{x}, \ddot{y}, \ddot{z}$ are the components of the acceleration $(\bar{R}_{OA})_{EE}$ of the
center of gravity of the airplane referred to XYZ

\( \omega_{EA} \), \( \omega_{EAY} \), \( \omega_{EAZ} \) are the components of the angular velocity \( \omega_{EA} \) of the airplane axes \( X_A Y_A Z_A \), or system AA, referred to XYZ

\( X_{AP}, Y_{AP}, Z_{AP} \) are the components of the distance \( \bar{R}_{AP} \), the separation between the center of gravity of the airplane and the pivot, measured in XYZ

\( \dot{X}, \dot{Y}, \dot{Z} \) are the components of the velocity \( \dot{\bar{R}}_{OA} \) of the center of gravity of the airplane referred to XYZ

\( \bar{w}_{IE} \) is the daily angular velocity of the Earth referred to an inertial system, system II

\( \dot{\lambda} \) is the latitude on the Earth

The vector \( \bar{R}_{AP} \) whose XYZ components \( X_{AP}, Y_{AP}, Z_{AP} \) are given in Eq. (101) does not have a constant direction in XYZ if the airplane is executing maneuvers, from which the components \( X_{AP}, Y_{AP}, Z_{AP} \) have varying values. The components \( X_{AAP}, Y_{AAP}, Z_{AAP} \) of the vector \( \bar{R}_{AP} \) do have constant values in \( X_A Y_A Z_A \) since the pivot generally remains fixed relative to the airplane.

\( X_A \) is positive forward along the longitudinal axis of the airplane

\( Y_A \) is positive to the left wing of the airplane

\( Z_A \) is positive relative to \( X_A Y_A \) by the right hand rule

To transform the constant \( X_A Y_A Z_A \) components of \( \bar{R}_{AP} \) to the varying XYZ components required in Eq. (101) use a set of Euler's angles \( \varphi_A, \omega_A, \theta_A \) in the tensor components of Eqs. (18) and (19) giving

\[
X_{AP} = X_{AAP}(\cos\varphi_A \cos\omega_A - \sin\varphi_A \sin\omega_A \cos\theta_A) - \\
Y_{AAP}(\cos\varphi_A \sin\omega_A + \sin\varphi_A \cos\omega_A \cos\theta_A) + \\
Z_{AAP} \sin\varphi_A \sin\theta_A
\]
By means of Eqs. (101) and (102) it is possible to write the components of the force acting at the pivot in any kind of maneuver directly in terms of the easily calculated components. A further advantage of these two equations is the fact that the algebraic signs, a major cause of much grief in numerical work, are automatically all taken care of.

**Euler's equations applied to the pendulous element**

Expansion of Eq. (60) in terms of the xyz principal axes of the body

\[
\vec{M} = \begin{vmatrix}
1 & j & k \\
\ell_x & \ell_y & \ell_z \\
-F_x & -F_y & -F_z
\end{vmatrix}
= i(F_y\ell_z-F_z\ell_y)+j(F_z\ell_x-F_y\ell_z)+k(F_y\ell_x-F_x\ell_y)
\]  

(103)

The values of the right hand side of Eq. (103) now become the right hand side of Euler's equation of motion Eq. (93); adding in the torques due to damping, direct torques at the pivot, and Coulomb friction torques gives

\[
\begin{align*}
\dot{\omega}_x &= \left[(I_y-I_z)/I_x\right]\omega_y\omega_z+(c_x/I_x)(\omega_x-\omega_{xL})-(mg\ell_z/I_x)\left[(F_y-F_z\ell_y/\ell_z)/mg\right]-M_{dx}/I_x + M_{px}/I_x = 0 \\
\dot{\omega}_y &= \left[(I_x-I_z)/I_y\right]\omega_z\omega_x+(c_y/I_y)(\omega_y-\omega_{yL})+(mg\ell_z/I_y)\left[(F_y-F_z\ell_y/\ell_z)/mg\right]-M_{dy}/I_y + M_{py}/I_y = 0 \\
\dot{\omega}_z &= \left[(I_x-I_y)/I_z\right]\omega_x\omega_y+(c_z/I_z)(\omega_z-\omega_{zL})-(mg\ell_z/I_z)\left[(F_y-F_z\ell_y/\ell_z)/mg\ell_z\right]-M_{dz}/I_z + M_{pz}/I_z = 0
\end{align*}
\]  

(104)

where \( c(\omega-\omega_L) \) represents the damping (viscous or fluid) torque and \( \omega_L \) is the angular velocity of the fluid directly adjacent to the body

\( M_d \) represents the torque acting directly at the pivot, e.g. the magnetic torque in the case of the magnetic compass

\( M_p \) represents the Coulomb (dry or rubbing) friction torque at the pivot.
If an instrument is to be a satisfactory indicator of the vertical it will never deviate very far from the vertical. It follows that the angle $\theta$ between the $z$ and $Z$ axes of Fig. 38 may be treated as a small angle, so that $\cos \theta \approx 1$ and $\sin \theta \approx \theta$. Under these conditions $\psi+\psi \approx \alpha$, the precession about the $Z$ axis of the $x$ axis relative to the $X$ axis. Then the tensor components of Eq. (19) become

$$a_{11} = \cos \psi \cos \psi - \sin \psi \sin \psi \cos \theta \approx \cos \psi \cos \psi - \sin \psi \sin \psi \approx \cos (\psi+\psi) \approx \cos \alpha \quad \text{etc.} \quad (105)$$

and Eq. (100) becomes

$$F_x \approx F_x \cos \alpha + F_y \sin \alpha + F_z \theta \sin \psi$$
$$F_y \approx -F_y \sin \alpha + F_x \cos \alpha + F_z \theta \cos \psi \quad (106)$$
$$F_z \approx F_x \theta \sin \psi - F_y \theta \cos \psi + F_Z$$

There will in general be no torques acting directly at the pivot that can affect the performance of an indicator of the vertical; it is also assumed that the bearings at the pivot are so good that the dry friction torque is negligible (any instrument having too much friction to be neglected will be considered as being in need of repair, and will not be subject to the subsequent analysis). It follows that both $M_d$ and $M_p$ vanish from Eq. (104). The pendulous element will be taken as having an axis of symmetry (such a characteristic was shown to be desirable in Appendix G). Let the $z$ axis be the axis of symmetry, this assumption is compatible with the previous statement that $xyz$ are principal axes since it was shown in Appendix F that any axis of symmetry is a principal axis; then

$$I_x = I_y = I \quad \text{and} \quad c_x = c_y = c$$

Let the instrument be so mounted that the $z$ axis passes through the pivot and the effective center of gravity of the instrument (this condition is
automatically fulfilled by the condition of symmetry and principal axes); then \( \ell_x = \ell_y = 0 \). With the above conditions Eq. (104) becomes

\[
\begin{align*}
\dot{\omega}_x - (1-I_z/I)\omega_y\omega_z + (c/I)\omega_x - \omega_{xL} - (mg\ell_z/I)(F_y/mg) &= 0 \\
\dot{\omega}_y + (1-I_z/I)\omega_z\omega_x + (c/I)\omega_y - \omega_{yL} + (mg\ell_z/I)(F_x/mg) &= 0 \\
\dot{\omega}_z + (c_z/I_z)\omega_z - \omega_{zL} &= 0
\end{align*}
\]

(107)

Eq. (107) represents the situation known as case damping where the angular velocity \( \omega_L \) of the damping fluid equals the angular velocity of the case surrounding the active element. Case damping means that the damping torque is proportional to the relative velocity between the pendulous element and the case surrounding the element. Such kind of damping generally occurs only when the thickness of the fluid between the element and the case is small.

During many maneuvers the angular velocity \( \omega_L \) of the damping fluid is approximately zero. This gives to the situation known as space damping, where the damping torque is proportional to the spatial velocity of the pendulous element itself. Space damping is also obtained when the thickness of the damping fluid is sufficiently great that no motion of the case is transmitted to the pendulum.

Actually neither true space damping nor true case damping exists in practice, the damping being of some intermediate form; but the assumption that one of the two limiting cases (usually space damping) is present gives satisfactory answers to the great majority of problems presented, and is much easier to analyze.

For space damping Eq. (107) becomes

\[
\begin{align*}
\dot{\omega}_x - (1-I_z/I)\omega_y\omega_z + (c/I)\omega_x - (mg\ell_z/I)(F_y/mg) &= 0 \\
\dot{\omega}_y + (1-I_z/I)\omega_z\omega_x + (c/I)\omega_y + (mg\ell_z/I)(F_x/mg) &= 0 \\
\dot{\omega}_z + (c_z/I_z)\omega_z &= 0
\end{align*}
\]

(108)
The solution for the motion about the z axis in Eq. (108) is

\[ \omega_z = \omega_{z0} e^{-t/\tau} \]  \hspace{1cm} (109)

where \( \tau = I_z/c_z \)

If \( \omega_{z0} = 0 \) then \( \omega_z = 0 \) for all time; if by any chance a motion is started about the z axis Eq. (108) shows that it can not last since there is no exciting torque about the z axis, and Eq. (109) shows that any motion once started about the z axis is soon damped out. It follows that in general it is impossible to have any lasting motion about the z axis for space damping. With this condition Eq. (109) reduces to

\[ \dot{\omega}_x + (c/I)\omega_x - (mg Lz/I)(F_y/mg) = 0 \]
\[ \dot{\omega}_y + (c/I)\omega_y + (mg Lz/I)(F_x/mg) = 0 \]  \hspace{1cm} (110)

Eq. (110) will be used in the majority of the problems concerning the performance of the pendulous element as an indicator of the vertical discussed in the following appendix.
APPENDIX I

Performance of a Pendulous Element as an Indicator of the Vertical

In the previous appendices the laws governing the motion of a pendulous element supported by a pivot have been developed from the basic fundamentals. Eq. (110) represents the equation of motion for the rotation about its pivot of a space damped pendulous element; Eq. (101) gives all the components of the inertia forces that will act on a pendulous element during various maneuvers of the airplane carrying the element; and Eq. (100) shows how to transform the forces of Eq. (101) into the form required for the equation of motion Eq. (110).

In this appendix the response of a pendulous element to selected disturbances typical of the perturbing forces encountered in actual flight will be analyzed, and the necessary characteristics which a pendulous element must possess to act as a satisfactory indicator of the vertical under such circumstances pointed out. The typical maneuvers chosen for analysis are:

1. A perfectly banked turn about the vertical, i.e. the resultant force always parallel to the $Z_A$ axis of the airplane.

2. Perfectly banked S-turns about a straight line of flight, representative of the action of many airplanes when supposedly flying a straight flight path.

3. The initial stages of a perfectly banked turn, a study of the response of an instrument to transients.

4. Rolling about a single axis (a common maneuver for marine vessels but unusual for airplanes, where roll is always coupled with yaw, etc.), showing the effects of space and case damping.
It is felt that the above four maneuvers are fairly indicative of the general run of maneuvers encountered in flight; and other types of maneuvers can be thought of as being composed of combinations of the above four types.

A perfectly banked turn

In a perfectly banked turn it is assumed that the longitudinal axis $X_A$ of the airplane remains substantially horizontal, that the angle of bank $\theta_A$ remains constant and is limited to small angles, that the resultant force acting on the airplane is always parallel to the $Z_A$ axis, i.e. perpendicular to the general plane of the wings, and that the speed $v_A$ of the airplane is substantially constant.

The condition on the longitudinal axis $X_A$ means that the angle $\psi_A = 0$, so that $\cos \psi_A = 1$ and $\sin \psi_A = 0$.

The condition on the angle of bank means that $\theta_A$ is small, so that $\cos \theta_A \approx 1$ and $\sin \theta_A \approx \theta_A$, also that $\tan \theta_A \approx \theta_A$.

The condition of turning means that the heading of the airplane, i.e. the direction in the $XY$ plane of the longitudinal axis $X_A$ of the airplane, is given by the angle $\varphi_A$ where $\varphi_A = \dot{\varphi}_A t$. $\dot{\varphi}_A$ is the rate of turn of the airplane about the vertical $Z$ axis and carries an algebraic sign. $\dot{\varphi}_A = 2\pi / T_A$ where $T_A$ is the time required for the airplane to complete a $360^\circ$ turn.

The condition on the resultant force shows in Fig. 5 that the magnitude of the horizontal acceleration $a_H$ present in the airplane must be given by

$$a_H = g \tan \theta_A \approx g \theta_A$$

where $g$ is the acceleration of gravity. Vectorially

$$\vec{a}_H = \vec{\theta}_A \vec{\theta}$$

(112)
Now \( \bar{g} = kg \) in XYZ axes, and \( \bar{\theta}_A = \theta_A \) in \( X_A Y_A Z_A \) axes where \( \theta_A \) carries an algebraic sign. By use of Eqs. (18) and (19) \( \bar{\theta}_A = \theta_A \cos \phi_A + j \theta_A \sin \phi_A \) in XYZ axes. On this basis Eq. (112) becomes
\[
\bar{a}_H = (i \theta_A \cos \phi_A + j \theta_A \sin \phi_A) \bar{g} = i \theta_A \sin \phi_A - j \theta_A \cos \phi_A
\] (113)

The XYZ components of Eq. (113) are then
\[
\begin{align*}
\ddot{X} & = \theta_A \sin \phi_A \\
\ddot{Y} & = -\theta_A \cos \phi_A \\
\ddot{Z} & = 0
\end{align*}
\] (114)

for the accelerations of the center of gravity of the airplane.

The vector relation between the angle of bank \( \theta_A \) and the rate of turn \( \phi_A \) about the vertical axis \( Z \) is given by
\[
\bar{\theta}_A = k(\bar{\theta}_A / \bar{v}_A)(\bar{v}_A \cdot \bar{\theta}_A)
\]
(115)
where \( \bar{v}_A \) is the velocity of the airplane. The magnitude in Eq. (115) comes from the fact that the centripetal acceleration \( \bar{g} \theta_A = v_A \phi_A \).

The velocity \( \bar{v}_A \) will always be directed along the positive \( X_A \) axis, but the angle of bank \( \bar{\theta}_A \) may be directed along either the positive or negative \( X_A \) axis depending on the direction of turning. It is seen from Eq. (115) that when the rate of turning about the \( Z \) axis is positive \( \bar{\theta}_A \) must be anti-parallel to \( \bar{v}_A \), i.e. \( \bar{\theta}_A \) is directed along the negative \( X_A \) axis. Thus Eq. (99) determines the algebraic sign of \( \theta_A \) in Eq. (113).

The velocity \( v_A \) of the center of gravity of the airplane is \( \bar{v}_A = iv_A \) in \( X_A Y_A Z_A \) axes. By use of Eqs. (18) and (19) \( \bar{v}_A = iv_A \cos \phi_A + jv_A \sin \phi_A \) in XYZ axes, making
\[
\begin{align*}
\dot{X} & = v_A \cos \phi_A \\
\dot{Y} & = v_A \sin \phi_A \\
\dot{Z} & = 0
\end{align*}
\] (116)
It is further assumed that the pivot of the instrument is located substantially in the median longitudinal plane of the airplane, i.e. \( Y_{AAP} \approx 0 \).

On the basis of all the above considerations the XYZ components of the distance \( R_{AP} \) between the pivot P and the center of gravity A of the airplane given in Eq. (102) become

\[
\begin{align*}
X_{AP} & \approx X_{AAP} \cos \phi_A + Z_{AAP} \theta_A \sin \phi_A \\
Y_{AP} & \approx X_{AAP} \sin \phi_A - Z_{AAP} \theta_A \cos \phi_A \\
Z_{AP} & \approx Z_{AAP}
\end{align*}
\]

(117)

The angular motions of the \( X_A Y_A Z_A \) axes of the airplane referred to the XYZ axes on the Earth become for a turn

\[
\begin{align*}
\dot{\theta}_{EAX} & = \dot{\theta}_A \cos \phi_A = \dot{\theta}_A \cos \phi_A t \\
\omega_{EAX} & = -\phi_A \theta_A \sin \phi_A \\
\dot{\omega}_{EAX} & = -\phi_A^2 \theta_A \cos \phi_A \\
\theta_{EAY} & = \dot{\theta}_A \sin \phi_A = \dot{\theta}_A \sin \phi_A t \\
\omega_{EAY} & = \dot{\phi}_A \theta_A \cos \phi_A \\
\dot{\omega}_{EAY} & = -\dot{\phi}_A^2 \theta_A \sin \phi_A \\
\omega_{EAZ} & = \dot{\phi}_A \\
\dot{\omega}_{EAZ} & = 0
\end{align*}
\]

(118)

On the basis of all the above data it is now possible to write the accelerations in Eq. (101) for the case of a perfectly banked turn

\[
\begin{align*}
\frac{F_x}{mg} & = \theta_A (1 - 3Z_{AAP}^2 \phi_A^2 / g) \sin \phi_A t - [X_{AAP} \phi_A^2 (1 + \phi_A^2 / g) \cos \phi_A t] \\
\frac{F_y}{mg} & = -\theta_A (1 - 3Z_{AAP}^2 \phi_A^2 / g) \cos \phi_A t + [X_{AAP} \phi_A^2 (1 + \phi_A^2 / g) \sin \phi_A t] \\
\frac{F_z}{mg} & = 1 - Z_{AAP} \phi_A^4 \phi_A^2 / g
\end{align*}
\]

(119)

neglecting the Coriolis terms. In practical cases Eq. (119) is subject to some simplification, as follows. Assume a 360° turn in 120 seconds, i.e. \( \phi_A = 0.0523 \) rad/sec; let \( Z_{AAP} = 10 \) ft. Then \( 3Z_{AAP} \phi_A^4 / g \)

\[
3 \cdot 10 \cdot (0.0523)^2 / 32.2 = 0.0026 << 1; \text{ so this term can be neglected.}
\]
a speed of 120 mph \( \theta_A = 16^\circ = 0.28 \text{ rad} \), as shown in Fig. 7, for the above turn. Then \( Z_{AAP} \phi_A^2 \theta_A^2 /g = 10 \cdot (0.0523)^2 (0.28)^2 /32.2 = 0.00007 \ll 1 \); so this term can be neglected. Let \( X_{AAP} = 32.2 \text{ ft.} \); then for the above turn \( X_{AAP} \phi_A^2 (1+\theta_A^2) /g = 32.2 (0.0523)^2 (1+0.0784) /32.2 = 0.003 \ll 0.28 \); so this term can also be neglected.

The above treatment for a typical (standard blind flying) turn reduces Eq. (119) to

\[
\begin{align*}
F_x/mg & \approx \theta_A \sin \phi_A t \\
F_y/mg & \approx -\theta_A \cos \phi_A t \\
F_z/mg & \approx 1
\end{align*}
\]

(120)

showing that the effect of the position of the pivot in the airplane, i.e. the effect of \( X_{AAP} \) and \( Z_{AAP} \), is negligible in the care of a perfectly banked turn.

Substitution of Eq. (120) in Eq. (100) for the expression of the forces acting during a perfectly banked in terms of xyz axes gives

\[
\begin{align*}
F_x/mg & \approx \theta_A \sin \phi_A t \cos \alpha - \theta_A \cos \phi_A t \sin \alpha + \theta \sin \psi \\
F_y/mg & \approx -\theta_A \sin \phi_A t \sin \alpha - \theta_A \cos \phi_A t \cos \alpha + \theta \cos \psi \\
F_z/mg & \approx (\theta_A \sin \phi_A t) \sin \phi + (\theta_A \cos \phi_A t) \cos \phi + 1
\end{align*}
\]

(121)

The angle \( \alpha = \phi + \psi \), and it was shown in Eq. (109) that there is no motion about the z axis. It follows that \( \alpha \) is a constant; also \( \alpha = 0 \) since \( \psi = -\phi \). Then Eq. (121) becomes

\[
\begin{align*}
F_x/mg & \approx \theta_A \sin \phi_A t + \theta \sin \psi \\
F_y/mg & \approx -\theta_A \cos \phi_A t + \theta \cos \psi \\
F_z/mg & \approx \theta \cos \psi = \theta_x \\
\theta \sin \psi & = -\theta_y
\end{align*}
\]

(122)

Now the vector angle \( \theta \) is along the line of nodes, from which

\[
\begin{align*}
\theta \cos \psi & = \theta_x \\
\theta \sin \psi & = -\theta_y
\end{align*}
\]

(123)
Putting Eqs. (122) and (123) in Eq. (110) gives

\[
\begin{align*}
\dot{\omega}_x + \frac{c}{I}\omega_x - \frac{(mg\ell_2/I)(-\theta_A \cos \varphi_A + \theta_x)}{=0}\\
\dot{\omega}_y + \frac{c}{I}\omega_y + \frac{(mg\ell_2/I)(\theta_A \sin \varphi_A - \theta_y)}{=0}
\end{align*}
\]

(124)

Now it was shown in Eq. (63) that the quantity \( \ell_2 \) must be negative for stability. Now let

\[-\frac{mg\ell_2}{I} = \omega_n^2 \]

the square of the undamped natural angular frequency of the pendulous element mounted on the pivot

\[c/I = 2\xi \omega_n \]

where \( \xi \) is the damping ratio, i.e. the ratio of the actual damping coefficient to the coefficient required for critical damping

also \( \omega_x = \dot{\theta}_x \) and \( \omega_y = \dot{\theta}_y \)

\[\ddot{\omega}_x = \ddot{\theta}_x \quad \ddot{\omega}_y = \ddot{\theta}_y \]

so that Eq. (124) may be written in the symbolic non-dimensional form used by Draper (3) as

\[
\begin{align*}
\ddot{\theta}_x + \frac{2\xi \omega_n \dot{\theta}_x + \omega_n^2 \theta_x}{= \omega_n^2 \theta_A \cos \varphi_A t}\\
\ddot{\theta}_y + \frac{2\xi \omega_n \dot{\theta}_y + \omega_n^2 \theta_y}{= \omega_n^2 \theta_A \sin \varphi_A t}
\end{align*}
\]

(125)

Now let the \( xy \) plane be represented by the complex plane used in analysis where \( x \) is the axis of real quantities and \( y \) is the axis of imaginary quantities; then

\[\bar{\theta} = \theta_x + j\theta_y\]

(126)

where \( j = \sqrt{-1} \)

Add the two parts of Eq. (125) and use the values of Eq. (126)

\[\bar{\theta} + 2\xi \omega_n \dot{\bar{\theta}} + \omega_n^2 \bar{\theta} = \omega_n^2 \theta_A (\cos \varphi_A t + j \sin \varphi_A t)\]

(127)

The right hand side of Eq. (127) can be simplified to the complex exponential form by the fact that

\[\cos \varphi_A t + j \sin \varphi_A t = e^{j\varphi_A t}\]

(128)

where \( e \) is the base of natural logarithms
Putting Eq. (128) in Eq. (127) gives
$$\theta + 2\xi \omega_n \dot{\theta} + \omega_n^2 \theta = \omega_n^2 \theta_A e^{j \Phi_A t}$$
(129)

The steady state solution of Eq. (129) is (3)
$$\theta = \theta_A e^{j (\Phi_A t - \varphi_1)}$$
(130)

where $$\mu = 1/\sqrt{(1 - \beta^2) + (2 \beta \gamma)^2}$$ and $$\varphi_1 = \tan^{-1} 2 \zeta \beta / (1 - \beta^2)$$

$$\beta = \varphi_A / \omega_n$$ the frequency ratio

In terms of the x and y components Eq. (130) becomes
$$\theta_x = \theta_A e^{j (\Phi_A t - \varphi_1)}$$
$$\theta_y = \theta_A e^{j (\Phi_A t - \varphi_1)}$$
(131)

Eq. (131) gives the deviation from the vertical of a pendulous element referred to the xyz principal axes fixed in the element. It is desired to find the deviation from the vertical referred to the XYZ axes fixed on the Earth. For this transformation from xyz reference to XYZ reference use Eqs. (18) and (19) remembering that $$\varphi \dot{\psi} = 0$$ and that $$\dot{\theta}$$ is small. Then
$$\theta_x \approx \dot{\theta}_x$$
$$\theta_y \approx \dot{\theta}_y$$
(132)

On the basis of Eq. (132) Eqs. (131) and (130) become
$$\theta_x = \dot{\theta}_A e^{j (\Phi_A t - \varphi_1)}$$
$$\theta_y = \theta_A e^{j (\Phi_A t - \varphi_1)}$$
(133)

for the deviation from the vertical during a perfectly banked turn of a pendulous element, referred to the XYZ axes fixed on the Earth. Taking the magnitude only in the third equation of Eq. (133) gives
$$\frac{\theta_A}{\theta} = \mu$$
(134)

where $$\theta_A$$ is the amplitude such that $$\theta = \theta_A e^{j (\Phi_A t - \varphi_1)}$$
Eq. (134) is shown graphically in Fig. 6 where the logarithm of $\theta_a/\theta_A$ is plotted against the logarithm of the frequency ratio $\beta$ for different values of the damping ratio $\zeta$. Note: the frequency ratio $\beta$ may be expressed either in terms of the undamped angular frequencies $\phi_A/\omega_n$ or in terms of the undamped periods $T_n/T_A$ where

$$T_n = 2\pi/\omega_n = \text{undamped natural period of the pendulous element.}$$

It is seen in Fig. 6 that for $\theta_a$ to be less than 1% of the angle of bank $\theta_A$ it is necessary for the frequency ratio $\beta$ to be greater than 10. Now the standard blind flying turn requires 2 minutes to complete $360^\circ$, from which it follows that a pendulous element must have an undamped natural period of approximately 20 minutes to be a satisfactory indicator of the vertical in turning. It is also seen in Fig. 6 that the effect of damping is negligible in the regions of large $\beta$ where a satisfactory indicator of the vertical must operate.

Now the undamped natural period $T_n = 2\pi/\omega_n$, and it was shown earlier that $\omega_n^2 = mgIz/I$. Let $I = mk^2$ where $k$ is the radius of gyration of the body of mass $m$ having a moment of inertia $I$. Then $\omega_n^2 = gIz/k^2$, so $\omega_n^2/k^2 = 4\pi^2/T_n^2$ or $\omega_n^2 = 4\pi^2k^2/gT_n^2$ (135) For the above condition that $T_n$ be 20 minutes Eq. (135) becomes

$$T_n = 7.1 \cdot 10^{-8} k^2 \text{ where all units are inches}$$ (136)

For a radius of gyration $k$ of 2 inches, which is a reasonably sized body, i.e. a disc of about 3 inches radius, Eq. (136) becomes

$$T_n = 2.8 \cdot 10^{-7} \text{ inches}$$ (137)

Eq. (137) shows that a disc 3 inches in radius must be mounted so that its effective center of gravity is only three ten-millionths of an inch from the pivot - an engineering feat worthy of the highest praise - in order that such a pendulous element be a satisfactory indicator of the
vertical during a two minute turn.

If the pendulous element were of the type known as a *simple pendulum* where all the mass is concentrated at a point, i.e. $l_z = k$, then Eq. (136) becomes

$$l_z = 1.4 \times 10^7 \text{ inches} = 222 \text{ miles} \quad (138)$$

Eqs. (137) and (138) show that although a pendulous element will act as an indicator of the vertical during a turn it is not a practical instrument because

1. if the mass of the element is distributed to form the so-called *physical pendulum* the separation between the center of gravity of the element and the pivot must be of the order of millionths of an inch

2. if the mass of the element is concentrated, i.e. the simple pendulum, the separation between the mass and the pivot must be of the order of hundreds of miles

**Rotation about the vertical Z axis**

It was shown in Eq. (109) that there is no rotation about the z axis, i.e. the axis through the center of gravity of the element and the pivot, during a maneuver; but it has been experimentally observed by Draper\(^{(3)}\) that after the airplane has straightened out after a turn there is a steady rotation about the z axis, which now coincides with the Z axis. Such an effect can best be explained by the conservation of angular momentum about the Z axis.

1. During the turn the angular momentum about the x and y axes causes a component about the Z axis due to the fact that the instrument is inclined at an angle $\theta$ to the vertical. Now the angular momentum components are
\[ H_x = I\omega_x \]
\[ H_y = I\omega_y \]  
(139)

By differentiating Eq. (131) to get the angular velocities about the x and y axes

\[ \omega_x = \dot{\theta}_x = -\dot{\varphi}_A \theta_A \sin(\varphi_A t - \varphi_i) = -\dot{\varphi}_A \theta_y \]
\[ \omega_y = \dot{\theta}_y = \dot{\varphi}_A \theta_A \cos(\varphi_A t - \varphi_i) = \dot{\varphi}_A \theta_x \]  
(140)

Putting Eq. (140) in Eq. (139) gives

\[ H_x = -I\dot{\varphi}_A \theta_y \]
\[ H_y = I\dot{\varphi}_A \theta_x \]  
(141)

Now Eq. (18) shows that

\[ H_z = H_x \theta \sin \psi + H_y \theta \cos \psi \]  
(142)

but \( \theta \sin \psi = -\dot{\theta}_y \) and \( \theta \cos \psi = \theta \) as shown in Eq. (123), from which Eq. (142) becomes

\[ H_z = -H_x \dot{\theta}_y + H_y \dot{\theta}_x \]  
(143)

which becomes with the aid of Eq. (141)

\[ H_z = I\dot{\varphi}_A \theta_y + I\dot{\varphi}_A \theta_x = I\dot{\varphi}_A \theta_A \]  
(144)

which becomes with the use of Eq. (134)

\[ H_z = I\dot{\varphi}_A \mu^2 \theta_A \]  
(145)

for the angular momentum about the Z axis during a turn.

2. After straightening out after a turn the angular momentum about the Z axis will be given by

\[ H_z = I_z \omega_Z \]  
(146)

Since there has been no torque applied about the Z axis in the process of straightening out after a turn it follows that Eqs. (145) and (146) must be equal in order to conserve the angular momentum about the Z axis, so

\[ \omega_Z = (I/I_z) \dot{\varphi}_A \mu^2 \theta_A^2 \]  
(147)
for the rate of rotation about the Z axis after straightening out.

**Angle of bank during a turn**

It was shown in Eq. (95) that the horizontal acceleration $a_H$ in a perfectly banked turn is given by

$$a_H = g \tan \theta_A$$  \hspace{1cm} (111)

Eq. (111) gives the expression for the centripetal acceleration required to change the path of the airplane from a straight line to a circle. It is shown in physics texts, such as that written by Frank(27) that the centripetal acceleration is also given by

$$a_H = \frac{v_A^2}{R}$$  \hspace{1cm} (148)

where $R$ = radius of turning

Equating Eqs. (111) and (148) gives

$$\tan \theta_A = \frac{v_A^2}{gR}$$  \hspace{1cm} (149)

Now the distance travelled in a $360^\circ$ turn is

$$2\pi R = v_A T_A$$  \hspace{1cm} (150)

so

$$R = \frac{v_A T_A}{2\pi}$$  \hspace{1cm} (151)

and

$$\theta_A = \tan^{-1} \frac{2\pi v_A}{g T_A}$$  \hspace{1cm} (152)

$$\theta_A = \tan^{-1} \frac{0.286 v_A}{T_A}$$  \hspace{1cm} (153)

where $v_A$ is in miles per hour and $T_A$ is in seconds.

Eq. (153) is plotted in Fig. 7 where the angle of bank $\theta_A$ is plotted against the time $T_A$ required to make a $360^\circ$ turn as a function of the speed $v_A$ of the airplane. It is seen that $\theta_A = 16^\circ$ for the standard blind flying turn of $360^\circ$ in 120 seconds at 120 miles per hour.

**Perfectly Banked S-Turns About A Straight Line**

The S-turns about a straight line of flight involve an oscillation about both the axis of yaw, i.e. the $Z_A$ axis, and the axis of roll, i.e. the $X_A$ axis, at the same frequency of oscillation as is shown in
Fig. 8. The relation between the angle of bank $\theta_A$ and the rate of turn $\dot{\phi}_A$ is given by Eq. (115). It is assumed that the deviation in yaw from a straight flight path is small, that the longitudinal axis of the airplane is substantially horizontal, and that the angle of bank can be classified as a small angle.

The condition on the deviation in yaw means that $\varphi_A$ is a small angle such that $\cos \varphi_A \equiv 1$ and $\sin \varphi_A \equiv \varphi_A \equiv 0$.

The condition on the longitudinal axis of the airplane means that $\psi_A = 0$, so that $\cos \psi_A = 1$ and $\sin \psi_A = 0$.

The condition on the angle of bank means that $\theta_A$ is small, so that $\cos \theta_A \equiv 1$ and $\sin \theta_A \equiv \tan \theta_A \equiv \theta_A$.

The velocity of roll $\omega_{EAXA} \equiv \dot{\theta}_A$, the velocity of yaw $\omega_{EAYA} \equiv \dot{\varphi}_A$, and the velocity of pitch $\omega_{EAYA} = 0$ in $X_AY_AZ_A$ axes. With the above assumptions concerning $\varphi_A \psi_A \theta_A$ it is seen by Eqs. (18) and (19) that in $XYZ$ axes

$$
\begin{align*}
\omega_{EAX} &\equiv \omega_{EAXA} \equiv \dot{\theta}_A \\
\omega_{EAY} &\equiv \omega_{EAYA} \equiv 0 \\
\omega_{EAY} &\equiv \omega_{EAYA} \equiv \dot{\varphi}_A \\
\omega_{EAX} &\equiv \dot{\theta}_A \\
\omega_{EAY} &\equiv 0 \\
\omega_{EAY} &\equiv \dot{\varphi}_A \\
v_A &\equiv v_{AA} \\
\end{align*}
$$

(154)

The instrument is mounted in the median longitudinal plane of the airplane, i.e. $Y_{AAP} = 0$, so that Eq. (102) becomes

$$
\begin{align*}
x_{AP} &\approx x_{AAP} \\
y_{AP} &\approx 0 \\
z_{AP} &\approx z_{AAP} \\
\end{align*}
$$

(155)
Eq. (115) may be rewritten in non-vector form as

\[ \dot{\varphi}_A = -g\theta_A/v_A \]  \ ...(156)

where \( \theta_A \) and \( v_A \) both carry algebraic signs.

The acceleration of the center of gravity of the airplane becomes from Eq. (114)

\[ \ddot{X} = g\theta_A \sin\varphi_A \equiv g\theta_A \varphi_A \equiv 0 \]
\[ \ddot{Y} = -g\theta_A \cos\varphi_A \equiv -g\theta_A \]
\[ \ddot{Z} \equiv 0 \]  \ ...(157)

With all the above assumptions Eq. (101) gives

\[ F_X = m[\dot{\theta}_A \dot{\varphi}_A X_{AAP} - \dot{\theta}_A X_{AAP}] \]
\[ F_Y = m[-\dot{\theta}_A \dot{\varphi}_A X_{AAP} - \dot{\theta}_A Z_{AAP}] \]
\[ F_Z = m[\dot{\theta}_A \dot{\varphi}_A X_{AAP} - \dot{\theta}_A Z_{AAP}] \]  \ ...(158)

neglecting Coriolis effects.

The oscillating angles of roll and yaw becomes, letting \( t = 0 \) when \( \varphi_A \) has its maximum positive value

\[ \theta_A = \theta_{Aa} \sin 2\pi T_\varphi \]
\[ \dot{\theta}_A = (2\pi/T_\varphi) \theta_{Aa} \cos 2\pi T_\varphi \]
\[ \ddot{\theta}_A = -(2\pi/T_\varphi)^2 \theta_{Aa} \sin 2\pi T_\varphi \]
\[ \varphi_A = \varphi_{Aa} \cos 2\pi T_\varphi = (\dot{\varphi}_A/T_\varphi)(2\pi) \dot{\theta}_{Aa} \cos 2\pi T_\varphi \]
\[ \dot{\varphi}_A = -(\dot{\varphi}_A/T_\varphi) \theta_{Aa} \sin 2\pi T_\varphi \]
\[ \ddot{\varphi}_A = -(\dot{\varphi}_A/T_\varphi)(2\pi/T_\varphi) \theta_{Aa} \cos 2\pi T_\varphi \]  \ ...(159)

where \( T_\varphi \) = period of yaw

Putting these values of Eq. (159) in Eq. (158) gives

\[ F_X/mg = -[(2\pi \dot{\theta}_A)/v_A T_\varphi] \sin (2\pi T_\varphi) \cos 2\pi T_\varphi X_{AAP} - \]
\[ [(2\pi \dot{\theta}_A)/v_A T_\varphi] \sin 2\pi T_\varphi X_{AAP} \]
\[ F_Y/mg = -\dot{\theta}_A \sin 2\pi T_\varphi -[(2\pi \dot{\theta}_A)/v_A T_\varphi] \cos 2\pi T_\varphi X_{AAP} + \]
\[ [(4\pi \dot{\theta}_A)/v_A T_\varphi] \sin 2\pi T_\varphi X_{AAP} \]
\[ F_Z/mg = -\left( (2\pi Aa/\nu A T_\varphi) \sin(2\pi t/T_\varphi) \cos^22\pi t/T_\varphi \right) X_{AAP} + \left( 4\pi^2 T_\varphi^2 / g T_\varphi^2 \cos^22\pi t/T_\varphi \right) Z_{AAP} + 1 \]  

(160)

In general terms involving \( \theta_A^2 \) can be neglected since \( \theta_A^2 \) is a small angle. Eq. (160) then becomes

\[ \begin{align*}
F_X/mg &\approx 0 \\
F_Y/mg &\approx \theta_A \left\{ \left[ (2\pi X_{AAP}/\nu A T_\varphi) \cos2\pi t/T_\varphi \right] + \left[ 1 - 4\pi^2 Z_{AAP}/g T_\varphi^2 \sin2\pi t/T_\varphi \right] \right\} \\
F_Z/mg &\approx 1
\end{align*} \]  

(161)

Putting Eq. (161) in Eq. (100) to get the forces in \( xyz \) axes, remembering that \( \varphi + \omega = 0 \), gives

\[ \begin{align*}
F_X/mg &\approx -\theta_y \\
F_Y/mg &\approx \theta_A \left\{ \left[ (2\pi X_{AAP}/\nu A T_\varphi) \cos2\pi t/T_\varphi \right] + \left[ 1 - 4\pi^2 Z_{AAP}/g T_\varphi^2 \sin2\pi t/T_\varphi \right] \right\} + \theta_x \\
F_Z &\text{is not of any importance}
\end{align*} \]

Putting Eq. (162) in Eq. (110) for the equation of motion gives

\[ \begin{align*}
\dot{\omega}_x + \frac{c}{I} \omega_x - \frac{m g \ell z}{I} \theta_x = -\left( \frac{m g \ell z}{I} \right) \theta_A \left\{ \left[ (2\pi X_{AAP}/\nu A T_\varphi) \cos2\pi t/T_\varphi \right] \\
+ \left[ 1 - 4\pi^2 Z_{AAP}/g T_\varphi^2 \sin2\pi t/T_\varphi \right] \right\} \\
\dot{\omega}_y + \frac{c}{I} \omega_y - \frac{m g \ell z}{I} \theta_y = 0
\end{align*} \]  

(163)

It has been shown earlier that \(-m g \ell z/I = \omega_h^2\), that \( c/I = 2 \omega_h \).

With these substitutions the equation of motion about the \( y \) axis in Eq. (163) becomes

\[ \theta_y + 2 \omega_h \dot{\theta}_y + \omega_h^2 \theta_y = 0 \]  

(164)

The solution of Eq. (164) is

\[ \theta_y = e^{-t/\tau} (A \cos \omega t + B \sin \omega t) \]  

(165)

where \( \tau = 1/\omega_h, \omega = \omega_h \sqrt{1 - \xi^2} \), and \( A \) and \( B \) are arbitrary constants depending on the initial conditions. In other words Eq. (165) represents a transient, from which it is seen that there is in general no motion about the \( y \) axis, and what motion may be started is damped out.

For the solution of the motion about the \( x \) axis in Eq. (163) first
make the following changes for convenience. Let

\[ [2\pi X_{AAP}/Vs^A T^\varphi] \cos 2\pi t/T^\varphi + [1-4\pi^2 Z_{AAP}/\varepsilon^T_T] \sin 2\pi t/T^\varphi \]

become

\[ [\sigma_{X_A} \theta_A \sigma_{X^\varphi} \theta_{X^\varphi}] \cos 2\pi t/T^\varphi + [1-\sigma_{Z^\varphi}^2] \sin 2\pi t/T^\varphi \]

which becomes

\[ \sqrt{(\sigma_{X_A} \theta_A \sigma_{X^\varphi} \theta_{X^\varphi})^2 + (1-\sigma_{Z^\varphi}^2) \sin(2\pi t/T^\varphi + \psi_1)] \]

which becomes

\[ \mu_{XZ} \sin[(2\pi t/T^\varphi) + \psi_1] \]

where

\[ \hat{\beta} = T_n/T^\varphi, \sigma_{X_A} \theta_A = \varepsilon T_n/2\pi v_A, \sigma_X = 4\pi^2 X_{AAP}/\varepsilon^T_T, \sigma_Z = 4\pi^2 Z_{AAP}/\varepsilon^T_T, \]

\[ \mu_{XZ} = \sqrt{(\sigma_{X_A} \theta_A \sigma_{X^\varphi} \theta_{X^\varphi})^2 + (1-\sigma_{Z^\varphi}^2) \sin(2\pi t/T^\varphi + \psi_1)] \]

On the basis of 1. through 3. above, the equation of motion about the x axis in Eq. (163) becomes

\[ \dot{\theta} + \sigma_{X_A} \theta_A \sigma_{X^\varphi} \theta_{X^\varphi} = \omega_{X_A} \mu_{XZ} \sin[(2\pi t/T^\varphi) + \psi_1)] \]  

(166)

whose solution for the steady state is

\[ \theta_x = \dot{\theta}_{X_A} \mu_{XZ} \sin[(2\pi t/T^\varphi) - (\varphi - \psi_1)] \]  

(167)

where

\[ \mu = 1/\sqrt{(1-\beta)^2 + (2Z\beta)^2} \]

and

\[ \varphi = \tan^{-1} (\sigma_{X_A} \theta_A \sigma_{X^\varphi} \theta_{X^\varphi})/(1-\sigma_{Z^\varphi}^2) \]

The steady state solution for the amplitude alone changes Eq. (167) to

\[ \dot{\theta}_{X_A} = \mu_{XZ} \]  

(168)

Now find the order of magnitude of \( \mu_{XZ} \) encountered in practice. Let

\[ T^\varphi = 2 \text{ sec.}, X_{AAP} = 30 \text{ ft.}, Z_{AAP} = 10 \text{ ft.}, v_A = 300 \text{ mph}; \]

then

\[ \sigma_{X_A} \theta_A = 2\pi X_{AAP}/Vs^A T^\varphi = 2\pi \times 30/440 \times 2 = 0.214 \]

\[ \sigma_Z \beta^2 = 4\pi^2 Z_{AAP}/\varepsilon^T_T = 4\pi^2 \times 10/32.2 \times 4 = 3.07 \]

which are both of the order of unity, and therefore must be retained.

The expression given for \( \mu_{XZ} \) gives excellent results as long as \( \sigma_{X_A} \theta_A \), \( \sigma_X \), and \( \sigma_Z \) are constants. These parameters are generally kept constant by keeping \( T_n \) constant, i.e. a given instrument. Then the frequency ratio \( \beta \) changes by changing \( T^\varphi \). If \( T^\varphi \) is kept constant, i.e. a given maneuver, the frequency ratio changes by changing \( T_n \) and it is
better to use the following substitutions than $\omega_{xz}$. Let

$$[2\pi \omega_{zzvp}/T_{\phi}]\cos2\pi nt/T_{\phi}+[1-4n^2\omega_{zzvp}/T_{\phi}^2]\sin2\pi nt/T_{\phi}$$

become

$$[\sigma_{\omega_{zzvp}}\omega_{zzvf}\cos2\pi nt/T_{\phi}+[1-\sigma_{zf}]\sin2\pi nt/T_{\phi}$$

which becomes

$$\sqrt{(\sigma_{\omega_{zzvp}}\omega_{zzvf})^2+(1-\sigma_{zf})^2\sin[(2\pi nt/T_{\phi})+\psi_{zf}]$$

which becomes

$$\omega_{zzvf}\sin[(2\pi nt/T_{\phi})+\psi_{zf}]$$

where

$$\sigma_{\omega_{zzvp}}\omega_{zzvf}=\frac{g_{zf}}{2n\omega_{vp}}, \sigma_{zf}=4n^2\omega_{zzvp}/g_{zf}, \sigma_{ZF}=4n^2\omega_{zzvp}/g_{zf}$$

$$\omega_{zzvf}=\sqrt{(\sigma_{\omega_{zzvp}}\omega_{zzvf})^2+(1-\sigma_{zf})^2}, \text{ and } \psi_{zf}=\tan^{-1}(\sigma_{\omega_{zzvp}}\omega_{zzvf}/(1-\sigma_{zf}))$$

On the basis of 1. through 3. Eq. (168) becomes

$$\theta_{xa}/\theta_{za}=\mu_{zzvf}$$

(169)

Eqs. (168) and (169) give the ratio of the maximum deviation from the vertical of a pendulous element to the maximum angle of roll of the airplane when the airplane executes perfectly banked S-turns, since as in the case of a turn $\theta_x \equiv \theta_X$; Eq. (168) being used when the same instrument is used in a series of maneuvers where the period of the S-turns varies, and Eq. (169) being used when the same maneuver is applied to an instrument whose natural period is varied. Eqs. (168) and (169) are plotted in Figs. 9, 47, 48, 49, where the logarithm of the ratio $\theta_{xa}/\theta_{za}$ is plotted against the logarithm of the frequency ratio $\beta = T_n/T_{\phi}$ for different values of the product $\omega_{zzv}$ and $\omega_{zzvf}$.

In Fig. 9 the instrument is located at the center of gravity of the airplane, i.e. both $\omega_{zzv}$ and $\omega_{zzvf}$ equal unity, to show the effect of changing the damping ratio $\zeta$. It is seen in the figure that the damping has negligible effect except in the region of resonance, i.e. $\beta = 1$. For small values of $\beta$ the ratio $\theta_{xa}/\theta_{za}$ is unity, independent of the damping, and for large values of $\beta$ the ratio $\theta_{xa}/\theta_{za}$ becomes rapidly smaller as $\beta$ increases. Using Eq. (135) with a period of yaw of 2 seconds and a $\beta$ of 10 gives
STEADY STATE AMPLITUDE OF ANGULAR DEVIATION FROM THE VERTICAL OF A SPACE DAMPED PENDULOUS ELEMENT CARRIED BY AN AIRPLANE WHEN THE AIRPLANE EXECUTES PERFECTLY BANKED S-TURNS ABOUT A STRAIGHT LINE - INSTRUMENT MOUNTED IN MEDIAN LONGITUDINAL PLANE OF THE AIRPLANE ALONG A LINE THRU THE CENTER OF GRAVITY OF THE AIRPLANE MAKING A CONSTANT ANGLE WITH THE LONGITUDINAL AXIS OF THE AIRPLANE
Fig. 48

STEADY STATE AMPLITUDE OF ANGULAR DEVIATION FROM THE VERTICAL OF A SPACE DAMPED PENDULOUS ELEMENT CARRIED BY AN AIRPLANE WHEN THE AIRPLANE EXECUTES PERFECTLY BANKED S-TURNS ABOUT A STRAIGHT LINE - INSTRUMENT MOUNTED AT AN ARBITRARY POSITION IN MEDIAN LONGITUDINAL PLANE OF THE AIRPLANE
\[
\frac{\theta_a}{\theta_{Aa}} = \frac{\mu \chi z}{\tau_0}
\]

\(\gamma = 0.7\)
\(\sigma_z = 0\)
\(\sigma_\phi \theta_A = 0.5\)
\(\sigma_x = 10^{-1}\)

\[
\beta = \frac{T_n}{T_\phi} = \frac{\text{UNDAMPED NATURAL PERIOD OF PENDULOUS ELEMENT}}{\text{PERIOD OF YAW}}
\]

**Fig. 49**

STEADY STATE AMPLITUDE OF ANGULAR DEVIATION FROM THE VERTICAL OF A SPACE DAMPED PENDULOUS ELEMENT CARRIED BY AN AIRPLANE WHEN THE AIRPLANE EXECUTES PERFECTLY BANKED S-TURNS ABOUT A STRAIGHT LINE - INSTRUMENT MOUNTED ON LONGITUDINAL AXIS OF THE AIRPLANE
\[ l_z = 2.55 \cdot 10^{-4} \text{ inches} \quad (170) \]

For a radius of gyration \( k \) of 2 inches Eq. (170) becomes
\[ l_z = 10^{-3} = 0.001 \text{ inches} \quad (171) \]

Eq. (171) shows that for a pendulous element to give a maximum deviation from the vertical of 1% of the maximum angle of roll, e.g.
\[ \dot{\theta}_{AA} = 10^6 \text{ so } \dot{\theta}_{Xa} = 0.1^\circ, \] when the airplane is executing perfectly banked S-turns of a 2 second period, the separation between the pivot and the effective center of gravity of the element must be about one thousandth of an inch. Such tolerances are within the province of a skilled machinist, but require painstaking work.

Fig. 47 shows the plot of the logarithm of the ratio \( \dot{\theta}_{Xa}/\dot{\theta}_{AA} \) against the logarithm of the frequency ratio \( \beta \) for different values of the parameters \( \sigma_z \) and \( \sigma_{Zf} \) when the ratios \( \sigma_z/\sigma_X \) and \( \sigma_{Zf}/\sigma_{Xf} \) are kept constant and the damping ratio \( \zeta = 0.7 \), i.e. when the pivot is in an arbitrary position on a line (in the median longitudinal plane of the airplane) that makes a constant angle with the longitudinal axis of the airplane, through the airplane's center of gravity. The dotted lines in Fig. 47 represent the plots of \( \mu XZf \) for different values of \( \sigma_{Zf} \).

Since \( \mu XZf \) is independent of \( \beta \) the log-log plots of \( \mu XZf \) are of the same shape as the simple \( \mu \) curves. The full lines in Fig. 47 represent the plots of \( \mu XZ \) for different values of \( \sigma_Z \). The term \( \mu XZ \) depends on \( \beta \) and has a minimum when \( \sigma_Z \beta^2 = 1 \), the value of the minimum depends on the term \( \sigma_{\dot{\phi}A} \theta_{AA} \sigma_X^2 \) as is shown later. It is seen that for a given value of \( \sigma_Z \) the function \( \mu XZ \) becomes "saturated" for large values of \( \beta \). The use of Fig. 47 is outlined in the following; start for example at the point where
\[ \dot{\theta}_{Xa}/\dot{\theta}_{AA} = 0.01, \quad \beta = 100, \quad \sigma_Z = 10^{-2}, \quad \sigma_{Zf} = 10^2 \text{ then} \]
1. if the period of yaw $T_\psi$ is varied, the effect on $\theta_{xa}/\theta_{aA}$ is given by moving along the line of constant $\sigma_Z$.

2. if the natural period $T_n$ of the pendulous element is varied, the effect on $\theta_{xa}/\theta_{aA}$ is given by moving along the line of constant $\sigma_{Zf}$.

3. if the distance $Z_{AAP}$ from the axis of roll, or the distance $X_{AAP}$ from the axis of yaw since the ratio $\sigma_{Z}/\sigma_{X}$ is constant, is varied, the effect on $\theta_{xa}/\theta_{aA}$ is given by moving along the line of constant $\beta$.

Fig. 48 shows the plot of the logarithm of the ratio $\theta_{xa}/\theta_{aA}$ against the logarithm of the frequency ratio $\beta$ for different values of the ratio $\sigma_{Z}/\sigma_{X}$ when the parameter $\sigma_{Z} = 10^{-2}$ and the damping ratio $\zeta = 0.7$, i.e. the pivot is in an arbitrary position in the median longitudinal plane of the airplane. It is seen in the figure that the effect of changing the ratio $\sigma_{Z}/\sigma_{X}$ is important only in the region where the $\mu_{\omega_{XZ}}$ curve has its minimum.

Fig. 49 shows the plot of the logarithm of the ratio $\theta_{xa}/\theta_{aA}$ against the logarithm of the frequency ratio $\beta$ for different values of the parameter $\sigma_{X}$ when the parameter $\sigma_{Z} = 0$ and the damping ratio $\zeta = 0.7$, i.e. the pivot is in an arbitrary position on the longitudinal axis of the airplane. It is seen that the effect of changing $\sigma_{X}$ is important only when the pivot is located very far from the axis of yaw.

**Initial Stages of a Perfectly Banked Turn**

In studying the performance of a pendulous element as an indicator of the vertical during the transition time in which an airplane goes from straight line flight to a perfectly banked turn the same assumptions are made concerning the angles $\psi_A, \psi_A, \theta_A$ as were made in the case of perfectly banked S-turns, for the results of which see Eqs. (154) through (158).

Instead of the sinusoidal oscillations of $\theta_A$ and $\psi_A$ however it is assumed
assumed for the initial stages of going into a perfectly banked turn that the angle \( \theta_A \) is given by the following expression, which has been experimentally found to be satisfactory by Mykytow, Pope, and Rieser (21)

\[
\theta_A = \frac{1}{2} \theta_{Ab} (1 - \cos nt/T_\theta)
\]  

(172)

where \( \theta_{Ab} \) is the full angle of bank (equivalent to \( \theta_A \) in the study of the steady state turn).

\( T_\theta \) is the time required to attain the full angle of bank, Eq. (172) is shown diagramatically in Fig.10.

Expanding Eq. (172) to get the angles of roll and yaw and their derivatives gives

\[
\dot{\theta}_A = (\theta_{Ab}/2)(1 - \cos nt/T_\theta)
\]

\[
\ddot{\theta}_A = (\pi \theta_{Ab}/2T_\theta^2)(\sin nt/T_\theta)
\]

\[
\dddot{\theta}_A = (\pi \theta_{Ab}/2T_\theta^3)\cos nt/T_\theta
\]

(173)

Putting the values of Eq. (173) in Eq. (158) gives

\[
\frac{F_X}{mg} \cong -(\pi \theta_{Ab}^2 Z_{AAP}/4v_A T_\theta)(\sin (\pi nt/T_\theta)(1 - \cos nt/T_\theta) - \\
(\pi \theta_{Ab} X_{AAP}/4v_A^2)(1 - \cos nt/T_\theta)^2
\]

\[
\frac{F_Y}{mg} \cong -(\theta_{Ab}/2)(1 - \cos nt/T_\theta) - (\pi \theta_{Ab} X_{AAP}/2v_A T_\theta)(\sin nt/T_\theta) - \\
(\pi \theta_{Ab}^2 Z_{AAP}/2g T_\theta^2)\cos nt/T_\theta
\]

(174)

\[
\frac{F_Z}{mg} \cong -(\pi \theta_{Ab}^2 Z_{AAP}/4v_A T_\theta^2)(\sin (\pi nt/T_\theta)(1 - \cos nt/T_\theta) - \\
(\pi \theta_{Ab}^2 Z_{AAP}/2g T_\theta^2)\sin^2 nt/T_\theta + 1
\]

In general terms involving \( \theta_{Ab}^2 \) can be neglected, making Eq. (174) become

\[
\frac{F_X}{mg} \cong 0
\]

\[
\frac{F_Y}{mg} \cong -(\theta_{Ab}/2)\{1 + (\pi X_{AAP}/v_A T_\theta)(\sin nt/T_\theta) - \\
[1 - (\pi^2 Z_{AAP}/g T_\theta^2)\cos nt/T_\theta]\}
\]
\[ F_{Z/mg} = 1 \]  

(175)

Putting Eq. (175) in Eq. (100) to get the forces in xyz axes, remembering that

\[ \varphi + \psi = 0 \] gives

\[ F_x/mg = -\theta_y \]

\[ F_y/mg = -(\theta_A/2)\left\{1+(\pi X_{AAP}/\nu A)sin\eta T/\theta \right. \\
\left.[1-(\pi Z_{AAP}/gT)]cos\eta T/\theta + \theta_x \right\} \]  

(176)

\[ F_z \] is not of any importance

Putting Eq. (176) in Eq. (110) for the equation of motion gives

\[ \dot{\omega}_x + (c/I)\omega_x - (mg/l_z/I)\theta_x = -(\theta_A/2)(mg/l_z/I)\left\{1+(\pi X_{AAP}/\nu A)sin\eta T/\theta \right. \\
\left.[1-(\pi Z_{AAP}/gT)]cos\eta T/\theta \right\} \]  

(177)

It has been shown earlier that \(-mg/l_z/I = \omega_n^2\), that \(c/I = 2\omega_n\)

With these substitutions the equation of motion about the y axis in Eq. (177) is identical with Eq. (164), and the solution is given by Eq. (165)

\[ \theta_y = e^{t/\tau}(Acos\omega t + Bsin\omega t) \]  

(165)

showing that the motion about the y axis is generally zero since there is no disturbing torque.

For the solution of the motion about the x axis in Eq. (177) first make the following changes for convenience. Let

\( (\pi X_{AAP}/\nu A)sin\eta T/\theta \) become

\[ [\sigma A/\theta_A^{\tau/\eta} + \sigma X^{\tau/\eta}_T] [1-(\pi Z_{AAP}/gT)]cos\theta_T/\tau + \theta_T \] which becomes

\[ \sqrt{\left(\sigma A/\theta_A^{\tau/\eta} + \sigma X^{\tau/\eta}_T\right)^2 + \left[1-(\pi Z_{AAP}/gT)]^2 \right]} [1-(\pi Z_{AAP}/gT)]cos\left([\pi \theta_T/\tau] + \psi_T \right) \] which becomes

\[ \mu X_T \cos \left([\pi \theta_T/\tau] + \psi_T \right) \]

where

\[ \beta_T = \eta/\theta_T, \quad \sigma A/\theta_A^{\tau/\eta} = \gamma T/\nu A, \quad \sigma X_T = \pi X_{AAP}/gT^2, \quad \sigma Z_T = \pi Z_{AAP}/gT^2, \quad \mu X_T = \sqrt{\left(\sigma A/\theta_A^{\tau/\eta} + \sigma X^{\tau/\eta}_T\right)^2 + \left[1-(\pi X_T^{\tau/\eta}_T)^2 \right]} \]

\[ \psi_T = \tan^{-1}(\sigma A/\theta_A^{\tau/\eta} + \sigma X T^{\tau/\eta}_T)/(1-\sigma Z_{AAP}/gT^2) \]

and

\[ \tau = 1/\xi \omega_n = T_n/2\pi \xi \]

the characteristic time
On the basis of 1. through 3. above, the equation of motion about the x axis in Eq. (177) becomes

$$\ddot{\theta}_x+2\zeta\omega_n\dot{\theta}_x+\omega_n^2\theta_x = \omega_n^2(\theta_{A\beta}/2)(1-u_{XXZT}\cos[(\pi\xi t/\tau)+x])$$  (178)

whose complete solution, transient and steady state is, depending on the amount of damping present.

1. Oscillatory, i.e. $0<\xi<1$

$$\theta_x = (\theta_{A\beta}/2)(1-u_{XXZT}\cos[(\pi\xi t/\tau)-(\varphi_{-\psi})]) + e^{-t/\tau}[\cos((\sqrt{1-\xi^2}/\xi)(t/\tau))+B\sin((\sqrt{1-\xi^2}/\xi)(t/\tau))]$$  (179)

2. Critically damped, i.e. $\xi = 1$

$$\theta_x = (\theta_{A\beta}/2)(1-u_{XXZT}\cos[(\pi\xi t/\tau)-(\varphi_{-\psi})]) + e^{-t/\tau}[A+Bt]$$  (179)

3. Overdamped, i.e. $\xi > 1$

$$\theta_x = (\theta_{A\beta}/2)(1-u_{XXZT}\cos[(\pi\xi t/\tau)-(\varphi_{-\psi})]) + e^{-2t/(1+\nu)\tau_{+B\xi}-2vt/(1+\nu)\tau}$$

where

$$\mu_{t} = \frac{1}{1-(\pi\xi\beta_{t})^2\sqrt{1-2(\pi\xi\beta_{t})^2}}$$

$$\varphi_{t} = \tan^{-1}[\frac{2\xi(\pi\xi\beta_{t})}{1-(\pi\xi\beta_{t})^2}]$$

$$\nu = \frac{2\xi^2-1+\sqrt{\xi^2-1}}{2\xi}$$

Using the initial conditions that $\theta_x = \dot{\theta}_x = 0$ when $t = 0$ makes Eq. (179) become

$$\frac{\dot{\theta}_x}{\theta_{A\beta}/2} = 1-u_{XXZT}\cos[(\pi\xi t/\tau)-(\varphi_{-\psi})] + e^{-t/\tau}[(\mu_{t}u_{XXZT}\cos(\varphi_{-\psi})-1]\cos((\sqrt{1-\xi^2}/\xi)(t/\tau)) + \varphi_{t}/\sqrt{1-\xi^2}[\mu_{t}u_{XXZT}\cos(\varphi_{-\psi})+\mu_{t}u_{XXZT}\sin(\varphi_{-\psi})-1]]sin((\sqrt{1-\xi^2}/\xi)(t/\tau))$$

for the oscillatory case of $0<\xi<1$;

$$\frac{\dot{\theta}_x}{\theta_{A\beta}/2} = 1-u_{XXZT}\cos[(\pi\xi t/\tau)-(\varphi_{-\psi})] + e^{-t/\tau}[(\mu_{t}u_{XXZT}\cos(\varphi_{-\psi})-1]+ [\mu_{t}u_{XXZT}\cos(\varphi_{-\psi})+\mu_{t}\sin(\varphi_{-\psi})-1]t/\tau$$  (180)
for the critically damped case of $\zeta = 1$;

$$\frac{\theta_x}{\theta_{Ab}/2} = 1 - \mu \mu XZ \cos((n\pi/n - \tau) - (\psi - \psi \tau)) +$$

$$\left\{ \mu \mu XZ \left[ v \cos(\psi - \psi \tau) + (1/2)(v+1)\pi \pi \sin(\psi - \psi \tau) \right] -$$

$$\frac{1}{1/(v-1)} \right\}
$$

for the overdamped case of $\zeta > 1$.

As in the case of S-turns $\zeta \neq 1$.

The amplitude function $\mu \tau$ and the phase angle $\phi \tau$ are the analogies of the amplitude function $\mu$ and the phase angle $\phi$ used by Draper(3) in vibration analysis except that in the former case the frequency ratio $\beta \tau$ depends on both the undamped natural period $T_n$ and the damping ratio $\zeta$, i.e.

$$\beta \tau = \tau/T_f = (T_n/2n \zeta)/T_f$$

where $T_f$ is the period of the forcing function acting, whereas the frequency ratio $\beta$ depends only on the undamped natural period $T_n$, i.e.

$$\beta = T_n/T_f$$

Because the characteristic time $\tau$ depends on both the undamped natural period and the damping ratio the functions $\mu \tau$ and $\phi \tau$ are the most convenient and powerful functions to use when treating transients, while the functions $\mu$ and $\phi$, depending primarily on the undamped natural period alone, are the best functions to use when treating steady state problems. The functions $\mu$ and $\phi$ can be found in standard works on vibrations(3)(20)(28); the function $\mu$, which is also the response curve for an accelerometer to sinusoidal forcing functions in vibration study, is shown in Figs. 6 and 9. The logarithm of the function $\mu \tau$ is plotted against the logarithm of $\beta \tau$ in Fig. 50 where it is seen to be
\[ \mu_\tau = \frac{1}{\sqrt{[1-(\pi \zeta \beta_\tau)^2]^2 + (2\pi^2 \zeta^2 \beta_\tau)^2}} \]

\[ \beta_\tau = \frac{\tau}{\theta} = \frac{\text{CHARACTERISTIC TIME OF PENDULOUS ELEMENT}}{\text{TIME FOR AIRPLANE TO ATTAIN FULL BANK ANGLE}} \]

**Fig. 50**

THE FUNCTION \( \mu_\tau \) AS A FUNCTION OF THE CHARACTERISTIC TIME RATIO
analogous to the log $\mu$ vs. log $\beta$ curves of Fig. 6. The principal
difference between the $\mu\beta$ curves and the $\mu_{\tau}\beta_{\tau}$ curves being that the
$\mu\beta$ curves have a resonance at $\beta = 1$ regardless of the value of the
damping ratio $\zeta$ whereas the resonance of the $\mu_{\tau}\beta_{\tau}$ curves occurs at
increasingly larger values of $\beta_{\tau}$ for decreasing damping ratio $\zeta$, i.e.
resonance occurs at $\beta_{\tau} = 1/\pi\zeta$. The phase angle curves for $\psi\beta$
and $\psi_{\tau}\beta_{\tau}$, the latter being shown in Fig. 51, do not look too different
save that the $\psi\beta$ curves all have a value of $90^\circ$ at $\beta = 1$ whereas the
value of $90^\circ$ for a $\psi_{\tau}\beta_{\tau}$ curve occurs when $\beta_{\tau} = 1/\pi\zeta$ and therefore is
different for different values of $\zeta$.

The deviation of a pendulous element from the vertical in the
initial stages of a perfectly banked turn is shown in Fig. 11 as the
solution of Eq. (180). In Fig. 11 the ratio $\dot{\theta}_{\chi A}/\theta_{\chi A}$ of the deviation
from the vertical of the element to the full angle of bank is plotted
against the ratio $t/\tau$ of the time to the characteristic time of the
element for different values of the frequency ratio $\beta_{\tau}$ and the damping
ratio $\zeta$, when the element is located at the center of gravity of the
airplane, i.e. $\mu_{\chi Z_{\tau}} = 1$. It is seen in the figure that the deviation
from the vertical for a given value of $\beta_{\tau}$ is less as the damping ratio
is increased, and that the deviation from the vertical for a given
value of $\zeta$ is less as the frequency ratio is increased. When the
frequency ratio $\beta_{\tau} = 10$, i.e. the time taken to go from straight
flight to the full angle of bank is only one-tenth of the characteristic
time of the element, the deviation from the vertical is so small
for all values of damping that the forcing function can be regarded as
a step-function instead of as the half-cosine curve of Fig. 10.

The values of the logarithm of the deviation from the vertical
\[ \phi_\tau = \tan^{-1} \left[ 2 \zeta (\pi \beta_\tau) \right] / \left[ 1 - (\pi \beta_\tau)^2 \right] \]

\[ \beta_\tau = \frac{\tau}{\Theta} \]

**Fig. 51**

Phase angle \( \phi_\tau \) as a function of the characteristic time ratio.

Characteristic time of pendulous element

Time for airplane to attain full bank angle
when the full angle of bank has been attained by the airplane, i.e. when \( t = T_0 \) or \( t/\tau = 1/\beta_\tau \), are plotted in Fig. 52 against the logarithm of \( \beta_\tau \) for different values of the damping ratio \( \zeta \). It is seen in the figure that the forcing function can be regarded as a step-function at \( \beta_\tau = 1 \) for a damping ratio of 1, i.e. critical damping, whereas the step-function assumption cannot be used until \( \beta_\tau = 10 \) for a damping ratio of 0.1.

The effect of shifting the position of the pivot in the airplane on the response of the element to the initial stages of a turn is shown in Fig. 53 where the ratio of the deviation from the vertical to the full angle of bank is plotted against the ratio \( t/\tau \), of the time to the characteristic time of the element, for a damping ratio of 0.4, both when the element is located at the center of gravity of the airplane, i.e. \( \mu_{XZ_\tau} = 1 \); and when the element is not located at the center of gravity of the airplane, e.g. \( \sigma_{X_\tau} = 0.1, \sigma_{Z_\tau} = 0.02, \sigma_\phi_\theta_\tau = 0.268 \); for two different values of the frequency ratio \( \beta_\tau \), viz. 0.5 and 4.

It is seen in the figure that there is very little effect in moving the position of the pivot for \( \beta_\tau = 0.5 \), i.e. the time to attain the full angle of bank for the airplane is twice the characteristic time of the element, but that the deviation from the vertical is considerably less for the center of gravity mounting when the more rapid maneuver of attaining the full angle of bank is one-fourth the characteristic time of the element is executed.

Roll About a Single Axis - Space vs. Case Damping

It was shown in Appendix H that there were two limiting kinds of velocity damping available, viz.

1. Space damping where the damping is proportional to the
\[ \frac{\theta_t}{\theta_{ab}} = \frac{\text{characteristic time of pendulous element}}{\text{time for airplane to attain full bank angle}} \]

**Fig. 52**

Initial stage of perfectly banked turn from straight flight - bank angle of pendulous element as fraction of full bank angle of airplane at the instant full airplane bank is reached as a function of characteristic time ratio - pendulous element mounted at center of gravity of airplane.
\[
\theta_A = \left(\frac{\theta_{Ab}}{2}\right) \left(1 - \cos ^{\frac{\pi}{2}} \beta_{\tau} \frac{t}{\tau}\right)
\]

INSTRUMENT MOUNTED AWAY FROM CG
\[\sigma_{x\tau} = 0.1 \quad \sigma_{z\tau} = 0.02 \quad \sigma_{\varphi A \theta A} = 0.268\]

INSTRUMENT MOUNTED AT CG
\[\sigma_{x\tau} = \sigma_{z\tau} = 0 \quad \zeta = 0.4\]

\[\beta_{\tau} = 0.5\]

\[\beta_{\tau} = 0.4\]

\[\frac{t}{\tau} = \text{CHARACTERISTIC TIME OF PENDULOUS ELEMENT}\]

Fig. 53

INITIAL STAGE OF PERFECTLY BANKED TURN FROM STRAIGHT FLIGHT
- BANK ANGLE OF PENDULOUS ELEMENT AS FRACTION OF FULL BANK ANGLE OF AIRPLANE - COMPARISON OF BANK ANGLE OF PENDULOUS ELEMENT WHEN PENDULOUS ELEMENT IS MOUNTED AT CENTER OF GRAVITY OF AIRPLANE AND WHEN PENDULOUS ELEMENT IS MOUNTED AWAY FROM CENTER OF GRAVITY OF AIRPLANE
spatial velocity of the pendulous element, i.e. $\omega_L = 0$

2. Case damping where the damping is proportional to the difference in velocity between the element and the fluid immediately adjacent to the element, and the fluid has the velocity of the case surrounding the whole system.

For a further discussion on the effects of different types of damping the work of Cook\(^{(29)}\) is recommended.

**Space damping**

The equation for the forces acting in an oscillation about a single axis, e.g. the $X_A$ axis of roll, will be similar to the equation for the forces acting in perfectly banked S-turns save that the terms involving the angle of yaw $\varphi_A$ and its derivatives and the acceleration of the center of gravity of the airplane will be zero. With these simplifications Eq. (158) becomes

\[
F_X = 0
\]

\[
F_Y \approx m(-\dot{\theta}_A Z_{AAP})
\]

\[
F_Z \approx m(-\dot{\phi}_A Z_{AAP} + \dot{\varphi})
\]

which makes Eq. (162) become

\[
F_X/mg \approx -\dot{\Theta}
\]

\[
F_Y/mg \approx \dot{\Theta}_A [4\pi^2 Z_{AAP}/(\phi T_{\varphi})] \sin 2\pi T_{\varphi} + \dot{\Theta}_A
\]

\[
F_Z \text{ is not of any importance}
\]

Putting Eq. (184) in Eq. (110) for the equation of motion gives

\[
\dot{\omega}_x + (c/I)\omega_x - (mg\ell_z/I)\Theta = (mg\ell_z/I)\dot{\Theta}_A [4\pi^2 Z_{AAP}/(\phi T_{\varphi})] \sin 2\pi T_{\varphi}
\]

\[
\dot{\omega}_y + (c/I)\omega_y - (mg\ell_z/I)\Theta = 0
\]

The equation of motion about the y axis in Eq. (185) is identical with the equation of motion about the y axis in Eq. (163), hence its solution is given by Eq. (185) as
\[ y(t) = e^{-t/\tau}(A \cos \omega t + B \sin \omega t) \]  
(165)

showing that any motion about the y axis is a damped transient only.

For the equation of motion about the x axis let

\[ [4\pi^2 Z_{AAp}/\beta^2 T_\theta^2] = \sigma_Z^2 \]  
as defined in the study of the motion about the x axis during S-turns. Then Eq. (166) becomes

\[ \ddot{\theta}_x + 2\zeta \omega_n \dot{\theta}_x + \omega_n^2 \theta_x = -\omega_n^2 \theta_{Ax} \sigma_Z^2 \sin 2\pi t/T_\theta \]  
(186)

whose solution for the steady state is

\[ \theta_x = -\theta_{Ax} \mu \sin[(2\pi t/T_\theta) - \varphi_1] \]  
(187)

where \[ \mu = 1/\sqrt{(1-\beta^2)^2 + (2\zeta \beta)^2} \]  
and \[ \varphi_1 = \tan^{-1} 2\zeta \beta / (1 - \beta^2) \]

The steady state solution for the amplitude alone changes Eq. (187) to

\[ \frac{\theta_{xA}}{\theta_{Ax}} = -\mu \sigma_Z^2 \]  
(188)

which is just what is expected when the term \( \mu \chi_\theta \) of Eq. (168) is changed to account for motion about one axis only. As discussed for the case of S-turns Eq. (188) is satisfactory only when \( \beta \) changes by changing \( T_\theta \) with a constant undamped natural period \( T_n \) of the element. If \( \beta \) is changed by changing \( T_n \) with a constant \( T_\theta \) Eq. (188) becomes

\[ \frac{\theta_{xA}}{\theta_{Ax}} = -\mu \sigma_Z f \]  
(189)

as would be expected from the reduction of Eq. (189). In this case of motion about a single axis only \( \theta_x \neq \theta_X \) also.

In Fig. 54 the logarithm of the ratio \( \theta_{xA}/\theta_{Ax} \) of the maximum deviation from the vertical of a space damped pendulous element to the maximum angle of roll as given by Eq. (188) is plotted against the logarithm of the ratio \( T_n/T_\theta \) of the undamped natural period of the element to the period of roll for different values of the damping ratio \( \zeta \), with a fixed value of \( \sigma_Z = 0.1 \). It is seen in the figure that the deviation is very small for small values of \( \beta \), i.e. low
Fig. 54.2

Steady state amplitude of angular deviation from the vertical of a space-damped pendulous element carried by a moving base when the moving base executes a sinusoidal oscillation about its axis of roll - instrument not mounted on axis of roll but in median longitudinal plane of the moving base.
frequencies of forcing oscillations and hence low accelerations, and becomes saturated for large values of $\beta$. The damping ratio $\zeta$ is seen to have but little effect except near resonance. Fig. 54, except for the multiplying factor $\sigma Z$ is the standard response curve of a vibrometer encountered in the analysis of vibrations(3)(20)(28), i.e. the $\mu \beta^2$ curve.

Fig.55 shows the effect of moving the pivot away from the axis of roll with a fixed value of the damping ratio $\zeta = 0.7$. The dotted lines show the effect of changing the undamped natural period of the element when the period of roll remains constant, the full lines show the effect of changing the period of roll when the undamped natural period of the element remains fixed, and lines of constant $\beta$ show the effect of changing the separation of the pivot from the axis of roll when both the period of roll and the undamped natural period of the element remain constant.

Case damping

In order to show the difference in the response of a case damped pendulous element from the response of a space damped pendulous element subjected to the same maneuver the following analysis will be concerned with the treatment of a case damped element when the airplane carrying it executes rolling about a single axis.

The force equation will be identical with the force equation for the space damped element, viz. Eq. (184). For case damped motion the fundamental equation of motion will be Eq. (107) rather than Eq. (110). Putting Eq. (184) in Eq. (107) gives
EFFECT OF VARIATION IN $T_n$ GIVEN BY LINES OF CONSTANT $\sigma_Z$ ($T_n =$ CONST.)
EFFECT OF VARIATION IN $Z_{AAP}$ GIVEN BY LINES OF CONSTANT $\beta$ ($T_n =$ CONST.)
EFFECT OF VARIATION IN $T_n$ GIVEN BY LINES OF CONSTANT $\sigma_{zf}$ ($Z_{AAP} =$ CONST.)

STEADY STATE AMPLITUDE OF ANGULAR DEVIATION FROM THE VERTICAL OF A SPACE DAMPED PENDULOUS ELEMENT CARRIED BY A MOVING BASE WHEN THE MOVING BASE EXECUTES A SINUSOIDAL OSCILLATION ABOUT ITS AXIS OF ROLL — INSTRUMENT MOUNTED IN ARBITRARY POSITION IN MEDIAN LONGITUDINAL PLANE OF THE MOVING BASE
\[
\dot{\omega}_x - (1 - I_z/I)\omega_y \omega_z + (c/I)(\omega_x - \omega_{XL}) - (mg/I_2)\theta_x = (mg/I_2)\theta_A a [4\pi^2 ZAAP/gT]^2 \sin 2\pi t/T_0
\]
\[
\dot{\omega}_y + (1 - I_z/I)\omega_z \omega_x + (c/I)(\omega_y - \omega_{YL}) - (mg/I_2)\theta_y = 0
\]
\[
\dot{\omega}_z + (c_z/I_z)(\omega_{zL}) = 0
\]

Now the velocity of the liquid is identical with the velocity of the case, which is in turn identical with the velocity of the airplane about its axes. Therefore \(\omega_{XL} = \omega_EA_{XA} = \omega_{EAX}, \omega_{YL} = \omega_{EAY} = \omega_{EAY},\) \(\omega_{ZL} = \omega_{EAZA} = \omega_{EAZ} \). In Eq. (154) it is shown that \(\omega_{EAX} = \dot{\theta}_A, \omega_{EAY} = 0,\) \(\omega_{EAZ} = \dot{\theta}_A = 0\). On the basis of these considerations the third equation of Eq. (190) becomes

\[
\dot{\omega}_z + (c_z/I_z)\omega_z = (c_z/I_z)\omega_{EAZ} = 0 \tag{191}
\]

Eq. (191) is identical with the third equation of Eq. (108) and its solution is given by Eq. (109)

\[
\omega_z = \omega_{z0}e^{-t/\tau} \tag{109}
\]

showing that if \(\omega_{z0} = 0\) there will be no motion about the z axis, or that any motion about the z axis will be damped out. Hence there will in general be no motion about the z axis, so that \(\omega_z = 0\). On this Eq. (190) becomes for the motion about the x and y axes

\[
\ddot{\theta}_x + 2\tilde{\omega}_n \dot{\theta}_x + \omega_n^2 \theta_x = 2\tilde{\omega}_n \dot{\theta}_A - \omega_n^2 \theta_A + \sigma^2 \tilde{\omega}^2 \sin 2\pi t/T_0 \tag{192}
\]
\[
\ddot{\theta}_y + 2\tilde{\omega}_n \dot{\theta}_y + \omega_n^2 \theta_y = 2\tilde{\omega}_n \omega_{EAY} = 0
\]

where the meanings of \(\sigma, \tilde{\omega}, \omega_n,\) and \(\tilde{\omega}\) are as previously determined.

The equation of motion about the y axis in Eq. (192) is identical with the equation of motion about the y axis in Eq. (163) and the solution is given by Eq. (165)

\[
\dot{\theta}_y = e^{-t/\tau}(A\cos\omega t + B\sin\omega t) \tag{185}
\]

showing that any motion about the y axis is a damped transient only.
Now \( \dot{\theta}_A = \theta_{Aa}(2\pi/T_\theta)\cos2\pi t/T_\theta \) so that Eq. (192) for the \( x \) axis becomes

\[
\ddot{\theta}_x + 2\zeta \omega_n \dot{\theta}_x + \omega_n^2 \theta_x = \omega_n^2 \theta_{Aa} - \omega_n^2 \theta_{Aa} \zeta \beta^2 \sin2\pi t/T_\theta
\]  

(193)

whose solution is

\[
\theta_x = \theta_{Aa} e^{\sqrt{(\sigma^2 \zeta^2) + (2\zeta \beta)^2} \sin([2\pi t/T_\theta] - (\varphi + \psi))}
\]  

(194)

where \( \psi = \tan \frac{2\zeta \beta}{\sigma^2 \zeta^2} \).

The steady state solution for amplitude changes Eq. (194) to

\[
\frac{\theta_{xa}}{\theta_{Aa}} = e^{\sqrt{(\sigma^2 \zeta^2) + (2\zeta \beta)^2}}
\]  

(195)

for the situation where \( \beta \) is changed by changing \( T_\theta \) with a constant \( T_n \).

If \( \beta \) is changed by changing \( T_n \) with a constant \( T_\theta \) the solution of Eq. (190) for the motion about the \( x \) axis becomes

\[
\frac{\theta_{xa}}{\theta_{Aa}} = e^{\sqrt{(\sigma^2 \zeta^2) + (2\zeta \beta)^2}}
\]  

(196)

As in the case of space damping \( \theta_x \approx \theta_x \).

In Fig. 56 the logarithm of the ratio \( \theta_{xa}/\theta_{Aa} \) of the maximum deviation from the vertical of a case damped pendulous element to the maximum angle of roll is plotted against the logarithm of the ratio \( T_n/T_\theta \) of the undamped natural period of the element to the period of roll for a damping ratio of \( \zeta = 0.7 \) and different values of \( \sigma_Z \) and \( \sigma_Zf \). It is seen in the figure that for a constant \( T_n \) the value of the deviation from the vertical increases as the period of roll decreases, i.e. \( \beta \) increases, reaching a maximum at \( \beta = 1 \), decreasing for still further decreased \( T_\theta \) and finally becoming saturated the point where saturation begins and value of saturation being determined by \( \sigma_Z \). Such behavior is opposed to that of the space damped pendulous element of Fig. 56 where the deviation from the vertical increases with decreasing \( T_\theta \) until \( \beta = 1 \) and then saturates, the value of saturation depending on \( \sigma_Z \).
\[ \frac{\theta_{xa}}{\theta_{ha}} = \frac{T_n}{T_\theta} \]

\[ \beta = \frac{T_n}{T_\theta} \]

\[ \theta_{xa} = \mu \sqrt{(\sigma_Z^2)^2 + (2\nu\beta)^2} \]

\[ \zeta = 0.7 \]

**Fig. 56**

Effect of variation in \( T_\theta \) given by lines of constant \( \sigma_Z \) (\( T_n = \text{const.} \))

Effect of variation in \( Z_{AAP} \) given by lines of constant \( \beta \) (\( T_n = \text{const.} \))

Effect of variation in \( T_n \) given by lines of constant \( \sigma_Z \) (\( Z_{AAP} = \text{const.} \))

Steady state amplitude of angular deviation from the vertical of a case damped pendulous element carried by a moving base when the moving base executes a sinusoidal oscillation about its axis of roll — instrument mounted in arbitrary position in median longitudinal plane of the moving base
APPENDIX J

A System Controlled By a Loose Ballistic as an Indicator of the Vertical During a Perfectly Banked Turn

It was shown in Appendix I that the desideratum in a pendulous element as an indicator of the vertical is that the element have a slow rate of response to disturbances. In such elements the disturbing torque $ma$ (where "$a$" is the forcing acceleration) and the inertia $mk^2$ are both proportional to the mass $m$ of the element. A slow response element is then obtained by making the ratio $\ell/k^2$ small. The use of a loose ballistic, i.e. a ball free to move around in a race, introduces the possibility of obtaining a long period element by making the mass appearing in the disturbing torque considerably less than the mass appearing in the inertia, thereby increasing the practicability of obtaining a slow rate of response.

A system controlled by a loose ballistic is shown in Fig. 12 where a body of mass $m$ and radius of gyration $k$ is mounted so as to be non-pendulous, i.e. the body itself is entirely undisturbed by accelerations acting at the pivot, and so cannot be an indicator of the vertical per se since it will have no preferred orientation under the influence of gravity. In the body is a spherical race with radius of curvature $\bar{R}$, the center of curvature of the race being located at a distance $\bar{\ell}$ from the pivot. Let $\bar{R}$ be much larger than $\bar{\ell}$. A ball of mass $m_B$, i.e. the loose ballistic, is free to run around on the race. The motion of the ball on the race is then identical with the motion of a simple pendulum of length $\bar{R}$ free to rotate about a point.

To study the effect of the loose ballistic system carried in an airplane during a perfectly banked turn, let the element be mounted at the center of gravity of the airplane and let $\theta_B$ be the deviation of
the ball from the vertical. Then from Eq. (130) in Appendix I the ratio of the deviation of the ball from the vertical to the angle of bank is given by

\[
\frac{\theta_B}{\theta_A} = \mu_B e^{i(\phi_A - \phi_B)}
\]

(197)

where \(\mu_B = 1/\sqrt{(1-B_B)^2 + (2\zeta_B B_B)^2}\) and \(\phi_B = \tan^{-1} 2\zeta_B B_B/(1-B_B)\)

\[
\beta_B = \frac{T_nB}{T_A} \quad \text{and} \quad T_nB = 2\pi \sqrt{\mu}/g
\]

It is assumed that the ball is in the dimple shown in Fig. 12 at the bottom of the race so little of the time that the dimple can be neglected during maneuvers. The use of the dimple will be discussed later.

When the ball is at an angle \(\theta_B\) to the vertical its pressure against the race, being normal to the surface of the race, causes a reaction force to act along the line connecting the center of curvature of the race and the point of contact of the ball and the race as is shown in Fig. 13. Since the center of curvature of the race and the pivot are separated a torque tending to rotate the large element about the x axis is produced and has the value

\[
\bar{M}_B = \bar{x} \bar{F}_B
\]

(198)

where \(\bar{F}_B = \text{reaction force of ball on the race}\)

It is seen from Eq. (101) and Fig. 13 that the force \(\bar{F}_B\) is given by

\[
\bar{F}_B = -jmg \sec \theta_B \sin(\theta_B - \theta_L) + kmg \sec \theta_B \cos(\theta_B - \theta_L)
\]

\[
= -jmg(\theta_B - \theta_L) + kmg
\]

(199)

Thereby making the torque \(\bar{M}_B\) in Eq. (198) equal

\[
\bar{M}_B = -lmg \bar{\theta}_L (\theta_B - \theta_L)
\]

(200)

where \(\bar{\theta}_L = \text{deviation of the element controlled by loose ballistic from the vertical}\). Eq. (129) then becomes for the whole element

\[
\theta_L + \theta_L^1/I_L - m_B \bar{\theta}_L / I_L = -m_B \bar{\theta}_B / I_L
\]

(201)

Now \(I_L = mk^2\); let \(m_B/m = \Gamma\), \(c_L/I_L = 2\zeta_L \omega_n L\), \(-m_B \bar{\theta}_B / I_L = \omega_B^2\).
Then Eq. (201) becomes, with the aid of Eq. (197)

$$\ddot{\theta}_L + 2\zeta_L \omega_n L \dot{\theta}_L + \Gamma \omega_n L^2 \theta_L = \Gamma \omega_n L \mu_B \theta_A e^{\gamma (\Phi_A - \Phi_B)}$$

(202)

whose solution in the steady state is

$$\theta_L = \Gamma \mu_B \mu_L e^{\gamma [\Phi_A - (\Phi_L + \Phi_B)]}$$

(203)

where

$$\mu_L = 1/\sqrt{(\Gamma - \delta_L)^2 + (2\zeta_L \beta_L)^2} \quad \beta_L = T_{nL}/T_A, \quad T_{nL} = 2\pi \sqrt{k^2/g^2},$$

$$\Phi_L = \tan^{-1} 2\zeta_L \beta_L / (\Gamma - \delta_L^2)$$

Now compare the deviation from the vertical \(\theta_L/\theta_A\) of a loose ballistically controlled element with the deviation \(\theta/\theta_A\) of a standard pendulous element. Let the radius of gyration \(k\) of both elements be the same; let the distance \(l\) between the pivot and the center of curvature of the loose ballistically controlled element equal the distance \(l_z\) between the pivot and the center of gravity of the standard pendulous element. The deviation of the loose ballistically controlled element is given in Eq. (203), and the deviation of the pendulous element is given in Eq. (130).

Now \(\beta_L = \beta\) since

$$T_{nL} = T_n = 2\pi \sqrt{k^2/g^2}$$

but

$$\beta_B = \beta T_{nB}/T_{n} = \sigma \beta$$

where

$$\frac{T_{nB}}{T_n} = 2\omega \sqrt{g} \sqrt{g/l^2} / 2\pi = \sqrt{k^2/g^2} = \sigma$$

Therefore the ratio of deviations is given by

$$\frac{\theta_L}{\theta} = (\Gamma \mu_B \mu_L/\mu) e^{\gamma [\Phi_B - (\Phi_B + \Phi_L)]}$$

(204)

where the frequency ratios \(\beta\) and \(\beta_L\) for calculating the values of \(\mu\) and \(\mu_L\) are identical, but the frequency ratio \(\beta_B\) for calculating the values of \(\mu_B\) equals \(\sigma \beta\). For large values of \(\beta\) both \(\mu\) and \(\mu_L\) approach \(1/\beta^2\) so

$$\frac{\theta_L}{\theta} \to \Gamma \mu_B \to \Gamma/\sigma \beta^2$$

(205)

The logarithm of the ratio of the deviation of the loose ballistically
controlled system from the vertical to the deviation of the standard pendulous element is plotted in Fig. 14 against the logarithm of the frequency ratio $\beta$ for a damping ratio of 0.7 and for differing values of the mass ratio $\Gamma$ and the ratio of frequency ratios $\sigma$. It is seen in the figure that for a fixed value of the mass ratio the loose ballistic system's deviation is less than the pendulous element's deviation, and that the ratio of deviations becomes very small for large values of $\beta$. The value of the ratio of frequency ratios simply changes the value of $\beta$ at which the deviation ratio suddenly starts to become rapidly small, the value of $\beta$ increasing as the value of $\sigma$ decreases.

The dimple in the bottom of the race has the effect of considerably increasing the restoring torque on the element when the element is approximately vertical. This is due to the fact, as was shown in Eq. (198), that the torque acting on the element due to the ball depends on the separation $\ell$ of the pivot and the center of curvature of the race. The radius of curvature of the dimple is much smaller than the radius of the race, from which it is seen that the separation $\ell_d$ between the pivot and center of curvature of the dimple is quite large, thereby causing a large torque to act on the element. The net result of the race and dimple is to make a system that has a large restoring torque against small deviations of the element from the vertical, but which has a much smaller disturbing torque for forcing accelerations.

It has been shown above that a system controlled by a loose ballistic is inherently capable of holding the vertical more closely than is a pendulum having similar physical characteristics. However, mechanical instruction features tend to offset the superior performance of the loose ballistic system. This is because the center of curvature
of the race must be machined and assembled to within a few thousandths of an inch of the pivot in a system that is to be useful in practice. Such precision is difficult to obtain. On the other hand the center of gravity of a pendulum may be located near the pivot rather easily after assembly by means of adjustable weights.
Indication of the Vertical By the Magnitude of the Resultant Acceleration

The magnitude of the resultant acceleration present in an airplane due to the combination of the acceleration of gravity and linear accelerations is related to the magnitude of the acceleration of gravity by means of a function of the angle which the resultant acceleration makes with the vertical. It follows that the magnitude of the resultant acceleration in an airplane might be used to indicate the direction of the vertical.

The magnitudes of the resultant acceleration and the acceleration of gravity are related as shown in Fig. 15 by

\[ a_R = g \sec \theta \]  

(206)

where \( a_R \) = resultant acceleration. The value of the ratio of the resultant acceleration to the acceleration of gravity is plotted in Fig. 15 against the angle which the resultant acceleration makes with the vertical. It is seen from the figure that the magnitude of the resultant acceleration equals that of the acceleration of gravity when the two accelerations are coincident, increases to twice gravity for an angle of 60°, and further increases rapidly for angles larger than 60°.

An accelerometer properly mounted in the airplane would then give a reading indicative of the angle between the resultant acceleration and the vertical. If the indicating system of the accelerometer were set to balance out the effect of acceleration of gravity, i.e. to read zero when the airplane is flying straight and level, the instrument would then read the difference between the magnitudes of the resultant acceleration and the acceleration of gravity. Such an arrangement could be accomplished by using an inductance bridge pickup like that
described by Draper and Wrigley\(^{(20)}\) and is necessary in order to read small deviations from the vertical in the presence of gravity. The readings given by such an accelerometer would be

\[ a_R - g = g(\sec \theta - 1) \]  

(207)

The logarithm of the ratio of the difference between the magnitudes of the resultant acceleration and the acceleration of gravity to the acceleration of gravity is plotted in Fig. 16 against the angle between the two accelerations. It is seen in the figure that the difference ratio equals unity when the angle is \(60^\circ\) and decreases to very small values when the angle approaches zero.

The accelerations registered by an accelerometer are those components which are parallel to the axis of response of the instrument. It is therefore necessary that the orientation of the response axis of the accelerometer with respect to the direction of the resultant acceleration be known. Knowledge of this orientation may be difficult to obtain in an airplane. It would follow then that an accelerometer will register the resultant acceleration only in such special conditions as those in which the relative orientation given above is known. The case of perfect banking is such a special case, i.e. the direction of the resultant acceleration is the \(Z_A\) axis. Accordingly the accelerometer should be so mounted that its axis of response is the \(Z_A\) axis.

The angle of bank \(\theta_A\) represents the deviation of the \(Z_A\) axis of the airplane from the vertical. To an observer in the airplane the direction of the vertical would then lie somewhere in a cone, about the axis of response of the accelerometer, making a half-angle \(\theta_A\) with the \(Z_A\) axis as its central axis. The problem of indicating the vertical is then to identify just which of the generators of the cone is the
vertical. If the longitudinal axis $X_A$ of the airplane is horizontal, the vertical lies in the cross-ship plane in the direction given by the turn indicator.

It is seen from the above discussion that the magnitude of the resultant acceleration may be used to indicate the direction of the vertical only in restricted conditions such as that of a perfectly banked turn with the longitudinal axis of the airplane horizontal. In addition the magnitude of $(\sec \theta - 1)$ is so small for small values of $\theta$ that it would be very difficult to make an instrument which could indicate these angles with precision.
APPENDIX L

Equations of Motion of a Gyroscopic Element

A gyroscopic element is defined as a body having the following properties.

1. The system is free to rotate about a point; this is mechanically achieved by the use of a gimbal system.

2. The system is non-pendulous, i.e. \( \bar{\omega} = 0 \), so the center of gravity of the system is located at the pivot, i.e. the intersection of the lines of action of the gimbals.

3. The system has a rotor spinning at a high rate of speed about the axis of symmetry of the rotor, i.e. the z axis.

4. The rotor is spinning at constant speed, i.e. \( \dot{\omega}_z = 0 \).

A fifth condition will be added later.

Torque-free motion - nutation

Euler's equation for the motion of a rigid body in Eq. (93) becomes for a gyroscopic element

\[
\begin{align*}
I_x\ddot{\omega}_x - (I_y - I_z)\omega_y \omega_z &= M_x \\
I_y\ddot{\omega}_y - (I_z - I_x)\omega_z \omega_x &= M_y \\
I_z\ddot{\omega}_z - (I_x - I_y)\omega_x \omega_y &= M_z
\end{align*}
\]  

(93)

where the axes xyz are fixed in the gyro rotor. Apply parts 3. and 4. above, from which the third equation of Eq. (93) becomes identically zero since \( \dot{\omega}_z = 0 \) and \( I_x = I_y = I \). Now assume that the only torques acting about the x and y axes are the space damping torques \( -\omega_x \) and \( -\omega_y \). Then Eq. (93) reduces to

\[
\begin{align*}
I_x\ddot{\omega}_x + (I_z - I)\omega_y \omega_z + \omega_x &= 0 \\
I_y\ddot{\omega}_y - (I_z - I)\omega_x \omega_z + \omega_y &= 0
\end{align*}
\]  

(208)
Differentiating Eq. (208) gives

\[ I\omega_x^\prime + (I_z - I)\omega_y \omega_z + c \omega_x = 0 \]
\[ I\omega_y^\prime - (I_z - I)\omega_x \omega_z + c \omega_y = 0 \]  \hspace{1cm} (209)

since \( \omega_z = 0 \). Combining Eqs. (208) and (209) gives

\[ \omega_x^\prime + 2(c/I)\omega_x + [(1 - I/I_z)^2(H/I)^2 + (c/I)^2] \omega_x = 0 \]
\[ \omega_y^\prime + 2(c/I)\omega_y + [(1 - I/I_z)^2(H/I)^2 + (c/I)^2] \omega_y = 0 \]  \hspace{1cm} (210)

where \( H = I_z \omega_z \). Eq. (210) solves to

\[ \omega_x = \omega_x e^{-t/\tau} \cos(1 - I/I_z)(H/I)t \]
\[ \omega_y = \omega_y e^{-t/\tau} \sin(1 - I/I_z)(H/I)t \]  \hspace{1cm} (211)

where \( \tau = I/c \).

Eq. (211) shows that the angular frequency of nutation, as this torque-free oscillation is called, is independent of the amount of damping present, and is given by

\[ \omega_{nu} = (1 - I/I_z)(H/I) = (1 - I/I_z)(I_z \omega_z/I) \]  \hspace{1cm} (212)

Eq. (212) shows that the frequency of nutation increases as the ratio of the angular momentum \( H \) about the axis of symmetry to the moment of inertia \( I \) about the non-symmetrical axes increases. It also shows that there is no nutation if all the moments of inertia about different axes are equal, i.e. if \( I_z = I \). This condition is met in a sphere. It is also seen that the frequency of nutation depends only on the properties of the gyroscope itself, viz. angular momentum \( H \) and moments of inertia \( I \) and \( I_z \), and not on any external torque, not even the damping.

It is also seen implicitly in Eq. (212) that an angular velocity not an angular displacement is necessary to start nutation. If an angular velocity \( \omega_0 \) is applied to the gyro rotor Eq. (211) becomes

\[ \omega_x = \omega_0 e^{-t/\tau} \cos \omega_{nu} t \]
\[ \omega_y = \omega_0 e^{-t/\tau} \sin \omega_{nu} t \]  \hspace{1cm} (213)
If an initial thrust of energy $E$ is applied to the gyro rotor then $E = I\omega_n^2/2$, so Eq. (211) becomes

$$
\omega_x = \sqrt{2E/I} e^{-t/\tau} \cos \omega_n t \\
\omega_y = \sqrt{2E/I} e^{-t/\tau} \sin \omega_n t
$$

(214)

Eq. (214) shows that the amplitude of the angular velocity of nutation is proportional to the square root of the applied energy, but is independent of the angular momentum of the gyro rotor.

Integration of Eq. (214) gives the amplitude of nutation as

$$
\theta_x = -\left[1/(1-I/I_z)H\right] \sqrt{2EI} \left[1 + \left(c/(1-I/I_z)H\right)^2\right] e^{-t/\tau} \cos (\omega_n t + \psi_n u) \\
\theta_y = -\left[1/(1-I/I_z)H\right] \sqrt{2EI} \left[1 + \left(c/(1-I/I_z)H\right)^2\right] e^{-t/\tau} \sin (\omega_n t + \psi_n u)
$$

(215)

where $\psi_n u = \tan^{-1}(1-I/I_z)H/c$. Eq. (215) shows that the amplitude of nutation decreases as the angular momentum $H$ increases.

For an approximately disc-shaped body $I \neq I_z/2$, so the frequency of nutation given in Eq. (212) becomes

$$
\omega_n = 2\pi n_n u = (1-I/I_z)(H/I) = (1-I/I_z)(I_z\omega_z/I) \cong \omega_z \cong 2\pi n_z
$$

(216)

where $n_n u$ and $n_z$ are the frequencies in cycles or revolutions per second corresponding to the angular frequencies $\omega_n u$ and $\omega_z$ in radians per seconds. It is seen in Eq. (216) that the frequency of nutation approximately equals the rate of spin of the disc, which, if a gyro rotor, is generally of the order of several thousand revolutions per minute. It is further seen, from Eq. (212), that the frequency of nutation decreases as the disc becomes thicker, and eventually reaches zero for a sphere. The amplitude of nutation is seen in Eq. (215) to decrease as the angular momentum, which depends mainly on the rate of spin of the rotor, increases.

The conclusion is that the effects of nutation, which is an undesirable characteristic in an instrument gyro even though it lends itself to beautiful mathematical treatment, can be reduced by making the rate of
spin of the gyro rotor sufficiently high, e.g. of the order of several thousand revolutions per minute.

Angular momentum of gyro concentrated about the spin axis

A fifth condition is now added to the four given at the beginning of this appendix for the definition of a gyroscopic element, viz.

5. All the angular momentum of the gyro is concentrated about the spin axis, i.e. the z axis, of the rotor.

For the equation of motion in this case return to the theorem of Coriolis given in Eq. (22); and replace the vector $\dot{R}$ by the angular momentum vector $\vec{H}$

$$\frac{\dot{\vec{H}}}{(H)}_{\alpha} = (\vec{H})_{\gamma} + \vec{\omega}_E \times \vec{H}$$

Now let the system $\alpha_1$ become the system $EE$ of axes XYZ fixed on the Earth, and let system $\alpha_2$ become the system $GG$ of axes xyz fixed in the gyro (the axis z is the axis of symmetry as discussed before, but the x and y axes are now, and from now on, fixed in the case enclosing the rotor and so do not rotate with the rotor). Eq. (22) then becomes

$$\frac{\dot{\vec{H}}}{(H)}_{EE} = (\vec{H})_{GG} + \vec{\omega}_E \times \vec{H}$$

(217)

Condition 4. on the gyroscopic element shows that the magnitude of $(\vec{H})_{GG}$ must be zero, and condition 5. shows that the change in direction of $(\vec{H})_{GG}$ must be zero. Let $\vec{\omega}_E = \vec{\omega}_p$ the precessional velocity of the angular momentum vector, i.e. the z axis, of the gyro as seen on the Earth. Then Eq. (217) becomes

$$\frac{\dot{\vec{H}}}{(H)}_{EE} = \vec{\omega}_p \times \vec{H}$$

(218)

If the system $fs$ in Eq. (91), i.e. the vector form of Euler's equation of motion, is the inertial system $II$, and the system $ms$ is the system $EE$ of axes XYZ fixed on the Earth the combination of Eqs. (91) and (218) gives
\[
\dot{(H)}_{II} = \bar{\omega}_p X \bar{H} + \bar{\omega}_T E X \bar{H} = \bar{M}
\]
which gives for the Euler form for the equation of motion of a gyroscopic element in practical use
\[
\bar{\omega}_p X \bar{H} = \bar{M} - \bar{\omega}_T E X \bar{H}
\]

The constancy of the magnitude of the angular momentum means that there will be no unbalanced torque acting about the z axis (the axis of symmetry of the gyro), i.e. \( \bar{M}_{\text{driving}} - \bar{M}_{\text{resisting}} = 0 \). Hence the angular velocity of rotation of the gyro case about the axis of symmetry is zero, i.e. \( \dot{\psi} = 0 \), and the angular velocity of precession \( \bar{\omega}_p \) is composed only of \( \dot{\phi} \) about the vertical Z axis and \( \dot{\theta} \) about the line of nodes (see Appendix C on Euler's angles). The in XYZ axes
\[
\bar{\omega}_p = i \dot{\phi} \cos \psi + j \dot{\phi} \sin \psi + \dot{\theta}
\]
It is shown later on that \( \dot{\phi} \) represents the rate of precession of a gyro about the vertical Z axis and that \( \dot{\theta} \) represents the rate of erection of a gyro about the line of nodes (the axis about which a gyro is tilted from the vertical). By the use of Eqs. (220) and (221) it is seen that the equation of motion of a gyro is expressed in XYZ axes alone, and hence there is no need to transform the answer into another set of axes as was necessary in the case of a pendulous element.

That the assumption of all the angular momentum's being concentrated about the z axis is justified is shown in the following example of the Sperry Artificial Horizon,
\[
\omega_x = \omega_y = 10^o/\text{min} = \pi/1080
\]
\[
\omega_z = 12000 \text{ rpm} = 400\pi
\]
so \( \omega_x/\omega_z = 2.3 \times 10^{-6} \) \( \text{(222)} \)
In general \( I \approx I_z/2 \) so
\[
H_x/H_z = I_\omega x / I_z \omega_z = 1.15 \times 10^{-6} \text{ (223)}
\]
Eq. (223) shows that the angular momentum of a standard commercial instrument gyro about a non-symmetrical axis is only about one-millionth of the angular momentum about the axis of spin (symmetry).

**Effect of the rotation of the Earth on a gyro**

The settling or rest position of a gyroscopic element is the position the instrument will take up when it should be indicating the direction of the vertical under the influence of no disturbing torques. When this condition is fulfilled there will be zero precessional velocity of the gyro, from which \( \dot{\omega}_p = 0 \), making Eq. (220) become

\[
\vec{M} = \vec{\omega}_E \times \vec{H}
\]

(224)

showing that a torque must be applied to a gyro to keep the spin axis of the gyro from precessing when seen by an observer on the Earth. In XYZ axes the angular momentum is

\[
\vec{H} = iH\sin\theta\sin\phi - jH\sin\theta\cos\phi + kH\cos\theta
\]

(225)

From Eq. (39) the Earth's angular velocity is

\[
\vec{\omega}_E = i\omega_{E}\cos\lambda + k\omega_{E}\sin\lambda
\]

(39)

Putting Eqs. (39) and (225) in Eq. (224) gives

\[
\vec{M} = \begin{vmatrix}
  i & j & k \\
  \omega_{E}\cos\lambda & 0 & \omega_{E}\sin\lambda \\
  H\sin\theta\sin\phi - H\sin\theta\cos\phi & -H\sin\theta\cos\phi & H\cos\theta
\end{vmatrix}
\]

(226)

For an approximately vertical gyro \( \cos\theta \approx 1 \) and \( \sin\theta \approx \theta \approx 0 \), from which Eq. (226) becomes

\[
\vec{M} \approx -jH\omega_{E}\cos\lambda
\]

(227)

Eq. (227) shows that a torque must be directed about an east-west axis in order to keep a gyro fixed on the Earth. The magnitude of the torque depends on the cosine of the latitude (maximum at the equator.
and zero at the poles) and on the angular momentum of the gyro. When the angular momentum is directed vertically upwards the required torque is directed about the eastward axis.

_Gimbal mounting of gyroscopes_

For practical purposes a gyroscopic element is mounted in a system of gimbals in order to make the element free to rotate about a point. The relation between the gimbal axes and the axes xyz fixed in the gyroscopic element are as follows. In the treatment of nutation the axes xyz were taken fixed in the gyro rotor with the z axis being the axis of spin. In all other gyro problems the spin of the rotor itself will be forgotten, the spin appearing only in the angular momentum of the gyro. The gyro will be thought of as a rigid body possessing an angular momentum \( H \) about the spin axis, and the motion of the gyro will be represented by the motion of the vector \( \vec{H} \). The x and y axes will be taken in the customary right hand sense fixed in the gyro case, i.e. non-rotating axes relative to the gyro case. The gyro case is mounted in a two-way gimbal system mounted in a moving base as is shown in Fig. 24. The y axis will be taken along the inner gimbal axis, and the x axis taken by the right hand rule relative to the y and z axes. The outer gimbal axis will be taken along the \( X_A \) axis in the moving base.

The Euler's angle transformation \( \phi' \psi' \theta' \) between the axes xyz in the gyro case and \( X_A Y_A Z_A \) in the moving base carrying the gyro gives that \( \phi' + \psi' = 0 \) since there is no motion about the z axis, making

\[
\begin{align*}
\theta_x &\cong \theta_{XA} & \omega_x &\cong \omega_{XA} \\
\theta_y &\cong \theta_{YA} & \omega_y &\cong \omega_{YA} \\
-\omega_x \dot{\theta}_y + \omega_y \dot{\theta}_x &\cong \omega_{ZA}
\end{align*}
\]

(228)
APPENDIX M

Performance of a System Containing a Gyroscope as an Indicator of the Vertical

A gyroscope by virtue of its angular momentum acts as an averaging device when coupled with a pendulum to form an indicator of the vertical. Two fundamental methods of coupling a gyroscope and a pendulum are:

a. Pendulous gyro in which the pendulous torque acts directly on the gyroscope.

b. Servo-controlled gyro in which the gyroscope is controlled by a pendulum through a servo-mechanism.

Performance of a Pendulous Gyro as an Indicator of the Vertical

A pendulous gyro is a combination of a pendulous element and a gyroscopic element such that the pendulous torque acts directly on the gyroscopic element.

Erection of a Pendulous Gyro

Undamped pendulous gyro

The equation for the torque tending to rotate a pendulous element about its pivot is given in Eq. (54) as

\[ \bar{M} = \bar{a}x(-\bar{F}) \]  

(54)

When the gyro is fixed on the Earth the force equation Eq. (101) becomes

\[ \bar{F} = kmg \]  

(229)

The separation between the pivot and the center of gravity of the gyroscopic element is \( hl_z \) in \( xyz \) axes due to symmetry, which becomes in \( XYZ \) axes

\[ \bar{t} = \bar{t}_z \sin \theta \sin \phi - j \bar{t}_z \sin \theta \cos \phi + hl \cos \theta \]  

(230)

Putting Eqs. (229) and (230) in Eq. (54) gives for the torque acting
on a pendulous gyro

\[
\vec{M} = \begin{vmatrix}
\ell_z \sin \theta \sin \phi & -\ell_z \sin \theta \cos \phi & \ell_z \cos \theta \\
0 & 0 & -mg
\end{vmatrix}
\]

\[
= t \ell_z \sin \theta \cos \phi + j \ell_z \sin \theta \sin \phi
\]

(231)

showing that the pendulous torque is entirely in the horizontal plane and is directed about the line of nodes \( (t \cos + j \sin \phi) \).

The angular momentum \( \vec{H} \) is \( \hbar \vec{H}_z \), or just \( \hbar \vec{H} \), in xyz axes by definition, which becomes in XYZ axes

\[
\vec{H} = t \vec{H}_z \sin \theta \sin \phi - j \vec{H}_z \sin \theta \cos \phi + \hbar \vec{H} \cos \theta
\]

(232)

The relations between the angular momentum, pendulous torque, and the angles of deviation and precession for a pendulous gyro are shown in Fig. 22.

Putting Eqs. (221) and (232) in the left side of Eq. (220) gives

\[
\bar{\omega}_p \times \vec{H} = \begin{vmatrix}
\ell_z \sin \theta \sin \phi & -\ell_z \sin \theta \cos \phi & \ell_z \cos \theta \\
0 & 0 & -mg
\end{vmatrix}
\]

\[
= i (H \cos \theta \sin \phi + H \sin \theta \cos \phi) + j (H \sin \theta \sin \phi - H \cos \theta \cos \phi) -
\]

\[
k (H \sin \theta \cos^2 \phi + H \sin \theta \sin^2 \phi)
\]

(233)

Equating Eq. (233) with its equal, the right side of Eq. (220), assuming that the applied torque \( \vec{M} \) is much larger than the Earth rotation term \( \vec{\omega}_E \times \vec{H} \), gives from Eq. (231)

\[
H \cos \theta \sin \phi + H \sin \theta \cos \phi = mg \ell_z \sin \theta \cos \phi
\]

\[
H \sin \theta \sin \phi - H \cos \theta \cos \phi = mg \ell_z \sin \theta \sin \phi
\]

\[
-\dot{\theta} = 0
\]

(234)

The solution of the third equation of Eq. (234) gives

\[
\dot{\theta} = 0
\]

(235)

Putting Eq. (235) in either of the first two equations of Eq. (234) gives the same solution for both
\[ \dot{\phi} = mg \dot{\theta} / H \] (236)

Now \( \dot{\lambda}_z \) carries an algebraic sign, so \( \dot{\phi} \) has the same sign as \( \dot{\lambda}_z \), i.e. if the center of gravity of the system is below the pivot the precessional velocity has a negative sign, if the angular momentum is directed upwards.

Eq. (236) shows that a pendulous gyro precesses about the vertical with an angular velocity \( \dot{\phi} \) that varies directly as the pendulous torque and inversely as the angular momentum. Eq. (235) shows that the angular momentum vector, i.e. the spin axis of the gyro, makes a constant angle with the vertical. Therefore an undamped pendulous gyro rotates about the vertical in a cone making a constant angle with the vertical. Since \( \dot{\theta} = 0 \) the gyro is not erecting and so cannot be used as an indicator of the vertical.

**Damped pendulous gyro - pedestal mount**

The precessional velocity \( \dot{\phi} \) will give rise to a friction torque about the vertical axis if the gyro is mounted on a rotatable pedestal. Let this friction torque be \( \bar{M}_c \) in XYZ axes. Now it was previously shown that there is no unbalanced torque about the z axis of the gyro. This means that the component \( M_c \cos \theta \) parallel to the z axis is sterile; it cannot affect the gyro, and merely acts back on the base holding the gyro. The component \( M_c \sin \theta \) perpendicular to the z axis and in the vertical plane containing the z axis is the important part of the friction torque. These friction torque components are shown in Fig.23.

In XYZ axes the effective friction torque is

\[ \bar{M}_{ce} = -tM_c \sin \theta \cos \theta \sin \phi + jM_c \sin \theta \cos \phi + \bar{M}_c \sin^2 \theta \] (237)

Adding the friction torque of Eq. (237) to the pendulous torque of Eq. (231) gives the total torque \( \bar{M} \) for the right side of Eq. (220).
Equating the total torque to Eq. (233) gives for the equation of motion

\[ H \cos \theta \sin \varphi + H' \sin \theta \cos \varphi = mg_L \sin \theta \cos \varphi - M_c \sin \theta \cos \theta \sin \varphi \]

\[ H' \sin \theta \sin \varphi - H \cos \theta \cos \varphi = mg_L \sin \theta \sin \varphi + M_c \sin \theta \cos \theta \cos \varphi \] (238)

\[ -H \sin \theta = M_c \sin^2 \theta \]

The third equation of Eq. (238) gives

\[ \dot{\theta} = -\frac{M_c \sin \theta}{H} \] (239)

which solves to, when the angle \( \theta \) is small enough for \( \sin \theta \approx \theta \)

\[ \theta = \theta_0 e^{-\frac{M_c \tau}{H}} \] (240)

Putting Eq. (239) in the first two equations of Eq. (238) gives the same solution for both

\[ \dot{\phi} = mg_L / H \] (241)

where Eq. (241) is treated exactly the same as was Eq. (236).

Eq. (241) leads to the same precession as the undamped case, but Eq. (240) shows that the damped pendulous gyro approaches the vertical (by a decreasing spiral) if \( \overline{M}_c \) and \( \overline{H} \) have the same algebraic sign. If the pivot is above the center of gravity of the system, i.e. if \( \overline{L}_z \) is negative, \( \overline{\phi} \) will have the opposite sign to \( \overline{H} \); and \( \overline{\phi} \) always has the opposite sign to \( \overline{M}_c \). Therefore if the center of gravity is below the pivot the gyro erects. If the center of gravity is above the pivot \( \overline{\phi} \) has the same sign as \( \overline{H} \), and so the gyro increases its separation from the vertical. The last case is the case of a spinning top.

**Damped pendulous gyro – gimbal mount**

Generally a gyro is mounted in gimbals instead of on a pedestal, especially for instrument use. In this case the damping is about the gimbal axes, and for small deviations from the vertical is substantially in a horizontal plane. The damping torque is

\[ \overline{M}_D = -ic\dot{\phi}_x - jc\dot{\phi}_y - k\dot{\phi}_y \sin \theta \cos \varphi \] (242)
Now \( \theta_Y = \theta \cos \varphi \) so \( \dot{\theta}_X = -\varphi \theta \sin \varphi \)
\( \dot{\theta}_Y = \theta \sin \varphi \)
\( \dot{\theta}_Y = \dot{\varphi} \theta \cos \varphi \)

which put in Eq. (242) gives

\[
M_D = I \varphi \theta \sin \varphi - J \varphi \theta \cos \varphi - 2 \varphi \theta \sin \varphi \cos \varphi \quad (243)
\]

Eq. (243) gives a component \( \varphi \theta \) perpendicular to the line of nodes and in the horizontal plane, and a component \( -\varphi \theta \sin \theta \cos^2 \varphi \) about the vertical, as shown in Fig. 25. Adding these two components together and discarding the sterile part parallel to the angular momentum leaves the effective damping torque perpendicular to the angular momentum and in the vertical plane containing the angular momentum.

\[
M_{De} = \varphi \theta \cos \theta + \varphi \theta \sin^2 \theta \cos^2 \theta \approx \varphi \theta \quad (244)
\]

for small angles where \( \cos \theta \approx 1 \) and \( \sin \theta \approx \theta \). Resolving Eq. (244) into its XYZ components gives

\[
\tilde{M}_{De} = I \varphi \theta \sin \varphi - J \varphi \theta \cos \varphi - 2 \varphi \theta \sin \varphi \quad (245)
\]

Adding the friction torque of Eq. (245) to the pendulous torque of Eq. (231) gives the total torque \( \tilde{M} \) for the right side of Eq. (220). Equating the total torque to Eq. (233) gives for the equation of motion

\[
\begin{align*}
\dot{\theta} \cos \theta \sin \varphi + \dot{\varphi} \theta \sin \theta \cos \varphi &= mg \ell_z \sin \theta \cos \varphi + \varphi \theta \sin \varphi \\
\dot{\varphi} \sin \sin \varphi - \dot{\theta} \cos \theta \cos \varphi &= mg \ell_z \sin \sin \varphi - \varphi \theta \cos \varphi \\
-\dot{\varphi} \sin \theta &= -2 \varphi \theta \sin \varphi \quad (246)
\end{align*}
\]

The third equation of Eq. (246) gives for small angles

\[
\dot{\theta} = \varphi \theta / H \quad (247)
\]

which solves to

\[
\theta = \varphi \theta \varphi / H \quad (248)
\]

Putting Eq. (247) in the first two equations of Eq. (246) gives the same solution for both

\[
\varphi = mg \ell_z / H \quad (249)
\]
where Eq. (249) is treated exactly the same as were Eq. (236) and (241).

Eq. (249) leads to the same precession as the undamped case and the pedestal-mounted damped case, but Eq. (248) shows that the damped gyro in gimbals approaches the vertical (by a decreasing spiral) if $\bar{r}$ and $\bar{H}$ have opposite signs. The case is true only if the center of gravity of the system is below the pivot.

**Effect of Coulomb friction**

The damping torque due to Coulomb (dry or rubbing) friction cannot be treated analytically since the torque is constant in magnitude and always opposes the direction of the velocity. Coulomb friction can be conveniently handled in approximation, however, by an artifice based on energy dissipation per cycle(3)(28).

For viscous (velocity) damping the energy loss per cycle is

$$W_v = \int_0^{2\pi} -c\delta d\theta \quad (250)$$

If the motion of $\theta$ is sinusoidal, and the change in the magnitude of $\theta$ per cycle is small, then

$$\theta = \theta_0 \cos \omega t \quad \text{and} \quad d\theta = -\omega \theta_0 \sin \omega t$$

So Eq. (250) becomes

$$W_v = -c\theta_0^2 \int_0^{2\pi} \sin^2 \omega t \, dt(\omega t)$$

$$= -\pi c \omega \theta_0^2 \quad (251)$$

For Coulomb damping the loss per cycle is

$$W_c = 4\int_0^{\pi/2} M_c d\theta \quad (252)$$

where the integral is carried out for a quarter cycle and multiplied by four. For a sinusoidal motion of $\theta$ Eq. (252) becomes

$$W_c = 4M_c \int_0^{\pi/2} -\theta_0 \sin \omega t \, dt(\omega t) = 4M_c \theta_0 \cos \omega t \bigg|_0^{\pi/2}$$

$$= -4M_c \theta_0 \quad (253)$$

Letting Eq. (253) equal Eq. (251) and defining the coefficient $c$
in Eq. (251) as $c_e$ the effective damping coefficient for Coulomb damping gives
\[ c_e = \frac{4M_o}{\pi \omega_0} \] (254)
The coefficient $c_e$ can now be used for $c$ in Eq. (248) for the erection of a pendulous gyro in gimbals damped by Coulomb friction.

A viscous-damped gyro erects completely to the vertical. A Coulomb-damped gyro erects to an indeterminate position somewhere in a cone about the vertical having a half-angle
\[ \theta_F = \frac{M_c}{mg\ell_z} \] (255)
i.e. until the pendulous torque equals the friction torque.

**Effect of Earth's Rotation on a Pendulous Gyro**

In Eq. (227) it was shown that a gyro must have a torque of
\[ \bar{M} = -jH\omega I_E \cos \lambda \] (227)
exerted on it in order that the gyro not precess on the Earth due to the Earth's rotation. For a pendulous gyro this necessary torque can only be furnished by the pendulous torque produced when the gyro tilts from the vertical (damping torques are forgotten since they require motion to exist, and this is a static problem). The pendulous torque is given in Eq. (231) as
\[ \bar{M} = \text{img} \ell_z \sin \theta \cos \phi + jmg \ell_z \sin \theta \sin \phi \] (231)
Equating Eqs. (227) and (231) gives
\[ mg \ell_z \sin \theta \cos \phi = 0 \] (256)
\[ mg \ell_z \sin \theta \sin \phi = -H\omega I_E \cos \lambda \]
and for small angles where $\sin \theta \approx \theta$, remembering that $\theta_X = \theta \cos \phi$ and $\theta_Y = \theta \sin \phi$ makes Eq. (256) become
\[ \theta_X = 0 \]
\[ \theta_Y = -\omega I_E H \cos \lambda / mg \ell_z = -\omega I_E T \cos \lambda / 2\pi \] (257)
where $T_p$ is the period of precession required for the gyro to complete one complete revolution about the vertical.

It follows from Eq. (257) that the vertical indicated by the rest position of a pendulous gyro will deviate from the true vertical by the angle $\omega T_p \cos \omega / 2\pi$. This deviation is to the north for the top of a gyro having its angular momentum directed upwards and with its center of gravity below the pivot. The magnitude of the indication error depends on the cosine of the latitude (maximum at the equator and zero at the poles) and on the period of precession of the gyro. Eq. (257) is shown in Fig. 57 where the logarithm of the angular deviation error is plotted against the latitude for different values of the period of precession. It is seen that at $45^\circ$ latitude the error for a gyro having a period of precession of 100 minutes is about 200 minutes of arc, or about three degrees.

**Response of a Pendulous Gyro to Typical Airplane Maneuvers**

In all studies of the deviation from the vertical of a pendulous gyro it will be assumed that the deviation of the gyro from the vertical is sufficiently small that the effect of the gravity torque is negligibly small in comparison with the torques due to horizontal disturbing accelerations. This means that the angular momentum of the gyro in XYZ axes is given by

$$\mathbf{H} \simeq \kappa \mathbf{H} \tag{258}$$

Combining Eqs. (221) and (258) gives

$$\dot{\mathbf{P}} \times \mathbf{H} = \begin{vmatrix} t & \hat{\theta} \cos \varphi & \hat{\theta} \sin \varphi \\ \hat{\varphi} \cos \varphi & \hat{\varphi} \sin \varphi & \hat{\varphi} \\ 0 & 0 & H \end{vmatrix} = t\hat{\Theta} \sin \varphi - jH \hat{\cos \varphi} \tag{259}$$

Now $\dot{\theta}_x = \hat{\theta} \cos \varphi$ and $\dot{\theta}_y = \hat{\theta} \sin \varphi$, so Eq. (259) becomes

$$\dot{\mathbf{P}} \times \mathbf{H} = t\hat{\Theta}_y - jH \hat{\Theta}_x \tag{260}$$
\[ \dot{\theta}_y = \frac{\omega_1 \xi \cos \lambda}{2\pi/T_p} \]

\[ \dot{\theta}_y = \text{AMPLITUDE OF ANGULAR DEVIATION FROM THE VERTICAL} \]

\[ \text{OF AN APPROXIMATELY VERTICAL PENDULOUS GYROSCOPE} \]

\[ \text{DUE TO THE DAILY ROTATION OF THE EARTH AS A} \]

\[ \text{FUNCTION OF THE LATITUDE AND OF THE PERIOD} \]

\[ \text{OF PRECESSION OF THE GYROSCOPE} \]

\[ \text{(SOUTH)} \quad \dot{\lambda} = \text{LATITUDE} \quad \text{(NORTH)} \]

Fig. 57
It is also assumed that the damping torques are negligibly small in comparison with the disturbing torques. The effect of the rotation of the Earth is neglected in the analysis, but it must be remembered that the rotation of the Earth affects the rest position of the gyro. By equating Eq. (260) with the disturbing torques as the right side of Eq. (220) the angular deviation of a pendulous gyro from the vertical can be determined.

*A perfectly banked turn*

The condition existing and the forces present in a perfectly banked turn are given in Appendix I. The forces acting are given in Eq. (120) as

\[
\vec{F} = im\theta_A \sin \phi_A t - jm\theta_A \cos \phi_A t + kmg
\]  

(120)

If the gyro remains approximately vertical the separation between the pivot and the center of gravity in XYZ axes is

\[
\ell = -k\ell_z
\]  

(261)

when the center of gravity is below the pivot, a condition which was shown to be necessary.

Putting Eqs. (120) and (261) in Eq. (54) for the torque acting gives

\[
\vec{M} = \vec{r}_x (\vec{F}) = \begin{vmatrix} i & j & k \\ 0 & 0 & -\ell_z \\ -m\theta_A \sin \phi_A t & m\theta_A \cos \phi_A t & -mg \end{vmatrix} = im\ell_z \theta_A \cos \phi_A t + jm\ell_z \theta_A \sin \phi_A t
\]

(262)

Equating Eqs. (260) and (262) gives

\[
\begin{align*}
H_0 Y &= m\ell_z \theta_A \cos \phi_A t \\
-H_0 X &= m\ell_z \theta_A \sin \phi_A t
\end{align*}
\]

(263)

which integrates to, letting \( \theta_{XO} = \theta_{YO} = 0 \)

\[
\begin{align*}
\theta_X &= \left( m\ell_z \theta_A / H_0 \right) \cos \phi_A t - m\ell_z / H_0 \\
\theta_Y &= \left( m\ell_z \theta_A / H_0 \right) \sin \phi_A t
\end{align*}
\]

(264)
Now let $\beta = m g \ell_z / H \theta_A$ then Eq. (264) becomes

$$\frac{\theta_X}{\theta_A} = \left(1/\beta\right) (\cos \phi_A t - 1)$$

$$\frac{\theta_Y}{\theta_A} = \left(1/\beta\right) \sin \phi_A t$$

If the horizontal XY plane becomes the complex plane in analysis then

$$\tilde{\theta} = \theta_X + j \theta_Y$$

$$= \left(\theta_A / \beta\right) (\cos \phi_A t - 1 + j \sin \phi_A t)$$

from Eq. (265)

$$= \left(\theta_A / \beta\right) (e^{j \phi_A t} - 1)$$

The maximum value of Eq. (266) is

$$\frac{\theta_{\max}}{\theta_A} = 2/\beta$$

Eq. (267) is shown in Fig. 26 where the logarithm of the ratio of the maximum deviation from the vertical to the angle of bank is plotted against the logarithm of the ratio of the period of precession of the gyro to the time required to make a 360° turn. It is seen in the figure that the deviation decreases linearly with increasing $\beta$, having a value of 2% of the angle of bank when the period of precession is one-hundred times the time required to make a 360° turn.

For the standard blind flying turn of 360° in 2 minutes at 120 mph it will be necessary for a pendulous gyro to have a period of precession of about 200 minutes. Now $T_p = 2 \pi H / m g \ell_z$; assume a gyro rotor 2 inches in diameter rotating at 12000 rpm, let the mass of the gyro case and gimbal system equal the mass of the rotor. Then

$$k \equiv 0.7 \text{ in}$$

$$\omega_z = 12000 \text{ rpm} = 400\pi \text{ rad/sec}$$

$$g = 386 \text{ in/sec}^2$$

$$T_p = 200 \text{ min} = 12000 \text{ sec}$$
With these data the pivot-center of gravity separation \( \ell_z \) is
\[
\ell_z = \frac{2\pi H}{2mgT_p} = \frac{n\kappa \omega_x}{gT_p} = \pi \cdot 0.5 \cdot \frac{400\pi}{386 \cdot 12000} = 0.0043 \text{ in}
\]

Eq. (268) shows that for a pendulous gyro to give a deviation from the vertical of about 2% of the angle of bank in a perfectly banked 2 minute turn the center of gravity of the gyro system must be about four thousandths of an inch below the pivot, when the gyro rotor is about two inches in diameter and is spinning at twelve thousand revolutions per minute. Such dimensions require excellent machine work, but are within the province of a skilled worker; which is more than can be said for the requisite three ten-millionths of an inch in a pendulous element alone.

**Perfectly banked S-turns**

The conditions existing and the forces present in perfectly banked S-turns about a straight line of flight are given in Appendix I. The forces acting are given in Eq. (161) as
\[
\bar{F} = -jm\alpha \left\{ \left[ \frac{1}{\alpha X_{AP}} \right] \cos 2\pi n t / T_\phi + \left[ 1 - 4\pi^2 \frac{Z_{AP}}{gT^2_\phi} \right] \sin 2\pi n t / T_\phi \right\} + \kappa m g
\]

Putting Eqs. (161) and (261) in Eq. (54) for the torque acting gives
\[
\bar{M} = \begin{vmatrix}
 1 & j & k \\
 0 & 0 & -\ell_z \\
 0 & -F_Y & -mg
\end{vmatrix} = -i\ell_z F_Y
\]
\[
= im\ell_z \alpha \left\{ \left[ \frac{1}{\alpha X_{AP}} \right] \cos 2\pi n t / T_\phi + \left[ 1 - 4\pi^2 \frac{Z_{AP}}{gT^2_\phi} \right] \sin 2\pi n t / T_\phi \right\}
\]

Using the non-dimensional parameters introduced in Appendix I reduces Eq. (269) to
\[
\bar{M} = im\ell_z \alpha \omega_{XZ} \sin \left[ (2\pi / T_\phi) + \omega_1 \right]
\]
for the case where the precessional period of the gyro is constant, and
$$M = l_{\text{gz}} \theta_{Aa} u_{XZf} \sin[(2\pi t/T_\phi) + \psi_f]$$  \hspace{1cm} (271)$$

for the case where the forcing period of yaw is constant.

Equating Eqs. (260) and (270) gives

$$H \dot{\theta}_Y = m_{\text{gz}} \theta_{Aa} u_{XZf} \sin[(2\pi t/T_\phi) + \psi_f]$$

$$-H \dot{\theta}_X = 0$$  \hspace{1cm} (272)$$

which integrates to

$$\theta_Y = -\left(m_{\text{gz}} T_\phi/2\pi H\right) \theta_{Aa} u_{XZf} \cos[(2\pi t/T_\phi) + \psi_f]$$  \hspace{1cm} (273)$$

Now, $$2\pi m_{\text{gz}} T_\phi/HT_\phi = \beta$$, so Eq. (273) becomes

$$\frac{\dot{\theta}_Y}{\theta_{Aa}} = -\left(u_{XZf}/\beta\right) \cos[(2\pi t/T_\phi) + \psi_f]$$  \hspace{1cm} (274)$$

Equating Eqs. (260) and (271) gives

$$H \dot{\theta}_Y = m_{\text{gz}} \theta_{Aa} u_{XZf} \sin[(2\pi t/T_\phi) + \psi_f]$$

$$-H \dot{\theta}_X = 0$$  \hspace{1cm} (275)$$

which integrates to

$$\theta_Y = -\left(m_{\text{gz}} T_\phi/2\pi H\right) \theta_{Aa} u_{XZf} \cos[(2\pi t/T_\phi) + \psi_f]$$  \hspace{1cm} (276)$$

using the above-given values of \( \beta \) changes Eq. (276) to

$$\frac{\dot{\theta}_Y}{\theta_{Aa}} = -\left(u_{XZf}/\beta\right) \cos[(2\pi t/T_\phi) + \psi_f]$$  \hspace{1cm} (277)$$

Eqs. (274) and (277) are shown in Fig. 58 where the logarithm of the ratio of the maximum deviation of the gyro from the vertical to the maximum angle of roll is plotted against the logarithm of the ratio of the period of precession of the gyro to the period of yaw when the gyro is mounted at the center of gravity of the airplane. It is seen in the figure that the maximum deviation from the vertical equals 1% of the maximum angle of roll when the period of precession of the gyro equals 100 times the period of yaw of the airplane. For a period of yaw of five seconds the period of precession of the gyro would then have to have a minimum value of five hundred seconds. Which
STEADY STATE ANGULAR DEVIATION FROM THE VERTICAL OF A PENDULOUS GYROSCOPE CARRIED BY A MOVING BASE WHICH EXECUTES A SINEODAL OSCILLATION ABOUT ITS AXIS OF ROLL - INSTRUMENT MOUNTED AT CENTER OF GRAVITY OF THE AIRPLANE.

\[ \theta_{Ya} = \frac{\text{MAX. DEVIATION OF PENDULOUS GYRO FROM THE VERTICAL}}{\theta_{Ax}} \]

\[ \theta_{Ax} = \text{MAXIMUM ANGLE OF ROLL} \]

\[ \theta_{Ya} = \frac{1}{2} \cdot \frac{2nH/mg}{T_{o}^{2}} \]

\[ T_{o} = \frac{\sqrt{2} \cdot \theta_{Ax}}{\theta_{Ya}} \]
gives for the separation of the center of gravity from the pivot, for
the gyro used in Eq. (268)

\[ l = \pi \cdot 0.5 \cdot 400 \pi / 386 \cdot 500 = 0.103 \text{ in} \quad (278) \]

which is a very feasible dimension from the construction point of view.
The effect of placing the gyro in different positions about the air-
plane is similar to the effect of changing the position of a pendulous
element, which was treated at length in Appendix I.

**Summary of the Pendulous Gyro**

The following is a summary of the principal characteristics of a
pendulous gyro used as an indicator of the vertical

1. A pendulous gyro requires damping, either viscous or
Coulomb, for erection.

2. A pendulous gyro precesses when erecting; the precession
is necessary to obtain the damping.

3. The rate of erection of a pendulous gyro is generally
small, since
   a. The damping is small
   b. The period of precession under the influence of gravity
   is long, thereby causing a small value of \( \dot{\phi} \) in the damping term.
   c. Most gimbal systems have Coulomb friction, thereby
   causing the rest position of the gyro to lie in an indeterminate zone
   about the vertical.

The rate of erection of a pendulous gyro is given by

\[ \dot{\theta} = mg \cdot c \cdot \theta / H^2 \quad (247) \]

4. The rate of deviation from the vertical of a pendulous
gyro is independent of the damping and is not necessarily small. The
rate of deviation is given by
\[ \dot{\theta} = \frac{mg\zeta a}{gH} \]  \hspace{1cm} (263)
where \( a \) = disturbing horizontal acceleration.

5. A pendulous gyro must set in a rest position that deviates from the vertical in order to produce the pendulous torque necessary to keep the gyroscope stationary relative to the Earth in the presence of the Earth's daily rotation.

\[ \theta_Y = -\omega_{hE} T_p \cos \lambda / 2\pi \]  \hspace{1cm} (257)
which may amount to a few degrees in a long period gyro.

**Performance of a Servo-Controlled Gyro as an Indicator of the Vertical**

It has been shown earlier that any self-contained indicator of the vertical must contain a pendulous element somewhere in the system in order to identify the direction of gravity. Pendulous elements and pendulous gyros use the whole system itself as the pendulous indicator. Both methods were shown to work, but to contain inherent difficulties. However it was shown that the increased inertia of the pendulous gyro without an accompanying increase of mass made the pendulous gyro definitely a much more practical indicator of the vertical than a pendulous element alone. A logical step would then be to retain the slow rate-of-response characteristics of the gyroscopic element, and let the pendulous element control a servo-mechanism which in turn acts on the gyroscopic element.

**Erection of a Servo-Controlled Gyro**

The most efficient type of servo-controlled gyro in practice, and also the simplest type to analyze is the gyro whose erection torque lies in the plane of the gyro rotor. In xyz axes the erection torque
then is
\[ M = lM_x + jM_y \] (278)
which transform in XYZ axes to
\[ M = -IM\cos\theta\sin\phi + JM\cos\theta\cos\phi + kM\sin\theta \] (279)
since the torque lies in the zZ plane, i.e. perpendicular to the line of nodes.

Equating Eq. (279) to Eq. (233) gives for the equation of motion
\[ H\cos\theta\sin\phi + H\sin\theta\cos\phi = -M\cos\theta\sin\phi \]
\[ H\sin\theta\sin\phi - H\cos\theta\cos\phi = M\cos\theta\cos\phi \]
\[ -H\theta\sin\theta = M\sin\theta \] (280)
Solving the third equation of Eq. (280) gives
\[ \theta = -M/H \] (281)
Putting Eq. (281) in the first two equations of Eq. (280) gives the same solution
\[ \phi = 0 \] (282)
Eq. (282) shows that such a system leads to a radial erection, i.e. there is no precession of the gyro about the vertical. Eq. (281) shows that the rate of erection depends directly on the magnitude of the erecting torque and inversely as the magnitude of the angular momentum of the gyroscope. The relationship between the erecting torque and the tilt angle of the gyro generally falls into two regimes:

1. The torque is proportional to the tilt angle. This regime generally obtains in the region of the vertical and exists up to some critical angle at which the torque retains a constant magnitude.

2. The torque is constant in magnitude. This regime generally obtains for tilt angles greater than critical. In some methods of erection this regime is not present.
For regime 1. in which the torque is proportional to the tilt

Eq. (281) becomes

$$\dot{\theta}_{pro} = -\frac{M_o}{H} \frac{\theta}{\theta_c}$$  \hspace{1cm} (283)

for $0 < \theta < \theta_c$ where $\theta_c$ is the critical angle at which the torque attains its full value of $M_o$. Integration of Eq. (283) gives for the deviation of the gyro from the vertical in the proportional regime

$$\theta = \theta_0 e^{-\frac{M_o}{H \theta_c} t}$$  \hspace{1cm} (284)

Eq. (284) shows that a gyro approaches the vertical exponentially in the proportional regime.

For regime 2. in which the torque is constant at a value $M$ Eq. (281) remains as is. Integration of Eq. (281) gives for the deviation of the gyro from the vertical in the constant regime

$$\theta = \dot{\theta}_0 e^{-\frac{M}{H} t}$$  \hspace{1cm} (285)

or, using the identity of Eq. (281)

$$\theta = \theta_0 - \theta_{con} t$$  \hspace{1cm} (286)

where $\theta_{con}$ is the rate of erection of the gyro. Eq. (286) shows that a gyro erects at a constant rate in the constant torque regime (which is what is expected).

**Response of Servo-Controlled Gyro to Accelerations**

When a servo-controlled gyro is subject to accelerations the pendulum will operate to force the gyro to leave the vertical, at the same rate as which it erects. In many practical cases, e.g. the Sperry Artificial Horizon, the critical angle above which the erecting torque is substantially constant is about two degrees. Now the tangent of two degrees is thirty-five thousandths; which means that the critical angle is reached by the pendulum when the horizontal acceleration has a magnitude of only three and one-half percent of the gravity. Accelera-
tions greater than three and one-half percent of gravity are quite frequently encountered in flight. Accordingly it will be assumed that the gyro torque is in the constant regime during maneuvers.

For a gyro having a rate of erection of eight degrees per minute, and originally vertical, i.e. $\theta_0 = 0$, the maximum deviation from the vertical in a maneuver having a period $T_f$ of eight seconds will be, from Eq. (286)

$$\theta_{\text{max}} = \frac{\theta T_f}{4} = \frac{8}{60} \times \frac{8}{4} = 4/15$$

or sixteen minutes of arc.

A rate of erection of eight degrees per minute is typical of those used in practice, e.g. the Sperry Artificial Horizon. Eq. (287) shows that such an erected gyro holds the vertical to a fraction of a degree during maneuvers of a few seconds duration. When extreme accuracy is desired in holding the vertical the rate of erection is made slower, with an optional faster rate of erection for getting the gyro vertical initially or for returning the gyro to the vertical after the gyro has hit its stops and so lost its orientation relative to the vertical.

A detailed treatment of the behavior of the Sperry Horizon during turns is available in the literature(3)(4)(5). Both theoretical and experimental data are discussed in these articles.

**Effect of Earth's Rotation on a Servo-Controlled Gyro**

In Eq. (227) it was shown that a gyro must have a torque of

$$\vec{M} = -jH_\omega\vec{E}_E\cos \lambda$$

applied to it in order that the gyro not precess on the Earth due to the Earth's rotation. The torque in a servo-controlled gyro is seen from Eq. (281) to be

$$M = -H\dot{\theta}$$
Equating the torques in Eqs. (227) and (281), and letting the torque in Eq. (281) be directed about the Y-axis (east-west), i.e. \( \bar{M} = -j\dot{\theta} \), gives

\[ \dot{\theta} = \omega_\text{E} \cos \lambda \]  

Eq. (288) shows the rate of erection required to overcome the effect of the Earth's rotation. In the proportional regime the rate of erection is given from Eq. (283) as

\[ \dot{\theta} = \left( \frac{\theta}{\theta_\text{con}} \right) \dot{\theta}_\text{con} \]  

where \( \dot{\theta}_\text{con} \) is the constant regime rate of erection. Combining Eqs. (288) and (289) gives the gyro tilt required to produce the torque that offsets the effect of the Earth's rotation

\[ \theta = \theta_\text{c} \left( \frac{\omega_\text{E} \cos \lambda}{\dot{\theta}_\text{con}} \right) \]  

For a typical servo-controlled gyro, e.g. the Sperry Artificial Horizon, the critical angle is two degrees and the constant rate of erection eight degrees per minute. With this data Eq. (290) becomes, for the equator, i.e. \( \lambda = 0^\circ \)

\[ \theta = \frac{2 \cdot (1/4) \cdot 8}{\theta_\text{c}} \leq 4 \text{ min of arc} \]  

Eq. (291) shows that a servo-controlled gyro need tilt only a few minutes of arc to overcome the effect of the Earth's rotation. Such a tilt is known as the latitude error.

It is thus seen that the servo-controlled gyro is a satisfactory practical indicator of the vertical. The flexibility of choice of erection rate allows the designer and operator to adapt the system to particular problems of indicating the vertical.

A turn about the vertical is the severest test of an indicator of the vertical due to the length of the time intervals involved, i.e. of the order of minutes whereas most other maneuvers are of the order of
seconds. It is now proposed to remove the erection during such maneuvers as turns. Removal of erecting means is most feasible in the case of the servo-controlled gyro.

**Summary of the Servo-Controlled Gyro**

The following is a summary of the principal characteristics of a servo-controlled gyro used as an indicator of the vertical.

1. A servo-controlled gyro can be designed to erect in a straight line. The total torque producing erection can be designed to have no sterile nor precessing components, i.e. the torque is one-hundred percent available for erecting the gyro.

2. The rate of erection of a servo-controlled gyro is the same as its rate of leaving the vertical.

3. The latitude error of a servo-controlled gyro may be made of the order of only a few minutes of arc.
APPENDIX IV

Effects of the Coriolis Acceleration

It was shown in Appendix D that the acceleration of Coriolis, caused by the interaction of the daily rotation of the Earth with the velocity of a body moving over the Earth, gives rise to a horizontal acceleration that has the effect of deviating the apparent vertical as seen from a moving base from the true vertical. The practical effects to aircraft of this shift of the apparent vertical lie principally in the fields of celestial navigation.

Effect on Navigation

Since the effect of the Coriolis acceleration is to deviate the direction of the apparent vertical to the right (in the northern hemisphere) it follows that celestial navigation sights taken with a bubble sextant from an airplane will be in error, since the bubble acts as a pendulum. It is seen in Fig. 30 that the observed zenith distance $\theta$ is less than the true zenith distance $\theta_T$ when the celestial body is to the left of the heading of the airplane. This effect is zero dead ahead or astern, and $\theta$ is greater than $\theta_T$ when the celestial body is to the right of the course.

The magnitude of the deviation of the apparent vertical from the true vertical is given by the ratio of the horizontal Coriolis acceleration to the acceleration of gravity. Using Eq. (49) for the horizontal Coriolis acceleration gives for the angular deviation of the apparent vertical

$$\theta_C = 2v_A \omega E \sin \lambda / g$$  \hspace{1cm} (292)

Eq. (292) is plotted in Fig. 4 where it is seen that the deviation
\( \theta_c \) is about 5 minutes of arc for an airplane moving 300 mph in the latitude of 45°. Such a deviation can give rise to an error in the airplane's position, determined from celestial observations, of several miles.

The effect of the angular difference between the azimuth of the celestial body observed and the heading of the airplane is such as to introduce the sine of this angular difference to the equation for the difference between the observed zenith distance and the true zenith distance. The angular difference between the celestial body azimuth and airplane course shown in Fig. 31 is expressed as the heading \( H \) of the airplane minus the azimuth \( S \) of the celestial body, both angles measured from north through east.

On the basis of the above treatment the complete expression for the difference between the observed and true zenith distances as measured by a bubble sextant from an airplane moving over the Earth is

\[
\theta_{cc} = (2v_{A\phi E}/g) \sin \lambda \sin (H-S) = \theta_T - \theta
\]  

(293)

It is seen from Eq. (293) that \( \theta_{cc} \) acts as a correction to be added to \( \theta \) in order to get \( \theta_T \), and is positive in the northern hemisphere when the celestial body is to the left of the line of flight.

The magnitudes of the necessary corrections \( \theta_{cc} \) as a function of the speed of the airplane, the angle between the airplane heading and the celestial body azimuth, and the latitude have been worked out completely by Stewart(26).